About Pearson

Pearson is the world's learning company, with presence across 70 countries worldwide. Our unique insights and world-class expertise comes from a long history of working closely with renowned teachers, authors and thought leaders, as a result of which, we have emerged as the preferred choice for millions of teachers and learners across the world.

We believe learning opens up opportunities, creates fulfilling careers and hence better lives. We hence collaborate with the best of minds to deliver you class-leading products, spread across the Higher Education and K12 spectrum.

Superior learning experience and improved outcomes are at the heart of everything we do. This product is the result of one such effort.

Your feedback plays a critical role in the evolution of our products and you can contact us - reachus@pearson.com. *We look forward to it.*

1 0 0
0 1 0
0 0 1
2+2=4
3,14
I ♥
√MATH
(X+Y)²
y
x

JEE
ADVANCED

MATHEMATICS

Calculus

Pearson

Om Sharma

I dedicate this book to my respected parents
Mr. Giriraj Prasad and Sampati Sharma
and family for their love, endless support,
encouragement and sacrifices

Although the author and publisher have made every effort to ensure that the information in this book was correct at the time of editing and printing, the author and publisher do not assume and hereby disclaim any liability to any party for any loss or damage arising out of the use of this book caused by errors or omissions, whether such errors or omissions result from negligence, accident or any other cause. Further, names, pictures, images, characters, businesses, places, events and incidents are either the products of the author's imagination or used in a fictitious manner. Any resemblance to actual persons, living or dead or actual events is purely coincidental and do not intend to hurt sentiments of any individual, community, sect or religion.

In case of binding mistake, misprints or missing pages etc., the publisher's entire liability and your exclusive remedy is replacement of this book within reasonable time of purchase by similar edition/reprint of the book.

Copyright © 2023 Pearson India Education Services Pvt. Ltd

All rights reserved. This book is sold subject to the condition that it shall not, by way of trade or otherwise, be lent, resold, hired out, or otherwise circulated without the publisher's prior written consent in any form of binding or cover other than that in which it is published and without a similar condition including this condition being imposed on the subsequent purchaser and without limiting the rights under copyright reserved above, no part of this publication may be reproduced, stored in or introduced into a retrieval system, or transmitted in any form or by any means (electronic, mechanical, photocopying, recording or otherwise), without the prior written permission of both the copyright owner and the publisher of this book.

ISBN 978-93-544-9997-5

First Impression

Published by Pearson India Education Services Pvt. Ltd, CIN: U72200TN2005PTC057128.

Head Office:15th Floor, Tower–B, World Trade Tower, Plot No. 1, Block–C, Sector–16,
Noida 201 301, Uttar Pradesh, India.
Registered Office: 7th Floor, SDB2, ODC 7, 8 & 9, Survey No. 01 ELCOT IT/ ITES-SEZ,
Sholinganallur, Chennai – 600 119, Tamil Nadu, India.
Phone: 044-66540100
Website: in.pearson.com, Email: companysecretary.india@pearson.com

Compositor: SRS Global, Puducherry
Printer:

CONTENTS

Chapter Insight xi
Preface xv
About the Author xvi

CHAPTER 1 SET AND RELATION 1.1
Set 1.1
Representation of a Set 1.1
Notations of Standard Sets of Numbers 1.1
Cardinal Number of Set 1.1
Certain Useful Symbols 1.2
Types of Sets 1.2
Universal Set, Subset, Superset and Power Set 1.3
Venn Diagrams 1.5
Operations on Sets 1.5
Laws of Algebra of Sets 1.6
Important Results on Sets 1.7
Cartesian Product $A \times B$ 1.9
Relation 1.10
Inverse Relation 1.10
Types of Relations 1.11
Congruence Modulo (m) 1.13
Solved Problems *1.15*
Practice Exercises *1.17*
  Single Correct Choice Type Questions *1.17*
  Integer/Numerical Answer Type Questions *1.19*
  Archive: JEE Main *1.20*
Answer Key–Practice Exercises *1.22*

CHAPTER 2 FUNCTION AND ITS GRAPH 2.1
Function 2.1
Vertical Line Test to Identify a Function 2.1
Types of Functions 2.2
Modulus Function (Absolute Value Function) 2.4
Exponential Function 2.5
Logarithmic Function 2.5
Greatest Integer Function 2.6
Fractional Part Function 2.7
Signum Function 2.11
Graph Transformation 2.14
Graph Transformation Rules 2.14
Finding Domain and Range 2.25
Range of Function 2.26
Equal/Identical Function 2.35

Types of Function . . . 2.36
Periodic Function . . . 2.39
Classification of Function . . . 2.44
Composite Function . . . 2.50
Inverse Function ($f^{-1}(x)$) . . . 2.56
Standard Functional Equations . . . 2.60
Solved Problems . . . *2.63*
Practice Exercises . . . *2.68*
    Single Correct Choice Type Questions . . . *2.68*
    Multiple Correct Choice Type Questions . . . *2.75*
    Linked Comprehension Type Questions . . . *2.79*
    Matrix Match/Column Match Type Questions . . . *2.82*
    Integer/Numerical Answer Type Questions . . . *2.84*
    Archive: JEE Main . . . *2.86*
    Archive: JEE Advanced . . . *2.88*
Answer Key–Practice Exercises . . . *2.89*

CHAPTER 3 LIMIT OF FUNCTION . . . 3.1

Indeterminate Form . . . 3.4
Solving Limits of $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$ Indeterminate Form . . . 3.5
Algebra of Limits . . . 3.6
Solving Limits Using Substitution . . . 3.7
Standard Limits . . . 3.8
L-Hospital's Rule . . . 3.13
Series Expansion Formula . . . 3.15
Solving Limit of $\infty - \infty$ Indeterminate Form . . . 3.19
Solving Limit of $\frac{\infty}{\infty}$ Form . . . 3.20
Solving Limit of 1^∞ Indeterminate Form . . . 3.22
Solving Limit of 0^0 and ∞^0 form . . . 3.26
Limit of a Composite Function . . . 3.27
Comparison of Polynomials, Exponential, Logarithmic Function at Infinity . . . 3.30
Sandwich Theorem/Squeeze Theorem . . . 3.30
Limits in Two Variables . . . 3.34
Solved Problems . . . *3.37*
Practice Exercises . . . *3.41*
    Single Correct Choice Type Questions . . . *3.41*
    Multiple Correct Choice Type Questions . . . *3.47*
    Linked Comprehension Type Questions . . . *3.49*
    Matrix Match/Column Match Type Questions . . . *3.51*
    Integer/Numerical Answer Type Questions . . . *3.52*
    Archive: JEE Main . . . *3.54*
    Archive: JEE Advanced . . . *3.55*
Answer Key–Practice Exercises . . . *3.57*

CHAPTER 4 METHOD OF DIFFERENTIATION . . . 4.1

Differentiation by First Principle . . . 4.1
Differentiation of Standard Functions . . . 4.2
Algebra of Differentiation . . . 4.2
Derivatives of Composite Function: [Chain Rule] . . . 4.4
Differentiation of Implicit Functions . . . 4.8
Differentiation Using Logarithm . . . 4.9

Differentiation of Infinite Series . . . 4.11
Differentiation of Parametric Functions . . . 4.12
Differentiation of a Function with Respect to Another Function . . . 4.13
Differentiation of Determinants . . . 4.14
Differentiation of Inverse Function . . . 4.15
Differentiation of Inverse Trigonometric Functions . . . 4.16
Higher Order Derivatives . . . 4.19
Solved Problems . . . *4.23*
Practice Exercises . . . *4.28*
    Single Correct Choice Type Questions . . . *4.28*
    Multiple Correct Choice Type Questions . . . *4.32*
    Linked Comprehension Type Questions . . . *4.35*
    Matrix Match/Column Match Type Questions . . . *4.36*
    Integer/Numerical Answer Type Questions . . . *4.38*
    Archive: JEE Main . . . *4.39*
    Archive: JEE Advanced . . . *4.41*
Answer Key–Practice Exercises . . . *4.42*

CHAPTER 5 **CONTINUITY AND DIFFERENTIABILITY OF FUNCTION . . . 5.1**
Continuity of Function . . . 5.1
Types of Discontinuity . . . 5.2
Theorem on Continuous Function . . . 5.6
Continuity Test in an Interval . . . 5.7
Intermediate Mean Value Theorem . . . 5.12
Differentiability of Function at Point $x = \alpha$ . . . 5.14
Theorem on Differentiability . . . 5.19
Differentiability Test in an Interval . . . 5.20
Continuity and Differentiability of Composite Function . . . 5.28
Continuity of Derivative . . . 5.30
Extended Formula of Derivative . . . 5.30
Solving Functional Equation . . . 5.31
Solved Problems . . . *5.34*
Practice Exercises . . . *5.39*
    Single Correct Choice Type Questions . . . *5.39*
    Multiple Correct Choice Type Questions . . . *5.45*
    Linked Comprehension Type Questions . . . *5.48*
    Matrix Match/Column Match Type Questions . . . *5.50*
    Integer/Numerical Answer Type Questions . . . *5.53*
    Archive: JEE Main . . . *5.55*
    Archive: JEE Advanced . . . *5.57*
Answer Key–Practice Exercises . . . *5.59*

CHAPTER 6 **APPLICATION OF DERIVATIVES . . . 6.1**
Geometrical Application of Derivative: Tangent and Normal . . . 6.1
Tangent/Normal to Curve with Parametric Co-ordinates . . . 6.6
Tangent and Normal on Curve from External Point $P(x_1, y_1)$ . . . 6.8
Angle Between Intersecting Curves . . . 6.11
Orthogonal Curves . . . 6.11
Length of Tangent/Normal/Sub-tangent/Sub-normal . . . 6.14
Shortest Distance Between Two Curves . . . 6.15
Aproximation Using Derivative . . . 6.18
Physical Application of Derivative: Rate of Change . . . 6.19
Rolle’s Theorem . . . 6.21

Lagrange's Mean Value Theorem (LMVT) . . . 6.25
Cauchy's Mean Value Theorem . . . 6.25
Solved Problems . . . *6.28*
Practice Exercises . . . *6.32*
  Single Correct Choice Type Questions . . . *6.32*
  Multiple Correct Choice Type Questions . . . *6.36*
  Linked Comprehension Type Questions . . . *6.38*
  Matrix Match/Column Match Type Questions . . . *6.40*
  Integer/Numerical Answer Type Questions . . . *6.42*
  Archive: JEE Main . . . *6.44*
  Archive: JEE Advanced . . . *6.45*
Answer Key–Practice Exercises . . . *6.46*

CHAPTER 7 **MONOTONICITY, CONCAVITY AND MAXIMA-MINIMA OF FUNCTIONS . . . 7.1**

Monotonicity of a Function . . . 7.1
Monotonicity in an Interval . . . 7.3
Monotonicity of Composite Function . . . 7.8
Inequality Using Monotonicity . . . 7.11
Concavity and Convexity of Curve . . . 7.14
Point of Inflection . . . 7.15
Inequality Using Concavity [Jensen Inequality] . . . 7.18
Stationary Points and Critical Points . . . 7.21
Local or Relative Maxima/Minima . . . 7.21
Use of Derivative for Identifying Extrema . . . 7.23
Global or Absolute Maxima/Minima . . . 7.25
Application of Derivatives in Geometrical/Physical Problems . . . 7.31
Solved Problems . . . *7.37*
Practice Exercises . . . *7.41*
  Single Correct Choice Type Questions . . . *7.41*
  Multiple Correct Choice Type Questions . . . *7.47*
  Linked Comprehension Type Questions . . . *7.52*
  Matrix Match/Column Match Type Questions . . . *7.55*
  Integer/Numerical Answer Type Questions . . . *7.58*
  Archive: JEE Main . . . *7.60*
  Archive: JEE Advanced . . . *7.62*
Answer Key–Practice Exercises . . . *7.65*

CHAPTER 8 **INDEFINITE INTEGRATION . . . 8.1**

Basic Theorems on Integration . . . 8.1
Integration of Elementary Trigonometric Expression . . . 8.3
Integration Using Substitution . . . 8.7
Some Important Substitution . . . 8.7
Integration of $\frac{1}{Q}, \frac{1}{\sqrt{Q}}$ Form . . . 8.11
Integration by Parts . . . 8.16
Integration by Cancellation . . . 8.19
Integral of $\frac{L}{Q}$ or $\frac{L}{\sqrt{Q}}$ Form . . . 8.21
Integration Using Partial Fraction . . . 8.23
Integration of Irrational Function . . . 8.27

Integration of Trigonometric Functions . . . 8.35
Reduction Integration . . . 8.41
Euler's Substitution . . . 8.43
Solved Problems . . . 8.45
Practice Exercises . . . 8.50
  Single Correct Choice Type Questions . . . 8.50
  Multiple Correct Choice Type Questions . . . 8.57
  Linked Comprehension Type Questions . . . 8.60
  Matrix Match/Column Match Type Questions . . . 8.61
  Integer/Numerical Answer Type Questions . . . 8.64
  Archive: JEE Main . . . 8.66
  Archive: JEE Advanced . . . 8.68
Answer Key–Practice Exercises . . . 8.69

CHAPTER 9 **DEFINITE INTEGRATION . . . 9.1**
Mathematical Definition . . . 9.1
Geometrical Definition . . . 9.1
Properties of Definite Integration . . . 9.5
Definite Integration Using Substitution . . . 9.12
Definite Integration for Periodic Function . . . 9.33
Walli's Formula . . . 9.37
Integration Involving $F(x)$ and $F^{-1}(x)$ . . . 9.39
Definite Integration with Two Variables . . . 9.40
Newton–Leibnitz Rule . . . 9.42
Extended Newton–Leibnitz Theorem . . . 9.47
Definite Integration Using Limit of Sum . . . 9.50
Solving Limit Using Integration . . . 9.53
Estimation of Definite Integration . . . 9.57
Solved Problems . . . 9.61
Practice Exercises . . . 9.69
  Single Correct Choice Type Questions . . . 9.69
  Multiple Correct Choice Type Questions . . . 9.80
  Linked Comprehension Type Questions . . . 9.85
  Matrix Match/Column Match Type Questions . . . 9.89
  Integer/Numerical Answer Type Questions . . . 9.91
  Archive: JEE Main . . . 9.96
  Archive: JEE Advanced . . . 9.99
Answer Key–Practice Exercises . . . 9.103

CHAPTER 10 **AREA UNDER THE CURVE . . . 10.1**
Curve Tracing . . . 10.1
Asymptotes of Curve . . . 10.2
Area Enclosed by Curve $y = f(x)$ with x-axis . . . 10.4
Area Enclosed by Curve $y = f(x)$ with y-axis . . . 10.6
Area Between Two Curves . . . 10.10
Area Computation for Parametric Curves . . . 10.21
Concept Based on Variable Area . . . 10.23
Solved Problems . . . 10.27
Practice Exercises . . . 10.32
  Single Correct Choice Type Questions . . . 10.32
  Multiple Correct Choice Type Questions . . . 10.37
  Linked Comprehension Type Questions . . . 10.40

Matrix Match/Column Match Type Questions . . . 10.42
Integer/Numerical Answer Type Questions . . . 10.43
Archive: JEE Main . . . 10.45
Archive: JEE Advanced . . . 10.47
Answer Key–Practice Exercises . . . 10.49

CHAPTER 11 DIFFERENTIAL EQUATION . . . 11.1

Formation of Differential Equation . . . 11.2
Solution of Differential Equation . . . 11.6
Reducible to Variable Separable Form . . . 11.7
Solving Homogenous Differential Equation . . . 11.9
Reducible to Homogeneous Differential Equation . . . 11.11
Solving Exact Differential Equation . . . 11.14
Solving Differential Equation Using Polar Coordinates . . . 11.16
Linear Differential Equation . . . 11.17
Differential Equation Reducible to Linear Differential Equation . . . 11.21
Singular Solution . . . 11.23
Clairaut's Differential Equation . . . 11.23
Orthogonal Trajectory . . . 11.25
Geometrical and Physical Application of Differential Equation . . . 11.26
Solved Problems . . . 11.33
Practice Exercises . . . 11.36
Single Correct Choice Type Questions . . . 11.36
Multiple Correct Choice Type Questions . . . 11.43
Linked Comprehension Type Questions . . . 11.46
Matrix Match/Column Match Type Questions . . . 11.49
Integer/Numerical Answer Type Questions . . . 11.50
Archive: JEE Main . . . 11.52
Archive: JEE Advanced . . . 11.55
Answer Key–Practice Exercises . . . 11.57

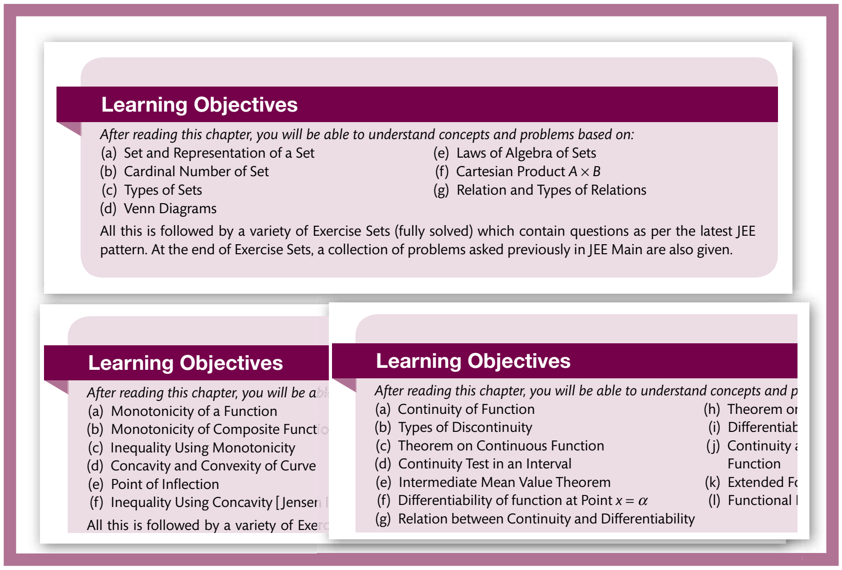
HINTS AND EXPLANATIONS

Chapter 1: Set and Relation . . . H.3
Chapter 2: Function and its Graph . . . H.10
Chapter 3: Limit of Function . . . H.50
Chapter 4: Method of Differentiation . . . H.82
Chapter 5: Continuity and Differentiability of Function . . . H.109
Chapter 6: Application of Derivatives . . . H.148
Chapter 7: Monotonicity, Concavity and Maxima-minima of Functions . . . H.181
Chapter 8: Indefinite Integration . . . H.229
Chapter 9: Definite Integration . . . H.268
Chapter 10: Area Under the Curve . . . H.343
Chapter 11: Differential Equation . . . H.378

CHAPTER INSIGHT

Learning Objectives

Help the students set an aim to achieve the major take-aways from a particular chapter.



Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Set and Representation of a Set
(b) Cardinal Number of Set
(c) Types of Sets
(d) Venn Diagrams
(e) Laws of Algebra of Sets
(f) Cartesian Product $A \times B$
(g) Relation and Types of Relations

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

Learning Objectives

After reading this chapter, you will be a

(a) Monotonicity of a Function
(b) Monotonicity of Composite Funct
(c) Inequality Using Monotonicity
(d) Concavity and Convexity of Curve
(e) Point of Inflection
(f) Inequality Using Concavity [Jensen

All this is followed by a variety of Exer

Learning Objectives

After reading this chapter, you will be able to understand concepts and p

(a) Continuity of Function
(b) Types of Discontinuity
(c) Theorem on Continuous Function
(d) Continuity Test in an Interval
(e) Intermediate Mean Value Theorem
(f) Differentiability of function at Point $x = \alpha$
(g) Relation between Continuity and Differentiability
(h) Theorem o
(i) Differentiab
(j) Continuity
Function
(k) Extended F
(l) Functional

Theory with Illustrations

Elaborative and simple theory helps the students to understand the illustrations supporting the theory. Please note that theory and problem solving techniques are based on simple learning program IF → THEN → ELSE. I would suggest you not to attempt the illustrations without going through the theory of that section.

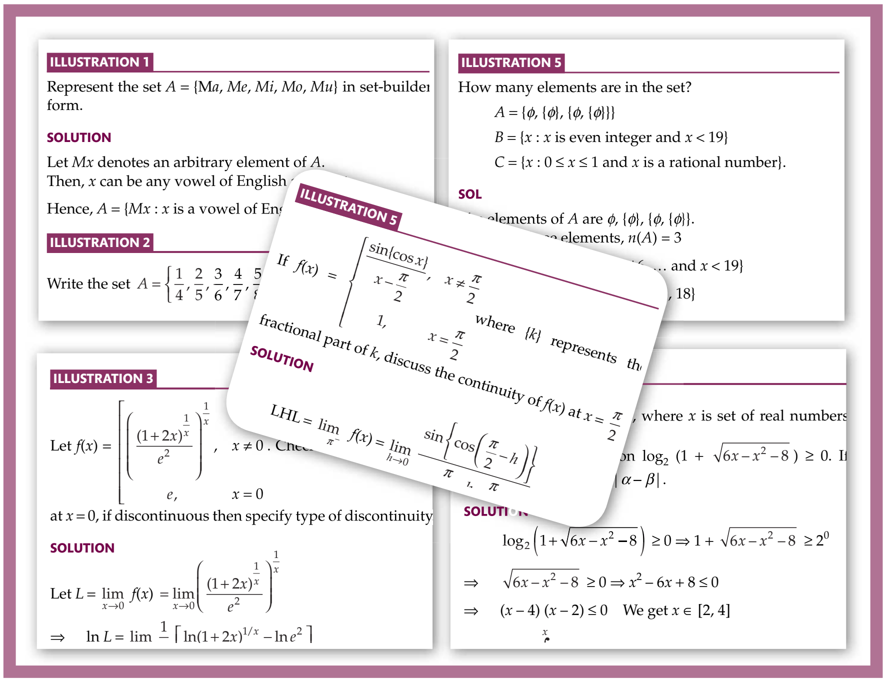


ILLUSTRATION 1

Represent the set $A = \{Ma, Me, Mi, Mo, Mu\}$ in set-builder form.

SOLUTION

Let Mx denotes an arbitrary element of A.
Then, x can be any vowel of English

Hence, $A = \{Mx : x$ is a vowel of Eng

ILLUSTRATION 2

Write the set $A = \left\{\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{}\right.$

ILLUSTRATION 3

Let $f(x) = \begin{cases} \left(\dfrac{(1+2x)^{\frac{1}{x}}}{e^2}\right)^{\frac{1}{x}}, & x \neq 0 \\ e, & x = 0 \end{cases}$. Chec

at $x = 0$, if discontinuous then specify type of discontinuity

SOLUTION

Let $L = \lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\dfrac{(1+2x)^{\frac{1}{x}}}{e^2}\right)^{\frac{1}{x}}$

$\Rightarrow \quad \ln L = \lim \frac{1}{} \left[\ln(1+2x)^{1/x} - \ln e^2\right]$

ILLUSTRATION 5

How many elements are in the set?

$A = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$

$B = \{x : x$ is even integer and $x < 19\}$

$C = \{x : 0 \leq x \leq 1$ and x is a rational number$\}$.

SOL

elements of A are $\phi, \{\phi\}, \{\phi, \{\phi\}\}$.
elements, $n(A) = 3$

… and $x < 19\}$

, 18}

ILLUSTRATION 5

If $f(x) = \begin{cases} \dfrac{\sin\{\cos x\}}{x - \frac{\pi}{2}}, & x \neq \frac{\pi}{2} \\ 1, & x = \frac{\pi}{2} \end{cases}$ where $\{k\}$ represents th fractional part of k, discuss the continuity of $f(x)$ at $x = \frac{\pi}{2}$

SOLUTION

$\text{LHL} = \lim_{\pi^-} f(x) = \lim_{h \to 0} \dfrac{\sin\left\{\cos\left(\frac{\pi}{2} - h\right)\right\}}{\pi \quad \pi}$

, where x is set of real numbers

on $\log_2 (1 + \sqrt{6x - x^2 - 8}) \geq 0$. If

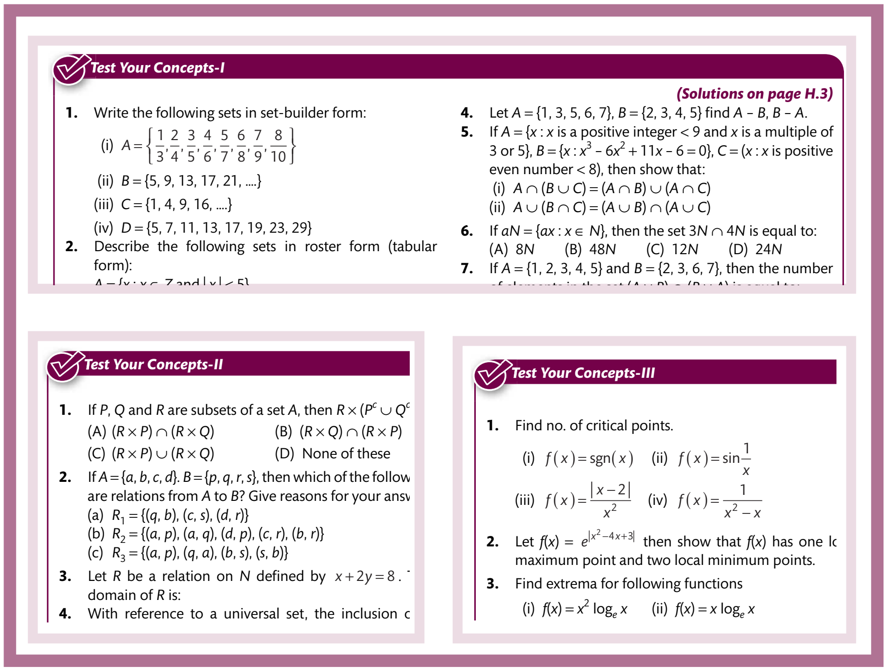
$|\alpha - \beta|$.

SOLUTI

$\log_2\left(1 + \sqrt{6x - x^2 - 8}\right) \geq 0 \Rightarrow 1 + \sqrt{6x - x^2 - 8} \geq 2^0$

$\Rightarrow \quad \sqrt{6x - x^2 - 8} \geq 0 \Rightarrow x^2 - 6x + 8 \leq 0$

$\Rightarrow \quad (x - 4)(x - 2) \leq 0 \quad$ We get $x \in [2, 4]$



Test Your Concepts-I

(Solutions on page H.3)

1. Write the following sets in set-builder form:
   (i) $A = \left\{\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}, \frac{7}{9}, \frac{8}{10}\right\}$
   (ii) $B = \{5, 9, 13, 17, 21, ...\}$
   (iii) $C = \{1, 4, 9, 16, ...\}$
   (iv) $D = \{5, 7, 11, 13, 17, 19, 23, 29\}$
2. Describe the following sets in roster form (tabular form):
   $A = \{x : x \in Z \text{ and } |x| < 5\}$
4. Let $A = \{1, 3, 5, 6, 7\}$, $B = \{2, 3, 4, 5\}$ find $A - B$, $B - A$.
5. If $A = \{x : x$ is a positive integer < 9 and x is a multiple of 3 or 5$\}$, $B = \{x : x^3 - 6x^2 + 11x - 6 = 0\}$, $C = (x : x$ is positive even number $< 8)$, then show that:
   (i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
   (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
6. If $aN = \{ax : x \in N\}$, then the set $3N \cap 4N$ is equal to:
   (A) $8N$ (B) $48N$ (C) $12N$ (D) $24N$
7. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 6, 7\}$, then the number

Test Your Concepts-II

1. If P, Q and R are subsets of a set A, then $R \times (P^c \cup Q^c$
   (A) $(R \times P) \cap (R \times Q)$ (B) $(R \times Q) \cap (R \times P)$
   (C) $(R \times P) \cup (R \times Q)$ (D) None of these
2. If $A = \{a, b, c, d\}$. $B = \{p, q, r, s\}$, then which of the follow are relations from A to B? Give reasons for your answ
   (a) $R_1 = \{(q, b), (c, s), (d, r)\}$
   (b) $R_2 = \{(a, p), (a, q), (d, p), (c, r), (b, r)\}$
   (c) $R_3 = \{(a, p), (q, a), (b, s), (s, b)\}$
3. Let R be a relation on N defined by $x + 2y = 8$. domain of R is:
4. With reference to a universal set, the inclusion c

Test Your Concepts-III

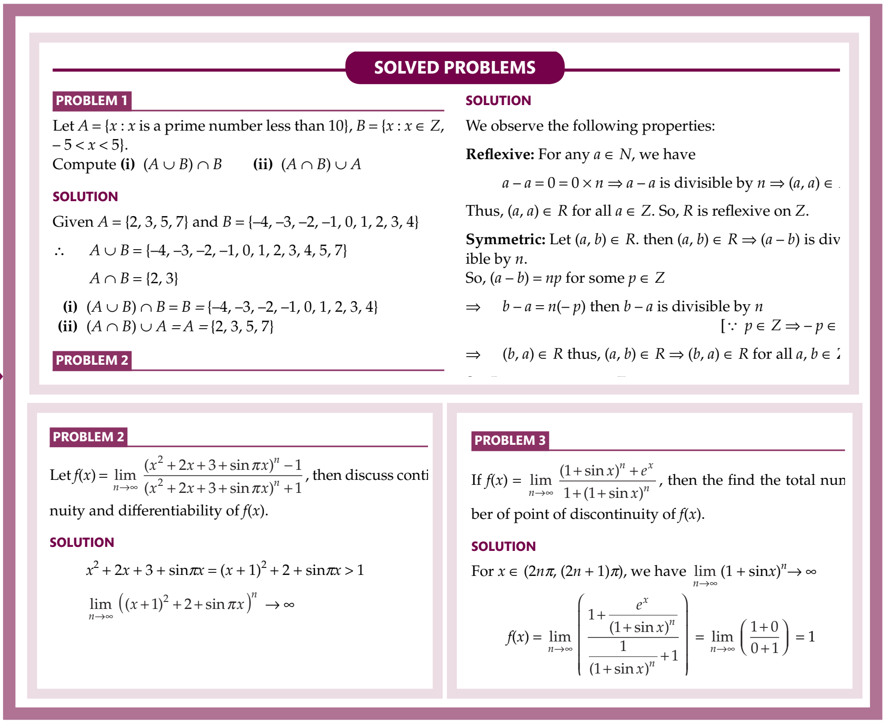
1. Find no. of critical points.
   (i) $f(x) = \text{sgn}(x)$ (ii) $f(x) = \sin\frac{1}{x}$
   (iii) $f(x) = \frac{|x-2|}{x^2}$ (iv) $f(x) = \frac{1}{x^2 - x}$
2. Let $f(x) = e^{|x^2 - 4x + 3|}$ then show that $f(x)$ has one lc maximum point and two local minimum points.
3. Find extrema for following functions
   (i) $f(x) = x^2 \log_e x$ (ii) $f(x) = x \log_e x$

Test Your Concepts

These topic based exercise sets are based on simple, single concept classification technique. These are meant for students practice after they study a particular topic and want to practice more on that topic learnt. Finally, in case of any difficulty they can refer to the hints and solutions to these exercise sets given at the end of the book.

Chapter End Solved Problems

These are based on multiple concept usage in a single problem approach so as to expose a student's brain to the ultimate throttle required to take the JEE examination.



SOLVED PROBLEMS

PROBLEM 1

Let $A = \{x : x$ is a prime number less than 10$\}$, $B = \{x : x \in Z, -5 < x < 5\}$.

Compute **(i)** $(A \cup B) \cap B$ **(ii)** $(A \cap B) \cup A$

SOLUTION

Given $A = \{2, 3, 5, 7\}$ and $B = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

$\therefore \quad A \cup B = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 7\}$

$A \cap B = \{2, 3\}$

(i) $(A \cup B) \cap B = B = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

(ii) $(A \cap B) \cup A = A = \{2, 3, 5, 7\}$

PROBLEM 2

SOLUTION

We observe the following properties:

Reflexive: For any $a \in N$, we have

$a - a = 0 = 0 \times n \Rightarrow a - a$ is divisible by $n \Rightarrow (a, a) \in$

Thus, $(a, a) \in R$ for all $a \in Z$. So, R is reflexive on Z.

Symmetric: Let $(a, b) \in R$. then $(a, b) \in R \Rightarrow (a - b)$ is divisible by n.

So, $(a - b) = np$ for some $p \in Z$

$\Rightarrow \quad b - a = n(-p)$ then $b - a$ is divisible by n

$[\because p \in Z \Rightarrow -p \in$

$\Rightarrow \quad (b, a) \in R$ thus, $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in$

PROBLEM 2

Let $f(x) = \lim\limits_{n\to\infty} \frac{(x^2 + 2x + 3 + \sin \pi x)^n - 1}{(x^2 + 2x + 3 + \sin \pi x)^n + 1}$, then discuss conti nuity and differentiability of $f(x)$.

SOLUTION

$x^2 + 2x + 3 + \sin\pi x = (x + 1)^2 + 2 + \sin\pi x > 1$

$\lim\limits_{n\to\infty} \left((x+1)^2 + 2 + \sin \pi x\right)^n \to \infty$

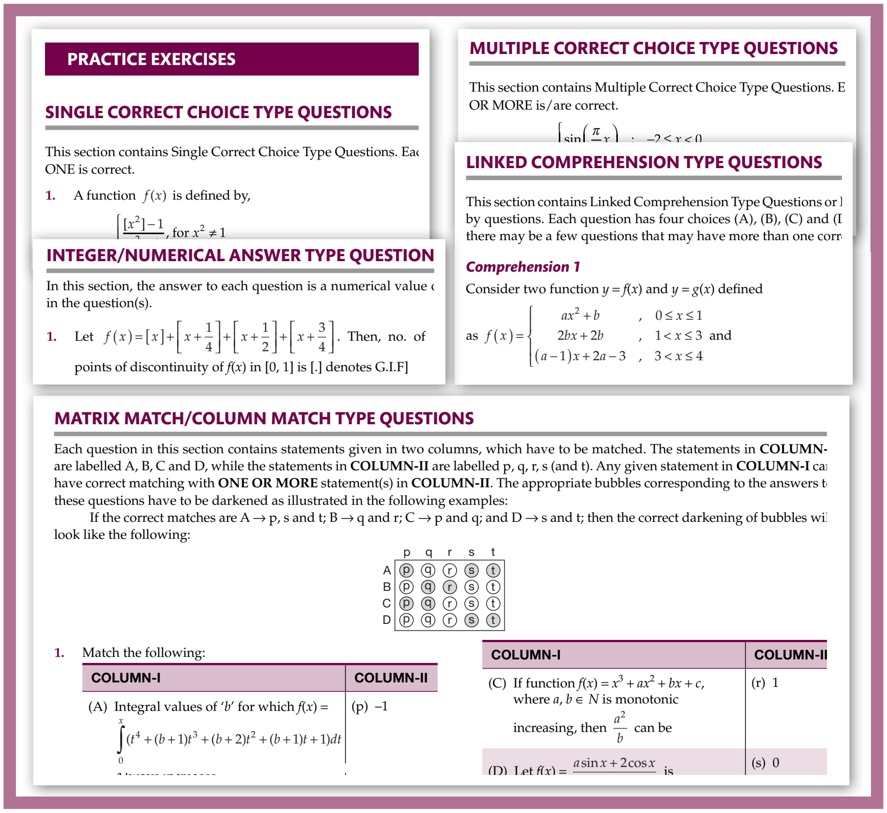
PROBLEM 3

If $f(x) = \lim\limits_{n\to\infty} \frac{(1 + \sin x)^n + e^x}{1 + (1 + \sin x)^n}$, then the find the total nun ber of point of discontinuity of $f(x)$.

SOLUTION

For $x \in (2n\pi, (2n + 1)\pi)$, we have $\lim\limits_{n\to\infty} (1 + \sin x)^n \to \infty$

$$f(x) = \lim_{n\to\infty} \left(\frac{1 + \frac{e^x}{(1+\sin x)^n}}{\frac{1}{(1+\sin x)^n} + 1} \right) = \lim_{n\to\infty} \left(\frac{1+0}{0+1}\right) = 1$$



PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Eac
ONE is correct.

1. A function $f(x)$ is defined by,

$\left[\frac{[x^2]-1}{\ldots}, \text{ for } x^2 \neq 1 \right.$

INTEGER/NUMERICAL ANSWER TYPE QUESTION

In this section, the answer to each question is a numerical value
in the question(s).

1. Let $f(x)=[x]+\left[x+\frac{1}{4}\right]+\left[x+\frac{1}{2}\right]+\left[x+\frac{3}{4}\right]$. Then, no. of points of discontinuity of $f(x)$ in [0, 1] is [.] denotes G.I.F]

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. E
OR MORE is/are correct.

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or
by questions. Each question has four choices (A), (B), (C) and (I
there may be a few questions that may have more than one corr

Comprehension 1

Consider two function $y = f(x)$ and $y = g(x)$ defined

as $f(x)=\begin{cases} ax^2+b & , \ 0 \le x \le 1 \\ 2bx+2b & , \ 1 < x \le 3 \\ (a-1)x+2a-3 & , \ 3 < x \le 4 \end{cases}$ and

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-**
are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** ca
have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers t
these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles wi
look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. Match the following:

COLUMN-I	COLUMN-II
(A) Integral values of 'b' for which $f(x) = \int_0^x (t^4+(b+1)t^3+(b+2)t^2+(b+1)t+1)dt$	(p) –1

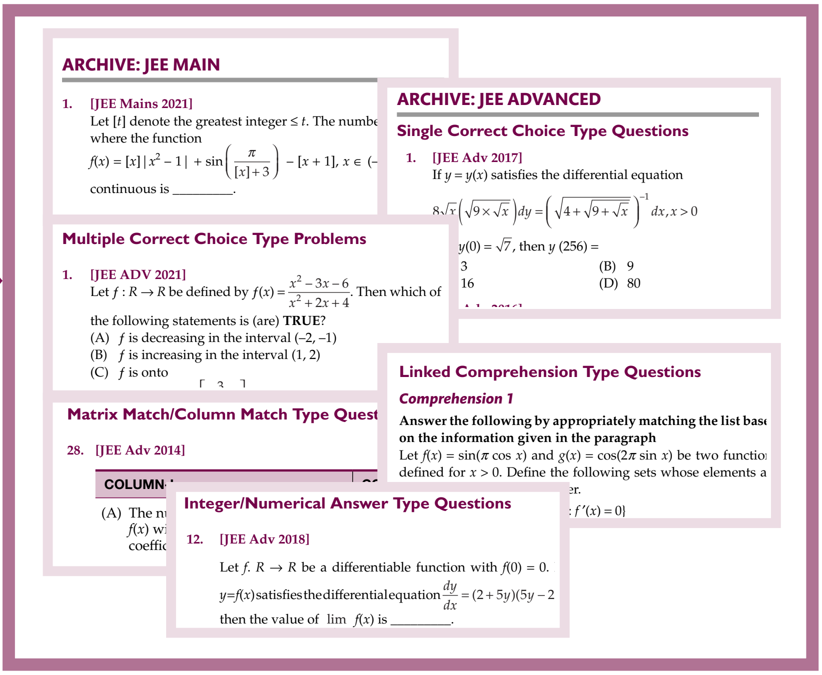
COLUMN-I	COLUMN-II
(C) If function $f(x) = x^3 + ax^2 + bx + c$, where $a, b \in N$ is monotonic increasing, then $\frac{a^2}{b}$ can be	(r) 1
(D) Let $f(x) = \frac{a\sin x + 2\cos x}{\ldots}$ is	(s) 0

Practice Exercise

Inclusion of all types of questions asked in JEE Advanced in adequate numbers helps you with enough practice

Archive: JEE Main and Advanced

From this fully updated section, students get to know the actual pattern of the problems asked in the past examinations.



ARCHIVE: JEE MAIN

1. **[JEE Mains 2021]**
Let $[t]$ denote the greatest integer $\le t$. The numbe
where the function
$f(x) = [x]\,|x^2 - 1| + \sin\left(\frac{\pi}{[x]+3}\right) - [x+1],\ x \in ($
continuous is ________.

Multiple Correct Choice Type Problems

1. **[JEE ADV 2021]**
Let $f : R \to R$ be defined by $f(x) = \frac{x^2-3x-6}{x^2+2x+4}$. Then which of the following statements is (are) **TRUE**?
(A) f is decreasing in the interval (–2, –1)
(B) f is increasing in the interval (1, 2)
(C) f is onto

Matrix Match/Column Match Type Quest

28. **[JEE Adv 2014]**

COLUMN-	
(A) The nu $f(x)$ wi coeffic	

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. **[JEE Adv 2017]**
If $y = y(x)$ satisfies the differential equation

$$8\sqrt{x}\left(\sqrt{9\times\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx, x > 0$$

$y(0) = \sqrt{7}$, then y (256) =

3 (B) 9
16 (D) 80

Linked Comprehension Type Questions

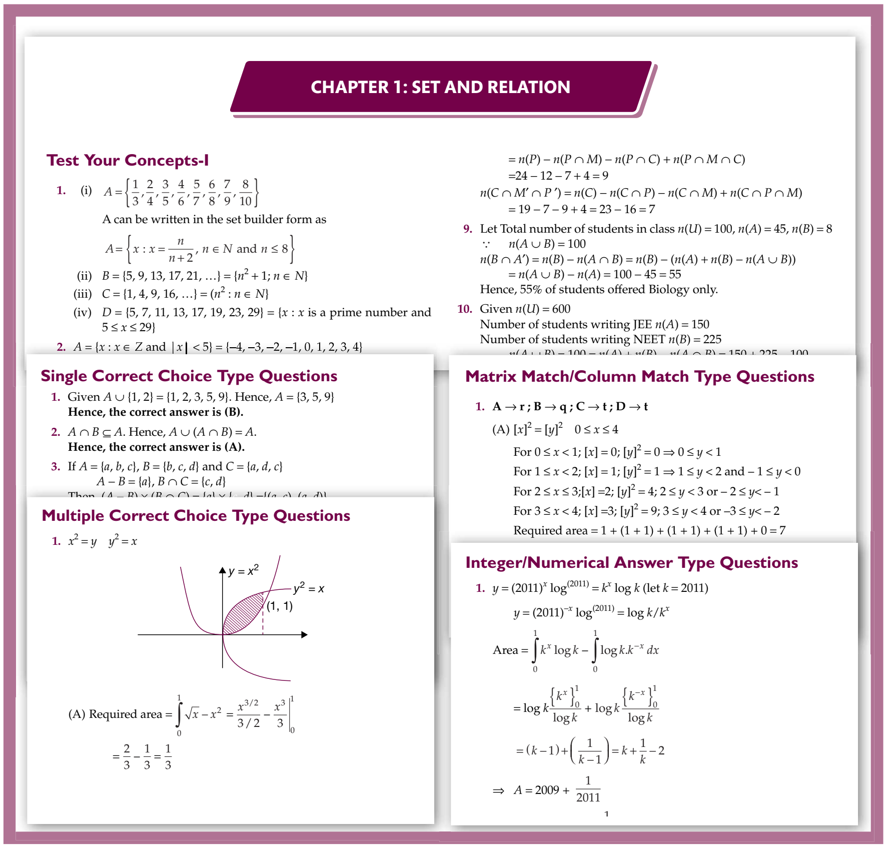
Comprehension 1

Answer the following by appropriately matching the list bas
on the information given in the paragraph
Let $f(x) = \sin(\pi \cos x)$ and $g(x) = \cos(2\pi \sin x)$ be two functio
defined for $x > 0$. Define the following sets whose elements a
er.
$: f'(x) = 0\}$

Integer/Numerical Answer Type Questions

12. **[JEE Adv 2018]**

Let $f.\ R \to R$ be a differentiable function with $f(0) = 0$.
$y=f(x)$ satisfies the differential equation $\frac{dy}{dx} = (2+5y)(5y-2$
then the value of $\lim f(x)$ is ________.



CHAPTER 1: SET AND RELATION

Test Your Concepts-I

1. (i) $A = \left\{\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}, \frac{7}{9}, \frac{8}{10}\right\}$

A can be written in the set builder form as

$$A = \left\{x : x = \frac{n}{n+2}, n \in N \text{ and } n \le 8\right\}$$

(ii) $B = \{5, 9, 13, 17, 21, \ldots\} = \{n^2 + 1; n \in N\}$

(iii) $C = \{1, 4, 9, 16, \ldots\} = (n^2 : n \in N\}$

(iv) $D = \{5, 7, 11, 13, 17, 19, 23, 29\} = \{x : x$ is a prime number and $5 \le x \le 29\}$

2. $A = \{x : x \in Z \text{ and } |x| < 5\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

$= n(P) - n(P \cap M) - n(P \cap C) + n(P \cap M \cap C)$

$= 24 - 12 - 7 + 4 = 9$

$n(C \cap M' \cap P') = n(C) - n(C \cap P) - n(C \cap M) + n(C \cap P \cap M)$

$= 19 - 7 - 9 + 4 = 23 - 16 = 7$

9. Let Total number of students in class $n(U) = 100$, $n(A) = 45$, $n(B) = 8$

$\because \quad n(A \cup B) = 100$

$n(B \cap A') = n(B) - n(A \cap B) = n(B) - (n(A) + n(B) - n(A \cup B))$

$= n(A \cup B) - n(A) = 100 - 45 = 55$

Hence, 55% of students offered Biology only.

10. Given $n(U) = 600$

Number of students writing JEE $n(A) = 150$

Number of students writing NEET $n(B) = 225$

Single Correct Choice Type Questions

1. Given $A \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$. Hence, $A = \{3, 5, 9\}$

Hence, the correct answer is (B).

2. $A \cap B \subseteq A$. Hence, $A \cup (A \cap B) = A$.

Hence, the correct answer is (A).

3. If $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{a, d, c\}$

$A - B = \{a\}$, $B \cap C = \{c, d\}$

Multiple Correct Choice Type Questions

1. $x^2 = y \quad y^2 = x$

(A) Required area $= \int_0^1 \sqrt{x} - x^2 = \left.\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right|_0^1$

$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

Matrix Match/Column Match Type Questions

1. **A → r ; B → q ; C → t ; D → t**

(A) $[x]^2 = [y]^2 \quad 0 \le x \le 4$

For $0 \le x < 1$; $[x] = 0$; $[y]^2 = 0 \Rightarrow 0 \le y < 1$

For $1 \le x < 2$; $[x] = 1$; $[y]^2 = 1 \Rightarrow 1 \le y < 2$ and $-1 \le y < 0$

For $2 \le x \le 3$;$[x] = 2$; $[y]^2 = 4$; $2 \le y < 3$ or $-2 \le y < -1$

For $3 \le x < 4$; $[x] = 3$; $[y]^2 = 9$; $3 \le y < 4$ or $-3 \le y < -2$

Required area $= 1 + (1 + 1) + (1 + 1) + (1 + 1) + 0 = 7$

Integer/Numerical Answer Type Questions

1. $y = (2011)^x \log^{(2011)} = k^x \log k$ (let $k = 2011$)

$y = (2011)^{-x} \log^{(2011)} = \log k / k^x$

$$\text{Area} = \int_0^1 k^x \log k - \int_0^1 \log k . k^{-x}\, dx$$

$$= \log k \frac{\{k^x\}_0^1}{\log k} + \log k \frac{\{k^{-x}\}_0^1}{\log k}$$

$$= (k-1) + \left(\frac{1}{k-1}\right) = k + \frac{1}{k} - 2$$

$$\Rightarrow \quad A = 2009 + \frac{1}{2011}$$

Hints and Explanations

Exhaustive solutions with shortcuts (where ever needed), help students enhance their problem-solving skills.

PREFACE

In the past few years, the IIT-JEE has evolved as an examination designed to check a candidate's true scientific skills. The examination pattern needs one to see those little details which others fail to see. These details tell us how much in-depth we should know to explain a concept in the right direction. Keeping the present-day scenario in mind, this series is written for students, to allow them not only to learn the tools but also to see why they work so nicely in explaining the beauty of ideas behind the subject.

The central goal of this series is to help the students develop a thorough understanding of Mathematics as a subject. This series stresses on building a rock-solid technical knowledge based on firm foundation of the fundamental principles followed by a large collection of formulae. The primary philosophy of this book is to guide the aspirants towards detailed groundwork for strong conceptual understanding and development of problem-solving skills like mature and experienced mathematicians.

Each book in this new Mathematics series is to help the aspirants prepare for both Advanced and Main levels of JEE conducted for IITs and other elite engineering institutions in India. The books in this series is to strengthen the basic understanding of the concepts and their applications. The books will be equally useful for the students preparing for Mathematics Olympiads.

This book is enriched with detailed exhaustive theory that introduces the concepts of Mathematics in a clear, concise, thorough and easy-to-understand language. A large collection of relevant problems is provided in eight major categories (including updated archive for JEE Advanced and JEE Main), for which the solutions are demonstrated in a logical and stepwise manner. These not only make students exam-ready but also helps them gain analytical skills.

We have carefully divided the series into five parts to make the learning of different topics seamless for the students. These parts are

- Coordinate Geometry
- Algebra
- Trigonometry
- Calculus
- 3-D Vector

Finally, I would like to thank all my teacher friends who had been a guiding source of light throughout the entire journey of writing this book.

I apologise in advance for the errors (if any) that might have inadvertently crept in the text. I would be grateful to the readers who bring errors of any kind to my attention. I truly welcome all comments, critiques and suggestions. I hope this book will nourish you with the concepts involved so that you get a great rank at JEE.

Wish you all the best!

Om Sharma

ABOUT THE AUTHOR

Om Sharma, an ex Research Fellow at BARC (Bhabha Atomic Research Centre), Mumbai, is an IIT Bombay (2007) Graduate. He has been a mathematics expert in the field of IIT JEE Mains and Advanced for more than 14 years from "Shiksha Ki Kaashi - Kota", Rajasthan. He has great exposure to teaching students in the offline and online mode of education.

He is also a motivational speaker and exceptional mentor, having skills to motivate students and ignite the spark in them for achieving success in targeting their goal. Throughout this journey, by the Grace of God, under his guidance and mentorship, many of his students have secured top ranks and become successful engineers and technocrats.

CHAPTER

1

Set and Relation

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Set and Representation of a Set
(b) Cardinal Number of Set
(c) Types of Sets
(d) Venn Diagrams
(e) Laws of Algebra of Sets
(f) Cartesian Product $A \times B$
(g) Relation and Types of Relations

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

SET

A set is collection of well-defined distinct objects or numbers. By the term 'distinct', we mean that no object is repeated. A set is represented by listing all its elements between braces { } and by separating them from each other by commas (If there are more than one element). Sets are usually denoted by capital letters of English alphabet.

If x is an element of a set A, we write $x \in A$ (read as 'x belongs to A'). If x is not an element of A, we write $x \notin A$ (read as 'x does not belong to A'). The symbol '$\in$' is called the membership relation.

EXAMPLE:

Let Set A = Single digit prime numbers = $\{2, 3, 5, 7\}$

Let Set B = Collection of all vowels in English alphabets, then $B = \{a, e, i, o, u\}$.

REPRESENTATION OF A SET

(i) Tabular Form or Roster Form: In this form elements are explicitly written in curly brackets after separating them by commas.

EXAMPLE:

If A is a set formed by natural number less than 5

$A = \{1, 2, 3, 4\}$

(ii) Set builder Form: In this form, in a set some logic/definition or mathematical constraints are written

$A = \{x \mid P(x)\}$ or $A = \{x : P(x)\}$

This signifies, A is the set of element x, such that x has the property P.

EXAMPLE:

The set $A = \{1, 2, 3, 4, 5\}$ can be written as $A = \{x \mid x \in N$ and $x \leq 5\}$

NOTATIONS OF STANDARD SETS OF NUMBERS

The set of all natural numbers (positive integers) is represented by $N = \{1, 2, 3, 4 \ldots\}$

Set of whole numbers is represented by $W = \{0, 1, 2, 3, 4 \ldots\}$

Set of all integers is represented by Z or $I = \{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$

Set of rational numbers is represented by $Q = \{p/q$ where $p, q \in I, q \neq 0\}$

Set of irrational number is represented by Q^C or $\mathbf{Q}'$.

Set of real numbers is represented by R.

Set of complex numbers is represented by C.

CARDINAL NUMBER OF SET

Cardinal number of Set $A = n(A)$ = Number of elements present in Set A.

Let $A = \{1, 3, 5\}$, then $n(A) = 3$.

Let $B = \{a, b, c, d, e, f, g, h\}$ then $n(B) = 8$

CERTAIN USEFUL SYMBOLS

Table 1.1 Some useful symbols

Symbol	Meaning
$\Rightarrow$	Implies
$\Leftrightarrow$	Implies and is implied by
$\in$	Belongs to
$\notin$	Does not belong to
: or |	Such that
Iff	If and only if
$\forall$	For every
$\exists$	There exists
$a \mid b$	a is a divisor of b

ILLUSTRATION 1

Represent the set $A = \{\text{M}a, Me, Mi, Mo, Mu\}$ in set-builder form.

SOLUTION

Let Mx denotes an arbitrary element of A.
Then, x can be any vowel of English alphabet.

Hence, $A = \{Mx : x \text{ is a vowel of English alphabet}\}$

ILLUSTRATION 2

Write the set $A = \left\{\frac{1}{4}, \frac{2}{5}, \frac{3}{6}, \frac{4}{7}, \frac{5}{8}, \frac{6}{9}\right\}$ in set-builder form.

SOLUTION

We observe that in every element of set A, numerator is a natural number 1 to 6 and denominator is three more than the numerator.

Hence, $A = \left\{x : x = \frac{n}{n+3}, n \in N, 1 \le n \le 6\right\}$.

ILLUSTRATION 3

Represent the set $A = \{x : x \text{ is an odd integer and } 3 \le x < 15\}$ in tabular form.

SOLUTION

Element of A is x which must have properties. It is an odd integer and is among 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14.
So, they are 3, 5, 7, 9, 11, 13.

Hence, $A = \{3, 5, 7, 9, 11, 13\}$

ILLUSTRATION 4

Match each of the sets on the left in the tabular (roster) form with the same set on the right given in the set-builder form:

(i) $\{7, 14, 21, \ldots.\}$	**(a)** $\{x : x$ is a letter of work IITJEE$\}$
(ii) {E, I, J, T}	**(b)** $\{x : x$ is an odd natural number less than 15$\}$
(iii) $\{1, 3, 5, 7, 9, 11, 13\}$	**(c)** $\{x : x = 7n, n \in N\}$
(iv) $\{2, 3\}$	**(d)** $\{x : x$ is a prime number and a divisor of 12$\}$

SOLUTION

In (c), given set is the set of all natural numbers which are multiples of 7 and therefore (i) matches (c).
In (a), there are six letters in the word IITJEE and I and E occur twice, therefore (ii) matches (a).
In (b), given set is the set of all odd natural numbers less than 15 and therefore (iii) matches (b).
Prime divisors of 12 are 2 and 3 and hence (iv) matches (d).

ILLUSTRATION 5

How many elements are in the set?

$A = \{\phi, \{\phi\}, \{\phi, \{\phi\}\}\}$

$B = \{x : x \text{ is even integer and } x < 19\}$

$C = \{x : 0 \le x \le 1 \text{ and } x \text{ is a rational number}\}$.

SOLUTION

The elements of A are $\phi, \{\phi\}, \{\phi, \{\phi\}\}$.
So, A has three elements, $n(A) = 3$

$B = \{x : x = 0, \pm 2, \pm 4, \pm 6, \ldots \text{ and } x < 19\}$

$B = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots, 18\}$

$\therefore$ B is an infinite set and hence, it has infinitely many elements.
C is also an infinite set because $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ are all elements of C.

TYPES OF SETS

Sets can be classified into many types. Each type of set has its own importance which we will discuss in the following definitions.

Null Set (Empty Set or Void Set)

A set having no element is called a null set or an empty set or void set. It is denoted by ϕ or { }, hence cardinal number $n(A) = 0$

EXAMPLES:

(i) Let A = The set of even numbers not divisible by 2 = { }

(ii) Let $B = \{x : x \in N \text{ and } 0 < x < 1\} = \{\ \}$
Here, B is a null set because there is no natural number lying between 0 and 1.

Note

The set {0} is not an empty set, it contains element 0.

Singleton Set

A set having single element is called a singleton set. It means set having cardinal number 1, are singleton set. $n(A) = 1$

EXAMPLES:

(i) A = The set of present President of India.
(ii) {5}, {0}, $\{\phi\}$.

Finite Set

A set having finite number of elements is called a finite set.

EXAMPLE:

(i) Let $A = \{1, 2, 3\}$. Here A is a finite set as it has 3 elements, $n(A) = 3$

(ii) Let B = Set of all odd positive integers $= \{1, 3, 5, 7, 9, \ldots\}$ where $n(B)$ is not finite.
Hence, B is not a finite set.

Infinite Set

A set containing infinite elements in it, is called an infinite set.

EXAMPLES:

(i) N = Set of all positive integers $= \{1, 2, 3, 4, \ldots\}$

(ii) A = Set of all even integers $= \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}$

(iii) Let Q = Set of all rational numbers $= \left\{\dfrac{p}{q} : p, q \in Z, q \neq 0\right\}$

Equivalent Set

Two finite sets A and B are said to be equivalent if they have the same cardinal number. Thus, sets A and B are equivalent iff $n(A) = n(B)$

EXAMPLE:
Let $A = \{1, 2, 3, 4, 5\}$, $B = \{a, e, i, o\ u\}$. Here $n(A) = n(B) = 5$. Therefore, sets A and B are equivalent.

Equal Set

Two sets A and B are said to be equal if each element of A is an element of B and each element of B is an element of A. Thus, two set A and B are equal if they have the same elements but the order in which the elements in the two sets have been written may differ. If sets A and B are equal, then we write $A = B$.

EXAMPLE:
Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{x : x \in N \text{ and } 1 \leq x < 7\}$
Here, A and B are equal sets.

Note

Equal sets are equivalent sets but equivalent sets may or may not be equal sets.

EXAMPLE:
Let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$
Here, A and B have the same number of elements and hence they are equivalent but they are not equal set as $1 \in A$ but $1 \notin B$.

UNIVERSAL SET, SUBSET, SUPERSET AND POWER SET

Universal Set

Any set which is superset of all the sets under consideration is called *universal set* and is denoted by S or U.

EXAMPLE:
Let $A = \{1, 2, 3\}$, $B = \{3, 4, 6, 9\}$ and $C = \{0, 1\}$
We can take $U = \{0, 1, 2, 3, 4, 6, 8, 9\}$ as universal set as all given three sets are subset of it.

Superset and Subset

Set A is said to be a subset of a set B, if each element of A is also an element of B and B is superset of A.
In other words, set formed by taking none, few or all elements of a given set, are subset of given set.

EXAMPLES:
(i) Let $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 5, 4\}$
Here, B is a subset of A, therefore, A is a superset of B.
(ii) Let N = set of all natural numbers and Z = set of all integers.
Z is a superset of N.

Proper Subsets

A set A is said to be a proper subset of a set B, if:

(i) Every element of A is an element of B
(ii) B has at least one element which is not an element of A.
This fact is expressed by writing $A \subset B$.
Every non-empty set is subset of itself, also known as **Improper subset**.

EXAMPLES:
(i) Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4, 1, 5\}$ then $A \subset B$.
(ii) Let $A = \{1, 2, 3\}$, $B = \{2, 3, 1\}$ then, A is improper subset of B.
Here, $A \subseteq B$ and $B \subseteq A$. Both are true, it means $A = B$.

EXAMPLE:
Let Set $A = \{1, 2, 3\}$
All subsets are { }, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} of set {1, 2, 3}

$\{1, 2, 3\}$ is **superset** of $\{\ \}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$
$\{\ \}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}$ are **proper subset** and $\{1, 2, 3\}$ is **improper subset** of A
we write $A \subset B$, where $\subset$ is symbol for proper subset.
$\{\ \} \subset \{1, 2, 3\}, \{1\} \subset \{1, 2, 3\}, \{1, 2, 3\} \subseteq \{1, 2, 3\}$

Conceptual Note(s)

(i) If A is non-empty set then null set and Set A itself are always subsets of Set A.
(ii) Null set is an improper subset of itself (since, it is equal to itself) but it is always a proper subset of any other non-empty set.
(iii) Each element has two choices, to be part of subset or not so if A has n elements, then total number of subsets are $= 2.2.2.2.......2 = 2^n$.

ILLUSTRATION 6

Find the number of proper subsets of the set $\{1, 2, 3, 4\}$

SOLUTION

Number of proper subsets of the set $\{1, 2, 3, 4\} = 2^4 - 1 = 15$
Hence, the number of proper subsets is 15.

Power Set

The set formed by all the possible subsets of a given set A is said to the power set of A and is denoted by $P(A)$.
Symbolically, $P(A) = \{X : X \subseteq A\}$ thus $X \in P(A) \Leftrightarrow X \subseteq A$.
Also, $\phi \in P(A)$ and $A \in P(A)$ for all sets A. Clearly, the elements of $P(A)$ are the subsets of A.

EXAMPLES:

(i) If $A = \{1\}$, then $P(A) = \{\phi, \{1\}\}$ where $\phi = \{\ \}$
(ii) If $A = \{1, 2\}$, then $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ where $\phi = \{\ \}$
(iii) If $A = \{1, 2, 3\}$, then $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$ where $\phi = \{\ \}$

Hence, if A has n elements then $P(A)$ has 2^n elements.

ILLUSTRATION 7

Which of the following sets are empty, singleton, pair and which of them are equal?

$$A = \{x : x^2 = 9 \text{ and } 2x = 7\}$$
$$B = \{x : x^2 - 5x + 6 = 0, 2x = 4\}$$
$$C = \{x : x^2 - 4x + 3 = 0\}$$
$$D = \{x : x^2 = 36\}$$
$$E = \{x : 2x = 6 \text{ or } x = 1\}$$

SOLUTION

$A = \{x : x^2 = 9 \text{ and } 2x = 7\} = \{x : (x = 3 \text{ or } x = -3) \text{ and } x = \frac{7}{2}\}$.

No, x satisfies both the conditions. So there is no x in A i.e., $A = \phi$.

$B = \{x : (x - 3)(x - 2) = 0 \text{ and } x = 2\}$
$= \{x : (x = 3 \text{ or } x = 2) \text{ and } x = 2\}$

Only $x = 2$ satisfies both the predicates. So x has only one value 2.

$\therefore \quad B = \{2\}$

$C = \{x : (x - 3)(x - 1) = 0\} = \{x : x = 3 \text{ or } x = 1\}$

3 and 1 both satisfy the statement $x = 3$ or $x = 1$

$\therefore \quad C = \{1, 3\}$

Similarly, $D = \{-6, 6\}$ and $E = \{1, 3\}$.
Hence, A is an empty set; B is a singleton set, C, D, E are pair sets. Also $C = E$.

ILLUSTRATION 8

$A = \{x : x = 4^n - 3n - 1 \text{ and } n \in N\}$, $B = \{y : y = 9(n - 1) \text{ and } n \in N\}$. Prove that $A \subset B$.

SOLUTION

$$x = 4^n - 3n - 1 = (1 + 3)^n - 3n - 1$$
$$= 1 + {}^nC_1 . 3 + {}^nC_2 \cdot 3^2 + {}^nC_3 \cdot 3^3 + \ldots {}^nC_n 3^n - 3n - 1$$
(Using binomial expansion)
$$= 9[{}^nC_2 + {}^nC_3 \cdot 3 + \ldots + 3^{n-2}]$$

Here, x is some multiple of 9, so

$$y = 9(n - 1), n \in N$$

But y is all the integral multiple of 9.

Hence, $A \subset B$.

ILLUSTRATION 9

Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3\}$ and $C = \{2, 4\}$. Find all sets X satisfying each pair of conditions:

(i) $X \subseteq A$, $X \subseteq B$ and $X \subseteq C$
(ii) $X \subseteq A$ and $X \not\subset C$

SOLUTION

(i) Since, X is a subset of all of A, B and C, therefore, only those elements will be in X which are in all the three sets A, B and C.
$A \cap B \cap C = \{2\}$
$\therefore \quad X = \phi$ or $\{2\}$

(ii) Since, X is a subset of A and X is not a subset of C, therefore, that element of A should belong to X which does not belong to C. Clearly such elements are 1 and 3.
Subsets of A containing 1 and not containing 3 are
$\{1\}, \{1, 2\}, \{1, 4\}, \{1, 2, 4\}$
Subsets of A containing 3 and not containing 1 are
$\{3\}, \{3, 2\}, \{3, 4\}, \{3, 2, 4\}$
Subsets of A containing both 1 and 3 are $\{1, 3\}, \{1, 3, 2\}, \{1, 3, 4\}$ $\{1, 2, 3, 4\}$
Hence, $X = \{1\}, \{3\}, \{1, 2\}, \{1, 4\}, \{3, 2\}, \{3, 4\}, \{1, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 2\}, \{1, 3, 4\}, \{1, 2, 3, 4\}$.

ILLUSTRATION 10

Find the power sets of the following sets.

$A = \{ \{ \phi \} \}, B = \{\phi, \{ \phi \} \}, C = \{a, b, \{c, d\} \}$

SOLUTION

$P(A) = \{X : X \subseteq A\} = \{X : X = \phi, X = A\} = \{\phi, A\} = \{\phi, \{ \{\phi\} \} \}.$

$P(B) = \{X : X \subseteq B\} = \{X : X = \phi, X = B, X = \{\phi\}, X = \{ \{\phi\} \} \}$

$= \{\phi, \mathrm{B}, \{\phi\}, \{ \{\phi\} \} \} = \{\phi, \{\phi\}, \{ \{\phi\} \}, \{ \phi, \{\phi\} \} \}$

$P(C) = \{X : X \subseteq C\}$

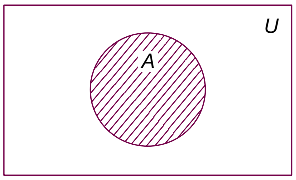
$= \{\phi, \{a\}, \{b\}, \{ \{c, d\} \}, \{a, b\}, \{a, \{c, d\} \}, \{b, \{c, d\} \}, C\}.$

VENN DIAGRAMS

Venn diagram is a pictorial representation of sets. The universal set U is represented by a rectangular region.

Set A

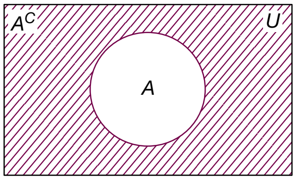
A set is represented by a circle or a closed geometrical figure inside the universal set and an element of a set A is represented by a point within the circle which represents A.



Set *A*

Complement of a Set (A' or A^c)

The complement of a set A is the set of all those elements of the universal set U which are not elements of A. It is denoted by A' or A^c. Clearly, A' or $A^c = U - A$.



$A^c = U - A$

Symbolically, A' or $A^c = \{x : x \in U \text{ and } x \notin A\}$

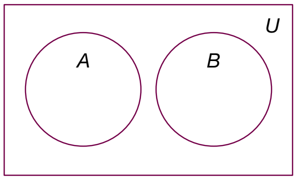
Thus, $x \in A' \Leftrightarrow x \notin A$.

Complement of a set can be represented by venn diagram as shown in the given picture.

Disjoint Sets

Two sets A and B are said to be disjoint sets if they have no common element, i.e., $A \cap B = \phi$.

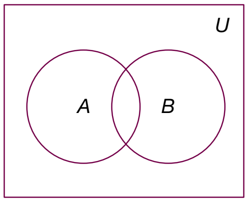
The disjoint sets can be represented by Venn diagram as shown in the figure.

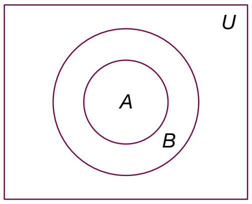


$A \cap B = \phi$

Subset

In the given figure (i) A and B are subset of universal set U and in figure (ii) B is a subset of universal set S and A is a subset of B.





OPERATIONS ON SETS

There are four main set operations which include set union, set intersection, set complement, and set difference.

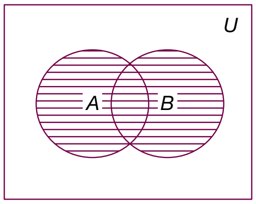
Union of Two Sets ($A \cup B$)

The union of two sets A and B is the set of all those elements which are either in A or in B or in both. This set is denoted by $A \cup B$ (read as 'A' union B').

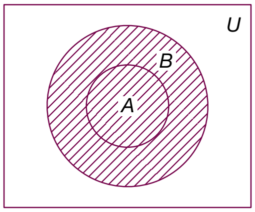
Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

The union of two sets can be represented by a venn diagram as shown in following figures.

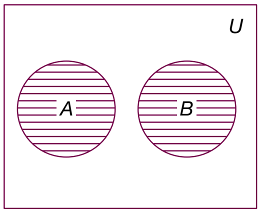
The shaded region represents $A \cup B$.



$A \cup B$



$A \cup B$ (when $A \subset B$)

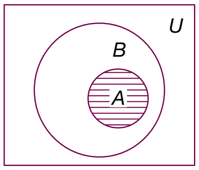


$A \cup B$ **when** A **and** B
are disjoint sets

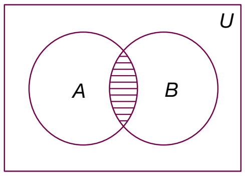
Intersection of Two Sets ($A \cap B$)

The intersection of two sets A and B is the set of all those elements which are common in A and B. This set is denoted by $A \cap B$ (read as 'A intersection B').

Symbolically, $A \cap B = \{x : x \in A \text{ and } x \in B\}$
The intersection of two sets can be presented by a venn diagram as shown in the given figures. The shaded region represents $A \cap B$.



$A \cap B$ (when $A \subset B$)

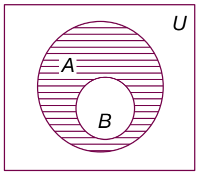


$A \cap B$

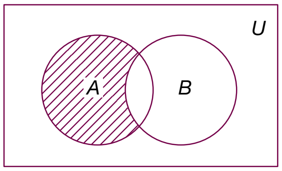
(i) If $A \subseteq B$, then $A \cap B = A$
(ii) For any two sets A and B
  (a) $(A \cup B) \cap A = A$ and $(A \cup B) \cap B = B$
  (b) $(A \cap B) \cup A = A$ and $(A \cap B) \cup B = B$

Difference of Two Sets $A - B$, $B - A$, $A \Delta B$

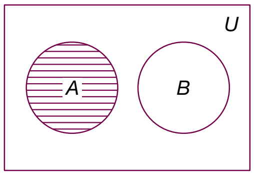
$A - B$: The difference of two sets A and B in this order (also called 'relative complement' of B in A) is the set of all those elements of A which are not elements of B.
Symbolically, $A - B = \{x : x \in A \text{ and } x \notin B\}$ and $B - A = \{x : x \in B \text{ and } x \notin A\}$.
A – B can be represented by venn diagram as shown in the given figure. The shaded region represents $A - B$.



$A - B$ when $B \subset A$

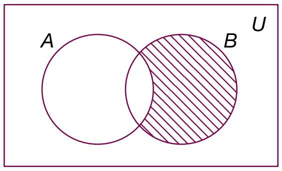


$A - B$



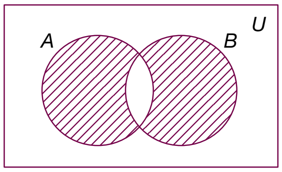
when A and B are Disjoint sets. Clearly $A - B = A$

$B - A$: Set of all those elements of B which are not elements of A.



$B - A$

$A \Delta B$: Symmetric difference of A and B



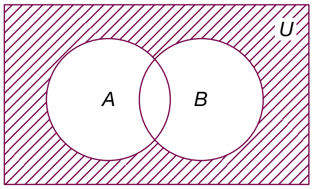
$A \Delta B$

If A and B are two sets, then:

(i) $A - B = A \cap B'$
(ii) $B - A = B \cap A'$
(iii) $A - B = A \Leftrightarrow A \cap B = \phi$
(iv) $(A - B) \cup B = A \cup B$
(v) $(A - B) \cap B = \phi$
(vi) $A \subseteq B \Leftrightarrow B' \subseteq A'$
(vii) $(A - B) \cup (B - A) = (A \cup B) - (B \cap A)$

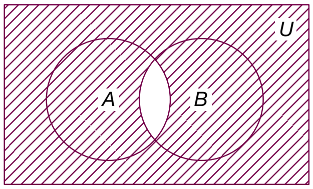
De-Morgan Law

(i) $(A \cup B)' = A' \cap B'$



$(A \cup B)'$

(ii) $(A \cap B)' = A' \cup B'$



$(A \cap B)'$

LAWS OF ALGEBRA OF SETS

(i) **Idempotent laws:** For any set A, we have:
  (a) $A \cup A = A$ (b) $A \cap A = A$
(ii) **Identity laws:** For any set A, we have:
  (a) $A \cup \phi = A$ (b) $A \cap U = A$
  i.e., ϕ and U are identity elements for union and intersection, respectively.
(iii) **Commutative laws:** For any two sets A and B, we have:
  (a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$
(iv) $A \Delta B = B \Delta A$ i.e., union, intersection and symmetric difference of two sets are commutative.
(v) $A - B \neq B - A$ i.e., difference of two sets are not commutative.
(vi) **Associative laws:** If A, B and C are any three sets, then:
  (a) $(A \cup B) \cup C = A \cup (B \cup C)$
  (b) $A \cap (B \cap C) = (A \cap B) \cap C$
  (c) $(A \Delta B) \Delta C = A \Delta (B \Delta C)$ i.e., union, intersection, and symmetric difference of two sets are associative.
  (d) $(A - B) - C \neq A - (B - C)$ i.e., difference of two sets is not associative.
(vii) **Distributive law:** If A, B and C are any three sets, then:
  (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  i.e., union, intersection are distributive over intersection and union respectively.

(viii) If A, B and C are any three sets, then:

- **(a)** $A \cap (B - C) = (A \cap B) - (A \cap C)$
- **(b)** $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$
- **(c)** $A - (B \cap C) = (A - B) \cup (A - C)$
- **(d)** $A - (B \cup C) = (A - B) \cap (A - C)$

IMPORTANT RESULTS ON SETS

If A, B and C are finite sets and U be the finite universal set, then use the following results whichever are required:

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow A, B$ are disjoint non-void sets

(iii) $n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

(iv) $n(A - B) = n(A) - n(A \cap B)$ i.e, $n(A) = n(A - B) + n(A \cap B)$

(v) $n(B) = n(B - A) + n(A \cap B)$

(vi) $n(A \Delta B)$ = Number of elements which belong to exactly one of A or B

$= n((A - B) \cup (B - A)) = n(A - B) + n(B - A)$

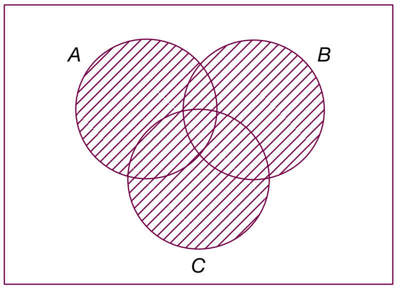
[$\because$ $(A - B)$ and $(B - A)$ are disjoint]

$= n(A) - n(A \cap B) + n(B) - n(A \cap B) = n(A) + n(B) - 2n(A \cap B)$

(vii) $n(A' \cup B') = n(A \cap B)' = n(U) - n(A \cap B)$

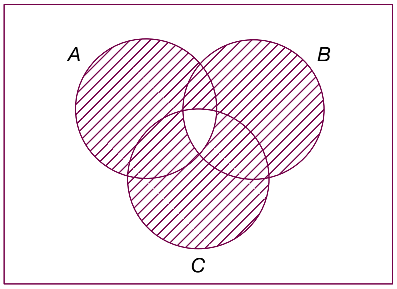
(viii) $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B)$

(ix) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$



(x) n(Number of elements in exactly two of the sets A, B, C)

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$



(xi) n(Number of elements in exactly one of the sets A, B, C)

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$

ILLUSTRATION 11

If $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 4, 6, 8\}$, $B = \{1, 3, 5, 7, 8\}$, $C = \{2, 3, 4, 5, 6, 7\}$.

Now, verify following:

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $(A \cup B)' = A' \cap B'$

(iii) $A - (B \cup C) = (A - B) \cap (A - C)$

SOLUTION

(i) $B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A \cap B = \{1, 8\}$, $A \cap C = \{2, 4, 6\}$

Now $A \cap (B \cup C) = \{x : x \in A$ and $x \in B \cup C\} = \{1, 2, 4, 6, 8\}$.

$(A \cap B) \cup (A \cap C) = \{1, 2, 4, 6, 8\}$

$\therefore$ $A \cap (B \cup C) = (A \cap B) \cap (A \cap C)$

(ii) $A \cup B = \{x : x \in A$ or $x \in B\} = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

$(A \cup B)' = \{x : x \in S$ and $x \notin A \cup B\} = \{9\}$

$A' = \{x : x \in S$ and $x \notin A\} = \{3, 5, 7, 9\}$

$B' = \{x : x \in S$ and $x \notin B\} = \{2, 4, 6, 9\}$

$A' \cap B' = \{9\}$

$\therefore$ $(A \cup B)' = A' \cap B'$

(iii) $A - (B \cup C) = \{1, 2, 4, 6, 8\} - \{1, 2, 3, 4, 5, 6, 7, 8\} = \phi$.

$A - B = \{1, 2, 4, 6, 8\} - \{1, 3, 5, 7, 8\} = \{2, 4, 6\}$

$A - C = \{1, 2, 4, 6, 8\} - \{2, 3, 4, 5, 6, 7\} = \{1, 8\}$

$\therefore$ $(A - B) \cap (A - C) = \phi$

$\therefore$ $A - (B \cup C) = (A - B) \cap (A - C)$

ILLUSTRATION 12

Shade the following sets in figure.

(i) $A' \cap (B \cup C)$ **(ii)** $A' \cap (C - B)$

SOLUTION

(i) In figure (i), shaded region represents $A' \cap (B \cup C)$

(ii) In figure (ii), shaded region represents $A' \cap (C - B)$

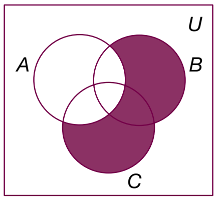


Fig. (i)

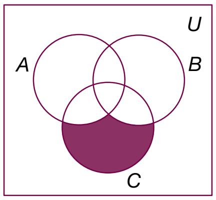


Fig. (ii)

ILLUSTRATION 13

The survey of 100 students stated that the number of students studying the various languages were: Sanskrit, Hindi and English–5; Hindi and Sanskrit–10; English and Sanskrit–8; Hindi and English–20; Sanskrit–30; English–50. Find the number of students studying only hindi.

SOLUTION

Let $A = \{x : x$ studies Sanskrit$\}$

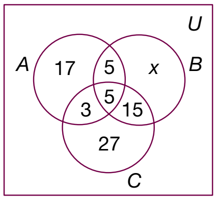
$B = \{x : x$ studies Hindi$\}$

$C = \{x : x \text{ studies English})$

Given, $n(A) = 30$, $n(C) = 50$,

$n(U) = 100$, $n(A \cap B \cap C) = 5$, $n(A \cap B) = 10$,

$n(A \cap C) = 8$, $n(B \cap C) = 20$



Let us denote the number of elements (students) contained in the bounded region as shown in the diagram according to the data given.
Now from venn-diagram

$$n(A \cup B \cup C) = 100 = 17 + x + 27 + 5 + 15 + 3 + 5$$

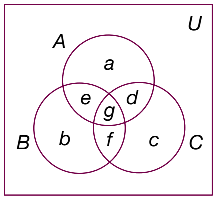
$$x + 22 = 50$$

We get $x = 23$, hence, there are 23 students studying only hindi.

ILLUSTRATION 14

A class has 175 students. Following is the description showing the number of students studying one or more of the following subjects in this class. Mathematics 100, Physics 70, Chemistry 46, Mathematics and Physics 30, Mathematics and Chemistry 28, Physics and Chemistry 23, Mathematics, Physics and Chemistry 18. How many students are enrolled in Mathematics alone, Physics alone and Chemistry alone? Are there students who have not offered any of these three subjects?

SOLUTION



Let A, B and C denote the sets of students studying Mathematics, Physics and Chemistry respectively. Let us denote the number of elements (students) contained in the bounded region as shown in the diagram by a, b, c, d, e, f and g, respectively.
Then,

$$a + d + g + e = 100$$
$$b + f + g + e = 70$$
$$c + f + g + d = 46$$
$$g + e = 30,\ g + d = 28,\ g + f = 23,\ g = 18.$$

Solving these, we get $g = 18$, $f = 5$, $d = 10$, $e = 12$, $c = 13$, $b = 35$, $a = 60$.

$$\therefore \quad a + b + c + d + e + f + g = 153.$$

So, the number of students who have not offered any of these three subjects = 175 – 153 = 22.
Number of students studying Mathematics only = $a = 60$
Number of students studying Physics only = $b = 35$
Number of students studying Chemistry only = $c = 13$.

Test Your Concepts-I

(Solutions on page H.3)

1. Write the following sets in set-builder form:

(i) $A = \left\{\frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \frac{6}{8}, \frac{7}{9}, \frac{8}{10}\right\}$

(ii) $B = \{5, 9, 13, 17, 21, ...\}$

(iii) $C = \{1, 4, 9, 16, ...\}$

(iv) $D = \{5, 7, 11, 13, 17, 19, 23, 29\}$

2. Describe the following sets in roster form (tabular form):
$A = \{x : x \in Z \text{ and } |x| < 5\}$
$B = \{x : x$ is a letter of the word 'upliftmaths'}
$C = \{x : x = 3n - 1 < 20 \text{ and } n \in N\}$
$D = \{x : x$ is a month of a year having 30 days}

3. Write the following sets in roster form:

(i) $A = (3^n - 2^n : n \in N \text{ and } 1 \le n \le 5\}$

(ii) $B = (a_n : n \in N, a_{n+1} = 2a_n \text{ and } a_1 = 3)$

(iii) $C = \{a_n : n \in N, a_{n+2} = a_{n+1} + a_n, a_1 = 1, a_2 = 2\}$

4. Let $A = \{1, 3, 5, 6, 7\}$, $B = \{2, 3, 4, 5\}$ find $A - B$, $B - A$.

5. If $A = \{x : x$ is a positive integer < 9 and x is a multiple of 3 or 5}, $B = \{x : x^3 - 6x^2 + 11x - 6 = 0\}$, $C = (x : x$ is positive even number $< 8)$, then show that:
(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

6. If $aN = \{ax : x \in N\}$, then the set $3N \cap 4N$ is equal to:
(A) $8N$ (B) $48N$ (C) $12N$ (D) $24N$

7. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 6, 7\}$, then the number of elements in the set $(A \times B) \cap (B \times A)$ is equal to:
(A) 4 (B) 5 (C) 10 (D) 20

8. In a class of 55 students, the number of students studying different subjects are 23 in mathematics, 24 in physics, 19 in chemistry, 12 in mathematics and physics, 9 is mathematics and chemistry, 7 in physics and chemistry and 4 in all the three subjects. Find the number of students who have taken exactly one subject.

9. If 45% of the students of a class have offered Mathematics and 85% of them Biology, find the percentage of students who offered Biology only?

10. In a survey of 600 students in a school, 150 students were found to be writing JEE and 225 writing NEET, 100 were taking both JEE and NEET. Find how many students are taking neither JEE nor NEET?

11. An investigator interviewed 100 students to determine their preferences for the three drinks: Milk (M), Coffee (C) and Tea (T). He reported the following: 10 students had all the three drinks M, C, T, 20 had M and C, 30 had C and T; 25 had M and T; 12 had M only; 5 had C only; 8 had T only. Using a venn diagram, find how many did not take any of the three drinks?

12. If $X = \{8^n - 7n - 1 \mid n \in N\}$ and $Y = \{49n - 49 \mid n \in N\}$. Then,

(A) $X \subset Y$ (B) $Y \subset X$

(C) $X = Y$ (D) $X \cap Y = \phi$

13. Given the sets $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, then $A \cup (B \cap C)$ is:

(A) $\{3\}$ (B) $\{1, 2, 3, 4\}$

(C) $\{1, 2, 4, 5\}$ (D) $\{1, 2, 3, 4, 5, 6\}$

Answer key

1. (i) $A = \left\{x : x = \dfrac{n}{n+2}, n \in N \text{ and } n \le 8\right\}$

(ii) $B = \{x : x = 4n + 1, n \in N\}$

(iii) $C = \{n^2 : n \in N\}$

(iv) $D = \{x : x \text{ is a prime number and } 5 \le x \le 29\}$

2. $A = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$, $B = \{u, p, l, i, f, t, m, a, h, s\}$,

$C = \{2, 5, 8, 11, 14, 17\}$,

$D = \{\text{April, June, September, November}\}$

3. $A = \{1, 5, 19, 65, 211\}$, $B = \{3, 6, 12, 24, \ldots\}$, $C = \{1, 2, 3, 5, 8, 13, \ldots\}$

4. $A - B = \{1, 6, 7\}$, $B - A = \{2, 4\}$

6. (C) **7.** (A) **8.** 7 **9.** 55% **10.** 325 **11.** 20 **12.** (A) **13.** (B)

CARTESIAN PRODUCT $A \times B$

$A \times B$ represents set consisting of all possible pairing of elements from set A to set B, known as cartesian product.

Let $A = \{x_1, x_2, x_3, \ldots, x_p)$ and $B = \{y_1, y_2, y_3, \ldots, y_q)$

then, $A \times B = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), \ldots, (x_1, y_q); (x_2, y_1), (x_2, y_2), (x_2, y_3), \ldots, (x_p, y_q)\}$.

Clearly each row has q ordered pairs and there are p such rows. So $A \times B$ has $p.q$ elements.

Hence, $n(A \times B) = n(A) \cdot n(B) = pq$

Similarly, $n(A \times B \times C) = n(A) \cdot n(B) \cdot n(C)$.

The elements of $A \times B$ are also called ordered pairs. The elements of $A \times B \times C$ are called ordered triplets.

Let $A = \{a, b, c\}$ and $B = \{p, q\}$.

Then, $A \times B = \{(a, p), (a, q), (b, p), (b, q), (c, p), (c, q)\}$

Also, $B \times A = \{(p, a), (p, b), (p, c), (q, a), (q, b), (q, c)\}$

Properties of Cartesian Product

If A and B are two non-empty sets, such that $n(A) = p$, $n(B) = q$, $n(A \times B) = r$.

(i) $A \times B \neq B \times A$ (Non-commutative)

(ii) $(A \times B) \times C \neq A \times (B \times C)$

(iii) $n(A \times B) = n(A)n(B) = p.q$ and cardinal number of power set formed by $A \times B$ $= n(P(A \times B)) = 2^{n(A)n(B)} = 2^{pq}$

(iv) $A = \phi$ and $B = \phi \Leftrightarrow A \times B = \phi$

(v) If A and B are having r elements in common then, $(A \times B)$ and $(B \times A)$ have r^2 elements in common.

(vi) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(vii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(viii) $A \times (B - C) = (A \times B) - (A \times C)$

(ix) if $A \times B = B \times A \Leftrightarrow A = B$

ILLUSTRATION 15

The Cartesian product $A \times A$ has 9 elements among which are found $(-2, 0)$ and $(0, 2)$. Find the set A and the remaining elements of $A \times A$.

SOLUTION

Let $n(A) = p$

Given, $n(A \times A) = 9$

$\Rightarrow \quad n(A) \cdot n(A) = 9$

$\Rightarrow \quad p \cdot p = 9 \Rightarrow p = 3$

$\therefore \quad n(A) = 3$

Now, $(-2, 0) \in A \times A \Rightarrow -2 \in A$ and $0 \in A$

Again, $(0, 2) \in A \times A \Rightarrow 0 \in A$ and $2 \in A$

Thus, $-2 \in A$, $0 \in A$ and $2 \in A$

$\therefore \quad -2, 0, 2 \in A$.

But A has exactly three elements, therefore $A = \{-2, 0, 2\}$

Hence, remaining elements of $A \times A$ are

$$(-2, -2), (-2, 2), (0, -2), (0, 0), (2, -2), (2, 0), (2, 2).$$

ILLUSTRATION 16

Let A and B be two sets such that $n(A) = 4$ and $n(B) = 2$. If a, b, c, d are distinct and $(a, 2)$, $(b, 3)$, $(c, 2)$, $(d, 3)$ are in $A \times B$. Find A and B.

SOLUTION

Since, a, b, c, d are distinct and $(a, 2)$, $(b, 3)$, $(c, 2)$, $(d, 3)$ are elements of $A \times B$, therefore $a, b, c, d \in A$.
As $n(A) = 4$

$\therefore \quad A = \{a, b, c, d\}$

Again $n(B) = 2$ and $(a, 2), (b, 3) \in A \times B$

$\therefore \quad B = \{2, 3\}$.

ILLUSTRATION 17

If $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, then find:

(i) $A \times (B \cap C)$ **(ii)** $(A \times B) \cup (B \times C)$
(iii) $(A \times B) \cap (B \times C)$

SOLUTION

$B \cap C = \{4\}$, $B \cup C = \{3, 4, 5, 6\}$

(i) $A \times (B \cap C) = \{(1, 4), (2, 4)\ (3, 4)\}$
$A \times B = \{(1, 3)\ (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$
$B \times C = \{(3, 4)\ (3, 5)\ (3, 6)\ (4, 4)\ (4, 5)\ (4, 6)\}$

(ii) $(A \times B) \cup (B \times C) = \{(1, 3)\ (1, 4)\ (2, 3), (2, 4), (3, 3), (3, 4), (3, 5), (3, 6), (4, 4), (4, 5), (4, 6)\}$

(iii) $(A \times B) \cap (B \times C) = \{(3, 4)\}$

ILLUSTRATION 18

If $n(A) = 7$, $n(B) = 8$ and $n(A \cap B) = 4$, then find the following:

(i) $n(A \cup B)$ **(ii)** $n(A \times B)$ **(iii)** $n((B \times A) \times A)$
(iv) $n((A \times B) \cap (B \times A))$ **(v)** $n(A \times B) \cup (B \times A)$

SOLUTION

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 7 + 8 - 4 = 11$
(ii) $n(A \times B) = n(A) \times n(B) = 7 \times 8 = 56 = n(B \times A)$
(iii) $n((B \times A) \times A) = n(B \times A) \cdot n(A) = 56 \times 7 = 392$
(iv) $n(A \times B) \cap (B \times A)) = (n((A \cap B))^2 = 4^2 = 16$
(v) $n(A \times B) \cup (B \times A)$
$= n(A \times B) + n(B \times A) - n(A \times B) \cap (B \times A)$
$= 56 + 56 - 16 = 96$

RELATION

Let A and B two non-empty sets then every subset of $A \times B$ defines a relation from A to B and every relation from A to B is a subset of cartesian product $A \times B$.
Let $R \subseteq A \times B$ and $(a, b) \in R$. Then we say that a is related to b by the relation R and write it as aRb or $(a, b) \in R$

Let A and B be two non-empty sets and $R : A \to B$ be a relation such that
$R = \{(a, b) \mid (a, b) \in R, a \in A \text{ and } b \in B\}$.

(i) 'b' is called image of 'a' under R.
(ii) 'a' is called pre-image of 'b' under R.
(iii) **Domain of *R*** = Collection of all elements of A which has a image in B
= set formed by first elements of ordered pair (a, b)
(iv) **Range of *R*** = Collection of all elements of B which has a pre-image in A
= set formed by second element of ordered pair (a, b)
(v) It is not necessary that each and every element of Set A has a image in Set B and each and every element of set B has a pre image in Set A
(vi) Elements of set A having image in B is not necessarily unique.
(vii) If $n(A) = p$, $n(B) = q$, $n(A \times B) = pq$
Total number of relations from set A to set $B = 2^{pq}$.

INVERSE RELATION

Let A, B be two sets and let R be a relation from a set A to a set B then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$
Clearly $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$.
Domain (R) = Range (R^{-1}) and Range (R) = Domain (R^{-1})

ILLUSTRATION 19

Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and $R = \{(a, 1), (a, 3), (b, 3), (c, 3)\}$. Find inverse relation and its domain and range.

SOLUTION

To find inverse relation, we need to interchange the order of ordered pair of R:

$$R^{-1} = \{(1, a), (3, a), (3, b), (3, c)\}$$

Domain of relation $R = \{a, b, c\}$ = Range of inverse relation R^{-1}
Hence, Range of relation $R = \{1, 3\}$ = Domain of inverse relation R^{-1}

ILLUSTRATION 20

$A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 5\}$,
A relation R is defined as $aRb \Rightarrow a$ and b are relatively prime or co-prime (i.e HCF is 1), find R.

SOLUTION

$R = \{(1, 2), (1, 4), (1, 5), (2, 5), (3, 2), (3, 4), (3, 5), (4, 5), (5, 2), (5, 4)\}$
Domain of $R = \{1, 2, 3, 4, 5\}$
Hence, Range of $R = \{2, 4, 5\}$

ILLUSTRATION 21

Let $A = \{1, 2, 3, 4, 5, 6\}$.
Find total number of distinct relations that can be defined from set A to A.

SOLUTION

$$n(A \times A) = n(A) \cdot n(A) = 6^2 = 36$$

Hence, the total number of subsets of $A \times A$ is 2^{36} and each subset of $A \times A$ is a relation over the set A.

ILLUSTRATION 22

Let A be the set of first eleven natural numbers and let R be a relation on a defined by $(x, y) \in R \Leftrightarrow x + 3y = 20$, i.e., $R = \{(x, y) : x \in A, y \in A$ and $x + 3y = 20\}$. Express R and R^{-1} as sets of ordered pairs and also find their domain and range.

SOLUTION

We have,

$$(x, y) \in R \Leftrightarrow x + 3y = 20 \Leftrightarrow y = \frac{20 - x}{3},\ x, y \in A$$

Where $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$.

Now, $x = 1 \Rightarrow y = \frac{20 - 1}{3} = \frac{19}{3} \notin A$.

This shows that 1 is not related to any element in A. Similarly, we can observe that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation.
Further we find that:

For $x = 2, y = \frac{20 - 2}{3} = 6 \in A$

$\therefore \quad (2, 6) \in R$

For $x = 5, y = \frac{20 - 5}{3} = 5 \in A$

$\therefore \quad (5, 5) \in R$

For $x = 8, y = \frac{20 - 8}{3} = 4 \in A$

$\therefore \quad (8, 4) \in R$

For $x = 11, y = \frac{20 - 11}{3} = 3 \in A$

$\therefore \quad (11, 3) \in R$

Thus, $R = \{(2, 6), (5, 5), (8, 4), (11, 3)\}$

$\Rightarrow \quad R^{-1} = \{(6, 2), (5, 5), (4, 8), (3, 11)\}$

Hence, Clearly, domain$(R) = \{2, 5, 8, 11\} =$ Range (R^{-1})

and, Range $(R) = \{3, 4, 5, 6\} =$ domain(R^{-1}).

ILLUSTRATION 23

A relation R is defined on the set Z of integers as follows:

$$(x, y) \in R \Leftrightarrow x^2 + y^2 = 25$$

Express R and R^{-1} as the sets of ordered pairs and hence find their respective domains.

SOLUTION

We have,

$$(x, y) \in R \Leftrightarrow x^2 + y^2 = 25 \Leftrightarrow y = \pm\sqrt{25 - x^2}$$

We observe that:

$x = 0 \Rightarrow y = \pm 5$

$\therefore \quad (0, 5) \in R$ and $(0, -5) \in R$

$x = \pm 3 \Rightarrow y = \pm\sqrt{25 - 9} = \pm 4$

$\therefore \quad (3, 4) \in R, (-3, 4) \in R, (3, -4) \in R$ and $(-3, -4) \in R$

$x = \pm 4 \Rightarrow y = \pm\sqrt{25 - 16} = \pm 3$

$\therefore \quad (4, 3) \in R, (-4, 3) \in R, (4, -3) \in R$ and $(-4, -3) \in R$

$x = \pm 5 \Rightarrow y = \sqrt{25 - 25} = 0$

$\therefore \quad (5, 0) \in R$ and $(-5, 0) \in R$

For any other integral value of x we notice that y is not an integer.

$\therefore \quad R = \{(0, 5). (0, -5), (3, 4), (-3, 4), (3, -4), (-3, -4), (4, 3), (-4, 3), (4, -3), (-4, -3), (5, 0), (-5, 0)\}$

$R^{-1} = \{(5, 0), (-5, 0), (4, 3), (4, -3), (-4, 3), (-4, -3), (3, 4), (3, -4), (-3, 4), (-3, -4), (0, 5), (0, -5)\}$

Hence, domain $(R) = \{0, 3, -3, 4, -4, 5, -5\} =$ domain (R^{-1}).

ILLUSTRATION 24

If R is the relation "less than" from $A = \{1, 2, 3, 4, 5\}$ to $B = \{1, 4, 5\}$, write down the set of ordered pairs corresponding to R. Find the inverse of R.

SOLUTION

It is given that $(x, y) \in R$

$\Leftrightarrow \quad x < y$, where $x \in A$ and $y \in B$.

For the elements of the given sets A and B, we find that

$1 < 4, 1 < 5, 2 < 4, 2 < 5, 3 < 4, 3 < 5$ and $4 < 5$

Thus, $R = \{(1, 4), (1, 5)\ (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$.
This shows that

$R^{-1} = \{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)\}$

$= \{(x, y) : x \in B, y \in A$ and $x > y\}$

TYPES OF RELATIONS

In this section we intend to define various types of relations on a given set A.

Void Relation

Let A be a non-empty set. Then, $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

Universal Relation

Let A be a non-empty set, then relation consisting of all elements of cartesian product is an universal relation on A

Note:

It is to note here that the void and the universal relations on a set A are respectively the smallest and the largest relations on A.

The relation $R_1 = \{\ \}$ is the void relation and $R_2 = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ is an universal relation on set $A = \{1, 2, 3\}$

Identity Relation

A relation R on a set A is an identity relation if every element of A is related to itself only.
Let A be a set then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the *identity relation on A*.

EXAMPLE:

The relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on set $A = \{1, 2, 3\}$

But relations $R_1 = \{(1, 1), (2, 2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R_1$ and in R_2 element 1 is related to elements 3.

Reflexive Relation

A relation R on a set A is said to be reflexive if every element of A is related to itself.
Thus, R is reflexive $\Leftrightarrow (a, a) \in R$ for all $a \in A$.
A relation R on a set A is not reflexive if there exists an element $a \in A$ such that $(a, a) \notin R$.

EXAMPLE:

Let $A = \{1, 2, 3\}$ be a set. Then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A.
But $R_1 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \notin R_1$.

Note

The identity relation on a non-void set A is always reflexive relation on A. However, a reflexive relation on A is not necessarily the identity relation on A.

The relation $R = \{(a, a), (b, b), (c, c), (a, b)\}$ is a reflexive relation on set $A = \{a, b, c\}$ but it is not the identity relation on A.

Symmetric Relation

A relation R on a set A is said to be a symmetric relation if $(a, b) \in R$ then $(b, a) \in R$ must exist in relation R for $(a, b) \in R$
i.e., aRb exists then bRa must exist.

Note:

(i) The identity and the universal relations on a non-void set are symmetric relations.
(ii) If R is symmetric then $R = R^{-1}$ and
Range of R = Domain of R

EXAMPLE:

Let $A = \{1, 2, 3, 4\}$ and let R_1 and R_2 be relations on A given by $R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2) (4, 1)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$.
Clearly, R_1 is a symmetric relation on A. However, R_2 is not Hence $(1, 3) \in R_2$ but $(3, 1) \notin R_2$.
A reflexive relation on a set A is not necessarily symmetric. For example, the relation $R = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ is a reflexive relation on set $A = \{1, 2, 3\}$ but it is not symmetric.

Transitive Relation

Let A be any set. A relation R on A is said to be a transitive relation if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$ must exist in Relation R.
i.e., aRb and $bRc \Rightarrow aRc$ for all $a, b\ c \in A$

Note:

The identity and the universal relations on a non-void set are transitive.

EXAMPLE:

On the set N of natural numbers, the relation R defined by $xRy \Rightarrow x$ is less than y is transitive, because for any $x, y, z \in N$ xRy and yRz exists. It means $x < y$ and $y < z \Rightarrow x < z$

$\Rightarrow$ xRy and $yRz \Rightarrow xRz$ exists.

Equivalence Relation

A relation R on a set A is said to be an equivalence relation on A iff it is reflexive, symmetric and transitive.

Anti Symmetric Relation

Let A be any set. A relation R on set A is said to be an anti-symmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$.

Conceptual Note(s)

(i) Singleton relation are always transitive Relations.
(ii) Null or void relations are always symmetric and transitive Relations.
(iii) Identity Relations are always reflexive, symmetric and transitive it means they are equivalence relations.

Trick

Always look for exceptions whenever we need to check for types of relations.

CONGRUENCE MODULO (*m*)

Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m if $a - b$ is divisible by m and we write $a \equiv b \pmod{m}$.
Thus $a \equiv b \pmod{m} \Leftrightarrow a - b$ is divisible by m.
For example, $28 \equiv 3 \pmod{5}$ because $28 - 3 = 25$ which is divisible by 5.
Similarly, $7 \equiv 23 \pmod{2}$ because $7 - 23 = -16$ which is divisible by 2.
But $35 \not\equiv 8 \pmod{4}$ because 4 is not a divisor of $35 - 8 = 27$.
The relation 'Congruence modulo m' is an equivalence relation.

ILLUSTRATION 25

Three relations R_1, R_2 and R_3 are defined on set $A = \{a, b, c\}$ as follows:

(i) $R_1 = \{(a, a), (a, b), (a, c), (b, b), (b, c), (c, a)\ (c, b), (c, c)\}$
(ii) $R_2 = \{(a, b), (b, a), (a, c), (c, a)\}$
(iii) $R_3 = \{(a, b), (b, c), (c, a)\}$

Find whether each of R_1, R_2 and R_3 is reflexive, symmetric, or transitive.

SOLUTION

(i) **Reflexive:** Clearly, $(a, a), (b, b), (c, c), \in R_1$.
So, R_1 is reflexive on A.
Symmetric: We observe that $(a, b) \in R_1$ but $(b, a) \notin R_1$. So, R_1 is not symmetric on A.
Transitive: We find that $(b, c) \in R_1$ and $(c, a) \in R_1$ but $(b, a) \notin R_1$. So, R is not transitive on A.

(ii) **Reflexive:** Since, (a, a), (b, b) and (c, c) are not in R_2. So, it is not a reflexive relation on A.
Symmetric: We find that the ordered pairs obtained by interchanging the components of ordered pairs in R_2 are also in R_2. So, R_2 is a symmetric relation on A.
Transitive: Clearly $(a, b) \in R_2$ and $(b, a) \in R_2$ but $(a, a) \notin R_2$. So, it is not a transitive relation on R_2.

(iii) **Reflexive:** Since, none of (a, a), (b, b) and (c, c) is an element of R_3. So, R_3 is not reflexive on A.
Symmetric: Clearly, $(b, c) \in R_3$ but $(c, b) \notin R_3$. So, R_3 is not symmetric on A.
Transitive: Clearly, $(a, b) \in R_3$ and $(b, c) \in R_3$ but $(a, c) \notin R_3$, So, R_3 is not transitive on A.

ILLUSTRATION 26

Find the number of reflexive relations of a set with four distinct elements.

SOLUTION

Total number of elements in given set = 4

Total number of relations $= 2^{4\times4} = 2^{16}$

In cartesian product, we have 16 ordered pairs in which 4 are compulsory to make relation reflexive.

For reflexive relation, out of 12 ordered pairs. We can take 0, 1, 2 or all remaining ordered pairs.

Hence, we can take ${}^{12}C_0 + {}^{12}C_1 + \ldots + {}^{12}C_{12} = 2^{12}$.

ILLUSTRATION 27

Categorize the following relation as reflexive and symmetric relations:

(i) aRb such that 'a' is brother of 'b'
(ii) aRb such that 'a' is a cousin of 'b'

SOLUTION

(i) $a\ R\ a$, a person can not be brother of himself so it is not reflexive.
aRb such that 'a' is brother of 'b' is not a symmetric relation as 'a' may be sister of 'b'.
For example, Om is brother of Seema but Seema is not brother of Om.

(ii) $a\ R\ a$, a person can not be cousin of himself so, it not reflexive.
aRb such that 'a' is a cousin of 'b'. This is a symmetric relation.
For example, Pawan is cousin of Seema then Seema is also cousin of Pawan, so it not reflexive but symmetric.

ILLUSTRATION 28

Let R be a relation defined in the set of real numbers by $aRb \Leftrightarrow 1 + ab > 0$. Then, R is equivalence relation or not?

SOLUTION

$aRa \Leftrightarrow 1 + a^2 > 0$ true for every real a hence R is reflexive.
$aRb \Leftrightarrow 1 + ab > 0 \Leftrightarrow 1 + ba > 0 \Leftrightarrow bRa$ hence R is symmetric.

$$\left.\begin{array}{r} -1R1/2 \Leftrightarrow 1+ab = 1-1/2 = 1/2 > 0 \\ 1/2R5 \Leftrightarrow 1+bc = 1+5/2 = 7/2 > 0 \end{array}\right]$$

Then, $-1\ R\ 5 = 1 - 5 = -4 > 0$ (False)
R is not transitive.
Hence, R is not an equivalence relation.

ILLUSTRATION 29

Let R be a relation on the set of all lines in a plane defined by $(l_1, l_2) \in R \Leftrightarrow$ angle between ℓ_1 and ℓ_2 is 0°. Show that R is an equivalence relation.

SOLUTION

Let L be the given set of all lines in a plane. Then, we observe the following properties.

Reflexive: For each line $l \in L$, we have angle between then as 0°.
$(l, l) \in R$ for all $l \in L \Rightarrow R$ is reflexive

Symmetric: Let $l_1, l_2 \in L$ such that $(l_1, l_2) \in R$. Then, if angle between l_1 and l_2 is 0° and angle between l_2 and l_3 is 0° then angle between l_1 and l_3 is also 0°.

$$(l_1, l_2) \in R \Rightarrow l_1 \,||\, l_2 \Rightarrow l_2 \,||\, l_1 \Rightarrow (l_2, l_1) \in R.$$

So, R is symmetric on L.

Transitive: Let $l_1, l_2, l_3 \in L$ such that $(l_1, l_2) \in R$ and $(l_2, l_3) \in R$. Then,

$$(l_1, l_2) \in R \text{ and } (l_2, l_3) \in R$$

$$\Rightarrow \quad l_1 \,\|\, l_2 \text{ and } l_2 \,\|\, l_3 \Rightarrow l_1 \,\|\, l_3$$

$$\Rightarrow \quad (l_1, l_3) \in R$$

Hence, it is an equivalence relation.

ILLUSTRATION 30

Check the following relation R, $a\ R\ b$ iff b is divisible by a for $a, b \in N$ for reflexivity, symmetry and transitivity.

SOLUTION

$a\ R\ b$ iff b is divisible by a for $a, b \in N$
We have, $aRb \Rightarrow a \mid b$ for all $a, b \in N$.
For any $a \in N$, we have $a \mid a \Rightarrow aRa$.
Thus, aRa for all $a \in N$. So, R is reflexive on N.
R is not symmetric because if a, b, then b may not divide a.
For example, $2 \mid 6$ but 2 is not divisible by 6.
Let $a, b, c \in N$ such that aRb and bRc.
Then, aRb and $bRc \Rightarrow a \mid b$ and $b \mid c \Rightarrow a \mid c \Rightarrow aRc$.
Hence, R is a transitive relation on N.

ILLUSTRATION 31

Check the following relations R for reflexivity, symmetry and transitivity.
$l_1\ R\ l_2$ iff l_1 is perpendicular to l_2 are straight lines in a plane.

SOLUTION

Let L be the set of all lines in a plane. We are given that $l_1\ S\ l_2 \Leftrightarrow l_1 \perp l_2$ for all $l_1, l_2 \in L$.
S is not reflexive because a line cannot be perpendicular to itself i.e., $l \perp l$ is not true.
Let $l_1, l_2 \in L$ such that $l_1\ R\ l_2 \Rightarrow l_1 \perp l_2 \Rightarrow l_2 \perp l_2 \Rightarrow l_2\ R\ l_1$. So, R is symmetric on L.
Hence, S is not transitive, because $l_1 \perp l_2$ and $l_2 \perp l_1$ does not imply that $l_1 \perp l_3$.

Test Your Concepts-II

(Solutions on page H.3)

1. If P, Q and R are subsets of a set A, then $R \times (P^c \cup Q^c)^c =$
(A) $(R \times P) \cap (R \times Q)$ (B) $(R \times Q) \cap (R \times P)$
(C) $(R \times P) \cup (R \times Q)$ (D) None of these

2. If $A = \{a, b, c, d\}$. $B = \{p, q, r, s\}$, then which of the following are relations from A to B? Give reasons for your answer.
(a) $R_1 = \{(q, b), (c, s), (d, r)\}$
(b) $R_2 = \{(a, p), (a, q), (d, p), (c, r), (b, r)\}$
(c) $R_3 = \{(a, p), (q, a), (b, s), (s, b)\}$

3. Let R be a relation on N defined by $x + 2y = 8$. The domain of R is:

4. With reference to a universal set, the inclusion of a subset in another, is relation, which is:
(A) symmetric only (B) equivalence relation
(C) reflective only (D) None of these

5. Two finite sets have m and n elements. Then total number of subsets of the first set is 56 more than that of the total number of subsets of the second. The values of m and n are:
(A) 7, 6 (B) 6, 3
(C) 5, 1 (D) 8, 7

6. Let R be the relation on the set R of all real numbers defined by $a\ R\ b$ iff $|a - b| \le 1$. Then R is:
(A) reflexive and symmetric (B) symmetric only
(C) transitive only (D) anti-symmetric only

7. The relation 'less than' in the set of natural numbers is
(A) only symmetric (B) only transistive
(C) only reflexive (D) equivalence relation

8. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1), (y, 2), (z, 1)$ are in $A \times B$, find A and B, where x, y, z are distinct elements.

9. If $A = \{a, b, c\}$ and some elements, of $A \times B$ are $(a, p), (b, q), (c, p)$. Write down the remaining elements of $A \times B$ if $n(A \times B) = 6$.

10. If $B = \{2, 3, 5\}$ and $(a, 2), (b, 3), (c, 5)$ are in $A \times B$, find A and the remaining elements of $A \times B$ such that $n(A \times B)$ is least.

11. A relation R is defined from $\{2, 3, 4, 5\}$ to $\{3, 6, 7, 10\}$ by $xRy \Leftrightarrow x$ is relatively prime to y. Then find domain of R.

12 If $A = \{2, 4\}$ and $B = \{3, 4, 5\}$, then find $(A \cap B) \times (A \cup B)$.

13. If $n(A) = 4$, $n(B) = 3$, $n(A \times B \times C) = 24$, then find $n(C)$.

Answer key

1. (A) **2.** R_2 **3.** $\{2, 4, 6\}$ **4.** (D) **5.** (B) **6.** (A) **7.** (B)
8. $A = \{x, y, z\}$, $B = \{1, 2\}$ **9.** $\{(a, q), (b, p), (c, q)\}$ **10.** $A = \{a, b, c\}$ $\{(a, 3), (a,5), (b, 2), (b, 5), (c, 2), (c, 3)\}$
11. $\{2, 3, 4, 5\}$ **12.** $\{(4, 2), (4, 3), (4, 4), (4, 5)\}$ **13.** 2

SOLVED PROBLEMS

PROBLEM 1

Let $A = \{x : x$ is a prime number less than 10$\}$, $B = \{x : x \in Z, -5 < x < 5\}$.
Compute **(i)** $(A \cup B) \cap B$ **(ii)** $(A \cap B) \cup A$

SOLUTION

Given $A = \{2, 3, 5, 7\}$ and $B = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

$$\therefore \quad A \cup B = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 7\}$$

$$A \cap B = \{2, 3\}$$

(i) $(A \cup B) \cap B = B = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
(ii) $(A \cap B) \cup A = A = \{2, 3, 5, 7\}$

PROBLEM 2

Let A and B be two sets such that $A \times B$ consists of 6 elements. If three elements of $A \times B$ are $(1, 4), (2, 6), (3, 6)$. Find $A \times B$ and $B \times A$.

SOLUTION

Since, $(1, 4), (2, 6)$ and $(3, 6)$ are elements of $A \times B$, it follows that 1, 2, 3 are elements of A and 4, 6 are elements of B. It is given that $A \times B$ has 6 elements. So, $A = \{1, 2, 3\}$ and $B = \{4, 6\}$. Hence,

$$A \times B = \{1, 2, 3\} \times \{4, 6\}$$

$$= \{(1, 4), (1, 6), (2, 4), (2, 6), (3, 4), (3, 6)\}$$

and $B \times A = \{4, 6\} \times \{1, 2, 3\}$

$$= \{(4, 1), (4, 2), (4, 3), (6, 1), (6, 2), (6, 3)\}$$

PROBLEM 3

A relation R is defined from a set $A = \{2, 3, 4, 5\}$ to a set $B = \{3, 6, 7, 10\}$ as follows $(x, y) \in R \Leftrightarrow x$ divides y. Express R as a set of ordered pairs and determine the domain and range of R. Also, find R^{-1}.

SOLUTION

Recall that $a \mid b$ stands for a divides b. For the elements of the given sets A and B, we find that $2 \mid 6$, $2 \mid 10$, $3 \mid 3$, $3 \mid 6$, and $5 \mid 10$.

$\therefore \quad (2, 6) \in R, (2, 10) \in R, (3, 3) \in R, (3, 6) \in R$ and $(5, 10) \in R$.

Thus, $R = \{(2, 6), (2, 10), (3, 3), (3, 6), (5, 10)\}$.

Hence, domain $(R) = \{2, 3, 5\}$ and range $(R) = \{3, 6, 10\}$

Also, $R^{-1} = \{(6, 2), (10, 2), (3, 3), (6, 3), (10, 5)\}$

PROBLEM 4

Prove that the relation R on the set Z of all integers defined by $(x, y) \in R \Leftrightarrow x - y$ is divisible by n is an equivalence relation on Z.

SOLUTION

We observe the following properties:

Reflexive: For any $a \in N$, we have

$$a - a = 0 = 0 \times n \Rightarrow a - a \text{ is divisible by } n \Rightarrow (a, a) \in R$$

Thus, $(a, a) \in R$ for all $a \in Z$. So, R is reflexive on Z.

Symmetric: Let $(a, b) \in R$. then $(a, b) \in R \Rightarrow (a - b)$ is divisible by n.
So, $(a - b) = np$ for some $p \in Z$

$\Rightarrow \quad b - a = n(-p)$ then $b - a$ is divisible by n
$[\because p \in Z \Rightarrow -p \in Z]$

$\Rightarrow \quad (b, a) \in R$ thus, $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in Z$.

So, R is symmetric on Z.

Transitive: Let $a, b, c \in Z$ such that $(a, b) \in R$ and $(b, c) \in R$. Then,
$(a, b) \in R \Rightarrow (a - b)$ is divisible by $n \Rightarrow a - b = np$ for some $p \in Z$
$(b, c) \in R \Rightarrow (b - c)$ is divisible by $n \Rightarrow b - c = nq$ for some $q \in Z$
Adding them $(a - b) + (b - c) = np + nq = n(p + q)$

$\Rightarrow \quad a - c$ is divisible by n $\quad [\because p, q, \in Z \Rightarrow p + q \in Z]$

$\Rightarrow \quad (a, c) \in R$

Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in Z$.
So, R is transitive relation on Z.
Hence, R being reflexive, symmetric and transitive, is an equivalence relation on Z.

PROBLEM 5

Prove that the relation R on the set $N \times N$ defined by (a, b) R $(c, d) \Leftrightarrow a + d = b + c$ for all $(a, b), (c, d) \in N \times N$ is an equivalence relation.

SOLUTION

We observe the following properties:

Reflexive: Let (a, b) be an arbitrary element of $N \times N$.
$(a, b) R (a, b) \Rightarrow a + b = b + a$, which is true for all $(a, b) \in N \times N$.
So, R is reflexive on $N \times N$.

Symmetric: Let $(a, b), (c, d) \in N \times N$ be such that (a, b) R (c, d). Then,

$$(a, b)\ R\ (c, d) \Rightarrow a + d = b + c \Rightarrow c + b = d + a$$

$\Rightarrow \quad (c, d)\ R\ (a, b)$ [By def. of R]

So, R is symmetric on $N \times N$.

Transitive Let $(a, b), (c, d), (e, f) \in N \times N$ such that

$$(a, b)\ R(c, d) \Rightarrow a + d = b + c$$

$$(c, d)R(e, f) \Rightarrow c + f = d + e$$

On addition $(a + d) + (c + f) = (b + c) + (d + e)$

$$\Rightarrow \quad a + f = b + e \Rightarrow (a, b)\ R\ (e, f)$$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$$\Rightarrow \quad (a, b)\ R\ (e, f) \text{ for all } (a, b), (c, d), (e, f) \in N \times N.$$

So, R is transitive on $N \times N$.
Hence, R being reflexive, symmetric and transitive, is an equivalence relation on $N \times N$.

PROBLEM 6

Let N be the set of all natural numbers and let R be a relation on $N \times N$, defined by $(a, b)\ R\ (c, d) \Leftrightarrow ad = bc$ for all $(a, b), (c, d) \in N \times N$. Show that R is an equivalence relation on $N \times N$.

SOLUTION

We observe the following properties:

Reflexive: Let (a, b) be an arbitrary element of $N \times N$. Then,

$$(a, b)\ R\ (a, b) \Rightarrow ab = ba, \text{ for all } (a, b) \in N \times N$$

So, R is reflexive on $N \times N$.

Symmetric: Let $(a, b), (c, d) \in N \times N$ be such that $(a, b)\ R\ (c, d)$. Then,

$$(a, b)\ R(c, d) \Rightarrow ad = bc \Rightarrow cb = da$$

$$\Rightarrow \quad (c, d)\ R\ (a, b) \text{ for all } (a, b), (c, d) \in N \times N.$$

So, R is symmetric on $N \times N$.

Transitive: Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$$\left.\begin{aligned} (a,b)\ R\ (c,d) &\Rightarrow ad = bc \\ (c,d)\ R\ (e,f) &\Rightarrow cf = de \end{aligned}\right\} \Rightarrow (ad)\ (cf) = (bc)\ (de)$$

$$\Rightarrow \quad af = be \Rightarrow (a, b)\ R\ (e, f)$$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$.
So, R is transitive on $N \times N$.
Hence, R being reflexive, symmetric and transitive, is an equivalence relation on $N \times N$.

PROBLEM 7

Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by

$$(a, b)\ R\ (c, d) \Leftrightarrow ad\ (b + c) = bc\ (a + d).$$

Check whether R is an equivalence relation on $N \times N$.

SOLUTION

We observe the following properties:

Reflexive: Let (a, b) be an arbitrary element of $N \times N$. Then,
$(a, b) \in n \times N \Rightarrow a, b \in N$
$(a, b)\ R\ (a, b) \Rightarrow ab\ (b + a) = ba\ (a + b)$, always true for $(a, b) \in N \times N$.
So, R is reflexive on $N \times N$.

Symmetric Let $(a, b), (c, d) \in N \times N$ be such that $(a, b)\ R\ (c, d)$. Then,

$$(a, b)\ R\ (c, d) \Rightarrow ad\ (b + c) = bc\ (a + d)$$

$$\Rightarrow \quad cb\ (d + a) = da(c + b) \Rightarrow (c, d)\ R\ (a, b)$$

Thus, $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in N \times N$.
So, R is symmetric on $N \times N$.

Transitive Let $(a, b), (c, d), (e, f) \in N \times N$ such that $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$. Then,

$$(a, b)\ R\ (c, d) \Rightarrow ad\ (b + c) = bc\ (a + d) \Rightarrow \frac{b + c}{bc} = \frac{a + d}{ad}$$

$$\Rightarrow \quad \frac{1}{b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{d} \qquad \ldots(1)$$

And, $(c, d)\ R\ (e, f) \Rightarrow cf\ (d + e) = de\ (c + f) \Rightarrow \dfrac{d + e}{de} = \dfrac{c + f}{cf}$

$$\Rightarrow \quad \frac{1}{d} + \frac{1}{e} = \frac{1}{c} + \frac{1}{f} \qquad \ldots(2)$$

Adding (1) and (2), we get

$$\left(\frac{1}{b} + \frac{1}{c}\right) + \left(\frac{1}{d} + \frac{1}{e}\right) = \left(\frac{1}{a} + \frac{1}{d}\right) + \left(\frac{1}{c} + \frac{1}{f}\right)$$

$$\Rightarrow \quad \frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f}$$

$$\Rightarrow \quad \frac{b + e}{be} = \frac{a + f}{af} \Rightarrow af(b + e) = be\ (a + f) \Rightarrow (a, b)\ R\ (e, f)$$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$.
So, R is transitive on $N \times N$.
Hence, R being reflexive, symmetric and transitive, is an equivalence relation on $N \times N$.

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. The smallest set A such that $A \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$

(A) $\{1, 3, 5, 9\}$ (B) $\{3, 5, 9\}$
(C) $\{3, 9\}$ (D) $\{2, 5, 9\}$

2. If A and B are any two sets, then $A \cup (A \cap B)$ is equal to:

(A) A (B) B
(C) A^c (D) B^c

3. If $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{a, d, c\}$, then, $(A - B) \times (B \cap C) =$

(A) $\{(a, c), (a, d)\}$ (B) $\{(a, b), (c, d)\}$
(C) $\{(c, a), (a, d)\}$ (D) $\{(a, d), (a, d)\ (b, d)\}$

4. Let $A = \{1, 2, 3, 4\}$, and let $R = \{(2, 2), (3, 3), (4, 4), (1, 2)\}$ be a relation on A. Then, R is:

(A) Reflexive (B) Symmetric
(C) Transitive (D) Equivalence

5. Let $R : A \to A$, A is set of all children in a family such that $xRy \Rightarrow x$ is sister of y (where $x, y \in A$), then, R is:

(A) Reflexive
(B) Symmetric
(C) Transitive
(D) Equivalence relation

6. Let R be a relation on a set A such that $R = R^{-1}$, then, R is:

(A) Reflexive (B) Symmetric
(C) Transitive (D) None of these

7. The relation $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1, 3)\}$ on the set $A = \{1, 2, 3\}$ is:

(A) Reflexive but not symmetric
(B) Reflexive but not transitive
(C) Symmetric and transitive
(D) Neither symmetric nor transitive

8. The relation R defined in $A = \{1, 2, 3\}$ by aRb if $|a^2 - b^2| \leq 5$. Which of the following is false?

(A) $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)\}$
(B) $R^{-1} = R$
(C) Domain of $R = \{1, 2, 3\}$
(D) Range of $R = \{5\}$

9. The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(x, y) : |x^2 - y^2| < 16\}$ is given by

(A) $\{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$
(B) $\{(2, 2), (3, 2), (4, 2), (2, 4)\}$
(C) $\{(3, 3), (3, 4), (5, 4), (4, 3), (3, 1)\}$
(D) None of these

10. If A is the set of even natural numbers less than 8 and B is the set of prime numbers less than 7, then the number of relations from A to B is:

(A) 2^9 (B) 9^2
(C) 3^2 (D) 2^{9-1}

11. The relation R is defined on the set of natural numbers as $\{(a, b) : a = 2b\}$. Then, R^{-1} is given by:

(A) $\{(2, 1), (4, 2), (6, 3), \ldots\}$
(B) $\{(1, 2), (2, 4), (3, 6), \ldots\}$
(C) R^{-1} is not defined
(D) None of these

12. If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$, which of the following is not a relations from A to B?

(A) $R_1 = \{(1, 4), (1, 5), (1, 6)\}$
(B) $R_2 = \{(1, 5), (2, 4), (3, 6)\}$
(C) $R_3 = \{(1, 4), (1, 5), (3, 6), (2, 6), (3, 4)\}$
(D) $R_4 = \{(4, 2), (2, 6), (5, 1), (2, 4)\}$

13. A survey shows that 63% of the Americans like cheese whereas 76% like apples. If x% of the Americans like both cheese and apples, then:

(A) $x = 39$ (B) $x = 63$
(C) $39 \leq x \leq 63$ (D) $x = 50$

14. Let R be a relation on the set N be defined by $\{(x, y) \mid x, y \in N, 2x + y = 41\}$. Then, R is:

(A) Reflexive (B) Symmetric
(C) Transitive (D) None of these

15. Let S = set of points inside the square, T = set of points inside the triangle and C = set of points inside the circle. If the triangle and circle intersect each other and are contained in a square, then:

(A) $S \cap T \cap C = \phi$
(B) $S \cup T \cup C = C$
(C) $S \cup T \cup C = S$
(D) $S \cup T = S \cap C$

16. Suppose $A_1, A_2, \ldots, A_{30}$ are thirty sets each having 5 elements and B_1, B_2, B_n are n sets each with 3 elements, let $\bigcup_{i=1}^{30} A_i = \bigcup_{i=1}^{n} B_j = S$ and each elements of S belongs to exactly 10 of the $A'_i s$ and exactly 9 of the $B'_j s$. Then, n is equal to:

(A) 15 (B) 3
(C) 45 (D) 35

17. If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$, $B = \{2, 4, \ldots, 18\}$ and N the set of natural numbers is the universal set, then $A' \cup ((A \cup B) \cap B')$

(A) ϕ (B) N
(C) A (D) B

18. The maximum number of equivalence relations on the set $A = \{1, 2, 3\}$ are:

(A) 1 (B) 2
(C) 3 (D) 5

19. Let R be the relation on the set R of all real numbers defined by $a\ R\ b$ iff $|a - b| \leq 1$. Then, R is:
(A) Reflexive and symmetric
(B) Symmetric only
(C) Transitive only
(D) Anti-symmetric only

20. Let R be a relation in N defined by $R = \{(1 + x, 1 + x^2) : x \leq 5, x \in N\}$. Which of the following is false?
(A) $R = \{(2, 2), (3, 5), (4, 10), (5, 17), (6, 25)\}$
(B) Domain of $R = \{2, 3, 4, 5, 6\}$
(C) Range of $R = \{2, 3, 4, 5, 6\}$
(D) None of these

21. The relation $R = \{(1, 1), (2, 2), (1, 2), (2, 3)\}$ on the set $A = \{1, 2, 3\}$ is:
(A) Reflexive but not symmetric
(B) Reflexive but not transitive
(C) Symmetric and transitive
(D) Neither symmetric nor transitive

22. If R be a relation $<$ from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$ i.e. $(a, b) \in R \Leftrightarrow a < b$, then, RoR^{-1} is:
(A) $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
(B) $\{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$
(C) $\{(3, 3), (3, 5), (5, 3), (5, 5)\}$
(D) $\{(3, 3), (3, 4), (4, 5)\}$

23. Let R be the real line. Consider the following subsets of the plane $R \times R$:
$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$
$T = \{(x, y) : x - y \text{ is an integer}\}$.
Which one of the following is true?
(A) Both S and T are equivalence relations on R
(B) S is an equivalence relation on R but T is not
(C) T is an equivalence relation on R but S is not
(D) Neither S nor T is an equivalence relation of R

24. Let $S = \{(a, b, c) \in N \times N \times N : a + b + c = 21, a \leq b \leq c\}$ and $T = \{(a, b, c) \in N \times N \times N\ a, b, c$ are in A.P.$\}$, where N is the set of all natural numbers. What will be the number of elements in the set $S \cap T$?
(A) 6
(B) 7
(C) 13
(D) 14

25. Let X and Y be two non-empty sets such that $X \cap A = Y \cap A = \phi$ and $X \cup A = Y \cup A$ for some non-empty set A. Then:
(A) X is a proper subset of y
(B) y is a proper subset of X
(C) $X = Y$
(D) X and Y are disjoint sets

26. If $R = \{(x, y) \mid x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation in Z, then domain of R is:
(A) $\{0, 1, 2\}$
(B) $\{0, -1, -2\}$
(C) $\{-2, -1, 0, 1, 2\}$
(D) None of these

27. Let A and B be two sets containing four and two elements respectively. Then, the number of subsets of the set $A \times B$, each having at least three elements is:
(A) 219
(B) 256
(C) 275
(D) 510

28. Let F_1 be the set of parallelograms, F_2 the set of rectangles, F_3 the set of rhombuses, F_4 the set of squares and F_5 the set of trapezium in a plane. Then, F_1 may be equal to:
(A) $F_2 \cap F_3$
(B) $F_3 \cap F_4$
(C) $F_2 \cup F_5$
(D) $F_2 \cup F_3 \cup F_4 \cup F_1$

29. If A and B are two sets, then $A \cap (A \cup B)$ equal to:
(A) A
(B) B
(C) ϕ
(D) $A \cap B$

30. If X and Y are two sets and X' denotes the complement of X, then $X \cap (X \cup Y)'$ is equal to:
(A) X
(B) Y
(C) ϕ
(D) $X \cap Y$

31. If $A = \{x : x$ is a multiple of $4\}$ and $B = \{x : x$ is a multiple of $6\}$ then $A \cap B$ consists of all multiples of:
(A) 16
(B) 12
(C) 8
(D) 4

32. If $A = \{(x, y) : x^2 + y^2 = 25\}$ and $B = \{(x, y) : x^2 + 9y^2 = 144\}$, then, $A \cap B$ contains:
(A) One point
(B) Three points
(C) Two points
(D) Four points

33. For real numbers x and y, we write $xRy \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then, the relation R is:
(A) Reflexive
(B) Symmetric
(C) Transitive
(D) Equivalence

34. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12)\ (6, 12)\ (3, 9)\ (3, 12), (3, 6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$. The relation is:
(A) An equivalence relation
(B) Reflexive and symmetric only
(C) Reflexive and transitive only
(D) Reflexive only

35. The relation 'less then' in the set of natural number is:
(A) Only symmetric
(B) Only transitive
(C) Only reflexive
(D) Equivalence relation

36. Let R_1 be relation defined by $R_1 = \{(a, b) \mid a \geq b, a, b \in R\}$. Then, R_1 is:
(A) An equivalence relation on R
(B) Reflexive, transitive but not symmetric
(C) Symmetric, Transitive but not reflexive
(D) Neither transitive nor reflexive but symmetric.

37. Let $A = \{2, 3, 4, 5\}$ and let $R = \{(2, 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)\}$ be a relation in A. Then, R is:
(A) Reflexive and transitive
(B) Reflexive and symmetric
(C) Reflexive and anti symmetric
(D) Equivalence

38. Let $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$. The relation R is:
(A) Transitive
(B) Not symmetric
(C) Reflexive
(D) A function

39. Given $n(U) = 20$, $n(A) = 12$, $n(B) = 9$, $n(A \cap B) = 4$, where U is the universal set, A and B are subsets of U, then $n((A \cup B)^c) =$

(A) 17 (B) 9
(C) 11 (D) 3

40. If A and B are two given sets, then $A \cap (A \cap B)^c$ is equal to:

(A) A (B) B
(C) ϕ (D) $A \cap B^c$

41. In a class of 80 students numbered 1 to 80, all odd numbered students opt for Cricket, student whose numbers are divisible by 5 opt for Football and those whose numbers are divisible by 7 opt for Hockey. The number of students who do not opt any of the three games, is:

(A) 13 (B) 24
(C) 28 (D) 52

42. On the set N of all natural numbers define the relation R by aRb if and only if the G.C.D. of a and b is 2, then R is:

(A) Reflexive, but not symmetric
(B) Symmetric only
(C) Reflexive and transitive
(D) Reflexive, symmetric and transitive

43. For any two real numbers a and b, we define aRb if and only if $\sin^2 a + \cos^2 b = 1$. The relation R is:

(A) Reflexive but not symmetric
(B) Symmetric but not transitive
(C) Transitive but not Reflexive
(D) An equivalence relation

44. If R be the set of points inside a rectangle of sides a and b ($a, b > 1$) with two sides along the positive direction of x-axis and y-axis, then:

(A) $R = \{(x, y) : 0 \le x \le a, 0 \le y \le b\}$
(B) $R = \{(x, y) : 0 \le x < a, 0 \le y \le b\}$
(C) $R = \{(x, y) : 0 \le x \le a, 0 < y < b\}$
(D) $R = \{(x, y) : 0 < x < a, 0 < y < b\}$

45. In a town of 840 persons, 450 persons read Hindi, 300 read English and 200 read both. Then, the number of persons who read neither, is:

(A) 210 (B) 290
(C) 180 (D) 260

46. Let T be the set of all triangles in the Euclidean plane and let a relation R on T be defined as aRb, If a is congruent to b, $\forall a, b \in T$. Then, R is:

(A) Reflexive but not transitive
(B) Transitive but not symmetric
(C) Equivalence
(D) None of these

47. If $A = \{x : x$ is a multiple of 3$\}$ and $B = \{x : x$ is a multiple of 5$\}$, then $A - B$ is: ($\bar{A}$ means complement of A)

(A) $\bar{A} \cap B$ (B) $A \cap \bar{B}$
(C) $\bar{A} \cap \bar{B}$ (D) $\overline{A \cap B}$

48. If $A = \{x : x^2 - 5x + 6 = 0\}$, $B = \{2, 4\}$, $C = \{4, 5\}$, then $A \times (B \cap C)$ is:

(A) $\{(2, 4), (3, 4)\}$ (B) $\{(4, 2), (4, 3)\}$
(C) $\{(2, 4), (3, 4), (4, 4)\}$ (D) $\{(2, 2), (3, 3), (4, 4), (5, 5)\}$

49. Given the sets $A = \{1, 2, 3\}$, $B = \{3, 4\}$, $C = \{4, 5, 6\}$, then $A \cup (B \cap C)$ is:

(A) $\{3\}$ (B) $\{1, 2, 3, 4\}$
(C) $\{1, 2, 4, 5\}$ (D) $\{1, 2, 3, 4, 5, 6\}$

50. The relation R defined on the set of natural numbers as $\{(a, b) : a$ differs from b by 3$\}$, is given by:

(A) $\{(1, 4), (2, 5), (3, 6), \ldots\}$ (B) $\{(4, 1), (5, 2), (6, 3), \ldots\}$
(C) $\{(1, 3), (2, 6), (3, 9), \ldots\}$ (D) None of these

51. Let r be a relation from R (set of real numbers) to R defined by $r = \{(a, b) \mid a, b \in R$ and $a - b + \sqrt{3}$ is an irrational number$\}$. The relation r is:

(A) An equivalence relation
(B) Reflexive only
(C) Symmetric only
(D) Transitive only

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. Let $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$, then $n(A^c \cap B^c) =$

2. A class has 175 students. The following data shows the number of students obtaining one or more subjects. Mathematics 100, Physics 70, Chemistry 40; Mathematics and Physics 30, Mathematics and Chemistry 28, Physics and Chemistry 23; Mathematics, Physics and Chemistry 18. How many students have offered Mathematics alone.

3. If $n(A)$ denotes the number of elements in set A and if $n(A) = 4$, $n(B) = 5$ and $n(A \cap B) = 3$, then $n[(A \times B) \cap (B \times A)] =$

4. Out of 800 boys in a school, 224 played cricket, 240 played hockey and 336 played basketball. Of the total, 64 played both basketball and hockey; 80 played cricket and basketball and 40 played cricket and hockey; 24 played all the tree games. The number of boys who did not play any game is ____________.

5. There are 100 students in a class. In a examination, 50 of them failed in Mathematics, 45 failed in Physics, 40 failed in Biology and 32 failed in exactly two of three subjects. Only one student passed in all the subjects. Then the number of students failing in all the three subjects is ____________.

6. Given the relation $R = \{(2, 3), (3, 4)\}$ on the set $\{2, 3, 4\}$. The number of minimum number of ordered pairs to be added to R so that R is reflexive an symmetric -

7. The minimum number of elements that must be added to the relation $R = \{(1, 2), (2, 3)\}$ on the set $\{1, 2, 3\}$, so that it is equivalence is ____________.

8. If $n(A) = 3$, $n(B) = 6$ and $A \subseteq B$. Then the number of elements in $A \cup B$ is equal to ____________.

9. In a battle 70% of the combatants lost one eye, 80% an ear, 75% an arm, 85% a leg, x% lost all the four limbs. The minimum values of x is ____________.

10. The number of non-empty subsets of the set $\{1, 2, 3, 4\}$ is ____________.

11. The number of elements in the set $\{(a, b) : 2a^2 + 3b^2 = 35, a, b \in Z\}$, where Z is the set of all integers, is ____________.

12. Let $S = \{1, 2, 3, 4\}$. The total number of unordered pairs of disjoint subsets of S is equal to ____________.

13. Let $X = \{1, 2, 3, 4, 5\}$. The number of different ordered pairs (Y, Z) that can formed such that $Y \subseteq X$, $Z \subseteq X$ and $Y \cap Z$ is empty, is ____________.

14. Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$ ____________.

15. In a town of 10,000 families it was found that 40% family buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C, 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three newspapers, then number of families which buy A only is ____________.

16. If $S = \{x \mid x \text{ is } a \text{ positive multiple of 3 less than 100}\}$ and $P = \{x \mid x \text{ is prime number less than 20}\}$. Then, $n(S) + n(P)$ is equal to ____________.

ARCHIVE: JEE MAIN

1. **[JEE Mains 2021]**
Let $A = \{n \in N : n \text{ is 3-digit number}\}$
$B = \{9k + 2 : k \in N\}$
and $C : \{9k + l : k \in N\}$ for some l $(0 < l < 9)$
If the sum of all the elements of the set $A \cap (B \cup C)$ is 274×400, then l is equal to _______.

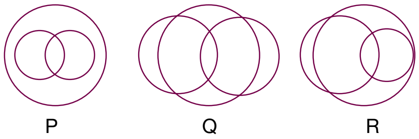
2. **[JEE Mains 2021]**
If $A = \{x \in R: |x - 2| > 1\}$, $B = \left\{x \in R : \sqrt{x^2 - 3} > 1\right\}$, $C = \{x \in R: |x - 4| \geq 2\}$ and Z is the set of all integers, then the number of subsets of the set $(A \cap B \cap C)^C \cap Z$ is __________.

3. **[JEE Mains 2021]**
Let $A = \{n \in N \mid n^2 \leq n + 10{,}000\}$, $B = \{3k + 1 \mid k \in N\}$ and $C = \{2k \mid k \in N\}$, then the sum of all the elements of the set $A \cap (B - C)$ is equal to _________.

4. **[JEE Mains 2021]**
The number of elements in the set $\{x \in R: (|x| - 3)|x + 4| = 6\}$ is equal to
(A) 3
(B) 2
(C) 4
(D) 1

5. **[JEE Mains 2021]**
Let $A = \{2, 3, 4, 5, \ldots, 30\}$ and '$\simeq$' be an equivalence relation on $A \times A$. defined by $(a, b) \simeq (c, d)$, if and only if $ad = bc$. Then the number of ordered pairs which satisfy this equivalence relation with ordered pair $(4, 3)$ is equal to:
(A) 5
(B) 6
(C) 8
(D) 7

6. **[JEE Mains 2021]**
In a school, there are three types of games to be played. Some of the students play two types of games, but none play all the three games. Which Venn diagrams can justify the above statement?



(A) P and Q
(B) P and R
(C) Q and R
(D) None of these

7. **[JEE Mains 2021]**
Let N be the set of natural numbers and a relation R on N be defined by $R = \{(x, y) \in N \times N : x^3 - 3x^2y - xy^2 + 3y^3 = 0\}$. Then the relation R is:
(A) Reflexive but neither symmetric nor transitive
(B) An equivalence relation
(C) Reflexive and symmetric, but not transitive
(D) Symmetric but neither reflexive nor transitive

8. **[JEE Mains 2020]**
Let $X = \{n \in N : 1 \leq n \leq 50\}$. If $A = \{n \in X : n \text{ is a multiple of 2}\}$ and $B = \{n \in X : n \text{ is a multiple of 7}\}$, then the number of elements in the smallest subset of X containing both A and B is ________.

9. **[JEE Mains 2020]**
Set A has m elements and Set B has n elements. If the total number of subsets of A is 112 more than the total number of subsets of B, then the value of $m.n$ is ________.

10. **[JEE Mains 2020]**
If $A = \{x \in R : |x| < 2\}$ and $B = \{x \in R : |x - 2| \geq 3\}$; then:
(A) $A \cup B = R - (2, 5)$
(B) $A - B = [-1, 2)$
(C) $A \cap B = (-2, -1)$
(D) $B - A = R - (-2, 5)$

11. **[JEE Mains 2020]**
If $R = \{(x, y) : x, y \in Z, x^2 + 3y^2 \leq 8\}$ is a relation on the set of integers Z, then the domain of R^{-1} is:
(A) $\{-2, -1, 1.2\}$
(B) $\{-1, 0, 1\}$
(C) $\{-2, -1, 0, 1, 2\}$
(D) $\{0, 1\}$

12. [JEE Mains 2020]

Let R_1 and R_2 be two relations defined as follows
$R_1 = \{(a, b) \in R^2 : a^2 + b^2 \in Q\}$ and $R_2 = \{(a, b) \in R^2 : a^2 + b^2 \notin Q\}$, where Q is the set of all rational numbers then:

(A) R_2 is transitive but R_1 is not transitive
(B) R_1 is transitive but R_2 is not transitive
(C) R_1 and R_2 are both transitive
(D) Neither R_1 nor R_2 is transitive

13. [JEE Mains 2020]

Let $\bigcup_{i=1}^{50} X_i = \bigcup_{i=1}^{n} Y_i = T$. Where each X_i contains 10 elements and each Y_i contains 5 elements. If each element of the set T is an element of exactly 20 of sets X_i's and exactly 6 of sets Y_i's, then n is equal to:

(A) 45
(B) 15
(C) 50
(D) 30

14. [JEE Mains 2020]

A survey shows that 63% of the people in a city read newspaper A whereas 76% read newspaper B. If x% of the people read both the newspapers, then a possible value of x can be:

(A) 65
(B) 37
(C) 29
(D) 55

15. [JEE Mains 2020]

A survey shows that 73% of the persons working in an office like coffee, whereas 65% like tea. If x denotes the percentage of them, who like both coffee and tea, then x cannot be:

(A) 63
(B) 38
(C) 54
(D) 36

16. [JEE Mains 2019]

In a class of 140 students numbered 1 to 140 all even numbered students opted Mathematics course, those whose number is divisible by 3 opted Physics course and those whose number is divisible by 5 opted Chemistry course. Then the number of students who did not opt for any of the three courses is:

(A) 42
(B) 102
(C) 1
(D) 38

17. [JEE Mains 2019]

Let $S = \{1, 2, 3, \ldots, 100\}$. The number of non-empty subsets A of S such that the product of elements in A is even is:

(A) $2^{50} - 1$
(B) $2^{50}(2^{50} - 1)$
(C) $2^{100} - 1$
(D) $2^{50} + 1$

18. [JEE Mains 2019]

Let Z be the set of integers. If $A = \{x \in Z : 2^{(x+2)(x^2-5x+6)} = 1\}$ and $B = \{x \in Z : -3 < 2x - 1 < 9\}$, then the number of subsets of the set $A \times B$, is:

(A) 2^{12}
(B) 2^{18}
(C) 2^{10}
(D) 2^{15}

19. [JEE Mains 2019]

Two newspapers A and B are published in a city. It is known that 25% of the city population reads A and 20% reads B while 8% reads both A and B. Further, 30% of those who read A but not B look into advertisements and 40% of those who read B but not A also look into advertisements, while 50% of those who read both A and B look into advertisements. Then the percentage of the population who look into advertisements is:

(A) 13.9
(B) 13.5
(C) 12.8
(D) 13

20. [JEE Mains 2019]

Let A, B and C be sets such that $\phi \neq A \cap B \subseteq C$. Then which of the following statements is not true?

(A) If $(A - B) \subseteq C$, then $A \subseteq C$
(B) $B \cap C \neq \phi$
(C) $(C \cup A) \cap (C \cup B) = C$
(D) If $(A - C) \subseteq B$, then $A \subseteq B$

21. [JEE Mains 2018]

Two sets A and B are as under.
$A = \{(a, b) \in R \times R : |a - 5| < 1$ and $|b - 5| < 1\}$ and $B = \{(a, b) \in R \times R : 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}$. Then:

(A) $A \subset B$
(B) $A \cap B = \phi$ (an empty set)
(C) Neither $A \subset B$ nor $B \subset A$
(D) $B \subset A$

22. [JEE Mains 2015 and 2013]

Let A and B be two sets containing four and two elements, respectively. Then the number of subsets of the set $A \times B$, each having at least three elements is:

(A) 275
(B) 510
(C) 219
(D) 256

23. [JEE Mains 2014]

If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n - 1) : n \in N\}$, where N is the set of natural numbers, then $X \cup Y$ is equal to:

(A) X
(B) Y
(C) N
(D) $Y - X$

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. B	**2.** A	**3.** A	**4.** C	**5.** C	**6.** B	**7.** A	**8.** D	**9.** D	**10.** A
11. B	**12.** D	**13.** C	**14.** D	**15.** C	**16.** C	**17.** B	**18.** D	**19.** A	**20.** A
21. D	**22.** C	**23.** C	**24.** B	**25.** C	**26.** C	**27.** A	**28.** D	**29.** A	**30.** C
31. B	**32.** D	**33.** A	**34.** C	**35.** B	**36.** B	**37.** B	**38.** B	**39.** D	**40.** D
41. C	**42.** B	**43.** D	**44.** D	**45.** B	**46.** C	**47.** B	**48.** A	**49.** B	**50.** B
51. B									

Integer/Numerical Answer Type Questions

1. 300	**2.** 60	**3.** 9	**4.** 160	**5.** 2	**6.** 5	**7.** 7	**8.** 6	**9.** 10	**10.** 15
11. 8	**12.** 41	**13.** 243	**14.** 6	**15.** 3300	**16.** 41				

ARCHIVE: JEE MAIN

1. 5	**2.** 256	**3.** 832	**4.** B	**5.** D	**6.** D	**7.** A	**8.** 29.00	**9.** 28.00	**10.** D
11. B	**12.** D	**13.** D	**14.** D	**15.** D	**16.** D	**17.** B	**18.** D	**19.** A	**20.** D
21. A	**22.** C	**23.** B							

CHAPTER 2

Function and its Graph

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Function Definition
(b) Various Types of Functions
(c) Graph Transformation
(d) Domain and Range
(e) Equal/Identical Function
(f) Odd, Even, Periodic Function
(g) Classification of Function
(h) Composite Function
(i) Inverse Function
(j) Standard Functional Equations

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

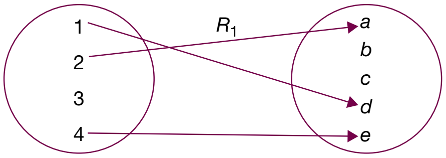
FUNCTION

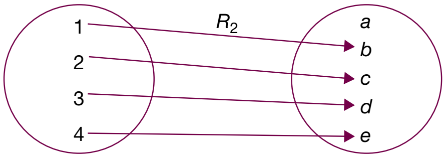
Relation from set A to set B is known as a function if each element from set A is uniquely connected with set B.

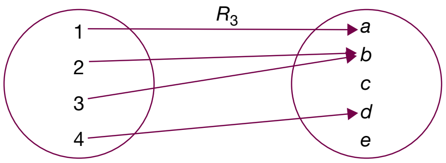
Every function is a relation but every relation is not a function.

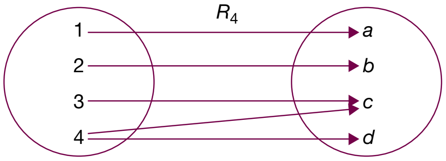
EXAMPLE:

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d, e\}$ be two sets and let R_1, R_2, R_3 and R_4 be relations from set A to set B. These mappings are shown as follows:





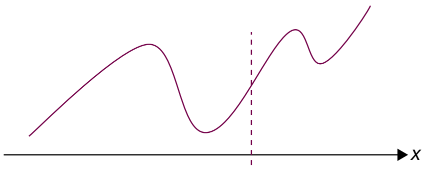




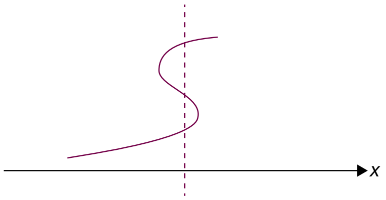
Now see that R_1 is not function from set A to set B, since there is an element $3 \in A$ which is not associated with any element of B. But R_2 and R_3 are the functions from A to B, because under R_2 and R_3 each element in A is associated with a unique element in B. But R_4 is not a function from A to B because an element $4 \in A$ is associated with two elements c and d in B.

VERTICAL LINE TEST TO IDENTIFY A FUNCTION

If we are able to draw even a single vertical line (parallel to y-axis) which cuts the graph of curve more than one point, then it is not a function.



Yes, function

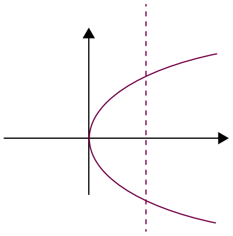


Not function

EXAMPLES:

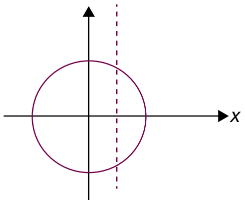
(i) $y^2 = 4x$

There exists a vertical line which cuts the graph at more than one point, then it is not a function.



(ii) $x^2 + y^2 = 4$

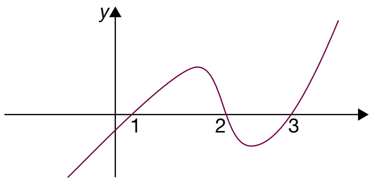
There exists a vertical line which cuts the graph at more than one point, then it is not a function.



(iii) $y = x^3 - 6x^2 + 11x - 6$

$y = (x - 1)(x - 2)(x - 3)$

We cannot draw even a single vertical line which cuts the graph at more than one point, so it is a function.



Let function $f: A \to B$

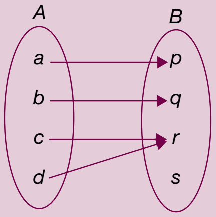
Domain of function f = Set A = All possible values of x for which $f(x)$ exists.

Co-domain of function f = Set B.

Range of function f = subset of Set B = set formed by images of $f(x)$.

For example by considering the following mapping defined from Set A to set B.

- Domain = $\{a, b, c, d\} = A$
- Co-domain = $\{p, q, r, s\} = B$
- Range = $\{p, q, r\}$



TYPES OF FUNCTIONS

Algebraic Functions

(i) Polynomial function: A function having the form $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots a_nx^n$, where $a_0, a_1, a_2 \dots a_n$ are real constant, $a_n \neq 0$ and $n \in W$ called rational integral function or polynomial of degree n.

(ii) Rational Function: The ratio of two polynomial is called fraction rational function.

E.g., $y = \dfrac{x^{12} + x^2 - 1}{x^6 + x^4 + 1}$

(iii) Irrational Function: Functions with operations of addition, subtraction, multiplication, division and raising to power with non-integral rational exponent are called irrational functions.

E.g., $y = \sqrt{x}$, $y = \dfrac{\sqrt{x^3 + 1} - \sqrt{x^{11}}}{\sqrt{x^2 + x + 1}}$

Such types of functions are called irrational functions.

Table 2.1 Domain and range of the algebraic functions

Function ($y = f(x)$)	**Domain (values taken by x)**	**Range (values taken by $f(x)$)**
x^n, $(n \in N)$	R = (set of real numbers)	R, if n is odd $R^+ \cup \{0\}$, if n is even
$\dfrac{1}{x^n}$, $(n \in N)$	$R - \{0\}$	$R - \{0\}$, if n is odd R^+, if n is even
$x^{1/n}$, $(n \in N)$	R, if n is odd $R^+ \cup \{0\}$, if n is even	R, if n is odd $R^+ \cup \{0\}$, if n is even
$\dfrac{1}{x^{1/n}}$, $(n \in N)$	$R - \{0\}$, if n is odd R^+, if n is even	$R - \{0\}$, if n is odd R^+, if n is even

Transcendental Function

All those functions which are infinite terms while expanded are called transcendental functions.

For example, all trigonometrical functions, inverse trigonometrical function, exponential function, logarithmic function, etc.

E.g., $f(x) = \sin x$, $y = \cos^{-1} x$, $y = \log_e x$, $y = \sqrt{\log_e x - \sin^{-1} x}$

Explicit Function

A function is said to be explicit if it can be expressed directly in terms of the independent variable.

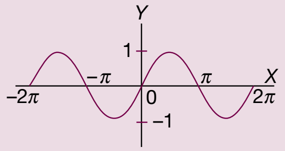
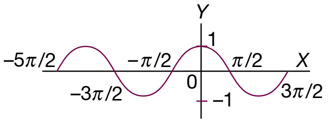
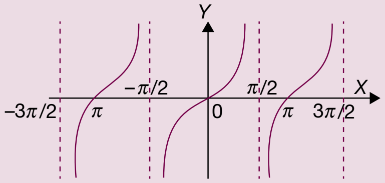
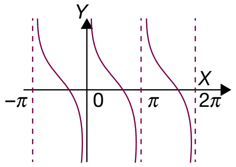
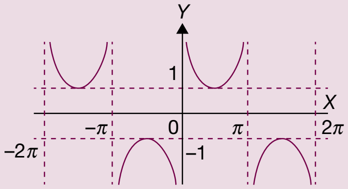
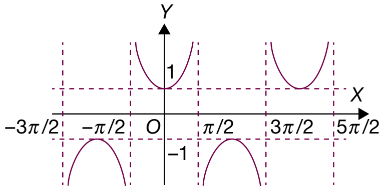
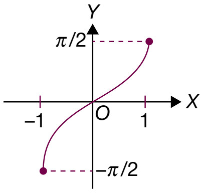
$y = f(x)$ or $x = \phi(y)$

Implicit Function

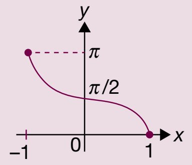
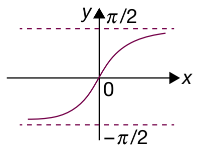
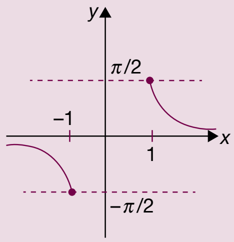
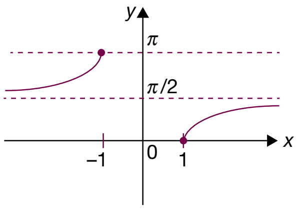
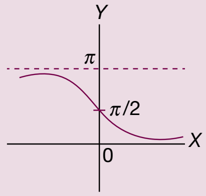
A function is said to be implicit if it cannot be expressed directly in terms of the independent variable.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Table 2.2 Domain, range and graphs of the functions

Function ($y = f(x)$)	**Domain** (i.e. values taken by x)	**Range** (i.e. values taken by $f(x)$)	**Graph**
Trigonometric Functions			
$\sin x$	R	$[-1, 1]$	
$\cos x$	R	$[-1, 1]$	
$\tan x$	$R - \{(2k+1)\frac{\pi}{2}\}$ where $k \in I$	R	
$\cot x$	$R - \{k\pi\}$, where $k \in I$	R	
$\operatorname{cosec} x$	$R - \{k\pi\}$, where $k \in I$	$(-\infty, -1] \cup [1, \infty)$	
$\sec x$	$R - \{(2k+1)\frac{\pi}{2}\}$, where $k \in I$	$(-\infty, -1] \cup [1, \infty)$	
Inverse Trigonometric Functions			
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	

(Continued)

Function ($y = f(x)$)	**Domain (i.e. values taken by x)**	**Range (i.e. values taken by $f(x)$)**	**Graph**
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$	
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$	
$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$	
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$	
$\cot^{-1} x$	R	$(0, \pi)$	

MODULUS FUNCTION (ABSOLUTE VALUE FUNCTION)

By definition, square root of a non-negative real number is always a non-negative real number.

$$\sqrt{x^2} = |x| \text{ where } |x| \geq 0,$$

Geometrically, it represents a distance between number x and origin.

(i) $|-x| = |x|$ or $|x - a| = |a - x|$

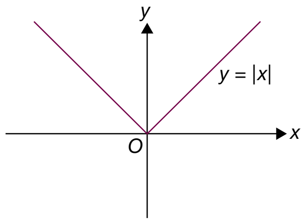
(ii) $|x|^2 = x^2$

(iii) $|xy| = |x|\,|y|$

(iv) Modulus function cannot be a negative quantity.

$$y = |x| = \begin{cases} x & ; \quad x \geq 0 \\ -x & ; \quad x < 0 \end{cases}$$

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$



(v) $|x| \le a \Rightarrow -a \le x \le a$ if $a > 0$

(vi) $|x| \ge a \Rightarrow x \le -a$ or $x \ge a$ if $a > 0$

(vii) For any two real numbers a and b, the following modulus inequality always holds.

- $||a| - |b|| \le |a+b| \le |a| + |b|$
- $||a| - |b|| \le |a-b| \le |a| + |b|$

Let us discuss when equality holds:

CASE 1:
$|a+b| = |a| + |b|$; $||a| - |b|| = |a-b|$ then a and b are of the same sign, it means $ab \ge 0$.

CASE 2:
$|a-b| = |a| + |b|$; $||a| - |b|| = |a+b|$ then a and b are of opposite sign, it means $ab \le 0$.

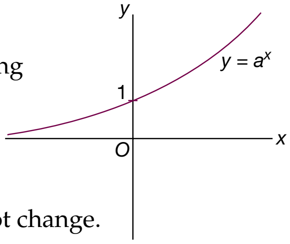
EXPONENTIAL FUNCTION

$y = a^x$ is defined for every real x where $a > 0, a \ne 1$ is known as an exponential function.
It has two types of behaviour depending upon the values of a.

CASE 1: Base $a > 1$
$y = a^x$ where $a > 1$ for $\forall\ x \in R$
As we can see it has an increasing pattern, as we increase the values of x, a^x also increases:

$$x_1 < x_2 \Rightarrow a^{x_1} < a^{x_2}$$

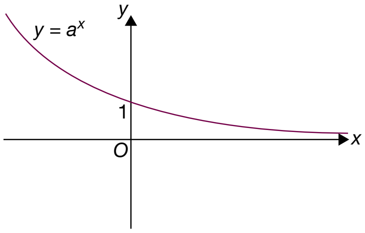
Here, sign of inequality does not change.



CASE 2: Base $a < 1$
$y = a^x$ where $0 < a < 1$ for $\forall\ x \in R$
As we can see it has a decreasing pattern, as we increase the values of x, a^x decreases:

$$x_1 < x_2 \Rightarrow a^{x_1} > a^{x_2}$$

Here, sign of inequality changes.



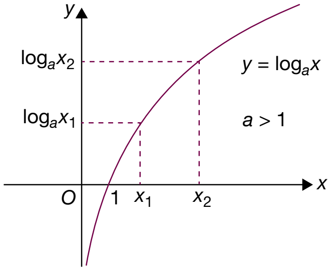
LOGARITHMIC FUNCTION

As we know that exponential function $a^x = N$; where $a > 0$, $a \ne 1$ and $N > 0$.

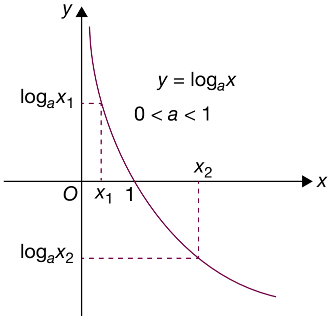
$$x = \log_a N$$

It is known as logarithmic function.
$f(x) = \log_a x$ is defined for positive real x and it is read as 'Logarithm of x to the base a'.
It tells about the index power given to base 'a' to become 'x'.

CASE 1: Base $= a > 1$,
$\log_a x$ is an increasing function (no sign change in inequality).
If $x_1 < x_2 \Rightarrow \log_a x_1 < \log_a x_2$



CASE 2: $0 < a < 1$,
$\log_a x$ is decreasing function (sign changes in inequality).
If $x_1 < x_2 \Rightarrow \log_a x_1 > \log_a x_2$



The function $y = \log_{g(x)} f(x)$ is defined only if $f(x) > 0$, $g(x) > 0$ and $g(x) \ne 1$.

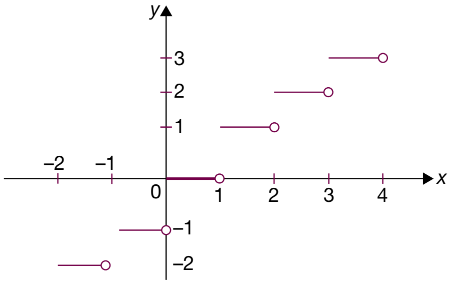
(i) $\log_a x > \log_a y \Leftrightarrow \begin{cases} x > y, & \text{if } \quad a > 1 \\ x < y, & \text{if } \quad 0 < a < 1 \end{cases}$

(ii) $\log_a x > b \Leftrightarrow \begin{cases} x > a^b, & \text{if } \quad a > 1 \\ x < a^b, & \text{if } \quad 0 < a < 1 \end{cases}$

Solving equations and inequalities involving modulus functions, exponential functions and logarithm functions are discussed in detail in the Algebra book, students are advised to go through that.

GREATEST INTEGER FUNCTION

A function is said to be greatest integer function if it is of the form of $f(x) = [x]$ where $[x]$ = greatest integer equal or less than x. E.g: $[4.2] = 4$, $[-4.\,4] = -5$



$[\pi] = 3, [e] = 2$

$[-\pi] = -4, [-e] = -3, \left[\dfrac{1}{e}\right] = 0$

$[e^2] = 7, [\pi^2] = 9, [-e^2] = -8, [-\pi^2] = -10$

(i) $x - 1 < [x] \le x$, if x is an integer
(ii) $[x \pm m] = [x] \pm m$, if m is an integer
(iii) $[x + y] \ge [x] + [y]$
(iv) If $[\phi(x)] \ge m$, then $\phi(x) \ge m$ if $m \in I$
(v) If $[\phi(x)] \le m$, then $\phi(x) < m + 1$ if $m \in I$
(vi) If $[\phi(x)] > m$, then $\phi(x) \ge m + 1$ if $m \in I$
(vii) If $[\phi(x)] < m$, then $\phi(x) < m$ if $m \in I$

(viii) $[x]+[-x]=\begin{cases} 0; & x \in I \\ -1; & x \notin I \end{cases}$

(ix) $[x + y] = [x] + [y + x - [x]]\ \forall\ x, y \in R$

(x) $[[x]] = [x]$

(xi) $[x] + \left[x+\dfrac{1}{n}\right]+\left[x+\dfrac{2}{n}\right]+\ldots+\left[x+\dfrac{n-1}{n}\right] = [nx]; n \in N$

(xii) $[x] = p$

CASE 1: If $p \notin I$, then $x \notin \phi$

CASE 2: $p \in I$, then $p \le x < p + 1$

EXAMPLE:

- If $[x] = 2 \Rightarrow 2 \le x < 3$
- If $[x] = 5 \Rightarrow 5 \le x < 6$
- If $[x] = -4 \Rightarrow -4 \le x < -3$
- If $[x] = -10 \Rightarrow -10 \le x < -9$

ILLUSTRATION 1

Compute x where [.] is GIF

(i) $[3x] = -10$ **(ii)** $\left[-\dfrac{x}{2}\right] = -5$

(iii) $\left[-\dfrac{x}{3}+\dfrac{1}{2}\right] = \dfrac{13}{6}$ **(iv)** $\left[x+\dfrac{3}{10}\right] = 5.1$

(v) $\left[\dfrac{x}{2}+\dfrac{5}{3}\right] = 2$ **(vi)** $[\log_2 x] = 1$

(vii) $[\sin x] = 0$ **(viii)** $[x^2 - 3x] = 2$

SOLUTION

(i) $[3x] = -10$

$\Rightarrow -10 \le 3x < -9$

$\Rightarrow -\dfrac{10}{3} \le x < -3$ or $x \in \left[-\dfrac{10}{3}, -3\right)$

(ii) $\left[-\dfrac{x}{2}\right] = -5 \Rightarrow -5 \le -\dfrac{x}{2} < -4$

$\Rightarrow -10 \le -x < -8 \Rightarrow 10 \ge x > 8$

$\Rightarrow x \in (8, 10]$

(iii) $\left[-\dfrac{x}{3}+\dfrac{1}{2}\right] = \dfrac{13}{6}$ is not an integer

$\Rightarrow x \in \phi$

(iv) $\left[x+\dfrac{3}{10}\right] = 5.1$ is not an integer

$\Rightarrow x \in \phi$

(v) $\left[\dfrac{x}{2}+\dfrac{5}{3}\right] = 2 \Rightarrow 2 \le \dfrac{x}{2} + \dfrac{5}{3} < 3$

$\Rightarrow 2 - \dfrac{5}{3} \le \dfrac{x}{2} < 3 - \dfrac{5}{3} \Rightarrow \dfrac{1}{3} \le \dfrac{x}{2} < \dfrac{4}{3}$

$\Rightarrow \dfrac{2}{3} \le x < \dfrac{8}{3}$

(vi) $[\log_2 x] = 1$

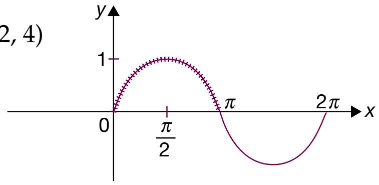
$\Rightarrow 1 \le \log_2 x < 2 \Rightarrow 2' \le x < 2^2$

$\Rightarrow 2 \le x < 4$ or $x \in [2, 4)$

(vii) $[\sin x] = 0$

$\Rightarrow 0 \le \sin x < 1$

$x \in [0, \pi] - \left\{\dfrac{\pi}{2}\right\}$



$x \in [2n\pi, 2n\pi + \pi] - \left\{2n\pi + \dfrac{\pi}{2}\right\}$ where $n \in I$

(viii) If $[x^2 - 3x] = 2$ where [.] is GIF

$\Rightarrow 2 \le x^2 - 3x < 3$

$\Rightarrow 2 \le x^2 - 3x$ and $x^2 - 3x < 3$

$\Rightarrow x^2 - 3x - 2 \ge 0 \Rightarrow \left(x - \dfrac{3}{2}\right)^2 - 2 - \dfrac{9}{4} \ge 0$

$\Rightarrow \left(x - \dfrac{3}{2}\right)^2 - \left(\dfrac{\sqrt{17}}{2}\right)^2 \ge 0$

$\Rightarrow \left(x - \dfrac{3}{2} + \dfrac{\sqrt{17}}{2}\right)\left(x - \dfrac{3}{2} - \dfrac{\sqrt{17}}{2}\right) \ge 0$

$$\Rightarrow x \in \left(-\infty, \frac{3-\sqrt{17}}{2}\right] \cup \left[\frac{3+\sqrt{17}}{2}, \infty\right) \qquad \ldots(1)$$

Now for second condition $x^2 - 3x < 3$

$$\Rightarrow x^2 - 3x - 3 < 0 \Rightarrow \left(x - \frac{3}{2}\right)^2 - 3 - \frac{9}{4} < 0$$

$$\Rightarrow \left(x - \frac{3}{2} + \frac{\sqrt{21}}{2}\right)\left(x - \frac{3}{2} - \frac{\sqrt{21}}{2}\right) < 0$$

$$\Rightarrow x \in \left(\frac{3-\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}\right) \qquad \ldots(2)$$

Eq. (1) ∩ (2), we get

$$x \in \left(\frac{3-\sqrt{21}}{2}, \frac{3-\sqrt{17}}{2}\right] \cup \left[\frac{3+\sqrt{17}}{2}, \frac{3+\sqrt{21}}{2}\right)$$

ILLUSTRATION 2

Find t and x if $([3t] - 4)^2 + ([7x] - 21)^2 = 0$ where [.] is GIF.

SOLUTION

As $([3t] - 4)^2 + ([7x] - 21)^2 = 0$, it implies $([3t] - 4)^2 = 0$ and $([7x] - 21)^2 = 0$

Here, $[3t] = 4 \Rightarrow 4 \le 3t < 5 \Rightarrow \frac{4}{3} \le t < \frac{5}{3}$

Now $[7x] = 21$

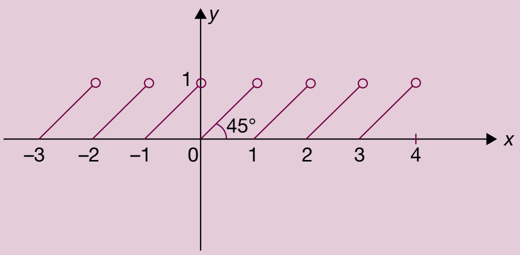
$$\Rightarrow \quad 21 \le 7x < 22 \Rightarrow 3 \le x < \frac{22}{7} \Rightarrow x \in \left[3, \frac{22}{7}\right)$$

Therefore, $\frac{4}{3} \le t < \frac{5}{3}$ and $x \in \left[3, \frac{22}{7}\right)$

FRACTIONAL PART FUNCTION

$x = [x] + \{x\}$ where [.] is greatest integer function and {.} is fractional part function.

We get $\{x\} = x - [x]$



(i) $0 \le \{x\} < 1$, for all $x \in R$

(ii) $\{x\}$ is periodic function with period 1

(iii) $\{x \pm m\} = \{x\}$, if m is an integer

(iv) $\{x\} = 0$ if $x \in I$

(iv) $\{x\} + \{-x\} = \begin{cases} 0; & x \in I \\ 1; & x \notin I \end{cases}$

(v) $\{x\} = p$

CASE 1: If $p \notin [0, 1)$, then $x \notin \phi$

CASE 2: If $p \in [0, 1)$, then $x = n + p$ where $n \in I$

$$\{\pi\} = \pi - [\pi] = \pi - 3, \{e\} = e - [e] = e - 2$$

$$\{-\pi\} = -\pi - [-\pi] = -\pi - (-4) = 4 - \pi$$

$$\{-e\} = -e - [-e] = -e - (-3) = 3 - e$$

$$\left\{\frac{1}{e}\right\} = \frac{1}{e}$$

$$\{e^2\} = e^2 - [e^2] = e^2 - 7, \{\pi^2\} = \pi^2 - [\pi^2] = \pi^2 - 9$$

$$\{-e^2\} = -e^2 - [-e^2] = -e^2 - (-8) = 8 - e^2$$

$$\{-\pi^2\} = -\pi^2 - [-\pi^2] = -\pi^2 - (-10) = 10 - \pi^2$$

ILLUSTRATION 3

Solve for x where {.} is fractional part function.

(i) $\{x\} = 0.1$ **(ii)** $\{2x - 1\} = \frac{1}{2}$ **(iii)** $\{x\} = \frac{1}{7}$

(iv) $\{3x\} = 0.3$ **(v)** $\{2x + 1\} = \frac{1}{2}$ **(vi)** $\left\{8x - \frac{2}{3}\right\} = \frac{1}{2}$

(vii) $\{3x\} = \sqrt{2}$ **(viii)** $\{7x - 100\} = \frac{1}{\sqrt{2}}$

SOLUTION

(i) $\{x\} = 0 \cdot 1$

$x = p + 0 \cdot 1$ where $p \in I$

$x = \ldots, -1{\cdot}9, -1{\cdot}9, -0 \cdot 1, 1 \cdot 1, 2 \cdot 1, 3 \cdot 1, \ldots$

(ii) $\{2x - 1\} = \frac{1}{2} \Rightarrow \{2x\} = \frac{1}{2}$

$\Rightarrow 2x = p + \frac{1}{2}$ where $p \in I$

$\Rightarrow x = \frac{p}{2} + \frac{1}{4}$

$x = \ldots, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \ldots$

(iii) $\{x\} = \frac{1}{7}$

$x = p + \frac{1}{7}$ when $p \in I$

(iv) $\{3x\} = 0 \cdot 3$

$\Rightarrow 3x = p + 0 \cdot 3$

$\Rightarrow x = \frac{p}{3} + 0 \cdot 1$ where $p \in I$

(v) $\{2x+1\} = \dfrac{1}{2}$

$$\Rightarrow\ 2x+1 = p + \frac{1}{2} \Rightarrow 2x = p - \frac{1}{2}$$

$$\Rightarrow\ x = \frac{p}{2} - \frac{1}{4} \text{ where } p \in I$$

(vi) $\left\{8x - \dfrac{2}{3}\right\} = \dfrac{1}{2}$

$$\Rightarrow\ 8x - \frac{2}{3} = p + \frac{1}{2}$$

$$\Rightarrow\ 8x = p + \frac{1}{2} + \frac{2}{3}$$

$$\Rightarrow\ x = \frac{p}{8} + \frac{7}{48} \text{ where } p \in I$$

(vii) $\{3x\} = \sqrt{2}$

$x \in \phi$ as $\sqrt{2} > 1$

(viii) $\{7x - 100\} = \dfrac{1}{\sqrt{2}}$

$$\Rightarrow\ \{7x\} = \frac{1}{\sqrt{2}} \Rightarrow 7x = p + \frac{1}{\sqrt{2}}$$

$$\Rightarrow\ x = \frac{p}{7} + \frac{1}{7\sqrt{2}} \text{ where } p \in I$$

ILLUSTRATION 4

If $[x] + [-x] = x^2 - 3x - 1$. Find x, where $[\cdot]$ is GIF.

SOLUTION

$$[x] + [x] = \begin{cases} 0 & x \in I \\ -1 & x \notin I \end{cases}$$

CASE 1: $x \in I$

$0 = x^2 - 3x - 1$

$\Rightarrow x = \dfrac{3 \pm \sqrt{9+4}}{2}$

$x = \dfrac{3 \pm \sqrt{13}}{2}$ but $x \in I$

$x \in \phi$

CASE 2: $x \notin I$

$-1 = x^2 - 3x - 1$

$\Rightarrow x^2 - 3x = 0$

$x = 0, 3$ but $x \notin I$

$x \in \phi$

Hence, by CASE 1 $\cup$ CASE 2 we get $x \in \phi$

ILLUSTRATION 5

If $\{x\} + \{-x\} = x^2 - 3x + 2$. Find x, where $\{\cdot\}$ is fpf.

SOLUTION

$$\because\quad \{x\} + \{-x\} = \begin{cases} 0; & x \in I \\ 1; & x \notin I \end{cases}$$

CASE 1: $x \in I$

$\{x\} = 0$

$0 + 0 = x^2 - 3x + 2$

$\Rightarrow (x-1)(x-2) = 0$

$\Rightarrow x = 1, 2$ but $x \in I$

acceptable

CASE 2: $x \notin I$

$\{x\} + \{-x\} = 1 = x^2 - 3x + 2$

$x^2 - 3x + 1 = 0$

$x = \dfrac{3 \pm \sqrt{5}}{2}$ but $x \notin I$; $x = \dfrac{3 \pm \sqrt{5}}{2}$

acceptable

Hence, $x \in \left\{1, 2, \dfrac{3 \pm \sqrt{5}}{2}\right\}$

ILLUSTRATION 6

If $2[x] + 3x = 4 - 5\{x\}$ where $[\cdot]$ is GIF and $\{\cdot\}$ is fpf. Find x.

SOLUTION

$$2[x] + 3x = 4 - 5\{x\}$$

$$\Rightarrow\quad 2[x] + 5\{x\} + 3x = 4$$

$$(\because\ x = [x] + \{x\};\ 0 \le \{x\} < 1 \Rightarrow [x] = x - \{x\})$$

$$2(x - \{x\}) + 5\{x\} + 3x = 4$$

$$\Rightarrow 5x + 3\{x\} = 4 \Rightarrow \{x\} = \frac{4-5x}{3}$$

$$\because\quad 0 \le \{x\} < 1 \Rightarrow 0 \le \frac{4-5x}{3} < 1$$

$$\Rightarrow\quad 0 \le 4 - 5x < 3 \Rightarrow -4 \le -5x < -1 \Rightarrow \frac{1}{5} < x \le \frac{4}{5}$$

$$\Rightarrow\quad 0.2 < x \le 0.8 \qquad \therefore\ [x] = 0$$

$$x = [x] + \{x\};\ x = 0 + \frac{4-5x}{3}$$

$$\Rightarrow\quad 3x = 4 - 5x \Rightarrow 8x = 4 \qquad \text{Hence, } x = \frac{1}{2}$$

ILLUSTRATION 7

If $g(x) = [x] + \displaystyle\sum_{a=1}^{2012} \frac{x + a - [x+a]}{2012}$ where [ ] is greatest integer function then find the value of $g(100)$.

SOLUTION

$$g(x) = [x] + \sum_{a=1}^{2012} \frac{(x+a) - [x+a]}{2012}$$

$$\Rightarrow\quad g(x) = [x] + \frac{1}{2012} \sum_{a=1}^{2012} \{x+a\}$$

For as x and a both are integer; $\{x+a\} = 0$

$$g(100) = [100] + \frac{1}{2012} \times 0$$

$$\therefore\quad g(100) = 100$$

ILLUSTRATION 8

Solve the following equation for x

$|x - 1| = 2[x] - 3\{x\}$ where $[x]$ and $\{x\}$ denotes integral and fractional part of x.

SOLUTION

$$|x-1| = 2[x] - 3\{x\}$$

CASE 1: If $x \ge 1$, then $x - 1 = 2[x] - 3\{x\}$

$$[x] + \{x\} - 1 = 2[x] - 3\{x\}$$

$$4\{x\} = 1 + [x]$$

$\because \quad 0 \le \{x\} < 1 \Rightarrow 0 \le 1 + [x] < 4$

$\therefore \quad -1 \le [x] < 3$ but $x \ge 1$

$\therefore$ Possible value of $[x]$ are 1 and 2.

If $[x] = 1$, then $\{x\} = \frac{1}{2}$

$\therefore \quad x = [x] + \{x\} = 1 + \frac{1}{2} = \frac{3}{2}$

If $[x] = 2$, then $\{x\} = 3/4$

$\therefore \quad x = [x] + \{x\} = 2 + \frac{3}{4} = \frac{11}{4}$

CASE 2: $x < 1$, then the equation becomes:

$$1 - x = 2[x] - 3\{x\}$$

$$1 - [x] - \{x\} = 2[x] - 3\{x\}$$

$$2\{x\} = 3[x] - 1$$

$\because \quad 0 \le \{x\} < 1 \Rightarrow 0 \le 2\{x\} < 2 \Rightarrow 0 \le 3[x] - 1 < 2$

i.e. $\quad 1 \le 3[x] < 3$

i.e. $\quad \frac{1}{3} \le [x] < 1$ which is not possible

$\therefore \quad x = \frac{3}{2}, \frac{11}{4}$ are the only solutions

ILLUSTRATION 9

If the fractional part of $\frac{1}{x}$ and x^2 for some $x \in (\sqrt{2}, \sqrt{3})$ are equal, then find the value of $x^4 - \frac{3}{x}$.

SOLUTION

$\sqrt{2} < x < \sqrt{3} \Rightarrow 2 < x^2 < 3 \qquad \therefore [x^2] = 2$

$\Rightarrow \quad \frac{1}{\sqrt{2}} > \frac{1}{x} > \frac{1}{\sqrt{3}} \qquad \therefore \left[\frac{1}{x}\right] = 0$

Given $\left\{\frac{1}{x}\right\} = \{x^2\} \Rightarrow \frac{1}{x} - \left[\frac{1}{x}\right] = x^2 - [x^2]$

$\Rightarrow \quad \frac{1}{x} - 0 = x^2 - 2 \Rightarrow x^3 - 2x = 1$

$\Rightarrow \quad x^3 - 2x - 1 = 0 \Rightarrow x^3 + x^2 - x^2 - x - x - 1 = 0$

$\Rightarrow \quad (x+1)(x^2 - x - 1) = 0$

When $x \ne -1$ then $x^2 - x - 1 = 0 \Rightarrow x = 1 + \frac{1}{x} \Rightarrow -\frac{1}{x} = 1 - x$

Now again $x^2 = x + 1 \Rightarrow x^4 = x^2 + 1 + 2x$

$\because \quad x^4 - \frac{3}{x} = (x+1)^2 + 3(1-x) = x^2 + 2x + 1 + 3 - 3x$

$$= x^2 - x + 4 = 1 + 4 = 5$$

ILLUSTRATION 10

Let $[k]$ denotes the greatest integer less than or equal to k. If number of positive integral solutions of the equation $\left[\frac{x}{[\pi^2]}\right] = \left[\frac{x}{\left[11\frac{1}{2}\right]}\right]$ is n, then find the value of n.

SOLUTION

$$\left[\frac{x}{[\pi^2]}\right] = \left[\frac{x}{\left[11\frac{1}{2}\right]}\right] \Rightarrow \left[\frac{x}{9}\right] = \left[\frac{x}{11}\right]$$

CASE 1: $0 \le \frac{x}{9} < 1$ and $0 \le \frac{x}{11} < 1 \Rightarrow 0 \le x < 9$ and $0 \le x < 11$.

Hence, common positive integral values of x are $\{1, 2, 3, \ldots, 8\}$ 8 values.

CASE 2: $1 \le \frac{x}{9} < 2$ and $1 \le \frac{x}{11} < 2$

$\Rightarrow \quad 9 \le x < 18$ and $11 \le x < 22$

$\Rightarrow \quad x \in \{11, 12, \ldots, 17\}$ Hence, 7 values.

CASE 3: $2 \le \frac{x}{9} < 3$ and $2 \le \frac{x}{11} < 3$

$\Rightarrow \quad 18 \le x < 27$ and $22 \le x < 33$

$\Rightarrow \quad x \in \{22, 23, \ldots, 26\}$ Hence, 5 values.

CASE 4: $3 \le \frac{x}{9} < 4$ and $3 \le \frac{x}{11} < 4$

$\Rightarrow \quad 27 \le x < 36$ and $33 \le x < 44$

$\Rightarrow \quad x \in \{33, 34, 35\}$ Hence, 3 values.

CASE 5: $4 \le \frac{x}{9} < 5$ and $4 \le \frac{x}{11} < 5$

$\Rightarrow \quad x = 44$ (1 value)

$\therefore \quad$ Hence, the total positive integer values of $x = n = 8 + 7 + 5 + 3 + 1 = 24$.

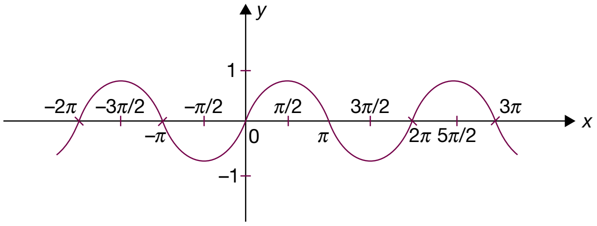
ILLUSTRATION 11

Plot the following function where [.] is greatest integer function and {.} is fractional part function.

(i) $y = [\sin x]$
(ii) $y = [\cos x]$
(iii) $y = \{\sin x\}$
(iv) $y = \{\cos x\}$
(v) $y = [x^2]$
(vi) $y = [\sqrt{x}\,]$
(vii) $y = \{\sqrt{x}\,\}$

SOLUTION

(i) $y = \sin x$



$y = [\sin x]$ it can have only integral values as shown in the given diagram.

$$x = \frac{\pi}{2} : [\sin x] = 1$$

If $x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

$0 \le \sin x < 1$

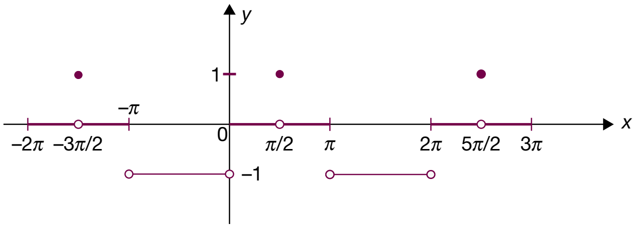
$y = [\sin x] = 0$

If $\pi < x < 2\pi$

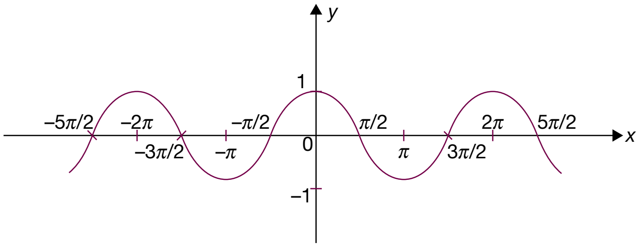
$-1 \le \sin x < 0$

$y = [\sin x] = -1$

$[\sin x]$ is a periodic function with 2π as period



(ii) $y = \cos x$



$x = 0 : [\cos x] = 1$

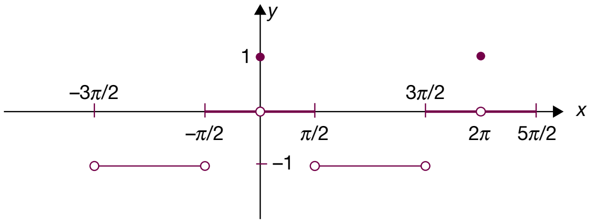
$0 < x \le \frac{\pi}{2}; 0 \le \cos x < 1; [\cos x] = 0$

$\frac{\pi}{2} < x \le \frac{3\pi}{2}; -1 \le \cos x < 0; [\cos x] = -1$

$\frac{3\pi}{2} \le x < 2\pi; 0 \le \cos x < 1; [\cos x] = 0$

$y = [\cos x]$ is periodic function with 2π as period

$y = [\cos x]$



(iii) $y = \{\sin x\}$

$= \sin x - [\sin x]$ periodic function with as 2π as period

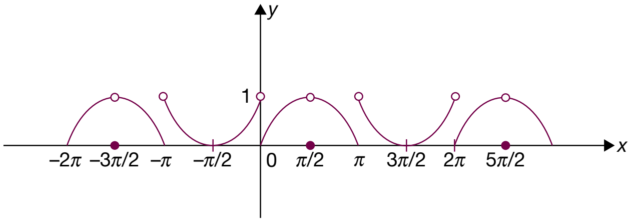
$x = \frac{\pi}{2}; y = \left\{\sin\frac{\pi}{2}\right\} = \{1\} = 0$

$x \in [0, \pi] - \left\{\frac{\pi}{2}\right\}; 0 \le \sin x < 1; [\sin x] = 0$

$y = \{\sin x\} = \sin x$

$x \in (\pi, 2\pi); -1 \le \sin x < 0; [\sin x] = -1$

$y = \sin x - [\sin x] = 1 + \sin x$



(iv) $y = \{\cos x\}$ period function with 2π as period

$x = 0 : y = \{\cos 0\} = \{1\} = 0$

If $0 < x \le \frac{\pi}{2}; 0 \le \cos x < 1; [\cos x] = 0$

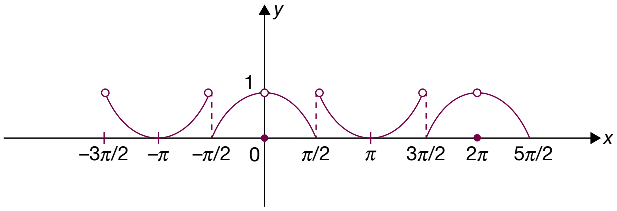
$y = \{\cos x\} = \cos x$

If $\frac{\pi}{2} < x < \frac{3\pi}{2} : -1 \le \cos x < 0 : [\cos x] = -1$

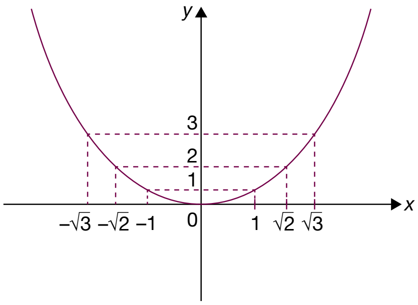
$y = \cos x - [\cos x] = 1 + \cos x$

If $\frac{3\pi}{2} \le x < 2\pi : 0 \le \cos x < 1 : [\cos x] = 0$

$y = \cos x$



(v) $y = x^2$

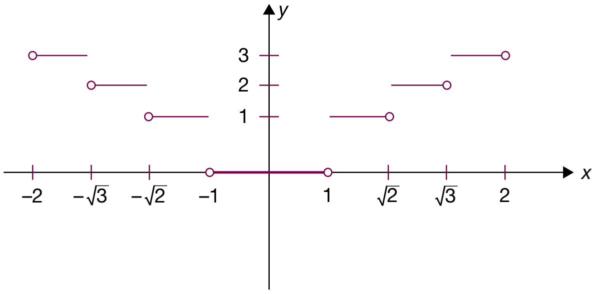


Graph of $y = x^2$ breaks at everyplace where it becomes an integer

$$x^2 = 0, 1, 2, 3, 4, \ldots$$

$$x = 0, \pm 1, \pm\sqrt{2}, \pm\sqrt{3}, 2, \ldots$$

$$y = [x^2]$$

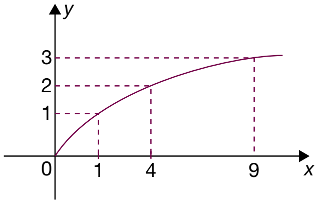


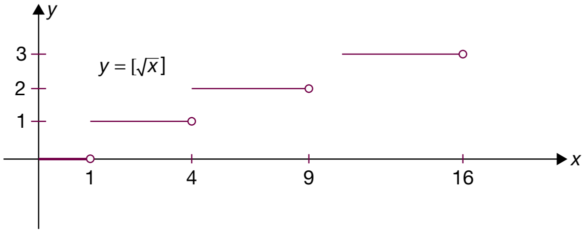
(vi) $y = \sqrt{x}$

Graph of $y = \sqrt{x}$ breaks where $\sqrt{x}$ becomes an integers.

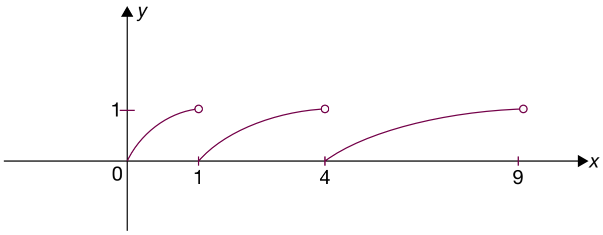
$$\sqrt{x} = 0, 1, 2, 3, 4, \ldots$$

$$x = 0, 1, 4, 9, 16, \ldots$$





(vii) $y = \{\sqrt{x}\} = \sqrt{x} - [\sqrt{x}]$



SIGNUM FUNCTION

The signum function f is defined as:

$$\text{sgn}(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

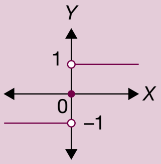


ILLUSTRATION 12

Find domain and range of the following function.

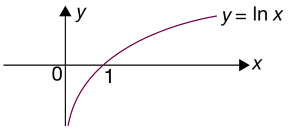
(i) $y = \text{sgn}(\ln x)$ **(ii)** $y = \text{sgn}(\sin x)$

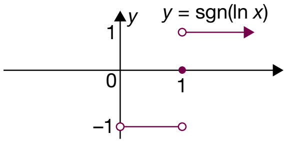
(iii) $y = \text{sgn}(\{x\} + \{-x\})$ **(iv)** $y = \text{sgn}([x] + [-x])$

(v) $y = \text{sgn}(e^x)$

SOLUTION

(i) $y = \text{sgn}(\ln x)$

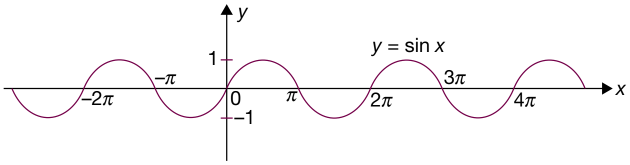


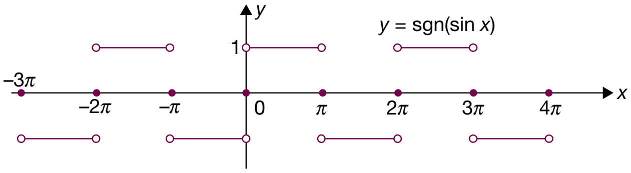


$$y = \text{sgn}(\ln x) = \begin{cases} +1 & x > 1 \\ 0 & x = 1 \\ -1 & 0 < x < 1 \end{cases}$$

Therefore, Domain $x \in (0, \infty)$, Range $\in \{-1, 0, 1\}$

(ii) $y = \text{sgn}(\sin x)$





$$y = \text{sgn}(\ln x) = \begin{cases} +1 & 2n\pi < x < (2n+1)\pi \\ 0 & x = n\pi \\ -1 & (2n-1)\pi < x < 2n\pi \end{cases}$$

Therefore, Domain $x \in R$, Range $\in \{-1, 0, 1\}$

(iii) $y = \text{sgn}(\{x\} + \{-x\})$

Domain $x \in R$

$$\{x\} + \{-x\} = \begin{cases} 0, & x \in I \\ 1, & x \notin I \end{cases}$$

$$y = \text{sgn}(\{x\} + \{-x\}) = \begin{cases} 0, & x \in I \\ 1, & x \notin I \end{cases}$$

Therefore, Domain $x \in R$, Range $\in \{0, 1\}$

(iv) $y = \text{sgn}([x] + [-x])$

Domain $x \in R$

$$[x] + [-x] = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$$

$$y = \text{sgn}([x]+[-x]) = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$$

Therefore, Domain $x \in R$, Range $\in \{-1, 0\}$

(v) $y = \text{sgn}(e^x)$

$0 < e^x < \infty$ for all real x

Domain $x \in R$

$$y = \text{sgn}(e^x) = 1$$

Domain $x \in R$, Range $\in \{1\}$, Singleton range

ILLUSTRATION 13

Find the number of integral values of x satisfying the equation $\text{sgn}\left(\left[\frac{15}{1+x^2}\right]\right) = [1 + \{2x\}]$; where [.] is GIF and {.} is fpf

SOLUTION

$$\text{sgn}\left(\left[\frac{15}{1+x^2}\right]\right) = [1+\{2x\}] \qquad (\because\ 0 \le \{2x\} < 1)$$

$$\Rightarrow \quad \text{sgn}\left(\left[\frac{15}{1+x^2}\right]\right) = 1$$

$$\Rightarrow \quad \left[\frac{15}{1+x^2}\right] > 0 \Rightarrow \left[\frac{15}{1+x^2}\right] \ge 1 \Rightarrow \frac{15}{1+x^2} \ge 1$$

$$\Rightarrow \quad 1 + x^2 \le 15 \Rightarrow x^2 \le 14$$

$$\Rightarrow \quad x \in \left[-\sqrt{14}, \sqrt{14}\right]$$

Here, Integral $x \in \{-3, -2, -1, 0, 1, 2, 3\}$. Hence, number of integral values of x are 7.

ILLUSTRATION 14

$$\text{Let} f(x) = \text{sgn}\left(\left(\sin\theta - \frac{\sqrt{3}}{2}\right)x^2 + \left(\cos\theta - \frac{1}{2}\right)x + (\tan\theta - \sqrt{3})\right)$$

If $f(x)$ is identically zero for every $x \in R$, then find values of θ in $[-2\pi, 2\pi]$

SOLUTION

As $f(x)$ is identically zero for every $x \in R$ then the equation

$$\left(\left(\sin\theta - \frac{\sqrt{3}}{2}\right)x^2 + \left(\cos\theta - \frac{1}{2}\right)x + (\tan\theta - \sqrt{3})\right) = 0 \text{ must}$$

be an identity in x.

$$\therefore \quad \sin\theta = \frac{\sqrt{3}}{2} \text{ and } \cos\theta = \frac{1}{2} \text{ and } \tan\theta = \sqrt{3}$$

$$\Rightarrow \quad \theta = \frac{\pi}{3} \text{ or } \frac{-5\pi}{3} \text{ in } [-2\pi, 2\pi]$$

Table 2.3 Domain and range of different functions

Function **($y = f(x)$)**	**Domain** **(i.e. values taken by x)**	**Range** **(i.e. values taken by $f(x)$)**
Modulus Function		
$\lvert x \rvert$	R	$R^+ \cup \{0\}$
$\frac{1}{\lvert x \rvert}$	$R - \{0\}$	R^+
Exponential Function		
e^x	R	R^+
$e^{1/x}$	$R - \{0\}$	$R^+ - \{1\}$
$a^x, a > 0$	R	R^+
$a^{1/x}, a > 0$	$R - \{0\}$	$R^+ - \{1\}$
Logarithmic Function		
$\log_a x$, $(a > 0)$ $(a \ne 1)$	R^+	R
$\log_x a = \frac{1}{\log_a x}$ $(a > 0)$ $(a \ne 1)$	$R^+ - \{1\}$	$R - \{0\}$
Greatest Integer Function		
$[x]$	R	I

(*Continued*)

Function ($y = f(x)$)	Domain (i.e. values taken by x)	Range (i.e. values taken by $f(x)$)
$\frac{1}{[x]}$	$R - [0, 1)$	$\left\{\frac{1}{n}, n \in I - \{0\}\right\}$
Fractional part Function		
$\{x\}$	R	$[0, 1)$
$\frac{1}{\{x\}}$	$R - I$	$(1, \infty)$
Signum Function		
$\operatorname{sgn}(x) = \begin{cases} \frac{\lvert x\rvert}{x}, x \neq 0 \\ 0, x = 0 \end{cases}$	R	$\{-1, 0, 1\}$

Test Your Concepts-I

(Solutions on page H.10)

1. Find which of the following relations are functions.
(i) $y = x^2 + 2$ (ii) $3x^2 + y^2 = 12$;
(iii) $xy^2 = 16$; (iv) $2x + 3x^2y = 6y$

2. Find the sum of solution(s) of the equation $2[x] + x = 6\{x\}$.

3. The equation $4\{x\} = x + [x]$ has k real roots (where $\{x\}$ and $[x]$ denotes fractional and Integral part of real numbers x, respectively), then find the value of k.

4. Find no. of solution of equation $x^2 - 12x + 35 = [x] + [-x]$ (where $[x]$ denotes the largest integer less than or equal to x).

5. Find the sum of all possible values of n, where $n \in N$ and $10 < n \leq 100$ such that the equation $[2x^2] + x - n = 0$ has a solution.

6. If $[x]^2 + [x - 2] < 0$ and $\{x\} = \frac{1}{2}$, then find the number of possible values of x.

7. Find number of integral value(s) of 'x' satisfying the equation

$$\log_2(3 - x) + \log_{\frac{1}{2}}\left(\frac{\sin\frac{9\pi}{4}}{5 - x}\right) = \cos\frac{11\pi}{3} - \log_{\frac{1}{2}}(x + 7).$$

8. Find the number of integral values of x satisfying the equation $|x^2 - 3x - 4| + |x^2 - 5x - 6| = |2x + 2|$.

9. Find the number of solution of the equation $2[x] = x + \{x\}$. Where $[y]$ denotes the greatest integer less than or equal to y and $\{y\}$ denotes fractional part of y.

10. Find the number of solutions of the equation $\operatorname{sgn}(\{x\}) = |1 - x|$ where $\{\,\}$ denotes fractional part function.

11. The total number of solutions of $\sin\{x\} = \cos\{x\}$ (where $\{.\}$ denotes the fractional part) in $[0, 2\pi]$ is equal to:

12. If $[x]$ denotes the integral part of x for real x, then the value of $\left[\frac{1}{4}\right] + \left[\frac{1}{4} + \frac{1}{200}\right] + \left[\frac{1}{4} + \frac{1}{100}\right] + \left[\frac{1}{4} + \frac{3}{200}\right] + \ldots + \left[\frac{1}{4} + \frac{199}{200}\right]$

Answer key

1. (i) Function; (ii) Not function; (iii) Not function (iv) Function

2. $\frac{8}{5}$ **3.** 2 **4.** 2 **5.** 190 **6.** 2 **7.** 1

8. 4 **9.** 2 **10.** 1 **11.** 6 **12.** 50

GRAPH TRANSFORMATION

If we are able to make graph of given function then it always benefits in solving questions, in this section we learn modification of graph of $f(x)$ with some rules.

Conceptual Note(s)

Equation $f(x) = g(x)$ has number of real solutions

= Number of points of intersection of graphs $y = f(x)$ and $y = g(x)$

From below figure, we get no. of solution of equation $f(x) = g(x)$ as 3, solutions are $x = \alpha, \beta, \gamma$.

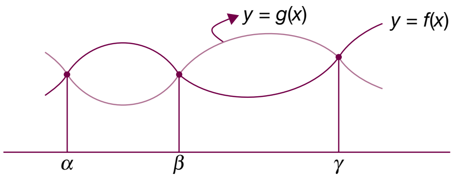


ILLUSTRATION 15

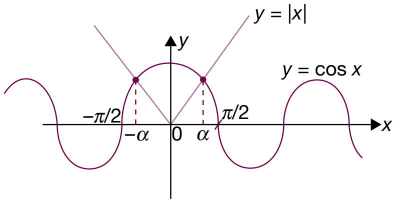
Find the number of solutions of the following equation.

(i) $\cos x = |x|$ **(ii)** $|x| = \sin x$

(iii) $e^x = \{x\}$ where {.} is fractional part function

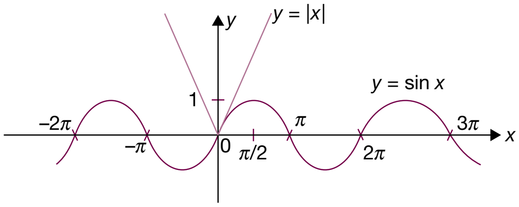
SOLUTION

(i) $y = \cos x$ and $y = |x|$



Hence, number of solutions of the following equation $\cos x = |x|$ are two.

(ii) $y = |x|$ and $y = \sin x$



Hence, no. of solution of equation $|x| = \sin x$ is one and solution is $x = 0$.

(iii) $y = e^x$ and $y = \{x\}$

There are ∞ solutions (all are negative) of equation $e^x = \{x\}$.

Hence, no. of positive real solution = 0

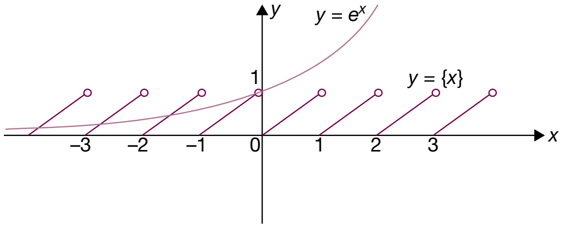


ILLUSTRATION 16

Find value of k if equation

$|x+2| - 2|x-7| + 3|x-4| = k$ has:

(i) One solution **(ii)** Two solution

(iii) 3 solution **(iv)** No solution **(v)** ∞ solution

SOLUTION

$$y = f(x) = |x+2| - 2|x-7| + 3|x-4|$$

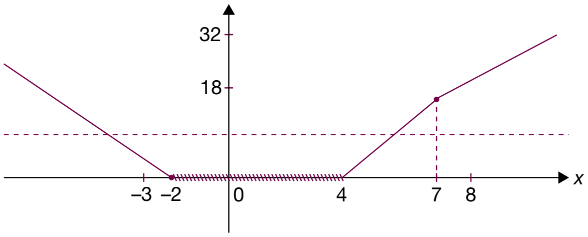
$$f(-2) = 0 - 18 + 18 = 0$$

$$f(4) = 6 - 6 + 0 = 0$$

$$f(7) = 9 - 0 + 9 = 18$$

$$f(-3) = 1 - 20 + 21 = 2$$

$$f(8) = 10 - 2 + 12 = 20$$



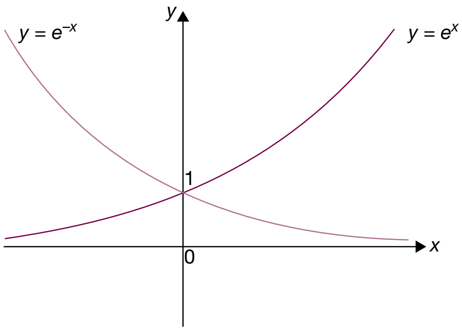
Now for equation $f(x) = k$, we have plotted $y = f(x)$ and we will observe it point of intersection with line $y = k$ for different values of k:

(i) One solution	$k \in \phi$
(ii) 2 solution	$k > 0$
(iii) 3 solution	$k \in \phi$
(iv) No solution	$k < 0$
(v) ∞ solution	$k = 0$

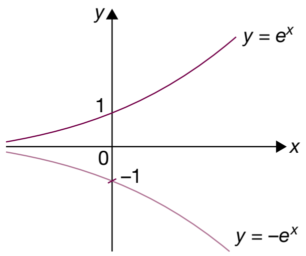
GRAPH TRANSFORMATION RULES

(a) $y = f(-x)$: Take image of $f(x)$ about y-axis

Consider function $y = e^{-x}$, we get graph $y = e^{-x}$ by taking the image of e^x about y-axis



(b) $y = -f(x)$: Take the image of $f(x)$ about x-axis



Consider function $y = -e^x$, we get graph $y = -e^x$ by taking the image of e^x about the x axis

ILLUSTRATION 17

Plot the following:

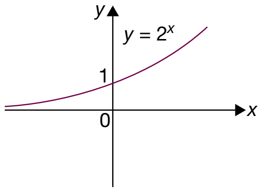
(i) $y = -2^{-x}$ **(ii)** $y = \sin(-x)$

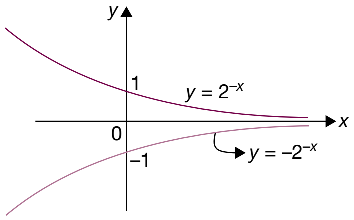
(iii) $y = -\{-x\}$ where $\{.\}$ is fractional part function

SOLUTION

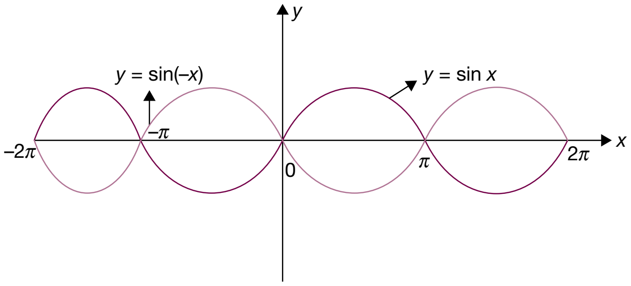
(i) $y = -2^{-x}$

Sequence of transformation $y = 2^x \rightarrow y = 2^{-x} \rightarrow y = -2^{-x}$





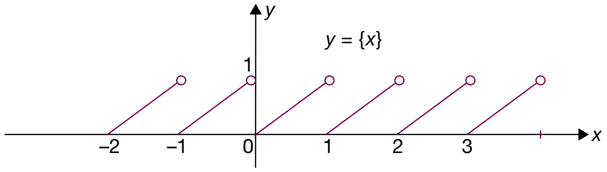
(ii) $y = \sin(-x) = -\sin x$



(iii) $y = -\{-x\}$ where $\{.\}$ is fractional part function

Sequence of transformation

$y = \{x\} \rightarrow y = \{-x\} \rightarrow y = -\{-x\}$



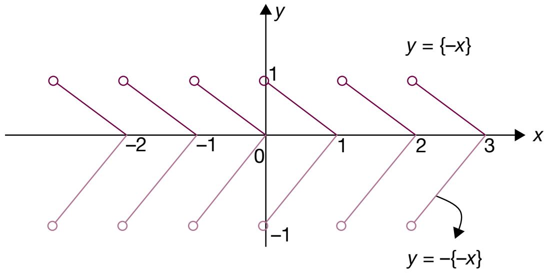
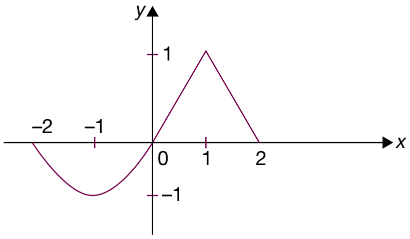


ILLUSTRATION 18

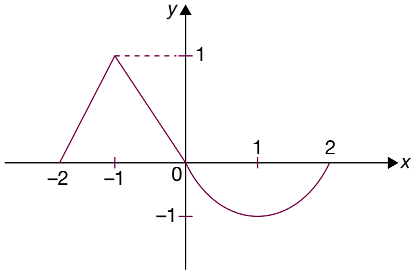
If $y = f(x)$ as drawn in diagram, then plot

(i) $f(-x)$ **(ii)** $-f(x)$

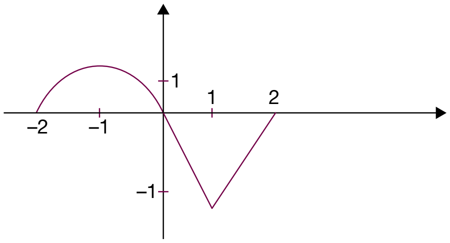


SOLUTION

(i) $y = f(-x)$



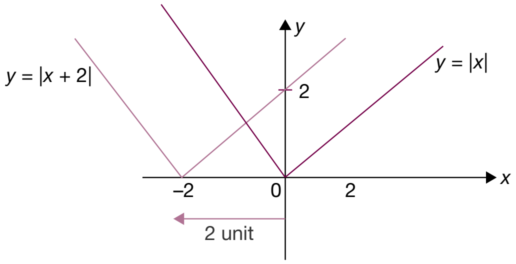
(ii) $y = -f(x)$



(c) $y = f(x + a)$; Shift the graph of $y = f(x)$ by 'a units' left side along x-axis

Using graph of $y = |x|$ to transform into graph of $y = |x + 2|$

Shift the graph of $|x|$ by –2 units along x-axis



(d) $y = f(x - a)$; Shift the graph of $y = f(x)$ by 'a units' right side along x-axis

$y = |x - 2|$

Shift the graph of $|x|$ by +2 units along x-axis

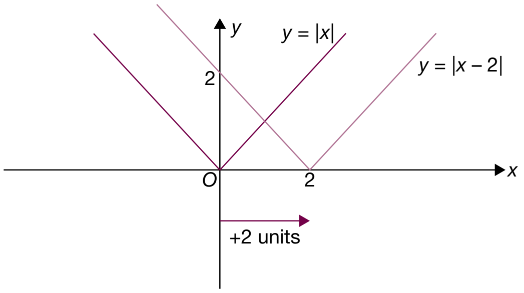


ILLUSTRATION 19

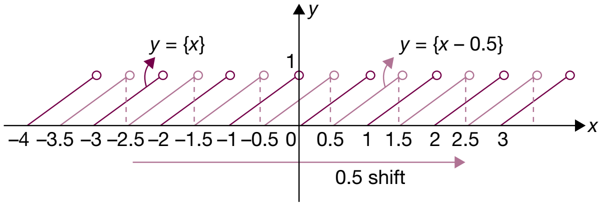
Plot the following functions:

(i) $y = \{x - 0.5\}$

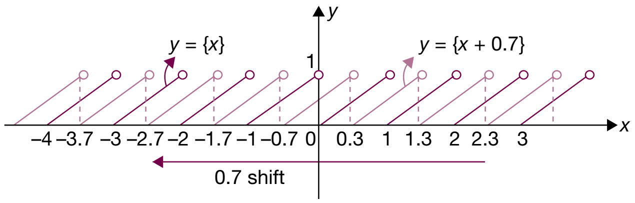
(ii) $y = \{x + 0.7\}$ where {.} is fractional part function

SOLUTION

(i) Sequence of transformation from $y = \{x\}$ to $y = \{x - 0.5\}$, by shifting graph of $y = \{x\}$ by 0.5 units right side along x-axis.

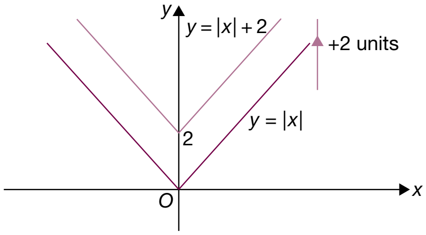


(ii) Sequence of transformation from $y = \{x\}$ to $y = \{x + 0.7\}$, by shifting graph by 0.7 units left side along x-axis.



(e) $y = f(x) + a$: Shift the graph of $y = f(x)$ by 'a units' upward along y-axis.

Let us consider the graph $y = |x|$ which we want to convert into a graph of $y = |x| + 2$, by shifting it by 2 units upward along the y-axis.



(f) $y = f(x) - a$: Shift the graph of $y = f(x)$ by 'a units' downward along the y-axis.

Let consider the graph $y = |x|$ which we want to convert into graph of $y = |x| - 2$, by shifting it by 2 units downward along the y-axis.

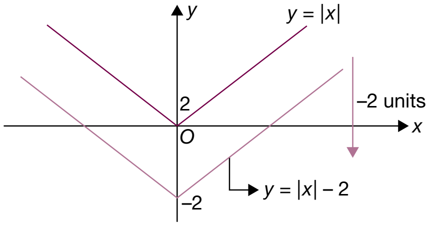


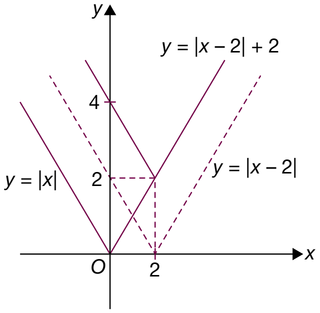
ILLUSTRATION 20

Plot $y = |x - 2| + 2$

SOLUTION

Sequence of transformation

$$y = |x| \to y = |x - 2| \to y = |x - 2| + 2.$$



(g) $y = f(kx)$: Graph of $f(x)$ shrinks/expands $\frac{1}{k}$ times along x-axis.

If $\frac{1}{k} > 1$, then the graph stretches (expands)

If $\frac{1}{k} < 1$, then the graph shrinks (contracts).

ILLUSTRATION 21

Plot the following functions:

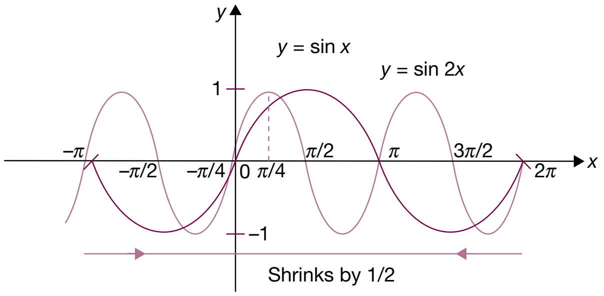
(i) $y = \sin 2x$ **(ii)** $y = \sin\frac{x}{2}$;

SOLUTION

(i) $y = \sin 2x$

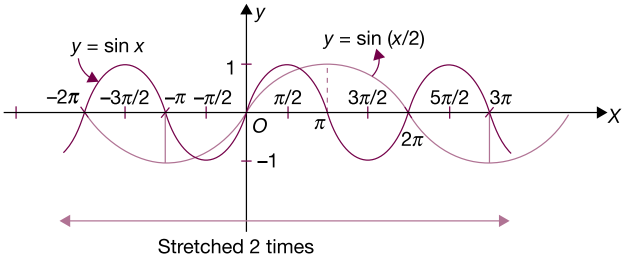
where $k = 2$; $\frac{1}{k} = \frac{1}{2}$ times

It means graph of sin x shrinks by 1/2 along the x-axis.



(ii) $y = \sin\frac{x}{2}$; where $k = \frac{1}{2}$; $\left(\frac{1}{k} = 2 \text{ times}\right)$

Graph of sin x stretches two times along the x-axis



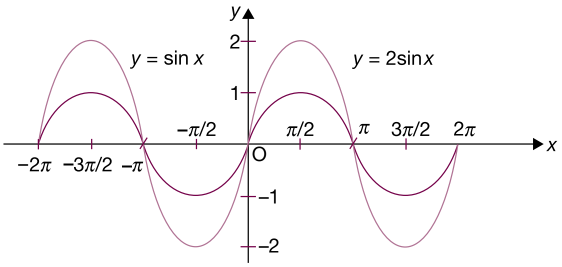
(h) $y = kf(x)$; graph of $y = f(x)$ becomes k times along the y-axis.

ILLUSTRATION 22

Plot **(i)** $y = 2\sin x$ **(ii)** $y = 2\left\{\frac{x}{2}\right\}$

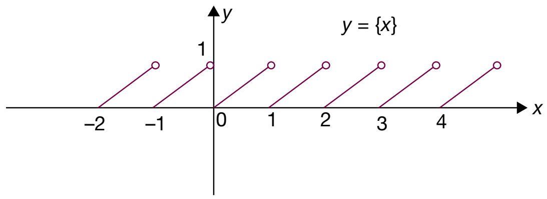
SOLUTION

(i) $y = 2\sin x$; $-1 \le \sin x \le 1$; $-2 \le 2\sin x \le 2$



(ii) $y = 2\left\{\frac{x}{2}\right\}$

The sequence of transformation $\{x\} \rightarrow \left\{\frac{x}{2}\right\} \rightarrow 2\left\{\frac{x}{2}\right\}$



Graph of $\{x\}$ is stretched 2 times along the x-axis, then it is again stretched 2 times along the y-axis.

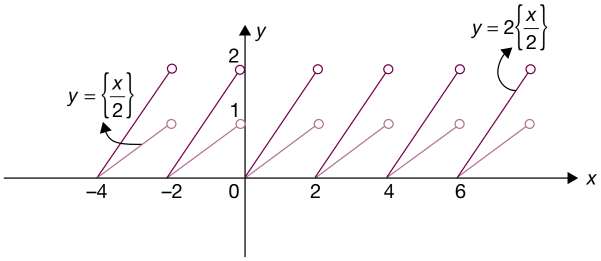
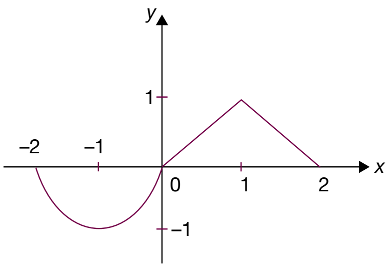


ILLUSTRATION 23

$y = f(x)$ is given as shown



Plot the following:

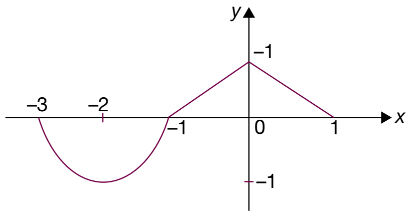
(i) $y = f(x + 1)$ **(ii)** $y = f(x - 2)$

(iii) $y = f(2x)$ **(iv)** $y = 2f(x)$

SOLUTION

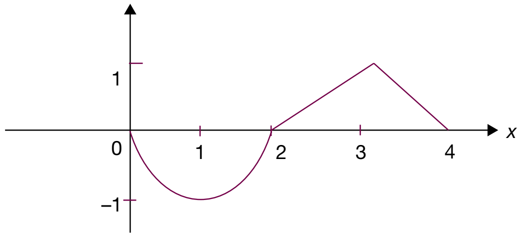
(i) $y = f(x + 1)$

Graph of $f(x)$ is shifted by 1 unit on left side along the x-axis.



(ii) $y = f(x-2)$

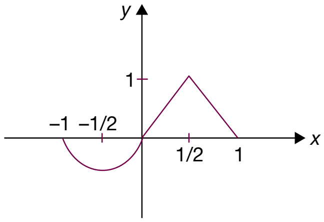
Graph of $f(x)$ is shifted by 2 units on right side along the x-axis.



(iii) $y = f(2x)$

$k = 2$ then $\dfrac{1}{k} = \dfrac{1}{2}$ times

Graph of $f(x)$ shrinks by 1/2 times along the x-axis.



(iv) $y = 2f(x)$

Graph of $f(x)$ expands by 2 times along the y-axis.

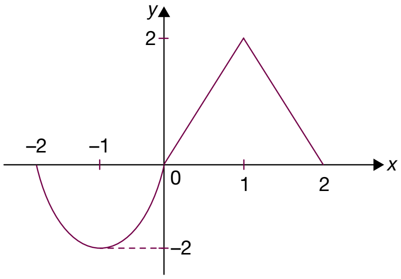


ILLUSTRATION 24

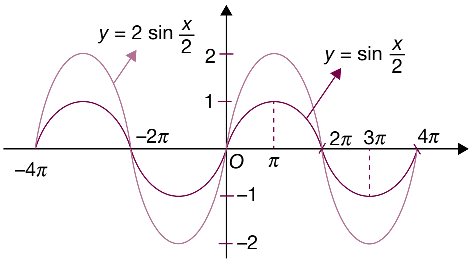
Plot **(i)** $y = 3 - 2\sin\left(\dfrac{x}{2}\right)$ **(ii)** $y = \tan\left(\dfrac{\pi}{4} - x\right) + 3$

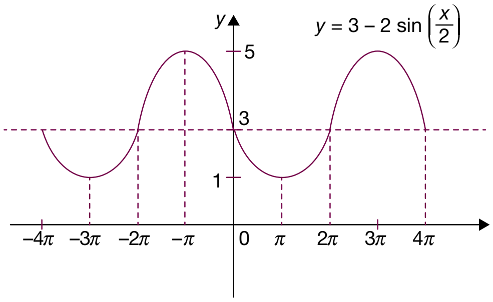
SOLUTION

(i) $y = 3 - 2\sin\left(\dfrac{x}{2}\right)$

Sequence of transformations

$$\sin x \to \sin\frac{x}{2} \to 2\sin\frac{x}{2} \to -2\sin\frac{x}{2} \to 3 - 2\sin\frac{x}{2}$$

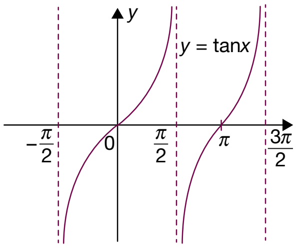


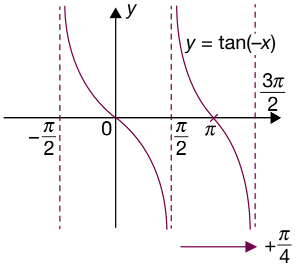


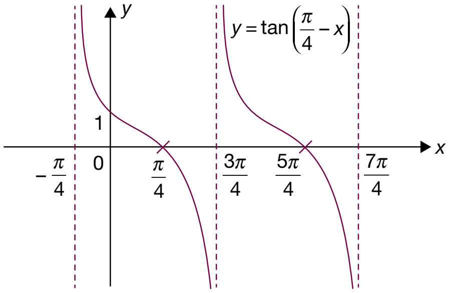
(ii) $y = \tan\left(\dfrac{\pi}{4} - x\right) + 3$

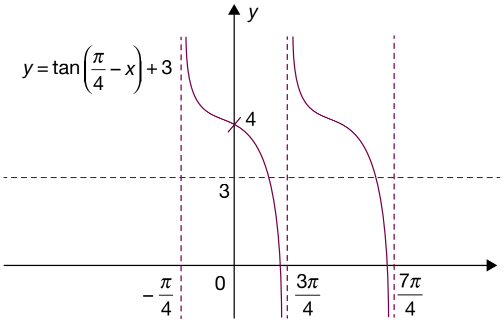
Sequence of transformations

$$y = \tan x \to \tan(-x) \to \tan\left(\frac{\pi}{4} - x\right) + 3$$









(i) $y = f(kx \pm a)$

Sequence of transformation

$$f(x)\xrightarrow{\substack{\text{Make graph of}\\ f(x)\,1/k\text{ times}\\ \text{along } x\text{-axis}}} f(kx)\xrightarrow{\substack{\text{shift graph of}\\ f(kx)\text{ by }\mp\left(\frac{a}{k}\right)\\ \text{units along } x\text{-axis}}} f(kx\pm a)$$

ILLUSTRATION 25

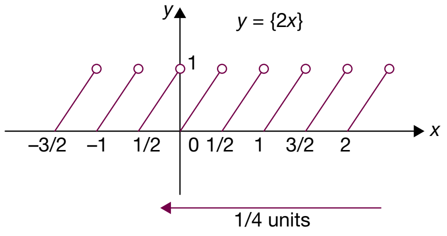
Plot **(i)** $y = \left\{2x + \frac{1}{2}\right\}$ where {.} is fractional part function

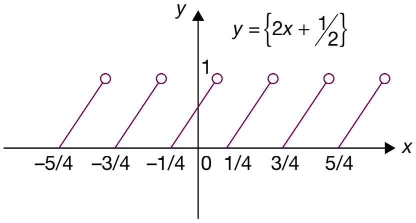
(ii) $y = \tan\left(2x - \frac{\pi}{4}\right)$

SOLUTION

(i) Sequence of transformation $\{x\} \to \{2x\} \to \left\{2x + \frac{1}{2}\right\}$

$$-\frac{a}{k} = -\frac{\frac{1}{2}}{2} = -\frac{1}{4}$$

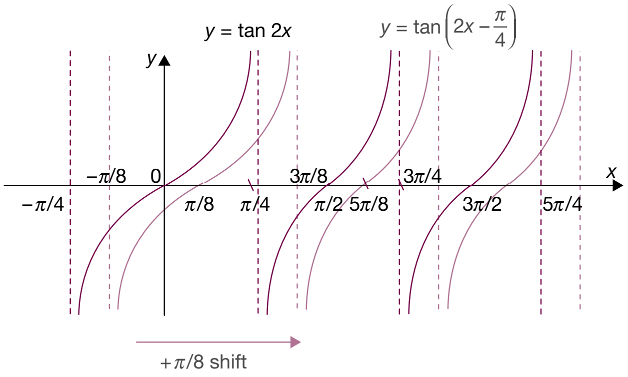




(ii) $y = \tan\left(2x - \frac{\pi}{4}\right)$

Sequence of transformation

$y = \tan x \to y = \tan 2x \to y = \tan\left(2x - \frac{\pi}{4}\right)$



(j) $y = kf(x) \pm a$

Sequence of transformation

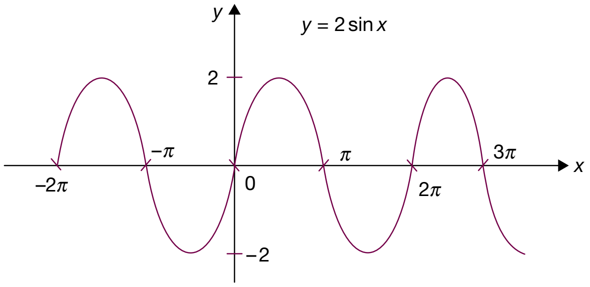
$$f(x)\xrightarrow{\substack{\text{Make graph of } f(x)\\ k\text{ times along}\\ y\text{-axis}}} kf(x)\xrightarrow{\substack{\text{shift graph of}\\ kf(x)\text{ by }\pm a\text{ units}\\ \text{along } y\text{-axis}}} kf(x)\pm a$$

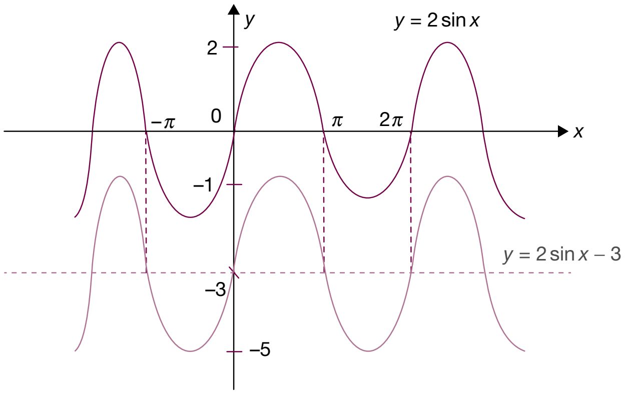
ILLUSTRATION 26

Plot $y = 2\sin x - 3$

SOLUTION

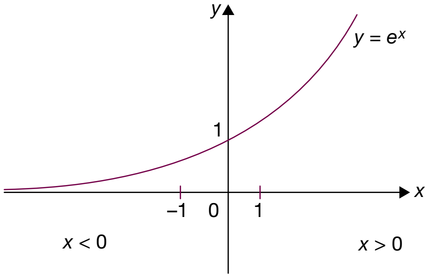
Sequence of transformation $\sin x \to 2\sin x \to 2\sin x - 3$

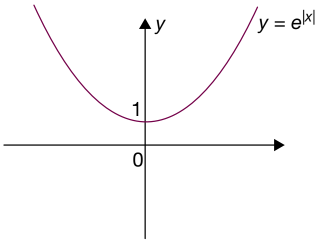




(k) $y = f(|x|)$: Remove the left part ($x < 0$) of $f(x)$ and take the image of the right part ($x \geq 0$) of $f(x)$ about y-axis.

Let consider function $y = e^{|x|}$, first the left part of $y = f(x) = e^x$ is removed and now take the image of the right part of $f(x)$ about y-axis.





(I) $y = f(-|x|)$: Remove the right part ($x > 0$) of $f(x)$ and take the image of left part ($x \le 0$) of $f(x)$ about the y-axis.

Let consider function $y = e^{-|x|}$, first the right part of $y = f(x) = e^x$ is removed and now take image of left part of $f(x)$ about y-axis.

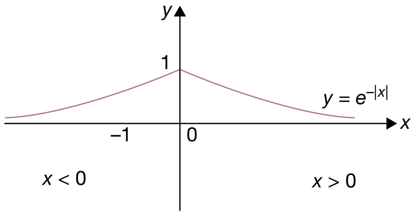


ILLUSTRATION 27

Plot the following:

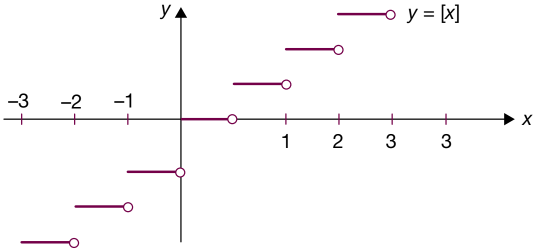
(i) $y = [|x|]$ **(ii)** $y = [-|x|]$ **(iii)** $y = \{|x|\}$

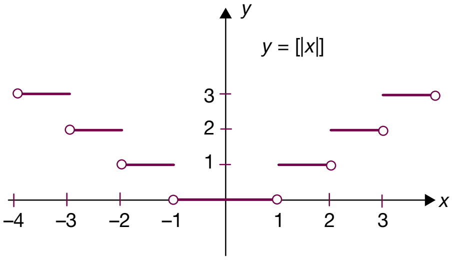
(iv) $y = \{-|x|\}$ **(v)** $y = \ln|x|$

where [.] is greatest integer function and {.} is fractional part function.

SOLUTION

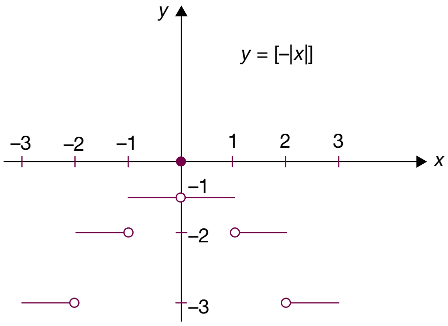
(i) $y = [|x|]$; Remove the left part of $[x]$ and take the image of right part of $[x]$ about the y-axis.





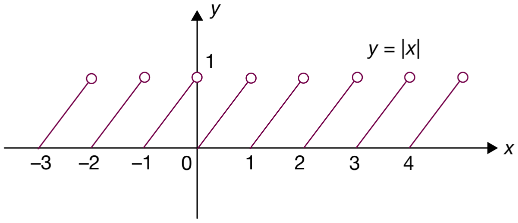
(ii) $y = [-|x|]$

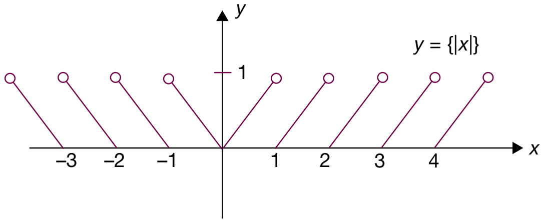
Remove the right part of $[x]$ and take the image of left part of $[x]$ about the y-axis.



(iii) $y = \{|x|\}$

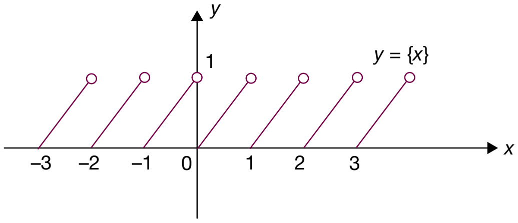
Remove the left part of $\{x\}$ and take the image of right part of $\{x\}$ about the y-axis.

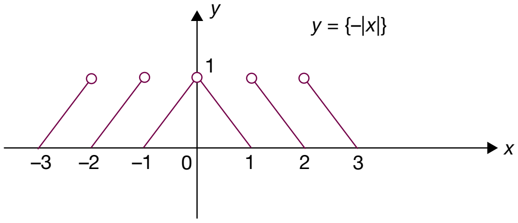




(iv) $y = \{-|x|\}$

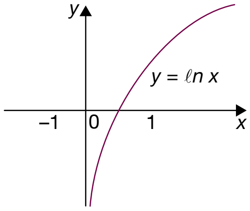
Remove the right part of $\{x\}$ and take the image of the left part of $\{x\}$ about the y-axis.

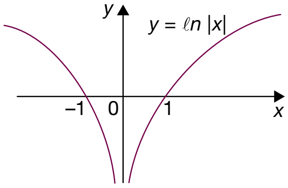




(v) $y = \ln|x|$

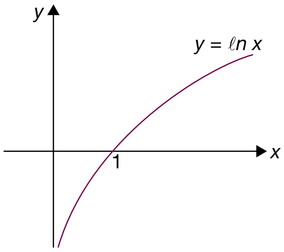
Remove the left part of $\ln x$ and take the image of the right part of $\ln x$ about the y-axis.

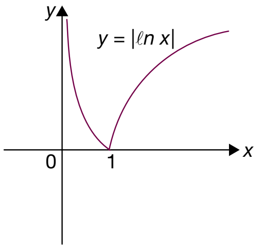




(m) $y = |f(x)|$; Only take the image of negative part of $f(x)$ (below x axis) about x-axis while keeping positive part (above x-axis) as unchanged.

Let us consider $y = |\ln x|$, only take the image of negative part (below x axis) of $\ln x$ about x-axis while keeping positive part (above x axis) as unchanged.



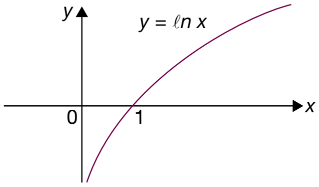


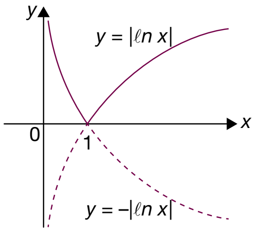
(n) $y = -|f(x)|$

$$f(x) \xrightarrow[\text{the } x\text{-axis while keeping positive part as unchanged}]{\text{Only take the image of negative part of } f(x) \text{ about}}$$

$$|f(x)| \xrightarrow[\text{about the } x\text{-axis}]{\text{Take the image of } |f(x)|} -|f(x)|$$

Let consider $y = -|\ln x|$,





(o) $y = |f(|x|)|$

Sequence of Transformation

$$f(x) \xrightarrow{\substack{\text{Remove left}\\ \text{part of } f(x)\\ \text{and take the}\\ \text{image of right}\\ \text{part about the}\\ y\text{-axis}}} f(|x|) \xrightarrow{\substack{\text{Only take the}\\ \text{image of negative}\\ \text{part of } f(x) \text{ about}\\ \text{the } x\text{-axis while}\\ \text{keeping positive}\\ \text{part as unchanged}}} |f(|x|)|$$

(p) $y = |f(-|x|)|$

Sequence of Transformation

$$f(x) \xrightarrow{\substack{\text{Remove right}\\ \text{part of } f(x)\\ \text{and take the}\\ \text{image of left}\\ \text{part about}\\ \text{the } y\text{-axis.}}} f(-|x|) \xrightarrow{\substack{\text{Only take the}\\ \text{image of negative}\\ \text{part of } f(x) \text{ about}\\ \text{the } x\text{-axis while}\\ \text{keeping positive}\\ \text{part as unchanged}}} |f(-|x|)|$$

ILLUSTRATION 28

If $|\ln|x|| = p$ find p.

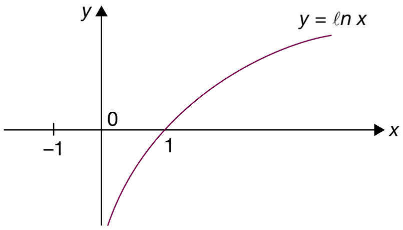
(i)	No solution	**(ii)**	2 solution
(iii)	4 solution	**(iv)**	∞ solution

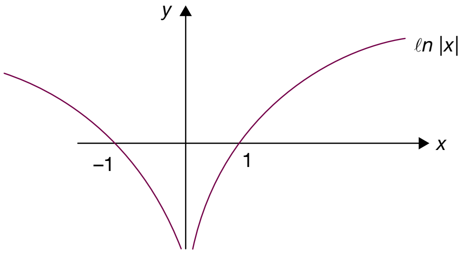
SOLUTION

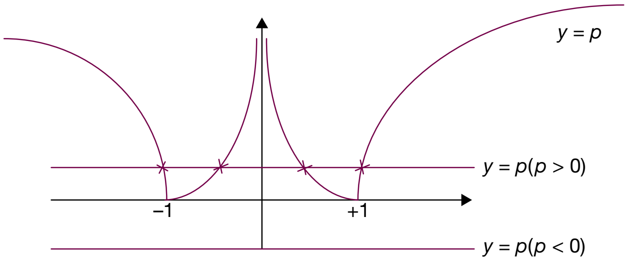
Let $y = |\ln|x||$ and $y = p$

Sequence of graph transformation

$$\ln x \rightarrow \ln|x| \rightarrow |\ln|x||$$







$|\ln|x|| = p$

Hence, For no solution $\Rightarrow p < 0$
2 solution $\Rightarrow p = 0$; 4 solution $\Rightarrow p > 0$; ∞ solution $\Rightarrow p \in \phi$

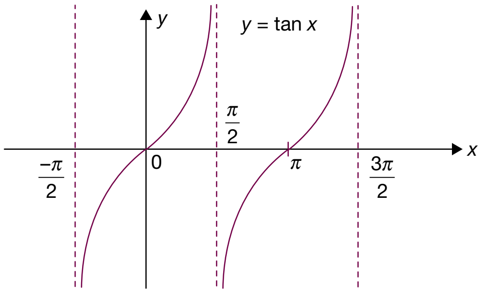
ILLUSTRATION 29

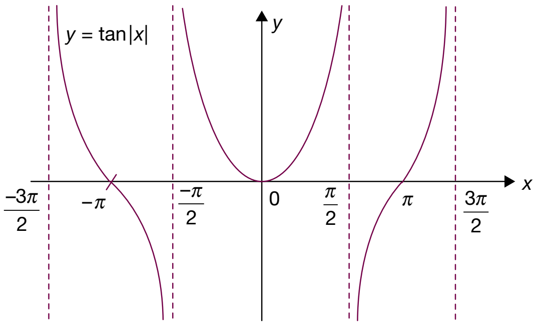
Plot $y = |\tan|x||$

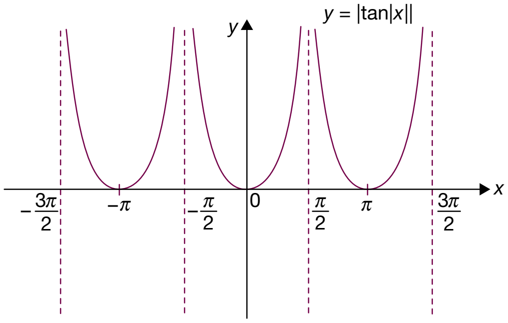
SOLUTION

Sequence of graph transformation

$$\tan x \to \tan|x| \to |\tan|x||$$







(p) $|y| = f(x)$: Remove the negative part (below x-axis) of $f(x)$ and take the image of positive part (above x-axis) of $f(x)$ about the x-axis.

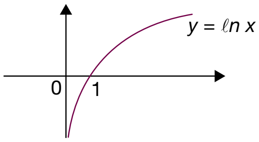
ILLUSTRATION 30

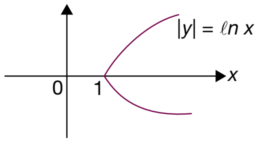
Plot the following.

(i) $|y| = \ln x$ **(ii)** $|y| = x^2 - 3x + 2$ **(iii)** $|y| = e^x$
(iv) $|y| = \sin x$ **(v)** $|y| = x^2 - 6x + 11x - 6$

SOLUTION

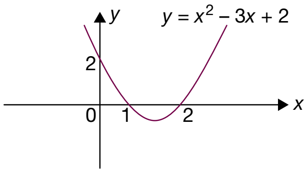
(i) $|y| = \ln x$

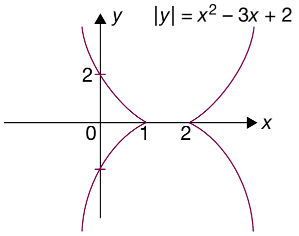




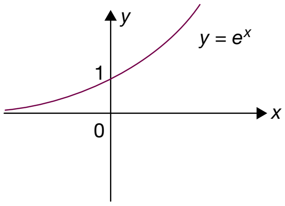
(ii) $|y| = x^2 - 3x + 2$

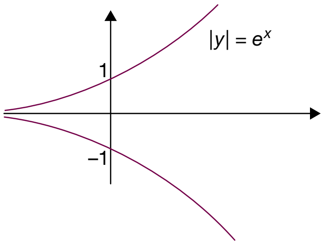
Here, $y = f(x) = x^2 - 3x + 2 = (x - 1)(x - 2)$



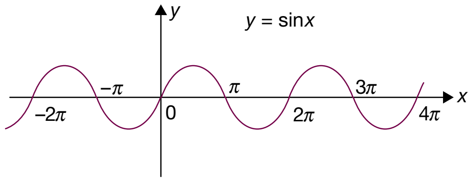


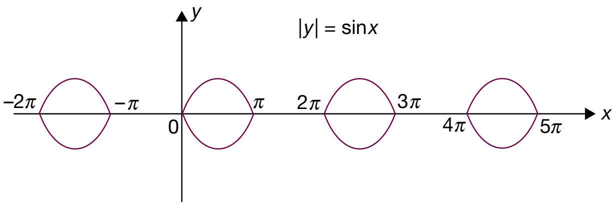
(iii) $|y| = e^x$





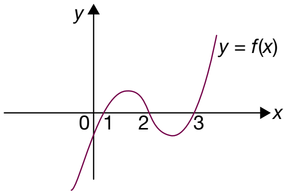
(iv) $|y| = \sin x$





(v) $|y| = x^2 - 6x + 11x - 6$

Here, $y = f(x) = (x - 1)(x - 2)(x - 3)$



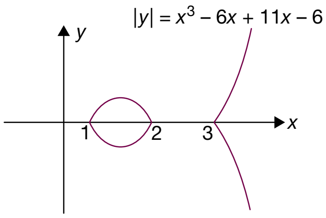
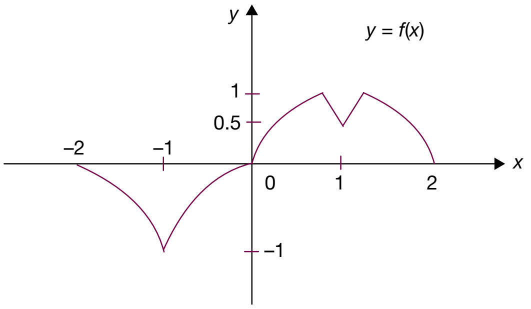


ILLUSTRATION 31

The graph of function f, $y = f(x)$ is shown below:



Plot the following graphs

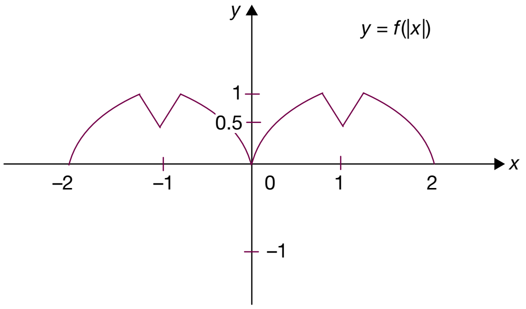
(i) $y = f(|x|)$ **(ii)** $y = f(-|x|)$ **(iii)** $y = |f(x)|$

(iv) $y = |f(|x|)|$ **(v)** $y = |f(-|x|)|$ **(vi)** $y = -|f(x)|$

SOLUTION

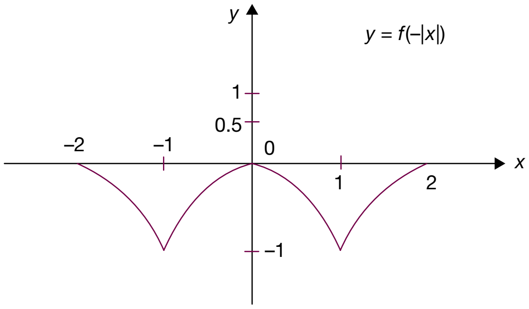
(i) $y = f(|x|)$

Remove the left side ($x < 0$) part of $f(x)$ and take the image of the right side ($x \geq 0$) about the y-axis.



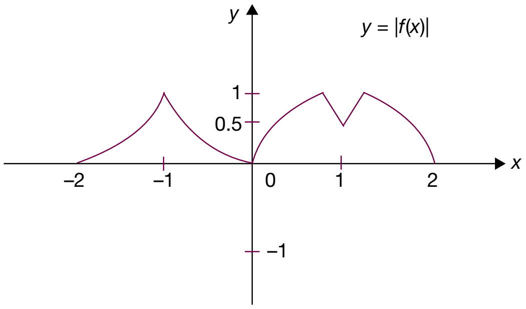
(ii) $y = f(-|x|)$

Remove the right side ($x > 0$) part of $f(x)$ and take the image of left side about the y-axis.



(iii) $y = |f(x)|$

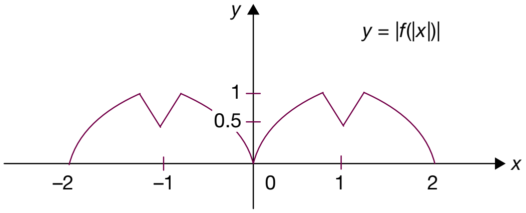
Remove the lower part ($y < 0$) part of $f(x)$ and take the image of upper side ($y \geq 0$) about the x-axis.



(iv) $y = |f(|x|)|$

Sequence of graph transformation

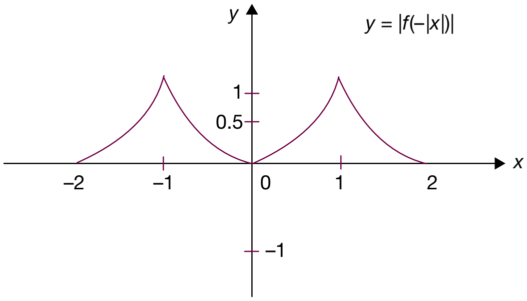
$f(x) \rightarrow f(|x|) \rightarrow |f(|x|)|$



(v) $y = |f(-|x|)|$

Sequence of graph transformation

$f(x) \rightarrow f(-|x|) \rightarrow |f(-|x|)|$



(vi) $y = -|f(x)|$

Sequence of graph transformation

$f(x) \rightarrow |f(x)| \rightarrow -|f(x)|$

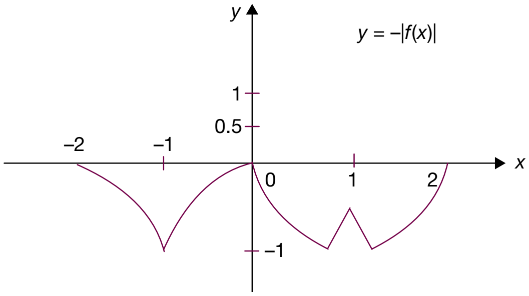


ILLUSTRATION 32

Plot the following expressions:

(i) $y = x^2 - 3x + 2$

(ii) $y = x^2 + 3x + 2$

(iii) $y = x^2 - 3|x| + 2$

(iv) $y = x^2 + 3|x| + 2$

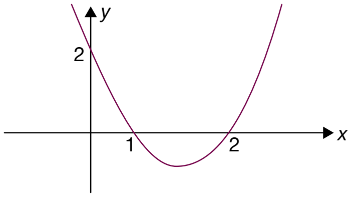
(v) $y = |x^2 - 3|x| + 2|$

(vi) $y = |x^2 + 3|x| + 2|$

SOLUTION

(i) $y = x^2 - 3x + 2$

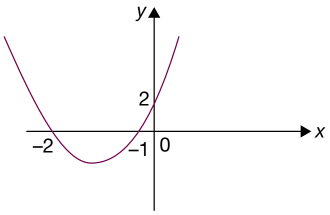
$\Rightarrow y = (x-1)(x-2) = f(x)$



(ii) $y = x^2 + 3x + 2$

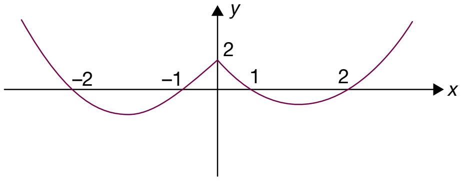
$\Rightarrow y = (x+1)(x+2) = f(-x)$

[Take the image of $f(x)$ about the y-axis]



(iii) $y = x^2 - 3|x| + 2$ $(\because \sqrt{x^2} = |x| \Rightarrow x^2 = |x|^2)$

$\Rightarrow y = |x|^2 - 3|x| + 2 = f(|x|)$



(iv) $y = x^2 + 3|x| + 2$

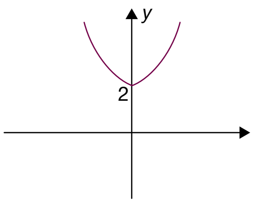
$(f(x) = x^2 - 3x + 2)$

$\Rightarrow y = (-|x|)^2 - 3(-|x|) + 2$

$\Rightarrow y = f(-|x|)$

Sequence of transformation

$f(x) \to f(-|x|)$



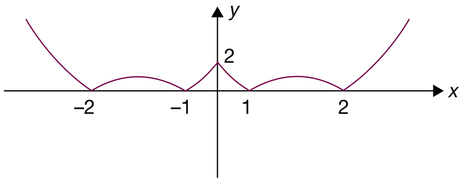
(v) $y = |x^2 - 3|x| + 2|$

$f(x) = x^2 - 3x + 2$

$\Rightarrow y = |x^2 - 3|x| + 2| = |f(|x|)|$

Sequence of transformation

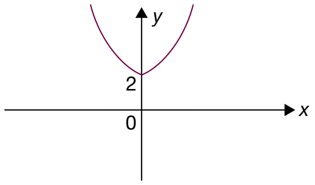
$f(x) \to f(|x|) \to |f(|x|)|$



(vi) $y = |x^2 + 3|x| + 2|$

$f(x) = x^2 - 3x + 2$

$\Rightarrow y = |x^2 + 3|x| + 2| = |f(-|x|)|$



Test Your Concepts-II

(Solutions on page H.11)

1. Find the number of solution $\{3x - 1\} - \frac{1}{2} = 0$, where {.} is fractional part function.
2. Find the number of solution of equation $\ln(2 - x) = e^x$.
3. Find the number of solution of $|x - 3| = e^{|x|}$.
4. Find the number of real solution of equation $\ln(x + 2) = e^x$.
5. If the equation $||x + 3| - 2| = p$, has exactly three distinct solutions, then find the number of integral values of p, is
6. If $|x^2 - 5|x| + 6| = k$, has 8 distinct solution, then 'k' lies in:
7. Plot the following function:
   (i) $y = 2 - e^x$;
   (ii) $y = 2 + \{2 - x\}$ where {.} is fractional part function
   (iii) $y = \ln(2 - x)$;
   (iv) $y = 3\sin\left(x - \frac{\pi}{4}\right)$
8. If the equation $|2 - x| - |x + 1| = k$ has exactly one solution, then find number of integral values of k.

Answer key

1. ∞ **2.** 1 **3.** 2 **4.** 0 **5.** 1 **6.** $\left(0, \frac{1}{4}\right)$ **8.** 5

FINDING DOMAIN AND RANGE

Let consider function $y = f(x)$,
Domain of $f(x)$ = values of x for which function is defined
Range of $f(x)$ = values of y which function can take

Working Rule

(i) $\sqrt{f(x)}$ square root of a number is real if $f(x) \geq 0$

(ii) $\frac{1}{f(x)} \Rightarrow f(x) \neq 0$ is finite at each point of domain of $f(x)$ except when denominator $f(x)$ is zero.

(iii) Let $f(x) = f_1(x) \pm f_2(x) \pm \ldots \pm f_n(x)$
or $f(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \ldots f_{n-1}(x) f_n(x)$
If $f_1(x), f_2(x), f_3(x) \ldots f_n(x)$ have domains $D_1, D_2, D_3, \ldots D_n$, respectively.
Then Domain of $f(x) = D_1 \cap D_2 \ldots \cap D_n$

(iv) Let $h(x) = \frac{f(x)}{g(x)}$, Then if domain of $f(x)$ and $g(x)$ is D_1 and D_2, respectively, then domain of $h(x) = D_1 \cap D_2 - \{x : g(x) = 0\}$.

(v) If $g(x) = \log_a f(x)$, then the function $g(x)$ is defined if $f(x) > 0$ and $a > 0, a \neq 1$.

(vii) $f(x) = {}^nP_r$ or ${}^nC_r$
where n and r may be function of x.
I. $n, r \in W$ where W is a set of whole numbers.
II. $n \geq 1$ and $0 \leq r \leq n$.

ILLUSTRATION 33

Find the domain of the function.

(i) $f(x) = \sqrt{x-1} + \sqrt{7-x}$ **(ii)** $f(x) = \frac{\sqrt{x-3}}{x^2 - 9x + 20}$

(iii) $f(x) = \log_4\left(x^2 - 1\right)$ **(iv)** $f(x) = \sin^{-1}\left(\log_2 x\right)$

(v) $f(x) = {}^{(3x-1)}P_{(20-4x)}$

SOLUTION

(i) $f(x) = \sqrt{x-1} + \sqrt{7-x}$

$f_1(x) = \sqrt{x-1} \Rightarrow D_1 = x \geq 1$

$f_2(x) = \sqrt{7-x} \Rightarrow D_2 = 7 - x \geq 0 \Rightarrow x \leq 7$

Thus, domain of $f(x) = D_1 \cap D_2 = [1, 7]$

(ii) $f(x) = \frac{\sqrt{x-3}}{x^2 - 9x + 20}$

$\sqrt{x-3} \Rightarrow x \geq 3 \Rightarrow [3, \infty)$

Also $x^2 - 9x + 20 \neq 0$

$\Rightarrow (x-4)(x-5) \neq 0 \Rightarrow x \neq 4, 5$

Thus, domain of $f(x) = [3, \infty) - \{4, 5\}$

$= [3, 4) \cup (4, 5) \cup (5, \infty)$

(iii) $f(x) = \log_4\left(x^2 - 1\right)$

$x^2 - 1 > 0$

$\Rightarrow (x-1)(x+1) > 0$

Thus, $x \in (-\infty, -1) \cup (1, \infty)$

(iv) $f(x) = \sin^{-1}\left(\log_2 x\right)$

$-1 \leq \log_2 x \leq 1$

$2^{-1} \leq x \leq 2$

[$\because$ Base $2 > 1$, sign of inequality remain same]

Thus, the domain of $f(x) = \left[\frac{1}{2}, 2\right]$

(v) $f(x) = {}^{(3x-1)}P_{(20-4x)}$

$3x - 1 \geq 20 - 4x$

$7x \geq 21 \Rightarrow x \geq 3$ ...(1)

Also $3x - 1 \geq 0 \Rightarrow x \geq \frac{1}{3}$ ...(2)

and $20 - 4x \geq 0 \Rightarrow x \leq 5$ ...(3)

Which is possible only if x is an integer taking the above into account we find x should be an integer. Thus, the domain of $f(x) = \{3, 4, 5\}$.

ILLUSTRATION 34

Find the domain of the following functions:

(i) $f(x) = \frac{1}{\sqrt{|x| - x}}$ **(ii)** $f(x) = \frac{\cos^{-1} x}{[x]}$ where [.] is GIF

(iii) $f(x) = \frac{1}{x} + 2^{\sin^{-1} x} + \frac{1}{\sqrt{x-2}}$ **(iv)** $f(x) = \sqrt{\log_2 \sin x}$

(v) $f(x) = \sqrt{\cos(\sin x)}$ **(vi)** $f(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$

SOLUTION

(i) Let $f(x) = \frac{1}{\sqrt{|x| - x}}$.

Clearly, $f(x)$ is defined for

$|x| - x > 0 \Rightarrow |x| > x \Rightarrow x < 0$

Hence, domain $(f) = (-\infty, 0)$.

(ii) Let $f(x) = \frac{\cos^{-1} x}{[x]}$, $g(x) = \cos^{-1} x$ and $h(x) = [x]$.

Then $f(x) = \frac{g(x)}{h(x)}$

$\Rightarrow$ Domain (f) = Domain $(g) \cap$ Domain $h - \{x \mid h(x) = 0\}$
Clearly, domain $(g) = [-1, 1]$, domain $h = R$, and $\{x \mid h(x) = 0\} = \{x \mid [x] = 0\} = [0, 1)$.
Thus, domain $(f) = [-1, 1] \cap R - [0, 1) = [-1, 0) \cup \{1\}$

(iii) $f(x)=\dfrac{1}{x}+2^{\sin^{-1}x}+\dfrac{1}{\sqrt{x-2}}$ is

Let $f(x) = k(x) + g(x) + h(x)$, where

$$k(x)=\frac{1}{x},\ g(x)=2^{\sin^{-1}x} \text{ and } h(x)=\frac{1}{\sqrt{x-2}}.$$

Clearly, domain $(k) = R - \{0\}$, domain $(g) = [-1, 1]$ and domain $h = (2, \infty)$.

Domain $(f) = R - \{0\} \cap [-1, 1] \cap (2, \infty) = \phi$ (null set).

Thus, $f(x)$ is not defined for any $x \in R$.

(iv) $f(x)=\sqrt{\log_2 \sin x}$

$f(x)$ exists iff $\log_2 (\sin x) \geq 0 \Rightarrow \sin x \geq 2^0$

$\sin x \geq 1$ but $-1 \leq \sin x \leq 1$

$\Rightarrow \sin x = 1 \Rightarrow x = (4n + 1)\, \pi/2, n \in I$

(v) $f(x)=\sqrt{\cos(\sin x)}$ is defined if

$\cos (\sin x) \geq 0$ …(1)

As we know, $-1 \leq \sin x \leq 1$ for all x

$\Rightarrow \cos (-1 \text{ to } 1) \geq 0$

i.e., $\cos (\sin x) \geq 0$ for all x

Thus, domain $f(x) \in R$

(vi) $f(x)=\sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$ is defined for,

$$-1 \leq \frac{1+x^2}{2x} \leq 1 \text{ or } \left|\frac{1+x^2}{2x}\right| \leq 1$$

$\Rightarrow |1 + x^2| \leq |2x|$, for all x

$\Rightarrow 1 + x^2 \leq |2x|$, for all x. $\{\text{As } 1 + x^2 > 0\}$

$\Rightarrow x^2 - 2|x| + 1 \leq 0$ $\{\text{As } x^2 = |x|^2\}$

$\Rightarrow (|x| - 1)^2 \leq 0$

But $(|x| - 1)^2 \geq 0$.

Thus, $(|x| - 1)^2 = 0$

$\Rightarrow |x| = 1 \Rightarrow x = \pm 1$

Thus, domain for $f(x)$ is $\{-1, 1\}$

ILLUSTRATION 35

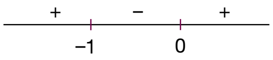
Find domain of **(i)** $f(x)=\sqrt{\dfrac{1-5^x}{7^{-x}-7}}$

(ii) $f(x)=\sqrt{\log_{1/3}(\log_4([x]^2-5))}$ where $[\cdot]$ is GIF

SOLUTION

(i) $f(x)=\sqrt{\dfrac{1-5^x}{7^{-x}-7}}$; $\dfrac{1-5^x}{7^{-x}-7} \geq 0$

$\Rightarrow \dfrac{(1-5^x)}{(1-7^{x+1})} \cdot 7^x \geq 0$



Here, $1 - 5^x = 0 \Rightarrow 5^x = 1 = 5^0 \Rightarrow x = 0$

So, $1 = 7^{x+1} = 7^0 \Rightarrow x = -1$

Hence, $x \in (-\infty, -1) \cup [0, \infty)$

(ii) $f(x)=\sqrt{\log_{1/3}(\log_4([x]^2-5))}$ where $[\cdot]$ is GIF

$\log_{1/3}(\log_4([x]^2 - 5)) \geq 0$

$\Rightarrow 0 < \log_4([x]^2 - 5) \leq \left(\dfrac{1}{3}\right)^0$

$\Rightarrow 0 < \log_4([x]^2 - 5) \leq 1$

$\Rightarrow 4^o < [x]^2 - 5 \leq 4^1 \Rightarrow 1 < [x]^2 - 5 \leq 4$

$\Rightarrow 6 < [x]^2 \leq 9$

Here, $[x]$ = Integer; $[x]^2 = 9$ is only possible case

So, $[x] = 3$ or $[x] = -3$

$\Rightarrow 3 \leq x < 4$ or $-3 \leq x < -2$

$\Rightarrow x \in [3, 4)$ or $x \in [-3, -2)$

Hence, $x \in [-3, -2) \cup [3, 4)$

RANGE OF FUNCTION

Let function $y = f(x)$

Range of function = set formed by values of y

Range of Trigonometric Functions

(i) $f(x) = a\sin x + b\cos x$.

$$-\sqrt{a^2+b^2} \leq a\sin x + b\cos x \leq \sqrt{a^2+b^2}$$

Range of $f(x)=\left[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}\right]$

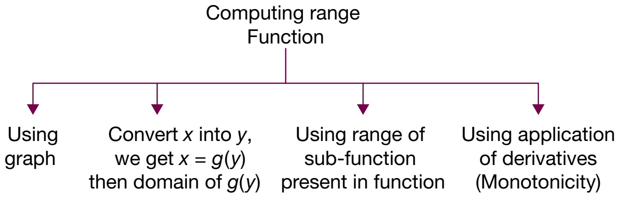
(ii) If $f(x) = \sin^{2n} x + \cos^{2n} x$, $n \in N - \{1\}$

Range $\in \left[\dfrac{1}{2^{n-1}},\ 1\right]$

(iii) If $f(x) = \sin^{2n+1} x + \cos^{2n+1} x$, $n \in N$ then

Range $\in [-1, 1]$

There are many approaches for computing range depending upon the function we are dealing.


Computing range
Function
Using graph
Convert x into y, we get x = g(y) then domain of g(y)
Using range of sub-function present in function
Using application of derivatives (Monotonicity)

Monotonicity of Function

Increasing Function

If $x_1 > x_2 \Rightarrow f(x_1) \geq f(x_2)$ for all such x_1 and x_2 in domain and if $f(x)$ is a continuous and differentiable function, then $f'(x) \geq 0$ for increasing function.

Decreasing Function

If $x_1 > x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all such x_1 and x_2 in domain and if $f(x)$ is continuous and differentiable function, then $f'(x) \leq 0$ for decreasing function.

ILLUSTRATION 36

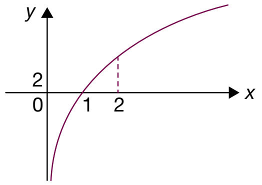
Find range of function $f(x) = \ln(x - 1)$.

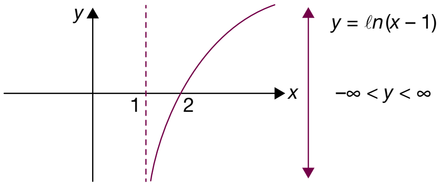
SOLUTION

Ist Method: [Using Graph]

$y = \ln x$

Graph of $\ln x$ is shifted right side by 1 unit along the x-axis.





IInd Method: [converting x into $g(y)$]

Given $y = \ln(x - 1)$

$\Rightarrow \quad e^y = x - 1 \Rightarrow x = 1 + e^y \quad y \in R$

IIIrd Method: [using sub-function present in given function]

$y = \ln(x - 1)$ where $0 < x - 1 < \infty$

$\Rightarrow \quad \ln 0^+ < \ln(x - 1) < \infty$

$\Rightarrow \quad -\infty < \ln(x - 1) < \infty$

$\Rightarrow \quad -\infty < y < \infty$

IVth Method: [using monotonicity]

$y = \ln(x - 1)$

Domain $x - 1 > 0 \Rightarrow x > 1$

$$\frac{dy}{dx} = \frac{1}{x-1} > 0 \Rightarrow \ln(x-1) \text{ increasing function}$$

Minimum $x \to 1^+ \quad \Rightarrow y = \ln(1^+ - 1) = \ln 0^+ \to -\infty$

Maximum $x \to \infty \quad \Rightarrow y = \ln(x - 1) \to \infty$

Thus, Range $y \in (-\infty, \infty)$

ILLUSTRATION 37

Find range

(i) $f(x) = \left(\frac{1}{3}\right)^{2\{x\}-1}$ where {.} is fractional part function

(ii) $f(x) = x^{[x]}$ where $1 \le x \le 3$ where [·] is GIF

SOLUTION

(i) $f(x) = \left(\frac{1}{3}\right)^{2\{x\}-1}$

For finding range, we use the range of sub function $\{x\}$

As $0 \le \{x\} < 1$

$\Rightarrow 0 \le 2\{x\} < 2 \Rightarrow -1 \le 2\{x\} - 1 < 2 - 1$

$\Rightarrow -1 \le 2\{x\} - 1 < 1$

$$\Rightarrow \left(\frac{1}{3}\right)^{-1} \ge \left(\frac{1}{3}\right)^{2\{x\}-1} > \left(\frac{1}{3}\right)^{1}$$

$$\Rightarrow 3 \ge y > \frac{1}{3} \Rightarrow y \in \left(\frac{1}{3}, 3\right]$$

(ii) $f(x) = x^{[x]}$ where $1 \le x \le 3$ where [·] is GIF

$$y = f(x) = \begin{cases} x^1 & 1 \le x < 2 & \Rightarrow 1 \le y < 2 \\ x^2 & 2 \le x < 3 & \Rightarrow 4 \le x^2 < 9 \Rightarrow 4 \le y < 9 \\ 3^3 & x = 3 & \Rightarrow y = 27 \end{cases}$$

Therefore, range $y \in [1, 2) \cup [4, 9) \cup \{27\}$

ILLUSTRATION 38

Find range of the following functions:

(i) $f(x) = |x+1| + |2x+1| + |2x-1| + |x-1|$

(ii) $f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$

(iii) $f(x) = x^2 + 3x + 2$

(iv) $f(x) = \sin^2 x + 3\sin x + 5$

(v) $f(x) = 3\sin x + 2\cos x + 3$

(vi) $f(x) = 3\sin^2 x + \sin 2x + 3$

SOLUTION

(i) $f(x) = |x+1| + |2x+1| + |2x-1| + |x-1|$

$|x+1| = 0 \Rightarrow x = -1, \ |2x+1| = 0 \Rightarrow x = -1/2$

$|2x-1| = 0 \Rightarrow x = 1/2, \ |x-1| = 0 \Rightarrow x = 1$

$f(x)$ is summation of four modulus with positive coefficients when $x \in [-1/2, 1/2]$, two modulus opens with positive sign and other two with negative sign. Hence, minimum value of $f(x)$ lies in it. x_0 is some value in this interval so range $\in [f(x_0), \infty)$. Generally, we choose value of x_0 at which calculations are easy like we can take $x_0 = 0$.

$f(0) = |0+1| + |0+1| + |2.0-1| + |0-1| = 4$

Hence, range $\in [4, \infty)$

(ii) $f(x) = \log_2\left(\frac{\sin x - \cos x + 3\sqrt{2}}{\sqrt{2}}\right)$

$$= \log_2\left(\sin\left(\pi - \frac{\pi}{4}\right) + 3\right) = y \qquad \text{(let)}$$

$$\because \ -1 \le \sin\left(x - \frac{\pi}{4}\right) \le 1 \Rightarrow 2 \le \sin\left(x - \frac{\pi}{4}\right) + 3 \le 4$$

$$\Rightarrow \log_2 2 \le \log_2\left(\sin\left(x-\frac{\pi}{4}\right)+3\right) \le \log_2 4$$

$$\Rightarrow 1 \le y \le 2$$

Hence, range of $f(x)$ is [1, 2].

(iii) $y = x^2+3x+2$

$$y = x^2+3x+2 = \left(x+\frac{3}{2}\right)^2+2-\frac{9}{4}$$

$$y = \left(x+\frac{3}{2}\right)^2-\frac{1}{4}$$

As $\left(x+\frac{3}{2}\right)^2 \ge 0 \quad \forall \ x \in R$

Thus, $\left(x+\frac{3}{2}\right)^2-\frac{1}{4} \ge -\frac{1}{4}$

Hence, range of $f(x) \in \left[-\frac{1}{4},\ \infty\right)$

(iv) $y = \sin^2 x+3\sin x+5$

Given function $y = \sin^2 x+3\sin x+5$

$$\Rightarrow y = \left(\sin x+\frac{3}{2}\right)^2+\frac{11}{4}$$

Since, $-1 \le \sin x \le 1 \Rightarrow \frac{1}{2} \le \sin x+\frac{3}{2} \le \frac{5}{2}$

$$\Rightarrow \frac{1}{4} \le \left(\sin x+\frac{3}{2}\right)^2 \le \frac{25}{4}$$

$$\Rightarrow 3 \le \left(\sin x+\frac{3}{2}\right)^2+\frac{11}{4} \le 9$$

Hence, the range of $f(x) \in [3, 9]$

(v) $f(x) = 3\sin x+2\cos x+3$

$$-\sqrt{3^2+2^2} \le 3\sin x+2\cos x \le \sqrt{3^2+2^2}$$

$$-\sqrt{13} \le 3\sin x+2\cos x \le \sqrt{13}$$

$$3-\sqrt{13} \le \sqrt{13}\sin(x+\phi)+3 \le 3+\sqrt{13}$$

Hence, the range $f(x) \in \left[3-\sqrt{13}, 3+\sqrt{13}\right]$

(vi) $f(x) = 3\sin^2 x+\sin 2x+3 = 3\left(\frac{1-\cos 2x}{2}\right)+\sin 2x+3$

$$\Rightarrow f(x) = \sin 2x-\frac{3}{2}\cos 2x+\frac{9}{2}$$

$$\Rightarrow -\sqrt{1+\frac{9}{4}} \le \sin 2x-\frac{3}{2}\cos 2x \le \sqrt{1+\frac{9}{4}}$$

$$\Rightarrow -\frac{\sqrt{13}}{2} \le \sin 2x-\frac{3}{2}\cos 2x \le \frac{\sqrt{13}}{2}$$

$$\Rightarrow -\frac{\sqrt{13}}{2}+\frac{9}{2} \le \sin 2x-\frac{3}{2}\cos 2x+\frac{9}{2} \le \frac{\sqrt{13}}{2}+\frac{9}{2}$$

Hence, the range of $f(x) \in \left[\frac{9-\sqrt{13}}{2}, \frac{9+\sqrt{13}}{2}\right]$

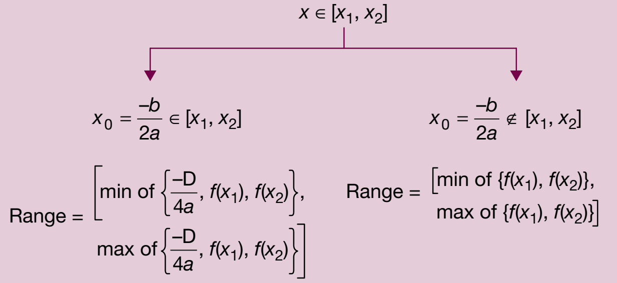
Range of quadratic expression $y = f(x) = ax^2+bx+c$

(i) For all real values of x

- If $a > 0$, then $y \in \left[-\frac{D}{4a}, \infty\right)$
- If $a < 0$, then $y \in \left(-\infty, -\frac{D}{4a}\right]$

(ii) x is in bounded interval $x \in [x_1, x_2]$

x-coordinate of vertex $x_0 = -\frac{b}{2a}$



$x \in [x_1, x_2]$

$x_0 = \frac{-b}{2a} \in [x_1, x_2]$

Range $= \left[\text{min of}\left\{\frac{-D}{4a}, f(x_1), f(x_2)\right\}, \text{max of}\left\{\frac{-D}{4a}, f(x_1), f(x_2)\right\}\right]$

$x_0 = \frac{-b}{2a} \notin [x_1, x_2]$

Range $= [\text{min of } \{f(x_1), f(x_2)\}, \text{max of } \{f(x_1), f(x_2)\}]$

ILLUSTRATION 39

Let the function $f(x) = -x^2+4x-6$, find range if:

(i) $x \in R$ **(ii)** $x \in (0, 4]$ **(iii)** $x \in [-3, -1)$

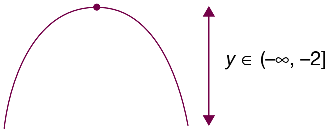
SOLUTION

(i) $x \in R$

$$D = B^2-4AC = 16-4(-1)(-6)$$

Vertex $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$$= \left(\frac{-4}{-2}, \frac{8}{-4}\right) = (2, -2)$$



Required range $y \in (-\infty, -2]$

(ii) $x_0 = -\frac{b}{2a} = 2 \in (0, 4]$

$$f(2) = -2;\ f(0) = -6$$

$$f(4) = -16+16-6 = -6$$

Required range $y \in [-6, -2]$

(iii) $x \in [-3, -1)$

$$f(x) = -x^2+4x-6$$

$$x_0 = -\frac{b}{2a} = 2 \notin [-3, -1)$$

$$f(-3) = -9-12-6 = -27$$

$$f(-1) = -1-4-6 = -11$$

Required range $y \in [-27, -11)$

ILLUSTRATION 40

Find the domain and range of function.

(i) $f(x) = \sin^2 x - \sin x + 1$ **(ii)** $f(x) = x^4 - 4x^2 + 1$

(iii) $f(x) = \ln(x^2 - 2x + 3)$ **(iv)** $f(x) = e^{2x} - 16e^x + 8$

(v) $f(x) = \dfrac{1}{\sqrt{x^2 - x}}$

SOLUTION

(i) $f(x) = \sin^2 x - \sin x + 1.$

Let $\sin x = t$, domain $x \in R$, $t \in [-1, 1]$

$f(t) = 4 = t^2 - t + 1$ where $t \in [-1, 1]$

$$t_0 = -\frac{B}{2A} = \frac{-(-1)}{2} = \frac{1}{2} \in [-1, 1]$$

So, $f(-1) = 1 + 1 + 1 = 3$

$f(1) = 1 - 1 + 1 = 1$

$$f\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} + 1 = \frac{3}{4}$$

Required domain $x \in R$ and range $y \in \left[\dfrac{3}{4}, 3\right]$

(ii) $f(x) = x^4 - 4x^2 + 1$

Domain $x \in R$

Put $x^2 = t$, $-\infty < x < \infty \Rightarrow 0 \leq x^2 < \infty \Rightarrow t \in [0, \infty)$

$y = t^2 - 4t + 1$ where $t \in [0, \infty)$

$$t_0 = -\frac{B}{2A} = \frac{-(-4)}{2(1)} = 2 \in [0, \infty)$$

So, $f(0) = 0 - 0 + 1 = 1$

$f(2) = 4 - 8 + 1 = -3$

$t \to \infty, f(t) \to \infty$

Required domain $x \in R$ and range $y \in [-3, \infty)$

(iii) $f(x) = \ln(x^2 - 2x + 3)$

$x^2 - 2x + 3 > 0$

$\Rightarrow (x - 1)^2 + 2 > 0 \ \forall \ x \in R$ (domain)

$0 \leq (x - 1)^2 < \infty$

$\Rightarrow 2 + 0 \leq (x - 1)^2 + 2 < \infty$

$\Rightarrow 2 \leq x^2 - 2x + 3 < \infty$

$\Rightarrow \ln 2 \leq \ln(x^2 - 2x + 3) < \infty$

$\Rightarrow \ln 2 \leq y < \infty$

Required domain $x \in R$ and range $y \in [\ln 2, \infty]$

(iv) $f(x) = e^{2x} - 16e^x + 8$

Domain $x \in R$

Let $e^x = t$

where $0 < e^x < \infty \Rightarrow 0 < t < \infty$

$y = f(t) = t^2 - 16t + 8$ where $t \in (0, \infty)$

$$t_0 = -\frac{B}{2A} = \frac{16}{2} = 8 \in (0, \infty)$$

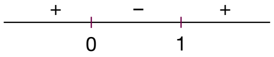
So, $f(0) = 0 - 0 + 8 = 8$

$f(8) = 64 - 8 \times 16 + 8 = -56$

$t \to \infty \ f(t) \to \infty$

Required domain $x \in R$ and range $y \in [-56, \infty]$

(v) $f(x) = \dfrac{1}{\sqrt{x^2 - x}}$



For computing domain

$x^2 - x > 0 \Rightarrow x(x - 1) > 0$

$\Rightarrow x \in (-\infty, 0) \cup (1, \infty)$

For range computation

$0 < x^2 - x < \infty \Rightarrow 0 < \sqrt{x^2 - x} < \infty$

$$\Rightarrow \infty > \frac{1}{\sqrt{x^2 - x}} > 0 \Rightarrow y \in (0, \infty)$$

Required domain $x \in (-\infty, 0) \cup (1, \infty)$ and range $y \in (0, \infty)$

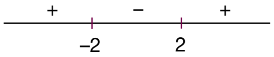
ILLUSTRATION 41

Let consider a function $f(x) = x + \dfrac{1}{x}$, compute its range.

SOLUTION

$y = f(x) = x + \dfrac{1}{x}$, domain $x \neq 0$

$$y = \frac{x^2 + 1}{x} \Rightarrow xy = x^2 + 1 \Rightarrow x^2 - xy + 1 = 0$$



As x is real, $D \geq 0$` $\Rightarrow (-y)^2 - 4 \geq 0$

$\Rightarrow \quad (y - 2)(y + 2) \geq 0$

Range $y \in (-\infty, -2] \cup [2, \infty)$

Alternate I: (Using AM-GM)

$y = x + \dfrac{1}{x}; \qquad x \neq 0$

CASE 1: $x > 0$

Since, AM $\geq$ GM

$$\Rightarrow \frac{x + \frac{1}{x}}{2} \geq \sqrt{x \frac{1}{x}}$$

$\Rightarrow x + \dfrac{1}{x} \geq 2$

$\Rightarrow y \geq 2$

Thus $y \in [2, \infty)$ ...(1)

CASE 2: $x < 0$

Let $x = -\alpha < 0$; $\alpha > 0$

$$y = x + \frac{1}{x} = -\left(\alpha + \frac{1}{\alpha}\right)$$

Now using AM $\geq$ GM

$$\frac{\alpha + \frac{1}{\alpha}}{2} \geq \sqrt{\alpha \cdot \frac{1}{\alpha}}$$

$\Rightarrow \alpha + \dfrac{1}{\alpha} \geq 2$ Put $\alpha = -x$.

$\Rightarrow -x - \dfrac{1}{x} \geq 2 \Rightarrow y = x + \dfrac{1}{x} \leq -2$

Thus, $y \in (-\infty, -2]$ ...(2)

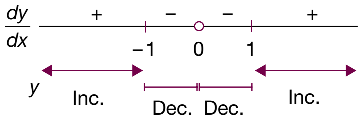
Hence, final range $y \in (-\infty, -1] \cup [2, \infty)$

Alternate II: (Using monotonicity)

$$y = x + \frac{1}{x} \text{ where } x \neq 0 \qquad \left[\because \frac{d}{dx}(x^{-1}) = -\frac{1}{x^2}\right]$$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x-1)(x+1)}{x^2}$$



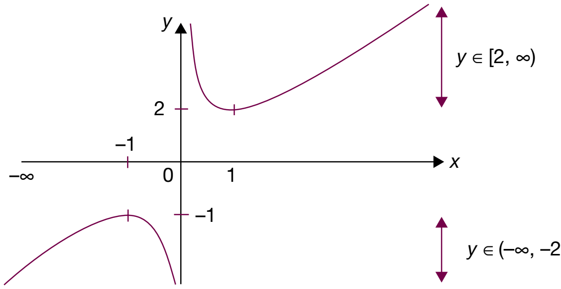
Clearly $f(x)$ is increasing $\frac{dy}{dx} \geq 0$ for $x \in (-\infty, -1] \cup [1, \infty)$

and $f(x)$ is decreasing $\frac{dy}{dx} \leq 0$ for $x \in (-1, 0) \cup (0, 1)$

$y = x + \frac{1}{x}$ is discontinuous at $x = 0$

$$\left.\begin{aligned} &x \to -\infty; && y \to -\infty + \frac{1}{-\infty} \to -\infty \\ &x = -1; && y = -1 - 1 = -2 \\ &x \to 0^-; && y \to 0^- + \frac{1}{0^-} \to -\infty \end{aligned}\right] \; y \in (-\infty, -2]$$

$$\left.\begin{aligned} &x \to 0^+; && y \to 0^+ + \frac{1}{0^+} \to +\infty \\ &x = 1; && y = 1 + 1 = 2 \\ &x \to \infty; && y \to \infty + 1/\infty \to \infty \end{aligned}\right] \; y \in [2, \infty)$$



From the graph, we can say the range: $y \in (-\infty, -2] \cup [2, \infty)$

ILLUSTRATION 42

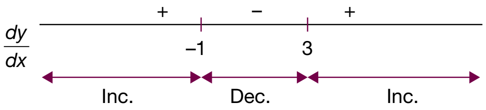
Let function $f(x) = x^3 - 3x^2 - 9x + 2$, find range if:

(i) $\forall\, x \in R$ **(ii)** $x \in [-2, 5]$ **(iii)** $x \in [0, 6)$

SOLUTION

$$y = x^3 - 3x^2 - 9x + 2$$

$$\frac{dy}{dx} = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$$



$f(x)$ is increasing $x \in (-\infty, -1) \cup (3, \infty)$

$f(x)$ is decreasing $x \in (-1, 3)$

(i) $x \in R$

$$y = f(x) = x^3 - 3x^2 - 9x + 2$$

as $x \to -\infty;\ y \to -\infty$

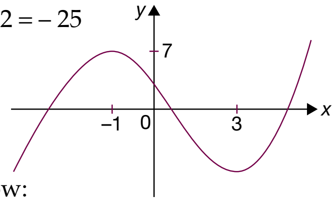
So, $f(-1) = -1 - 3 + 9 + 2 = 7$

$f(3) = 27 - 27 - 27 + 2 = -25$

as $x \to \infty;\ y \to +\infty$

Thus, $y \in (-\infty, \infty)$

Using the above information we can plot $f(x)$ as shown below:



(ii) $x \in [-2, 5]$

$$y = f(x) = x^3 - 3x^2 - 9x + 2$$

$$\frac{dy}{dx} = 3(x + 1)(x - 3) = 0 \text{ For } x = -1, 3$$

Here, $-1, 3 \in [-2, 5]$

both critical points are in interval

So, $f(-1) = 7$;

$f(3) = -25$;

$f(-2) = -8 - 3(4) + 18 + 2 = 0$;

$f(5) = 125 - 75 - 45 + 2 = 7$;

As $x \in [-2, 5]$ we get range $y \in [-25, 7]$

(iii) $x \in [0, 6)$

$$\frac{dy}{dx} = 3(x + 1)(x - 3) = 0 \text{ For } x = -1, 3$$

Here, $-1 \notin [0, 6)$ and $3 \in [0, 6)$

So, $f(0) = 2$;

$f(6) = 216 - 36 \times 3 - 54 + 2 = 56$;

$f(3) = -25$

As $x \in [0, 6]$ we get range $y \in [-25, 56)$

ILLUSTRATION 43

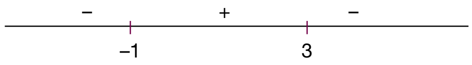
$f(x) = -x^3 + 3x^2 + 9x - 7$, find range if:

(i) $x \in (-2, 5)$ **(ii)** $x \in (-2, 0]$ **(iii)** $x \in (0, 2]$

SOLUTION

$$\frac{dy}{dx} = -3x^2 + 6x + 9 = -3(x^2 - 2x - 3) = -3(x - 3)(x + 1)$$

$$\frac{dy}{dx} = 0 \text{ at } x = -1, 3 \text{ critical points}$$



Clearly $f(x)$ is increasing $\frac{dy}{dx} \geq 0$ for $x \in [-1, 3]$

$f(x)$ is decreasing $\frac{dy}{dx} \leq 0$ for $x \in (-\infty, -1] \cup [3, \infty)$

(i) $x \in (-2, 5)$

Critical points $x = -1, 3 \in (-2, 5)$

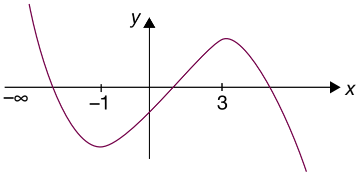
$f(-2) = +8 + 12 - 18 - 7 = -5;$

$f(5) = -125 + 75 + 45 - 7 = -12;$

$f(-1) = +1 + 3 - 9 - 7 = -12;$

$f(3) = 0 + 27 - 7 = 20$

As $x \in (-2, 5)$ we have $y \in [-12, 20]$



(ii) $x \in (-2, 0]$

$x = -1$ lies in given interval but $x = 3 \notin (-2, 0]$

So, $f(-2) = -5;$

$f(0) = -7;$

$f(-1) = -12$

As $x \in (-2, 0]$, we have $y \in [-12, -5)$

(iii) $x \in (0, 2]$

Both critical points $x = -1, 3 \notin (0, 2]$

So, $f(0) = -7;$

$f(2) = -8 + 12 + 18 - 7 = 15$

As x $(0, 2]$, we have $y \in (-7, 15]$

ILLUSTRATION 44

Find the domain and range of the following:

(i) $f(x) = \cos^{-1}\sqrt{\log_{[x]}\frac{|x|}{x}}$, where [.] denotes the greatest integer function.

(ii) $f(x) = \log_{[x-1]}\sin x$, where [ ] denotes greatest integer function.

(iii) $f(x) = \log_3\{\log_{1/2}(x^2 + 4x + 4)\}$.

(iv) $f(x) = \sqrt{2-x} + \sqrt{1+x}$.

SOLUTION

(i) For domain

$[x] > 0$ and $[x] \neq 1$

So, $[x] \geq 2$, so $x \in [2, \infty)$

For range, if $x \in [2, \infty)$, then $\frac{|x|}{x} = 1$

So, $f(x) = \cos^{-1} 0 = \frac{\pi}{2}$

Range of $f(x) = \frac{\pi}{2}$

Thus, Domain: $[2, \infty)$; Range: $\{\pi/2\}$

(ii) $f(x) = \log_{[x-1]}\sin x$

$\sin x > 0 \Rightarrow x \in (2n\pi, (2n+1)\pi)$

Here, $[x - 1] > 0$ and $[x - 1] \neq 1$

$\Rightarrow x \in [3, \infty]$

Domain $x \in [3, \pi] \cup \bigcup_{n=1}^{\infty} (2n\pi, (2n+1)\pi)$.

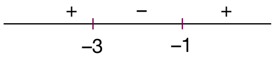
For range, $\sin x \in (0, 1]$ and $[x - 1] \in [2, \infty)$

So, range $\in (-\infty, 0]$

(iii) First, to find the domain

$\log_3\{\log_{1/2}(x^2 + 4x + 4)\}$ exists if,

$\log_{1/2}(x^2 + 4x + 4) > 0$



$\Rightarrow x^2 + 4x + 4 < \left(\frac{1}{2}\right)^0$

[using $\log_a x < b \Rightarrow x > a^b$, if $0 < a < 1$]

$\Rightarrow x^2 + 4x + 4 < 1 \Rightarrow x^2 + 4x + 3 < 0$

$\Rightarrow (x + 1)(x + 3) < 0 \Rightarrow -3 < x < -1$ …(1)

and $x^2 + 4x + 4 > 0 \Rightarrow (x + 2)^2 > 0$

which is always true except for $x = -2$ …(2)

Thus, from (1) and (2), we have

Hence, domain $x \in (-3, -2) \cup (-2, -1)$

Now we find out the range,

Since, $0 < \log_{1/2}(x^2 + 4x + 4) < \infty$

$\Rightarrow -\infty < \log_3\{\log_{1/2}(x^2 + 4x + 4)\} < \infty$

Thus, range $(y) \in R$

(iv) $f(x) = \sqrt{2-x} + \sqrt{1+x}$

Domain of $f(x) = \{x \mid 2 - x \geq 0 \text{ and } 1 + x \geq 0\}$

$\therefore$ Domain of $f(x) = [-1, 2]$

Again, $\{f(x)\}^2 = \left(\sqrt{2-x} + \sqrt{1+x}\right)^2$

$$= 3 + 2\sqrt{(2-x)(1+x)} = 3 + 2\sqrt{\frac{9}{4} - \left(x - \frac{1}{2}\right)^2}$$

$\therefore$ The greatest value of $\{f(x)\}^2$

$= 3 + 2.\sqrt{\frac{9}{4}} = 6$, when $x = \frac{1}{2}$

The least value of $\{f(x)\}^2 = 3 + 0 = 3$,

when $x - \frac{1}{2} = \frac{3}{2}$, i.e. $x = 2$

$\therefore$ The greatest value of $f(x) = \sqrt{6}$

and the least value of $f(x) = \sqrt{3}$

$\therefore$ Range of $f(x) = [\sqrt{3}, \sqrt{6}]$

ILLUSTRATION 45

Find the domain and the range of the following functions where [.] is GIF and {.} is fpf.

(i) $f(x)=\dfrac{\{x\}}{\{x\}+1}$

(ii) $f(x)=\sin^{-1}(\ln[x])+\ln(\sin^{-1}[x])$,

(iii) $f(x)=\ln(\{x\}^2+3\{x\}+2)$

SOLUTION

(i) $y=\dfrac{\{x\}}{\{x\}+1}$ domain R, always defined

$$\Rightarrow\ y\{x\}+y=\{x\}\ \Rightarrow\ \{x\}=\frac{y}{1-y}$$

As $0\le\{x\}<1$, thus $0\le\dfrac{y}{1-y}<1$

$$\frac{y}{1-y}\ge 0\ \Rightarrow\ \frac{y}{y-1}\le 0\ \Rightarrow y\in[0,1) \qquad \ldots(1)$$

$$\Rightarrow\ \frac{y}{1-y}-1<0\ \Rightarrow\ \frac{2y-1}{y-1}>0$$

$$\Rightarrow y\in\left(-\infty,\frac{1}{2}\right)\cup(1,\infty) \qquad \ldots(2)$$

Common of condition (1) and (2), we get

Hence, the range of $f(x)\in\left[0,\dfrac{1}{2}\right)$

(ii) $f(x)=\sin^{-1}(\ln[x])+\ln(\sin^{-1}[x])$,

$\sin^{-1}\ln[x]$ is defined if only if

$$-1\le\ln[x]\le 1\ \Rightarrow\ e^{-1}\le[x]\le e^{1}\ \Rightarrow [x]=1,2$$

$\ln\sin^{-1}[x]$ is defined if and only if $\sin^{-1}[x]>0$ then $[x]=1$ is only possibility

$f(x)$ is defined if $[x]=1$ only, so domain of $f(x)$ is $x\in[1,2)$

$$f(x)=\sin^{-1}\ln 1+\ln\sin^{-1}1$$

$$=\sin^{-1}(0)+\ln\left(\frac{\pi}{2}\right)=\ln\left(\frac{\pi}{2}\right)$$

Hence, the range of $f(x)=\left\{\ln\dfrac{\pi}{2}\right\}$ as it has only one element, It is also known as singleton range.

(iii) $f(x)=\ln(\{x\}^2+3\{x\}+2)$

Domain is R

As $\ln x$ is strictly increasing function it is sufficient to find range of $\{x\}^2+3\{x\}+2$

$$\{x\}^2+3\{x\}+2=\left(\{x\}+\frac{3}{2}\right)^2-\frac{1}{4}$$

Since, $0\le\{x\}<1\Rightarrow\dfrac{3}{2}\le\left(\{x\}+\dfrac{3}{2}\right)<\dfrac{5}{2}$

$$\Rightarrow\ \frac{9}{4}\le\left(\{x\}+\frac{3}{2}\right)^2<\frac{25}{4}\ \Rightarrow\ 2\le\left(\{x\}+\frac{3}{2}\right)^2-\frac{1}{4}<6$$

$$\Rightarrow\ \ln 2\le\ln\left(\{x\}^2+3\{x\}+2\right)<\ln 6$$

Hence, the range of $f(x)\in=[\ln 2,\ln 6)$

ILLUSTRATION 46

If $f(x)=\left[\dfrac{2x^2+2x+11}{x^2+x+4}\right]$ and $g(x)=\log_2\left(\dfrac{x^2+1}{2x}\right)$, where $[x]$ denotes the greatest integer function less than or equal to x, then find the range of the function $h(x)=\log_{g(x)}f(x)$

SOLUTION

We have, $f(x)=\left[\dfrac{2x^2+2x+11}{x^2+x+4}\right]=\left[2+\dfrac{3}{x^2+x+4}\right]$

$$=2+\left[\frac{3}{\left(x+\frac{1}{2}\right)^2+\frac{15}{4}}\right]=2\text{ as }\frac{3}{\left(x+\frac{1}{2}\right)^2+\frac{15}{4}}\in\left(0,\frac{12}{15}\right]$$

$\therefore\quad f(x)=2$

$\therefore\quad g(x)=\log_2\left(\dfrac{1}{2}\left(x+\dfrac{1}{x}\right)\right)$ Clearly $x>0$

Now for the function $h(x)=\log_{g(x)}f(x)$, $f(x)>0$, $g(x)>0$ and $g(x)\ne 1$.

$$\left\{\because\frac{1}{2}\left(x+\frac{1}{x}\right)>1\Rightarrow x^2+1>2x\Rightarrow(x-1)^2>0\Rightarrow x\ne 1\right.$$

$$\left.\text{and }\frac{1}{2}\left(x+\frac{1}{x}\right)\ne 2\Rightarrow x^2-4x+1\ne 0\Rightarrow x\ne 2\pm\sqrt{3}\right\}$$

As $g(x)>0$ and $g(x)\ne 1$

$\therefore\quad$ Domain of $h(x)$ is $R^+-\{2-\sqrt{3},1,2+\sqrt{3}\}$

Range of $h(x)=\log_{g(x)}2=\dfrac{1}{\log_2 g(x)}$ is $(-\infty,\infty)-\{0\}$

ILLUSTRATION 47

For $x\ne\dfrac{n\pi}{2}$ where $n\in I$, find the range of function

$f(x)=\text{sgn}(\sin x)+\text{sgn}(\cos x)+\text{sgn}(\tan x)+\text{sgn}(\cot x)$

SOLUTION

$f(x)$ periodic function with period 2π.

$$f(x)=\begin{cases}1+1+1+1 & ;\ 0<x<\dfrac{\pi}{2}\\[2mm] 1-1-1-1 & ;\ \dfrac{\pi}{2}<x<\pi\\[2mm] -1-1+1+1 & ;\ \pi<x<\dfrac{3\pi}{2}\\[2mm] -1+1-1-1 & ;\ \dfrac{3\pi}{2}<x<2\pi\end{cases}$$

$$f(x) = \begin{cases} 4 & ; \quad 0 < x < \frac{\pi}{2} \\ -2 & ; \quad \frac{\pi}{2} < x < \pi \\ 0 & ; \quad \pi < x < \frac{3\pi}{2} \\ -2 & ; \quad \frac{3\pi}{2} < x < 2\pi \end{cases}$$

$\therefore$ Range of function = {–2, 0, 4}.

ILLUSTRATION 48

If the range of function $f(x) = \dfrac{x^2+x+c}{x^2+2x+c}$, $x \in R$ is $\left[\dfrac{5}{6}, \dfrac{3}{2}\right]$, then find c.

SOLUTION

Let $y = \dfrac{x^2+x+c}{x^2+2x+c} \Rightarrow (y-1)\,x^2 + (2y-1)\,x + c(y-1) = 0$

As x is real, so $D \geq 0$

$\Rightarrow \quad (2y-1)^2 \geq 4c(y-1)^2$

$\Rightarrow \quad 4y^2 + 1 - 4y \geq 4cy^2 - 8cy + 4c$

$\Rightarrow \quad 4(c-1)\,y^2 + 4\,(1-2c)\,y + (4c-1) \leq 0 \qquad \ldots(1)$

But we are given $y \in \left[\dfrac{5}{6}, \dfrac{3}{2}\right]$

$(6y-5)\,(2y-3) \leq 0 \Rightarrow 12y^2 - 28y + 15 \leq 0 \qquad \ldots(2)$

$\therefore$ On comparing (1) and (2), we get:

$$\frac{c-1}{3} = \frac{1-2c}{-7} = \frac{4c-1}{15} \Rightarrow c = 4$$

ILLUSTRATION 49

If function $f(x) = (x+2)\,e^{\ln(x+2)}$ and $g(x) = \dfrac{e^{\left(\frac{1}{-\log_x e}\right)}}{x} - \dfrac{2x}{e^{-\ln 2}}$, then find the smallest positive integer in the range of $h(x) = f(x) + g(x)$.

SOLUTION

$$h(x) = f(x) + g(x) = (x+2)\,e^{\ln(x+2)} + \frac{e^{\left(\frac{1}{-\log_x e}\right)}}{x} - \frac{2x}{e^{-\ln 2}}$$

$$= (x+2)^2 + \frac{1}{x^2} - 2x.\,e^{\ln 2} = x^2 + 4x + 4 + \frac{1}{x^2} - 4x$$

$$= x^2 + \frac{1}{x^2} + 4 \text{ [domain of } h(x) \text{ is } x \in R^+ - \{1\}]$$

$$\left(\because x^2 + \frac{1}{x^2} > 2;\ x \neq 1\right)$$

$\Rightarrow \quad h(x) > 6\ \forall\ x \in R^+ - \{1\}$

$\therefore$ Least integral value in the range of $h(x)$ is 7.

ILLUSTRATION 50

If $f(x) = \pi\left(\dfrac{\sqrt{x+7}-4}{x-9}\right)$, then find the range of function $y = \sin(2f(x))$.

SOLUTION

Given, $f(x) = \pi\left(\dfrac{\sqrt{x+7}-4}{x-9}\right)$

Clearly, domain of $f(x) = [-7, \infty) - \{9\}$.

Now $f(x) = \pi\dfrac{\left(\sqrt{x+7}-4\right)\left(\sqrt{x+7}+4\right)}{(x-9)\left(\sqrt{x+7}+4\right)}$

$= \dfrac{\pi(x+7-16)}{(x-9)\left(\sqrt{x+7}+4\right)}$ (rationalise)

$= \dfrac{\pi}{\left(\sqrt{x+7}+4\right)}$

So, range of $f(x)$ is $\left(0, \dfrac{\pi}{4}\right] - \left\{\dfrac{\pi}{8}\right\}$

$2f(x) \in \left(0, \dfrac{\pi}{2}\right] - \left\{\dfrac{\pi}{4}\right\}$

$y = \sin\left(\dfrac{2\pi}{\sqrt{x+7}+4}\right) = \sin 2f(x)$

$\Rightarrow \quad y \in \left(0, \dfrac{1}{\sqrt{2}}\right) \cup \left(\dfrac{1}{\sqrt{2}}, 1\right]$

Hence, the range of $y = \sin(2f(x))$ is $(0, 1] - \left\{\dfrac{1}{\sqrt{2}}\right\}$.

ILLUSTRATION 51

If $f(x) = 2\sin^2 x + \cos^4 x + 3\ \forall\ x \in R$, then find the number of integers in range of $f(x)$.

SOLUTION

$f(x) = 2\sin^2 x + \cos^4 x + 3\ \forall\ x \in R$

$\Rightarrow \quad f(x) = \cos^4 x + 2(1 - \cos^2 x) + 3$

$\Rightarrow \quad f(x) = \cos^4 x - 2\cos^2 x + 5$

$\Rightarrow \quad f(x) = (\cos^2 x - 1)^2 + 4$

$0 \leq (\cos^2 x - 1)^2 \leq 1$

$\therefore \quad f(x) \in [4, 5]$

Number of integers in the range of $f(x) = 2$.

ILLUSTRATION 52

Find the number of prime numbers in the range of the function $f(x) = \dfrac{96}{7(x^4 + 3x^2 + 1)}$.

SOLUTION

$x^4 + 3x^2 + 1$ is defined for $\forall\ x \in R$

$$1 \le x^4 + 3x^2 + 1 < \infty$$

$$\Rightarrow \quad 1 \ge \frac{1}{x^4 + 3x^2 + 1} > 0$$

$$\Rightarrow \quad 0 < \frac{96}{7(x^4 + 3x^2 + 1)} \le \frac{96}{7}$$

$$\Rightarrow \quad 0 < f(x) \le \frac{96}{7}$$

Range of the function is $\left(0, \frac{96}{7}\right]$.

Thus, Integers in the range are 1, 2, 3, … 13.

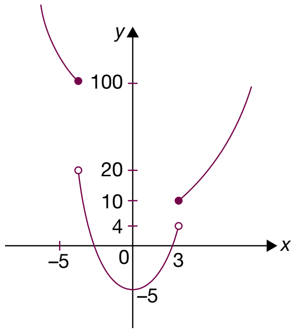
Prime numbers in the range = 2, 3, 5, 7, 11, 13.

ILLUSTRATION 53

Let $f(x) = \begin{cases} 2x^2 - 10x, & -\infty < x \le -5 \\ x^2 - 5, & -5 < x < 3 \\ x^2 + 1, & 3 \le x < \infty \end{cases}$

Find the number of negative integers in the range of the function $f(x)$.

SOLUTION



Clearly, from above graph, $R_f = [-5, \infty)$.

So, the number of negative integers in the range of $f(x)$ are five (i.e., –5, –4, –3, –2, –1).

ILLUSTRATION 54

Find the range of the function

$f(x) = \sum_{n=1}^{3}\left((\sin^2 x)^n + (\cos^2 x)^n\right)$ for all $x \in R$ is equal to

SOLUTION

$$f(x) = \sum_{n=1}^{3}\left((\sin^2 x)^n + (\cos^2 x)^n\right)$$

$$\Rightarrow \quad f(x) = (\sin^2 x + \cos^2 x) + (\sin^4 x + \cos^4 x) + (\sin^6 x + \cos^6 x)$$

$$\Rightarrow \quad f(x) = 1 + 1 - 2\sin^2 x\cos^2 x + 1 - 3\sin^2 x\cos^2 x$$

$$\Rightarrow \quad f(x) = 3 - 5\sin^2 x\cos^2 x$$

$$\Rightarrow \quad f(x) = 3 - \frac{5}{4}\sin^2(2x) \le 3 \ \forall\ x \in R$$

$$\because \quad 0 \le \sin^2 2x \le 1 \Rightarrow 0 \le \frac{5}{4}\sin^2 2x \le \frac{5}{4}$$

$$\therefore \quad \text{Range of } f(x) \in \left[\frac{7}{4}, 3\right]$$

Test Your Concepts-III

(Solutions on page H.12)

1. Find the domain of the following functions:

(i) $f(x) = \sqrt{\log_2\left(\frac{x-2}{3-x}\right)}$; (ii) $f(x) = \frac{\sqrt{x+2}}{x^2 - 9}$;

(iii) $f(x) = \sqrt{\log_2\left(\frac{1}{\log_2 x}\right)}$;

(iv) $f(x) = \sqrt{\log_e \frac{1}{|\sin x - 1|}}$

(iv) $f(x) = \sqrt{10 - \sqrt{x^4 - 21x^2}}$

(v) $f(x) = In\left(\sqrt{\log_2 x - 1} + \frac{1}{2}\log_{\frac{1}{2}} x^3 + 2\right)$;

(vi) $f(x) = \frac{1}{\sqrt{\text{lncot}^{-1} x}}$; (vii) $\frac{\log_2(x+3)}{x^2 + 3x + 2}$

(viii) $f(x) = \log_{(x^2 - x + 1)}(2x^2 - 7x + 9)$

2. Find the domain of the following functions.

(i) $\sqrt{\sin(\cos x)}$ (ii) $f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right)$

(iii) $f(x) = \left[9^x + 27^{\frac{2}{3}(x-2)} - 219 - 3^{2(x-1)}\right]^{1/4}$

3. Find the range of the following functions.

(i) $f(x) = \frac{3}{5 + 4\sin 3x}$

(ii) $f(x) = \frac{\tan\left(\pi[x^2 - x]\right)}{1 + \sin(\cos x)}$ where [ ] denotes the greatest integer function.

(iii) $y = \dfrac{x^2 - 6x - 7}{x^2 + 2x - 63}$ (iv) $f(x) = \dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$

(v) $f(x) = \dfrac{x^2 + x + 2}{x^2 + x + 1}$ (vi) $f(x) = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$

(vii) $f(x) = 3 \tan \sqrt{\dfrac{\pi^2}{9} - x^2}$

(viii) $g(x) = \sin\left(x + \dfrac{\pi}{6}\right) + \cos\left(x - \dfrac{\pi}{6}\right)$

4. Find the domain and range of the following function.

(i) $f(x) = e^{\sin x}$ (ii) $f(x) = e^{\cos^2 x}$

(iii) $f(x) = \left(e^{\sin x}\right)^{\sec x}$ (iv) $f(x) = \dfrac{3^{\sin x}}{\left(\sqrt{3}\right)^{2\cos x}}$

(v) $f(x) = e^{\sin x - \sqrt{3}\cos x}$

5. If $f : R \to R$ is defined by $f(x) = \dfrac{\sin[x]\pi + \tan[x]\pi}{1 + [x]^2}$, then the range of f = (where $[x]$ denotes integral part of x).

6. Find the domain and range of $f(x)$:

(i) $y = \ln \sin x$ (ii) $f(x) = \ln \cos x$

(iii) $f(x) = \ln(3 - \sin 4x)$

7. $f(x) = x^3 - 3x^2 + 3x + 6$ if $x \in [0, 1)$, then find the range.

Answer key

1. (i) $x \in \left[\dfrac{5}{2}, 3\right)$; (ii) $[-2, 3) \cup (3, \infty)$; (iii) $\{x : 1 < x \le 2\}$; (iv) $n\pi + (-1)^n \alpha$; where n is any integer and $\alpha \in \left[0, \dfrac{\pi}{2}\right)$

2. (i) $\left[2n\pi - \dfrac{\pi}{2}, 2n\pi + \dfrac{\pi}{2}\right], n \in I$ (ii) $[-2, -1] \cup [1, 2]$ (iii) $\left[\dfrac{5}{2}, \infty\right)$ (iv) $\left[-5, -\sqrt{21}\right] \cup \left[\sqrt{21}, 5\right] \cup \{0\}$

(v) $[2, 4)$; (vi) $(-\infty, \cot 1)$; (vii) $(-3, \infty) - \{-1, -2\}$ (viii) $R - \{0, 1\}$

3. (i) $\left[\dfrac{1}{3}, 3\right]$ (ii) 0 (iii) $R - \left\{\dfrac{1}{2}, 1\right\}$ (iv) $\left[\dfrac{1}{7}, 7\right]$ (v) $\left(1, \dfrac{7}{3}\right]$ (vi) $\left[\dfrac{1}{3}, 3\right]$ (vii) $\left[0, 3\sqrt{3}\right]$

(viii) $\left[\dfrac{-\left(\sqrt{3}+1\right)}{\sqrt{2}}, \dfrac{\sqrt{3}+1}{\sqrt{2}}\right]$

4. (i) $x \in R$; $y \in \left[\dfrac{1}{e}, e\right]$ (ii) $x \in R$ and $y \in [1, e]$ (iii) $x \in R - \left\{(2x+1)\dfrac{\pi}{2}\right\}, x \in I$ and $y \in (0, \infty)$

(iv) $x \in R, y \in \left[\dfrac{1}{3^{\sqrt{2}}}, 3^{\sqrt{2}}\right]$ (v) $x \in R$ and $y \in \left[\dfrac{1}{e^2}, e^2\right]$

5. $\{0\}$

6. (i) $x \in (2n\pi, 2n\pi + \pi)$; where $n \in I$, $y \in (-\infty, 0]$ (ii) $x \in \left(2n\pi - \dfrac{\pi}{2}, 2n\pi + \dfrac{\pi}{2}\right)$; where $n \in$ I, $y \in (-\infty, 0]$

(iii) $x \in R$ and $y \in [\ln 2, \ln 4]$

7. $y \in [6, 7)$

EQUAL/IDENTICAL FUNCTION

$y = f(x)$ and $y = g(x)$ are equal functions if they are defined for same set values of x and having unique and equal images at each x belonging to their domains. In simple words

(i) Domain of $f(x)$ = Domain of $g(x)$

(ii) $f(x) = g(x)\ \forall\ x \in$ domain.

ILLUSTRATION 55

Are they equal functions?

(i) $f(x) = 2 \ln x; g(x) = \ln x^2$ **(ii)** $f(x) = x; g(x) = \dfrac{x^2}{x}$

(iii) $f(x) = \dfrac{1}{x}; g(x) = \dfrac{x}{x^2}$ **(iv)** $f(x) = x; g(x) = \sqrt{x^2}$

(v) $f(x) = \sec^2 x - \tan^2 x, g(x) = \mathrm{cosec}^2 x - \cot^2 x$

(vi) $f(x) = [\{x\}]; g(x) = \{[x]\}$ where [.] is GIF and {.} is fpf

SOLUTION

(i) $f(x) = 2 \ln x; x > 0;$

$g(x) = \ln x^2 \Rightarrow x^2 > 0 \Rightarrow x \ne 0$

So, $f(-2)$ not defind

$g(2) = \ln [-2]^2 = \ln 4$

As both functions are having different domains they are not equal functions.

(ii) $f(x) = x\ \forall\ x \in R$ and $g(x) = \dfrac{x^2}{x} = x\ (x \ne 0)$

Both are not equal functions as the domains are different.

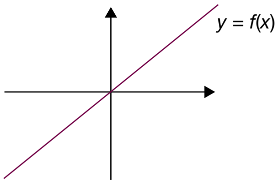
(iii) $f(x) = \frac{1}{x}$ for $x \neq 0$ and $g(x) = \frac{x}{x^2} = \frac{1}{x}$ for $x \neq 0$

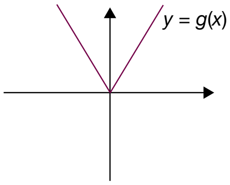
$y = \frac{1}{x} = \frac{1}{x}$ yes, both are equal functions.

(iv) $f(x) = x;\ g(x) = \sqrt{x^2}$

Here, domain are the same for both functions $x \in R$.

$f(x) = x : g(x) = |x|$





(v) $f(x) = \sec^2 x - \tan^2 x$, $g(x) = \text{cosec}^2 x - \cot^2 x$

Domain of $f(x)$, $x \in R - \left\{(2n+1)\frac{\pi}{2}\right\}$ we have $f(x) = 1$

Domain of $g(x)$, $x \in R - \{n\pi\}$ we have $g(x) = 1$

Both functions are not equal functions.

(vi) $f(x) = [\{x\}]$; $g(x) = \{[x]\}$

Domain $x \in R$; Domain $x \in R$

$0 \leq \{x\} < 1$; $[x] \in I$

$f(x) = [\{x\}] = 0$; $g(x) = \{[x]\} = 0$

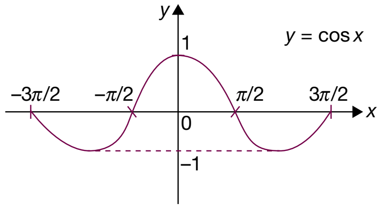
Hence, both functions are equal functions.

TYPES OF FUNCTION

Even Function

If $f(-x) = f(x)$, $\forall\ x \in$ domain, then $f(x)$ is even
Function is symmetric about y-axis

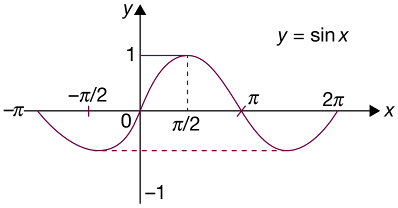
The function x^2, $\cos x$, $\sec x$ etc., are even functions.
In the given diagram we can observe that $f(x) = \cos x$ is an even function and symmetric about the y-axis.



Odd Function

If $f(-x) = -f(x)$, $\forall\ x \in$ domain, then $f(x)$ is called odd function.
Function is symmetric about origin or opposite quadrant.

The function x, $\sin x$, $\text{cosec}\ x$, etc., are odd functions.
In the given diagram we can observe that $f(x) = \sin x$ is an odd function and symmetric about origin.



Even and Odd Function

$f(x) = 0$ is the only function defined in R which is even as well as odd function.

Neither Even Nor Odd Function

If function $f(-x) \neq \pm f(x)$, $\forall\ x \in$ domain, then $f(x)$ is neither even nor odd function.

Even Extension

If a function $f(x)$ is defined on the interval $[0, a]$, $0 \leq x \leq a$

We define $f(x)$ in the $[-a, 0]$ such that $f(x) = f(-x)$.

Let $k(x) = \begin{cases} f(x) & : x \in [0, a] \\ f(-x) & : x \in [-a, 0] \end{cases}$

Odd Extension

If a function $f(x)$ is defined on the interval $[0, a]$, $0 \leq x \leq a$

$\therefore$ $x \in [-a, 0]$, we define $f(x) = -f(-x)$

Let I be the odd extension then

$$k(x) = \begin{cases} f(x), & x \in [0, a] \\ -f(-x), & x \in [-a, 0] \end{cases}$$

(i) Each function can be expressed as the sum or difference of an even and an odd function.

$$f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$$

$= f_1(x) + f_2(x)$ (say)

where $f_1(x)$ is an even function, whereas $f_2(x)$ is an odd function.

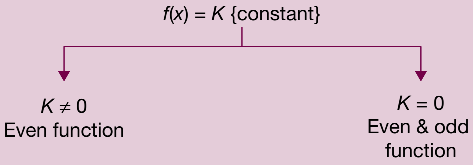
$$f(x) = \frac{f(x)+f(-x)}{2} - \frac{f(-x)-f(x)}{2}$$

$= f_1(x) - f_2(x)$ (say)

where $f_1(x)$ is an even function, whereas $f_2(x)$ is an odd function.

(ii) If $f(x)$ is odd, then $f'(x)$ is even and if $f(x)$ is even, then $f'(x)$ is odd.

(iii)



(iv) If $f(x)$ is a combination of more than one function, then we can use the following table:

Table 2.4 Nature of the composition of functions

F	*G*	$f \pm g$	*fg* or *f*/*g*	*f*(*g*(*x*))	*g*(*f*(*x*))
Odd	Odd	Odd	Even	Odd	Odd
Odd	Even	Neither Even Nor Odd	Odd	Even	Even
Even	Even	Even	Even	Even	Even
Even	Odd	Neither Even Nor Odd	Odd	Even	Even

ILLUSTRATION 56

Identify the following as even or odd function.

(i) $f(x) = \text{cosec}\, x(\sin x + \tan x)$ **(ii)** $f(x) = (7^x + 1)^2/7^x$

(iii) $f(x) = \dfrac{x^5 + x^3 - 1}{x^2}$ **(iv)** $f(x) = \ln\left(\dfrac{3-x}{3+x}\right)$

(v) $f(x) = \dfrac{x}{10^x - 1} + \dfrac{x}{2} + 5$

SOLUTION

(i) $f(x) = \text{cosec}\, x(\sin x + \tan x)$

$f(-x) = \text{cosec}(-x)\ (\sin(-x) + \tan(-x))$

$= -\text{cosec}\, x\ (-\sin x - \tan x)$

$= \text{cosec}\, x\ (\sin x + \tan x) = f(x)$ even function

(ii) $f(x) = (7^x + 1)^2/7^x$

$$f(-x) = \frac{(7^{-x}+1)^2}{7^{-x}} = \frac{\left(\frac{1}{7^x}+1\right)^2}{\frac{1}{7^x}} = \frac{(1+7^x)^2}{7^{2x}} \cdot 7^x$$

$f(-x) = \dfrac{(1+7^x)^2}{7^x} = f(x)$ even function.

(iii) $f(x) = \dfrac{x^5 + x^3 - 1}{x^2}$

$$f(-x) = \frac{-x^5 - x^3 - 1}{x^2}$$

$= -\left(\dfrac{x^5 + x^3 + 1}{x^2}\right) \neq \pm f(x)$ Neither even nor odd

(iv) $f(x) = \ln\left(\dfrac{3-x}{3+x}\right)$

$$f(-x) = \ln\left(\frac{3+x}{3-x}\right) = \ln\left(\frac{3-x}{3+x}\right)^{-1}$$

$$= -\ln\left(\frac{3-x}{3+x}\right)$$

$f(-x) = -f(x)$ odd function.

(v) $f(x) = \dfrac{x}{10^x - 1} + \dfrac{x}{2} + 5$

$$f(-x) = \frac{-x}{10^{-x}-1} - \frac{x}{2} + 5 = \frac{-x.10^x}{1-10^x} - \frac{x}{2} + 5$$

$$= \frac{x(1-10^x-1)}{1-10^x} - \frac{x}{2} + 5$$

$$f(-x) = x\,.-\frac{x}{1-10^x} - \frac{x}{2} + 5 = \frac{x}{10^x-1} + \frac{x}{2} + 5 = f(x)$$

even function

ILLUSTRATION 57

Identify $f(x) = \dfrac{2x^5(\sin x + \tan x)}{2\left[2+\dfrac{x}{\pi}\right]-3}$; where [.] is G.I.F. as odd or even function.

SOLUTION

$$2\left[2+\frac{x}{\pi}\right] - 3 = 2\left(2+\left[\frac{x}{\pi}\right]\right) - 3 = 1 + 2\left[\frac{x}{\pi}\right]$$

$$f(x) = \frac{2x^5(\sin x + \tan x)}{1+2\left[\frac{x}{\pi}\right]}$$

$$f(x) = \begin{cases} 0 & ; \quad \dfrac{x}{\pi} = k \Rightarrow x = k\pi \\ \dfrac{2x^5(\sin x + \tan x)}{1+2\left[\frac{x}{\pi}\right]} & ; \quad \dfrac{x}{\pi} \neq k \Rightarrow x \neq \pm k\pi \end{cases}$$

$$\because \left[\frac{x}{\pi}\right] + \left[\frac{-x}{\pi}\right] = \begin{cases} 0 & ; \quad x = k\pi \\ -1 & ; \quad x \neq k\pi \end{cases}$$

For $x = k\pi, f(-x) = 0$;

For $x \neq k\pi$

$$f(-x) = \frac{-2x^5(-\sin x + tan(-x))}{1+2\left[\frac{-x}{\pi}\right]} = \frac{2x^5(\sin x + \tan x)}{1+2\left(-1-\left[\frac{x}{\pi}\right]\right)}$$

$$f(x) = \frac{-2x^5(\sin x + \tan x)}{1+2\left[\frac{x}{\pi}\right]} = -f(x)$$

$\therefore$ $f(x)$ is an odd function.

ILLUSTRATION 58

Let $f(x) = (a+2)x^3 + a^2 + 3a + 2$ find '*a*' if $f(x)$ is:

(i) Even only **(ii)** Odd only **(iii)** Even and Odd both

SOLUTION

(i) Even only

$f(-x) = f(x)$

$\Rightarrow -(a+2)x^3 + a^2 + 3a + 2 = (a+2)x^3 + a^2 + 3a + 2$

$\Rightarrow\ 2(a+2)x^3 = 0$

$\Rightarrow\ a = -2$

$f(x) = 0$ it is even and odd both for $a = -2$

for $f(x)$ is even only, $a \in \phi$,

(ii) Odd function only

$$f(-x) = -f(x)$$

$\Rightarrow\ -(a+2)x^3 + a^2 + 3a + 2 = -(a+2)x^3 - (a^2 + 3a + 2)$

$\Rightarrow\ 2(a^2 + 3a + 2) = 0$

$\Rightarrow\ 2(a+1)(a+2) = 0$

$\Rightarrow\ a = -1, -2$

For $a = -2$, $f(x) = 0$ (even and odd function)

For $a = -1$, $f(x) = x^3$

Odd function only, for $a = -1$

(iii) For $a = -2$, $f(x) = 0$ then $f(x)$ is odd and even function both.

ILLUSTRATION 59

Find out whether the given function is even, odd or neither even nor odd.

$$f(x) = \begin{cases} x\,|x| & ; \quad x \le -1 \\ [1+x] + [1-x] & ; \quad -1 < x < 1 \\ -x^2 & ; \quad x \ge 1 \end{cases}$$

where [.] represents greatest integer function.

SOLUTION

{using definition of modulus and greatest integer function} The given function can be written as:

$$f(x) = \begin{cases} -x^2 & ; \quad x \le -1 \\ 2 + [x] + [-x] & ; \quad -1 < x < 1 \\ -x^2 & ; \quad x \ge 1 \end{cases}$$

$$\Rightarrow \quad f(x) = \begin{cases} -x^2 & ; \quad x \le -1 \\ 1 & ; \quad -1 < x < 0 \\ 2 & ; \quad x = 0 \\ 1 & ; \quad 0 < x < 1 \\ -x^2 & ; \quad x \ge 1 \end{cases}$$

which is clearly even as $f(-x) = f(x)$. Thus, $f(x)$ is even.

ILLUSTRATION 60

Let $f(x) = x^2 + x$ be defined on the interval [0, 2]. Find the odd and even extensions of $f(x)$ in the interval [–2, 2].

SOLUTION

Odd extension.

$$f(x) = \begin{cases} f(x) & ; \quad 0 \le x \le 2 \\ -f(-x) & ; \quad -2 \le x < 0 \end{cases}$$

$$= \begin{cases} x^2 + x & ; \quad 0 \le x \le 2 \\ -x^2 + x & ; \quad -2 \le x < 0 \end{cases}$$

Even extension

$$f(x) = \begin{cases} f(x) & ; \quad 0 \le x \le 2 \\ f(-x) & ; \quad -2 \le x < 0 \end{cases}$$

$$= \begin{cases} x^2 + x & ; \quad 0 \le x \le 2 \\ x^2 - x & ; \quad -2 \le x < 0 \end{cases}$$

ILLUSTRATION 61

If $h(x) = Ax^5 + B\sin x + C\ln\left(\dfrac{1+x}{1-x}\right) + 7$, where A, B, C are non-zero real constant and $h\left(-\dfrac{1}{2}\right) = 6$, then find the value of $h\left(\dfrac{\operatorname{sgn}(e^{-x})}{2}\right)$.

SOLUTION

Given $h\left(-\dfrac{1}{2}\right) = 6$, $\qquad (\because\ e^{-x} > 0 \Rightarrow \operatorname{sgn}(e^{-x}) = 1)$

$$h\left(\frac{\operatorname{sgn}(e^{-x})}{2}\right) = h\left(\frac{1}{2}\right)$$

$$h(x) = Ax^5 + B\sin x + C\ln\left(\frac{1+x}{1-x}\right) + 7$$

$$\Rightarrow \quad h(-x) = -Ax^5 - B\sin x + C\ln\left(\frac{1-x}{1+x}\right) + 7$$

$$\therefore \quad h(x) + h(-x) = 14 \Rightarrow h\left(\frac{1}{2}\right) + h\left(-\frac{1}{2}\right) = 14$$

$$\therefore \quad h\left(\frac{1}{2}\right) + 6 = 14 \Rightarrow h\left(\frac{1}{2}\right) = 8$$

ILLUSTRATION 62

Find the smallest natural number k for which

$f(x) = \ln\left(x^3 + \sqrt{x^6+1}\right) + \sin 5x + \left[\dfrac{x^2}{k}\right]$ is an odd function

$\forall\ x \in [-2\pi, 2\pi]$, is ([$y$] denotes largest integer $\le y$).

SOLUTION

$$f(x) = \ln\left(x^3 + \sqrt{x^6+1}\right) + \sin 5x + \left[\frac{x^2}{k}\right]$$

Since, $f(x)$ is an odd function and $\ln(x^3 + \sqrt{x^6+1}) + \sin 5x$ is an odd function.

$$\Rightarrow \quad \left[\frac{x^2}{k}\right] = 0\ \forall\ x \in [-2\pi, 2\pi]$$

$$\Rightarrow \quad 0 \le \frac{x^2}{k} < 1\ \forall\ x \in [-2\pi, 2\pi]$$

Hence $k > 4\pi^2 = 39.44$

$\therefore$ The smallest natural number $k = 40$.

ILLUSTRATION 63

For $n \in N$, show that the function

$$g(x) = \begin{cases} \dfrac{x^{4n}}{(x^{4n}\operatorname{sgn} x)^{4n+1}}\left(\dfrac{e^{\frac{\pi}{x}} - e^{\frac{-\pi}{x}}}{e^{\frac{\pi}{x}} + e^{\frac{-\pi}{x}}}\right) & x \neq 0 \\ 10, & x = 0 \end{cases}$$

is an even function

SOLUTION

$$g(-x) = \frac{(-x)^{4n}\left(e^{-\frac{\pi}{x}} - e^{\frac{\pi}{x}}\right)}{\left((-x)^{4n}\operatorname{sgn}(-x)\right)^{4n+1}\left(e^{-\pi/x} + e^{\pi/x}\right)}$$

$$= \frac{-x^{4n}\left(e^{\pi/x} - e^{-\pi/x}\right)}{-\left(x^{4n}\operatorname{sgn}(x)\right)^{4n+1}\left(e^{\pi/x} + e^{-\pi/x}\right)}$$

$$= \frac{x^{4n}\left(e^{\pi/x} - e^{-\pi/x}\right)}{\left(x^{4n}\operatorname{sgn}(x)\right)^{4n+1}\left(e^{\pi/x} + e^{-\pi/x}\right)}$$

As $g(-x) = g(x)\ \forall\ x \in R$

$\Rightarrow$ $g(x)$ is an even function

ILLUSTRATION 64

Let $f(x) = \left[\begin{array}{lll} 0 & ; & \text{if } x \text{ is rational} \\ x & ; & \text{if } x \text{ is irrational} \end{array}\right.$ and

$$g(x) = \left[\begin{array}{lll} 0 & ; & \text{if } x \text{ is irrational} \\ x & ; & \text{if } x \text{ is rational} \end{array}\right. .$$

Identify function $(f - g)x$ as odd or even.

SOLUTION

$$f(x) - g(x) = F(x) = \left[\begin{array}{lll} -x & ; & \text{if } x \text{ is rational} \\ x & ; & \text{if } x \text{ is irrational} \end{array}\right.$$

$$F(-x) = \left[\begin{array}{lll} -(-x) & ; & \text{if } -x \text{ is rational} \Rightarrow x \text{ is rational} \\ -x & ; & \text{if } -x \text{ is irrational} \Rightarrow x \text{ is irrational} \end{array}\right.$$

$$F(-x) = \left[\begin{array}{lll} x & ; & \text{if } x \in Q \\ -x & ; & \text{if } x \notin Q \end{array}\right.$$

$\therefore$ $F(x) + F(-x) = 0$

$f(x) - g(x)$ is an odd function

ILLUSTRATION 65

$$\text{Let } f(x) = \left[\begin{array}{lll} x^2 - 3 & ; & x \le -5 \\ x + \lambda & ; & -5 < x < -1 \\ (\mu - 7)(|1 - x| + |1 + x|) & ; & -1 \le x \le 1 \\ x + 6 & ; & 1 < x < 5 \\ 3 - x^2 & ; & x \ge 5 \end{array}\right.$$

If $f(x)$ is an odd function, then find the value of $(\lambda + \mu)$.

SOLUTION

$$f(x) = \left[\begin{array}{lll} x^2 - 3 & ; & x \le -5 \\ x + \lambda & ; & -5 < x < -1 \\ (\mu - 7)(|1 - x| + |1 + x|); & & -1 \le x \le 1 \\ x + 6 & ; & 1 < x < 5 \\ 3 - x^2 & ; & x \ge 5 \end{array}\right.$$

$$\therefore\ f(-x) = \left[\begin{array}{lll} x^2 - 3 & ; & -x \le -5 \Rightarrow x \ge 5 \\ -x + \lambda & ; & -5 < -x < -1 \Rightarrow 1 < x < 5 \\ (\mu - 7)(|1 + x| + |1 - x|); & & -1 \le -x \le 1 \Rightarrow -1 \le x \le 1 \\ -x + 6 & ; & 1 < -x < 5 \Rightarrow -5 < x < -1 \\ 3 - x^2 & ; & -x \ge 5 \Rightarrow x \le -5 \end{array}\right.$$

$$\Rightarrow\quad f(x) + f(-x) = \left[\begin{array}{lll} 0 & ; & x \le -5 \\ \lambda + 6 & ; & -5 < x < -1 \\ 2(\mu - 7)(|1 - x| + |1 + x|); & & -1 \le x \le 1 \\ \lambda + 6 & ; & 1 < x < 5 \\ 0 & ; & x \ge 5 \end{array}\right.$$

$\therefore$ For $f(x)$ to be an odd function,

$f(x) + f(-x) = 0\ \forall\ x \in R$

$\therefore$ $\lambda = -6$ and $\mu = 7$

$\Rightarrow$ $(\lambda + \mu) = 1.$

ILLUSTRATION 66

If the function f satisfies the relation $f(x + y) + f(x - y) = 2f(x) \cdot f(y)$ for all $x, y \in R$ and $f(0) \neq 0$, prove that $f(x)$ is an even function.

SOLUTION

Given $f(x + y) + f(x - y) = 2f(x) \cdot f(y)$ ...(1)

Replacing x by y and y by x in (1), we have:

$f(y + x) + f(y - x) = 2f(y) \cdot f(x)$

$\Rightarrow$ $f(x + y) + f(y - x) = 2f(x) \cdot f(y)$ ...(2)

From (1) and (2), we have $f(x - y) = f(y - x)$.

Now, putting $y = 2x$ we get $f(-x) = f(x)$.

Hence, $f(x)$ is an even function.

PERIODIC FUNCTION

A function is said to be periodic function if its each value is repeated after a definite interval. So a function $f(x)$ will be periodic if a positive real number T exist such that, $f(x + T) = f(x),\ \forall\ x \in$ Domain

Here the least positive value of T is called fundamental period of the function.

A function which is non-periodic, also known as aperiodic.

EXAMPLE:

sin x is a periodic function with period 2π, 4π, 6π, or $10000\pi\ldots$, where fundamental period is 2π.

$\{x\}$, cos x, tan x are periodic functions with fundamental period 1, 2π and π, respectively, where $\{\cdot\}$ is fractional part function.

Table 2.5 Functions and their period

$f(x)$	Period
$\sin^n x, \cos^n x, \operatorname{cosec}^n x, \sec^n x$	π if n is even, 2π if n is odd.
$\tan^n x, \cot^n x$	π for all values of n.
$\lvert\sin x\rvert^n, \lvert\cos x\rvert^n, \lvert\tan x\rvert^n, \lvert\cot x\rvert^n$	π for all values of n.
$f(x) = c$	periodic with no fundamental period.

Conceptual Note(s)

(i) If function $f(x)$ has fundamental period as T then multiple of fundamental period are also period of $f(x)$
For $f(x) = \tan x$ periods are $\pi, 2\pi, 3\pi, 4\pi, 5\pi, \ldots, 400\pi$

(ii) If function $f(x)$ has period T then $f(ax + b)$ has period $\dfrac{T}{|a|}$

(iii) If $f(x)$ is a periodic function with T then $f(-x)$, $|f(x)|$, $f(x \pm a)$, $f(x) \pm a$, $\dfrac{1}{f(x)}, \sqrt{f(x)}$ is also periodic function with period T.

(iv) If the period of $f(x)$ and $g(x)$ are same (T) then the period of $af(x) + bg(x)$ will also be T.

(v) Periodic function $f(x) = f_1(x) + \lambda f_2(x) + \mu f_3(x)$ where function $f_1(x)$, $f_2(x)$, $f_3(x)$ are having time period as T_1, T_2 and T_3, respectively.
Time period of $f(x)$ = LCM of $\{T_1, T_2, T_3\}$ (May or may not be fundamental period).

Note

(i) LCM of rational and irrational does not exist.

(ii) if a, b, c, l, m, n are natural numbers then LCM of $\left\{\dfrac{a}{l}, \dfrac{b}{m}, \dfrac{c}{n}\right\} = \dfrac{\text{LCM of } \{a, b, c\}}{\text{HCF of } \{l, m, n\}}$

ILLUSTRATION 67

Find fundamental period of the following functions if they are periodic:

(i) $f(x) = 3 \sin 2x$ **(ii)** $f(x) = 4 \tan\left(\dfrac{7x}{5} - 6\right)$

(iii) $f(x) = \sqrt{\dfrac{5\cos(7x-1)+3}{6}}$

(iv) $f(x) = \tan x + \tan 2x + \tan 3x$

(v) $f(x) = \sin 3x + \cos 4x$

SOLUTION

(i) $f(x) = 3 \sin 2x$

Period of $\sin x = 2\pi$

Period of $\sin 2x = \dfrac{2\pi}{2} = \pi$

$f(\pi + x) = 3 \sin (2\pi + 2x) = 3 \sin 2x$

$\therefore$ Fundamental period of $f(x) = \pi$

(ii) $f(x) = 4 \tan\left(\dfrac{7x}{5} - 6\right)$

Period of $\tan x = \pi$

Period of $f(x) = \dfrac{\pi}{7/5} = \dfrac{5\pi}{7}$

(iii) $f(x) = \sqrt{\dfrac{5\cos(7x-1)+3}{6}}$

Period of $\cos x = 2\pi$

Period of $f(x) = \dfrac{2\pi}{7}$

(iv) $f(x) = \tan x + \tan 2x + \tan 3x$

Period of sub-function present in $f(x)$

$\tan x$, $\tan 2x$, $\tan 5x$ are $\pi, \dfrac{\pi}{2}, \dfrac{\pi}{5}$ respectively.

Period of $f(x)$ = LCM of $\left\{\pi, \dfrac{\pi}{2}, \dfrac{\pi}{5}\right\}$

$= \dfrac{\text{LCM of } \{\pi, \pi, \pi\}}{\text{HCF } \{1, 2, 5\}} = \dfrac{\pi}{1} = \pi$

Fundamental period of $f(x) = \pi$

(v) $f(x) = \sin 3x + \cos 4x$

Period of $\sin 3x = \dfrac{2\pi}{3}$

Period of $\cos 4x = \dfrac{2\pi}{4} = \dfrac{\pi}{2}$

Period of $f(x)$ = LCM of $\left\{\dfrac{2\pi}{3}, \dfrac{\pi}{2}\right\}$

$= \dfrac{\text{LCM } \{2\pi, \pi\}}{\text{HCF } \{3, 2\}} = \dfrac{2\pi}{1} = 2\pi$

Checking for smaller period

$f(\pi + x) = \sin(3\pi + 3x) + \cos(4\pi + 4x) = -\sin 3x + \cos 4x \neq f(x)$

Period $\neq \pi$

Hence, 2π is fundamental period of $f(x)$.

ILLUSTRATION 68

Find fundamental period of the following function if they are periodic:

(i) $f(x) = \sin 2x + \cos 3x - \tan 2x + \left\{\frac{3x}{\pi}\right\}$ where {.} is fpf

(ii) $f(x) = |\sin x| + |\cos x|$

(iii) $f(x) = \cos^2 x + \sin^2 x$

(iv) $f(x) = \cos x \cdot \cos 3x$

SOLUTION

(i) $f(x) = \sin 2x + \cos 3x - \tan 2x + \left\{\frac{3x}{\pi}\right\}$

Period of sub-function present in $f(x)$

$\sin 2x$, $\cos 3x$, $\tan 2x$, $\left\{\frac{3x}{\pi}\right\}$ are as π, $\frac{2\pi}{3}$, $\frac{\pi}{2}$ and $\frac{\pi}{3}$ respectively.

Period of $f(x) = \text{LCM of } \left\{\pi, \frac{2\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}\right\}$

$$= \frac{\text{LCM of } \{\pi, 2\pi, \pi, \pi\}}{\text{HCF of } \{1, 3, 2, 3\}} = \frac{2\pi}{1} = 2\pi$$

Thus, period of $f(x) = 2\pi$

(ii) $f(x) = |\sin x| + |\cos x|$

$|\sin x|$ and $|\cos x|$ both has period π.

Checking for fundamental period.

$$f\left(\frac{\pi}{2}+x\right) = \left|\sin\left(\frac{\pi}{2}+x\right)\right| + \left|\cos\left(\frac{\pi}{2}+x\right)\right|$$

$$(\because |-x| = |x|)$$

$$= |\cos x| + |-\sin x| = |\sin x| + |\cos x| = f(x)$$

Thus, period of $f(x) = \frac{\pi}{2}$

(iii) $f(x) = \cos^2 x + \sin^2 x$

$f(x) = 1$

$f(x + T) = 1 = f(x)\ T > 0$

Thus, period of period $= \frac{\pi}{2}, \pi, \sqrt{2}, \sqrt{3}, 7, 100020$

$f(x)$ is constant periodic function with no fundamental period.

(iv) $f(x) = \cos x \cdot \cos 3x$

$\cos x$ and $\cos 3x$ has period 2π and $\frac{2\pi}{3}$ respectively.

Period of $f(x) = \text{LCM of } 2\pi \text{ and } \frac{2\pi}{3} = 2\pi$

Checking for fundamental period.

$$f(\pi + x) = \cos(\pi + x)\cos(3\pi + 3x) = (-\cos x)(-\cos 3x)$$

$$f(\pi + x) = \cos x \cos 3x = f(x)$$

Thus, period of $f(x) = \pi$

ILLUSTRATION 69

Find fundamental period of the following functions if they are periodic:

(i) $f(x) = \cos(\sin x) + \cos(\cos x)$ **(ii)** $f(x) = x + \sin x$

(iii) $f(x) = \cos(x + \sin x)$ **(iv)** $f(x) = \cos^2 x + \sin^4 x$

SOLUTION

(i) $f(x) = \cos(\sin x) + \cos(\cos x)$

Clearly, we can see that 2π is a period of given function:

$$f(\pi + x) = \cos(\sin(\pi + x)) + \cos(\cos(\pi + x))$$

$$= \cos(-\sin x) + \cos(-\cos x) = \cos(\sin x) + \cos(\cos x)$$

We have received smaller period π.

$$f\left(\frac{\pi}{2}+x\right) = \cos\left(\sin\left(\frac{\pi}{2}+x\right)\right) + \cos\left(\cos\left(\frac{\pi}{2}+x\right)\right)$$

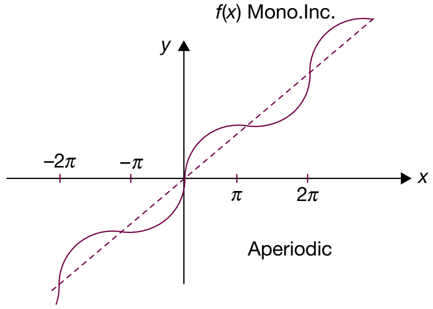
$$= \cos(\cos x) + \cos(-\sin x) = \cos(\cos x) + \cos(\sin x) = f(x)$$

Thus, fundamental period of $f(x)$ is $\frac{\pi}{2}$

(ii) $f(x) = x + \sin x$

$f'(x) = 1 + \cos x \geq 0$

Clearly, $f(x)$ is increasing so it cannot be periodic.



(iii) $f(x) = \cos(x + \sin x)$

$$f(2\pi + x) = \cos(2\pi + x + \sin(2\pi + x))$$

$$= \cos(2\pi + x + \sin x) = \cos(x + \sin x)$$

Thus, periodic of $f(x) = 2\pi$

(iv) $f(x) = \cos^2 x + \sin^4 x$

$\cos x$ and $\sin x$ both has period 2π,

Clearly seen that 2π is the period of $f(x)$.

Let check for the period $= \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}+x\right) = \cos^2\left(\frac{\pi}{2}+x\right) + \sin^4\left(\frac{\pi}{2}+x\right)$$

$$= (-\sin x)^2 + (\cos x)^4 = \sin^2 x + \cos^4 x$$

$$= \sin^2 x + (1 - \sin^2 x)^2 = \sin^2 x + 1 + \sin^4 x - 2\sin^2 x$$

$$= \sin^4 x + 1 - \sin^2 x = \sin^4 x + \cos^2 x = f(x)$$

The fundamental period of $f(x) = \pi/2$.

ILLUSTRATION 70

Find the fundamental period of $f(x)=\dfrac{\sin x+\sin 3x}{\cos x+\cos 3x}$, Also plot it.

SOLUTION

$$f(x)=\frac{\sin x+\sin 3x}{\cos x+\cos 3x}=\frac{2\sin\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}{2\cos\left(\frac{x+3x}{2}\right)\cos\left(\frac{x-3x}{2}\right)}$$

$$=\frac{\sin 2x\cos x}{\cos 2x\cos x}=\tan 2x;\ (\cos x\neq 0,\ x\neq(2n+1)\pi/2)$$

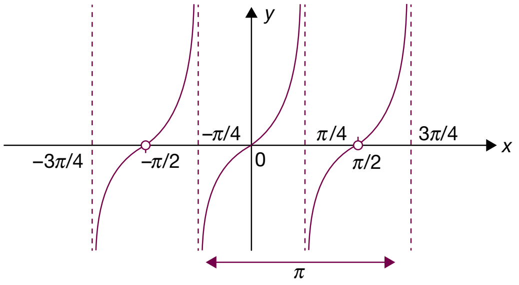
$T=\dfrac{\pi}{2}$ seems to be period of $f(x)$ let cross-check it

$$f\left(\frac{\pi}{2}+x\right)=\frac{\sin\left(\pi/2+x\right)+\sin\left(\frac{3\pi}{2}+3x\right)}{\cos\left(\pi/2+x\right)+\cos\left(\frac{3\pi}{2}+3x\right)}$$

$$f\left(\frac{\pi}{2}+x\right)=\frac{\cos x-\cos 3x}{-\sin x+\sin 3x}\neq f(x)$$

Clearly, $\dfrac{\pi}{2}$ is not period of $f(x)$

Let draw $f(x)=\tan 2x$, $x\neq(2n+1)\pi/2$



From graph, it is clearly visible that $f(x)$ has fundamental period as π not $\pi/2$.

ILLUSTRATION 71

$f(x)=\{x\}+\tan x$; where $\{\cdot\}$ is fractional part function.

SOLUTION

Period of $\{x\}$ is 1 and $\tan x$ as π

LCM of $\{1,\pi\}\to$ not defined

$\because$ LCM of (rational and irrational) = Does not exist

$f(x)=\{x\}+\tan x$ is aperiodic

ILLUSTRATION 72

Find the period of $f(x)=\sin 3x+\cos \pi x$.

SOLUTION

$\sin x$ has period $2\pi/3$ and $\cos \pi x$ has period 2

$$\text{Period of } f(x)=\text{LCM}\left\{\frac{2\pi}{3},2\right\}$$

$$\left(\because \frac{2\pi}{3}\text{ is irrational and 2 is rational}\right).$$

As LCM of irrational and rational not possible.

Hence, $f(x)$ is non periodic.

ILLUSTRATION 73

Find the fundamental period of the following functions, where {.} is fractional part function and [.] is greatest integer function:

(i) $f(x)=\sin\{x\}$ **(ii)** $f(x)=\{\sin x\}$

(iii) $f(x)=\sin[x]$ **(iv)** $f(x)=\sin[x]$

(v) $f(x)=\sin 12\pi\{x\}$ **(vi)** $f(x)=\sin 11\pi\{x\}$

SOLUTION

(i) $f(x)=\sin\{x\}$

Fundamental period is 1

$f(x+1)=\sin\{x+1\}=\sin\{x\}$

(ii) $f(x)=\{\sin x\}$

$f(2\pi+x)=\{\sin(2\pi+x)\}=\{\sin x\}$

Fundamental period is 2π

(iii) $f(x)=\sin[x]$

It is non-periodic function

(iv) $f(x)=[\sin x]$

$f(x+2\pi)=[\sin(2\pi+x)]=[\sin x]$

(v) $f(x)=\sin 12\pi\{x\}$

$f(x+1)=\sin 12\pi\{x+1\}=\sin 12\pi\{x\}$

Clearly $f(x)$ has period 1

$f(x)=\sin 12\pi\{x\}=\sin 12\pi\left(x-[x]\right)$

$=-\sin\left(12\pi[x]-12\pi x\right)$

$\because$ $[x]$ is greatest integer function $12\pi[x]$ is multiple of 2π.

Using $\sin(2n\pi-\theta)=-\sin\theta\}$

$f(x)=\sin 12\pi x$ which has a period $\dfrac{2\pi}{12\pi}=\dfrac{1}{6}$

$\sin 12\pi\{x\}$ has fundamental time period 1/6

(vi) $f(x)=\sin 11\pi\{x\}$ where {.} is fractional part function

$f(x+1)=\sin 11\pi\{x+1\}=\sin 11\pi\{x\}$

Clearly, $f(x)$ has period as 1

$f(x)=\sin 11\pi(x-[x])=-\sin(11\pi[x]-11\pi x)$ can give two values depending upon $[x]$ is odd or even

So fundamental period of $\sin 11\pi\{x\}$ is 1 only

ILLUSTRATION 74

If $f(x+1)+f(x-1)=f(x)$. Prove that $f(x)$ is periodic function.

SOLUTION

$$f(x+1)+f(x-1)=f(x)\qquad\ldots(1)$$

Replace $x\to x+1$

$$f(x+1+1)+f(x+1-1)=f(x+1)$$

$$\Rightarrow\quad f(x+2)+f(x)=f(x+1)\qquad\ldots(2)$$

Add (1) + (2), we get

$$f(x+1)+f(x-1)+f(x+2)+f(x)=f(x)+f(x+1)$$

$$f(x+2)=-f(x-1)$$

Replace $x \to x+1$

$$f(x+1+2)=-f(x+1-1)=-f(x)$$

$$\Rightarrow \quad f(x+3)=-f(x)$$

Replace $x \to x+3$

$$f(x+3+3)=-f(x+3)=-(-f(x))$$

$$\Rightarrow \quad f(x+6)=f(x)$$

$$\Rightarrow \quad T=6$$

$\therefore$ $f(x)$ is periodic function with period 6.

ILLUSTRATION 75

If the function $f(x)$ satisfies the equation $f(x+1)+f(x-1)=\sqrt{3}\ f(x)$ for all $x \in R$,

(i) Show that $f(x)$ is a periodic function and find its period.

(ii) If $f(2)=9$, find the sum $\sum_{r=0}^{9} f(2+12r)$.

SOLUTION

(i) We have,

$$f(x+1)+f(x-1)=\sqrt{3}\ f(x), \text{ for all } x \in R \qquad \ldots(1)$$

Replacing x by $x-1$ and $x+1$ successively in (1), we get

$$f(x)+f(x-2)=\sqrt{3}\ f(x-1) \qquad \ldots(2)$$

and $f(x+2)+f(x)=\sqrt{3}\ f(x+1) \qquad \ldots(3)$

Adding (2) and (3), we get

$$2f(x)+f(x-2)+f(x+2)=\sqrt{3}\ \{f(x-1)+f(x+1)\}$$

$$=\sqrt{3}\cdot\sqrt{3}\ f(x);\ \text{[from (1)]}$$

$$=3f(x)$$

$$\Rightarrow f(x+2)+f(x-2)=f(x) \qquad \ldots(4)$$

Replacing x by $x+2$ in (4), we get

$$f(x+4)+f(x)=f(x+2) \qquad \ldots(5)$$

Adding (4) and (5), we get

$$f(x+4)+f(x-2)=0 \qquad \ldots(6)$$

$$f(x+4)=-f(x-2)$$

replace x by $x+2$

$$f(x+6)=-f(x)$$

replacing x by $x+6$

$$f(x+12)=-f(x+6)=f(x)$$

we get $f(x+12)=f(x)$.

Hence $f(x)$ is a periodic function with period 12

(ii) $\sum_{r=0}^{9} f(2+12r)=$

$$f(2)+f(2+12)+f(2+24)+\ldots+f(2+12\times 9)$$

$=f(2)+f(2)+f(2)+\ldots$ to 10 terms, since $f(x)$ is periodic with period 12.

$$=10\times f(2)=10\times 9=90.$$

Test Your Concepts-IV

(Solutions on page H.14)

1. Which of the following pair(s) of functions are identical, mark as yes/no?

(i) $f(x)=\cos(2\tan^{-1}x)$, $g(x)=\dfrac{1-x^2}{1+x^2}$

(ii) $f(x)=\dfrac{2x}{1+x^2}$, $g(x)=\sin(2\cot^{-1}x)$

(iii) $f(x)=\sin^{-1}(\sin x)$, $g(x)=\cos^{-1}(\cos x)$

(iv) $f(x)=e^{\ln(\operatorname{sgn}\cot^{-1}x)}$ $g(_x)=e^{\ln[1+\{x\}]}$

Where sgn(.), [.], {.} denotes signum, the greatest integer and fractional part functions, respectively)

2. Find which of the following function are even, odd or neither odd nor even.

(i) $f(x)=\log\left(x+\sqrt{1+x^2}\right)$ (ii) $f(x)=\left(\dfrac{a^x+1}{a^x-1}\right)$

(iii) $f(x)=\dfrac{\cos x}{\left[\dfrac{x}{\pi}\right]+\dfrac{1}{2}}; x\neq\pi$, where [.] is GIF

(iv) $f(x)=\sec^3\left(\log\left(x+\sqrt{x^2+1}\right)\right)$ is

3. Find the period of the following function:

(i) $f(x)=\operatorname{cosec}^2 3x+\cot 4x$

(ii) $f(x)=\dfrac{1}{2}\left(\dfrac{|\sin x|}{\cos x}+\dfrac{\sin x}{|\cos x|}\right)$

(iii) $f(x)=\tan\dfrac{\pi}{2}[x]$, where [.] denotes greatest integer function

(iv) $f(x)=|\cos \pi x|+|\cos 2\pi x|+\ldots+|\cos n\pi x|+e^{x-[x]}$ where [.] is greatest integer function.

(v) $f(x)=\operatorname{sgn}([x]+[-x])$ where [.] denotes greatest integer function

(vi) $f(x)=\exp(x-[x]+\sqrt{x-[x]}+(x-[x])^2+|\sin\pi x|+|\cos\pi x|+|\tan\pi x|)$ where [.] is GIF

(vii) $f(x)=\sin 3x\ \cos[3x]-\cos\ 3x\ \sin[3x]$, where $[\cdot]$ denotes the greatest integer function

4. A real valued function f satisfies $f(10+x)=f(10-x)$ and $f(20-x)=-f(20+x)$, for all $x \in R$ show that f is an odd function as well as periodic function.

5. Let $G(x) = \left(\frac{1}{a^x-1}+\frac{1}{2}\right) F(x)$, where '$a$' is a positive real number not equal to 1 and $F(x)$ is an odd function, Identify $G(x)$ as odd or even.

6. Find the value of $n \in I$ for which the function $f(x) = \frac{\sin nx}{\sin(x/n)}$ has 4π as its period.

7. Find the period of the function $f(x) = [x] + [2x] + [3x] + [4x] + \ldots + [nx] - \frac{n(n+1)x}{2}$, where $n \in N$ and [ ] denotes the greatest integer function.

8. If $f(2 + x) = a + [1 - f(x) - a)^4]^{1/4}$ for all $x \in R$, then find period of $f(x)$.

9. Let g be a function defined by $g(x) = \sqrt{\{x\}} + \sqrt{1-\{x\}}$, where {.} denotes fractional part function, then g is odd/even?

10. If the period of $f(x)=\frac{1}{2}\cos(\sin x)+\cos(\cos x)$ is $\frac{\pi}{2k}$, then find the value of k.

11. If $f(x) = \cos x + \cos ax$ is periodic function, then a must be a rational number (T/F).

Answer key

1. (i) Yes (ii) Yes (iii) No (iv) Yes
2. (i) Odd function (ii) Odd function (iii) Odd function (iv) Even function
3. (i) π (ii) 2π (iii) 2 (iv) 1 (v) 1 (vi) 1 (vii) 1/3
5. Even **6.** 2 **7.** 1 **8.** 4 **9.** Even **10.** 0.5 **11.** True

CLASSIFICATION OF FUNCTION

One-One Function (Injective Function)

A function $f: A \to B$ is said to be one-one if distinct elements from set A have distinct images in set B.

Many-One Function

A function f: $A \to B$ is called many-one, if atleast two or more distinct elements of A have the same image in B.

Conceptual Note(s)

Identify One-one/Many-one

(i) $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ only then one-one function.
$f(x_1) = f(x_2) \Rightarrow x_1 \neq x_2$ then many-one function.

(ii) By observation or mathematically if we get two distinct x having the same value of $f(x)$, then it is many-one.

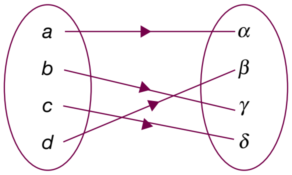
(iii) **Graphical Approach:** If we can draw any line parallel to x-axis which cuts the graph at more than one point then, the mapping is many-one otherwise one-one.

Onto Function (Surjective Function)

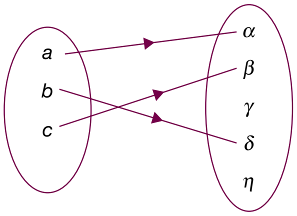
A function $f: A \to B$ is onto if the each element of B has its pre-image in A.
In simple words, range of f = co-domain of f

Into Function

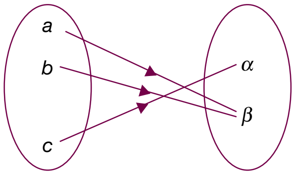
A function $f : A \to B$ is into if there exist atleast one element in B which has no pre-image in set A.
In other words, range of $f \subset$ Co-domain of f



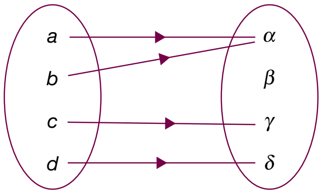
One-one and onto
Bijective mapping



One-one and Into



Many-one and onto



Many-one-into

Note

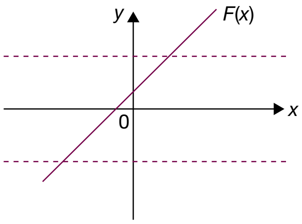
Identify Onto/Into function
Find the range of the function if it is equal to co-domain then the function is onto otherwise into.

ILLUSTRATION 76

Classify as one-one or many-one.

(i) $F(x) = 4x + 15$

(ii) $F(x) = 7x^2 + 4x + 10$



SOLUTION

(i) $F(x) = 4x + 15$

$F(x_1) = F(x_2)$

$\Rightarrow 4x_1 + 15 = 4x_2 + 15$

$\Rightarrow 4x_1 = 4x_2$

$\Rightarrow x_1 = x_2$ one-one

Graphical approach:
We are not able to draw a horizontal line which is intersecting given graph more than one point hence, it is one-one function.

(ii) $F(x) = 7x^2 + 4x + 10$

$F(x_1) = F(x_2)$

$\Rightarrow 7x_1^2 + 4x_1 + 10 = 7x_2^2 + 4x_2 + 10$

$\Rightarrow 7(x_1^2 - x_2^2) + 4(x_1 - x_2) = 0$

$\Rightarrow (x_1 - x_2)\,[7(x_1 + x_2) + 4] = 0$

$x_1 = x_2$ or $7\,(x_1 + x_2) + 4 = 0$

Many-one function
By observation, we can $F(0) = 10 = F(-4/7)$
But $0 \neq -4/7$, hence it is many-one

Graphical approach:
We are able to draw a horizontal line and it is intersecting given graph more than one point hence, it is many-one function.

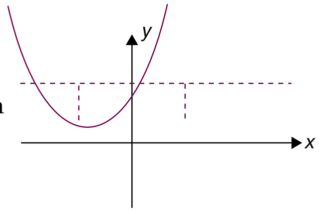


ILLUSTRATION 77

Categorise following defined function $f(x) = \sin x$, as one-one/many-one or onto/into. If domain and co-domain of function f are as follows:

(i) $f : R \to R$

(ii) $f : R \to [-1, 1]$

(iii) $f : [-\pi/2, \pi/2] \to R$

(iv) $f : \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \to [-1, 1]$

(v) $f : \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \to [-1, 1]$

SOLUTION

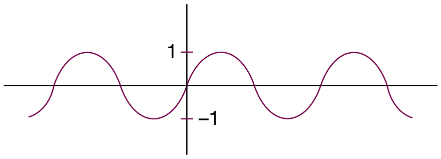
In this illustration function is same but its domain and co-domain are changed, due to this classification may change.

(i) $f : R \to R$

$f(x) = \sin x$

Range $\in [-1, 1] \subset R$

Into function



Thus, $f(0) = f(\pi) = 0$ Many-one.

$f(x)$ is many-one and into function.

(ii) $f : R \to [-1, 1]$

$f(x) = \sin x$

Range $= [-1, 1] =$ Co-domain so it is an onto function

$f(\pi) = f(2\pi) = 0$ Many-one

$f(x)$ is many-one and onto function

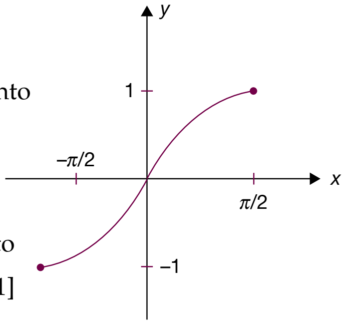
(iii) $f : [-\pi/2, \pi/2] \to R$

$f(x) = \sin x$

Range $\in [-1, 1] \subset R$, into function

$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$f(x)$ is one-one and into



(iv) $f : [-\pi/2, \pi/2] \to [-1, 1]$

$f(x) = \sin x$

Range $\in [-1, 1] =$ Codomain

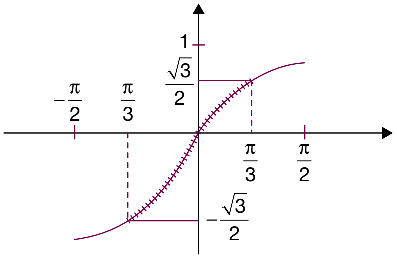
$[-1, 1]$ onto function

Thus, $f(x)$ is bijective as $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(v) $f : \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] \to [-1, 1]$:

$f(x) = \sin x$

Range $\in \left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right] \subset [-1, 1]$



Thus, $f(x)$ is one-one and into

ILLUSTRATION 78

Classify the following function as one-one and onto.

(i) $f: R \to R, f(x) = x^2 - 2x + 3$

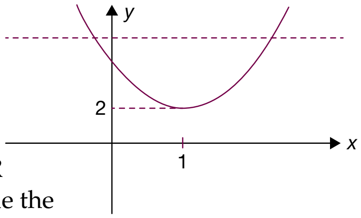
(ii) $f: [1, \infty) \to [2, \infty), f(x) = x^2 - 2x + 3$

SOLUTION

(i) $f: R \to R$

$f(x) = x^2 - 2x + 3$

$y = (x - 1)^2 + 2$



Range $y \in [2, \infty) \subset R$

We can also conclude the same from graph of $f(x)$.

As a horizontal line is cutting the graph more than one point so it is many-one function.

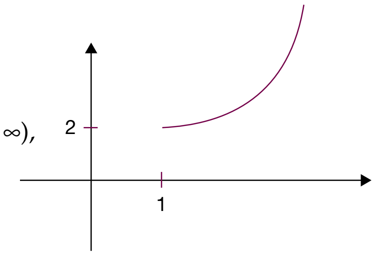
Graph of $f(x)$ is also not having complete real values so range $\subset R$(co-domain). Hence, into function

Many-one and into function

(ii) $f: [1, \infty) \to [2, \infty)$

$f(x) = x^2 - 2x + 3$

$y = (x - 1)^2 + 2$



For domain $x \in [1, \infty)$, $f(x)$ is increasing function

Range $y \in [2, \infty) =$ Co-domain

Onto function and one-one function

We cannot draw a horizontal line which cuts the graph more than one point so it is one-one function.

Graph of $f(x)$, range $= [2, \infty) =$ co-domain, hence, it is onto function.

$f(x)$ is bijective function

Analysis of Cubic

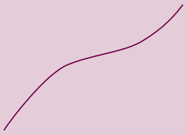
Let consider a cubic.

$$f(x) = y = ax^3 + bx^2 + cx + d \;\forall\; x \in R$$

CASE 1:

For $a > 0$, $\frac{dy}{dx} \geq 0$

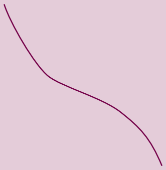
Always increasing (one-one and onto function)



CASE 2:

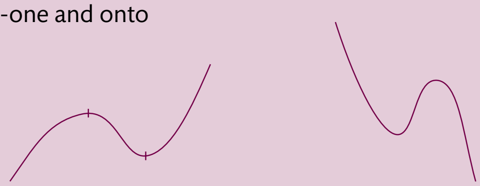
For $a < 0$, $\frac{dy}{dx} \leq 0$

Always decreasing function one-one and onto



CASE 3:

If $\frac{dy}{dx}$ changes sign then $f(x)$ is non-monotonic so cubic is many-one and onto



In above cases $f(x)$ is surjective as $x \in R$

Conceptual Note(s)

1. For cubic $f(x) = ax^3 + bx^2 + cx + d$ defined in R is one-one or many one.

$f'(x) = 3ax^2 + 2bx + c$

$D > 0$ — $f(x)$ Many-one

$D \leq 0$ — $f(x)$ one-one

2. Every polynomial for odd degree is always surjective in real number.

$$f(x) = ax^{2n+1} + bx^{2n} + cx^{2n-1} + dx^{2n-2} + \ldots + ex + f$$

ILLUSTRATION 79

Classify following function defined as $f: R \to R$

(i) $f(x) = x^3 - 4x^2 + 12x - 1$ **(ii)** $f(x) = -2x^3 + 3x^2 + 6x - 5$

SOLUTION

(i) $f(x) = x^3 - 4x^2 + 12x - 1$

$\Rightarrow f'(x) = 3x^3 - 8x + 12$

$D = (-8)^2 - 4(3)(12)$

$= 64 - 12 \times 12 = 64 - 144 < 0$

$\Rightarrow f'(x) = 3x^2 - 8x + 12 > 0$ for all x:

$f(x)$ is monotonically increasing

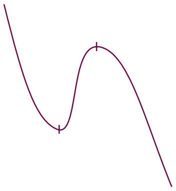
Hence, $f(x)$ is one-one onto

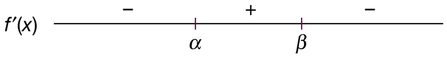
(ii) $f(x) = -2x^3 + 3x^2 + 6x - 5$

$\frac{dy}{dx} = f'(x) = -6x^2 + 6x + 6$

$= -6(x^2 - x - 1) = -6(x - \alpha)(x - \beta)$

$D = 1 - 4(-1) = 5 > 0$





$\Rightarrow f'(x)$ is changing its sign so $f(x)$ is non-monotonic

Hence, we can conclude that $f(x)$ is onto and many-one.

ILLUSTRATION 80

Classify following function defined as:

(i) $f(x) = x^2 + \ln x$ **(ii)** $f(x) = 9^{|x|} + 9^{2-x}$

SOLUTION

(i) $f(x) = x^2 + \ln x$, domain $x > 0$

$$\frac{dy}{dx} = 2x + \frac{1}{x} = \frac{2x^2+1}{x} > 0$$

$f(x)$ is continuous and strictly increasing then $f(x)$ is one-one.

$$x \longrightarrow 0^+ \text{ then } y \longrightarrow \ln 0^+ \longrightarrow -\infty$$

$$x \longrightarrow \infty \text{ then } y \longrightarrow \infty$$

Range $\in (-\infty, \infty)$ = Co-domain then surjective function $f(x)$ is bijective

(ii) $f(x) = 9^{|x|} + 9^{2-x}$

As $9^{|x|} > 0$, $9^{2-x} > 0$

Clearly, $f(x) = 9^{|x|} + 9^{2-x}$ cannot have negative values
Hence, the range $\neq R$, Into function.
By observation, we obtain

$$f(0) = 9^0 + 9^{2-0} = 1 + 81 = 90$$

$$f(2) = 9^2 + 9^0 = 81 + 1 = 90$$

$\Rightarrow f(2) = f(0)$: many-one function.

Hence, $f(x)$ is many-one and Into function

ILLUSTRATION 81

Find the values of 'p' for which $f(x) = px + \sin x$ is bijective.

SOLUTION

Given, $f(x) = px + \sin x$

As $f(x)$ is sum of line $y = px$ and bounded function $y = \sin x$. Linear function keeps its range as R for non-zero p. Therefore, for $p \neq 0$ onto function

$$f'(x) = p + \cos x$$

For one-one function $f(x)$ should be monotonic

Monotonic

Case I: Increasing	Case II: Decreasing
$f'(x) = p + \cos x$	$f'(x) = p + \cos x$
for $f'(x) = p + \cos x \geq 0, \forall x \in R$	for $f'(x) = p + \cos x \leq 0, \forall x \in R$
$p - 1 \geq 0 \Rightarrow p \geq 1$	$p + 1 \leq 0$ $p \leq -1$

CASE 1 ∪ CASE 2

$p \in (-\infty, -1] \cup [1, \infty)$ for $f(x)$ to be one-one
Final set of values of p for $f(x)$ to be bijective
$p \in (-\infty, -1] \cup [1, \infty)$

ILLUSTRATION 82

$f : [0, 3] \to A$ such that $f(x) = x^2 - 2x + 5$
Find A if $f(x)$ is onto?

SOLUTION

As $x \in [0, 3]$; $f(x) = x^2 - 2x + 5$

$$x_0 = -\frac{B}{2A} = -\frac{(-2)}{2} = 1 \in [0, 3]$$

So, $f(0) = 0 - 0 + 5 = 5$

$$f(1) = 1 - 2 + 5 = 4$$

$$f(3) = 9 - 6 + 5 = 8$$

Thus, range $\in [4, 8]$
For $f(x)$ to be onto, Range = co-domain (A)
We have set $A = [4, 8]$

ILLUSTRATION 83

Let $f : X \to Y$ be a function defined by $f(x) = a \sin\left(x + \frac{\pi}{4}\right) + b \cos x + c$. If f is both one-one and onto, find sets X and Y.

SOLUTION

$$f(x) = a \sin\left(x + \frac{\pi}{4}\right) + b \cos(x) + c$$

$$\Rightarrow \quad f(x) = a\left\{\sin x \cos\frac{\pi}{4} + \cos x \sin\frac{\pi}{4}\right\} + b \cos x + c \qquad \ldots(1)$$

$$\Rightarrow \quad f(x) = \frac{a}{\sqrt{2}} \sin x + \left(\frac{a}{\sqrt{2}} + b\right) \cos x + c$$

Let $\left(\frac{a}{\sqrt{2}}\right) = r \cos \alpha$, $\left(\frac{a}{\sqrt{2}} + b\right) = r \sin \alpha$

$$\therefore \quad f(x) = r[\cos\alpha \sin x + \sin\alpha \cos x] + c = r \sin(x + \alpha) + c$$

where $r = \sqrt{a^2 + \sqrt{2}ab + b^2}$ and $\alpha = \tan^{-1}\left(\frac{a + b\sqrt{2}}{a}\right)$ …(2)

For f to be one-one, we must have

$$-\pi/2 \leq x + \alpha \leq \pi/2$$

Thus, domain $\in \left[-\frac{\pi}{2} - \alpha, \frac{\pi}{2} - \alpha\right]$ and range $\in [c - r, c + r]$

where $r = \sqrt{a^2 + \sqrt{2}ab + b^2}$ and $\alpha = \tan^{-1}\left(\frac{a + b\sqrt{2}}{a}\right)$

ILLUSTRATION 84

Let $f : R \to R$ be defined as $f(x) = \begin{cases} 2x + 3, & x \leq 1 \\ p^2 x + 1, & x > 1 \end{cases}$

If $f(x)$ is onto function, then number of integral values of p is equal to:

SOLUTION

$\because$ $f(x)$ is onto function

$\Rightarrow$ Range of f is R.

$\Rightarrow$ $f(1) \geq f(1^+) \Rightarrow 5 \geq p^2 + 1$

$\Rightarrow$ $p^2 \leq 4 \Rightarrow p \in [-2, 2]$

Thus, integral $p \in \{-2, -1, 1, 2\}$

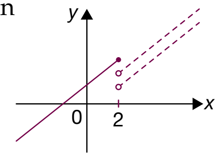
ILLUSTRATION 85

Let a function f defined from $R \rightarrow R$ as $f(x) = \begin{cases} x + p^2, & \text{for } x \leq 2 \\ px + 5, & \text{for } x > 2 \end{cases}$. If the function is surjective, then find the sum of all possible integral values of p in $[-100, 100]$.

SOLUTION

For f to be surjective range = co-domain

$$f(x) = \begin{cases} x + p^2, & \text{for } x \leq 2 \\ px + 5, & \text{for } x > 2 \end{cases}$$



For $x \leq 2$, range of f is $(-\infty, 2 + p^2]$ and

For $x > 2$, range of f is $(2p + 5, \infty)$.

[Note that $p > 0$]

For range of f to be R, $f(2^+) \leq f(2) \Rightarrow 2p + 5 \leq 2 + p^2$

$$p^2 - 2p - 3 \geq 0 \Rightarrow (p - 3)(p + 1) \geq 0$$

$\Rightarrow$ $p \in (-\infty, -1] \cup [3, \infty)$ but $p > 0$.

Hence, $p \in [3, \infty)$ given $p \in [-100, 100]$

Sum of acceptable values of p

$$S = 3 + 4 + \ldots + 100 = 5050 - 3 = 5047.$$

ILLUSTRATION 86

If $f : R \rightarrow R$ be an injective mapping and p, q, r are non-zero distinct real quantities satisfying $f\left(\frac{p}{r}\right) = f\left(\frac{p-q}{q-r}\right)$ and $f\left(\frac{q}{r}\right) = f\left(\frac{r}{p}\right)$. If the graph of $g(x) = px^2 + qx + r$ passes through $M(1, 6)$, then find the value of q.

SOLUTION

f is one-one function

$$f\left(\frac{p}{r}\right) = f\left(\frac{p-q}{q-r}\right) \Rightarrow \frac{p}{r} = \frac{p-q}{q-r}$$

$\Rightarrow$ $pq - pr = pr - rq \Rightarrow 2pr = (p + r)q$ …(1)

$$f\left(\frac{q}{r}\right) = f\left(\frac{r}{p}\right) \Rightarrow p = \frac{r^2}{q} \text{ from (1)}$$

$\Rightarrow$ $\dfrac{2r^2}{q} = \left(\dfrac{r}{q} + 1\right)q \Rightarrow 2r^2 = rq + q^2$ …(2)

$$g(x) = px^2 + qx + r$$

Here, $M(1, 6)$ lies on it, we get

$$6 = p + q + r$$

$\Rightarrow$ $6 = \dfrac{r^2}{q} + q + r \Rightarrow 6q = r^2 + q^2 + rq = r^2 + 2r^2$

$\Rightarrow$ $r^2 = 2q \Rightarrow q = \dfrac{r^2}{2}$

From (2), $2r^2 = rq + q^2$

$\Rightarrow$ $2r^2 = r\dfrac{r^2}{2} + \dfrac{r^4}{4} \Rightarrow 2 = \dfrac{r}{2} + \dfrac{r^2}{4} \Rightarrow r^2 + 2r - 8 = 0$

$\Rightarrow$ $(r + 4)(r - 2) = 0 \Rightarrow r = 2, -4$

For $r = 2$; $q = 2$ but they must be distinct so not acceptable and $r = -4$; $q = 8$ (acceptable).

ILLUSTRATION 87

Let $f : R \rightarrow [0, \infty)$ be defined as $f(x) = \log\left(\sqrt{9x^2 - 12x + \lambda} + 1\right)$ be an onto function where λ is a real parameter belong to $(0, 10)$. Find number of integral value of λ.

SOLUTION

$$f(x) = \log(\sqrt{9x^2 - 12x + \lambda} + 1)$$

$\Rightarrow$ $9x^2 - 12x + \lambda \geq 0$

Given $f : R \rightarrow [0, \infty)$

As range of $f(x)$ is $[0, \infty)$, we get

$$9x^2 - 12x + \lambda = 0$$

Since, $D = 0$

$\Rightarrow$ $144 - 4(9)\lambda = 0$

Hence, $\lambda = 4$ only. $\therefore$ one value of λ.

ILLUSTRATION 88

Function $f : A \rightarrow B$ where $n(A) = 4$, $n(B) = 4$

(i) Total number of functions
(ii) Number of one-one functions
(iii) Number of many-one functions
(iv) Number of onto functions
(v) Number of into functions

SOLUTION

(i) Total number of Functions

$= 4.4.4.4 = 4^4 = 256$

(ii) Number of one-one functions

$= 4.3.2.1 = 4! = 24$

(iii) Number of many-one functions

$=$ Total–one-one $= 256 - 24 = 232$

(iv) Number of onto functions

$= 4.3.2.1 = 4!$

Alternate:

Number of onto function = Total possible function – Exactly one element in range – Exactly two elements in range – Exactly three elements in range

$$= 4^4 - {}^4C_1 - {}^4C_2(2^4-2) - {}^4C_3(3^4 - {}^3C_1 - {}^3C_2(2^4-2))$$
$$= 24$$

(v) Number of into functions

$= \text{Total-onto} = 256 - 24 = 232$

ILLUSTRATION 89

Function $f : A \to B$ where $n(A) = 4$, $n(B) = 6$

(i) Total number of Functions
(ii) Number of one-one functions
(iii) Number of many-one functions
(iv) Number of onto functions
(v) Number of into functions

SOLUTION

Function $f : A \to B$ $n(A) = 4$, $n(B) = 6$

(i) Total number of functions

$= 6.6.6.6 = 6^4 = 1296$

(ii) Number of one-one functions

$= 6.5.4.3 = 30 \times 12 = 360$

(iii) Number of many – one functions

$= 6^4 - 360 = 1296 - 360 = 936$

(iv) (4 elements can not be mapped to all 6 elements as function definition says each element can have only unique image)

number of onto functions = 0

(v) Number of into functions

$= 6^4 = 1296$

Total no. of function no. of onto function

ILLUSTRATION 90

Function $f : A \to B$ where $n(A) = 6$, $n(B) = 4$

(i) Total number of Functions
(ii) Number of one-one functions
(iii) Number of many-one functions
(iv) Number of onto functions
(v) Number of into functions

SOLUTION

$$f : A \to B : n(A) = 6, n(B) = 4$$

(i) Total number of functions $= 4^6$

(ii) Set B has only 4 elements and domain has 6 elements, as definition of function says that each element from domain has unique image in set B, so all 6 elements can not have 6 different images in set B so no injective mapping is possible Number of one-one functions = 0

(iii) Number of many – one functions $= 4^6 - 0 = 4^6$ [all possible functions are many-one]

(iv) Number of onto functions = no. of ways of 6 distinct things are distributed among 4 persons such that each received atleast one things.

= Group of 1, 1, 1, 3 toys + group of 1, 1, 2, 2 toys

$$= \left(\frac{6!}{3!(1!)^3 3!}\right)4! + \left(\frac{6!}{(2!)^2(1!)^2 2!} \times \frac{1}{2!}\right)4!$$

$$= 4!\left[\frac{6\cdot5\cdot4}{6} + \frac{6\cdot5\cdot4\cdot3}{2\times4}\right]$$

$$= 24[20 + 45] = 24(65) = 1560$$

Alternate:

Number of onto functions = Total functions – Exactly one element in range – Exactly two elements in range – Exactly three elements in range

$$= 4^6 - {}^4C_1 - {}^4C_2(2^6-2) - {}^4C_3(3^6 - {}^3C_1 - {}^3C_2(2^6-2))$$
$$= 4096 - 4 - 6(62) - 4(729 - 3 - 3(62))$$
$$= 1560$$

(v) Number of into functions = Total no. of function – No. of onto function

$= 4^6 - 1560 = 2536$

Test Your Concepts-V

(Solutions on page H.15)

1. Let $f : N \to N$ be defined as $f(n) = \begin{cases} \dfrac{n+1}{2}, & \text{if } n \text{ is odd} \\ \dfrac{n}{2}, & \text{if } n \text{ is even} \end{cases}$

for all $n \in N$. Then show that f is surjective but not injective.

2. Classify following function as one-one/many-one and onto/into:

(i) $f : R \to R$ $f(x) = \dfrac{e^{|x|} - e^{-x}}{e^x + e^{-x}}$

(ii) $f : R \to R$ $f(x) = 2x + \sin x$

(iii) $f : R \to R$ $f(x) = x^3 + x^2 + 100x + 5\sin x$,

(iv) $f : R \to R, f(x) = x|x|$

(v) $f : \left[0, \sqrt{3}\right] \to \left[0, \dfrac{\pi}{3} + \log_e^2\right]$,

$f(x) = \log_e \sqrt{x^2+1} + \tan^{-1} x$

3. Classify following function as injective or surjective:

(i) $f : [-1, 3] \to [-8, 72]$ be defined as $f(x) = 4x^3 - 12x$,

(ii) $f : R \to \{x \in R : -1 < x < 1\}$, $f(x) = \dfrac{x}{1+|x|}$

(iii) $f: [-1, 3] \to [-37, 27]$ $f(x) = 2x^3 - 6x^2 - 18x + 17$

(iv) $f: R \to R$, $f(x) = e^{x^2} + \cos x$

(v) $f: R \to R$, $f(x) = 2x - \sin x$,

4. Number of surjective function $f: A \to B$, where set A contains 4 elements and set B contains 3 elements.

5. Let $f: R \to \left[0, \frac{\pi}{2}\right)$ defined by $f(x) = \tan^{-1}\left(x^2 - x + a\right)$, then find the set of values of 'a' for which f is onto.

6. Consider the function $f: [1, \infty) \to (0, 1]$, $f(t) = \frac{1}{1+t^2}$.
Then f is one-to-one but not onto. Mark true or false.

7. Let S be the set of all triangles and R^+ be the set of positive real numbers.
Then show that the function, $f: S \to R^+$, $F(\Delta) = \text{area of } \Delta$, where $\Delta \in S$ surjective but not injective.

Answer key

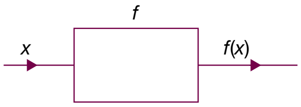
2. (i) many-one and into (ii) one-one and onto (iii) one-one and onto
(iv) one-one and onto (v) one-one and onto

3. (i) many-one and onto (ii) one-one and onto (iii) one-one and onto
(iv) many-one and into (v) one-one and onto

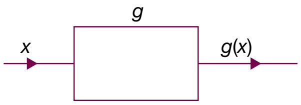
4. 36 **5.** $a \in \left\{\frac{1}{4}\right\}$ **6.** True

COMPOSITE FUNCTION

Let consider a function f defined as f: $A \to B$ where A is domain of $f(x)$ and B is co-domain of $f(x)$.



where $x \in A, f(x) \in B$
Let consider a function g defined as g: $C \to D$ where C is domain of $f(x)$ and D is co-domain of $f(x)$.

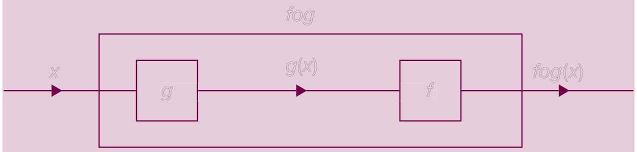


where $x \in C, g(x) \in D$

$D_{f(x)}$: Domain of $f(x)$, $R_{f(x)}$: Range of $f(x)$

$D_{g(x)}$: Domain of $g(x)$, $R_{g(x)}$: Range of $g(x)$

$y = fog(x) = f(g(x))$ flow of formation of $f(g(x))$ is shown as below.

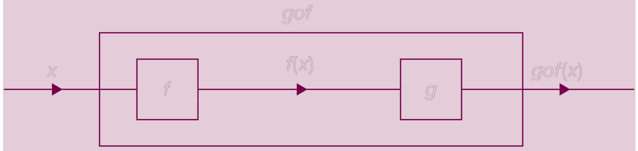


$fog(x)$ exists if $R_{g(x)} \cap D_{f(x)} \neq \phi$

Domain of $fog(x)$ = set of values of x (subset of domain of $g(x)$) which are connected to $R_{g(x)} \cap D_{f(x)}$

Range of $fog(x)$ = Range of $f(x)$ where $x \in R_{g(x)} \cap D_{f(x)}$

$y = gof(x) = g(f(x))$ flow of formation of $g(f(x))$ is shown as below.

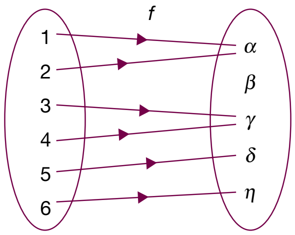


$gof(x)$ exists if $\text{Range}_{f(x)} \cap \text{Domain}_{g(x)} \neq \phi$

Domain of $gof(x)$ = set of values of x (subset of domain of $f(x)$) which are connected to $R_{f(x)} \cap D_{g(x)}$

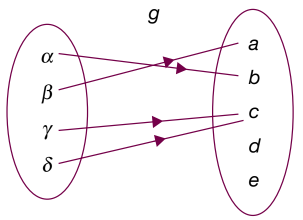
Range of $gof(x)$ = Range of $g(x)$ when $x \in \text{Range}_{f(x)} \cap \text{Domain}_{g(x)}$

Consider following mapping:



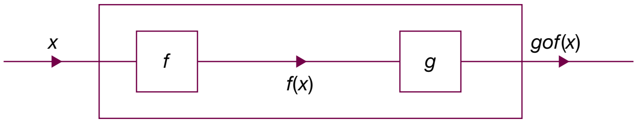
Domain of $f(x) = D_{f(x)} = \{1, 2, 3, 4, 5, 6\}$

Range of $f(x) = R_{f(x)} = \{\alpha, \gamma, \delta, \eta\}$



Domain of $f(x) = D_{g(x)} = \{\alpha, \beta, \gamma, \delta\}$

Range of $f(x) = R_{g(x)} = \{a, b, c\}$



$$R_{f(x)} \cap D_{g(x)} = \{\alpha, \gamma, \delta\}$$

Domain of $gof(x)$ = Subset of domain of $f(x)$ which are connected to $\text{Range}_{f(x)} \cap \text{Domain}_{g(x)}$

$$= \{1, 2, 3, 4, 5\}$$

Range of $gof(x)$ = Range of $g(x)$ when $x \in \text{Range}_{f(x)} \cap \text{Domain}_{g(x)} = \{b, c\}$

$$y = fog(x) = f(g(x)) \neq f(x)\, g(x), fog(x) \neq gof(x)$$

E.g., $f(x) = x^2$, $g(x) = \sin x$

$$y = fog(x) = f(g(x)) = (g(x))^2 = (\sin x)^2 = \sin^2 x$$

$$y = gof(x) = g(f(x)) = \sin f(x) = \sin x^2 \neq \sin^2 x$$

Clearly $fog(x) \neq gof(x)$

Properties of Composite Function

(i) If f and g are two functions, then for composite of two functions $fog \neq gof$.
It means composite functions in not commutative

(ii) Composite functions obeys the property of associativity i.e., $fo\,(goh) = (fog)\,oh$.

ILLUSTRATION 91

$f(x) = \{x\}$ where $\{\cdot\}$ is fpf, $g(x) = \sin x$; $h(x) = \sqrt{x}$

Compute the following function:

(i) $y = hogof(x)$, **(ii)** $y = gofoh(x)$
(iii) $y = fohog(x)$, **(iv)** $y = hofog(x)$

SOLUTION

(i) $y = hogof(x)$

$$y = h(g(f(x))$$

$$y = \sqrt{\sin\{x\}}$$

(ii) $y = gofoh(x)$

$$y = g(f(h(x)))$$

$$y = \sin\{\sqrt{x}\}$$

(iii) $y = fohog(x)$

$$= f(h(g(x)))$$

$$= \left\{\sqrt{\sin x}\right\}$$

(iv) $y = hofog(x)$

$$= h(f(g(x)))$$

$$= \sqrt{\{\sin x\}}$$

ILLUSTRATION 92

Let $f(x) = \sqrt{x}$; $g(x) = \sqrt{2-x}$, find domain of the following function:

(i) $fog(x)$, **(ii)** $gof(x)$ **(iii)** $fof(x)$ **(iv)** $gog(x)$

SOLUTION

(i) $fog(x)$

$$y = f(g(x)) = \sqrt{g(x)}$$

$$= \sqrt{\sqrt{2-x}}$$

For domain, $2 - x \geq 0$

$\Rightarrow x \leq 2$

Hence, $x \in (-\infty, 2]$

(ii) $gof(x)$

$$y = gof(x) = g(f(x))$$

$$= \sqrt{2 - f(x)} = \sqrt{2 - \sqrt{x}}$$

For domain, $2 - \sqrt{x} \geq 0$

$\Rightarrow \sqrt{x} \leq 2 \Rightarrow x \leq 4$

as $x \geq 0$

So we get $0 \leq x \leq 4$

(iii) $fof(x)$

$$y = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}}$$

$$x \geq 0 \qquad \therefore x \in [0, \infty)$$

(iv) $gog(x) = g(g(x))$

$$y = \sqrt{2 - g(x)} = \sqrt{2 - \sqrt{2-x}}$$

$$2 - \sqrt{2-x} \geq 0 \Rightarrow 2 \geq \sqrt{2-x} \quad [x \leq 2]$$

$\Rightarrow 4 \geq 2 - x \Rightarrow 2 + x \geq 0 \Rightarrow x \leq -2$

Hence, final domain $-2 \leq x \leq 2$

ILLUSTRATION 93

$f(x) = x^2 + 1 : g(x) = \sin x - 2$ find domain and range of $fog(x)$ and $gof(x)$.

SOLUTION

(i) $fog(x) = f(g(x)) = (g(x))^2 + 1 = (\sin x - 2)^2 + 1$

Domain of $fog(x) = x \in R$

$-1 \leq \sin x \leq 1 \Rightarrow -3 \leq \sin x - 2 \leq -1$

$\Rightarrow 1 \leq (\sin x - 2)^2 \leq 9$

$\Rightarrow 2 \leq (\sin x - 2)^2 + 1 \leq 10$

Range of $fog(x) \in [2, 10]$

(ii) $gof(x) = g(f(x)) = \sin f(x) - 2 = \sin(x^2 + 1) - 2$

Domain of $gof(x) = x \in R$

$-1 \le \sin(x^2 + 1) \le 1$

$\Rightarrow -3 \le \sin(x^2 + 1) - 2 \le -1$

Range of $gof(x) \in [-3, -1]$

ILLUSTRATION 94

If domain of $f(x)$ is $(0, 1]$, find domain of:

(i) $f(e^x)$ **(ii)** $f(\ln x)$ **(iii)** $f(\sin x)$ **(iv)** $f(\tan x)$

SOLUTION

(i) $f(e^x)$

$f(x)$ where $x \in (0, 1]$ or $0 < x \le 1$

$f(e^x)$; $0 < e^x \le 1$

$\ln 0^+ < x \le \ln 1$

$\Rightarrow -\infty < x \le 0$

(ii) $f(\ln x)$

$0 < \ln x \le 1$

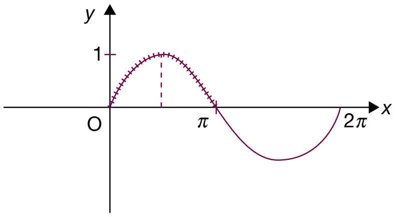
$\Rightarrow e^0 < x \le e^1 \quad \Rightarrow \quad 1 < x \le e$

$x \in (1, e]$

(iii) $f(\sin x)$

Domain of $f(x)$ is $0 < x \le 1$

$0 < \sin x \le 1$, $x \in (0, \pi)$

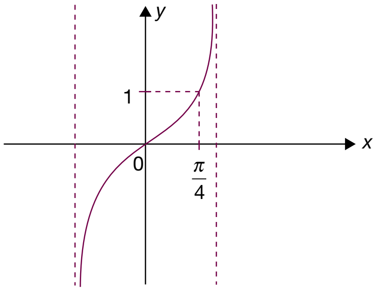


As $\sin x$ has period 2π.

Then complete domain $x \in (2n\pi, 2n\pi + \pi)$.

(iv) $f(\tan x)$; $0 < \tan x \le 1$

$$\Rightarrow x \in \left(0, \frac{\pi}{4}\right]$$



As $\tan x$ is periodic with π as period.

Then complete domain $x \in \left(n\pi, n\pi + \frac{\pi}{4}\right]$.

ILLUSTRATION 95

Let $f(x)$ has domain $x \in \left(\frac{1}{2}, 1\right]$ find domain of:

(i) $f(\sin 2x)$, **(ii)** $f([x])$ where $[\cdot]$ is GIF

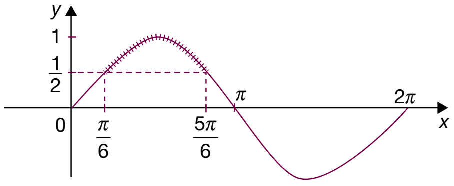
SOLUTION

(i) $f(\sin 2x)$

$\frac{1}{2} < \sin 2x \le 1 \quad$ Let $t = 2x$

$\Rightarrow \frac{1}{2} < \sin t \le 1 \qquad T = 2\pi$

$\Rightarrow t \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$



So, $2x = t \in \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$

$\Rightarrow x \in \left(n\pi + \frac{\pi}{12}, n\pi + \frac{5\pi}{12}\right)$

(ii) $f([x])$

$\frac{1}{2} < [x] \le 1$

$[x] = 1 \Rightarrow 1 \le x < 2 \Rightarrow x \in [1, 2)$

ILLUSTRATION 96

Let $f(x) = ax + b$, where a and b are integers. If $f(f(0)) = 0$ and $f(f(f(4))) = 9$, then find the value of $f(f(f(f(10))))$.

SOLUTION

$f(x) = ax + b$; $f(0) = b$

$f(f(0)) = a.b + b = 0$

$\Rightarrow \quad (a + 1)\, b = 0 \Rightarrow a = -1$ or $b = 0$

$f(f(f(4))) = 9$

$f(f(4a + b)) = 9$

$\Rightarrow \quad f(a(4a + b) + b) = 9$

$\Rightarrow \quad f(4a^2 + ab + b) = 9$

$\Rightarrow \quad a(4a^2 + ab + b) + b = 9$

$\Rightarrow \quad 4a^3 + a^2b + ab + b = 9$

when $b = 0$, $4a^3 = 9 \Rightarrow a^3 = \frac{9}{4}$

$\Rightarrow \quad a$ is not an integer

when $a = -1$, $-4 + b - b + b = 9 \Rightarrow b = 13$

$\therefore \quad f(x) = -x + 13$

$f(f(x)) = -(-x + 13) + 13 = x$

$\therefore \quad f(f(f(f(10)))) = f(f(10)) = 10.$

ILLUSTRATION 97

If $g(x) = \left(4\cos^4(x) - 2\cos(2x) - \frac{1}{2}\cos 4(x) - x^7\right)^{\frac{1}{7}}$, then find the value of $g(g(100))$.

SOLUTION

We have $4\cos^4 x - 2\cos 2x - \frac{1}{2}\cos 4x - x^7$

$$= 4\cos^4 x - 2(2\cos^2 x - 1) - \frac{1}{2}(2\cos^2 2x - 1) - x^7$$

$$= 4\cos^4 x - 2(2\cos^2 x - 1) - \frac{1}{2}(2(2\cos^2 x - 1)^2 - 1) - x^7$$

$$= 4\cos^4 x - 4\cos^2 x + 2 - (2\cos^2 x - 1)^2 + \frac{1}{2} - x^7$$

$$= \frac{3}{2} - x^7$$

$\therefore$ We get $g(x) = \left(\frac{3}{2} - x^7\right)^{\frac{1}{7}}$

$$\Rightarrow \quad g(g(x)) = \left(\frac{3}{2} - (g(x))^7\right)^{\frac{1}{7}} = \left(\frac{3}{2} - \left(\frac{3}{2} - x^7\right)\right)^{\frac{1}{7}} = x$$

Hence, $g(g(100)) = 100$.

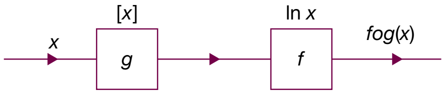
ILLUSTRATION 98

If $f: R^+ \to R, f(x) = \ln x$ and $g = R \to R, g(x) = [x]$ where [.] is greatest integer function. Compute $fog(x)$ and $gof(x)$.

SOLUTION

$f: R^+ \to R$	$g = R \to R$
$f(x) = \ln x$	$g(x) = [x]$
Domain of $f(x) = D_{f(x)} = R^+$	Domain of $g(x) = D_g = R$
Range of $f(x) = R_{f(x)} = R$	Range of $g(x) = R_g = I$

(i) $fog(x)$



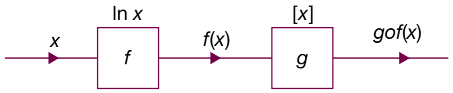
$y = fog(x) = \ln[x]$

$fog(x)$ exists as $R_g \cap D_f = I \cap R^+ = N$

Domain of $fog(x)$ we have $[x] > 0$, we get $x \in [1, \infty)$

Range of $fog(x)$ $y \in \{0, \ln 2, \ln 3, \ldots\}$

(ii) $gof(x) = [\ln x]$



$gof(x)$ exists as $R_f \cap D_g = R \cap R = R$

As we can see that range of $f(x)$ is the same as domain of $g(x)$.

So domain of $gof(x) = \text{domain of } f(x) = x \in R^+$

and range of $gof(x) = \text{range of } g(x) = I$

$gof(x) = [\ln x]$ and $fog(x) = \ln [x]$

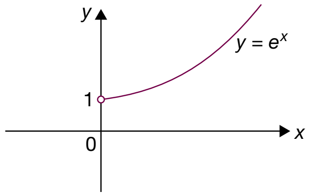
$gof(x) \neq fog(x)$

ILLUSTRATION 99

$f: R^+ \to R, f(x) = e^x$ and $g: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1], g(x) = \sin x$

Compute (i) $fog(x)$ and (ii) $gof(x)$ and their domain and range.

SOLUTION



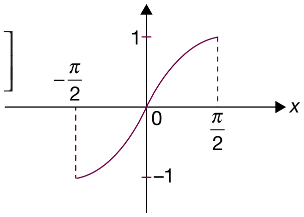
$f: R^+ \to R, f(x) = e^x$

Domain of $f(x) = D_f = (0, \infty)$

Range of $f(x) = R_f = (1, \infty)$

Clearly we can observe that $f(x)$ is defined for subset of domain of e^x, so we need to be vigilant

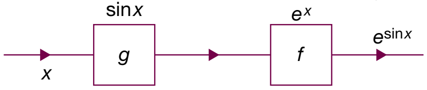
$g: \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1], g(x) = \sin x$



Domain of $g(x) = D_g = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Range of $g(x) = R_g = [-1, 1]$

$fog(x) = e^{\sin x}$



$R_{g(x)} \cap D_{f(x)} = [-1, 1] \cap (0, \infty) = (0, 1]$

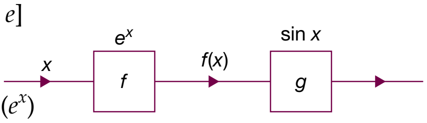
$fog(x) = e^{\sin x}$

Domain of $fog(x) = x$ for which range of $g(x)$ is $(0, 1] = x \in \left(0, \frac{\pi}{2}\right]$

Range $fog(x)$ = values of $f(x)$ when $x \in (0, 1]$

$y = fog(x) \in (1, e]$

(ii) $gof(x)$



$y = gof(x) = \sin(e^x)$

$R_f \cap D_g = (1, \infty) \cap \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] = \left(1, \frac{\pi}{2}\right]$

Domain of $gof(x) = x$ for which range of $f(x)$ is $\left(1, \frac{\pi}{2}\right]$.

$$1 < e^x \leq \frac{\pi}{2} \Rightarrow \ln 1 < x \leq \ln\frac{\pi}{2} \Rightarrow 0 < x \leq \ln\frac{\pi}{2}$$

Domain of $gof(x) = x \in \left(0, \ln\frac{\pi}{2}\right]$

Range of $gof(x)$ = values of $g(x)$ when $x \in \left(1, \frac{\pi}{2}\right]$

$y \in (\sin 1, 1]$

ILLUSTRATION 100

Let $f : R^+ \to R, f(x) = e^x, g : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], g(x) = \sin^{-1}x$

Compute $fog(x)$ and $gof(x)$.

SOLUTION

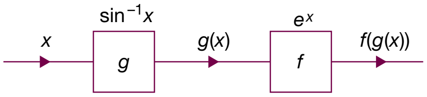
$$f : R^+ \to R, f(x) = e^x$$

Domain of $f(x) = D_f = R^+$, Range of $f(x) = R_f = (1, \infty)$

$$g : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], g(x) = \sin^{-1}x$$

Domain of $g(x) = D_g = [-1, 1]$, Range of $g(x) = R_g = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$fog(x)$ can be shown as



$$fog(x) = e^{\sin^{-1}x}$$

$$R_{g(x)} \cap D_{f(x)} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \cap R^+ = \left(0, \frac{\pi}{2}\right]$$

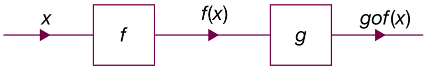
Domain of $fog(x) = x$ for which range of $g(x)$ is $(0, 1]$ $= x \in (0, 1]$.

As $0 < \sin^{-1}x \le \frac{\pi}{2} \Rightarrow 0 < x \le 1$

Range of $fog(x)$ = values of $f(x)$ when $x \in \left(0, \frac{\pi}{2}\right]$

$$0 < x \le \frac{\pi}{2} \Rightarrow e^0 < e^x \le e^{\frac{\pi}{2}} \Rightarrow y \in \left(1, e^{\frac{\pi}{2}}\right]$$

$gof(x)$ can be shown as



$$R_{f(x)} \cap D_{g(x)} = (1, \infty) \cap [-1, 1] = \phi$$

$gof(x)$ does not exist

Clearly, we can see that $fog(x)$ exists but $gof(x)$ does not.

ILLUSTRATION 101

Let $f(x) = \begin{cases} x+1 \; ; & x \le 1 \\ 2x+1; & 1 < x \le 2 \end{cases}$ and $g(x) = \begin{cases} x^2 \; ; & -1 \le x < 2 \\ x+2; & 2 \le x \le 3 \end{cases}$

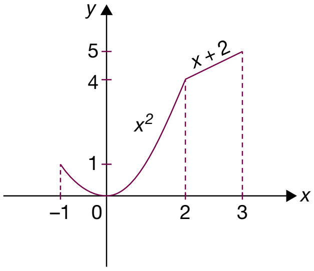
Find fog and gof.

SOLUTION

(i) $f(g(x)) = \begin{cases} g(x)+1 \; ; & g(x) \le 1 \\ 2g(x)+1; & 1 < g(x) \le 2 \end{cases}$

Graphical Approach

Here, $g(x)$ becomes the variable that means we should draw the graph.



It is clear that $g(x) \le 1;\ \forall\ x \in [-1, 1]$, where $g(x) = x^2$

and $1 < g(x) \le 2;\ \forall\ x \in (1, \sqrt{2}]$, where $g(x) = x^2$

$$\Rightarrow f(g(x)) = \begin{cases} x^2+1 \; ; & -1 \le x < 1 \\ 2x^2+1; & 1 < x \le \sqrt{2} \end{cases}$$

Alternate (Mathematical Approach):

$$f\{g(x)\} = \begin{cases} g(x)+1 \; ; & g(x) \le 1 \\ 2g(x)+1; & 1 < g(x) \le 2 \end{cases}$$

$$f(g(x)) =$$

$$\begin{cases} x^2+1 \; ; & x^2 \le 1 \Rightarrow x \in [-1,1] \\ & \text{but } -1 \le x < 2 \Rightarrow -1 \le x \le 1 \\ x+2+1 \; ; & x+2 \le 1 \Rightarrow x \le -1 \\ & \text{but } 2 \le x \le 3 \Rightarrow x \in \phi \\ 2x^2+1 \; ; & 1 < x^2 \le 2 \Rightarrow x \in [-\sqrt{2}, -1) \cup (1, \sqrt{2}] \\ & \text{but } -1 \le x < 2 \Rightarrow x \in (1, \sqrt{2}] \\ 2(x+2)+1; & 1 < x+2 \le 2 \Rightarrow -1 < x \le 0 \\ & \text{but } 2 \le x \le 3 \Rightarrow x \in \phi \end{cases}$$

(i) $f(g(x)) = \begin{cases} x^2+1; & -1 \le x \le 1 \\ 2x^2+1; & 1 < x \le \sqrt{2} \end{cases}$

(ii) $g\{f(x)\} = \begin{cases} (f(x))^2; & -1 \le f(x) < 2 \\ f(x)+2; & 2 \le f(x) \le 3 \end{cases}$

$$= \begin{cases} (x+1)^2 \; ; & -1 \le x+1 < 2 \Rightarrow -2 \le x < 1 \\ & \text{but } x \le 1 \Rightarrow -2 \le x < 1 \\ (2x+1)^2 \; ; & -1 \le 2x+1 < 2 \Rightarrow -1 \le x < 1/2 \\ & \text{but } 1 < x \le 2 \Rightarrow x \in \phi \\ (x+1)+2 \; ; & 2 \le x+1 \le 3 \Rightarrow 1 \le x \le 2 \\ & \text{but } x \le 1 \Rightarrow x = 1 \\ (2x+1)+2 \; ; & 2 \le 2x+1 \le 3 \Rightarrow 1/2 \le x \le 1 \\ & \text{but } 1 < x \le 2 \Rightarrow x \in \phi \end{cases}$$

$$gof(x) = (x+1)^2, -2 \le x < 1$$

ILLUSTRATION 102

Let $f(x)=\begin{cases}1-x & ; \quad x\le 0\\ x^2 & ; \quad x>0\end{cases}; g(x)=\begin{cases}-x & ; \quad x<1\\ 1-x\ ; & \quad x\ge 1\end{cases}$

Compute **(i)** $fog(x)$ **(ii)** $gof(x)$ **(iii)** $fof(x)$ **(iv)** $gog(x)$

SOLUTION

(i) $y=fog(x)=\begin{cases}1-g(x) & g(x)\le 0\\ (g(x))^2 & g(x)>0\end{cases}$

$$=\begin{cases}1-(-x) & ;\ \text{for } -x\le 0 & \Rightarrow x\ge 0 & \text{but } x<1, x\in[0,1)\\ 1-(1-x) & ;\ \text{for } 1-x\le 0 & \Rightarrow x\ge 1 & \text{but } x\ge 1, x\ge 1\\ (-x)^2 & ;\ \text{for } -x>0 & \Rightarrow x<0 & \text{but } x<1, x<0\\ (1-x)^2 & ;\ \text{for } 1-x>0 & \Rightarrow x<1 & \text{but } x\ge 1, x\in\phi\end{cases}$$

$$y=fog(x)=\begin{cases}1+x & ; & x\in[0,1)\\ x & ; & x\ge 1\\ x^2 & ; & x<0\end{cases}$$

(ii) $gof(x)=\begin{cases}-f(x) & f(x)<1\\ 1-f(x) & f(x)\ge 1\end{cases}$

$gof(x)=$

$$\begin{cases}-(1-x)\ ; & \text{for } 1-x<1\Rightarrow x>0\\ & \text{but } x\le 0, x\in\phi\\ -x^2 \quad ; & \text{for } x^2<1\Rightarrow(x-1)(x+1)<0\Rightarrow x\in(-1,1)\\ & \text{but } x>0, x\in(0,1)\\ 1-(1-x); & \text{for } 1-x\ge 1\Rightarrow x\le 0\\ & \text{but } x\le 0, \therefore x\le 0\\ & \text{for } x^2\ge 1\Rightarrow(x-1)(x+1)\ge 0\\ 1-x^2 \quad ; & \Rightarrow x\in(-\infty,-1]\cup[1,\infty)\\ & \text{but } x>0 \quad x\in[1,\infty)\end{cases}$$

$$gof(x)=\begin{cases}-x^2 & ; & x\in(0,1)\\ x & ; & x\le 0\\ 1-x^2 & ; & x\in[1,\infty)\end{cases}$$

(iii) $y=fof(x)$

$$f(x)=\begin{cases}1-x & ; & x\le 0\\ x^2 & ; & x>0\end{cases}$$

$$fof(x)=\begin{cases}1-f(x) & ; & f(x)\le 0\\ (f(x))^2 & ; & f(x)>0\end{cases}$$

$$=\begin{cases}1-(1-x); & \text{for } 1-x\le 0 & \Rightarrow x\ge 1 & \text{but } x\le 0, x\in\phi\\ 1-x^2 & ;\ \text{for } x^2\le 0 & \Rightarrow x=0 & \text{but } x>0, x\in\phi\\ (1-x)^2 & ;\ \text{for } 1-x>0 & \Rightarrow x<1 & \text{but } x\le 0, x\le 0\\ (x^2)^2 & ;\ \text{for } x^2>0 & \Rightarrow x\ne 0 & \text{but } x>0, x>0\end{cases}$$

$$fof(x)=\begin{cases}(1-x)^2 & ; & x\le 0\\ x^4 & ; & x>0\end{cases}$$

(iv) $y=gog(x)$

$$g(x)=\begin{cases}-x & ; & x<1\\ 1-x & ; & x\ge 1\end{cases}$$

$$gog(x)=\begin{cases}-g(x) & ; & g(x)<1\\ 1-g(x) & ; & g(x)\ge 1\end{cases}$$

$$=\begin{cases}-(-x) & ;\ \text{for } -x<1 & \Rightarrow x>-1 & \text{but } x<1, -1<x<1\\ -(1-x) & ;\ \text{for } 1-x<1 & \Rightarrow x>0 & \text{but } x\ge 1, x\ge 1\\ 1-(-x) & ;\ \text{for } -x\ge 1 & \Rightarrow x\le -1 & \text{but } x<1, x\le -1\\ 1-(1-x); & \text{for } 1-x\ge 1 & \Rightarrow x\le 0 & \text{but } x\ge 1, x\in\phi\end{cases}$$

$$gog(x)=\begin{cases}x & -1<x<1\\ x-1 & x\ge 1\\ 1+x & x\le -1\end{cases}$$

ILLUSTRATION 103

If $f(x)$ is defined on [0, 1] by the rule

$f(x)=\begin{cases}x, & \text{if } x\in Q\\ 1-x, & \text{if } x\notin Q\end{cases}$ then, for all $x\in[0,1]$, then compute $f(f(x))$.

SOLUTION

We have

$$f(x)=\begin{cases}x & ; & \text{if } x\in Q\\ 1-x & ; & \text{if } x\notin Q\end{cases}$$

As $x\in Q, f(x)=x$ so $f(x)\in Q$

As $x\notin Q$ then $1-x\notin Q$ it means $f(x)\notin Q$

$$\therefore \quad f(f(x))=\begin{cases}f(x), & f(x)\in Q\\ 1-f(x), & f(x)\notin Q\end{cases}$$

$$=\begin{cases}x, & x\in Q\\ 1-(1-x), & x\notin Q\end{cases}=\begin{cases}x, & x\in Q\\ x, & x\notin Q\end{cases}$$

$f(f(x))=x$ for all $x\in[0,1]$.

ILLUSTRATION 104

If f, g, h are functions from R to R such that $f(x)=x^2-1$, $g(x)=\sqrt{x^2+1}$, $h(x)=\begin{cases}0 & x<0\\ x & x\ge 0\end{cases}$ then find the composite function $ho(fog)$ and determine whether the function fog is invertible and the function h is the identity function.

SOLUTION

Here, $f(x)=x^2-1$ for all x and $g(x)=\sqrt{x^2+1}$ for all x

$$\therefore \quad f\{g(x)\}=\{g(x)\}^2-1$$

$$=x^2+1-1=x^2 \text{ for all } x$$

$$\therefore \quad h\{f(g(x))\}=h(x^2)=x^2$$

Because $x^2 \geq 0$ [from definition of $h(x)$.]

Now, $f\{g(x)\} = x^2$ for all x

As $x^2 \geq 0$, $(fog)(x)$ cannot be negative.

So *fog* is not an onto function.

Hence, *fog* is not invertible.

Again, $h(x) = x$ for $x \geq 0$.

But, by definition $h(x) \neq x$ for $x < 0$.

Hence, h is not the identity function.

Test Your Concepts-VI

(Solutions on page H.16)

1. If $f(x)$ is defined on (0, 1), then the domain of definition of $f(e^x) + f(\ln|x|)$ is:

2. The function $f(x)$ is defined in [0, 1]. Find the domain of $f(\cot x)$.

3. Let $f(x) = \tan x,\ x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ and $g(x) = \sqrt{1-x^2}$.

Determine *fog* and *gof*.

4. Let $f(x) = \begin{cases} 1+x, & 0 \leq x \leq 2 \\ 3-x, & 2 < x \leq 3 \end{cases}$, find $fof(x)$.

5. Let $f(x) = \dfrac{1}{1-x}$, $g(x) = fofofofofofof(x)$ and $h(x) = \tan^{-1}(g(-x^2 - x))$, then find $h(0)$.

6. If domain of $f(x)$ is [0, 1], then find domain of $f(\{x\}^3 + 1)$, where {.} represents fractional part function,

7. Let $f(x) = 2^x$ and $g(x) = x^2$, then find number of solution of $fog = gof$.

8. Let $f(x)$ be a function defined on [0, 1] such that

$$f(x) = \begin{cases} x; & x \in Q \\ 1-x; & x \notin Q \end{cases}.$$

If $fof(x) = p + qx,\ \forall\, x \in [0, 1]$ where $p, q \in I$, then find the value of $(p^2 + 2q)$.

9. Let $f(x) = x^2 + 1$ and $g(x) = \dfrac{1}{x-1}$, then find:

(i) $f(g(2))$ (ii) $f(g(3))$ (iii) $f(g(4))$ (iv) $g\left(f\left(\dfrac{1}{3}\right)\right)$

10. If $f(x) = [x]$ and $g(x) = |x|$ then find:

(i) If $x \in (0, 1)$, $\text{sgn}(fog(x)) + \text{sgn}(gof(x))$

(ii) $fog\left(\dfrac{7}{2}\right) + gof\left(\dfrac{-3}{2}\right)$

11. Let $f(x) = \dfrac{1}{x}$, $g(x) = \dfrac{1}{4x^2 - 1}$ and $h(x) = \dfrac{5x}{x+2}$ be three functions and $k(x) = h(g(f(x)))$. If domain and range of $k(x)$ are $R - A$ respectively where R is the set of real numbers, then find number of integers in set A.

12. Let $f(x) = \begin{cases} x+1; & -1 \leq x \leq 0 \\ 3-x; & 0 < x \leq 1 \end{cases}$ and $g(x) = \begin{cases} 3x+1; & 0 \leq x \leq 1 \\ 4x; & -1 \leq x < 0 \end{cases}$,

then number of integers in domain of $gof(x)$ is:

13. Let $f(x) = \sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$ and $g(\sin 2t) = \sin t + \cos t\ \ \forall\, t \in \left[\dfrac{-\pi}{4}, \dfrac{\pi}{4}\right]$. If the range of $g(f(x))$ is $[a, b]$, then find the value of $(a^2 + b^2)$.

Answer key

1. $(-e, -1)$ **2.** $x \in \left[n\pi + \dfrac{\pi}{4}, n\pi + \dfrac{\pi}{2}\right]$ **3.** $(fog)x = \tan\sqrt{1-x^2}$; $(gof)x = \sqrt{1-\tan^2 x}$

4. $fof(x) = \begin{cases} 2+x, & 0 \leq x \leq 1 \\ 2-x, & 1 < x \leq 2 \\ 4-x, & 2 < x \leq 3 \end{cases}$ **5.** $\dfrac{\pi}{4}$ **6.** $x \in I$ **7.** 2 **8.** 2

9. (i) $f(g(2)) = 2$ (ii) $f(g(3)) = \dfrac{5}{4}$ (iii) $f(g(4)) = \dfrac{5}{16}$ (iv) $g\left(f\left(\dfrac{1}{3}\right)\right) = 9$

10. (i) 0, (ii) 5 **11.** (i) 5, (ii) 7 **12.** 2 **13.** 2

INVERSE FUNCTION ($f^{-1}(x)$)

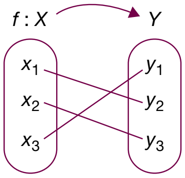
$f : X \to Y$ be a bijective (one-one and onto) function then there exist a unique function $g : Y \to X$ such that $f(x) = y \Leftrightarrow g(y) = x\ \ \forall\, x \in X$ and $y \in Y$. Then g is called Inverse function of f (inverse mapping of f).

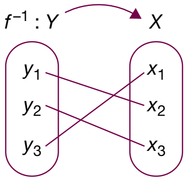
We do write $g = f^{-1}: Y \to X$

It may be clear from mapping diagram.

Domain of $f = \{x_1, x_2, x_3\}$ = Range of f^{-1}

Range of $f = \{y_1, y_2, y_3\}$ = domain of f^{-1}





To Find out Inverse Function:

- Check that $y = f(x)$ is bijective (one-one and onto)
- Find x in terms of y.
- Replace x by $f^{-1}(x)$ and y by x (because x is always taken as independent variable)

Properties of Inverse Function

(i) The inverse of a bijection is unique and the inverse of a bijection is also a bijection.

(ii) All even functions cannot have their inverse defined as they are many-one.

(iii) All periodic functions are many-one so they are non-invertible function.

(iv) If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are identity functions on the sets A and B, respectively.

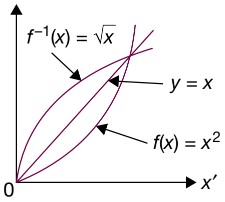
(v) Note that the graphs of f and g are the mirror images of each other in the line $y = x$.

(vi) If f and g are two bijections $f : A \to B$, $g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$

EXAMPLE:

$$f : [0, \infty) \to [0, \infty) \text{ and } f(x) = x^2$$

$$f^{-1}: [0, \infty) \to [0, \infty)\ f^{-1}(x) = \sqrt{x}$$



As shown in the figure, clearly we can say that f and f^{-1} are mirror image of each other with respect to line $y = x$.

ILLUSTRATION 105

$f : (-\infty, 2] \to A$, $f(x) = x^2 - 4x$ if $f(x)$ is invertible find A, also find $f^{-1}(x)$.

SOLUTION

$$y = f(x) = x^2 - 4x \Rightarrow x^2 - 4x - y = 0$$

$$\Rightarrow \quad x = \frac{4 \pm \sqrt{16 + 4y}}{2} = 2 \pm \sqrt{4 + y}$$

As $x \in (-\infty, 2]$ or $x \le 2$, we pick negative sign

$$x = 2 - \sqrt{4 + y}\text{, where } y + 4 \ge 0$$

$$\Rightarrow \quad y \ge -4 = \text{Range} = \text{co-domain } (A)$$

we get $A = [-4, \infty)$

$$x = 2 - \sqrt{4 + y} \text{ replacing } x \text{ by } f^{-1} \text{ and } y \text{ by } x$$

$$\Rightarrow \quad f^{-1} : [-4, \infty) \to (-\infty, 2]$$

$$f^{-1}(x) = 2 - \sqrt{4 + x}$$

ILLUSTRATION 106

$f(x) = \ln\left(x + \sqrt{1 + x^2}\right)$ if $f^{-1}(x)$ exists then find it.

SOLUTION

$$y = \ln\left(x + \sqrt{1 + x^2}\right)$$

$$\Rightarrow \quad e^y = x + \sqrt{1 + x^2} \Rightarrow e^y - x = \sqrt{1 + x^2}$$

$$\Rightarrow \quad e^{2y} + x^2 - 2xe^y = 1 + x^2 \Rightarrow e^{2y} - 1 = 2xe^y$$

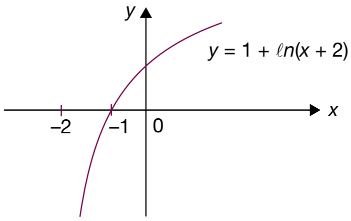
$$\Rightarrow \quad x = \frac{e^{2y} - 1}{2e^y} = \frac{1}{2}[e^y - e^{-y}]$$

Thus, $f^{-1}(x) = \frac{1}{2}(e^x - e^{-x})$

ILLUSTRATION 107

$f : (-2, \infty) \to R$, $f(x) = 1 + \ln(x + 2)$ find $f^{-1}(x)$ if exists.

SOLUTION



$$f(x) = 1 + \ln(x + 2)$$

Range $\in (-\infty, \infty)$

By observing the graph of $f(x)$ is one-one and onto

$$y = 1 + \ln(x + 2) \Rightarrow \ln(x + 2) = y - 1$$

$$\Rightarrow \quad x + 2 = e^{y-1} \Rightarrow x = e^{y-1} - 2$$

Thus, $f^{-1}(x) = e^{x-1} - 2$

ILLUSTRATION 108

$f : R \to R$, $f(x) = x^3 + (a + 2)x^2 + 3ax + 5$ is $f(x)$ an invertible function find a.

SOLUTION

Any odd degree polynomial is always surjective in R

Domain $x \in R$, Range $= R$

$f(x)$ is always onto for $\forall\ a \in R$

$f'(x) = 3x^2 + 2(a + 2)x + 3a \ge 0\ \forall\ a \in R$ for one-one

Graph of $f(x)$ should be as shown.

Discriminant of $f'(x)$, $D \le 0$

$$4(a + 2)^2 - 4(3)3a \le 0$$

$$\Rightarrow \quad 4[a^2 + 4 + 4a - 9a] \le 0$$

$$\Rightarrow \quad a^2 - 5a + 4 \le 0$$

$$\Rightarrow \quad (a - 1)(a - 4) \le 0$$

$$\Rightarrow \quad a \in [1, 4] \text{ one-one}$$

$\therefore f(x)$ is bijective $a \in [1, 4]$

ILLUSTRATION 109

Find the inverse of the function $f(x) = \begin{cases} x & ; \ x < 1 \\ x^2 & ; \ 1 \le x \le 4 \\ 8\sqrt{x} & ; \ x > 4 \end{cases}$

SOLUTION

Let $y = \begin{cases} x & ; \ x < 1 \\ x^2 & ; \ 1 \le x \le 4 \\ 8\sqrt{x} & ; \ x > 4 \end{cases} = \begin{cases} x & ; \ x < 1 \\ x^2 & ; \ 1 \le x^2 \le 16 \\ 8\sqrt{x} & ; \ 8\sqrt{x} > 16 \end{cases}$

$$\Rightarrow \quad x = \begin{cases} y & ; \ y < 1 \\ \sqrt{y} & ; \ 1 \le y \le 16 \\ y^2/64 ; & y > 16 \end{cases}$$

Let $f(x) = y$

$\therefore \quad x = f^{-1}(y)$

$$\therefore \quad f^{-1}(x) = \begin{cases} x & ; \ x < 1 \\ \sqrt{x} & ; \ 1 \le x \le 16 \\ x^2/64 ; & x > 16 \end{cases}$$

ILLUSTRATION 110

Let $f : N \to N$ where $f(x) = x + (-1)^{x-1}$. Find the inverse of f.

SOLUTION

We have, $f(x) = x + (-1)^{x-1}$

$$y = \begin{cases} x - 1, & x \text{ is even} \\ x + 1, & x \text{ is odd} \end{cases}$$

$$x = \begin{cases} y + 1, & y \text{ is odd} \\ y - 1, & y \text{ is even} \end{cases}$$

$$f^{-1}(x) = x + (-1)^{x-1}; \ x \in N.$$

Hence, the inverse of f is f itself.

Alternate:

We have, $f(x) = x + (-1)^{x-1}$

Putting successively $x = 1, 2, 3, \ldots$, we get:

$f(1) = 1 + 1 = 2; f(2) = 2 + (-1) = 1;$

$f(3) = 3 + (-1)^2 = 3 + 1 = 4; f(4) = 4 + (-1)^3 = 3$

$f(5) = 5 + 1 = 6; f(6) = 6 - 1 = 5; \ldots$

From above, we see that the graph of $f(x)$ consists of the points:

$(1, 2), (2, 1), (3, 4), (4, 3), (5, 6), (6, 5), \ldots$.

Thus, if (a, b) is a point on the graph, then (b, a) is also a point on the graph.

Hence, the inverse of is f itself, i.e.,

$$f^{-1}(x) = x + (-1)^{x-1}; \ x \in N.$$

ILLUSTRATION 111

If the functions $f(x)$ and $g(x)$ are defined on $R \to R$ such that $f(x) = \cos^2 x + \cos^2\left(x + \frac{2\pi}{3}\right) + \cos^2\left(x - \frac{2\pi}{3}\right) + \cos^2 \frac{5\pi}{4}$ and $g(2) = -1$, then find the value of $\cot^{-1}(g(f(1))) + \cos^{-1}(g(f(2)))$

SOLUTION

$$f(x) = \cos^2 x + \cos^2\left(x + \frac{2\pi}{3}\right) + \cos^2\left(x - \frac{2\pi}{3}\right) + \cos^2 \frac{5\pi}{4}$$

$$\Rightarrow \quad f(x) = \cos^2 x + \left(\frac{-\cos x}{2} - \frac{\sin x\sqrt{3}}{2}\right)^2 + \left(\frac{\cos x}{-2} + \frac{\sin x\sqrt{3}}{2}\right)^2 + \frac{1}{2}$$

$$\Rightarrow \quad f(x) = \cos^2 x + \frac{1}{4}[\cos^2 x + 3\sin^2 x + 2\sqrt{3}\ \sin x \cos x + \cos^2 x + 3\sin^2 x - 2\sqrt{3}\sin x \cos x] + \frac{1}{2}$$

$$\Rightarrow \quad f(x) = \frac{3}{2}\cos^2 x + \frac{3}{2}\sin^2 x + \frac{1}{2}$$

$$\Rightarrow \quad f(x) = 2 \ \forall \ x \in R.$$

So, $\cot^{-1}(-1) + \cos^{-1}(-1) = \frac{3\pi}{4} + \pi = \frac{7\pi}{4}$

ILLUSTRATION 112

Let $f(x) = \frac{3}{2} + \sqrt{x - \frac{7}{4}}$ and $g(x)$ be the inverse function of $f(x)$, then find the value of $(f^{-1}og^{-1})$ (17).

SOLUTION

$$f(x) = \frac{3}{2} + \sqrt{x - \frac{7}{4}}$$

$$\Rightarrow \quad (f^{-1}og^{-1})(x) = f^{-1}(g^{-1}(x)) = f^{-1}(f(x)) = x$$

$$\therefore \quad (f^{-1}og^{-1}(17) = 17.$$

ILLUSTRATION 113

If $f(x) = \frac{x|x|}{1+x^2}$ then show that $f^{-1}(x) = \text{sgn}(x) \cdot \sqrt{\frac{|x|}{1-|x|}}$; $-1 < x < 1$

SOLUTION

$$f(x) = \frac{x|x|}{1+x^2}$$

For $x \ge 0$, $y = \frac{x|x|}{1+x^2} = \frac{x^2}{1+x^2} \Rightarrow x^2(y - 1) = -y \Rightarrow x^2 = \frac{y}{1-y}$

$$x = \sqrt{\frac{y}{1-y}}$$

when $x < 0$, $y = \frac{x|x|}{1+x^2} = \frac{-x^2}{1+x^2}$

$$\Rightarrow \quad x^2(y+1) = -y \Rightarrow x^2 = -\frac{y}{1+y} \Rightarrow x = -\sqrt{\frac{y}{1+y}}$$

$$\Rightarrow \quad x = \begin{cases} \sqrt{\dfrac{y}{1-y}}, 0 \le y < 1 \\ -\sqrt{\dfrac{-y}{1+y}}, -1 < y < 0 \end{cases}$$

$$\Rightarrow \quad f^{-1}(x) = \begin{cases} \sqrt{\dfrac{x}{1-x}}; 0 \le x < 1 \\ -\sqrt{\dfrac{-x}{1+x}}; -1 < x < 0 \end{cases}$$

$$f^{-1}(x) = \text{sgn}(x).\sqrt{\frac{|x|}{1-|x|}}; -1 < x < 1$$

ILLUSTRATION 114

Let $f : [1, \infty) \to [3, \infty)$ be defined as $f(x) = (\log_2 x)^2 + 2(\log_2 x) + 3$. If f^{-1} is the inverse function of f, then find $f^{-1}(x)$.

SOLUTION

$$f(x) = (\log_2 x)^2 + 2(\log_2 x) + 3$$

$$\Rightarrow \quad f(x) = ((\log_2 x) + 1)^2 + 2$$

$$\Rightarrow \quad y - 2 = (\log_2 x + 1)^2 \Rightarrow \log_2 x = -1 \pm \sqrt{y-2}$$

$$\Rightarrow \quad x = 2^{-1 \pm \sqrt{y-2}}$$

We take positive sign as $x \in [1, \infty]$

$$\Rightarrow \quad f^{-1}(x) = 2^{-1+\sqrt{x-2}}$$

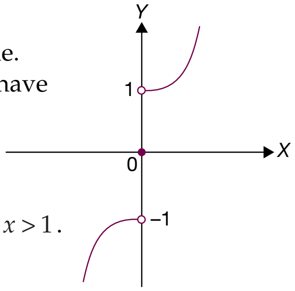
ILLUSTRATION 115

Find the inverse of the function. $f(x) = (1 + x^2)\ \text{sgn}(x)$. (where sgn is signum function)

SOLUTION

We have, $y = \left(1+x^2\right)\text{sgn}(x) = \begin{cases} 1+x^2 & ,x > 0 \\ 0 & ,x = 0 \\ -\left(1+x^2\right) & ,x < 0. \end{cases}$

The graph of the function is shown as below:



From the graph, we see that f is one-one and hence invertible.

Now, for the interval $x > 0$, we have

$y = 1 + x^2 \Rightarrow x^2 = y - 1$

$\Rightarrow x = \sqrt{y-1}; y > 1.$

Hence, its inverse is $y = \sqrt{x-1}; x > 1$.

For the interval $x < 0$

We have $y = -\left(1+x^2\right); y < -1 \Rightarrow x^2 + 1 = -y \Rightarrow x^2 = -y - 1$

$$\Rightarrow \quad x = -\sqrt{-(y+1)};\ y < -1.$$

Hence, its inverse is $y = -\sqrt{-(x+1)}; x < -1$.

Thus, the inverse of the given function is

$$f^{-1}(x) = \begin{cases} \sqrt{x-1} & ; x > 1 \\ 0 & ; x = 0 \\ -\sqrt{-(x+1)} & ; x < -1. \end{cases}$$

The graph of the inverse of $f(x)$ is the mirror image of the given function in the line $y = x$ as shown below:

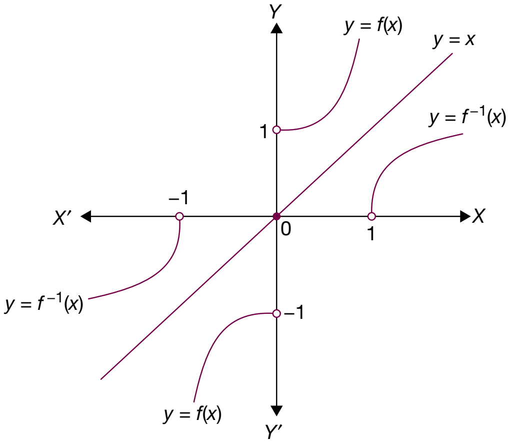


ILLUSTRATION 116

Solve $x^2 + 2x = -1 + \sqrt{1+x}$ if $x \ge -1$

SOLUTION

$$x^2 + 2x = -1 + \sqrt{1+x}$$

$$\Rightarrow \quad x^2 + 2x + 1 = \sqrt{1+x} \qquad \text{if } x \ge -1$$

$$\Rightarrow \quad (x+1)^2 = \sqrt{1+x}$$

$$\Rightarrow \quad \sqrt{1+x}\ ((1+x)^{3/2} - 1)) = 0$$

$$\sqrt{1+x} = 0 \Rightarrow x = -1$$

$$\Rightarrow \quad (1+x)^{3/2} - 1 = 0$$

$$\Rightarrow \quad (1+x)^{3/2} = 1, 1 + x = 1, x = 0$$

Hence, final solution $x = 0, -1$

Alternate:

Let $y = f(x) = x^2 + 2x$

$$\Rightarrow \quad x^2 + 2x - y = 0$$

$$\Rightarrow \quad x = \frac{-2 \pm \sqrt{4+4y}}{2} = -1 \pm \sqrt{1+y} \text{ given } x \ge -1$$

$$\Rightarrow \quad x = -1 + \sqrt{1+y}$$

$$f^{-1}(x) = -1 + \sqrt{1+x}$$

$$x^2 + 2x = -1 + \sqrt{1+x}$$

Given equation is $f(x) = f^{-1}(x)$: then solution of given equations are the same as solution of equation.

$$f(x) = f^{-1}(x) = x$$

$\Rightarrow \quad x^2 + 2x = x \Rightarrow x^2 + x = 0$

$\Rightarrow \quad x = -1, 0$

Hence, final solution $x \in \{-1, 0\}$

STANDARD FUNCTIONAL EQUATIONS

If x, y are independent variables then:

(i) If $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$

(ii) If $f(x \,.\, y) = f(x) + f(y)$, then $\Rightarrow f(x) = k \log x$

(iii) If $f\left(\dfrac{x}{y}\right) = f(x) - f(y) \Rightarrow f(x) = k \log x$

(iv) If $f(xy) = f(x) \,.\, f(y)$, then $\Rightarrow f(x) = x^n$, $n \in R$

(v) If $f(x + y) = f(x) \,.\, f(y) \Rightarrow f(x) = a^{kx}$

(vi) If $f(x + y) = f(x) + f(y) \Rightarrow f(x) = k$, here k is constant

(vii) By considering $f(x)$ a general n^{th} degree polynomial and $f(x) \,.\, f\left(\dfrac{1}{x}\right) = f(x) + f\left(\dfrac{1}{x}\right) \Rightarrow f(x) = 1 \pm x^n$

ILLUSTRATION 117

If $f(x + y) = f(x) + f(y)\ \forall\ x, y \in R$ $f(1) = 3$, then find $\sum_{r=1}^{10} f(r)$.

SOLUTION

$$f(x) = kx;\ f(1) = k(1) = 3$$

$$f(x) = 3x$$

$$\sum_{r=1}^{10} f(r) = \sum_{r=1}^{10} 3r = 3\left(\sum_{r=1}^{10} r\right)$$

$$= 3(1 + 2 + 3 + \ldots + 9 + 10) = 3\left(\frac{10 \cdot 11}{2}\right) = 165$$

ILLUSTRATION 118

If $f(x)$ is polynomial and satisfies $f(x) + f\left(\dfrac{1}{x}\right) = f(x)\, f\left(\dfrac{1}{x}\right)$ and $f(3) = -26$, then find $f(4)$.

SOLUTION

$$f(x) = 1 \pm x^n$$

$\Rightarrow \quad f(3) = 1 \pm 3^n = -26 = 1 - 27$ (negative is taken)

$\Rightarrow \quad 3^n = 27 = 3^3 \Rightarrow x = 3$

$\therefore \quad f(x) = 1 - x^3$

$\Rightarrow \quad f(4) = 1 - 4^3 = 1 - 64 = -63$

ILLUSTRATION 119

If $3\,f(x) - f\left(\dfrac{1}{x}\right) = \log_e x^4$ for $x > 0$, then find $f(e^x)$.

SOLUTION

$$3f(x) - f\left(\frac{1}{x}\right) = 4 \ln x \qquad \ldots(1)$$

Replace $x \rightarrow \dfrac{1}{x}$

$$3f\left(\frac{1}{x}\right) - f(x) = -4 \ln x \qquad \ldots(2)$$

Eq. (1) × 3 + Eq. (2), we get

$$9f(x) - 3f\left(\frac{1}{x}\right) + 3f\left(\frac{1}{x}\right) - f(x) = 12 \ln x - 4 \ln x$$

$\Rightarrow \quad 8f(x) = 8 \ln x \Rightarrow f(x) = \ln x$

Now $f(e^x) = \ln e^x = x$

ILLUSTRATION 120

If $f(x) = \dfrac{9^x}{9^x + 3}$ for all $x \in R$, then find the sum $f\left(\dfrac{1}{1996}\right) + f\left(\dfrac{2}{1996}\right) + \ldots + f\left(\dfrac{1995}{1996}\right)$

SOLUTION

$$f(x) = \frac{9^x}{9^x + 3} \Rightarrow f(1 - x) = \frac{3}{3 + 9^x}$$

$\therefore \quad f(x) + f(1 - x) = 1$

Given expression

$$\left[f\left(\frac{1}{1996}\right) + f\left(\frac{1995}{1996}\right)\right] + \left[f\left(\frac{2}{1996}\right) + f\left(\frac{1994}{1996}\right)\right] + \ldots + \left[f\left(\frac{997}{1996}\right) + f\left(\frac{999}{1996}\right)\right] + \left(\frac{998}{1996}\right)$$

$$= [1 + 1 + \ldots \text{ upto 997 times}] + \frac{1}{2} = 997.5$$

ILLUSTRATION 121

If $g(x)$ is a polynomial function satisfying $g(x)\, g(y) = g(x) + g(y) + g(xy) - 2$ for all $x, y \in R$ and $g(2) = 5$, then find $g(5)$.

SOLUTION

Put $x = 1$; $y = 2$

$\Rightarrow \quad g(1)\, g(2) = g(1) + g(2) + g(2) - 2$

$\Rightarrow \quad 5g(1) = g(1) + 5 + 5 - 2$

$\Rightarrow \quad 4g(1) = 8 \Rightarrow g(1) = 2$

In given relation, put $y = \frac{1}{x}$

$$g(x)\,g\left(\frac{1}{x}\right) = g(x) + g\left(\frac{1}{x}\right) + g(1) - 2$$

$$\Rightarrow \quad g(x)\,g\left(\frac{1}{x}\right) = g(x) + g\left(\frac{1}{x}\right)$$

$$\Rightarrow \quad g(x) = 1 \pm x^n$$

$$\Rightarrow \quad g(2) = 5 = 1 \pm 2^n,$$

Thus, $g(x) = 1 + x^2$

$$\Rightarrow \quad g(5) = 1 + 5^2 = 26$$

ILLUSTRATION 122

If $f(x) = \cos(\log x)$, then compute $f(x)f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\}$

SOLUTION

$$f(x)\,f(y) - \frac{1}{2}\left\{f\left(\frac{x}{y}\right) + f(xy)\right\}$$

$$= \cos(\ln x)\cos(\ln y) - \frac{1}{2}\{\cos(\ln x - \ln y) + \cos(\ln x + \ln y)\}$$

$$= \cos(\ln x)\cos(\ln y) - \frac{1}{2}\{2\cos(\ln x)\cos(\ln y)\} = 0$$

ILLUSTRATION 123

A function $f: R \to R$ is defined by $f(x+y) - kxy = f(x) + 2y^2 \ \forall x, y \in R$ and $f(1) = 2$ where 'k' is some real constant then find $f(x+y).f\left(\frac{1}{x+y}\right)$

SOLUTION

$$f(x+y) - kxy = f(x) + 2y^2 \ \forall\ x, y \in R$$

Put $x = 0 \Rightarrow f(y) = f(0) + 2y^2$

$$\Rightarrow \quad f(x) = 2x^2 + f(0)$$

Now $f(1) = 2 \Rightarrow f(0) = 0$

$$\therefore \quad f(x) = 2x^2$$

Hence $f(x+y) \times f\left(\frac{1}{x+y}\right) = 2(x+y)^2 \cdot \frac{2}{(x+y)^2} = 4$

ILLUSTRATION 124

If $f: R \to R$ satisfying $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$, for all $x, y \in R$, then find $f(10)$.

SOLUTION

Given,

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1 \qquad \ldots(1)$$

Putting $x = 0$ and $f(y) = 0$, we get

$$f(0) = f(0) + f(0) - 1 \Rightarrow f(0) = 1 \qquad \ldots(2)$$

Putting $x = a$ and $f(y) = a$, we get

$$f(0) = f(a) + a^2 + f(a) - 1$$

$$\Rightarrow \quad 1 = a^2 + 2f(a) - 1 \Rightarrow f(a) = 1 - \frac{a^2}{2}$$

$$\Rightarrow \quad f(x) = 1 - \frac{x^2}{2} \qquad \text{[Putting } x \text{ in place of } a\text{]}$$

$$\therefore \quad f(10) = 1 - \frac{10^2}{2} = -49$$

ILLUSTRATION 125

Let $f: N \to R$ be such that:

(i) $f(1) = 1$

(ii) $f(1) + 2f(2) + 3f(3) + \ldots + nf(n) = n(n+1)f(n)$, for $n \geq 2$, then find $\frac{1}{f(2010)}$

SOLUTION

Given,

$$f(1) + 2f(2) + 3f(3) + \ldots + nf(n) = n(n+1)f(n) \quad \ldots(1)$$

Putting $(n+1)$ is place of n, we get

$$f(1) + 2f(2) + \ldots + nf(n) + (n+1)f(n+1) = (n+1)(n+2)f(n+1) \qquad \ldots(2)$$

Eq. (2) – (1), we get

$$(n+1)f(n+1) = (n+1)[(n+2)f(n+1) - nf(n)]$$

$$\Rightarrow \quad f(n+1) = (n+2)f(n+1) - nf(n)$$

$$\Rightarrow \quad (n+1)f(n+1) - nf(n) = 0$$

$$\Rightarrow \quad nf(n) = (n+1)f(n+1)$$

$$\Rightarrow \quad 2f(2) = 3f(3) = \ldots = nf(n)$$

From (1).

$$f(1) + [nf(n) + nf(n) \ldots \text{to } (n-1) \text{ terms}] = n(n+1)f(n)$$

$$\Rightarrow \quad f(1) + (n-1)nf(n) = n(n+1)f(n) \Rightarrow f(1) = 2n\,f(n)$$

$$\Rightarrow \quad \frac{1}{f(n)} = 2n \qquad \therefore \quad \frac{1}{f(2010)} = 4020$$

ILLUSTRATION 126

Let $f(x + y + 1) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$ for all $x, y \in R$ and $f(0) = 1$. Find $f(x)$.

SOLUTION

Given $f(x + y + 1) = \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$

Putting $y = 0$, we get $f(x+1) = \left(\sqrt{f(x)} + 1\right)^2$;

Putting $x = 0$, we have $f(1) = (1+1)^2 = 2^2$;

Putting $x = 1$, $f(2) = (2+1)^2 = 3^2$;

Putting $x=2$, $f(3)=\left(\sqrt{f(2)}+1\right)^2=(3+1)^2=4^2$ and so on.

Proceeding in this way, we get $f(x)=(x+1)^2$.

ILLUSTRATION 127

If a polynomial function f satisfies the relation

$$\log_2(f(x))=\log_2\left(2+\frac{2}{3}+\frac{2}{9}+\ldots\infty\right)\cdot\log_3\left(1+\frac{f(x)}{f\left(\frac{1}{x}\right)}\right) \text{ and}$$

$f(10)=1001$, then find the value of $f(20)$.

SOLUTION

$$\log_2(f(x))=\log_2\left(2+\frac{2}{3}+\frac{2}{9}+\ldots\infty\right)\cdot\log_3\left(1+\frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$$

$$\Rightarrow \quad \log_2(f(x))=\log_2\left(\frac{2}{1-\frac{1}{3}}\right)\cdot\log_3\left(1+\frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$$

$$\Rightarrow \quad \log_2(f(x))=\log_2 3\cdot\log_3\left(1+\frac{f(x)}{f\left(\frac{1}{x}\right)}\right)$$

$$\Rightarrow \quad \log_2(f(x))=\log_2\left(1+\frac{f(x)}{f\left(\frac{1}{x}\right)}\right)\Rightarrow f(x)=1+\frac{f(x)}{f\left(\frac{1}{x}\right)}$$

$$\Rightarrow \quad f(x)\times f\left(\frac{1}{x}\right)=f\left(\frac{1}{x}\right)+f(x)$$

$$\Rightarrow \quad f(x)=1\pm x^n$$

$$\therefore \quad f(10)=1\pm 10^n=1001$$

$$\Rightarrow \quad \pm 10^n=1000$$

$\Rightarrow$ Positive sign must be taken and $n=3$.

$$\therefore \quad f(x)=1+x^3$$

$$f(20)=1+(20)^3=8001.$$

Test Your Concepts-VII

(Solutions on page H.17)

1. If $f:(3,6)\to(2,5)$ is a function defined by $f(x)=x-\left[\frac{x}{3}\right]$ (where [.] denotes the greatest integer function), then find $f^{-1}(x)$.
2. If $f(x)=1+\alpha x, \alpha\neq 0$ is the inverse of itself, then $[|\alpha|]=$ ([.] denotes greatest integer function):
3. If $x^4f(x)-\sqrt{1-\sin(2\pi x)}=|f(x)|-2f(x)$, then $f(-2)$ equals:
4. Let $f:R\to R$, $f(x)=2x^3-3(k+2)x^2+12kx-7$, $-4\le k\le 6$, $k\in I$, then find the value of k for which $f(x)$ is invertible.
5. Let $f(x)=\ln\left(x+\sqrt{1+x^2}\right)$, then find the value of $f^{-1}(\ln 7)$.
6. If $x=\log_4\left(\frac{2f(x)}{1-f(x)}\right)$, then find $f(2010)+f(-2009)$.
7. Let $f(x)$ be a function such that $2f(x)+f(-x)=\cos x$ for every real number 'x'. Find the value of $f(\pi)$.
8. If $3f(x)-f\left(\frac{1}{x}\right)=\log_e x^4$ for $x>0$; then find $f(e^x)$.
9. If $f(x)=\frac{2015x+123}{138x-2015}$ and $g(x)=f(f(x))+f\left(f\left(\frac{1}{x}\right)\right)$, $\forall x$ then find least value of $g(x)$ is:
10. Let $f:R-\left\{\frac{1}{2}\right\}\to R-\left\{\frac{1}{2}\right\}$, $f(x)=\frac{x-2}{2x-1}$ be a function such that $x=m$ is the solution of $f(x)+2f^{-1}(x)+2=(f(f(x))$, then find m.
11. Let $f(x)$ be an even function satisfying $f(x)+f(x+2)=f(x+1)$ $\forall x\in R$, where $f(-13)=4$, then find $f(-5)+f(1)$.
12. If $f(x)=\frac{2}{4^x+2}$ for real numbers x, then find $f\left(\frac{1}{2011}\right)+f\left(\frac{2}{2011}\right)+\ldots+f\left(\frac{2010}{2011}\right)$

Answer key

1. $x+1$ **2.** 1 **3.** $\frac{1}{17}$ **4.** 2 **5.** $\frac{24}{7}$ **6.** 1 **7.** $\frac{-1}{3}$ **8.** x

9. 2 **10.** 2 **11.** 8 **12.** 1005

SOLVED PROBLEMS

PROBLEM 1

If $f(x)=\begin{cases} x^3+1, & x<0 \\ x^2+1, & x\ge 0\end{cases}$ and $g(x)=\begin{cases}(x-1)^{1/3}, & x<1 \\ (x-1)^{1/2}, & x\ge 1\end{cases}$

Compute $gof(x)$ and deduce that $f(x)$ and $g(x)$ are symmetric about line $y = x$.

SOLUTION

We have

$$gof(x)=g(f(x))=\begin{cases}(f(x)-1)^{1/3}, & f(x)<1 \\ (f(x)-1)^{1/2}, & f(x)\ge 1\end{cases}$$

$$=\begin{cases}(x^3+1-1)^{1/3}, & x^3+1<1 \Rightarrow x<0 \\ (x^2+1-1)^{1/3} & x^2+1<1\Rightarrow x^2<0 \\ & \therefore x\in\phi \\ (x^3+1-1)^{1/2}, & x^3+1\ge 1\Rightarrow x\ge 0 \text{ but } x<0 \\ & \therefore x\in\phi \\ (x^2+1-1)^{1/2} & x^2+1\ge 1\Rightarrow x^2\ge 0\ \forall x\in R \text{ but } x\ge 0 \\ & \therefore x\ge 0\end{cases}$$

$$=\begin{cases}x, & x<0 \\ x, & x\ge 0\end{cases}=x \text{ for all } x.$$

Hence, $gof(x) = x$ for all x.

It should be true because if we closely observe $g(x)=f^{-1}(x)$ it means they are symmetric about straight line $y = x$

PROBLEM 2

Let $f_1(x)=\dfrac{x}{3}+10$ for all $x\in R$, and $f_n(x)=f_1(f_{n-1}(x))$ for $n\ge 2$. Then find $f_n(x)$.

SOLUTION

We have

$$f_n(x)=f_1(f_{n-1}(x)),\ n\ge 2$$

$$\Rightarrow\quad f_2(x)=f_1(f_1(x))=\frac{1}{3}f_1(x)+10$$

$$=\frac{1}{3}\left(\frac{x}{3}+10\right)+10=\frac{x}{3^2}+\frac{10}{3}+10$$

$$f_3(x)=f_1(f_2(x))=\frac{1}{3}f_2(x)+10$$

$$=\frac{1}{3}\left(\frac{x}{3^2}+\frac{10}{3}+10\right)+10$$

$$=\frac{x}{3^3}+\frac{10}{3^2}+\frac{10}{3}+10$$

$$f_4(x)=f_1(f_3(x))=\frac{1}{3}f_3(x)+10$$

$$=\frac{1}{3}\left(\frac{x}{3^3}+\frac{10}{3^2}+\frac{10}{3}+10\right)+10$$

$$=\frac{x}{3^4}+\frac{10}{3^3}+\frac{10}{3^2}+\frac{10}{3}+10$$

Continuing in this manner, we obtain

$$f_n(x)=\frac{x}{3^n}+\frac{10}{3^{n-1}}+\frac{10}{3^{n-2}}+\ldots+\frac{10}{3}+10$$

$$=\frac{x}{3^n}+10\left(\frac{1-\frac{1}{3^n}}{1-\frac{1}{3}}\right)$$

$$=\frac{x}{3^n}+15\left(1-\frac{1}{3^n}\right)=\frac{x-15}{3^n}+15$$

PROBLEM 3

Let $f(x)=ax^3+bx^2+cx+d\,\sin x$. Find the condition that $f(x)$ is always one-one function.

SOLUTION

Given: $f(x)=ax^3+bx^2+cx+d\,\sin x$

For $f(x)$ to be one-one, f should be strictly monotonic.

CASE 1: Either f is strictly increasing

Differentiating (1), we get $f'(x)=3ax^2+2bx+c+d\cos x$

$$\ge 3ax^2+2bx+c-|d|;\ \text{for } |\cos x|\le 1.$$

If $f'(x)\ge 0$ for all $x\in R$, we must have $a>0$ and

$$D=4b^2-4.3a(c-|d|)\le 0$$

(From the quadratic expression)

$\Rightarrow\quad b^2\le 3a(c-|d|).$

$\therefore\quad f(x)$ is strictly increasing if $a>0$ and $b^2\le 3a(c-|d|)$.

CASE 2: $f(x)$ strictly decreasing.

Also, $f'(x)=3ax^2+2bx+c+d\cos x\le 3ax^2+2bx+c+|d|$

If $f'(x)\ge 0$ for all $x\in R$, $a<0$ and

$$D=4b^2-12a(c+|d|)\le 0\Rightarrow b^2\le 3a(c+|d|).$$

Hence, the required condition is (i) $a>0$, $b^2\le 3a(c-|d|)$ or (ii) $a<0$, $b^2\le 3a(c-|d|)$.

Final condition $b^2\le 3a(c-|d|)$ and $a\ne 0$.

PROBLEM 4

Determine a function f satisfying the functional relation $f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}$.

SOLUTION

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2}{x}-\frac{2}{1-x} \quad \ldots(1)$$

Replacing x by $\frac{1}{1-x}$ in relation (1)

$$\therefore \quad f\left(\frac{1}{1-x}\right)+f\left(\frac{x-1}{x}\right)=2(1-x)+2\left(\frac{1-x}{x}\right) \quad \ldots(2)$$

Replacing x by $\frac{x-1}{x}$ in relation (1)

$$\therefore \quad f\left(\frac{x-1}{x}\right)+f(x)=\frac{2x}{x-1}-2x \quad \ldots(3)$$

Now relation (1) + (3) – (2)

$$\therefore \quad 2f(x)=\frac{2}{x}-\frac{2}{1-x}-2(1-x)-\frac{2(1-x)}{x}+\frac{2x}{x-1}-2x$$

$$=\frac{2x}{x-1}+\frac{2}{x-1}=2\left(\frac{x+1}{x-1}\right)$$

$$\therefore \quad f(x)=\frac{x+1}{x-1}$$

PROBLEM 5

If for all real values of u and v, $2f(u)\cos v=f(u+v)+f(u-v)$, prove that for all real values of x:

(i) $f(x)+f(-x)=2a\cos x$ **(ii)** $f(\pi-x)+f(-x)=0$

(iii) $f(\pi-x)+f(x)=2b\sin x$

Deduce that $f(x)=a\cos x+b\sin x$ where a and b are arbitrary constants.

SOLUTION

Given $f(u+v)+f(u-v)=2f(u)\cos v$ ...(1)

Putting $u=0$ and $v=x$ in (1), we get

$$f(x)+f(-x)=2f(0)\cos x \quad \ldots(2)$$

$$=2a\cos x \text{ where } a=f(0).$$

This proves (i).

Again putting $u=\frac{\pi}{2}-x$ and $v=\frac{\pi}{2}$ in (1), we get

$$f(\pi-x)+f(-x)=0; \qquad \because \cos\frac{\pi}{2}=0.$$

This proves (ii).

Again, putting $u=\frac{\pi}{2}$ and $v=\frac{\pi}{2}-x$ in (1), we get

$$f(\pi-x)+f(x)=2f\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}-x\right)=2b\sin x$$

where $f(\pi/2)=b$ ...(3)

This proves (iii),

Now, adding (2) and (3), we get

$$2f(x)+f(\pi-x)+f(-x)=2a\cos x+2b\sin x$$

$$\Rightarrow \quad 2f(x)+0=2a\cos x+2b\sin x; \text{ from (3)}$$

Hence, $f(x)=a\cos x+b\sin x$.

PROBLEM 6

Let $f(x)$ be a function such that $f(x-1)+f(x+1)=\sqrt{3}\,.\,f(x)\,\forall x\in R$. If $f(5)=10$. Then find the value of $g'(x)$ where $g(x)=\sum_{r=0}^{99}f(5+12r)$.

SOLUTION

$$f(x-1)+f(x+1)=\sqrt{3}.f(x) \quad \ldots(1)$$

Replace x by $x+1$

$$f(x)+f(x+2)=\sqrt{3}.f(x+1) \quad \ldots(2)$$

Again replace x by $x+1$

$$f(x+1)+f(x+3)=\sqrt{3}.f(x+2) \quad \ldots(3)$$

Add equation (1) and (3)

$$f(x-1)+2f(x+1)+f(x+3)=\sqrt{3}\,(f(x)+f(x+2))$$

$$\Rightarrow \quad f(x-1)+f(x+3)=f(x+1) \quad \ldots(4)$$

Replace x by $x+2$

$$f(x+1)+f(x+5)=f(x+3) \quad \ldots(5)$$

Adding (4) and (5)

We get

$$f(x-1)+f(x+5)=0$$

$$f(x+6)=-f(x)$$

Now $f(x+12)=-f(x+6)=f(x)$

$f(x)$ is a periodic function with period '12'

Then $g(x)=\sum_{r=0}^{99}f(5+12r)=100\times f(5)$

$=1000$

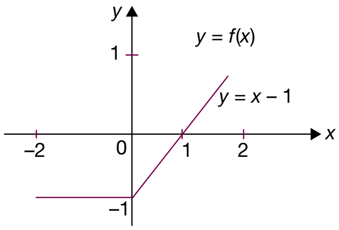
$\Rightarrow \quad g(x)=$ a constant function. Hence $g'(x)=0$

PROBLEM 7

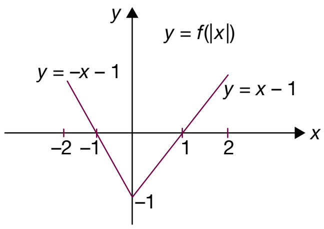
If $f(x)=\begin{cases}-1 & -2\le x\le 0\\ x-1 & 0<x\le 2\end{cases}$ Compute $g(x)=f(|x|)+|f(x)|$

SOLUTION

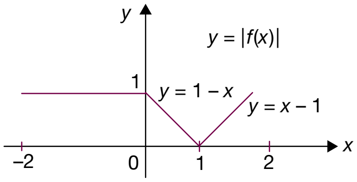
Function $y=f(x)$ is given to us and we take plot it as shown below.



Now for graph of $f(|x|)$, remove the left side of graph of $f(x)$ and take the image of the positive side; we get a graph of $f(|x|)$as shown below.



For graph $|f(x)|$, first we should take the image of portion of graph of $f(x)$ which lies below x axis about x axis.



$$g(x) = f(|x|) + |f(x)| = \begin{cases} -x-1+1 & ; \quad -2 \le x \le 0 \\ x-1+1-x & ; \quad 0 < x < 1 \\ x-1+x-1 & ; \quad 1 \le x \le 2 \end{cases}$$

$$g(x) = \begin{cases} -x & ; \quad -2 \le x \le 0 \\ 0 & ; \quad 0 < x < 1 \\ 2x-2 & ; \quad 1 \le x \le 2 \end{cases}$$

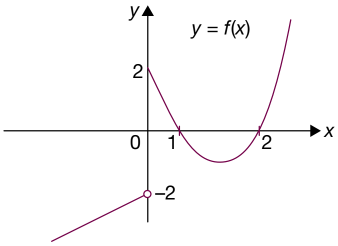
PROBLEM 8

If $f(x) = \begin{cases} x-2 & x < 0 \\ x^2-3x+2 & x \ge 0 \end{cases}$ compute $g(x) = f(|x|) + |f(x)|$

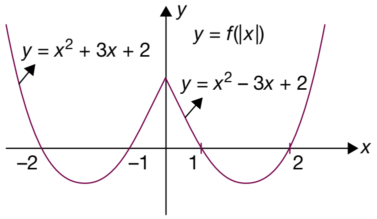
SOLUTION

$$f(x) = x^2 - 3x + 2 = (x-1)(x-2)$$

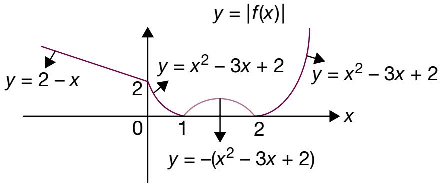
Function $y = f(x)$ is given to us and we take plot it as shown



Now for the graph of $f(|x|)$, remove the left side of the graph of $f(x)$ and take the image of the positive side; we get graph of $f(|x|)$as shown below:



For graph $|f(x)|$, first we should take image of portion of graph of $f(x)$ which lies below x axis about x axis.



$$g(x) = f(|x|) + |f(x)| = \begin{cases} x^2+3x+2+2-x & ; \quad x < 0 \\ 2(x^2-3x+2) & ; \quad 0 \le x \le 1 \\ 0 & ; \quad 1 < x < 2 \\ 2(x^2-3x+2) & ; \quad x \ge 2 \end{cases}$$

$$g(x) = \begin{cases} x^2+2x+4 & ; \quad x < 0 \\ 2x^2-6x+4 & ; \quad 0 \le x \le 1 \text{ and } x \ge 2 \\ 0 & ; \quad 1 < x < 2 \end{cases}$$

PROBLEM 9

Let $f : D \to R$ be defined as $f(x) = \dfrac{x^2+2x+a}{x^2+4x+3a}$ where D and R denote the domain of f and the set all real numbers respectively. If f is surjective mapping, then find a.

SOLUTION

Let $y = \dfrac{x^2+2x+a}{x^2+4x+3a}$ …(1)

$\Rightarrow \quad (y-1)\,x^2 + (4y-2)x + (3ay-a) = 0$ …(2)

If range of eq. (1) is R, then roots of eq. (2) must be real for all $y \in R$.

$$D \ge 0 \Rightarrow (4y-2)^2 - 4(y-1)\,(3ay-a) \ge 0 \;\forall\; y \in R.$$

$\Rightarrow \quad 16y^2 - 16y + 4 - 4(3ay^2 - ay - 3ay + a) \ge 0$

$\Rightarrow \quad (4-3a)y^2 - 4(1-a)\,y + (1-a) \ge 0 \;\forall\; y \in R.$ …(3)

Now, eq. (3) is true $\forall\; y \in R$,

If $4 - 3a > 0 \Rightarrow a < \dfrac{4}{3}$ and $16(1-a)^2 - 4(4-3a)\,(1-a) \le 0$

$\Rightarrow \quad (4(1-a)-4+3a)(1-a) \le 0$

$\Rightarrow \quad a(a-1) \le 0 \Rightarrow 0 \le a \le 1$

[If $x^2+2x+a=0$ and $x^2+4x+3a=0$ have a common root, then $a=0, 1$. So range of f is not equal to R.]

$$\frac{x^2}{6a-4a} = \frac{x}{a-3a} = \frac{1}{4-2} \Rightarrow x = -a$$

Put value of x in any equation to get $a = 0, 1$
Hence, we get $0 < a < 1$.

PROBLEM 10

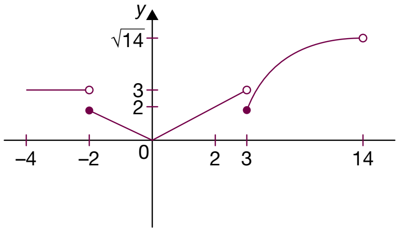
Let d be the number of integers in the range of the function

$$f(x) = \begin{cases} 3, & \text{if } -4 \le x < -2 \\ |x|, & \text{if } -2 \le x < 3 \\ \sqrt{x}, & \text{if } 3 \le x < 14 \end{cases}.$$

Also roots of $p(x) = x^2 + mx - 4m + 20$ are α and β. If $\alpha < \frac{d}{4} < \frac{d}{2} < \beta$ then find the smallest integral value of m.

SOLUTION

Range of $f(x)$ is $[0, \sqrt{14}\,]$



Integers in range are 0, 1, 2, 3
Hence, $d = 4$
Now, one root of $P(x) = x^2 + mx - 4m + 20$ is less than 1 and other root greater than 2.
Hence, $P(1) < 0 \Rightarrow 21 - 3m < 0 \Rightarrow m > 7$
and $P(2) < 0 \Rightarrow 24 - 2m < 0 \Rightarrow m > 12$
Hence, $m > 12$.
$\therefore$ Least integral value of m is 13

PROBLEM 11

Let $P_k = \begin{bmatrix} k, & \text{if } k \text{ is odd} \\ \dfrac{1}{k-1}, & \text{if } k \text{ is even} \end{bmatrix}$ and $\prod_{k=1}^{14} (P_k \cdot \log_{(k+1)}(k+2)) = N$, then find the number of solution of the equation $\tan^2 x + \cot^2 x + 2 = N \sin^2(2x)$ in $x \in [0, 2\pi]$.

SOLUTION

$$N = \prod_{k=1}^{14} (P_k \cdot \log_{(k+1)}(k+2))$$

$$= \left(\prod_{k=1}^{14} P_k\right)\left(\prod_{k=1}^{14} \log_{(k+1)}(k+2)\right)$$

$$= \left(1.\frac{1}{2-1}.3.\frac{1}{3}\ldots 13.\frac{1}{13}\right)\left(\log_2 3.\log_3 4\ldots\log_{15}{}^{16}\right)$$

$$= 1 \cdot \log_2 16 = 4$$

Now $\tan^2 x + \cot^2 x + 2 = N \sin^2 2x$

$\therefore \quad (\tan x - \cot x)^2 + 4 = 4\sin^2 2x,$

$(\tan x - \cot x)^2 + 4\cos^2 2x = 0$

$\tan x - \cot x = 0$ and $\cos^2 2x = 0$

$\Rightarrow \quad \tan^2 x = 1$

$\Rightarrow \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$\therefore$ Number of solution is 4.

PROBLEM 12

Find the natural number c for which $\sum_{r=1}^{n} f(c+r) = 16(2^n - 1)$ where the function satisfies the relation $f(x+y) = f(x) \,.\, f(y)$ for all $x, y \in N$ (natural numbers) and $f(1) = 2$.

SOLUTION

From the given functional equation $f(x+y) = f(x) \,.\, f(y)$, we have

$$f(x) = e^{kx} \qquad \ldots(1)$$

Where k is a constant.

Putting $x = 1$, $f(1) = e^k \Rightarrow 2 = e^k$. Hence, from (1), $f(x) = 2^x$.

Now, $\sum_{r=1}^{n} f(c+r) = f(c+1) + f(c+2) + \ldots + f(c+n)$

$$= 2^{c+1} + 2^{c+2} + \ldots + 2^{c+n} = 2^c.2 + 2^c.2^2 + \ldots + 2^c.2^n$$

$$= 2^c 2.\ \{1 + 2 + 2^2 + \ldots \text{ to } n \text{ terms}\}$$

$$= 2^c.2.\frac{1(2^n - 1)}{2-1} = 2^c.2(2^n - 1)$$

$\Rightarrow \quad 16(2^n - 1) = 2^{c+1}(2^n - 1) \Rightarrow 2^{c+1} = 16 = 2^4 \Rightarrow c + 1 = 4$

$\therefore \quad c = 3$.

PROBLEM 13

If the function $f(x) = \dfrac{x-1}{c - x^2 + 1}$ does not take any value in the interval $\left[-1, -\frac{1}{3}\right]$, then find the largest integral value that c can attain.

SOLUTION

Let $y=f(x)=\dfrac{x-1}{c-x^2+1}$

Take $y=-t$, where $t\in\left[\dfrac{1}{3},1\right]$

$$\therefore \quad -t=\frac{x-1}{c-x^2+1}$$

$$\Rightarrow \quad x^2-c-1=\frac{x-1}{t}\Rightarrow x^2-\frac{1}{t}x+\frac{1}{t}-c-1=0$$

As $-t\in\left[-1,-\dfrac{1}{3}\right]$, hence the above must not possess real solution.

$$\therefore \quad \left(\frac{1}{t}\right)^2-4\left(\frac{1}{t}-c-1\right)<0\Rightarrow\frac{1}{t^2}-\frac{4}{t}+4<-4c$$

$$\Rightarrow \quad c<-\frac{1}{4}\left(\frac{1}{t}-2\right)^2$$

Now, $\dfrac{1}{3}\le t\le 1\Rightarrow -1\le\dfrac{1}{t}-2\le 1\Rightarrow -\dfrac{1}{4}\le-\dfrac{1}{4}\left(\dfrac{1}{t}-2\right)^2\le 0$

Hence, $c\in\left(-\infty,-\dfrac{1}{4}\right)$

PROBLEM 14

Let f be a real valued function defined on R (the set of all real numbers) such that $f(1)=f(-1)=5$ and whose graph off is symmetrical with respect to the lines $x=\pm 1$.

Given $\displaystyle\sum_{r=50}^{99} f\left(\frac{r}{100}\right)=10$ and $\displaystyle\sum_{r=50}^{99} f\left(\frac{-r}{100}\right)=5$. If $S=\displaystyle\sum_{r=50}^{150} f\left(\frac{r}{100}\right)+\sum_{r=50}^{150} f\left(\frac{-r}{100}\right)$. Then find S.

SOLUTION

$\therefore$ f is symmetrical about $x=\pm 1$

$\Rightarrow$ $f(1+x)=f(1-x)$ and $f(-1+x)=f(-1-x)$

$$\text{Put } x=\frac{1}{100}\Rightarrow f\left(\frac{101}{100}\right)=f\left(\frac{99}{100}\right),f\left(\frac{-99}{100}\right)=f\left(\frac{-101}{100}\right)$$

$$x=\frac{2}{100}\Rightarrow f\left(\frac{102}{100}\right)=f\left(\frac{98}{100}\right),f\left(\frac{-98}{100}\right)=f\left(\frac{-102}{100}\right)$$

$$x=\frac{50}{100}\Rightarrow f\left(\frac{150}{100}\right)=f\left(\frac{50}{100}\right),f\left(\frac{-50}{100}\right)=f\left(\frac{-150}{100}\right)$$

$$\text{Again } \sum_{r=50}^{150} f\left(\frac{r}{100}\right)=f\left(\frac{50}{100}\right)+\left(\frac{51}{100}\right)+f\left(\frac{52}{100}\right)+\ldots+$$

$$\ldots+f\left(\frac{149}{100}\right)+f\left(\frac{150}{100}\right)$$

$$\therefore \quad f\left(\frac{50}{100}\right)+f\left(\frac{51}{100}\right)+\ldots+\left(\frac{99}{100}\right)+f(1)+f\left(\frac{99}{100}\right)+$$

$$\left(\frac{51}{100}\right)+f\left(\frac{50}{100}\right)$$

$$=2\left[f\left(\frac{50}{100}\right)+f\left(\frac{51}{100}\right)+\ldots+f\left(\frac{99}{100}\right)\right]+f(1)$$

$$=2.(10)+5=25$$

$$\text{Similarly } \sum_{r=50}^{99} f\left(\frac{-r}{100}\right)=2.5+5=15$$

Now required $S=40$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. Domain of the definition $f(x)=\dfrac{\log_4(5-[x-1]-[x]^2)}{x^2+x-2}$ (where [.] denotes greatest integer function), is:

(A) $(-3,-2)\cup(1,2)$ (B) $(-4,-3)\cup(2,3)\cup(3,4)$

(C) $\left(-\frac{4}{3},-\frac{3}{4}\right)\cup\left(\frac{2}{3},\frac{3}{2}\right)$ (D) $(-2,1)\cup(1,2)$

2. The range of the function $f(x)=4\cos^3 x-8\cos^2 x+1$ is:

(A) $[-11,1]$ (B) R

(C) $[-1,1]$ (D) $[-11,-3]$

3. If $0\le x\le\frac{\pi}{3}$, then range of $f(x)=\sec\left(\frac{\pi}{6}-x\right)+\sec\left(\frac{\pi}{6}+x\right)$ is

(A) $\left(\frac{4}{\sqrt{3}},\infty\right)$ (B) $\left[\frac{4}{\sqrt{3}},\infty\right)$

(C) $\left(0,\frac{4}{\sqrt{3}}\right]$ (D) $\left(0,\frac{4}{\sqrt{3}}\right)$

4. If $[x]$ denotes the integral part of x, then domain of the function $f(x)=\dfrac{\sqrt{3-x}}{(x-1)(x-2)(x-3)}+\sin^{-1}\left[\dfrac{3x-2}{2}\right]$ is:

(A) $[0,2)$ (B) $[0,2)-\{1\}$

(C) $(-\infty,3)-\{1,2\}$ (D) None of these

5. Let $f(x)=\sqrt{|x|-\{x\}}$ (where {.} denotes the fractional part of x) and X, Y be its domain and range respectively. Then:

(A) $X=\left(-\infty,\frac{1}{2}\right]$

(B) $X=\left(-\infty,-\frac{1}{2}\right]$ and $Y=\left[\frac{1}{2},\infty\right)$

(C) $X=\left(-\infty,-\frac{1}{2}\right]\cup[0,\infty)$ and $Y=[0,\infty)$

(D) $\left(-\infty,-\frac{1}{2}\right]$ and $[0,\infty)$

6. The range of the function $y=[x^2]-[x]^2$, $x\in[0,2]$ where [.] denotes the integral part, is:

(A) $\{0\}$ (B) $\{0,1\}$

(C) $\{1,2\}$ (D) $\{0,1,2\}$

7. The domain of the function $\log_e\left(\text{sgn}\left(9-x^2\right)\right)+\sqrt{[x]^3-4[x]}$ (where sgn is signum function and [.] is step function)

(A) $[-2,1]\cup[2,3]$ (B) $[-2,1]\cup[2,3)$

(C) $(-2,1]\cup[2,3)$ (D) $[-2,1)\cup[2,3)$

8. If $[x]$ denotes the integral part of x, then the domain of $f(x)=\cos^{-1}(x+[x])$ is:

(A) $(0,1)$ (B) $[0,1)$

(C) $[0,1]$ (D) $[-1,1]$

9. Range of $f(x)={}^{16-x}C_{2x-1}+{}^{20-3x}C_{4x-5}$ is:

(A) $[728, 1474]$ (B) $\{728, 1617\}$

(C) $\{0, 728\}$ (D) None of these

10. If $f:(0,\pi/n)\to R$, defined by $f(x)=\sum_{k=1}^{n}[1+\sin kx]$, where $[x]$ denotes the integral part of x, then range of $f(x)$ is:

(A) $\{n-1,n+1\}$ (B) $\{n-1,n,n+1\}$

(C) $\{n,n+1\}$ (D) None of these

11. The domain of the function

$$y=\sqrt{\frac{2x-1}{2x^3+3x^2+x}}+\sqrt{\sin^{-1}(\log_2 x)}\text{ is:}$$

(A) $\left(\frac{1}{2},\infty\right)$ (B) $\left(\frac{1}{2},2\right]$

(C) $[1,2]$ (D) $(1,\infty)$

12. If $[x]$ denotes the integral part of x and $f(x)$ has domain $\left[-\frac{5}{2},2\right]$, the domain of $f([\,|x|\,])$ is:

(A) $[-3,2]$ (B) $(-3,2]$

(C) $[-2,2]$ (D) $(-3,3)$

13. The domain of the function $y=\sqrt{\sin x+\cos x}+\sqrt{7x-x^2-6}$ is:

(A) $[1,6]$ (B) $\left[1,\frac{3\pi}{4}\right]\cup\left[\frac{7\pi}{4},6\right]$

(C) $[1,\pi]\cup\left[\frac{7\pi}{4},6\right]$ (D) None of these

14. The range of the function $f(x)=[\sin x+\cos x]$ (where [.] denotes the greatest integer function) is:

(A) $[-2,1]$ (B) $\{-2,-1,0,1\}$

(C) $\{-1,1\}$ (D) $\{-2,-1,1\}$

15. If $f(x)=\dfrac{x^3+x-2}{x^3-1}$ and $g(x)=\dfrac{x^2+x+2}{x^2+x+1}$ then (D_f represents domain and R_f represents range of $f(x)$):

(A) $D_f=R-\{1\},\ R_f=\left(1,\frac{7}{3}\right]-\left\{\frac{4}{3}\right\}$

$D_g=R,\quad R_g=\left(1,\frac{7}{3}\right]$

(B) $D_f = R - \{1\},\ R_f = \left(1, \frac{7}{3}\right]$

$D_g = R, \qquad R_g = \left[1, \frac{7}{3}\right]$

(C) $D_f = R - \{1\},\ R_f = \left[1, \frac{7}{3}\right]$

$D_g = R, \qquad R_g = \left[1, \frac{7}{3}\right]$

(D) $D_f = R - \{1\},\ R_f = \left(1, \frac{7}{3}\right]$

$D_g = R, \qquad R_g = \left(1, \frac{7}{3}\right]$

16. If $f(x) = \sin^{-1}x$ and $g(x) = [\sin x(\cos x)] + [\cos(\sin x)]$, then range of $f(g(x))$ is:

(A) $\left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$ (B) $\left\{-\frac{\pi}{2}, 0\right\}$

(C) $\left\{0, \frac{\pi}{2}\right\}$ (D) $\left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$

17. If $f : R \to R$ defined by $f(x) = \begin{cases} x, & x > 2 \\ 5x - 2, & x \le 2 \end{cases}$ then f is:

(A) not a function (B) one-one

(C) onto (D) one-one onto

18. Let Z denote the set of all integers, defined as $f : Z \to Z$ by $f(x) = \begin{cases} x/2 & x \text{ is even} \\ 0 & x \text{ is odd} \end{cases}$ then f is:

(A) onto but not one-one (B) one-one but not onto

(C) one-one and onto (D) neither one-one nor onto

19. Range of the function 'f' defined by $f(x) = \left[\frac{1}{\cos\{x\}}\right]$, where [.] and {.} denotes the greatest integer and fractional part function respectively is …

(A) set of integers (B) set of all natural numbers

(C) {2, 3, 4, …} (D) {1}

20. The range of the function $f(x) = \sqrt{\tan^{-1}\sqrt{x^2 - 1} - \sec^{-1}x}$ is:

(A) $\left[0, \sqrt{\pi}\right]$ (B) {0}

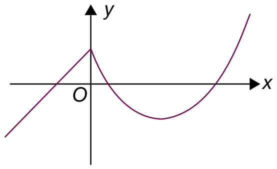
(C) $\left[0, \sqrt{\frac{\pi}{2}}\right]$ (D) $\left[0, \sqrt{\frac{\pi}{4}}\right]$

21. Let $f : [-1, 1] \to R$ be a function defined as $y = f(x)$ and $g(x) = \log_{2e}[x]$; ([.] denotes greatest integer function) then number of integers in the domain of $f(g(x))$ is/are:

(A) 4 (B) 5

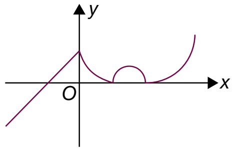
(C) 6 (D) 7

22. The graph of $y = f(x)$ is shown in figure:

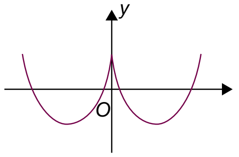


Which of the following best represent the graph of $y = f(-|-x|)$?

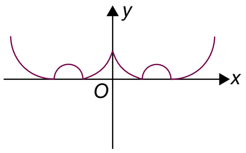
(A)



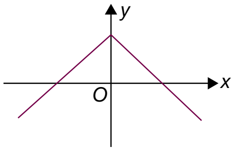
(B)



(C)



(D)

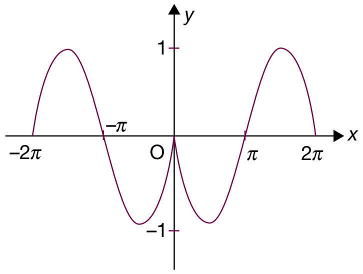


23. The range of the function $f(x) = \sqrt{4 - x^2} + \sqrt{x^2 - 1}$ is:

(A) $\left[\sqrt{3}, \sqrt{7}\right]$ (B) $\left[\sqrt{3}, \sqrt{5}\right]$

(C) $\left[\sqrt{2}, \sqrt{3}\right]$ (D) $\left[\sqrt{3}, \sqrt{6}\right]$

24. Which function corresponds to the given graph?



(A) $y = |\sin x|$ (B) $y = \sin|x|$

(C) $y = -|\sin x|$ (D) $y = -\sin|x|$

25. Let $f : A \to B$ and $g : B \to C$ be two functions and $g \circ f : A \to C$ is defined. Then which of the following statement(s) is true?

(A) If *gof* is onto then *f* must be onto.

(B) If *f* is into and *g* is onto then *gof* must be onto function.

(C) If *gof* is one-one then *g* is not necessarily one-one.

(D) If *f* is injective and *g* is surjective then *gof* must be bijective mapping.

26. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{\pi x}{2}\right); & |x| \le 1 \\ x|x|; & |x| > 1 \end{cases}$ then $f(x)$, is:

(A) an even function

(B) an odd function

(C) a decreasing function

(D) periodic function

27. The sum of all values of λ for which the equation $4\lambda[x]^2 = \lambda + 12$ has a solutions in $[1, \infty)$ is:

(A) 8 (B) 3

(C) 4 (D) 6

28. Let $S = \{1, 2, 3, 4\}$. Then number of functions $f : S \to S$ such that $f(k) \le 2k$ for all $k \in S$, is:

(A) 32 (B) 64

(C) 128 (D) 156

29. Let $f(x)$ is an odd function defined on R such that $f(1) = 2$, $f(3) = 5$ and $f(-5) = -1$. The value of $\frac{f(f(f(-3))) + f(f(0))}{3f(1) - 2f(3) - f(5)}$ is:

(A) $\frac{-2}{3}$ (B) $\frac{-2}{5}$

(C) $\frac{2}{5}$ (D) $\frac{2}{3}$

30. The domain of the function $f(x) = (x+0.5)^{\log(0.5+x)\left(\frac{x^2+2x-3}{4x^2-4x-3}\right)}$ is:

(A) $(-\infty, -3) \cup \left(\frac{-1}{2}, 1\right) \cup \left(\frac{3}{2}, \infty\right)$

(B) $(-\infty, -3) \cup \left(\frac{-1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, \infty\right)$

(C) $\left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, \infty\right)$

(D) $\left(\frac{-1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, \infty\right)$

31. If $[x]^2 + [x] - 2 < 0$ and $\{x\} = \frac{1}{2}$, then the number of possible real values of x, is:

(A) 1 (B) 2

(C) 3 (D) 4

32. Let $f(x) = \frac{\sin 2x - \cos 2x}{4}$, $g(x) = x^2 - 4$. If $g(f(x))$ is surjective, then it is invertible for $x \in$:

(A) $\left[-\frac{\pi}{2}, 0\right]$ (B) $\left[-\frac{\pi}{2}, \pi\right]$

(C) $\left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$ (D) $\left[0, \frac{\pi}{2}\right]$

33. The number of integral value(s) of 'α' for which expression $\log(\log_{1/3}(\log_7(\sin x + \alpha)))$ is meaningful $\forall\ x \in R$, is:

(A) 1 (B) 2

(C) 3 (D) 5

34. Let $f(x) = x^2 - 2x - 3$; $x \geq 1$ and $g(x) = 1 + \sqrt{x+4}$; $x \geq -4$, then the number of real solutions of equation $f(x) = g(x)$ is/are

(A) 0 (B) 1

(C) 2 (D) 4

35. Let $f(x) = \ln\left(\frac{1-x}{1+x}\right)$. The set of values of α for which $f(\alpha) + f(\alpha^2) = f\left(\frac{\alpha}{\alpha^2 - \alpha + 1}\right)$ is satisfied is:

(A) $(-\infty, -1) \cup (1, \infty)$ (B) $(-1, 1)$

(C) $(0, 1)$ (D) None of these

36. Consider $f(x) = \frac{x^2 - 3x}{x^2 - 3x + 4}$; $g(y) = \frac{3y - 3}{y^2 - 3y + 4}$; $h(z) = \frac{2}{z^2 - 3z + 4}$; where $x, y, z \in R$.

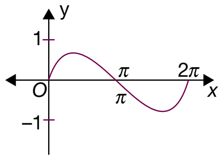
If $\alpha = f(x) + g(y) + h(z) - [f(x)] - [g(y)] - [h(z)]$, then possible value of α is (where [.] denotes greatest integer function)

(A) 5 (B) 2

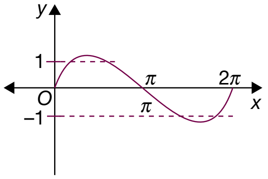
(C) 3 (D) 4

37. Which of the following best represents the graph of $f(x) = \tan(\sin x,)\ x > 0$?

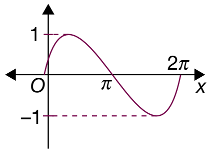
(A)



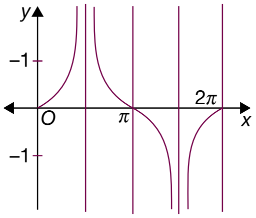
(B)



(C)



(D)



38. Let $f(x) = \frac{\log x}{x}$, and solution of $f(x) = k$ be denoted by $g(x)$ {where $k < 0$}, $g : (-\infty, 0) \to (0, 1)$, then $y = g(x)$ will be:

(A) bijective function (B) many-one function

(C) periodic function (D) into function

39. If $f : f\{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$, $y = f(x)$ be a function such that $|f(\alpha) - \alpha| \leq 1$, for $\alpha \in \{1, 2, 3, 4\}$ then total number of such function are:

(A) 81 (B) 36

(C) 54 (D) None of these

40. If $f: N \to N, f(x) = (x+1)^2 + x - \left[\sqrt{(x+1)^2 + (x+1)}\right]^2$, then $f(x)$ will be ([.] is GIF)

(A) bijective function (B) many-one function

(C) periodic function (D) into function

41. The domain of the function $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - \sin 2x}}}$ is:

(A) $\left[2n\pi\frac{\pi}{2}, 2n\pi + \frac{\pi}{6}\right]$ (B) $\left[n\pi - \frac{\pi}{2}, \pi + \frac{\pi}{6}\right]$

(C) R (D) $\left[n\pi - \frac{\pi}{6}, n\pi + \frac{\pi}{2}\right]$

42. The range of unction $f(x) = \cos^2(\sin x) + \sin^2(\cos x)$ is:

(A) $[1 + \sin^2 1, 2 + \cos^2 1]$ (B) $[\sin^2 1, 1 + \cos^2 1]$

(C) $(1 + \sin^2 1, 2 - \cos^2 1]$ (D) $[\cos^2 1, 1 + \sin^2 1]$

43. If f : rational numbers → real numbers, $f(x) = \sin x$, then f is:

(A) One to one, into (B) Many to one, into

(C) Bijective (D) Not a mapping

44. $f(x) = \frac{x}{\ln x}$ and $g(x) = \frac{\ln x}{x}$. Then identify the CORRECT statement.

(A) $\frac{1}{g(x)}$ and $f(x)$ are identical functions

(B) $\frac{1}{f(x)}$ and $g(x)$ are identical functions

(C) $f(x) \cdot g(x) = 1\ \forall\ x > 0$

(D) $\frac{1}{f(x) \cdot g(x)} = 1\ \forall\ x > 0$

45. Which of the following function is surjective but not injective?

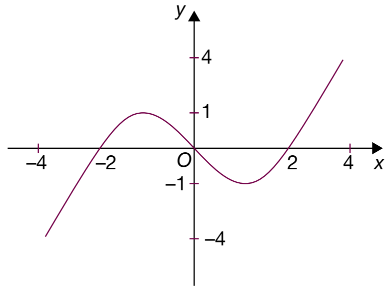
(A) $f: R \to R, f(x) = x^4 + 2x^3 - x^2 + 1$

(B) $f: R \to R, f(x) = x^3 + x + 1$

(C) $f: R \to R^+, f(x) = \sqrt{1+x^2}$

(D) $f: R \to R, f(x) = x^3 + 2x^2 - x + 1$

46. The graph of the function $y = g(x)$ is shown. The number of solutions of the equation $||g(x)| - 1| = \frac{1}{2}$, is:



(A) 4 (B) 5

(C) 6 (D) 8

47. Range of function $f(x) = \tan^{-1}\left(\sqrt{[x]+[-x]}\right) + \sqrt{2-|x|} + \frac{1}{x^2}$ is: (where [*] is the greatest integer function.)

(A) $\left[\frac{1}{4}, \infty\right)$ (B) $\left\{\frac{1}{4}\right\} \cup [2, \infty)$

(C) $\left\{\frac{1}{4}, 2\right\}$ (D) $\left[\frac{1}{4}, 2\right]$

48. Which of the following statements are incorrect?

I. If $f(x)$ and $g(x)$ are one to one, then $f(x) + g(x)$ is also one to one.

II. If $f(x)$ and $g(x)$ are one-one, then $f(x) \cdot g(x)$ is also one-one.

III. If $f(x)$ is odd, then it is necessarily one to one.

(A) I and II only (B) II and III only

(C) III and I only (D) I, II and III

49. The function $f(x)$ is defined by $f(x) = \cos^4 x + K \cos^2 2x + \sin^4 x$, where K is a constant. If the function $f(x)$ is a constant function, then the value of K is:

(A) –1 (B) $-\frac{1}{2}$

(C) 0 (D) $\frac{1}{2}$

50. Let two functions $f(x)$ and $g(x)$ are defined on $R \to R$ such that $f(x) = \begin{cases} x^2 & x \in \text{irrational} \\ 2 - x^2 & x \in \text{rational} \end{cases}$ and

$g(x) = \begin{cases} 2 - x^2 & x \in \text{irrational} \\ x^2 & x \in \text{rational} \end{cases}$.

Then the function $f + g : R \to R$ is:

(A) injective as well as surjective.

(B) injective but not surjective

(C) surjective but not injective.

(D) neither surjective nor injective.

51. For the function $f(x) = \frac{e^x + 1}{e^x - 1}$, if $n(d)$ denotes the number of integers which are not in its domain and $n(r)$ denotes the number of integers which are not in its range, then $n(d) + n(r)$ is equal to:

(A) 2 (B) 3

(C) 4 (D) infinite

52. If $f(x) = \pi\left(\frac{\sqrt{x+7} - 4}{x - 9}\right)$, then the range of function $y = \sin(2f(x))$ is:

(A) [0, 1] (B) $\left(0, \frac{1}{\sqrt{2}}\right]$

(C) $\left(0, \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right]$ (D) (0, 1]

53. Let $f(x) = (x+2)e^{\ln(x+2)}$ and $g(x) = \frac{e^{\left(\frac{1}{-\log_x e}\right)}}{x} - \frac{2x}{e^{-\ln 2}}$. If $h(x) = f(x) + g(x)$, then the smallest positive integer in the range of $h(x)$ is:

(A) 2 (B) 3

(C) 6 (D) 7

54. Number of elements in the domain of the function $f(x) = \frac{1}{\sqrt{{}^{10}C_{x-1} - 3 \cdot {}^{10}C_x}}$, is:

(A) 2 (B) 3

(C) 4 (D) 10

55. Let $f : [0, 2] \to [2, 5]$ be defined as $f(x) = 3x^2 - 6x + 5$, then $f(x)$ is:

(A) injective but not surjective

(B) surjective but not injective

(C) injective as well as surjective

(D) neither injective nor surjective

56. Set A consists of 4 distinct and set–B consists of 3 distinct elements. Number of functions that can be defined from a set $A \to B$ which are not onto is:

(A) 35 (B) 45

(C) 63 (D) 9

57. If $[2 \sin x] + [\cos x] = -3$, then the range of the function $f(x) = \sin x + \sqrt{3} \cos x$ in $[0, 2\pi]$ is (where [.] denotes the greatest integer function)

(A) $[-2, -1)$ (B) (–2, –1)

(C) $\left(-1, -\frac{1}{2}\right)$ (D) None of these

58. The range of α such that range of $f(x) = \frac{(x^3 - 3x + 6)(x - \alpha)}{2(x - \alpha)}$ is a proper subset of R is:

(A) $R - [-2, 2]$ (B) $R - (-2, 2)$

(C) [–2, 2] (D) (–2, 2)

59. $f : R^+ \to R, f(x) = |x - 1|$ and $g : [-1, \infty) \to R, g(x) = e^x$

If the function $fog(x)$ is defined, then its domain and range respectively are:

(A) $(0,\infty)$ and $[0,\infty)$ (B) $[-1,\infty)$ and $[0,\infty)$

(C) $[-1,\infty)$ and $\left[1-\frac{1}{e},\infty\right)$ (D) $[-1,\infty)$ and $\left[\frac{1}{e}-1,\infty\right)$

60. If 'f' is a real valued function not identically zero, satisfying $f(x+y)+f(x-y)=2f(x)\,.\,f(y)$ $\forall x,y\in R$, then $f(x)$ is:

(A) odd (B) even

(C) neither even nor odd (D) can't say

61. If $f(x)+f(x+a)+f(x+2a)+\ldots+f(x+na)=$ constant; $\forall\ x\in R$ and $a>0$ and $f(x)$ is periodic, then period of $f(x)$, is:

(A) $(n+1)a$ (B) $e^{(n+1)a}$

(C) na (D) e^{na}

62. If $F(x)$ and $G(x)$ are even and odd extensions of the functions $f(x)=x|x|+\sin|x|+xe^x$, where $x\in(0,1)$, $g(x)=\cos|x|+x^2-x$, where $x\in(0,1)$, respectively to the interval (–1, 0) then $F(x)+G(x)$ in $(-1,0)$ is:

(A) $\sin x+\cos x+xe^{-x}$

(B) $-\left(\sin x+\cos x+xe^{-x}\right)$

(C) $-\left(\sin x+\cos x+x+xe^{-x}\right)$

(D) $-\left(\sin x+\cos x+x^2+xe^{-x}\right)$

63. If $q^2-4pr=0$ $p>0$, then the domain of the function $f(x)=\log\left\{px^3+(p+q)x^2+(q+r)x+r\right\}$ is:

(A) $R-\left\{-\frac{q}{2p}\right\}$ (B) $R-\left[(-\infty,-1]\cup\left\{-\frac{q}{2p}\right\}\right]$

(C) $R-\left[(-\infty,-1]\cap\left\{-\frac{q}{2p}\right\}\right]$ (D) None of these

64. The domain of the function $f(x)=\sqrt{x^{12}-x^9+x^4-x+1}$ is:

(A) (–1, 1) (B) $(-\infty,-1)$

(C) $(1,\infty)$ (D) $(-\infty,\infty)$

65. The domain of definition of the function $f(x)=\sqrt{2^{2x}+(64)^{(x-2)/3}-2^{-1}\left(72+2^{2x}\right)}$ is:

(A) $(-\infty,\infty)$ (B) $(-\infty,-3]$

(C) $[3,\infty)$ (D) $\left[\frac{1}{9},1\right)$

66. Consider the function $f(x)=\text{sgn}(x)$ and $g(x)=x\,(1-x^2)$ then which of the following does not hold good?

(A) $(gof)\,(x)$ is even as well as odd

(B) $(fog)(x)$ is an odd function

(C) $(fog)(x)$ is neither odd nor even

(D) None of these

67. If $f(x)$ is an even function and satisfies the relation $x^2f(x)-2f\left(\frac{1}{x}\right)=g(x)$, where $g(x)$ is an odd function, then $f(10)$ equals:

(A) 0 (B) $\frac{50}{79}$

(C) $\frac{49}{75}$ (D) 1

68. Let $f(x)=\frac{\sin^{101}x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}$, where $[x]$ denotes the integral part of x is:

(A) an odd function

(B) an even function

(C) neither odd nor even function

(D) both odd and even function

69. Period of $f(x)=\sin\frac{\pi x}{(n-1)!}+\cos\frac{\pi x}{n!}$ is:

(A) $n!$ (B) $2\,(n!)$

(C) $2(n-1)!$ (D) None of these

70. Let $f:R\rightarrow R$ be a periodic function such that $f(T+x)=1+[1-3f(x)+3(f(x))^2-(f(x))^3]^{1/3}$ where T is a fixed positive number, then period of $f(x)$ is:

(A) T (B) $2T$

(C) $3T$ (D) None of these

71. If $f(g(x))=|\cos x|$, $g(f(x))=\cos^2\sqrt{x}$, then:

(A) $f(x)$ is a periodic function and $g(x)$ is a non-periodic function

(B) $f(x)$ is a non-periodic function and $g(x)$ is a periodic function

(C) both $f(x)$ and $g(x)$ is a periodic functions

(D) neither $f(x)$ nor $g(x)$ is a periodic function

72. If $f(x)$ satisfies the relation $f\left(\frac{5x-3y}{2}\right)=\frac{5f(x)-3f(y)}{2}\,\forall x,y\in R$, and $f(0)=3$ and $f'(0)=2$, then the period of $\sin(f(x))$ is:

(A) 2π (B) π

(C) 3π (D) 4π

73. For real valued functions $f_i(x)=\sin 2p_ix,\forall i=1,2,3;\ p_i\in N$ such that period of $f_1(x)-f_2(x)+3f_3(x)$ is π, period of $f_1(x)+2f_2(x)$ is $\frac{\pi}{2}$ and period of $f_1(x)+10f_3(x)$ is $\frac{\pi}{3}$, then min $(p_1+p_2+p_3)$ is:

(A) 5 (B) 10

(C) 11 (D) 8

74. Let $f(x)=x^3+ax^2+6x+\cos 3x$, then f^{-1} will exist if:

(A) $a<-3$ (B) $-3\le a\le 3$

(C) $a>3$ (D) $4<a<5$

75. $f(x)=x^2-x+1,\ x\ge\frac{1}{2}$ and $g(x)=\frac{1}{2}+\sqrt{x-\frac{3}{4}}$ are mutually inverse, then the number of solutions of the equation $x^2-x+1=\frac{1}{2}+\sqrt{\left(x-\frac{3}{4}\right)}$ is:

(A) 0 (B) 2

(C) 1 (D) ∞

76. Let f be a function satisfying of x, then $f(xy)=\frac{f(x)}{y}$ for all positive real numbers x and y; if $f(30) = 20$, then the value of $f(40)$ is:

(A) 15
(B) 20
(C) 40
(D) 60

77. If $f(x+y)=f(x.y)\ \forall\, x, y\in R$ and $f(2011)=2010$, then $f(2012)=$

(A) 2000
(B) 2010
(C) 2012
(D) 2011

78. If $f(x)+f(x+4)=f(x+2)+f(x+6)\ \ \forall\, x\in R$, and $f(5) = 10$, then $\sum_{r=1}^{100} f(5+8r)$ equal to:

(A) 1000
(B) 100
(C) 10000
(D) 10

79. Let $g(x)$ be a function defined on $[-1,1]$. If the area of the equilateral triangle with two of its vertices at (0, 0) and $(x, g(x))$ is $\sqrt{3}/4$, then the function $g(x)$ is:

(A) $g(x)=\pm\sqrt{4-x^2}$
(B) $g(x)=\sqrt{1-x^2}$
(C) $g(x)=-\sqrt{1+x^2}$
(D) $g(x)=\sqrt{1+x^2}$

80. If $f(x)=\frac{(x-a)(x-b)}{x}$ and

$$\frac{f(x)}{(x-y)(x-z)}+\frac{f(y)}{(y-z)(y-x)}+\frac{f(z)}{(z-x)(z-y)}=\frac{k}{xyz}$$ then $k=$

(A) a
(B) b
(C) ab
(D) $3ab$

81. $f: N\to Q$ such that $f(1) = 1$ and $\sum_{r=1}^{n} r\ f(r)=n(n+1)\ f(n)$, then $48\,f(12) =$

(A) 1
(B) 2
(C) 4
(D) 6

82. If the following functions are defined from [–1, 1] to [–1, 1], identify these which are bijective.

(A) $\sin(\sin^{-1}x)$
(B) $\frac{2}{\pi}\sin^{-1}(\sin x)$
(C) $\mathrm{sgn}(x)\ln(e^x)$
(D) $x^3(\mathrm{sgn}(x))$

83. Let $f(x)=e^x+\sin x$ be defined on the interval [–4, 0], the odd extension of $f(x)$ in the interval [–4, 4]:

(A) $e^{-x}+\sin x,\ x\in(0,4)$
(B) $-e^{-x}+\sin x,\ x\in(0,4)$
(C) $e^{-x}-\sin x,\ x\in(0,4)$
(D) $-e^{-x}-\sin x,\ x\in(0,4)$

84. If the functions of f and g are defined by $f(x) = 3 - x$, $g(x) = 2 + 3x$ for $x \in R$ respectively, then $g^{-1}(f^{-1}(2)) =$

(A) 1
(B) $-\frac{1}{3}$
(C) $-\frac{4}{3}$
(D) $\frac{1}{4}$

85. If for nonzero x, $2f(x^2)+3f\left(\frac{1}{x^2}\right)=x^2-1$, then $f(x^2)=$

(A) $\frac{3+2x^4-x^2}{5x^2}$
(B) $\frac{3-2x^4+x^2}{5x^2}$
(C) $\frac{3-2x^4-x^2}{5x^2}$
(D) $\frac{3+2x^4+x^2}{5x^2}$

86. Let $f(x)=\frac{2\alpha x}{2x+1}, x\neq-\frac{1}{2}$. Then $f(f(x))=x$ if $\alpha=$

(A) –1
(B) $-\frac{1}{2}$
(C) –2
(D) 2

87. A function f from integers to integers is defined as

$$f(x)=\begin{cases} x+3;\ x\in\text{odd} \\ \frac{x}{2}\quad;\ x\in\text{even}\end{cases}$$. Suppose $k\in$ odd and $f(f(f(k)))=27$

then the sum of digits of k is:

(A) 3
(B) 6
(C) 9
(D) 12

88. Define a real valued function of a real variable x given by $f(x)=a\sin x+b\sqrt[3]{x}+6$, where a and b are real numbers. If $f(\log_{30}(\log_{15}30))=7$, the value of $f(\log_{30}(\log_{30}15))$ is:

(A) 5
(B) 7
(C) –5
(D) –7

89. The total number of solution of $[x]^2=x+2\{x\}$, where [.] and {.} denote greatest integer function and fractional part, respectively is equal to:

(A) 2
(B) 4
(C) 6
(D) 8

90. Let $f:R\to R$, and $g:R\to R$ be two one-one and onto functions such that they are the mirror images of each other about the line $y = a$. If $h(x)=f(x)+g(x)$, then $h(x)$ is:

(A) One-One and onto
(B) Only one-one and onto
(C) Only Onto but not one-one
(D) Neither one-one nor onto

91. The 'x' for which $\sin x(\sin x+\cos x)=[x]$ where [.] denotes G.I.F is:

(A) $[0,1)\cup[1,3)$
(B) $[0,1)\cup[2,3)$
(C) $[-1,1)\cup[2,3)$
(D) $[0,1)\cup[1,2)$

92. If $f:[-20,20]\to R$ defined by $f(x)=\left[\frac{x^2}{a}\right]\sin x+\cos x$ is an even function, then the set of values of 'a' is [.] G.I.F

(A) $a\in(500,\infty)$
(B) $a\in(400,\infty)$
(C) $a\in(-400,0)$
(D) $a\in(-300,900)$

93. If $f:[-4,4]-\{-\pi,0,\pi\}\to R$, such that

$f(x)=\cot(\sin x)+\left[\frac{x^2}{|a|}\right]+\frac{\sin 2x}{x^2}$, ([.] denotes greatest integer function) is an odd function, then the complete set of values of 'a', is:

(A) $[-\infty,-4]\cup[4,\infty)$ (B) $(-\infty,-16)\cup(16,\infty)$

(C) $[-16,-16]$ (D) $(-\infty,-16]\cup[16,\infty)$

94. The number of functions f from the set $A = \{0, 1, 2\}$ in to the set $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) \le f(j)$ for $i < j$ and $i, j \in A$ is:

(A) ${}^{8}C_3$ (B) ${}^{8}C_3 + 2({}^{8}C_2)$

(C) ${}^{10}C_3$ (D) ${}^{10}C_4$

95. If $0 < x < 1000$ and $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = \frac{31}{30}x$, where $[x]$ is the greatest integer less than or equal to x, the number of possible values of x is:

(A) 34 (B) 33

(C) 32 (D) None of these

96. The function $f(x)$ is defined on $[0, 1]$ as following $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$ then for all $x \in [0, 1]$ $f\,f(x)$ is equal to:

(A) 0 (B) $1 + x$

(C) x (D) 1

97. Which of the following is always correct?

(A) Sum of two injective function is also injective

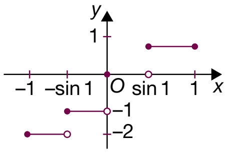
(B) If $f(x)$ and $g(x)$ are inverse to each other, then if $f(x) > x \ \forall\, x \in R$, then $g(x) < x \ \ \forall\, x \in R$

(C) If $f(x)$ is a periodic function, then $f(|x|)$ with also be periodic.

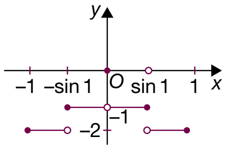
(D) If $f(x)$ and $g(x)$ are two aperiodic functions, then function $h(x) = f(x) \cdot g(x)$ will also be aperiodic.

98. If $f(x) = [\sin^{-1}x]$, where $[\cdot]$ is GIF then graph of $f(-|x|)$ is:

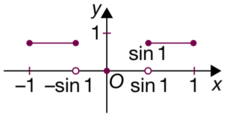
(A)



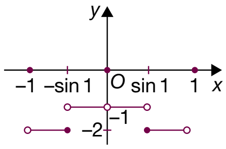
(B)



(C)



(D)



99. Domain of definition

$$f(x) = \sec^{-1}(x^2 - 15x + 55) + \sqrt{-\log_{\left(\frac{x}{8}\right)}(x^2+1)} - \frac{1}{\sqrt{|x^2 - 6x + 5|}}$$

is:

(A) $(0,6)\cup[7,8)-\{1,5\}$ (B) $(0,5|\cup(5,6)\cup(7,8)$

(C) $(6,1]\cup[7,8)$ (D) $(1,5]\cup(7,8)$

100. If $f(x) = \sqrt[3]{\dfrac{9}{\log_2(3-2x)} - 1}$, then value of '$a$' which satisfies $f^{-1}(2a-4) = \dfrac{1}{2}$

(A) 1 (B) 2

(C) 3 (D) 4

101. The function $f : R \to R, f(x) = x(x-1)(x-2)(x-3), g : R \to R, g(x) = f'(x)$, then $g(x)$ is:

(A) one-one and onto function

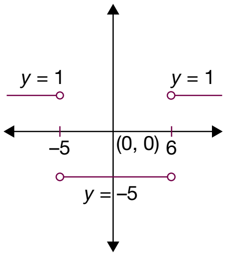
(B) one-one and into function

(C) many-one and onto function

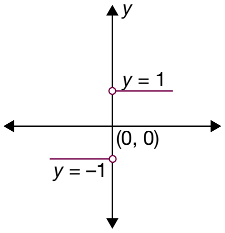
(D) many-one and into function

102. Which one of the following best represents the graph of function $f(x) = \dfrac{|x^2 - x - 30|}{x^2 - x - 30}$?

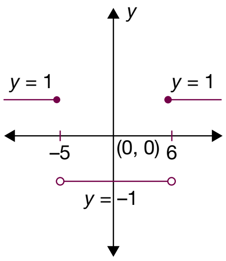
(A)



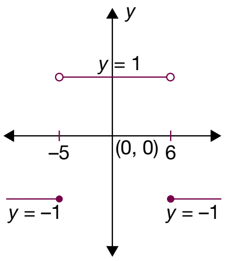
(B)



(C)



(D)



103. Let $f ; X \to Y$ be defined as $f(x) = \sin x + \cos x + 2\sqrt{2}$. If f is invertible then $X \to Y$, is:

(A) $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right] \to \left[\sqrt{2}, 3\sqrt{2}\right]$

(B) $\left[\frac{-\pi}{4}, \frac{-3\pi}{4}\right] \to \left[\sqrt{2}, 3\sqrt{2}\right]$

(C) $\left[\frac{-3\pi}{4}, \frac{3\pi}{4}\right] \to \left[\sqrt{2}, 3\sqrt{2}\right]$

(D) $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \to \left[\sqrt{2}, 3\sqrt{2}\right]$

104. For a real number x, let $[x]$ denote the greatest integer less than or equal to x. Let $f : R \to R$ be defined as $f(x) = 2x + [x] + \sin x \cdot \cos x$ then f is:

(A) one-one but not onto

(B) onto but not one-one

(C) both one-one and onto

(D) neither one-one nor onto

105. If the following functions are defined from R to R then identify the function which is bijective.

(A) $f(x) = \dfrac{e^x + e^{-x}}{2}$

(B) $f(x) = x^4 - 3x^3 + 1$

(C) $f(x) = 18x^3 - 21x^2 + 8x - 1$

(D) $f(x) = x^3 - 4x^2 + 16x + 17$

106. If the functions $f(x)$ and $g(x)$ are defined on $R \to R$, such that

$$f(x) = \begin{cases} x+3, & x \in \text{rational and} \\ 4x, & x \in \text{irrational} \end{cases};$$

$$g(x) = \begin{cases} x+\sqrt{5}, & x \in \text{irrational} \\ -x, & x \in \text{rational} \end{cases}$$ then $(f-g)(x)$ is:

(A) one-one and onto (B) neither one-one nor onto
(C) one-one but not onto (D) onto but not one-one

107. Number of functions $f: \{a, b, c\} \to \{1, 2, 3, 4, 5, 6\}$ which are not injective, is equal to:

(A) 96 (B) 120
(C) 216 (D) 240

108. Let $f: R \to R$ be defined as $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + ax + b$. The least value of '$a$' for which $f(x)$ is injective function is:

(A) $\frac{1}{4}$ (B) $\frac{1}{8}$
(C) $\frac{1}{2}$ (D) 1

109. Let $f: R \to R$ be a function defined by $f(x) = \begin{cases} x - \frac{1}{x}, & x > 0 \\ e^x, & x \le 0 \end{cases}$ then f is:

(A) both injective and surjective
(B) injective but not surjective
(C) surjective but not injective
(D) neither injective non surjective

110. Let $f_1(x) = \begin{cases} x & 0 \le x \le 1 \\ 1 & x > 1 \\ 0 & \text{otherwise} \end{cases}$ and $\begin{cases} f_2(x) = f_1(-x) & \text{all } x \\ f_3(x) = -f_2(x) & \text{all } x \\ f_4(x) = f_3(-x) & \text{all } x \end{cases}$

Which of the following is necessarily true?

(A) $f_4(x) = f_1(x)$ for all x
(B) $f_1(x) = -f_3(-x)$ for all x
(C) $f_2(-x) = f_4(x)$ for all x
(D) $f_1(x) + f_3(x) = 0$ for all x

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. $f(x) = \cos^2 x + \cos^2\left(\frac{\pi}{3} + x\right) - \cos x.\cos\left(\frac{\pi}{3} + x\right)$ is:

(A) an odd function
(B) an even function
(C) a period function
(D) $f(0) = f(1)$

2. Let $f: N \to N$ satisfying the conditions

(i) $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right]$

(ii) $1900 < f(1990) < 2000$, where $[x]$ denotes the integral part of x. Then which of the following are true

(A) $f(1990) = 1990$ (B) $f(1990) = 1994$
(C) $f(1990) = 2004$ (D) $f(1990) = 1904$

3. $A = \{1,2,3,4,5\}, B = \{1,2,3,4\}$ and $f: A \to B$ is a function, then:

(A) number of onto functions if $n(f(A)) = 4$ is 240
(B) number of onto functions if $n(f(A)) = 3$ is 600
(C) number of onto functions if $n(f(A)) = 2$ is 180
(D) number of onto functions if $n(f(A)) = 1$ is 5

4. If the function f satisfies the relation $f(x+y) + f(x-y) = 2f(x)f(y) \forall x, y \in R$ and $f(0) \ne 0$, then:

(A) $f(x)$ is an even function
(B) $f(x)$ is an odd function
(C) If $f(2) = a$ then $f(-2) = a$
(D) If $f(4) = b$ then $f(-4) = -b$

5. Let $f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \ge 3 \end{cases}$ and

$g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \ge 4 \end{cases}$ then, which of the following is/are true?

(A) $(f+g)(3.5) = 0$ (B) $f(g(3)) = 3$
(C) $(fg)(2) = 1$ (D) $(f-g)(4) = 0$

6. $f(x) = x^2 - 2ax + a(a+1), f: [a, \infty) \to [a, \infty)$. If one of the solutions of the equation $f(x) = f^{-1}(x)$ is 5049, then the other may be:

(A) 5051 (B) 5048
(C) 5052 (D) 5050

7. If $f: R \to R$, where $f(x) = \frac{Ax + B}{Cx + D}$ is inverse of itself, then:

(A) $A + D = 0 = C; B \in R, A, D \ne 0$
(B) $A = D$ and $B = C = 0$
(C) $B = C = 0$ and $\left|\tan^{-1}\left(\frac{A}{D}\right)\right| = \frac{\pi}{4}$
(D) $2\sin^{-1}C = \pi = \cos^{-1} D$ and $\sin^{-1}A + \sin^{-1}B = \pi$

8. Which of the following is/are pair of identical functions in their common domain?

(A) $f(x) = \sin^{-1} x; g(x) = \cos^{-1}\sqrt{1-x^2}$
(B) $f(x) = \cos^{-1}(1-2x^2); g(x) = 2\sin^{-1} x$

(C) $f(x)=\tan^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right);\ g(x)=\dfrac{1}{2}\cos^{-1}x$

(D) $f(x)=\tan^{-1}\left(\dfrac{\sqrt{1-x^2}-1}{x}\right);\ g(x)=\dfrac{1}{2}\tan^{-1}x$

9. Let $f:R\to R$ be a function defined by $f(x+1)=\dfrac{f(x)-5}{f(x)-3}$ $\forall x\in R$. Then which of the following statement(s) is/are true?

(A) $f(2008)=f(2004)$ (B) $f(2006)=f(2010)$

(C) $f(2006)=f(2002)$ (D) $f(2006)=f(2018)$

10. Let f be a function such that $f(x) = f(2 - x)$ $\forall$ $x \in R$ and $g(x) = f(1 + x)$, then:

(A) $g(x)$ is an odd function

(B) $g(x)$ is an even function

(C) $g(x)$ is neither odd nor even function

(D) graph for $f(x)$ is symmetrical w.r.t line $x = 1$

11. Which of the following statement(s) is/are correct?

(A) Equation $\text{sgn}(x) = \sin^{-1}x$ has one solution

(B) Equation $|x^2 - 4|x| + 3| = 1$ has six solutions

(C) If α is solution of equation $\log_{10}x^2 - \log_{10}(-x) = 4$, then $\text{sgn}(\alpha) = 1$

(D) Equation $(\sin^{-1}x)^3 + (\cos^{-1} x)^3 + (\tan^{-1} x)^3 = 3\sin^{-1}x \cdot \cos^{-1}x \cdot \tan^{-1}x$ has no solution.

(where sgn(.) denotes signum function)

12. Let us consider a function $f(x) = \sqrt{a\{x\}^2+b\{x\}}$ (where $\{x\}$ denotes fractional part of x), then:

(A) If $a > 0 > b$ and $|b| > |a|$, then domain of $f(x)$ is I.

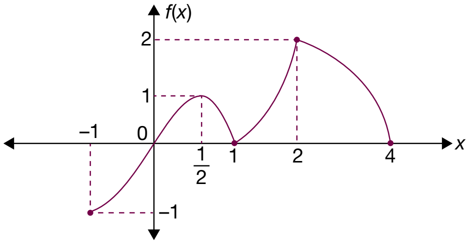
(B) If $a < 0 < b$ and $|b| > |a|$, then domain of $f(x)$ is R.

(C) If $a > 0$ and $b > 0$ then domain of $f(x)$ is R.

(D) If $a < 0$ and $b < 0$, then domain of $f(x)$ is I.

(I denote set of integers, R denote set of real numbers)

13. If the graph of a function $f(x)$ which is defined on [–1, 4] is shown in the adjacent figure, then:



(A) domain of $f(|x| - 1)$ is [–5, 5]

(B) range of $f(|x| + 1)$ is [0, 2]

(C) range of $f(-|x|)$ is [–1, 0]

(D) domain of $f(|x|)$ is [–3, 3]

14. Let $f: R \to R$ be any function. Let $g : R \to R$ be the function defined by $g(x) = |f(x)|$ then function $g(x)$ is:

(A) Onto if f is onto

(B) one-one if f is bijection

(C) Many-one if f is onto

(D) Into if f is onto

15. Which of the following statement are **correct**? Where [·] is GIF

(A) The solution set of $[\sin^{-1} x] > [\cos^{-1} x]$ contains exactly one integer.

(B) Range of $|\sin x| + |\cos x|$ contains exactly two integers

(C) Range of $\left[x+\dfrac{1}{2}\right]+\left[x-\dfrac{1}{2}\right]+2[-x]$ contains exactly two integers.

(D) Range of $\left[\dfrac{1}{[x-3]}\right]$ contains exactly three integers

16. If $f(g(x)) = x$ and $g(f(x)) = x$ then which of the following may be the functions $f(x)$ and $g(x)$?

(A) $f(x) = g(x) = (7 - x^{1/3})^3$

(B) $f(x)=\dfrac{8x-7}{5x+8}\ x\neq-\dfrac{8}{5};\ g(x)=\dfrac{8x+7}{8-5x}\ x\neq\dfrac{8}{5}$

(C) $g(x)=f(x)=\begin{cases} x, & x\in Q\\ -x, & x\notin Q\end{cases}$

(D) $f(x) = \log_e(x - 2), x > 2; g(x) = e^x + 2, x \in R$

17. The range of $f(x)=\left[\dfrac{1}{\sin\{x\}}\right]$ (where [ ] and { } respectively denote the greatest integer and fractional part function) is subset of:

(A) I, set of integers

(B) N, set of natural numbers

(C) W, set of whole numbers

(D) $\{2, 3, 4, \ldots\}$

18. Let $f(x) = x^2 - x + 1$ $\forall$ $x \in R$ and $g(x)$ is also quadratic polynomial such that $[f(x)] = [g(x)]$ $\forall$ $x \in R$ (where [.] denotes greatest integer function), then which of the following must be true for all possible $g(x)$ satisfying condition?

(A) $f(2) - g(1) = 2$ (B) $f(2)-g(2)=\dfrac{2}{3}$

(C) $g(3) - f(2) = 4$ (D) $g(3)-f(3)=\dfrac{1}{2}$

19. Let f, g and h are quadratic polynomials such that $f(x) f(x + 1) = g(h(x))$. If α_1, β_1 are roots of $f(x) = 0$, α_2, β_2 are roots of $g(x) = 0$ and α_3, β_3 are roots of $h(x) = 0$ then which of the following is/are true?

(A) $\alpha_1+\beta_1=\alpha_2+\beta_2$ (B) $\alpha_1+\beta_1=\alpha_3+\beta_3$

(C) $\alpha_1\beta_1=\alpha_2\beta_2$ (D) $\alpha_1\beta_1=\alpha_3\beta_3$

20. Let $f(x)=\sqrt{(\sin^{-1}x-\cos^{-1}x)}$, $g(x)=\sqrt{(\tan^{-1}x-\cot^{-1}x)}$ and $h(x)=\sqrt{(\sec^{-1}x-\text{cosec}^{-1}x)}$ then correct statement(s) is (are):

(A) Domain of $f(x) + g(x)$ is $\{1\}$

(B) Domain of $g(x) + h(x)$ is $[\sqrt{2}, \infty]$

(C) Domain of $h(x) + f(x)$ is $\left[\dfrac{1}{\sqrt{2}},1\right]$

(D) Domain of $f(x) + g(x) + h(x)$ is ϕ

21. Let f be a real valued function defined for all real numbers x such that $f(x + 3) = 4 + \sqrt{8f(x)-(f(x))^2}$, then which is always correct?

(A) $f(x)$ is non periodic
(B) $f(x)$ is periodic
(C) $f(13) - 2f(61) + f(103) = 0$
(D) $f(6) - 2f(10) + f(26) = 0$

22. Let $f_n\theta = \sum_{k=0}^{n} \frac{1}{4^k}\sin^4(2^k\theta)$. Which of the following is/are correct?

(A) $f_2\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ (B) $f_3\left(\frac{\pi}{8}\right) = \frac{2-\sqrt{2}}{4}$
(C) $f_4\left(\frac{3\pi}{2}\right) = 1$ (D) $f_5(\pi) = 0$

23. If $f(x) = \left([\{x\}]\tan^{-1}\left(\frac{x^2-3x-1}{x^2-3x+5}\right) + 3 - x^7\right)^{1/7}$, where $[k]$ and $\{k\}$ denotes greatest integer and fractional part functions of k respectively, then:

(A) $f^{-1}(50) = f(50)$ (B) $f(f(100)) = 100$
(C) $f^{-1}(50) \neq f(50)$ (D) $f(f(100)) = 10$

24. Let $f : [-1, 1] \to \left[\frac{-3\pi}{2}, \frac{3\pi}{2}\right]$ be a real-valued function defined as $f(x) = \sin^{-1} x + |\sin^{-1}x| + \sin^{-1}|x|$, then:

(A) $f(x)$ is neither injective nor surjective
(B) the number of solutions of the equation $f(x) = x$ is one.
(C) $\lim_{x\to 0} \frac{f(x)}{x} = 3$
(D) the number of solutions of the equation $f(x) = \frac{\pi}{2}$ are two.

25. If $f : R \to [2, 4]$, $f(x)$ be periodic function such that the equation $f(x) = 1 - \cos\pi x$ has a unique solution then period of $f(x)$ can be:

(A) π (B) e
(C) $\sqrt{3}$ (D) 1

26. Let $f : R \to$ R be defined as $f(x) = \sqrt[3]{x} + \sin 2x$ and $g : R \to R$ is defined by $g(x) = \frac{1-x^2}{1+x^2}$, then which of the following statement(s) is (are) correct?

(A) f is surjective
(B) $f(x) + g(x)$ is neither even nor odd
(C) range of $g(x)$ is $[-1, 1]$
(D) f is bounded and g is unbounded

27. Which of the following statement(s) is (are) correct?

(A) If f is a one-one mapping from set A to A, then f is onto
(B) If f is an onto mapping from set A to A, then f is one-one
(C) Let f and g be two functions defined from $R \to R$ such that gof is injective, then f must be injective.
(D) If set A contains 3 elements while set B contains 2 elements, then total number of functions from A to B is 8.

28. Let $f : R \to R$ and $g : R \to R$ be mapping such that $f(g(x))$ is an injective mapping, then which of the following statement(s) is (are) correct?

(A) $g(x)$ must be an injective mapping
(B) If $g(x)$ is surjective, then $f(x)$ may not be injective mapping
(C) If $g(x)$ is not surjective, then $f(x)$ may be injective or many-one mapping
(D) $f(x)$ may not be an injective mapping.

29. Which of the following pair of functions have the same graph? [**NOTE:** $[k]$, $\{k\}$ and sgn k denote the largest integer less than or equal to k, fractional part of k and signum function of k, respectively.]

(A) $f(x) = \ln[1 + \{x\}]$ and $g(x) = \ln(1 + [\{x\}])$.
(B) $f(x) = \frac{1}{1+\tan^2 x} + \frac{1}{1+\cot^2 x}$ and $g(x) = 2\sin^2 x + \cos 2x$
(C) $f(x) = 2^{\text{sgn}(x^2+3x+3)}$ and $g(x) = 2\ \text{sgn}\ (x^2 + 3x + 3)$
(D) $f(x) = \frac{\text{sgn}\, x}{\text{sgn}\, x}$ and $g(x) = \frac{|x|}{|x|}$

30. Identify the statement(s) which is/are incorrect?

(A) The function $f(x) = \cos(\cos^{-1} x)$ is neither odd nor even.
(B) The fundamental period of $f(x) = \cos(\sin x) + \cos(\cos x)$ is π.
(C) The range of the function $f(x) = \cos(3 \sin x)$ is $[-1, 1]$.
(D) The function $f(x, y) = \sqrt{x^2 - y^2} + x$ is a homogeneous function of degree 1.

31. Let $f(x) = \begin{cases} x^2 - 4 & \text{if } |x| \le 3 \\ 5\,\text{sgn}\,|x-3|, & \text{if } |x| > 3 \end{cases}$ and $g(x) = 2\tan^{-1}(e^x) - \frac{\pi}{2}$ for all $x \in R$, then which of the following is (are) correct?

[**NOTE:** sgn (k) denotes the signum function of k.]

(A) $fog(x)$ is an even function
(B) $gof(x)$ is an even function
(C) $gog(x)$ is an odd function
(D) $fof(x)$ is an odd function

32. Let $f : \{x, y, z\} \to \{1, 2, 3\}$ is a one-one function. It is given that exactly one of the statements is true and the other two are false. $S_1 : f(x) = 1$, $S_2 : f(y) \neq 1$, $S_3 : f(z) \neq 2$. Then:

(A) $f(x) = 2$ (B) $f(y) = 1$
(C) $f(z) = 3$ (D) $f^{-1}(1) = y$

33. The function 'g' defined by $g(x) = \sin(\alpha) + \cos(\alpha) - 1$ where $\alpha = \sin^{-1}\sqrt{\{x\}}$ where { } denotes functional part function is:

(A) an even function
(B) periodic function
(C) odd function
(D) Neither even nor odd

34. If $2f(x) + xf\left(\frac{1}{x}\right) - 2f\left(\left|\sqrt{2}\sin\pi\left(x+\frac{1}{4}\right)\right|\right) = 4\cos^2\left(\frac{\pi x}{2}\right) + x\cos\frac{\pi}{x}$, $\forall x \in R - \{0\}$ then which of the following statement(s) is/are true?

(A) $f(2) + f\left(\frac{1}{2}\right) = 1$ (B) $f(2) + f(1) = 0$
(C) $f(2) + f(1) = f\left(\frac{1}{2}\right)$ (D) $f(1)\,f\left(\frac{1}{2}\right)f(2) = 1$

35. Let $f(x)=\dfrac{x}{1+x^2}$ and $g(x)=\dfrac{e^{-x}}{1+[x]}$, where $[\cdot]$ is the greatest integer less than or equal to x then:

(A) Domain $(f+g)=R-[-2,0)$

(B) Domain $(f-g)=R-[-1,0)$

(C) Range $f\cap$ Range $g=\left[-2,\dfrac{1}{2}\right]$

(D) Range $g=R-\{0\}$

36. The function $f(x)=x^3+(a+2)x^2+3ax+5,\ f:R\to R$ is:

(A) invertible for infinitely many real values of a

(B) invertible for exactly four integral values of a

(C) invertible for no integral value of a

(D) invertible for $a=2$

37. If $f(x)=-1+|x-2|, 0\le x\le 4, g(x)=2-|x|,-1\le x\le 3$ then:

(A) $fog(x)=\begin{cases}-1-x, & -1\le x\le 0\\ x-1, & 0<x\le 1\\ -1+x, & 1<x\le 2\end{cases}$

(B) $fog(x)=\begin{cases}-1-x, & -1\le x\le 0\\ x-1, & 0<x\le 2\end{cases}$

(C) $gof(x)=\begin{cases}1+x, & 0\le x<1\\ 3-x, & 1\le x\le 2\end{cases}$

(D) $gof(x)=\begin{cases}x-1, & 2<x\le 3\\ 5-x, & 3<x\le 4\end{cases}$

38. Let $f:[-2,2]\to R$ be a function. If $x\in[0,2]$,

$f(x)=\begin{cases}x\tan x; & 0<x\le\dfrac{\pi}{2}\\ \dfrac{\pi}{2}[x]\ ; & \dfrac{\pi}{2}<x\le 2\end{cases}$ then:

(A) If f is even, for $x\in[-2,0]\,f(x)=\begin{cases}-x\tan x; & \dfrac{-\pi}{2}\le x<0\\ \dfrac{-\pi}{2}[-x]; & -2\le x<\dfrac{-\pi}{2}\end{cases}$

(B) If f is odd, for $x\in[-2,0]\,f(x)=\begin{cases}-x\tan x; & \dfrac{-\pi}{2}\le x<0\\ \dfrac{-\pi}{2}[-x]; & -2\le x\le\dfrac{-\pi}{2}\end{cases}$

(C) If f is even function for

$x\in[-2,0]\,f(x)=\begin{cases}x\tan x; & \dfrac{-\pi}{2}\le x<0\\ \dfrac{\pi}{2}[-x]; & -2\le x<\dfrac{-\pi}{2}\end{cases}$

(D) If f is odd function for

$x\in[-2,0]\,f(x)=\begin{cases}x\tan x; & \dfrac{-\pi}{2}\le x<0\\ \dfrac{\pi}{2}[-x]; & -2\le x\le\dfrac{-\pi}{2}\end{cases}$

39. The function $f(x)=\cos^{-1}\left(\dfrac{2\left[|\sin x|+|\cos x|\right]}{\sin^2 x+2\sin x+\dfrac{11}{4}}\right)$ is defined if x belongs to: (where [.] represents greatest integer function)

(A) $\left[0,\dfrac{7\pi}{6}\right]$ (B) $\left[0,\dfrac{\pi}{6}\right]$

(C) $\left[\dfrac{11\pi}{6},2\pi\right]$ (D) $[\pi,2\pi]$

40. If $\dfrac{y+3}{2y+5}=\sin^2 x+2\cos x+1$, then the value of y lie in the interval:

(A) $\left(-\infty,-\dfrac{8}{3}\right]$ (B) $\left[-\dfrac{12}{5},\infty\right)$

(C) $\left[-\dfrac{8}{3},-\dfrac{12}{5}\right]$ (D) $\left(-\infty,-\dfrac{8}{3}\right]\cup\left[-\dfrac{12}{5},\infty\right)$

41. Let $f: X\to Y$ and $g: Y\to Z$. Suppose g is one-one and onto. If $h=gof$, then which of the following is/are true?

(A) If h is one- one, then f is one- one

(B) If f is one-one, then h is one-one and onto

(C) If f is not onto, then h is not onto

(D) If f is not one-one, then h is not one-one

42. Let $f:\ A\to B$, where $A=\{1,2,3,4\}$, $B=\{3,5,7,9,11,13,15,17\}$ and $f(2)=7$

(A) number of one-one function $f:\ A\to B$ is 25

(B) number of strictly increasing function $f:\ A\to B$ is 20

(C) number of strictly decreasing function $f:\ A\to B$ is 5

(D) number of function $f:\ A\to B$ is 8^3

43. $f(x)=\sin^{-1}\left[e^x\right]+\sin^{-1}\left[e^{-x}\right]$, where [.] greatest integer function then:

(A) domain of $f(x)$ is $(-\ln 2, \ln 2)$

(B) range of $f(x)=\{\pi\}$

(C) $f(x)$ has removable discontinuity at $x=0$

(D) $f(x)=\cos^{-1}x$ has only one solution

44. Given functional equation

$$f(x)+\frac{3}{2}f\left(\frac{2x+29}{x-2}\right)=50x+40\{x\ne 2\}.$$

Now if $f(x)=A-Bx+\dfrac{5C}{x-2}$, then:

(A) $\left(2000-\dfrac{B}{2}\right)=5C$ (B) $17A=3B$

(C) $99A=34C$ (D) $5A=17B$

45. Let f be a real valued function such that for any real x, $f(15+x)=f(15-x)$ and $f(30+x)=-f(30-x)$. Then which of the following statement is true?

(A) f is odd (B) f is even

(C) f is periodic (D) f is not periodic

46. If a function satisfies

$$(x-y)f(x+y)-(x+y)f(x-y)\quad=2\left(x^2y-y^3\right):\quad x,y\in R$$

and $f(1)=2$, then:

(A) $f(x)$ must be polynomial function

(B) $f(3)=12$

(C) $f(0)=0$

(D) $f(x)$ may not be differentiable

47. Let $f(x)$ be an onto function defined from $[-3,2]$ to $[2,7]$ then:

(A) Domain of $f(\sin x - \cos x)$ is R

(B) Range of $f(x)$ and $f(x+7)$ is same

(C) Domain of $f(x)$ and $f(x-2)$ is same

(D) Range of $\dfrac{5}{f(x-2)+3}$ is $\left[\dfrac{1}{2},1\right]$

48. Let $f: R \to [\alpha,\infty), f(x) = x^2 + 3ax + b, g(x) = \sin^{-1}\dfrac{x}{4}, (\alpha \in R)$ then:

(A) The number of possible integral values of ‘a’ for which $f(x)$ is many to one in $[-3,5]$ is 5

(B) If $a=-1$ and $gof(x)$ is defined for $x \in [-1,1]$ then number of possible integral values of ‘b’ can be 3.

(C) If $a=2, \alpha=-8$ then the value of ‘b’ for which $f(x)$ is surjective is 1

(D) If $a=1, b=2$ then exact number of integers in the range of $fog(x)$ is 3

49. Consider the real valued function satisfying $2f(\sin x)+f(\cos x)=x$, then:

(A) domain of $f(x)$ is R

(B) domain of $f(x)$ is [–1 1]

(C) range of $f(x)$ is $\left[\dfrac{-2\pi}{3},\dfrac{\pi}{6}\right]$

(D) range of $f(x)$ is $\left[\dfrac{-2\pi}{3},\dfrac{\pi}{3}\right]$

50. Let $f(x)=\begin{cases} x^2-4x+3; & x<3 \\ x-4 \quad ; & x \ge 3 \end{cases}$ and

$g(x)=\begin{cases} x-3 \quad ; & x<4 \\ (x+1)^2+1\,; & x \ge 4 \end{cases}$ then:

(A) $(f-g)\left(\dfrac{7}{2}\right)=-1$ (B) $fog(3)=3$

(C) $(fg)(2)=1$ (D) $(f+g)\left(\dfrac{7}{2}\right)-(f-g)(4)=26$

51. A function $f: R \to R$ is defined by

$f(x+y) - kxy = f(x) + 2y^2 \ \forall x.y \in R$ and $f(1)=2; f(2)=8$,

where k is some constant, then $f(x+y).f\left(\dfrac{1}{x+y}\right)$ equal to: (where $x+y \ne 0$)

(A) 1 (B) 4

(C) k (D) $f(1)$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Consider the function $f(x) = \left|\dfrac{1}{|x|-1}\right|, g(x) = \{x\}$

Where {.} represents fractional part function.

1. Number of solutions of equation $f(x) = g(x)$ in [–3, 3] is:

(A) 1 (B) 4

(C) 2 (D) 0

2. Number of integral solutions of $f(x) = g(x)$ is:

(A) 1 (B) 2

(C) 0 (D) ∞

Comprehension 2

Let $f(x) = x^2 - 2x - 1 \ \forall \ x \in R$. Let $f: (-\infty, a] \to [b, \infty]$, where ‘$a$’ is the largest real number for which $f(x)$ is bijective.

3. The value of $(a + b)$ is equal to:

(A) –2 (B) –1

(C) 0 (D) 1

4. Let $f: R \to R, g(x) = f(x) + 3x - 1$, then the least value of function $y = g(|x|)$ is:

(A) $-\dfrac{9}{4}$ (B) $-\dfrac{5}{4}$

(C) –2 (D) –1

Comprehension 3

Let $f_1(x) = \sin \pi x, f_2(x) = \cos \pi x, f_3(x) = [x], f_4(x) = \{x\}, f_5(x) = -x, f_6(x) = |x|$.

5. Two functions $f_a(x)$ and $f_b(x)$ are selected at random, then the probability that $f_a(f_b(x))$ is a constant function $(a \ne b)$ is:

(A) $\dfrac{11}{{}^6C_2}$ (B) $\dfrac{{}^4C_2\,{}^4C_1}{{}^6C_2}$

(C) $\dfrac{{}^4C_2}{{}^6C_2}$ (D) $\dfrac{3}{{}^6C_2}$

6. Two functions are selected at random and added; if it is known that resulting is periodic, then the probability of that period 2 is:

(A) $\dfrac{1}{2}$ (B) $\dfrac{3}{4}$

(C) $\dfrac{1}{4}$ (D) 1

Comprehension 4

Let $f: [2, \infty) \to [1, \infty)$ defined by $f(x) = 2^{x^4-4x^2}$ and $g: \left[\dfrac{\pi}{2},\pi\right] \to A$ defined by $g(x) = \dfrac{\sin x + 4}{\sin x - 2}$ be two invertible functions, then:

7. $f^{-1}(x)$ is equal to:

(A) $-\sqrt{2+\sqrt{4+\log_2 x}}$
(B) $\sqrt{2+\sqrt{4+\log_2 x}}$
(C) $\sqrt{2-\sqrt{4+\log_2 x}}$
(D) None of these

8. The set A is equal to:

(A) $[-5, -2]$
(B) $[2, 5]$
(C) $[-5, 2]$
(D) $[-3, -2]$

9. The domain of $f^{-1}g^{-1}(x)$ is:

(A) $[-5, -2]$
(B) $\left[-5, \frac{\sin 1}{2-\sin 1}\right]$
(C) $\left[-5, -\frac{(4+\sin 1)}{2-\sin 1}\right]$
(D) $\left[-\frac{(4+\sin 1)}{2-\sin 1}, -2\right]$

Comprehension 5

Let $f: N \times N \to N$ be a function such that $f(1, 1) = 2$,

$f(\alpha + 1, \beta) = f(\alpha, \beta) + \alpha$ and

$f(\alpha, \beta + 1) = f(\alpha, \beta) - \beta$ for all $\alpha, \beta \in N$

and $f(a, b) = 2001$.

Answer the following using the given information.

10. The number of ordered pairs (a, b) is:

(A) 1
(B) 2
(C) 3
(D) Infinite

11. The maximum value of $a + b$ is:

(A) 4000
(B) 3999
(C) 2999
(D) 2000

12. The minimum value of $a + b$ is:

(A) 2999
(B) 1999
(C) 2000
(D) 1000

Comprehension 6

Let f be an even function satisfying $f(x-2) = f\left(x + \left[\frac{6x^2+13}{x^2+2}\right]\right)$ $\forall\ x \in R$ and $f(x) = \begin{cases} 3x, & 0 \le x < 1 \\ 4-x, & 1 \le x \le 4 \end{cases}$ where $[\cdot]$ is GIF

13. The area bounded by the graph of $f(x)$ and the x-axis from $x = -1$ to $x = 9$ is:

(A) $\frac{31}{2}$
(B) 15
(C) 12
(D) $\frac{15}{2}$

14. The value of $f(-89) - f(67) + f(46)$ is equal to

(A) 4
(B) 5
(C) 6
(D) 7

Comprehension 7

Let $f(x) = \begin{cases} |x|; & -3 \le x \le 7 \\ 2x-7; & 7 < x \le 10 \end{cases}$ and $g(x) = 2x + 1, -3 \le x \le 10$.

15. Range of $f(g(x))$ is:

(A) $[-3, 10]$
(B) $[0, 5]$
(C) $[0, 13]$
(D) $[-3, 13]$

16. If the equation $f(|x|) + g(|x|) = k$, has exactly two distinct real solutions, then the value of k cannot be:

(A) 32
(B) 33
(C) 34
(D) 35

Comprehension 8

Consider function $f: R \to R$ defined as

$f(x) = \begin{cases} x+3; & x \in \text{rational} \\ 4x\ ; & x \in \text{irrational} \end{cases}$ and $g: R \to R$ defined as

$g(x) = \begin{cases} x+\sqrt{5}; & x \in \text{irrational} \\ -x\ ; & x \in \text{rational} \end{cases}$

Also $h: [1, \infty) \to [3, \infty)$ be defined as $h(x) = (\log_2 x)^2 + 2(\log_2 x) + 3$.

17. The function $(f - g)(x)$ is:

(A) injective but not surjective.
(B) surjective but not injective.
(C) injective as well as surjective.
(D) neither injective nor surjective.

18. The function $h^{-1}(x)$ is equal to:

(A) $2^{\sqrt{x-3}}$
(B) $4^{\sqrt{x-3}}$
(C) $2^{-1-\sqrt{x-2}}$
(D) $2^{-1+\sqrt{x-2}}$

Comprehension 9

Let the function f be defined in (0, 1). Two more functions are given as $g(x) = e^x$ and $h(x) = \ln|x|$.

19. The number of solution of the equation $g(x) \cdot h(g(x)) = 1$ is:

(A) 0
(B) 1
(C) 2
(D) 3

20. Domain of $f(g(x)) + f(h(x))$ is:

(A) $(-e, -1)$
(B) $(-e, -1) \cup (1, e)$
(C) $\left(\sqrt{e}, e\right)$
(D) $(1, e)$

21. Let f be one-one function with range (1, 2) and $gof\ (x)$ is defined then domain of $f^{-1}og^{-1}(x)$ is:

(A) (0, 1)
(B) (e, e^2)
(C) $\left(\sqrt{e}, e\right)$
(D) $(-\infty, 0)$

Comprehension 10

If a function $y = f(x); f: A \to B$ then the set A is called as domain of the function and B is called co-domain of the function. For all $x \in A$, the values of y thus obtained comprise the set 'C' where C is called as range of function.

22. The domain of the function $f(x) = \frac{1}{\ln\left[\cos^{-1} x\right]}$ is: (where [.] indicates greatest integer function)

(A) $[0, 1]$
(B) $[-1, \cos 2]$
(C) $[-1, \cos 3) \cup (\cos 3, \cos 4)$
(D) $[-1, \cos 3) \cup (\cos 3, \cos 2)$

23. The domain of the function $f(x)=\sqrt{\cos(\sin x)}+\sqrt{\log_x\{x\}}$ is: (where {.} indicates fractional part function).

(A) $[1,\pi)$ (B) $(0,2\pi)-[1,\pi)$

(C) $\left(0,\frac{\pi}{2}\right)-\{1\}$ (D) $(0,1)$

24. The range of the function of $f(x)=\sin^{-1}\sqrt{x^2+x+1}$ is:

(A) $\left[0,\frac{\pi}{2}\right]$ (B) $\left[0,\frac{\pi}{3}\right]$

(C) $\left[\frac{\pi}{3},\frac{\pi}{2}\right]$ (D) $\left[\frac{\pi}{6},\frac{\pi}{2}\right]$

Comprehension 11

$f(x)=\frac{x+a}{bx^2+cx+2}$ where $a, b, c \in R$, $f(-1)=0$ and $y = 1$ is an asymptotes of $y = f(x)$, $y = f^{-1}(x)$ is inverse function of $f(x)$.

25. $f(x)$ is equal to:

(A) $f(x)=\frac{x+1}{x+2}$ (B) $f(x)=\frac{x+1}{x^2+x+2}$

(C) $f(x)=\frac{x+1}{-x+2}$ (D) None of these

26. $f^{-1}: R-\{1\}\to R$ then $f^{-1}(x)$ is:

(A) one-one onto (B) one-one into

(C) many-one into (D) many-one onto

27. Area bounded between the asymptotes curve $f(x)$ and $f^{-1}(x)$:

(A) 4 (B) 9

(C) 16 (D) 25

Comprehension 12

Let $f: R\to[-1,1]$, $g: R\to[-1,1]$ are defined by:

$f(x)=1-2\sin^2 x$, $g(x)=\cos 2x$ and

$F(x)=f(x)+g(x)$, $G(x)=\frac{f(x)}{g(x)}$

28. Domain and range of $G(x)$ are respectively:

(A) R and $\{1\}$

(B) R and $\{0,1\}$

(C) $R-\left\{(2n+1)\frac{\pi}{2}\right\}$ and $\{0,1\}$, $n\in I$

(D) $R-\left\{(2n+1)\frac{\pi}{4}\right\}$ and $\{1\}$, $n\in I$

29. If $F: R\to[-2,2]$, then then $f(x)$ is:

(A) one-one function (B) onto function

(C) a constant function (D) into function

30. Which statement is correct?

(A) periods of $f(x)$, $g(x)$ and $F(x)$ are in A.P with common difference $\frac{\pi}{3}$

(B) periods of $f(x)$, $g(x)$ and $F(x)$ are the same and is equal to 2π

(C) sum of the periods of $f(x)$, $g(x)$ and $F(x)$ is 3π

(D) sum of the periods of $f(x)$, $g(x)$ and $F(x)$ is 6π

Comprehension 13

Let $f: R-\{0,1\}\to R$ be a function satisfying the relation $f(x)+f\left(\frac{x-1}{x}\right)=x$ for all $x\in R-\{0,1\}$. Based on this, answer the following questions.

31. $f(x)$ is equal to:

(A) $\frac{1}{2}\left[x+\frac{1}{1-x}-\frac{x-1}{x}\right]$ (B) $\frac{1}{2}\left[x-\frac{1}{1-x}+\frac{x-1}{x}\right]$

(C) $\frac{1}{2}\left[x-\frac{1}{1-x}-\frac{x-1}{x}\right]$ (D) $\frac{1}{2}\left[x+\frac{1}{1-x}+\frac{x-1}{x}\right]$

32. $f(-1)\ f\left(\frac{1}{2}\right)$ is equal to:

(A) $\frac{3}{16}$ (B) $\frac{-15}{16}$

(C) $\frac{25}{16}$ (D) $\frac{-35}{16}$

Comprehension 14

Let f be a real valued function such that (i) $f(0)=\frac{1}{2}$ $f(x+y)=f(x)f(a-y)+f(y)f(a-x)$ for all $x,y\in R$, for some $a\in R$, then:

33. $f(x^2+1)+f(x-1)=$

(A) 0 (B) $2x^2+x$

(C) 1 (D) x^2-x+2

34. $f(1)+f(2)+f(3)+\ldots+f(100)=$

(A) 5050 (B) 50

(C) 100 (D) 5000

35. If $g(x)=\frac{x^2+2x}{2x^2+1}$ then $(fog)\left(\frac{1}{4}\right)+gof(4)=$

(A) 0 (B) 3/4

(C) 4/3 (D) 1

Comprehension 15

Let $f(x)=\begin{cases}2x+a & : & x\ge -1\\ bx^2+3 & : & x<-1\end{cases}$ and $g(x)=\begin{cases}x+4 & : & 0\le x\le 4\\ -3x-2 & : & -2<x<0\end{cases}$ functions:

36. $g(f(x))$ is not defined if:

(A) $a\in(10,\infty)$, $b\in(5,\infty)$ (B) $a\in(4,10)$, $b\in(5,\infty)$

(C) $a\in(6,\infty)$, $b\in(1,\infty)$ (D) $a\in(4,10)$, $b\in(1,5)$

37. If domain of $g(f(x))$ is $[-1, 4]$, then:

(A) $a=1$, $b>5$ (B) $a=2$, $b>7$

(C) $a=2$, $b>10$ (D) $a=0$, $b\in R$

38. If $a=2$ and $b=3$, then range of $g(f(x))$ is:

(A) $(-2, 8]$ (B) $(0, 8]$

(C) $[4, 8]$ (D) $[-1, 8]$

Comprehension 16

If $f:(0,\infty)\to(0,\infty)$ satisfy $f(xf(y)) = x^2y^a\,(a\in R)$, then:

39. Value of a is:
(A) 4
(B) 2
(C) $\sqrt{2}$
(D) 1

40. $\sum\limits_{r=1}^{n} f(r)\,{}^nC_r$ is:
(A) $n2^{n-1}$ (B) $n(n-1)2^{n-2}$
(C) $n2^{n-1}+n(n-1)2^{n-2}$ (D) 0

41. Number of solutions of $2f(x)=e^x$ is:
(A) 1 (B) 2
(C) 3 (D) 4

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	q	r	●	●
B	p	●	●	s	t
C	●	●	r	s	t
D	p	q	r	●	●

1. Consider three functions, $f(x) = x^3+x^2+x+1$, $g(x) = \dfrac{2x}{1+x^2}$ and $h(x) = \sin^{-1}x - \cos^{-1}x + \tan^{-1}x - \cot^{-1}x$.
Match COLUMN-I with COLUMN-II and select the correct answer using the code given below the list.

COLUMN-I	COLUMN-II
(A) If range of $f(g(x))$ is $[a, b]$, then $(a+b)$ is equal to:	(p) 1
(B) The number of integers in the range of $g(f(x))$ is equal to:	(q) 3
(C) The maximum value of $g(h(x))$ is equal to:	(r) 4
(D) If the minimum value of $h(g(f(x)))$ is $\dfrac{k\pi}{2}$, then $\lvert k\rvert$ is equal to:	(s) 5

2. Match the following:

COLUMN-I	COLUMN-II
(A) Let $f: R\to R$ satisfies $f(x+y)+f(x-y) = 2f(x)f(y)$ $\forall\ x, y\in R$ and $f(0)\neq 0$, then $f(x)$ is:	(p) Even
(B) Let $f: R\to R$ is defined by $f(x) = \dfrac{e^{\lvert x\rvert}-e^{-x}}{e^x+e^{-x}}$, then $f(x)$ is:	(q) Odd
(C) Let $f: R\to R$ be polynomial function satisfying $f(x)$. $f\left(\dfrac{1}{x}\right) = f(x)+f\left(\dfrac{1}{x}\right)$ and $f(3) = 28$ then $f(x)$ is:	(r) Into
(D) Let $f: R\to R$ is defined by $f(x) = 2x+\sin x$, then $f(x)$ is:	(s) Many-one
	(t) Bijective

(Continued)

3. Match the following:

COLUMN-I	COLUMN-II
(A) The function $f(x) = \dfrac{3x-2}{x+4}$ has an inverse that can be written in the form $f^{-1}(x) = \dfrac{x+b}{cx+d}$. The value of $(b+c+d)$ is:	(p) Even
(B) Let $f(x) = \dfrac{x}{1+x}$ and let $g(x) = \dfrac{rx}{1-x}$. Let S be the set of all real numbers r such that $f(g(x)) = g(f(x))$ for infinitely many real number x. The number of elements in set S is:	(q) Odd

(Continued)

COLUMN-I	COLUMN-II
(C) If $f(x) = 2x + 1$ then the value of x satisfying the equation $f(x) + f(f(x)) + f(f(f(x))) + f(f(f(f(f(x))))) = 116$, is:	(r) Prime
	(s) Neither prime nor composite

4. Column Matching:

COLUMN-I	COLUMN-II
(A) Let $f: R \to R$ defined as $f(x) = e^{\operatorname{sgn} x} + e^{x^2}$, where sgn x denotes signum function of x, then $f(x)$ is:	(p) Odd
(B) Let $f: (-1, 1) \to R$ defined as $f(x) = x[x^4] + \dfrac{1}{\sqrt{1-x^2}}$ where$[x]$ denotes greatest integer less then $f(x)$ is:	(q) Even
(C) Let $f: R \to R$ defined as $f(x) = \dfrac{x(x+1)(x^4+1)+2x^4+x^2+2}{x^2+x+1}$ then $f(x)$ is:	(r) Neither odd nor even
(D) Let $f: R \to R$ defined as $f(x) = x + 3x^3 + 5x^5 + \ldots + 101x^{101}$ then $f(x)$ is:	(s) One-One
	(t) Many-One

5. Match the following

COLUMN-I	COLUMN-II
(A) $f(x) = \begin{cases} x+1, & x<0 \\ x^2-1, & x \geq 0 \end{cases}$ then $f(f(x))$ for $-1 \leq x < 0$ is:	(p) $x + 1$
(B) If $f\left(\dfrac{2\tan x}{1+\tan^2 x}\right) = \dfrac{(\cos 2x+1)(\sec^2 x+2\tan x)}{2}$ then $f(x)$ is:	(q) $\dfrac{x-3}{2}$
(C) If $f(x+y+1) = \left(\sqrt{f(x)}+\sqrt{f(y)}\right)^2$ $\forall x, y \in R$ and $f(0) = 1$, then $f(x)$ is:	(r) $x^2 + 2x$
(D) If $4 < x < 5$ and $f(x) = \left[\dfrac{x}{4}\right] + 2x + 2$ where [.] denotes G.I.F. then $f^{-1}(x)$ is:	(s) $(x + 1)^2$

(Continued)

6. Let $A = \{1, 3, 5, 7\}$ and $B = \{2, 4, 6, 8\}$ and $f : A \to B$, then the number of functions 'f' possible for:

COLUMN-I	COLUMN-II
(A) $i + f(i) < 10, \forall i \in \{1, 3, 5, 7\}$	(p) 16
(B) $f(i) - i > 2, \forall i \in \{1, 3, 5, 7\}$	(q) 24
(C) $f(i) \geq 6, \forall i \in \{1, 3, 5, 7\}$	(r) 0
(D) $f(i) \neq i + 1, \forall i \in \{1, 3, 5, 7\}$	(s) 81

7. Match the following:

COLUMN-I	COLUMN-II
(A) Solution of $\lvert x^2 - 1 + \sin x \rvert = \lvert x^2 - 1 \rvert + \lvert \sin x \rvert$ in $[-2\pi, 2\pi]$	(p) $(-\infty, 1)$
(B) Domain of $f(x) = \dfrac{1}{\sqrt{\log_{1/2}(x^2 - 7x + 13)}}$	(q) $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$
(C) Domain of single valued function $y = f(x)$ given by $10^x + 10^y = 10$ is:	(r) $(3, 4)$
(D) Let $x \in \left(0, \dfrac{\pi}{2}\right)$, then solution of $f(x) = \dfrac{1}{\sqrt{-\log_{\sin x} \tan x}}$ is:	(s) $x \in [-2\pi, -\pi] \cup [-1, 0] \cup [1, \pi] \cup \{2\pi\}$

8. Let $f(x) = g(x) + h(x)$ Where $g(x) = \dfrac{1}{\pi}$ $(\sin^{-1}x + \tan^{-1}x)$ and $h(x) = \dfrac{x+1}{x^2+2x+5}$

COLUMN-I	COLUMN-II
(A) If domain of $f(x)$ is $[a, b]$, then $a + b$ equals:	(p) 4
(B) maximum value of $g(x)$ is:	(q) 1/4
(C) Maximum value of $h(x)$ is:	(r) 0
(D) If range of $f(x)$ is $[l, m]$, then $l + m$ equals:	(s) 3/4

9. Match the following:

COLUMN-I	COLUMN-II
(A) The number of possible values of k if fundamental period of $\sin^{-1}(\sin kx)$ is $\frac{\pi}{2}$, is:	(p) 1
(B) Numbers of elements in the domain of $f(x) = \tan^{-1}x + \sin^{-1}x + \sec^{-1}x$ is:	(q) 2
(C) Period of the function $f(x) = \sin\left(\frac{\pi x}{2}\right)\cdot \operatorname{cosec}\left(\frac{\pi x}{2}\right)$ is:	(r) 3
(D) If the range of the function $f(x) = \cos^{-1}[5x]$ is $\{a, b, c\}$ and $a + b + c = \frac{\lambda\pi}{2}$, then λ is equal to (where [.] denotes greatest integer)	(s) 4
	(t) 0

(*Continued*)

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. Find the number of integers in the domain of

$$f(x) = \frac{1}{\sqrt{\ln \cos^{-1} x}}.$$

2. Let $f(x) = \frac{x+1}{x-1}$ for all $x \neq 1$. Let $f^{-1}(x) = f(x)$; $f^2(x) = f(f(x))$ and $f^n(x) = f(f^{n-1}(x))$ for $n > 1$. Let $P = f^1(2)\,f^2(3)\,f^3(4)\,f^4(5)$. Find the number of divisors of P.

3. If $f(x) + f\left(1 - \frac{1}{x}\right) = 1 + x \ \forall\ x \in R - \{0, 1\}$. Find the value of $4f(2)$.

4. Let $f(x) = [\sec\{x\}]$ where $[x]$ and $\{x\}$ denote greatest integer and fractional part of x respectively and $g(x) = 2x^2 - 3x(k + 1) + k(3k + 1)$. Find the number of integral values of k is $g(f(x)) < 0 \ \forall\ x \in R$.

5. If $f(x) = \cos(2010\ \{x^3\}\ (2011^{[x^2]} + 2012x))\ x \in R$ where $\{\cdot\}$ denotes fractional part function and [.] denotes greatest integer function, then find the value of $f_{\max}$ is equal to:

6. Let f be defined on the natural numbers as follow : $f(1) = 1$ and for $n > 1$, $f(n) = f(f(n-1) + f(n - f(n-1)))$, then the value of $\frac{1}{30}\sum_{r=1}^{20} f(r)$ is:

7. Let $f : R \to R$ defined as $f(x) = x^3 + x + 1$, $1 \le x \le 2$. The graph of $y = g(x)$ is the reflection of graph of $y = f(x)$ through the line $y = x$. If the domain of $g(x)$ is $[a, b]$, then $|a - b|$ is:

8. If domain of $f(x) = \dfrac{\sin^{-1}(\sin x)}{\sqrt{-\log_{\left(\frac{x+4}{2}\right)} \log_2\left(\frac{2x-1}{3+x}\right)}}$ is $(a, b) \cup (c, \infty)$, then find the value of $a + b + 3c$.

9. If $f(x) = \dfrac{2010x + 163}{165x - 2010}, x > 0$ and $x \neq \dfrac{2010}{165}$ then the least value of $f(f(x)) + f\left(f\left(\frac{4}{x}\right)\right)$ is _________.

10. Number of solutions of the equation $f(x - 1) + f(x + 1) = \sin\alpha$, $0 < \alpha < \frac{\pi}{2}$ where $f(x) = \begin{cases} 1 - |x|; & |x| \le 1 \\ 0; & |x| > 1 \end{cases}$ is:

11. The number of linear functions satisfying $f[x + f(x)] = x + f(x)\ \forall x \in R$ is:

12. $f(x) = \dfrac{ax + b}{cx + d}\ \forall\ x \in R - \left\{\dfrac{-d}{c}\right\}$ and if $f(5) = 5$, $f(13) = 13$ and $f[f(x)] = x$ for all x. Then range of $f(x) = R - \{x\}$, then $x =$

13. $A = \{1, 2, 3\}$, $B = \{1, 3, 5, 7, 9\}$, then

$$\frac{\text{No. of one-one functions}}{\text{No. strictly monotonic functions}} =$$

14. If $n(A) = 4$, $n(B) = 5$ and number of functions from A to B such that range contains exactly 3 elements is k then $k/60 =$

15. If a and b be constants $f(x) = a\sin x + bx\cos x + 2x^2$. If $f(2) = 15$ then $f(-2) =$

16. If $f(x) = 3[2x] + 1000\sum_{r=1}^{2008} \dfrac{x + r - [x + r]}{2008}$, then $f(3)$ where [.] is GIF

17. Let $f(n)$ denote the square of the sum of the digits of natural number n, where $f^2(n)$ denote $f(f(n))$, $f^3(n)$ denote $f(f(f(n)))$ and so on. The value of $\dfrac{f^{2011}(2011) - f^{2010}(2011)}{f^{2013}(2011) - f^{2012}(2011)} =$

18. Find number of solutions of equation $[\sin x] + \left[\dfrac{x}{2\pi}\right] + \left[\dfrac{2x}{5\pi}\right] = \dfrac{9x}{10\pi}$ in the interval (30, 40)

(where [ ] is GIF)

19. If the range of the function $f(x) = \sqrt{24\{x\} - 5 - 16\{x\}^2}$, ({.} denotes fractional part of x) is $[a, b]$ then the value of $b^2 - a$ is equal to ___________.

20. The number of solutions of the equation $e^{-\sqrt{|\ln\{x\}|}}-\{x\}^{1/\sqrt{|\ln\{x\}|}}=[\operatorname{sgn}(x)]$ (where [.] is greatest integer function and {.} is fractional part function) is/are:

21. The number of solution of equation $8\left[x^2-x\right]+4[x]=13+12[\sin x]$ where [.] GIF is ______ .

22. If $(a,b]$ is the domain of the function $f(x)=\sqrt{\log_{0.5}\left(\dfrac{2x-8}{x-2}\right)}$ then value of $b-a$ is _____ .

23. Let $f:A\to B$ be an onto function where A = {1, 2, 3, 4}, B = {x, y, z} also $f(1)=x$. Let M be total such functions then find the value of $\dfrac{M}{2}$.

24. The sum of all the real values of x satisfying the equation $[x]^2-5[x]+5-2\cos x=0$, where [.] denotes GIF and $x\in(0,\pi)$ is $\dfrac{\mu\pi}{\lambda}$, then the value of $|\mu+\lambda|=$ (Where μ,λ are prime to each other)

25. If $f(x^2-6x+6)+f(x^2-4x+4)=2x\ \forall x\in R$, then $f(-3)+f(9)-5f(1)=$ ____ .

26. Let $f:\ R\to R$ is a function satisfying $f(10-x)=f(x)$ and $f(2-x)=f(2+x)\ \forall x\in R$. If $f(0)=101$. Then the minimum possible number of values of x satisfying $f(x)=101\ \forall x\in[0,25]$ is _____________ .

27. If $f(x)=\sin^2 x+\sin^2\left(x+\dfrac{\pi}{3}\right)+\cos x\cos\left(x+\dfrac{\pi}{3}\right)$ and $g(x)=[x]\operatorname{sgn}(x)$ where [·] is GIF, then $(gof)(x)$ is

28. If the maximum possible set B, where $f:R\to B$ and $f(x)=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}+2$ is onto is (a,b) and if c is the arithmetic mean of a and b, then $f^{-1}(c)$ equal:

29. $A=\{1,2,3,4,5\}$, $B=\{1,2,3,4\}$. The number of functions from A to B where $n(f(A))=3$ is K, then $\dfrac{k}{100}=$

30. $f:[-2,2]\to R$ where $f(x)=x^3+\tan x+\left[\dfrac{x^2+1}{\lambda}\right]$ is a odd function where [·] is GIF. Then the least positive integer of λ is.

31. Sum of all possible values of range of $[|\sin x|+\pi]$ is, Where [.] denotes G.I.F.

32. Number of functions defined from $f:\{1,2,3,4,5,6\}\longrightarrow\{7,8,9,10\}$ such that the sum $f(1)+f(2)+f(3)+f(4)+f(5)+f(6)$ is odd, is , 2^p, then sum of the digits of p is:

33. The total number of ordered pairs (x, y) satisfying $|x|+|y|=4$ and $\sin\left(\dfrac{\pi x^2}{3}\right)=1$ is equal to:

34. Number of distinct real solutions of $\sin\pi x=\left|\ln|x|\right|$ is:

35. f is a function such that $f(x)+2f(1-x)=x^2+1$. then find $f(3)$.

36. If n is a natural and $1\le n\le 100$, then the number of solutions of $\left[\dfrac{n}{2}\right]+\left[\dfrac{n}{3}\right]+\left[\dfrac{n}{5}\right]=\dfrac{n}{2}+\dfrac{n}{3}+\dfrac{n}{5}$ (where [.] denotes the greatest integer function).

37. Let $f\left(x+\dfrac{1}{y}\right)+f\left(x-\dfrac{1}{y}\right)=2f(x)f\left(\dfrac{1}{y}\right)\forall x,y\in R,\quad y\ne 0$ and $f(0)=0$, then the value of $f(1)+f(2)=$

38. For non negative integers m,n define a function as follows

$$f(m,n)=\begin{cases} n+1 & \text{if} \quad m=0\\ f(m-1,1) & \text{if} \quad m\ne 0, n=0\\ f(m-1,f(m,n-1)) & \text{if} \quad m\ne 0, n\ne 0\end{cases}$$

Then the value of $f(1,1)$ is:

39. Number of real values of x, satisfying the equation $[x]^2-5[x]+6-\sin x=0$ [.] denoting the greatest integer function is:

40. Let $f:[0,1]\to[0,1]$ defined by $f(x)=\dfrac{1-x}{1+x}$, for $0\le x\le 1$ and let $g:[0,1]\to[0,1]$ defined by $g(x)=4x(1-x), 0\le x\le 1$. If range of $fog(x)$ is $[\alpha,\beta]$, then $\alpha+\beta=$

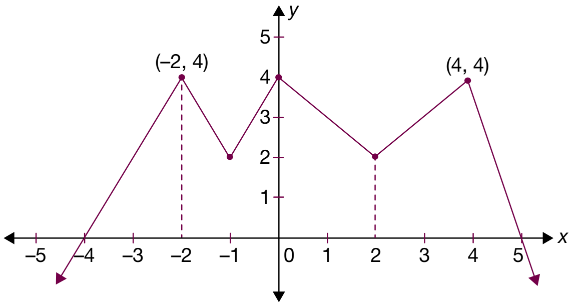
41. If $f(x+y,x-y)=xy$, then the arithmetic mean of $f(x,y)$ and $f(y,-x)$ is _______ .

42. $f:R\to R$ is given by $f(x)=\dfrac{a^x}{a^x+\sqrt{a}}\forall\ x\in R$, then $f\left(\dfrac{1}{11}\right)+f\left(\dfrac{2}{11}\right)+....+f\left(\dfrac{9}{11}\right)+f\left(\dfrac{10}{11}\right)=$

43. If the function $f(x)=\begin{cases}x+1 \text{ if } & x\le 1\\ 2x+1 \text{ if } & 1<x\le 2\end{cases}$ and $g(x)=\begin{cases}x^2 \ ; & -1\le x\le 2\\ x+2; & 2\le x\le 3\end{cases}$ then the number of roots of the equation $f(g(x))=2$ is:

44. If $f(x)+f(y)+f(xy)=2+f(x)\cdot f(y)$, for all real values of x and y and $f(x)$ is a polynomial function with $f(4)=17$, then find the value of $f(5)\cdot(f(1)\ne 1)$.

45. Consider the graph of a real-valued continuous function $f(x)$ defined on R (the set of all real numbers) as shown below.



Find the number of real solutions of the equation $f(f(x))=4$.

46. Let the equation $(a-1)x^2 = x(2b+3)$ be satisfied by three distinct values of x, where $a, b \in R$.
If $f(x) = (a-1)x^3 + (2b+3)x^2 + 2x + 1$, and $f(g(x)) = 6x - 7$ where $g(x)$ is a linear function then find the value of $g'(2012)$.

47. Consider $g : D \to R$ defined as $g(x) = \dfrac{tx^2 + 3x - 4}{t + 3x - 4x^2}$ where D is the domain of function and R is set of all real numbers. If $g(x)$ is surjective, then find the number of integral values of t.

48. Consider all function $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ which are one-one, onto and satisfy the property: if $f(k)$ is odd then $f(k+1)$ is even, $k = 1, 2, 3$. If number of such functions is p, then find $(p - 7)$.

49. Let $f(x) = \left[\dfrac{1}{\cos\{x\}}\right]$ where $[y]$ and $\{y\}$ denote greatest integer and fractional part function, respectively and $g(x) = 2x^2 - 3x(k+1) + k(3k+1)$. If $g(f(x)) < 0\ \forall\ x \in R$, then find the number of integral values of k.

50. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 2, 3, 4\}$. If N be the number of 4 element subsets Y of X such that $10 \in Y$ and the intersections of Y and A is not empty, then find the value of $\dfrac{(N-4)}{10}$.

51. Find the sum of all integral values of a where $a \in [-10, 10]$ such that the graph of the function $f(x) = \left|\,|x-2| - a\right| - 3$ has exactly three x-intercepts.

52. Find the smallest value of $\left(\dfrac{-8p}{7}\right)$ for which

$|x^2 - 5x + 7 - p| = 6 + |x^2 - 5x + 1 - p|$ for all $x \in [-1, 3]$.

53. Let the range of the function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ assumes exactly 3 distinct values. If the number of such functions is N, then find the value of $\dfrac{N}{300}$.

54. Find the number of integral values of k for which the equation $\left|x^2 - 5|x| + 6\right| = 2k$, has exactly four real and distinct solutions.

55. Let f be a function such that $4f(x^{-1} + 1) + 8f(x + 1) = \log_{12} x$, then find the value of $4(f(10) + f(13) + f(17))$.

56. Find the sum of the integral values of 'a' where $a \in [20, 100]$ for which $[\cot x]^2 + \cot x - a = 0$ has real roots? Where $[\cdot]$ is GIF.

57. Set A consists of 5 distinct elements and set B consists of 4 distinct elements. Number of functions that can be defined from A to B so that exactly one element of the set B is not image of any element of the set A.

58. Find the sum of all solution of the equation $\cot \dfrac{\pi x}{2} = \log_2\{x\}$ in $x \in (0, 100)$ where $\{\cdot\}$ is fractional part function.

59. Find the number of integers in the range of the function $f(x) = \cos x(\sin x + \sqrt{\sin^2(x) + 3}\,)$.

60. Let f be a real valued invertible function such that

$f\left(\dfrac{2x-3}{x-2}\right) = 5x - 2,\ x \neq 2$ Find $f^{-1}(13)$.

61. Find the largest integral value of 'a' for which every solution of the equation, $x([x] - 5) + 2\{x\} + 6 = 0$ satisfies the inequality $(a-3)x^2 + 2(a+3)x - 8a \leq 0$, where $[y]$ and $\{y\}$ denote greatest integer and fractional part functions, respectively.

62. Find the number of integral values of a in interval $[0, 100]$ so the range of function $f(x) = \dfrac{x+a}{x^2+1}$ contains the interval $[0, 1]$.

63. $f : R \to R, f(x) = \dfrac{3x^2 + mx + n}{x^2 + 1}$. If the range of this function is $[-4, 3)$, then find the value of $m^2 + n^2$.

64. Find the smallest integral value of a such that
$|x + a - 3| + |x - 2a| = |2x - a - 3|$ is true $\forall\ x \in R$.

ARCHIVE: JEE MAIN

1. **[JEE MAINS 2021]**
The number of the real roots of the equation
$(x+1)^2 + |x-5| = \dfrac{27}{4}$ is _______.

2. **[JEE MAINS 2021]**
If $a + \alpha = 1, b + \beta = 2$ and $af(x) + \alpha f\left(\dfrac{1}{x}\right) = bx + \dfrac{\beta}{x}, x \neq 0$, then the value of the expression $\dfrac{f(x) + f\left(\dfrac{1}{x}\right)}{x + \dfrac{1}{x}}$ is _______.

3. **[JEE MAINS 2021]**
A function $f(x)$ is given by $f(x) = \dfrac{5^x}{5^x + 5}$, then the sum of the series $f\left(\dfrac{1}{20}\right) + f\left(\dfrac{2}{20}\right) + f\left(\dfrac{3}{20}\right) + \ldots + f\left(\dfrac{39}{20}\right)$ is equal to:

(A) $\dfrac{19}{2}$ (B) $\dfrac{49}{2}$
(C) $\dfrac{39}{2}$ (D) $\dfrac{29}{2}$

4. **[JEE MAINS 2021]**
Let $f : R \to R$ be defined as $f(x) = 2x - 1$ and $g : R - \{1\} \to R$ be defined as $g(x) = \dfrac{x - \frac{1}{2}}{x - 1}$.
Then the composition function $f(g(x))$ is:
(A) onto but not one-one (B) one-one but not onto
(C) both one-one and onto (D) neither one-one nor onto

5. **[JEE MAINS 2021]**
Let $A = \{1, 2, 3, \ldots, 10\}$ and $f : A \to A$ be defined as
$f(k) = \begin{cases} k+1, & k \in \text{odd} \\ k, & k \in \text{even} \end{cases}$.

Then the number of possible functions $g : A \to A$ such that $gof = f$ is:

(A) ${}^{10}C_5$ (B) $5!$

(C) 5^5 (D) 10^5

6. **[JEE MAINS 2020]**

Let $A = (a, b, c)$ and $B = \{1, 2, 3, 4\}$. Then the number of elements in the set $C = \{f : A \to B \mid 2 \in f(A)$ and f is not one-one$\}$ is ________.

7. **[JEE MAINS 2020]**

Suppose that a function $f : R \to R$ satisfies $f(x + y) = f(x)\, f(y)$ for all $x, y \in R$ and $f(1) = 3$.

If $\sum_{i=1}^{n} f(i) = 363$, then n is equal to ________.

8. **[JEE MAINS 2020]**

If $g(x) = x^2 + x - 1$ and $(gof)(x) = 4x^2 - 10x + 5$, then $f\left(\frac{5}{4}\right)$ is equal to:

(A) $\frac{3}{2}$ (B) $-\frac{1}{2}$

(C) $-\frac{3}{2}$ (D) $\frac{1}{2}$

9. **[JEE MAINS 2020]**

The inverse function of $f(x) = \frac{8^{2x} - 8^{-2x}}{8^{2x} + 8^{-2x}}$, $x \in (-1, 1)$, is

(A) $\frac{1}{4}(\log_8 e)\log_e\left(\frac{1-x}{1+x}\right)$ (B) $\frac{1}{4}\log_e\left(\frac{1-x}{1+x}\right)$

(C) $\frac{1}{4}(\log_8 e)\log_e\left(\frac{1+x}{1-x}\right)$ (D) $\frac{1}{4}\log_e\left(\frac{1+x}{1-x}\right)$

10. **[JEE MAINS 2020]**

Let S be the set of all real roots of the equation, $3^x(3^x - 1) + 2 = |3^x - 1| + |3^x - 2|$. Then S:

(A) contains at least four elements.

(B) is an empty set.

(C) contains exactly two elements.

(D) is a singleton.

11. **[JEE MAINS 2020]**

Let $f : (1, 3) \to R$ be a function defined by $f(x) = \frac{x[x]}{1+x^2}$, where $[x]$ denotes the greatest integer $\le x$. Then the range of f is:

(A) $\left(\frac{2}{5}, \frac{3}{5}\right] \cup \left(\frac{3}{4}, \frac{4}{5}\right)$ (B) $\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]$

(C) $\left(\frac{3}{5}, \frac{4}{5}\right)$ (D) $\left(\frac{2}{5}, \frac{4}{5}\right]$

12. **[JEE MAINS 2019]**

If the function $f : R - \{1, -1\} \to A$ defined by $f(x) = \frac{x^2}{1-x^2}$, is surjective, then A is equal to:

(A) $R - \{-1\}$ (B) $R - [-1, 0)$

(C) $R - (-1, 0)$ (D) $[0, \infty)$

13. **[JEE MAINS 2019]**

The domain of the definition of the function

$f(x) = \frac{1}{4 - x^2} + \log_{10}(x^3 - x)$ is:

(A) $(1, 2) \cup (2, \infty)$ (B) $(-2, -1) \cup (-1, 0) \cup (2, 0)$

(C) $(-1, 0) \cup (1, 2) \cup (2, \infty)$ (D) $(-1, 0) \cup (1, 2) \cup (3, \infty)$

14. **[JEE MAINS 2019]**

Let $f(x) = x^2$, $x \in R$, For any $A \subseteq R$, define $g(A) = \{x \in R : f(x) \in A\}$. If $S = [0, 4]$, then which one of the following statements is not true?

(A) $g(f(S)) \ne S$ (B) $f(g(S)) = S$

(C) $g(f(S)) \ne f(S)$ (D) $g(f(S)) = g(S)$

15. **[JEE MAINS 2019]**

The number of real roots of the equation $5 + |2^x - 1| = 2^x(2^x - 2)$ is:

(A) 2 (B) 1

(C) 3 (D) 4

16. **[JEE MAINS 2019]**

For $x \in (0, 3/2)$, let $f(x) = \sqrt{x}$, $g(x) = \tan x$ and $h(x) = \frac{1-x^2}{1+x^2}$

If $\phi(x) = ((hof)og)(x)$, then $\phi\left(\frac{\pi}{3}\right)$ is equal to:

(A) $\tan\frac{7\pi}{12}$ (B) $\tan\frac{11\pi}{12}$

(C) $\tan\frac{\pi}{12}$ (D) $\tan\frac{5\pi}{12}$

17. **[JEE MAINS 2019]**

For $x \in R$, let $[x]$ denote the greatest integer $\le x$, then the sum of the series:

$$\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \left[-\frac{1}{3} - \frac{2}{100}\right] + \ldots + \left[-\frac{1}{3} - \frac{99}{100}\right]$$

(A) -135 (B) -153

(C) -133 (D) -131

18. **[JEE MAINS 2019]**

If $[x]$ denotes the greatest integer $\le x$, then the system of linear equations

$[\sin\theta]x + [-\cos\theta]y = 0$, $[\cot\theta]x + y = 0$

(A) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and have infinitely many solutions if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$

(B) have infinitely many solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and has a unique solution if $\theta \in \left(\pi, \frac{7\pi}{6}\right)$

(C) have infinitely many solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

(D) has a unique solution if $\theta \in \left(\frac{\pi}{2}, \frac{2\pi}{3}\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

19. **[JEE MAINS 2017]**

The function $f : R \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$, is:

(A) neither injective nor surjective

(B) invertible

(C) injective but not surjective

(D) surjective but not injective

20. **[JEE MAINS 2017]**

Let, $a, b, c \in R$. If $f(x) = ax^2 + bx + c$ is such that $a + b + c = 3$ and $f(x + y) = f(x) + f(y) + xy, \forall\, x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to:

(A) 255 (B) 330

(C) 165 (D) 190

21. **[JEE MAINS 2016]**

If $f(x) + 2f\left(\frac{1}{x}\right) = 3x, x \neq 0$ and $S = \{x \in R : f(x) = f(-x)\}$; then S

(A) contains exactly one element

(B) contains exactly two elements

(C) contains more than two elements

(D) is an empty set

22. **[JEE (MAIN)-2014]**

If $a \in R$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where $[x]$ denotes the greatest integer $\leq x$) has no integral solution, then all possible values of a lie in the interval:

(A) $(-1, 0) \cup (0, 1)$ (B) $(1, 2)$

(C) $(-2, -1)$ (D) $(-\infty, -2) \cup (2, \infty)$

ARCHIVE: JEE ADVANCED

1. **[JEE Adv 2020]**

If the function $f : R \to R$ is defined by $f(x) = |x|\,(x - \sin x)$, then which of the following statements is TRUE?

(A) f is one-one, but NOT onto

(B) f is onto, but NOT one-one

(C) f is BOTH one-one and onto

(D) f is NEITHER one-one NOR onto

2. **[JEE Adv 2020]**

Let the function $f : [0, 1] \to R$ be defined by $f(x) = \dfrac{4^x}{4^x + 2}$.

Then the value of $f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + f\left(\frac{3}{40}\right) + \ldots + f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right)$ is ____

3. **[JEE Adv 2018]**

Let $E_1 = \left\{x \in R : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$ and

$E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{ is a real number}\right\}$

(Here, the inverse trigonometric function $\sin^{-1}x$ assumes value in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

Let $f : E_1 \to R$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$ and $g : E_2 \to R$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$.

COLUMN-I	COLUMN-II
(A) The range of f is:	(p) $\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
(B) The range of g contains:	(q) $(0, 1)$
(C) The domain of f contains:	(r) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(D) The domain of g is:	(s) $(-\infty, 0) \cup (0, \infty)$
	(t) $\left(-\infty, \frac{e}{e-1}\right]$
	(u) $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

(*Continued*)

The correct option is:

(A) A → (s); B → (q); C → (p); D → (p)

(B) A → (r); B → (r); C → (u); D → (t)

(C) A → (s); B → (q); C → (p); D → (u)

(D) A → (s); B → (r); C → (u); D → (t)

4. **[JEE Adv 2018]**

Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is is the number of one-one functions from X to Y and β is the number of onto functions from Y to x, then the value of $\frac{1}{5!}(\beta - \alpha)$ is ________.

5. **[JEE Adv 2015]**

Let $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x) = \frac{\pi}{2}\sin x$ for all $x \in R$. Let $(fog)(x)$ denote $f(g(x))$ and $(gof)(x)$ denote $g(f(x))$. Then which of the following is (are) true?

(A) Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(B) Range of fog is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(C) $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{\pi}{6}$

(D) There is an $x \in R$ such that $(gof)(x) = 1$

6. **[JEE Adv 2014]**

Let $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R$ be given by $f(x) = (\log(\sec x + \tan x))^3$ then:

(A) $f(x)$ is an odd function

(B) $f(x)$ is a one-one function

(C) $f(x)$ is an onto function

(D) $f(x)$ is an even function

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. D	**2.** A	**3.** B	**4.** B	**5.** C	**6.** D	**7.** D	**8.** B	**9.** B	**10.** C
11. C	**12.** D	**13.** B	**14.** B	**15.** D	**16.** D	**17.** C	**18.** A	**19.** D	**20.** B
21. B	**22.** D	**23.** D	**24.** D	**25.** C	**26.** B	**27.** D	**28.** C	**29.** C	**30.** D
31. B	**32.** C	**33.** C	**34.** B	**35.** B	**36.** B	**37.** B	**38.** A	**39.** B	**40.** A
41. C	**42.** D	**43.** A	**44.** A	**45.** D	**46.** D	**47.** C	**48.** D	**49.** B	**50.** D
51. C	**52.** C	**53.** D	**54.** A	**55.** B	**56.** B	**57.** B	**58.** A	**59.** B	**60.** B
61. A	**62.** C	**63.** B	**64.** D	**65.** C	**66.** C	**67.** A	**68.** B	**69.** B	**70.** B
71. B	**72.** B	**73.** C	**74.** B	**75.** C	**76.** A	**77.** B	**78.** A	**79.** B	**80.** C
81. B	**82.** A	**83.** B	**84.** B	**85.** C	**86.** B	**87.** B	**88.** A	**89.** B	**90.** D
91. D	**92.** B	**93.** B	**94.** C	**95.** B	**96.** C	**97.** B	**98.** B	**99.** A	**100.** C
101. C	**102.** C	**103.** D	**104.** A	**105.** D	**106.** B	**107.** A	**108.** A	**109.** C	**110.** B

Multiple Correct Choice Type Questions

1. B, C, D	**2.** B, D	**3.** A, B, C, D	**4.** A, C	**5.** A, B, C
6. B, D	**7.** A, B, C	**8.** C, D	**9.** A, B, C, D	**10.** B, D
11. B, D	**12.** A, B, C, D	**13.** A, B, C	**14.** C, D	**15.** A, C, D
16. A, B, C, D	**17.** A, B, C	**18.** A, C	**19.** A, C, D	**20.** A, B, D
21. B, C	**22.** B, C, D	**23.** A, B	**24.** A, B, D	**25.** A, B, C
26. A, B	**27.** C, D	**28.** A, C, D	**29.** A, C, D	**30.** A, B, C
31. A, B, C	**32.** A, B, C, D	**33.** A, B	**34.** A, B, C	**35.** B, D
36. A, B, D	**37.** B, C, D	**38.** B, C	**39.** A, B, C	**40.** A, B, D
41. A, C, D	**42.** B, C, D	**43.** A, C	**44.** A, C, D	**45.** A, C
46. A, B, C	**47.** A, B, D	**48.** A, B, C	**49.** B, D	**50.** A, B, C, D
51. B, C				

Linked Comprehension Type Questions

1. A	**2.** C	**3.** B	**4.** C	**5.** D	**6.** D	**7.** B	**8.** A	**9.** A	**10.** B
11. B	**12.** C	**13.** B	**14.** A	**15.** C	**16.** D	**17.** D	**18.** D	**19.** B	**20.** A
21. B	**22.** B	**23.** D	**24.** C	**25.** A	**26.** B	**27.** B	**28.** D	**29.** B	**30.** C
31. A	**32.** D	**33.** C	**34.** B	**35.** C	**36.** C	**37.** D	**38.** C	**39.** A	**40.** C
41. C									

Matrix Match/Column Match Type Questions

1. A → (r)	B → (q)	C → (p)	D → (s)
2. A → (p, r, s)	B → (r, s)	C → (t)	D → (q, t)
3. A → (q, s)	B → (p, r)	C → (q, r)	
4. A → (r, t)	B → (q, t)	C → (r, t)	D → (p, s)
5. A → (r)	B → (p)	C → (s)	D → (q)
6. A → (q)	B → (r)	C → (p)	D → (s)
7. A→ (s)	B→(r)	C→ (p)	D→ (q)
8. A → (r)	B → (s)	C → (q)	D → (q)
9. A → (q)	B → (q)	C → (q)	D → (r)

Integer/Numerical Answer Type Questions

1. 2	**2.** 6	**3.** 3	**4.** 1	**5.** 1	**6.** 7	**7.** 8	**8.** 5	**9.** 4	**10.** 4
11. 2	**12.** 9	**13.** 3	**14.** 6	**15.** 1	**16.** 18	**17.** 1	**18.** 1	**19.** 4	**20.** 0
21. 0	**22.** 4	**23.** 6	**24.** 2	**25.** 9	**26.** 9	**27.** 1	**28.** 0	**29.** 6	**30.** 6
31. 7	**32.** 2	**33.** 12	**34.** 6	**35.** 0	**36.** 3	**37.** 0	**38.** 3	**39.** 1	**40.** 1
41. 0	**42.** 5	**43.** 2	**44.** 26	**45.** 7	**46.** 3	**47.** 8	**48.** 5	**49.** 1	**50.** 7
51. 3	**52.** 6	**53.** 5	**54.** 3	**55.** 3	**56.** 310	**57.** 600	**58.** 2525	**59.** 5	**60.** 3
61. 1	**62.** 100	**63.** 16	**64.** 1						

ARCHIVE: JEE MAIN

1. 2	**2.** 2	**3.** C	**4.** B	**5.** D	**6.** 19.00	**7.** 5.00	**8.** B	**9.** C	**10.** D
11. B	**12.** B	**13.** C	**14.** D	**15.** B	**16.** B	**17.** C	**18.** B	**19.** D	**20.** B
21. B	**22.** A								

ARCHIVE: JEE ADVANCED

1. C	**2.** 19	**3.** A	**4.** 119	**5.** A, B, C	**6.** A, B, C

CHAPTER

3

Limit of Function

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Indeterminate Form
(b) Solving Limits Using Substitution Method
(c) Standard Limits
(d) L-hospital's Rule
(e) Series Expansion for Limit
(f) Solving Limit of $\infty - \infty$, 1^∞ Indeterminate Form
(g) Solving Limit of 0^0 and ∞^0 form
(h) Limit of a Composite Function
(i) Sandwich Theorem/Squeeze Theorem
(j) Limits in Two Variables

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

In this chapter, we discuss behaviour of function in neighbourhood of a point. Let consider a function $y = f(x)$, we are studying function's behavior around a real number α, if we think of finding an interval for values of x in which α lies, let $(\alpha - \delta, \alpha + \delta)$ is such interval in which α lies, to make is smallest interval there is no measure to get such least δ.

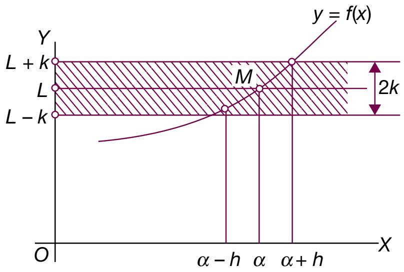
The set $(\alpha - \delta, \alpha)$ is called as left neighborhood of α and $(\alpha, \alpha + \delta)$ is called as right neighborhood of α. Clearly, x is not equal to α but very close to α, then, we get two values, one slightly less than α and other slightly greater than α. In other words, as $x \to \alpha$ and $x = \alpha$ are two different things, $x = \alpha$ mean x has exact value α but $x \to \alpha$ means x is tending (moving/approaching) towards α and α can be approached from two side one from left side and other from right side of it, The function $y = f(x)$ approaches to a definite value L $(y \to L)$ as x approaches α.

If for every positive number h arbitrarily small, we are able to indicate very small $k > 0$ such that for all such x, close to α and satisfying the inequality $|x - \alpha| < h$.

we have the inequality $|f(x) - L| < k$ then,

$$\lim_{x \to \alpha} f(x) = L$$

or, $\quad f(x) \to L$ as $x \to \alpha$.



Graph: $y = f(x)$

Fig. 3.1

If $f(x) \to L$ as $x \to \alpha$, this is illustrated on the graph of the function $y = f(x)$ as shown. Since, from the inequality $|x - \alpha| < h$; there follows the inequality $|f(x) - L| < k$, this means that for all points x that are not more distant from the point α than h, the points M of the graph of the function $y = f(x)$ lie within a band of width $2k$ bonded by the lines $y = L - k$ and $y = L + k$.

To summarize above, we can conclude that,

$x \to \alpha$ and $x = \alpha$ are two different things, $x = \alpha$ mean x has exact value α but $x \to \alpha$ means x is tending to α and α can be approached from two side one from left side and other from right side of it.

Left hand limit (LHL) = $\lim\limits_{x \to \alpha^-} f(x) = \lim\limits_{h \to 0} f(\alpha - h)$.

Right hand limit (RHL) = $\lim\limits_{x\to\alpha^+} f(x) = \lim\limits_{h\to 0} f(\alpha + h)$.

$\lim\limits_{x\to\alpha} f(x)$ exists if $\lim\limits_{x\to\alpha^-} f(x) = \lim\limits_{x\to\alpha^+} f(x) =$ finite

Or $f(\alpha^-) = f(\alpha^+) =$ finite

Note

For existing of limit, value of function at $x = \alpha$, $f(\alpha)$ has no role to decide existence of limit.

In given function represented in figure 3.2, we can observe that $\lim\limits_{x\to\alpha^-} f(x) = \lim\limits_{x\to\alpha^+} f(x) =$ finite, hence, $\lim\limits_{x\to\alpha} f(x)$ exists.

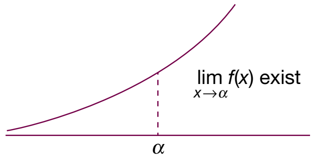


Fig. 3.2

In given function represented in figure 3.3, we can observe that $\lim\limits_{x\to\alpha^-} f(x) \neq \lim\limits_{x\to\alpha^+} f(x)$.

LHL and RHL both are having different values, hence, $\lim\limits_{x\to\alpha} f(x)$ does not exists.

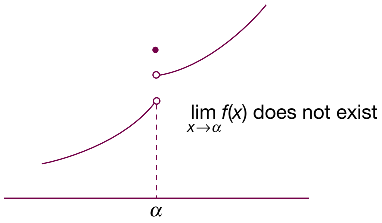


Fig. 3.3

In given function represented in figure 3.4, we can observe that $\lim\limits_{x\to\alpha^-} f(x) = \lim\limits_{x\to\alpha^+} f(x)$.

LHL and RHL both are having same definite values, hence, $\lim\limits_{x\to\alpha} f(x)$ exists.

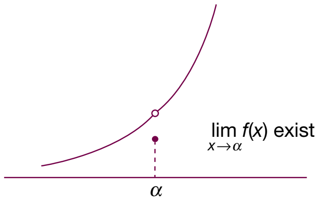
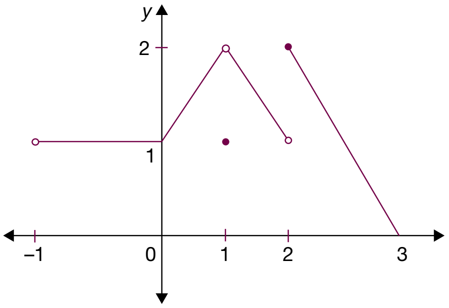


Fig. 3.4

Cleary, we can conclude that for function whose graph is not broken (continuous function) at that point, limit exists. If graph of function is broken at point, then, limit may not exist, in function 3.3 limit does not exist but in function 3.4 limit exists.

ILLUSTRATION 1

Function $f(x)$ is shown, discuss limit of $f(x)$ at points $x = 0$, 1, 2.



SOLUTION

At $x = 0$, LHL =RHL= 1, hence $\lim\limits_{x\to 0} f(x) = 1; f(0) = 1$

At $x = 1$, LHL = $\lim\limits_{x\to 1^-} f(x) = 2; f(1) = 1$, RHL = $\lim\limits_{x\to 1^+} f(x) = 2$

$\therefore \quad \lim\limits_{x\to 1} f(x) = 2 \neq f(1)$

At $x = 2$, LHL = $\lim\limits_{x\to 2^-} f(x) = 1; f(2) = 2$; RHL = $\lim\limits_{x\to 2^+} f(x) = 2$

Hence, $\lim\limits_{x\to 2} f(x)$ does not exist as LHL ≠ RHL

ILLUSTRATION 2

If $f(x) = \begin{cases} 5x - 4 & ; \ 0 < x \le 1 \\ 4x^3 - 3x; & 1 < x < 2 \end{cases}$, Show that $\lim\limits_{x\to 1} f(x)$ exists.

SOLUTION

We have, LHL of $f(x)$ at $x = 1$

$$= \lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h)$$

$$= \lim_{h\to 0} \{5(1-h) - 4\} = \lim_{h\to 0} (1 - 5h) = 1$$

RHL of $f(x)$ at $x = 1$

$$= \lim_{x\to 1^+} f(x) = \lim_{h\to 0} f(1+h)$$

$$= \lim_{h\to 0} \left\{4(1+h)^3 - 3(1+h)\right\} = 4(1)^3 - 3(1) = 1$$

Thus, RHL = LHL = 1

Hence, $\lim\limits_{x\to 1} f(x)$ exists and is equal to 1.

ILLUSTRATION 3

$$f(x) = \begin{cases} -x^2 + 3; & 0 \le x < 1 \\ 4x - 1; & 1 \le x < 2 \end{cases}, \text{ find } \lim_{x\to 1} f(x).$$

SOLUTION

Here, LHL = $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (-x^2 + 3)$

$$= \lim_{h\to 0} (-(1-h)^2 + 3) = 2$$

$$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4x-1)$$

$$= \lim_{h\to 0} (4(1+h)-1) = 3$$

LHL ≠ RHL. Hence, limit does not exist.

> {.} is fractional part function and [.] is greatest integer function and *h* is very small positive quantity, then,
>
> $\{h\} = h, \{-h\} = 1 - h$
>
> $[h] = 0, [-h] = -1$
>
> $\text{sgn}(h) = 1, \text{sgn}(-h) = -1$
>
> If *P* is an integer
>
> $\{P + h\} = \{h\} = h, \{P - h\} = \{-h\} = 1 - h$
>
> $[P + h] = P, [P - h] = P - 1$

ILLUSTRATION 4

$\lim_{x\to 3}\left[x^2-1\right]$ where [.] is greatest integer function.

SOLUTION

$$\text{RHL} = \lim_{x\to 3^+}\left[x^2-1\right] = \lim_{h\to 0}\left[(3+h)^2-1\right]$$

$$= \lim_{h\to 0}\,[9+h^2+6h-1] = \lim_{h\to 0}\,[8+h^2+6h] = 8$$

$$\text{LHL} = \lim_{x\to 3^-}\,[x^2-1] = \lim_{h\to 0}\left[(3-h)^2-1\right]$$

$$= \lim_{h\to 0}\,[8-6h+h^2] = 7$$

∴ h^2 is negligible in comparison to h. So, $8 - 6h + h^2 < 8$.

So, $[8 - 6h + h^2] = 7$ and LHL ≠ RHL.

Hence, $\lim_{x\to 3}\,[x^2-1]$ does not exists.

ILLUSTRATION 5

Evaluate $\lim_{x\to 0}\,\{x\}[x]$ where {.}is fractional part function and [.] is greatest integer function.

SOLUTION

$$\text{LHL} = \lim_{h\to 0}\,\{-h\}[-h] = \lim_{h\to 0}\,(1-h)\,(-1) = -1$$

$$\text{RHL} = \lim_{h\to 0}\,\{h\}[h] = \lim_{h\to 0}\,h(0) = 0$$

Hence, required limit does not exist.

ILLUSTRATION 6

Evaluate $\lim_{x\to 1}\,[x]\{x\}$ where {.}is fractional part function and [.] is greatest integer function.

SOLUTION

$$\text{LHL} = \lim_{h\to 0}\,[1-h]\,\{1-h\} = \lim_{h\to 0}\,0\,(1-h) = 0$$

$$\text{RHL} = \lim_{h\to 0}\,[1+h]\,\{1+h\} = \lim_{h\to 0}\,1(h) = 0$$

Hence, $\lim_{x\to 1}\,[x]\,\{x\} = 0$

ILLUSTRATION 7

Evaluate $\lim_{x\to 1} x.\,\text{sgn}\,(x-1)$ where sgn(.) is signum function.

SOLUTION

Put, $x = 1 - h$

$$\text{LHL} = \lim_{h\to 0}\,(1-h)\,\text{sgn}\,(1-h-1) = \lim_{h\to 0}\,(1-h)\,\text{sgn}\,(-h)$$

$$= \lim_{h\to 0}\,(1-h)\,(-1) = -1$$

Put, $x = 1 + h$

$$\text{RHL} = \lim_{h\to 0}\,(1+h)\,\text{sgn}\,(1+h-1) = \lim_{h\to 0}\,(1+h)\,\text{sgn}\,h$$

$$= \lim_{h\to 0}\,(1+h)\,1 = 1$$

Hence, required limit does not exist.

ILLUSTRATION 8

Evaluate $\lim_{x\to 0}\frac{\{x\}}{x}$ where {.}is fractional part function.

SOLUTION

Put, $x = 0 - h$

$$\text{LHL} = \lim_{h\to 0}\frac{\{-h\}}{-h}\,(-\,0.00001)$$

$$= \lim_{h\to 0}\frac{1-h}{-h} = \lim_{h\to 0} -\frac{1}{h} - 1 \to -\infty$$

Put, $x = 0 + h$

$$\text{RHL} = \lim_{x\to 0^+}\frac{\{x\}}{x} = \lim_{h\to 0}\frac{\{h\}}{h} = \lim_{h\to 0}\frac{h}{h} = 1$$

Hence, required limit does not exist.

ILLUSTRATION 9

Evaluate $f(x) = \lim_{x\to 2}\dfrac{1}{1+5^{\left(\frac{1}{2-x}\right)}}$

SOLUTION

For LHL of $f(x)$ at $x = 2$

Here, $x \to 2^-$, put $x = 2 - h$

$$\text{LHL} = \lim_{x\to 2^-}\frac{1}{1+5^{\frac{1}{2-x}}} = \lim_{h\to 0}\frac{1}{1+5^{\left(\frac{1}{2-(2-h)}\right)}}$$

$$= \lim_{h\to 0}\frac{1}{1+5^{\frac{1}{h}}}\;\left(h\to 0 \;\Rightarrow\; \frac{1}{h}\to\infty \;\Rightarrow\; 5^{1/h}\to\infty\right)$$

$$\therefore \quad \text{LHL} = \frac{1}{1+\infty} = 0$$

For RHL of $f(x)$ at $x = 2$, put $x = 2 + h$

$$\text{RHL} = \lim_{x\to 2^+}\frac{1}{1+5^{\frac{1}{2-x}}} = \lim_{h\to 0}\frac{1}{1+5^{\frac{1}{2-(2+h)}}}$$

$$= \lim_{h\to 0}\frac{1}{1+5^{(-1/h)}}\;\left(h\to 0 \;\Rightarrow -\frac{1}{h}\to -\infty \;\Rightarrow\; 5^{-1/h}\to 0\right)$$

$$\text{RHL} = \frac{1}{1+0} = 1$$

LHL ≠ RHL

Hence, limit does not exist.

ILLUSTRATION 10

Evaluate $\lim\limits_{x\to 0} \dfrac{e^{1/x}-1}{e^{1/x}+1}$.

SOLUTION

Let, $f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}$

LHL of $f(x)$ at $x = 0$

$$= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{e^{-1/h}-1}{e^{-1/h}+1}$$

$$= \lim_{h\to 0}\left(\frac{\frac{1}{e^{1/h}}-1}{\frac{1}{e^{1/h}}+1}\right) = \frac{0-1}{0+1} = -1$$

$$\left[\because\ h\to 0 \Rightarrow \frac{1}{h}\to\infty \Rightarrow e^{1/h}\to\infty \Rightarrow \frac{1}{e^{1/h}}\to 0\right]$$

RHL of $f(x)$ at $x = 0$

$$= \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1}$$

[Dividing Numerator and Denominator by $e^{1/h}$]

$$= \lim_{h\to 0}\left(\frac{1-\frac{1}{e^{1/h}}}{1+\frac{1}{e^{1/h}}}\right) = \frac{1-0}{1+0} = 1$$

Clearly, $\lim\limits_{x\to 0^-} f(x) \neq \lim\limits_{x\to 0^+} f(x)$.

Hence, $\lim\limits_{x\to 0} f(x)$ does not exist.

INDETERMINATE FORM

Lets consider following function and their values in neighbourhood of zero.

$\lim\limits_{x\to 0}\dfrac{x}{2x}$ is 0/0 form; $\lim\limits_{x\to 0}\dfrac{x}{2x} = \lim\limits_{x\to 0}\dfrac{1}{2} = \dfrac{1}{2}$

$\lim\limits_{x\to 0}\dfrac{4x}{3x}$ is 0/0 form; $\lim\limits_{x\to 0}\dfrac{4x}{3x} = \lim\limits_{x\to 0}\dfrac{4}{3} = \dfrac{4}{3}$

$\lim\limits_{x\to 0}\dfrac{2x^2}{7x^2}$ is 0/0 form; $\lim\limits_{x\to 0}\dfrac{2x^2}{7x^2} = \lim\limits_{x\to 0}\dfrac{2}{7} = \dfrac{2}{7}$

$\lim\limits_{x\to 0}\dfrac{4x^2}{9x}$ is 0/0 form; $\lim\limits_{x\to 0}\dfrac{4x^2}{9x} = \lim\limits_{x\to 0}\dfrac{4x}{9} = 0$

$\lim\limits_{x\to 0}\dfrac{7x}{11x^2}$ is 0/0 form; $\lim\limits_{x\to 0}\dfrac{7x}{11x^2} = \lim\limits_{x\to 0}\dfrac{7}{11x}$ $\begin{cases} \to +\infty \text{ if } x\to 0^+ \\ \to -\infty \text{ if } x\to 0^- \end{cases}$

As we can see that all these function are of $\dfrac{0}{0}$ form and we are getting different values as $\dfrac{1}{2}, \dfrac{4}{3}, \dfrac{2}{7}$, 0, ±∞

So, we can say that $\dfrac{0}{0}$ is an indeterminate form as we are not getting any definite value so these are known as indeterminate form. Value of these terms can be computed (if defined) once function is known to us.

Conceptual Note(s)

There are total 7 such indeterminate form as follows:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty, 1^{\infty}, \infty-\infty, 0^{\circ}, \infty^{0}$$

(i) All values are approaching values, not exact values.

(ii) Students often get confused with following terms but these are not indeterminate form.

$\infty+\infty\to\infty$; $\infty-0\to\infty$; $\infty\times\infty\to\infty$; $\infty^{\infty}\to\infty$; $0^{\infty}\to 0$; $\dfrac{0}{\infty}\to 0$

ILLUSTRATION 11

Identify type of indeterminate form if present

(i) $\lim\limits_{x\to 0}\dfrac{1}{x^2}-\dfrac{1}{x^4}$ **(ii)** $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x}+\dfrac{1}{x^3}\right)$ **(iii)** $\lim\limits_{x\to 0}(1)^{1/x^2}$

(iv) $\lim\limits_{x\to 0}(\cos x)^{1/x^2}$ **(v)** $\lim\limits_{x\to 0}(\sin x)^x$ **(vi)** $\lim\limits_{x\to\infty}\dfrac{\cos x}{x}$

(vii) $\lim\limits_{x\to\infty}\dfrac{x}{e^x}$ **(viii)** $\lim\limits_{x\to 1^+}([x])^{\frac{1}{x-1}}$

SOLUTION

(i) $\lim\limits_{x\to 0}\dfrac{1}{x^2}-\dfrac{1}{x^4}$ where indeterminate form $(\infty-\infty)$ is present.

(ii) $\lim\limits_{x\to 0^+}\left(\dfrac{1}{x}+\dfrac{1}{x^3}\right)$ where $\infty+\infty\to\infty$, it is not indeterminate form.

(iii) $\lim\limits_{x\to 0}(1)^{1/x^2}$ where (Exact 1)$^{\infty}$, it is not indeterminate form.

(iv) $\lim\limits_{x\to 0}(\cos x)^{1/x^2}$ where 1^{∞} indetermiante form is present.

(v) $\lim\limits_{x\to 0}(\sin x)^x$ where 0^0 indeterminate form is present.

(vi) $\lim\limits_{x\to\infty}\dfrac{\cos x}{x} = \dfrac{-1\text{ to }1}{\infty}\to 0$ it is not indeterminate form.

(vii) $\lim\limits_{x\to\infty}\dfrac{x}{e^x}$ where $\left(\dfrac{\infty}{\infty}\right)$ indeterminate form is present.

(viii) $\lim\limits_{x\to1^+}([x])^{\frac{1}{x-1}} = \lim\limits_{h\to0}(1)^{1/h}$ Put $x = 1 + h$

It is (Exact 1)$^{\infty}$, so it not indeterminate form.

SOLVING LIMITS OF $\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty$ INDETERMINATE FORM

Using Factorization and Rationalization

Factorization and rationalization methods are also used to remove indeterminate form of type $\frac{0}{0}, \frac{\infty}{\infty}, 0\times\infty$ form.

This is particularly used when either the numerator or denominator or both involve expression consisting of factors, square roots and on substituting the value of x.

ILLUSTRATION 12

If $\lim\limits_{x\to2}\left(\dfrac{4x^2-16x+16}{3x^3-9x^2+12}\right) = \dfrac{m}{n}$, where m and n are relatively prime natural numbers, then, compute $m + n$.

SOLUTION

$$\lim_{x\to2}\left(\frac{4x^2-16x+16}{3x^3-9x^2+12}\right) = \lim_{x\to2}\frac{4(x-2)^2}{3(x-2)^2(x+1)}$$

$$= \lim_{x\to2}\frac{4}{3(x+1)} = \frac{4}{9} \equiv \frac{m}{n}$$

Hence, $(m + n) = 13$.

ILLUSTRATION 13

Evaluate $\lim\limits_{x\to\cot^{-1}(-1)}\left(\dfrac{\tan^3 x - 2\tan x - 1}{\tan^5 x - 2\tan x - 1}\right)$.

SOLUTION

$$\lim_{x\to\cot^{-1}(-1)}\left(\frac{\tan^3 x-2\tan x-1}{\tan^5 x-2\tan x-1}\right)$$

$$= \lim_{x\to\frac{3\pi}{4}}\left(\frac{\tan^3 x-2\tan x-1}{\tan^5 x-2\tan x-1}\right) \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to\frac{3\pi}{4}}\left(\frac{(\tan x+1)(\tan^2 x-\tan x-1)}{(\tan x+1)(\tan^4 x-\tan^3 x+\tan^2 x-\tan x-1)}\right)$$

$$= \lim_{x\to\frac{3\pi}{4}}\left(\frac{\tan^2 x-\tan x-1)}{\tan^4 x-\tan^3 x+\tan^2 x-\tan x-1}\right)$$

$$= \frac{(-1)^2-(-1)-1}{(-1)^4-(-1)^3+(-1)^2-(-1)-1} = \frac{1}{3}$$

ILLUSTRATION 14

Evaluate $\lim\limits_{x\to3}\dfrac{\sqrt{x+1}-\sqrt{3x-5}}{x-3}$.

SOLUTION

$$\lim_{x\to3}\frac{\sqrt{x+1}-\sqrt{3x-5}}{x-3}$$

$$= \lim_{x\to3}\frac{\sqrt{x+1}-\sqrt{3x-5}}{(x-3)}\times\frac{\sqrt{x+1}+\sqrt{3x-5}}{\left(\sqrt{x+1}+\sqrt{3x-5}\right)}$$

$$= \lim_{x\to3}\frac{(x+1)-(3x-5)}{(x-3)\left(\sqrt{x+1}+\sqrt{3x-5}\right)}$$

$$= \lim_{x\to3}\frac{-2(x-3)}{(x-3)\left(\sqrt{x+1}+\sqrt{3x-5}\right)}$$

$$= \lim_{x\to3}\frac{-2}{\left(\sqrt{x+1}+\sqrt{3x-5}\right)} = \frac{-2}{2+2} = -\frac{1}{2}$$

ILLUSTRATION 15

Evaluate $\lim\limits_{x\to3}\dfrac{\sqrt{16x^2+112}-16}{3x^2-15x+18}$.

SOLUTION

$$\lim_{x\to3}\frac{\sqrt{16x^2+112}-16}{3x^2-15x+18}$$

$$= \lim_{x\to3}\frac{16x^2-144}{3(x-3)(x-2)}\times\frac{1}{\sqrt{16x^2+112}+16},$$

By rationalization,

$$= \lim_{x\to3}\frac{16(x-3)(x+3)}{3(x-3)(x-2)}\times\frac{1}{\sqrt{16x^2+112}+16}$$

$$= \lim_{x\to3}\frac{16(x+3)}{3(x-2)}\times\frac{1}{\left(\sqrt{16x^2+112}+16\right)}$$

$$= \frac{16\times6}{3.1.32} = 1$$

ILLUSTRATION 16

If $\lim\limits_{x\to1}\dfrac{a\sqrt{x+3}-b}{x-1} = \dfrac{1}{4}$, then, find a and b.

SOLUTION

$$\lim_{x\to1}\frac{a\sqrt{x+3}-b}{x-1} = \frac{1}{4}$$

limit exists so it must be $\frac{0}{0}$ from $\Rightarrow 2a - b = 0$.

$\Rightarrow \quad b = 2a$ …(1)

Now, $\lim\limits_{x\to 1}\dfrac{a\left(\sqrt{x+3}-2\right)}{(x-1)}=\lim\limits_{x\to 1}\dfrac{a\left(\sqrt{x+3}-2\right)\left(\sqrt{x+3}+2\right)}{(x-1)\left(\sqrt{x+3}+2\right)}$

$$\Rightarrow \quad \lim_{x\to 1}\frac{a(x-1)}{(x-1)\left[\sqrt{x+3}+2\right]}=\frac{1}{4}$$

$$\Rightarrow \quad \frac{a}{4}=\frac{1}{4}$$

Hence, $a = 1$ and $b = 2$.

ILLUSTRATION 17

Let $\lim\limits_{x\to 1}\dfrac{x^6+1+\frac{5}{3}(x^4+x^2)-\frac{8}{3}(x^5+x)}{x^5+x+6x^3-4x^4-4x^2}=\dfrac{p}{q}$ where $p, q \in N$. find the smallest value of $(p + q)$.

SOLUTION

$$\lim_{x\to 1}\frac{x^6+1+\frac{5}{3}(x^4+x^2)-\frac{8}{3}(x^5+x)}{x^5+x+6x^3-4x^4-4x^2}$$

$$=\frac{1}{3}\lim_{x\to 1}\frac{3x^6-8x^5+5x^4+5x^2-8x+3}{x(x-1)^4},$$

(Using factorization)

$$=\frac{1}{3}\lim_{x\to 1}\frac{(x-1)^4(3x^2+4x+3)}{x(x-1)^4},\ \left(\frac{0}{0}\text{form}\right)$$

$$=\frac{1}{3}\lim_{x\to 1}\frac{3x^2+4x+3}{x}=\frac{10}{3}$$

Hence, smallest value of $p + q = 13$.

Test Your Concepts-I

(Solutions on page H.50)

1. Evaluate $\lim\limits_{x\to 4}\dfrac{[x]-4}{x^2-5x+4}$ where [.] is GIF.

2. Evaluate $\lim\limits_{x\to 1}\dfrac{\{x\}}{x}$ where {.} is fractional part function.

3. Show that the $\lim\limits_{x\to a}\dfrac{1}{1-e^{1/(x-a)}}$ does not exist.

4. Evaluate $\lim\limits_{x\to 1^-}\dfrac{x^2-1}{[x-1]}$; where [ ] is the greatest integer function.

5. Find $\lim\limits_{x\to 1^-}[\sin^{-1}x]$ where [.] is GIF.

6. Evaluate $\lim\limits_{x\to \frac{1}{2}^+}[x^2-x+2]\dfrac{x}{[2x]}$ where [.] is GIF.

7. Let $f(x) = 2p[2x + 5] + q[3x - 7]$. If $\lim\limits_{x\to 1}f(x)$ exists, then, find $2p + q$.

8. Evaluate $\lim\limits_{x\to 1}\dfrac{x^2-4x+3}{x^3-6x^2+11x-6}$.

9. Evaluate $\lim\limits_{x\to 3}\dfrac{(x-7)(x^2+2x-15)}{x^3-11x^2+24x}$.

10. Evaluate $\lim\limits_{x\to 6}\dfrac{\sqrt{x-4}-\sqrt{2}}{\sqrt{x-2}-2}$.

11. Evaluate $\lim\limits_{x\to -2^-}\dfrac{\sqrt{x+2}}{\left(\sqrt{2+\sqrt{x+2}}-\sqrt{2}\right)}$.

12. Find the value of $\lim\limits_{x\to\infty}x^{3/2}\left\{\sqrt{x^3+1}-\sqrt{x^3-1}\right\}$ is:

Answer key

1. DNE **2.** DNE **4.** 0 **5.** 1 **6.** $\frac{1}{2}$ **7.** 0 **8.** -1 **9.** $\frac{32}{15}$ **10.** $\sqrt{2}$

11. $2\sqrt{2}$ **12.** 1

ALGEBRA OF LIMITS

(i) Let $\lim\limits_{x\to\alpha}f(x)=l$ and $\lim\limits_{x\to\alpha}g(x)=m$. If l and m exist, then,

(a) $\lim\limits_{x\to\alpha}(f\pm g)(x)=\lim\limits_{x\to\alpha}f(x)\pm\lim\limits_{x\to\alpha}g(x)=l\pm m$

(b) $\lim\limits_{x\to\alpha}(fg)(x)=\lim\limits_{x\to\alpha}f(x)\,.\,\lim\limits_{x\to\alpha}g(x)=lm$

(c) $\lim\limits_{x\to\alpha}\left(\dfrac{f}{g}\right)(x)=\dfrac{\lim\limits_{x\to\alpha}f(x)}{\lim\limits_{x\to\alpha}g(x)}=\dfrac{l}{m}$, provided $m \neq 0$.

(d) $\lim\limits_{x\to\alpha}kf(x)=k\,.\,\lim\limits_{x\to\alpha}f(x)=k\,l$, where k is constant.

(e) $\lim\limits_{x\to\alpha}\left(f(x)\right)^{g(x)}=\lim\limits_{x\to\alpha}f(x)^{\lim\limits_{x\to\alpha}g(x)}=l^{\,m}$ where l and m both are non zero finite.

(ii) If $\lim\limits_{x\to\alpha} f(x) = +\infty$ or $-\infty$, then, $\lim\limits_{x\to\alpha} \dfrac{1}{f(x)} = 0$.

(iii) If $\lim\limits_{x\to\alpha} (f(x)+g(x))$ exists then,

Both $\lim\limits_{x\to\alpha} f(x)$ and $\lim\limits_{x\to\alpha} g(x)$ exists.

Let $f(x) = x^2$; $g(x) = 5x$

$\lim\limits_{x\to1} (f(x)+g(x)) = \lim\limits_{x\to1}(x^2+5x) = 6$

Where $\lim\limits_{x\to1} f(x) = 1$ and $\lim\limits_{x\to1} g(x) = 5$ both exist.

Both $\lim\limits_{x\to\alpha} f(x)$ and $\lim\limits_{x\to\alpha} g(x)$ do not exist.

Let $f(x) = [x]$, $g(x) = \{x\}$

Where [.] is gif, {.} is fpf

$\lim\limits_{x\to1} [x]$ and $\lim\limits_{x\to1} \{x\}$ both does not exist.

But $\lim\limits_{x\to1} ([x]+\{x\}) = \lim\limits_{x\to1} x = 1$ exists.

(iv) If $\lim\limits_{x\to\alpha} f(x)\cdot g(x)$ exists then,

$\lim\limits_{x\to\alpha} f(x)$ and $\lim\limits_{x\to\alpha} g(x)$ both exists.

Let $f(x) = x$; $g(x) = x^2$

$\lim\limits_{x\to2} f(x) = 2$; $\lim\limits_{x\to2} g(x) = 4$

$\lim\limits_{x\to2} f(x)\,g(x) = \lim\limits_{x\to2} x^2 = 8$

Exactly one of $\lim\limits_{x\to\alpha} f(x)$ and $\lim\limits_{x\to\alpha} g(x)$ exists.

Let $f(x) = [x]$; $g(x) = 0$

$\lim\limits_{x\to0} [x]$ = Does not exist, $\lim\limits_{x\to0} g(x) = 0$

$\lim\limits_{x\to0} f(x)\,g(x) = \lim\limits_{x\to0} 0 = 0$

Both $\lim f(x)$ and $\lim\limits_{x\to\alpha}$ does not exist.

Let $f(x) = [x]$, $g(x) = \{-x\}$

$\lim\limits_{x\to0} f(x)$, $\lim\limits_{x\to0} g(x)$ both not exist.

but $\lim\limits_{x\to0} f(x)\,g(x) = 0$

ILLUSTRATION 18

Let $\lim\limits_{x\to a} (f(x).g(x)) = 9$ and $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = 4$. If $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ both exist then, find $\lim\limits_{x\to a} (f(x)+g(x))$.

SOLUTION

Let $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m$ then, $lm = 9$ and $\dfrac{l}{m} = 4$.

$\therefore\quad l^2 = 36 \Rightarrow l = +6$ or -6

If $l = 6$ then, $m = \dfrac{3}{2}$

$\therefore\quad \lim\limits_{x\to a} (f(x)+g(x)) = l + m = \dfrac{15}{2}$

And, if $l = -6$ then, $m = \dfrac{-3}{2}$

$\therefore\quad \lim\limits_{x\to a} (f(x)+g(x)) = l + m = \dfrac{-15}{2}$

SOLVING LIMITS USING SUBSTITUTION

ILLUSTRATION 19

Evaluate $\lim\limits_{x\to\pi/6} \dfrac{\sin\left(x-\dfrac{\pi}{6}\right)}{\sqrt{3}-2\cos x}$.

SOLUTION

$$\lim_{x\to\pi/6} \frac{\sin\left(x-\dfrac{\pi}{6}\right)}{\sqrt{3}-2\cos x}$$

Put $x = \dfrac{\pi}{6} + h$

$$= \lim_{h\to0} \frac{\sin h}{\sqrt{3}-2\cos(h+\pi/6)} = \lim_{h\to0} \frac{\sin h}{\sqrt{3}-\sqrt{3}\cos h+\sin h}$$

$$= \lim_{h\to0} \frac{2\sin(h/2)\cos(h/2)}{2\sqrt{3}\sin^2(h/2)+2\sin(h/2)\cos h/2}$$

$$= \lim_{h\to0} \frac{\cos\,(h/2)}{\sqrt{3}\sin(h/2)+\cos\dfrac{h}{2}} = 1$$

ILLUSTRATION 20

Find the following limit $\lim\limits_{\theta\to1/\sqrt{2}} \dfrac{\theta-\cos(\sin^{-1}\theta)}{1-\tan(\sin^{-1}\theta)}$.

SOLUTION

Let $\theta = \sin t$. When $\theta \to 1/\sqrt{2}$, $t \to \pi/4$.

We have $\lim\limits_{\theta\to1/\sqrt{2}} \dfrac{\theta-\cos(\sin^{-1}\theta)}{1-\tan(\sin^{-1}\theta)} = \lim\limits_{t\to\pi/4} \dfrac{\sin t-\cos t}{1-\tan t}$

$$= \lim_{t\to\pi/4} \frac{\cos t(\sin t-\cos t)}{\cos t-\sin t} = -\lim_{t\to\pi/4} \cos t = -\frac{1}{\sqrt{2}}$$

ILLUSTRATION 21

Evaluate $\lim\limits_{x \to 2} \dfrac{2^x + 2^{3-x} - 6}{\sqrt{2^{-x} - 2^{1-x}}}$.

SOLUTION

$$\lim_{x \to 2} \frac{2^x + 2^{3-x} - 6}{\sqrt{2^{-x} - 2^{1-x}}} = \lim_{x \to 2} \frac{2^x + \frac{2^3}{2^x} - 6}{\left(2^{-x}\right)^{\frac{1}{2}} - \frac{2}{2^x}}$$

$$= \lim_{x \to 2} \frac{(2^x)^2 + 8 - 6.2^x}{2^{x - x/2} - 2} \cdot \frac{2^x}{2^x}$$

$$= \lim_{x \to 2} \frac{(2^x)^2 - 6.2^x + 8}{\left(2^x\right)^{\frac{1}{2}} - 2}$$

Put $2^x = t$, As $x \to 2$, we get $t \to 4$

$$= \lim_{t \to 4} \frac{t^2 - 6t + 8}{\sqrt{t} - 2} = \lim_{t \to 4} \frac{(t-4)(t-2)}{\sqrt{t} - 2}$$

$$= \lim_{t \to 4} (\sqrt{t} + 2)\,(t - 2) = 4.2 = 8$$

ILLUSTRATION 22

Evaluate $\lim\limits_{x \to 0} \dfrac{(\cos x)^{1/3} - (\cos x)^{1/2}}{\sin^2 x}$.

SOLUTION

$$\lim_{x \to 0} \frac{(\cos x)^{1/3} - (\cos x)^{1/2}}{\sin^2 x} = \lim_{x \to 0} \frac{(\cos x)^{1/3} - (\cos x)^{1/2}}{1 - \cos^2 x}$$

Put $\cos x = t^6$, As $x \to 0$ then, $t^6 = \cos x \to 1$;

$$= \lim_{t \to 1} \frac{(t^6)^{1/3} - (t^6)^{1/2}}{1 - t^{12}}$$

$$= \lim_{t \to 1} \frac{t^2 - t^3}{1 - t^{12}} \qquad \left(\frac{0}{0} \text{form}\right)$$

$$= \lim_{t \to 1} \frac{t^2(1-t)}{(1+t^6)(1+t^3)(1+t+t^2)(1-t)}$$

$$= \frac{1}{2.2.3} = \frac{1}{12}$$

STANDARD LIMITS

(i) $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = \lim\limits_{x \to 0} \dfrac{\sin^{-1} x}{x} = 1$

(ii) $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2} = \dfrac{1}{2}$

(iii) $\lim\limits_{x \to 0} \dfrac{\tan x}{x} = \lim\limits_{x \to 0} \dfrac{\tan^{-1} x}{x} = 1$

(iv) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log_e a \ (a > 0)$

(v) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$

(vi) $\lim\limits_{x \to 0} \dfrac{\ell n(1+x)}{x} = 1, \lim\limits_{x \to 0} \dfrac{\log_a(1+x)}{x} = \log_a e$

(vii) $\lim\limits_{x \to \infty} \dfrac{\sin x}{x} = \lim\limits_{x \to \infty} \dfrac{\cos x}{x} = 0 \qquad \left(\because \dfrac{-1 \text{ to } 1}{\infty} = 0\right)$

(viii) $\lim\limits_{x \to \infty} \dfrac{\sin 1/x}{1/x} = 1$

As $x \to \alpha$, then, $f(x) \to 0$, each of following limits is $0/0$ form so we can use above standard limits in modified version as follows,

(i) $\lim\limits_{x \to \alpha} \dfrac{\sin f(x)}{f(x)} = \lim\limits_{x \to \alpha} \dfrac{\sin^{-1} f(x)}{f(x)} = 1$

(ii) $\lim\limits_{x \to \alpha} \dfrac{1 - \cos f(x)}{f(x)^2} = \dfrac{1}{2}$

(iii) $\lim\limits_{x \to \alpha} \dfrac{\tan f(x)}{f(x)} = \lim\limits_{x \to \alpha} \dfrac{\tan^{-1} f(x)}{f(x)} = 1$

(iv) $\lim\limits_{x \to \alpha} \dfrac{a^{f(x)} - 1}{f(x)} = \log_e a \ (a > 0)$

(v) $\lim\limits_{x \to \alpha} \dfrac{e^{f(x)} - 1}{f(x)} = 1$

(vi) $\lim\limits_{x \to \alpha} \dfrac{\ln(1 + f(x))}{f(x)} = 1, \lim\limits_{x \to \alpha} \dfrac{\log_a(1 + f(x))}{f(x)} = \log_a e$

ILLUSTRATION 23

Evaluate $\lim\limits_{n \to \infty} \left(\dfrac{\sin \frac{a}{n}}{\tan \frac{b}{n+1}} \right)$.

SOLUTION

$$\lim_{n \to \infty} \left(\frac{\sin \frac{a}{n}}{\tan \frac{b}{n+1}} \right) = \lim_{n \to \infty} \left(\frac{\sin \frac{a}{n}}{\frac{a}{n}} \right) \left(\frac{\frac{b}{n+1}}{\tan \frac{b}{n+1}} \right) \cdot \frac{a}{b} \left(\frac{n+1}{n} \right)$$

$$= (1)\,(1)\,\frac{a}{b} \lim_{n \to \infty} \left(\frac{n+1}{n} \right) = \frac{a}{b}$$

ILLUSTRATION 24

Evaluate $\lim\limits_{x \to 0} \dfrac{8^x - 4^x - 2^x + 1}{x^2}$.

SOLUTION

$$\lim_{x \to 0} \frac{4^x.2^x - 4^x - 2^x + 1}{x^2} = \lim_{x \to 0} \left(\frac{4^x - 1}{x} \right) \lim_{x \to 0} \left(\frac{2^x - 1}{x} \right)$$

$$= \log_e(4)\,.\,\log_e(2) = 2(\ln 2)^2.$$

ILLUSTRATION 25

Evaluate $\lim\limits_{x\to 3}\left[\dfrac{(x^3+27)\log(x-2)}{x^2-9}\right]$.

SOLUTION

$$\lim_{x\to 3}\left[\frac{(x^3+27)\log(x-2)}{x^2-9}\right] \text{(It is } \frac{0}{0} \text{ form)}$$

$$=\lim_{x\to 3}\left[\frac{(x+3)(x^2-3x+9)\log(x-2)}{(x+3)(x-3)}\right]$$

$$=\lim_{x\to 3}\left[\frac{(x^2-3x+9)\log(x-2)}{(x-3)}\right]$$

$$=\lim_{x\to 3}(x^2-3x+9).\lim_{x\to 3}\frac{\log(x-3+1)}{x-3}=9(1)=9$$

ILLUSTRATION 26

Evaluate $\lim\limits_{x\to 1}(1+\cos\pi x)\cot^2\pi x$.

SOLUTION

$\lim\limits_{x\to 1}[(1+\cos\pi x)\cot^2\pi x]$; It is of $(0\times\infty)$ form

$$=\lim_{x\to 1}\left[\frac{(1+\cos\pi x)}{\tan^2\pi x}\right]; \text{It is of } (\frac{0}{0} \text{ form})$$

Put $x=1+h$; As $x\to 1$, therefore $h\to 0$.

$$\text{Required limit}=\lim_{h\to 0}\frac{1+\cos(\pi+\pi h)}{\tan^2(\pi+\pi h)}=\lim_{h\to 0}\frac{1-\cos\pi h}{\tan^2\pi h}$$

$$=\lim_{h\to 0}\frac{2\sin^2\frac{\pi h}{2}\cos^2\pi h}{\sin^2\pi h}=\lim_{h\to 0}\frac{2\sin^2\frac{\pi h}{2}\cos^2\pi h}{4\sin^2\frac{\pi h}{2}\cos^2\frac{\pi h}{2}}$$

$$=\lim_{h\to 0}\frac{\cos^2\pi h}{2\cos^2\frac{\pi h}{2}}=\frac{1}{2}$$

ILLUSTRATION 27

Evaluate $\lim\limits_{x\to 0}\dfrac{\sin\left(\sin\left(\tan\frac{x^2}{2}\right)\right)}{\log\cos 3x}$.

SOLUTION

$$\lim_{x\to 0}\frac{\sin\left(\sin\left(\tan\frac{x^2}{2}\right)\right)}{\log\cos 3x}$$

$$=\lim_{x\to 0}\frac{\sin\left(\sin\left(\tan\frac{x^2}{2}\right)\right)}{\sin\left(\tan\frac{x^2}{2}\right)}\times\frac{\sin\left(\tan\frac{x^2}{2}\right)}{\tan\frac{x^2}{2}}\times\frac{\tan\frac{x^2}{2}}{\ln\cos 3x}$$

$$=\lim_{x\to 0}1\cdot 1\cdot\frac{\tan\frac{x^2}{2}}{\frac{x^2}{2}}\times\frac{\frac{x^2}{2}}{\ln(1+\cos 3x-1)}$$

$$=\lim_{x\to 0}\frac{(\cos 3x-1)}{\ln(1+\cos 3x-1)}\cdot\left(\frac{\frac{x^2}{2}}{\cos 3x-1}\right)$$

$$=\lim_{x\to 0}\frac{(3x)^2}{1-\cos 3x}\left(-\frac{1}{18}\right)=\left(-\frac{1}{18}\right)\times\left(\frac{2}{1}\right)=-\frac{1}{9}$$

ILLUSTRATION 28

Evaluate $\lim\limits_{h\to 0}\left[\dfrac{2\left\{\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)\right\}}{h\sqrt{3}\left(\sqrt{3}\cos h-\sin h\right)}\right]$.

SOLUTION

$$\lim_{h\to 0}\left[\frac{2\left\{\sqrt{3}\sin\left(\frac{\pi}{6}+h\right)-\cos\left(\frac{\pi}{6}+h\right)\right\}}{h\sqrt{3}\left(\sqrt{3}\cos h-\sin h\right)}\right]; \frac{0}{0}\text{ form}$$

$$=\lim_{h\to 0}\left[\frac{2\left\{\frac{\sqrt{3}}{2}\sin\left(\frac{\pi}{6}+h\right)-\frac{1}{2}\cos\left(\frac{\pi}{6}+h\right)\right\}}{h\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos h-\frac{1}{2}\sin h\right)}\right]$$

$$=\lim_{h\to 0}\left[\frac{-2\cos\left(\frac{\pi}{6}+h+\frac{\pi}{3}\right)}{h\sqrt{3}\cos\left(\frac{\pi}{6}+h\right)}\right]=\lim_{h\to 0}\left[\frac{-2\cos\left(\frac{\pi}{2}+h\right)}{h\sqrt{3}\cos\left(\frac{\pi}{6}+h\right)}\right]$$

$$=\lim_{h\to 0}\left[\frac{2\sin h}{h\sqrt{3}\cos\left(\frac{\pi}{6}+h\right)}\right]$$

$$=\lim_{h\to 0}\left(\frac{2}{\sqrt{3}}\right)\left(\frac{\sin h}{h}\right)\frac{1}{\cos\left(\frac{\pi}{6}+h\right)}=\frac{2}{\sqrt{3}}.(1).\frac{1}{\cos\frac{\pi}{6}}=\frac{4}{3}$$

ILLUSTRATION 29

If $\lim\limits_{x\to 0}\dfrac{\sin 2x+a\sin x}{x^3}=b$ (finite), then, find a and b.

SOLUTION

$$\lim_{x\to 0}\frac{\sin 2x+a\sin x}{x^3}=b$$

$$\Rightarrow\quad\lim_{x\to 0}\frac{2\sin x\cos x+a\sin x}{x^3}=b$$

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x\left(\cos x+\frac{a}{2}\right)}{x^3}=b$$

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x}{x}\times\frac{\left(\cos x+\frac{a}{2}\right)}{x^2}=b$$

For limit to finitely exist, If $\frac{a}{2}=-1$ we get $a=-2$.

$$\Rightarrow \lim_{x\to 0}\frac{2\sin x}{x}\left(\frac{\cos x-1}{x^2}\right)=b$$

$$\Rightarrow 2\left(-\frac{1}{2}\right)=b\Rightarrow b=-1$$

ILLUSTRATION 30

Evaluate the value of $\lim\limits_{x\to 3}\frac{\sqrt{x+6}-\sin(x-3)-3}{(x-3)\cos(x-3)}$.

SOLUTION

Put $x=3+h;\ h\to 0$

$$l=\lim_{h\to 0}\frac{\sqrt{9+h}-\sin h-3}{h\cos h}=\lim_{h\to 0}\frac{\sqrt{9+h}-3}{h}-\lim_{h\to 0}\frac{\tan h}{h}$$

$$=\lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}-1=\frac{1}{6}-1=\frac{-5}{6}$$

ILLUSTRATION 31

Evaluate $\lim\limits_{x\to 0^+}\left((\tan x)^{\frac{1}{\sin^2 x}}+\frac{\sin 2x+2\tan^{-1}(3x)+3x^2}{\ln(1+3x+\sin^2 x)+xe^x}\right)$.

SOLUTION

$$\because\ \lim_{x\to 0}(\tan x)^{\frac{1}{\sin^2 x}}=0\quad(\text{As } 0^{\infty}=0)$$

$$\text{Required limit}=\lim_{x\to 0}\frac{\sin 2x+2\tan^{-1}3x+3x^2}{\ln(1+3x+\sin^2 x)+xe^x}$$

$$=\lim_{x\to 0}\frac{2\left(\frac{\sin 2x}{2x}\right)+3\cdot 2\left(\frac{\tan^{-1}(3x)}{3x}\right)+3x}{\frac{3x+\sin^2 x}{x}\cdot\frac{\ln(1+3x+\sin^2 x)}{(3x+\sin^2 x)}+e^x}$$

$$=\frac{2+6+0}{(3+0)\cdot 1+1}=\frac{8}{4}=2$$

ILLUSTRATION 32

If $\lim\limits_{x\to 0}\frac{\tan^3 x+\tan^3 2x+\tan^3 3x}{\tan^3 4x+\tan^3 5x+\tan^3 6x}=\frac{p}{q}$ $(p, q\in N)$, then, find least value of $(p+q)$.

SOLUTION

$$\lim_{x\to 0}\frac{\tan^3 x+\tan^3 2x+\tan^3 3x}{\tan^3 4x+\tan^3 5x+\tan^3 6x}$$

$$=\lim_{x\to 0}\frac{\left[x^3\cdot\frac{\tan^3 x}{x^3}+8x^3\cdot\frac{\tan^3 2x}{(2x)^3}+27x^3\cdot\frac{\tan^3 3x}{(3x)^3}\right]}{\left[64x^3\cdot\frac{\tan^3 4x}{(4x)^3}+125x^3\cdot\frac{\tan^3 5x}{(5x)^3}+216x^3\cdot\frac{\tan^3 6x}{(6x)^3}\right]}$$

(Take x^3 common from both numerator and denominator)

$$=\frac{1+8+27}{64+125+216}=\frac{36}{405}=\frac{4}{45}$$

ILLUSTRATION 33

If $L=\lim\limits_{x\to 0}\frac{1+\cos(\pi\sec^2 x)}{\log(1+x^2).(\sin^{-1}x)^2}$, then, find the value of L.

SOLUTION

$$L=\lim_{x\to 0}\frac{1+\cos(\pi\sec^2 x)}{\log(1+x^2).(\sin^{-1}x)^2}$$

$$=\lim_{x\to 0}\frac{1+\cos(\pi+\pi\tan^2 x)}{\log(1+x^2).(\sin^{-1}x)^2}$$

$$L=\left(\frac{1-\cos(\pi\tan^2 x)}{\pi^2\tan^4 x}\right)\frac{\pi^2\frac{\tan^4 x}{x^4}}{\frac{\log(1+x^2)}{x^2}\times\left(\frac{\sin^{-1}x}{x}\right)^2}$$

Using standard limits, the value of $L=\frac{\pi^2}{2}$.

ILLUSTRATION 34

Evaluate $\lim\limits_{x\to 0}\left(\frac{\cos x-\sqrt{1+\sin^2 x}}{1-\sqrt{1+\tan^2 x}}\right)$.

SOLUTION

$$\lim_{x\to 0}\left(\frac{\cos x-\sqrt{1+\sin^2 x}}{1-\sqrt{1+\tan^2 x}}\right)$$

$$=\lim_{x\to 0}\left(\frac{\cos x-\sqrt{1+\sin^2 x}}{1-\sqrt{1+\tan^2 x}}\right)\left(\frac{\cos x+\sqrt{1+\sin^2 x}}{\cos x+\sqrt{1+\sin^2 x}}\right)$$

$$=\lim_{x\to 0}\left(\frac{\cos^2 x-(1+\sin^2 x)}{\cos x+\sqrt{1+\sin^2 x}}\right)\left(\frac{1+\sqrt{1+\tan^2 x}}{1-(1+\tan^2 x)}\right)$$

$$=\lim_{x\to 0}\left(\frac{\cos 2x-1}{-\tan^2 x}\right)\left(\frac{1+\sqrt{1+\tan^2 x}}{\cos x+\sqrt{1+\sin^2 x}}\right)$$

$$=\lim_{x\to 0}\frac{2\sin^2 x.\cos^2 x}{\sin^2 x}\cdot\frac{2}{2}=2$$

ILLUSTRATION 35

If $\lim\limits_{x\to0}\dfrac{1-\cos(ax)}{x^2} = \lim\limits_{x\to -a}\dfrac{\cos x-\cos a}{x^2-a^2}$, then, show that $a^3+\sin a=0$.

SOLUTION

$$\lim_{x\to0}\frac{1-\cos(ax)}{(ax)^2}a^2 = \lim_{x\to-a}\frac{\cos x-\cos a}{x^2-a^2}$$

$$\frac{a^2}{2} = -\lim_{x\to-a}\frac{2\sin\left(\frac{x+a}{2}\right)\cdot\sin\left(\frac{x-a}{2}\right)}{2\left(\frac{x+a}{2}\right)(x-a)}$$

$$\frac{a^2}{2}=\frac{-\sin a}{2a}$$

Hence, $a^3+\sin a=0$.

ILLUSTRATION 36

Let $a=\lim\limits_{x\to0}\dfrac{\ln(\cos 2x)}{3x^2}$, $b=\lim\limits_{x\to0}\dfrac{\sin^2 2x}{x(1-e^x)}$ and $c=\lim\limits_{x\to1}\dfrac{\sqrt{x}-x}{\ln x}$ then, arrange them in increasing order.

SOLUTION

$$a=\lim_{x\to0}\frac{\ln(\cos 2x)}{3x^2}$$

$$a=\lim_{x\to0}\frac{\ln(1+\cos 2x-1)}{(\cos 2x-1)}\,\frac{(\cos 2x-1)}{3x^2}$$

$$a=-\frac{1}{3}\lim_{x\to0}\frac{\ln(1+\cos 2x-1)}{(\cos 2x-1)}\cdot\frac{(1-\cos 2x)}{(2x)^2}4$$

$$a=-\frac{4}{3}\cdot(1)\left(\frac{1}{2}\right)\Rightarrow a=-\frac{2}{3}$$

$$b=\lim_{x\to0}\frac{\sin^2 2x}{x(1-e^x)}$$

$$b=-\lim_{x\to0}\left(\frac{\sin 2x}{2x}\right)^2\frac{4}{\left(\frac{e^x-1}{x}\right)}=-4,$$

we get, $b=-4$

$$C=\lim_{x\to1}\frac{\sqrt{x}-x}{\ln x},\ \text{put } x=t^2$$

$$C=\lim_{t\to1}\frac{t-t^2}{\ln t^2}=\lim_{t\to1}\frac{t(1-t)}{2\ln(1+t-1)}$$

$$=-\lim_{t\to1}\frac{t(t-1)}{2\ln(1+t-1)}=-\frac{1}{2}$$

Hence, $b<a<c$

ILLUSTRATION 37

If $f(x)$ is a quadratic expression such that $f(-2)=f(2)=0$ and $f(1)=6$ then, find $\lim\limits_{x\to0}\dfrac{\sqrt{f(x)}-2\sqrt{2}}{\ln(\cos x)}$.

SOLUTION

$$f(x)=\lambda(x^2-4)$$

$$f(1)=\lambda(1-4)=6\Rightarrow\lambda=-2$$

$$\therefore\quad f(x)=8-2x^2$$

$$\lim_{x\to0}\frac{\sqrt{f(x)}-2\sqrt{2}}{\ln(\cos x)}=\lim_{x\to0}\frac{\sqrt{8-2x^2}-2\sqrt{2}}{\ln(\cos x)}$$

$$=\lim_{x\to0}\frac{(8-2x^2-8)}{\dfrac{\ln(1+\cos x-1)}{\cos x-1}(\sqrt{8-2x^2}+2\sqrt{2})(\cos x-1)}$$

$$=\lim_{x\to0}\frac{(-2x^2)}{1.\left(4\sqrt{2}\right)(\cos x-1)}=\frac{4}{4\sqrt{2}}=\frac{1}{\sqrt{2}}$$

ILLUSTRATION 38

Let $f(x)=3x^3+bx^2+bx+3$ where $b\in R$. If $x=2$ is a real root of cubic equation $f(x)=0$, then, find the value of $\lim\limits_{x\to1/2}\left(\dfrac{e^{f(x)}-1}{2x-1}\right)$.

SOLUTION

$$3x^3+bx^2+bx+3=0$$

$$\Rightarrow\quad 3(x^3+1)+bx(x+1)=0,\ \text{has roots } -1,\ 2,\ \frac{1}{2}.$$

$$f(x)=3x^3+bx^2+bx+3=3(x+1)(x-2)(x-1/2)$$

$$\lim_{x\to\frac{1}{2}}\frac{e^{f(x)}-1}{2x-1},\ \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to\frac{1}{2}}\left(\frac{e^{f(x)}-1}{f(x)}\right)\cdot\left(\frac{f(x)}{2x-1}\right)=\lim_{x\to\frac{1}{2}}\frac{f(x)}{2x-1}$$

$$=\lim_{x\to1/2}\frac{3(x+1)(x-2)\left(x-\frac{1}{2}\right)}{2\left(x-\frac{1}{2}\right)}=\frac{3}{2}\times\frac{3}{2}\times\frac{-3}{2}=\frac{-27}{8}$$

ILLUSTRATION 39

Evaluate the value of $\lim\limits_{x\to0}\dfrac{1}{x}\left(\tan^{-1}\left(\dfrac{x+1}{2x+1}\right)-\dfrac{\pi}{4}\right)$.

SOLUTION

$$\lim_{x\to0}\frac{1}{x}\left(\tan^{-1}\left(\frac{x+1}{2x+1}\right)-\frac{\pi}{4}\right)$$

$$= \lim_{x\to 0}\left(\frac{\tan^{-1}\left(\frac{x+1}{2x+1}\right)-\tan^{-1}1}{x}\right)$$

$$= \lim_{x\to 0}\frac{\tan^{-1}\left(\frac{\left(\frac{x+1}{2x+1}\right)-1}{1+\left(\frac{x+1}{2x+1}\right)}\right)}{x}$$

$$= -\lim_{x\to 0}\frac{\tan^{-1}\left(\frac{-x}{3x+2}\right)}{\left(\frac{-x}{3x+2}\right)\cdot(3x+2)} = -\frac{1}{2}$$

ILLUSTRATION 40

Evaluate $\lim_{n\to\infty}\left(\frac{2}{\pi}\left(3n+\frac{22}{7}\right)\cot^{-1}\frac{1}{n}-3n\right)$.

SOLUTION

$$\lim_{n\to\infty}\left(\frac{2}{\pi}\left(3n+\frac{22}{7}\right)\cot^{-1}\frac{1}{n}-3n\right)$$

$$= \lim_{n\to\infty}\left(\frac{2}{\pi}\left(3n+\frac{22}{7}\right)\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{n}\right)-3n\right)$$

$$= \lim_{n\to\infty}\left(\left(3n+\frac{22}{7}\right)-\frac{2}{\pi}\left(3n+\frac{22}{7}\right)\tan^{-1}\frac{1}{n}-3n\right)$$

$$= \lim_{n\to\infty}\left(\frac{22}{7}-\frac{6}{\pi}n\cdot\tan^{-1}\frac{1}{n}-\frac{44}{7\pi}\tan^{-1}\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{22}{7}-\frac{6}{\pi}\left(\frac{\tan^{-1}\frac{1}{n}}{\frac{1}{n}}\right)-\frac{44}{7\pi}\tan^{-1}\frac{1}{n}\right) = \left(\frac{22}{7}-\frac{6}{\pi}\right)$$

ILLUSTRATION 41

If $\lim_{x\to -2}\frac{(2(a-3)(x+2)-6\sin^{-1}(x+2))\tan^{-1}(5x+10)}{(x+2)^2} = 0,$ then, find the value of a.

SOLUTION

$$\lim_{x\to -2}\frac{(2(a-3)(x+2)-6\sin^{-1}(x+2))\tan^{-1}(5x+10)}{(x+2)^2} = 0,$$

$$\Rightarrow \quad 5\lim_{x\to -2}\left(2(a-3)-\frac{6\sin^{-1}(x+2)}{(x+2)}\right)\cdot\frac{\tan^{-1}5(x+2)}{5(x+2)} = 0$$

$$\Rightarrow \quad 5(2(a-3)-6) = 0$$

$$\therefore \quad a-3 = 3$$

Hence, $a = 6$

ILLUSTRATION 42

Evaluate $e^{\lim_{n\to\infty}\sum_{r=1}^{n}\log_e\cos\left(\frac{x}{2^r}\right)}$.

SOLUTION

Let $A = e^{\lim_{n\to\infty}\sum_{r=1}^{n}\log_e\cos\left(\frac{x}{2^r}\right)}$

$$\Rightarrow \quad A = e^{\lim_{n\to\infty}\sum_{r=1}^{n}\log_e\cos\left(\frac{x}{2^r}\right)} = \lim_{n\to\infty} e^{\log_e\left(\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^n}\right)}$$

$$A = \lim_{n\to\infty}\cos\frac{x}{2}\cos\frac{x}{4}\ldots\cos\frac{x}{2^n},$$

($\because$ using the result $\cos x\cos 2x\cos 4x\ldots\cos 2^{n-1}x = \frac{\sin\left(2^n x\right)}{2^n.\sin(x)}$)

$$\text{Required limit} = A = \lim_{n\to\infty}\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$$

$$= \lim_{n\to\infty}\frac{\sin x}{x}\cdot\frac{\frac{x}{2^n}}{\sin\left(\frac{x}{2^n}\right)} = \frac{\sin x}{x}$$

ILLUSTRATION 43

If $f(x) = \lim_{a\to x}\left(\frac{e^{a^2+|x|-x^2}-e^{|x|}}{(a+x)\sin(a-x)}\right)$, $x \in R$ then, find number of roots of the equation $f(x)(|x^2-1|) = 1$.

SOLUTION

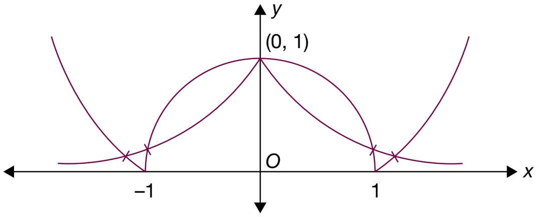
$$f(x) = \lim_{a\to x}\left(\frac{e^{a^2+|x|-x^2}-e^{|x|}}{(a+x)\sin(a-x)}\right)$$

$$= \lim_{a\to x}\frac{e^{|x|}(e^{a^2-x^2}-1)}{(a+x)\frac{\sin(a-x)}{(a-x)}\cdot(a-x)}$$

$$= \lim_{a\to x}\frac{e^{|x|}(e^{a^2-x^2}-1)}{(a^2-x^2)} = e^{|x|}$$

Now, $f(x)(|x^2-1|) = 1$

$$e^{|x|}(|x^2-1|) = 1 \Rightarrow |x^2-1| = e^{-|x|}$$



Hence, both the curves intersect at 5 distinct points.

Test Your Concepts-II

(Solutions on page H.50)

1. If $\lim\limits_{x\to 0}\dfrac{p\sin 2x+(1-\cos 2x)}{x+\tan x}=1$, then, find p.

2. Evaluate $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$.

3. Evaluate $\lim\limits_{x\to \frac{\pi}{2}}\dfrac{2^{-\cos x}-1}{x\left(x-\dfrac{\pi}{2}\right)}$.

4. Evaluate $\lim\limits_{x\to 0}\left[\dfrac{x2^x-x}{1-\cos x}\right]$.

5. Evaluate $\lim\limits_{x\to -\infty}\dfrac{x^2\tan\left(\dfrac{1}{x}\right)-x}{1-|x|}$.

6. Evaluate $\lim\limits_{x\to \frac{\pi}{2}}\left(\dfrac{\tan 2x+\sin 2x}{x-\dfrac{\pi}{2}}\right)$.

7. If $\lim\limits_{x\to 0}\dfrac{a^{\sin x}-1}{1+x-\cos x}=\lim\limits_{x\to 1}\dfrac{x-1}{\ln(x)}$ (where $a>0$), then, find a.

8. Evaluate $\lim\limits_{n\to\infty} n\left(1+\tan^2\left(\dfrac{\pi}{4n}\right)\right)\cos^3\dfrac{\pi}{4n}\sin\dfrac{\pi}{4n}$.

9. If $\lim\limits_{x\to 0}\dfrac{\sin\left(\dfrac{x+x^2}{a^4}\right)(e^x-1)^2}{\ln\left(1+\dfrac{x^3}{2}\right)}=\dfrac{2}{81}$, then, find positive value of a.

10. If $\lim\limits_{x\to a}\tan\left(\dfrac{\pi x}{2a}\right).\sin\left(\dfrac{x-a}{2}\right)=\dfrac{2}{\pi}$, then, find a.

11. Evaluate $\lim\limits_{x\to 0}\dfrac{(2^{\sin x}-1)\cdot\ln(1+\sin 2x)}{x(arc\tan x)}$.

12. Let $f(x)=\dfrac{2+e^{1/x}}{1+e^{4/x}}+\dfrac{\sin x}{|x|}$ find the value of $\lim\limits_{x\to 0}f(x)$

13. Evaluate $\lim\limits_{x\to 0}\dfrac{5(1-\cos x)+20\sin^{-1}x-\sin^2 x}{(10\tan^{-1}(x)-x^2+x^4)}$.

Answer key

1. 1 **2.** $\dfrac{1}{2}$ **3.** $\dfrac{2}{\pi}\ln 2$ **4.** $2\log_e 2$ **5.** 0 **6.** 0 **7.** e

8. $\dfrac{\pi}{4}$ **9.** 3 **10.** –2 **11.** ln 4 **12.** 1 **13.** 2

L-HOSPITAL'S RULE

By evaluating limits of 0/0 or ∞/∞ form, we can use L-Hospital rule if functions written in numerator and denominator are having continuous derivatives in neighborhood of α. Polynomial functions, log function, exponential functions and trigonometric function can be handled easily with L-Hospital rule.

Let $f(\alpha)=g(\alpha)=0$, and f and g are continuously differentiable at $x=\alpha$

$$\lim_{x\to\alpha}\frac{f(x)}{g(x)}=\lim_{x\to\alpha}\frac{f(x)-f(\alpha)}{g(x)-g(\alpha)}$$

$$=\lim_{x\to\alpha}\frac{\left(\dfrac{f(x)-f(\alpha)}{x-\alpha}\right)}{\left(\dfrac{g(x)-g(\alpha)}{(x-\alpha)}\right)}=\lim_{x\to\alpha}\frac{f'(x)}{g'(x)}$$

Clearly, we can see that limit remains same if we differentiate numerator and denominator, L-H rule can be reapplied if indeterminate form further exist provided $f(x)$ and $g(x)$ are n times differentiable. Good knowledge of differentiation requires to expertise this method.

$$\lim_{x\to\alpha}\frac{f(x)}{g(x)}=\lim_{x\to\alpha}\frac{f'(x)}{g'(x)}\qquad\text{(if 0/0 or }\infty/\infty\text{ exists, we can reapply L-H rule)}$$

$$=\lim_{x\to\alpha}\frac{f''(x)}{g''(x)}\qquad\text{(if 0/0 or }\infty/\infty\text{ exists, we can reapply L-H rule)}$$

$$=\lim_{x\to\alpha}\frac{f'''(x)}{g'''(x)}\qquad\text{(if 0/0 or }\infty/\infty\text{ exists, we can again reapply L-H rule)}$$

$$=\lim_{x\to\alpha}\frac{f^{n}(x)}{g^{n}(x)}\text{ exists}\qquad\text{(if }g^{n}(\alpha)\text{ is non- zero)}$$

***All standard limits can be easily proved using L-Hospital rule**

ILLUSTRATION 44

Evaluate $\lim\limits_{x\to1}\dfrac{\sqrt{x}+\sqrt{\sqrt{x}}+\sqrt{\sqrt{\sqrt{x}}}+\sqrt{\sqrt{\sqrt{\sqrt{x}}}}-4}{x-1}$.

SOLUTION

$$\lim_{x\to1}\frac{\sqrt{x}+\sqrt{\sqrt{x}}+\sqrt{\sqrt{\sqrt{x}}}+\sqrt{\sqrt{\sqrt{\sqrt{x}}}}-4}{x-1}$$

$$=\lim_{x\to1}\frac{x^{1/2}+x^{1/4}+x^{1/8}+x^{1/16}-4}{x-1}$$

$$=\lim_{x\to1}\left(\frac{x^{1/2}-1}{x-1}+\frac{x^{1/4}-1}{x-1}+\frac{x^{1/8}-1}{x-1}+\frac{x^{1/16}-1}{x-1}\right)$$

(Using L-H Rule)

$$=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{8+4+2+1}{16}=\frac{15}{16}$$

ILLUSTRATION 45

Evaluate $\lim\limits_{x\to a}\dfrac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}$.

SOLUTION

$$\lim_{x\to a}\frac{(x+2)^{5/3}-(a+2)^{5/3}}{x-a}\ \text{(0/0 form now using L-H Rule)}$$

$$=\lim_{x\to a}\frac{(5/3)(x+2)^{2/3}-0}{1}=\frac{5}{3}(a+2)^{2/3}$$

ILLUSTRATION 46

Evaluate $\lim\limits_{x\to1}\dfrac{\sqrt[3]{x+1}-\sqrt[3]{3x-1}}{x-1}$.

SOLUTION

$$\lim_{x\to1}\frac{\sqrt[3]{x+1}-\sqrt[3]{3x-1}}{x-1}\times\frac{\left[(x+1)^{2/3}+(3x-1)^{2/3}+(x+1)^{1/3}(3x-1)^{1/3}\right]}{\left[(x+1)^{2/3}+(3x-1)^{2/3}+(x+1)^{1/3}(3x-1)^{1/3}\right]}$$

$$=\lim_{x\to1}\frac{(x+1)-(3x-1)}{(x-1)\left[(x+1)^{2/3}+(3x-1)^{2/3}+(x+1)^{1/3}(3x-1)^{1/3}\right]}$$

$$=\lim_{x\to1}\frac{-2}{\left[(x+1)^{2/3}+(3x-1)^{2/3}+(x+1)^{1/3}(3x-1)^{1/3}\right]}=-\frac{2^{1/3}}{3}$$

Alternate:

$$\lim_{x\to1}\frac{\sqrt[3]{x+1}-\sqrt[3]{3x-1}}{x-1}\quad\text{(Using L-H Rule)}$$

$$=\lim_{x\to1}\frac{\frac{1}{3}(x+1)^{-2/3}-\frac{3}{3}(3x-1)^{-2/3}}{1-0}=\frac{2^{-2/3}}{3}-2^{-2/3}$$

$$=-\frac{2^{1/3}}{3}$$

ILLUSTRATION 47

Evaluate $\lim\limits_{h\to0}\left[\dfrac{\sqrt{x+h}-\sqrt{x}}{h}\right]$.

SOLUTION

$$\lim_{h\to0}\left[\frac{\sqrt{x+h}-\sqrt{x}}{h}\right]\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{h\to0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$=\lim_{h\to0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}=\lim_{h\to0}\left[\frac{1}{\sqrt{x+h}+\sqrt{x}}\right]=\frac{1}{2\sqrt{x}}$$

Alternate:

$$\lim_{h\to0}\frac{\sqrt{x+h}-\sqrt{x}}{h}\qquad\text{(Using L–H Rule)}$$

As limit is on variable *h*, so we should differentiate relative to *h*.

$$=\lim_{h\to0}\frac{\frac{1}{2\sqrt{x+h}}-0}{1}=\frac{1}{2\sqrt{x}}$$

ILLUSTRATION 48

Compute $\lim\limits_{x\to0^+}\dfrac{\ln(-\cos3x+\cos x)}{\ln\sin(x)}$.

SOLUTION

$$\lim_{x\to0^+}\frac{\ln(-\cos3x+\cos x)}{\ln\sin(x)}\ \left(\frac{\infty}{\infty}\text{ form}\right)$$

Using L-Hospital's Rule,

$$=\lim_{x\to0^+}\left(\frac{-\sin x+3\sin3x}{\cos x-\cos3x}\right)\frac{\sin x}{\cos x}$$

$$=\lim_{x\to0^+}\frac{(3(3\sin x-4\sin^3x)-\sin x)\sin x}{(\cos x-4\cos^3x+3\cos x)\cos x}$$

$$=\lim_{x\to0^+}\frac{4\sin^2x(2-3\sin^2x)}{4\cos^2x(1-\cos^2x)}$$

$$=\lim_{x\to0^+}\frac{2-3\sin^2x}{\cos^2x}=2$$

ILLUSTRATION 49

If $f(x)=\lim\limits_{t\to0}\tan^{-1}\left(\dfrac{e^{xt}-1}{t}\right)$, then, find $\lim\limits_{x\to0}\dfrac{f(x)-x}{x^3}$.

SOLUTION

$$f(x)=\lim_{t\to0}\tan^{-1}\left(\frac{e^{xt}-1}{t}\right)$$

$$= \lim_{t\to 0} \tan^{-1}\left(x\left(\frac{e^{xt}-1}{xt}\right)\right) = \tan^{-1}\left\{x.\lim_{t\to 0}\left(\frac{e^{xt}-1}{xt}\right)\right\}$$

$f(x) = \tan^{-1}x$

$$\therefore \quad \lim_{x\to 0}\frac{f(x)-x}{x^3} = \lim_{x\to 0}\left(\frac{\tan^{-1}x - x}{x^3}\right) \qquad \left(\frac{0}{0}\text{ Form}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{1}{1+x^2}-1}{3x^2}\right) \qquad \text{(Using L-H Rule)}$$

$$= \lim_{x\to 0}\frac{1-1-x^2}{(1+x^2)3x^2} = -\frac{1}{3}\lim_{x\to 0}\frac{1}{1+x^2} = -\frac{1}{3}$$

Test Your Concepts-III

(Solutions on page H.52)

1. Find the value of $\lim\limits_{x\to 1}\dfrac{(x+x^2+x^3+\ldots+x^n)-n}{x-1}$.

2. If $\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1} = \lim\limits_{x\to k}\dfrac{x^3-k^3}{x^2-k^2}$, then, find the value of k.

3. Find the value of $\lim\limits_{x\to\sqrt{2}}\left(\dfrac{x^4-4}{x^2+3x\sqrt{2}-8}\right)$.

4. Find the value of $\lim\limits_{x\to 2}\dfrac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$.

5. Find the value of $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2}-1}{x^2}$.

6. Find the value of $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-\sqrt{1-x}}{2x}$.

7. Find the value of $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2}-1}{\sqrt{16+x^2}-4}$.

8. Find the value of $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{7+x^2}-\sqrt[3]{1-2x}}{x+x^2}$.

9. Find the value of $\lim\limits_{x\to 2}\dfrac{\sqrt{x-2}+\sqrt{x}-\sqrt{2}}{\sqrt{x^2-4}}$.

10. Find the value of $\lim\limits_{x\to 1}\dfrac{4x^3-x^2+2x-5}{x^6+5x^3-2x-4}$.

Answer key

1. $\dfrac{n(n+1)}{2}$ **2.** $\dfrac{8}{3}$ **3.** $\dfrac{8}{5}$ **4.** $\dfrac{1}{8\sqrt{3}}$ **5.** $\dfrac{1}{2}$

6. $\dfrac{1}{2}$ **7.** 4 **8.** $\dfrac{3}{2}$ **9.** $\dfrac{1}{2}$ **10.** $\dfrac{12}{19}$

SERIES EXPANSION FORMULA

(i) $a^x = 1+(x\ln a)+\dfrac{(x\ln a)^2}{2!}+\dfrac{(x\ln a)^3}{3!}+\ldots+\dfrac{(x\ln a)^n}{n!}+\ldots$

(ii) $a^{-x} = 1-(x\ln a)+\dfrac{(x\ln a)^2}{2!}-\dfrac{(x\ln a)^3}{3!}+\ldots$

$+(-1)^n\dfrac{(x\ln a)^n}{n!}+\ldots$

(iii) (a) $e^x = 1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots+\dfrac{x^n}{n!}+\ldots$

(b) $e^{-x} = 1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\ldots+\dfrac{(-x)^n}{n!}+\ldots$

(iv) (a) $\ln(1+x) = x-\dfrac{x^2}{2!}+\dfrac{x^3}{3!}-\ldots$

(b) $\ln(1-x) = -x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\ldots$ where $-1 < x < 1$

(v) $\sin x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots$

(vi) $\cos x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots$

(vii) $\tan x = x+\dfrac{x^3}{3}+\dfrac{2}{15}x^5+\dfrac{17}{315}x^7+\dfrac{62}{2835}x^9\ldots$

(viii) $\sin^{-1}x = x+\dfrac{x^3}{6}+\dfrac{3x^5}{40}+\dfrac{5x^7}{112}+\dfrac{35x^9}{1152}\ldots$

(ix) $\cos^{-1}x = \dfrac{\pi}{2}-\left(x+\dfrac{x^3}{3!}+\dfrac{9x^5}{5!}+\ldots\right)$

(x) $\tan^{-1}x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}x^{2n+1} = x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$

(xi) $\sec^{-1}x = 1 + \frac{x^2}{2!} + \frac{5^2.x^4}{4!} + \frac{61x^6}{6!} + \ldots$

(xii) $(1+x)^{1/x} = e\left[1 - \frac{x}{2} + \frac{11}{24}x^2 \ldots\right]$

(xiii) Binomial Theorem

(a) For natural number n

$$(X+Y)^n = {}^nC_0X^nY^0 + {}^nC_1X^{n-1}Y + \ldots + {}^nC_rX^{n-r}Y^r + \ldots + {}^nC_nX^0Y^n$$

$$(X+Y)^n = \sum_{n-1}^{n} {}^nC_rX^{n-r}Y^r$$

$$(X+Y)^n = \sum_{n-1}^{n} {}^nC_r(\text{first term})^{n-r}(\text{second term})^r$$

(b) For fractional rational n, or negative integral n.

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$ infinite terms where $|x| < 1$.

ILLUSTRATION 50

Evaluate $\lim\limits_{x\to0} \frac{\sin^{-1}x - \sin x}{x^3}$.

SOLUTION

$$\lim_{x\to0} \frac{\sin^{-1}x - \sin x}{x^3}$$

$$= \lim_{x\to0} \frac{\left(x + \frac{1^2x^3}{3!} + \frac{1^2.3^2x^5}{5!}\ldots\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)}{x^3}$$

$$= \lim_{x\to0} \frac{\frac{2}{3!}x^3 + \frac{8}{5!}x^5 + \ldots}{x^3} = \frac{1}{3}$$

ILLUSTRATION 51

Evaluate $\lim\limits_{x\to0} \frac{(x-\sin x)^2}{(e^{x^2}-1)x^4}$.

SOLUTION

$$\lim_{x\to0} \frac{(x-\sin x)^2}{(e^{x^2}-1)x^4} = \lim_{x\to0}\left(\frac{x-\sin x}{x^3}\right)^2 \frac{x^6}{\left(\frac{e^{x^2}-1}{x^2}\right)x^6}$$

$$(\because \lim_{x\to0}\frac{x-\sin x}{x^3} = \lim_{x\to0}\left(\frac{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots.\right)}{x^3}\right) = \frac{1}{6})$$

Now, required limit $= \lim\limits_{x\to0}\left(\frac{x-\sin x}{x^3}\right)^2 \lim\limits_{x\to0}\frac{1}{\left(\frac{e^{x^2}-1}{x^2}\right)}$.

$$= \frac{1}{36} \times 1 = \frac{1}{36}$$

ILLUSTRATION 52

Evaluate $\lim\limits_{x\to0} \frac{(1+x)^{1/x} - e\left(1-\frac{x}{2}\right)}{(1-\cos x)}$.

SOLUTION

$$\lim_{x\to0} \frac{(1+x)^{1/x} - e\left(1-\frac{x}{2}\right)}{(1-\cos x)}$$

$$= \lim_{x\to0} \frac{e\left(1 - x/2 + \frac{11}{24}x^2 + \ldots.\right) - e\left(1-\frac{x}{2}\right)}{2\sin^2 x/2}$$

$$= \lim_{x\to0} \frac{\frac{11}{24}e\ \ x^2}{2\sin^2(x/2)} = \lim_{x\to0}\frac{11e}{12}\left(\frac{x/2}{\sin x/2}\right)^2 = \frac{11}{12}e$$

ILLUSTRATION 53

If $\lim\limits_{x\to0} \frac{e^{ax} + b\cos x + 2}{x^2}$ exists and is equal to 'ℓ', then, find a, b, ℓ.

SOLUTION

$$\ell = \lim_{x\to0} \frac{e^{ax} + b\cos x + 2}{x^2}$$

$$\ell = \lim_{x\to0} \frac{\left(1 + ax + \frac{a^2x^2}{2!} + \ldots\right) + b\left(1 - \frac{x^2}{2!} + \ldots\right) + 2}{x^2}$$

$$\ell = \lim_{x\to0} \frac{1 + b + 2 + x(a) + x^2\left(\frac{a^2}{2} - \frac{b}{2}\right) + \ldots}{x^2}$$

[For limit to exists $2 + 1 + b = 0 \Rightarrow b = -3$ and $a = 0$]

$$\ell = \frac{a^2}{2} - \frac{b}{2} = \frac{3}{2}$$

ILLUSTRATION 54

Solve $\lim\limits_{x\to0} \frac{a\sin x - bx + cx^2 + x^3}{2x^2\log(1+x) - 2x^3 + x^4}$, if it exists and is finite. Also, find a, b, c.

SOLUTION

Here, $\lim\limits_{x\to0} \frac{a\sin x - bx + cx^2 + x^3}{2x^2\log(1+x) - 2x^3 + x^4}$

$$= \lim_{x\to0} \frac{a\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right) - bx + cx^2 + x^3}{2x^2\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) - 2x^3 + x^4}$$

$$= \lim_{x\to0} \frac{(a-b)x + cx^2 + \left(1 - \frac{a}{3!}\right)x^3 + \frac{a}{5!}x^5 + \ldots}{\frac{2}{3}x^5 - \frac{1}{2}x^6 + \ldots}$$

For finite limit, all the term having less than 5 degree should be zero in numerator.

Thus, $a-b=0,\ c=0,\ 1-\frac{a}{3!}=0$ (i.e., $a=b=6,\ c=0$)

$$=\lim_{x\to 0}\frac{\frac{a}{5!}+\text{Higher power of } x}{\frac{2}{3}-\frac{1}{2}x+\ldots}=\frac{a}{5!}\cdot\frac{3}{2}=\frac{a}{80}=\frac{6}{80}=\frac{3}{40}$$

$$\therefore\quad \lim_{x\to 0}\frac{a\sin x-bx+cx^2+x^3}{2x^2\log(1+x)-2x^3+x^4}=\frac{3}{40}$$

Hence, $a=6=b,\ c=0$.

ILLUSTRATION 55

Find value of a, b and c such that,

$$\lim_{x\to 0}\frac{ax\,e^x-b\log(1+x)+c\,x\,e^{-x}}{x^2\sin x}=2$$

SOLUTION

$$\lim_{x\to 0}\frac{ax\,e^x-b\log(1+x)+c\,x\,e^{-x}}{x^2\sin x}=2$$

Using expansion, we get

$$=\lim_{x\to 0}\frac{ax\left(1+x+\frac{x^2}{2!}+\ldots\right)-b\left(x-\frac{x^2}{2}+\frac{x^3}{3}\ldots\right)+cx\left(1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots\right)}{x^3\left(\frac{\sin x}{x}\right)}$$

$$=\lim_{x\to 0}\frac{x(a-b+c)+x^2(a+\frac{b}{2}-c)+x^3(\frac{a}{2}-\frac{b}{3}+\frac{c}{2})+\ldots.}{x^3\,(1)}$$

As the limit is finite and non-zero, thus degree of numerator should be equal to degree of denominator.

i.e. coefficients of x and $x^2=0$ and coefficient of $x^3=2$

$$a-b+c=0;\ a+\frac{b}{2}-c=0;\ \frac{a}{2}-\frac{b}{3}+\frac{c}{2}=2$$

On solving, we get $a=3,\ b=12,\ c=9$.

ILLUSTRATION 56

Let $f(x)$ be a function such that $\lim\limits_{x\to 0}\frac{f(x)}{x}=1$ and $\lim\limits_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{(f(x))^3}=1$, then, find $b-3a$.

SOLUTION

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{(f(x))^3}=1$$

$$=\lim_{x\to 0}\frac{x+ax\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}-,\ldots\right\}-b\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}-,\ldots\right\}}{\{f(x)\}^3}$$

$$\Rightarrow\quad \lim_{x\to 0}\frac{\frac{1+a-b}{x^2}+\left(-\frac{a}{2!}+\frac{b}{3!}\right)+x^2\left(\frac{a}{4!}-\frac{b}{5!}\right)+\ldots\ldots}{\left\{\frac{f(x)}{x}\right\}^3}=1$$

$$\Rightarrow\quad 1+a-b=0 \text{ and } -\frac{a}{2!}+\frac{b}{3!}=1\Rightarrow -3a+b=6$$

$$\Rightarrow\quad a=-\frac{5}{2}\text{ and } b=-\frac{3}{2}.\ \text{Hence, } b-3a=6.$$

ILLUSTRATION 57

Evaluate $\lim\limits_{x\to 0}\frac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$.

SOLUTION

$$\lim_{x\to 0}\frac{\sqrt[3]{1+\tan^{-1}3x}-\sqrt[3]{1-\sin^{-1}3x}}{\sqrt{1-\sin^{-1}2x}-\sqrt{1+\tan^{-1}2x}}$$

$$=\lim_{x\to 0}\frac{\left(1+\frac{1}{3}\tan^{-1}3x\ldots\right)-\left(1-\frac{1}{3}\sin^{-1}3x\ldots\right)}{\left(1-\frac{1}{2}\sin^{-1}2x\ldots\right)-\left(1+\frac{1}{2}\tan^{-1}2x\ldots\right)}$$

$$=\lim_{x\to 0}\frac{\frac{1}{3}\left(\frac{\tan^{-1}3x}{x}+\frac{\sin^{-1}3x}{x}\right)+\frac{1}{x}(\text{Higher power of } \sin^{-1}2x \text{ and } \tan^{-1}2x)}{-\frac{1}{2}\left(\frac{\sin^{-1}2x}{x}+\frac{\tan^{-1}2x}{x}\right)+\frac{1}{x}(\text{Higher power of } \sin^{-1}2x\text{ and } \tan^{-1}2x)}$$

$$=\frac{1+1}{-1-1}=-1$$

ILLUSTRATION 58

Evaluate $\lim\limits_{n\to\infty} n^2\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}-2\right)$.

SOLUTION

$$\lim_{n\to\infty} n^2\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}-2\right)$$

$$=\lim_{n\to\infty} n^2\left(\left(1+\frac{1}{n}\right)^{\frac{1}{2}}+\left(1-\frac{1}{n}\right)^{\frac{1}{2}}-2\right)$$

$$=\lim_{n\to\infty} n^2\left(\left(1+\frac{1}{2}\cdot\frac{1}{n}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{1}{n^2}+\ldots\right)+\left(1-\frac{1}{2}\cdot\frac{1}{n}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\frac{1}{n^2}+\ldots\right)-2\right)$$

$$=\lim_{n\to\infty} n^2\left(\frac{-2}{8}\cdot\frac{1}{n^2}+\ldots\right)=-\frac{1}{4}$$

ILLUSTRATION 59

Find the value of $\lim\limits_{x\to 0}\dfrac{x\sin(\sin x)-\sin^2 x}{(\sin x)^6}$.

SOLUTION

$$\lim_{x\to 0}\frac{x\sin(\sin x)-\sin^2 x}{(\sin x)^6}$$

Put $\sin x = t$

$$\Rightarrow\quad \lim_{t\to 0}\frac{(\sin^{-1} t)(\sin t)-t^2}{t^6}$$

Now, $\sin^{-1}x = x+\dfrac{x^3}{6}+\dfrac{3}{40}x^5+\dfrac{5}{112}x^7\ldots$

And $\sin x = x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\ldots$

$$\lim_{t\to 0}\frac{\left(t+\frac{t^3}{6}+\frac{3}{40}t^5+\ldots\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\ldots\right)-t^2}{t^6}$$

$$\lim_{t\to 0}\left(\frac{t^2-\frac{t^4}{3!}+\frac{t^6}{5!}+\frac{t^4}{6}-\frac{t^6}{6\times 6}+\frac{3}{40}t^6-t^2}{t^6}\right)$$

$$=\frac{1}{120}-\frac{1}{36}+\frac{3}{40}=\frac{1}{18}$$

ILLUSTRATION 60

Evaluate $\lim\limits_{x\to\infty} x\ln\left(e\left(1+\dfrac{1}{x}\right)^{1-x}\right)$.

SOLUTION

$$\lim_{x\to\infty} x\left(1+(1-x)\ln\left(1+\frac{1}{x}\right)\right)$$

$$=\lim_{x\to\infty} x\left(1+\ln\left(1+\frac{1}{x}\right)-x\cdot\ln\left(1+\frac{1}{x}\right)\right)$$

Put $x=\dfrac{1}{t}$; as $x\to\infty$, $t\to 0^+$

$$=\lim_{t\to 0^+}\frac{1}{t}\left(1+\ln(1+t)-\frac{1}{t}\ln(1+t)\right)$$

$$=\lim_{t\to 0^+}\left(\frac{\ln(1+t)}{t}+\frac{t-\ln(1+t)}{t^2}\right)$$

$$=\lim_{t\to 0^+}\left(\frac{\ln(1+t)}{t}+\frac{t-\left(t-\frac{t^2}{2}+\frac{t^3}{3}\ldots\right)}{t^2}\right)$$

$$=\lim_{t\to 0^+}\left(\frac{\ln(1+t)}{t}+\frac{1}{2}-\frac{t}{3}+\ldots\right)=1+\frac{1}{2}=\frac{3}{2}$$

Test Your Concepts-IV

(Solutions on page H.52)

1. If $I=\lim\limits_{x\to 0}\dfrac{\sin(ax)-\sin(x)-x}{x^3}$ exists and finite then, the find I and a.

2. Evaluate $\lim\limits_{x\to 0}\dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$.

3. Find the value of $a\in R$ such that $\lim\limits_{x\to 1}\operatorname{cosec}^{-1}\left(\dfrac{a^2}{\ln x}-\dfrac{a^2}{x-1}\right)$ exists.

4. Find the values of a and b so that $\lim\limits_{x\to 0}\dfrac{(1+ax\sin x)-b\cos x}{x^4}$ may find to a definite limit, find limit.

5. If $\lim\limits_{x\to 0}\dfrac{x+\sin x-x\cos x-\tan x}{x^n}$ exists and is non-zero finite value, then, find the value of n.

6. If $\lim\limits_{x\to 1}\sec^{-1}\left(\dfrac{k}{\ell nx}-\dfrac{k}{x-1}\right)=\dfrac{\pi}{3}$, then, find the value of k.

7. Let a, b, c be sides of a triangle ABC such that $\lim\limits_{r\to\infty}\dfrac{\cos\frac{a}{r}-\cos\frac{b}{r}\cdot\cos\frac{c}{r}}{\sin\frac{b}{r}\cdot\sin\frac{c}{r}}=\dfrac{1}{2}$, then, the value of $\dfrac{b^2+c^2-a^2}{2bc}$.

8. Evaluate $\lim\limits_{x\to\infty} x\left(\dfrac{1}{e}-\left(\dfrac{x}{x+1}\right)^x\right)$.

Answer key

1. $\dfrac{-7}{6}, 2$ **2.** $\dfrac{1}{2}$ **3.** $a\in(-\infty,\sqrt{2}]\cup[\sqrt{2},\infty)$ **4.** $\dfrac{1}{24}$ **5.** 5 **6.** 4 **7.** $\dfrac{1}{2}$ **8.** $\dfrac{-1}{2e}$

SOLVING LIMIT OF ∞ – ∞ INDETERMINATE FORM

ILLUSTRATION 61

If $\lim\limits_{x\to\infty}(x+1-\sqrt{ax^2+x+3})$ exist and is equal to L, then, find $a-2L=0$.

SOLUTION

$\lim\limits_{x\to\infty}\left((x+1)-\sqrt{ax^2+x+3}\right)$ it is $\infty-\infty$ indeterminate form

$$L=\lim_{x\to\infty}\frac{\left((x+1)-\sqrt{ax^2+x+3}\right)\left(x+1+\sqrt{ax^2+x+3}\right)}{\left(x+1+\sqrt{ax^2+x+3}\right)}$$

$$L=\lim_{x\to\infty}\frac{(x^2+2x+1-ax^2-x-3)}{\left(x+1+\sqrt{ax^2+x+3}\right)}$$

$$L=\lim_{x\to\infty}\frac{(1-a)x^2+x-2}{(x+1)+\sqrt{x^2+x+3}} \qquad \text{(On Rationalisation)}$$

Highest index of x in denominator is one

$\therefore \quad a=1,$

$$L=\lim_{x\to\infty}\frac{\left(1-\frac{2}{x}\right)}{\left(1+\frac{1}{x}\right)+\sqrt{1+\frac{1}{x}+\frac{3}{x^2}}}$$

$\Rightarrow \quad L=\frac{1}{2}$ Now, we have $a-2L=0$

ILLUSTRATION 62

If $\lim\limits_{x\to\infty}(a\sqrt{2x^2+x+1}-bx)=2$ where $a,b\in R$ then, the value of a^2+b^2.

SOLUTION

$\lim\limits_{x\to\infty}(a\sqrt{2x^2+x+1}-bx)$ [It is $\infty-\infty$ indeterminate form]

Let $l=\lim\limits_{x\to\infty}x\left[a\sqrt{2+\frac{1}{x}+\frac{1}{x^2}}-b\right]$

Hence, limit would exist only if $b=\sqrt{2}a$

Now, $l=\lim\limits_{x\to\infty}a\left[\sqrt{2x^2+x+1}-\sqrt{2}x\right]$

$$l=\lim_{x\to\infty}\frac{a\left(\sqrt{2x^2+x+1}-\sqrt{2}x\right)\left(\sqrt{2x^2+x+1}+\sqrt{2}x\right)}{\sqrt{2x^2+x+1}+\sqrt{2}x}$$

$$\therefore \quad l=a\lim_{x\to\infty}\frac{(x+1)}{\left(\sqrt{2x^2+x+1}+\sqrt{2}x\right)}$$

$$=a\lim_{x\to\infty}\left(\frac{x+1}{x\sqrt{2+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{2}}\right)$$

$$l=a\lim_{x\to\infty}\frac{\left(1+\frac{1}{x}\right)}{\sqrt{2+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{2}}$$

$$\therefore \quad \frac{a}{2\sqrt{2}}=2\Rightarrow a=4\sqrt{2},\ b=8.$$

Hence, $a^2+b^2=32+64=96$

ILLUSTRATION 63

Evaluate $\lim\limits_{x\to-\infty}\left(\sqrt{x^2-\frac{4x}{3}}+2+x\right)$.

SOLUTION

$$\lim_{x\to-\infty}\left(\sqrt{x^2-\frac{4x}{3}}+2+x\right)\text{(It is }\infty-\infty\text{ indeterminate form)}$$

Replace $x\to-x$

$$=\lim_{x\to\infty}\left(\sqrt{x^2+\frac{4x}{3}}+2-x\right)=\lim_{x\to\infty}\left(x\sqrt{1+\frac{4}{3x}}+2-x\right)$$

$$=\lim_{x\to\infty}\left(x\left(1+\frac{4}{3x}\right)^{1/2}+(2-x)\right)$$

$$=\lim_{x\to\infty}\left\{x\left(1+\frac{1}{2}\left(\frac{4}{3x}\right)+\frac{1}{2!}\times\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{4}{3x}\right)^2+\ldots+2-x\right)\right\}$$

$$\lim_{x\to\infty}\left\{x+\frac{2}{3}+\left(\frac{-1}{8}\right)\frac{16}{9x}+\ldots+2-x\right\}=\frac{2}{3}+2=\frac{8}{3}.$$

ILLUSTRATION 64

For $n\in N$, Net $a_n=\sum\limits_{k=1}^{n}(2k)$ and $b_n=\sum\limits_{k=1}^{n}(2k-1)$ then, find $\lim\limits_{n\to\infty}(\sqrt{a_n}-\sqrt{b_n})$.

SOLUTION

We have, $a_n=2(1+2+3+\ldots+n)=n(n+1)$

And, $b_n=1+3+5+\ldots+(2n-1)=n^2$

Now, $\lim\limits_{n\to\infty}(\sqrt{a_n}-\sqrt{b_n})=\lim\limits_{n\to\infty}(\sqrt{n(n+1)}-\sqrt{n^2})$ ($\infty-\infty$ form)

$$=\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})\sqrt{n}$$

$$=\lim_{n\to\infty}\frac{\left(\sqrt{n+1}-\sqrt{n}\right)\left(\sqrt{n+1}+\sqrt{n}\right)\sqrt{n}}{\left(\sqrt{n+1}+\sqrt{n}\right)}$$

$$=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}=\lim_{n\to\infty}\left(\frac{1}{\sqrt{1+\frac{1}{n}}+1}\right)=\frac{1}{2}$$

SOLVING LIMIT OF $\frac{\infty}{\infty}$ FORM

We know that $\lim\limits_{x\to\pm\infty}\frac{c}{x}, \lim\limits_{x\to\pm\infty}\frac{c}{x^2}, \lim\limits_{x\to\pm\infty}\frac{c}{x^3}=0.$

Now, consider the limit of the form

$$\lim_{x\to\pm\infty}\frac{a_0x^n+a_1x^{n-1}+a_2x^{n-2}+....+a_{n-1}x+a_n}{b_0x^n+b_1x^{n-1}+a_2x^{n-2}+....+b_{n-1}x+b_n}.$$

This is $\frac{\infty}{\infty}$ form.

Divide numerator and denominator by highest common power of x occurring that is x^n,

We get $\lim\limits_{x\to\pm\infty}\dfrac{a_0+\frac{a_1}{x}+\frac{a_2}{x^2}+\ldots+\frac{a_{n-1}}{x^{n-1}}+\frac{a_n}{x^n}}{b_0+\frac{b_1}{x}+\frac{b_2}{x^2}+\ldots+\frac{b_{n-1}}{x^{n-1}}+\frac{b_n}{x^n}}$

$$=\frac{a_0+0+0+....+0}{b_0+0+0+....+0}=\frac{a_0}{b_0}$$

(i) $\lim\limits_{x\to\infty}a^x=\begin{cases}0 & \text{if } 0\le a<1\\ 1 & \text{if } a=1\\ \infty & \text{if } a>1\end{cases}$

(ii) $\lim\limits_{x\to\infty}\dfrac{a_0x^m+a_1x^{m-1}+\ldots}{b_0x^n+b_1x^{n-1}+\ldots}$

$$=\begin{cases}\frac{a_0}{b_0} & \text{for } m=n\\ 0 & \text{for } m<n\\ \infty & \text{when } m>n \text{ and } a_0b_0>0\\ -\infty & \text{when } m>n \text{ and } a_0b_0<0\end{cases}$$

ILLUSTRATION 65

Evaluate $\lim\limits_{x\to\infty}\dfrac{3x^2+x+1}{5x^2+1}.$

SOLUTION

$$\lim_{x\to\infty}\frac{3x^2+x+1}{5x^2+1}\quad\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{3x^2}{x^2}+\frac{x}{x^2}+\frac{1}{x^2}}{\frac{5x^2}{x^2}+\frac{1}{x^2}}$$

$$=\lim_{x\to\infty}\frac{3+\frac{1}{x}+\frac{1}{x^2}}{5+\frac{1}{x^2}}=\frac{3}{5}$$

ILLUSTRATION 66

Evaluate $\lim\limits_{n\to\infty}\left[\dfrac{(n+2)!+(n+1)!}{(n+3)!}\right].$

SOLUTION

$$\lim_{n\to\infty}\left[\frac{(n+2)!+(n+1)!}{(n+3)!}\right]\qquad\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$\lim_{n\to\infty}\left(\frac{(n+2)!}{(n+3)!}+\frac{(n+1)!}{(n+3)!}\right)$$

$$=\lim_{n\to\infty}\left[\frac{(n+2)!}{(n+3)(n+2)!}+\frac{(n+1)!}{(n+3)(n+2)(n+1)!}\right]$$

$$=\lim_{n\to\infty}\left[\frac{1}{n+3}+\frac{1}{(n+3)(n+2)}\right]=0$$

ILLUSTRATION 67

Evaluate $\lim\limits_{x\to-\infty}\dfrac{\sqrt{3x^2+2}}{x+2}.$

SOLUTION

$$\lim_{x\to-\infty}\frac{\sqrt{3x^2+2}}{x+2}\qquad\left(\because\sqrt{x^2}=|x|\right)$$

$$=\lim_{x\to-\infty}\frac{|x|\sqrt{3+2/x^2}}{x\left(1+2/x\right)}$$

$$=\lim_{x\to-\infty}\frac{-x\sqrt{3+2/x^2}}{x\left(1+2/x\right)}=\frac{-1\sqrt{3+0}}{1+0}=-\sqrt{3}$$

ILLUSTRATION 68

Evaluate $\lim\limits_{x\to-\infty}\dfrac{x^4\sin\frac{1}{x}+x^2}{1+|x|^3}.$

SOLUTION

$$\lim_{x\to-\infty}\frac{x^4\sin\frac{1}{x}+x^2}{1+|x|^3}=\lim_{x\to-\infty}\frac{x^4\sin\frac{1}{x}+x^2}{1-x^3}$$

$$=\lim_{x\to-\infty}\frac{x^3\left[x\sin\frac{1}{x}+\frac{1}{x}\right]}{x^3\left(\frac{1}{x^3}-1\right)}$$

$$=\lim_{x\to-\infty}\left(\frac{\frac{\sin 1/x}{1/x}+\frac{1}{x}}{1/x^3-1}\right)=\frac{1+0}{0-1}=-1$$

ILLUSTRATION 69

Evaluate $\lim\limits_{x\to-\infty} \dfrac{x^5\tan\dfrac{1}{\pi x^2}+3|x|^2+7}{|x|^3+7|x|+4}$.

SOLUTION

$$\text{Given limit} = \lim_{x\to-\infty}\frac{x^5\tan\dfrac{1}{\pi x^2}+3x^2+7}{-x^3-7x+4}$$

$$= \lim_{x\to-\infty}\frac{x^3\left[x^2\tan\dfrac{1}{\pi x^2}+\dfrac{3}{x}+\dfrac{7}{x^3}\right]}{x^3\left(-1-\dfrac{7}{x^2}+\dfrac{4}{x^3}\right)}$$

$$= \lim_{x\to-\infty}\frac{\left(\dfrac{\tan\dfrac{1}{\pi x^2}}{\dfrac{1}{\pi x^2}}\right)\dfrac{1}{\pi}+\dfrac{3}{x}+\dfrac{7}{x^3}}{-1-\dfrac{7}{x^2}+\dfrac{4}{x^3}} = \frac{\dfrac{1}{\pi}+0+0}{-1} = -\frac{1}{\pi}$$

ILLUSTRATION 70

If $\lim\limits_{x\to0^+}\dfrac{(1+a^3)+8e^{1/x}}{1+(1-b^3)e^{1/x}} = 2$, then, find a and b.

SOLUTION

We have

$$2 = \lim_{x\to0^+}\frac{(1+a^3)+8e^{1/x}}{1+(1-b^3)e^{1/x}} \qquad \left(\frac{\infty}{\infty}\text{ form}\right) \qquad \ldots(1)$$

[Using L' Hospital's Rule]

$$\Rightarrow \quad 2 = \lim_{x\to0^+}\frac{0+8e^{1/x}\left(-\dfrac{1}{x^2}\right)}{0+(1-b^3)e^{1/x}\left(-\dfrac{1}{x^2}\right)} = \frac{8}{1-b^3}$$

$\Rightarrow \quad 1-b^3 = 4 \Rightarrow b^3 = -3 \Rightarrow b = (-3)^{1/3}$

$\therefore \quad$ From (1), $\lim\limits_{x\to0}\dfrac{(1+a^3)+8e^{1/x}}{1+4e^{1/x}} = 2$

$\Rightarrow \quad 1+a^3 = 2 \Rightarrow a = 1$

Hence, $a = 1$ and $b = (-3)^{1/3}$.

ILLUSTRATION 71

Let $f(x)$, $g(x)$ and $h(x)$ be polynomials of degree m, n and p respectively such that there leading coefficients are one, where m, n and p are prime numbers ($m < n < p$). If $\lim\limits_{x\to\infty}\dfrac{h(x)}{x^{13}} = 1$ and $L = \lim\limits_{x\to\infty}\dfrac{h(10x)}{f(2x).g(5x)} = (a)^2(2)^b$ (where a and b are prime numbers), then, find the value of $|a-b|$.

SOLUTION

Given, $\lim\limits_{x\to\infty}\dfrac{h(x)}{x^{13}} = 1 \qquad \therefore$ Degree of $h(x)$ must be 13.

$$L = \lim_{x\to\infty}\frac{h(10x)}{f(2x).g(5x)} = (a)^2(2)^b$$

Also, $m < n < p$ and all are primes $\Rightarrow m = 2$, $n = 11$ and $p = 13$

Degree of $h(x) = p = 13$

Degree of $f(x) = m = 2$

Degree of $g(x) = n = 11$

$$L = \lim_{x\to\infty}\frac{h(10x)}{f(2x).g(5x)}$$

$$= \lim_{x\to\infty}\frac{(10x)^{13}+\ldots}{((2x)^2+\ldots)((5x)^{11}+\ldots)} = \frac{(10)^{13}}{2^2.5^{11}} = \frac{2^{13}.5^{13}}{2^2.5^{11}}$$

$$= 5^2\cdot(2)^{11} = a^2\cdot 2^b$$

We get $a = 5$, $b = 11 \quad \therefore \ |a-b| = 6$

ILLUSTRATION 72

Evaluate $\lim\limits_{x\to\infty}\sqrt{\dfrac{x+\sin^4 x+\sin^5 x}{x+\cos^4 x+\cos^5 x}}$.

SOLUTION

$$\lim_{x\to\infty}\sqrt{\frac{x+\sin^4 x+\sin^5 x}{x+\cos^4 x+\cos^5 x}} = \lim_{x\to\infty}\sqrt{\frac{1+\dfrac{\sin^4 x}{x}+\dfrac{\sin^5 x}{x}}{1+\dfrac{\cos^4 x}{x}+\dfrac{\cos 5x}{x}}}$$

$$= \lim_{x\to\infty}\sqrt{\frac{1+\dfrac{o\text{ to }1}{\infty}+\dfrac{-1\text{ to }1}{\infty}}{1+\dfrac{0\text{ to }1}{\infty}+\dfrac{-1\text{ to }1}{\infty}}} = \sqrt{\frac{1+0+0}{1+0+0}} = 1$$

ILLUSTRATION 73

If $a_1 = 3$ and $a_{n+1} = \sqrt{2+a_n}$, $n \geq 1$ then, find $\lim\limits_{n\to\infty} a_n$.

SOLUTION

Now, since $n\to\infty$, $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} a_n = k$ both will converge to same definite value

$\therefore \quad k = \sqrt{2+k}$

$\therefore \quad k^2 = k+2$

$\Rightarrow \quad k^2-k-2 = 0$

$\Rightarrow \quad (k-2)(k+1) = 0, k \neq -1$

$\Rightarrow \quad k = 2$

i.e. $\lim\limits_{n\to\infty} a_n = k = 2$

ILLUSTRATION 74

Let $f(x) = \left(\dfrac{2x^2+2x+1}{x^2+x+2}\right)^{\frac{6x+1}{3x+2}}$ then, find $\lim\limits_{x\to\infty}(1+f(x))^3$.

SOLUTION

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{2x^2+2x+1}{x^2+x+2}\right)^{\frac{6x+1}{3x+2}}$$

$$= \left\{ \lim_{x\to\infty} \left(\frac{2x^2+2x+1}{x^2+x+2} \right) \right\}^{\lim\limits_{x\to\infty} \frac{6x+1}{3x+2}}$$

$$= \left\{ \lim_{x\to\infty} \left(\frac{2+\frac{2}{x}+\frac{1}{x^2}}{1+\frac{1}{x}+\frac{2}{x^2}} \right) \right\}^{\lim\limits_{x\to\infty} \frac{6x+\frac{1}{x}}{3+\frac{2}{x}}} = 2^2 = 4$$

Hence, $\lim\limits_{x\to\infty} (1+f(x))^3 = (1+\lim\limits_{x\to\infty} f(x))^3 = (1+4)^3 = 125$

ILLUSTRATION 75

Evaluate $\lim\limits_{n\to\infty} \frac{1}{n^4}$ (1.4.7 + 2.5.8 + 3.6.9 + … to n terms).

SOLUTION

Given, $\lim\limits_{n\to\infty} \left[\frac{1.4.7+2.5.8+3.6.9+\ldots \text{to } n \text{ terms}}{n^4} \right]$ $(\frac{\infty}{\infty}$ form)

$$= \lim_{n\to\infty} \left[\frac{\sum\limits_{n=1}^{n} n(n+3)(n+6)}{n^4} \right]$$

$$= \lim_{n\to\infty} \left[\frac{\Sigma n^3 + 9\Sigma n^2 + 18\Sigma n}{n^4} \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^4} \left\{ \frac{n^2(n+1)^2}{4} + \frac{9n(n+1)(2n+1)}{6} + \frac{18n(n+1)}{2} \right\} \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n^4} \frac{n(n+1)}{2} \left\{ \frac{n(n+1)}{2} + 3(2n+1) + 18 \right\} \right]$$

$$= \lim_{n\to\infty} \left[\frac{(n+1)}{2n^3} \left\{ \frac{n^2+13n+42}{2} \right\} \right]$$

$$= \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{13}{n}+\frac{42}{n}\right)}{4} = \frac{(1+0)(1+0)}{4} = \frac{1}{4}$$

ILLUSTRATION 76

If $\alpha = \lim\limits_{n\to\infty} \left(\frac{1}{n^3+1} + \frac{4}{n^3+1} + \frac{9}{n^3+1} + \ldots + \frac{n^2}{n^3+1} \right)$ and $\beta = \lim\limits_{x\to 0} \frac{\tan 2x}{\tan 8x}$, then, find the quadratic equation whose roots are α, β.

SOLUTION

We have $\alpha = \lim\limits_{n\to\infty} \left(\frac{1^2+2^2+3^2+\ldots+n^2}{n^3+1} \right)$

$$= \lim_{n\to\infty} \left(\frac{n\cdot(n+1)\cdot(2n+1)}{6\cdot(n^3+1)} \right) = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6\left(1+\frac{1}{n^3}\right)}$$

$$= \frac{2}{6} = \frac{1}{3}$$

Also, $\beta = \lim\limits_{x\to 0} \frac{\tan 2x}{\tan 8x} = \lim\limits_{x\to 0} \frac{2x \cdot \left(\frac{\tan 2x}{2x} \right)}{8x \cdot \left(\frac{\tan 8x}{8x} \right)} = \frac{2}{8} = \frac{1}{4}.$

Now, $\alpha + \beta = \frac{1}{3} + \frac{1}{4} = \frac{4+3}{12} = \frac{7}{12}$

Also, $\alpha\beta = \left(\frac{1}{3}\right) \cdot \left(\frac{1}{4}\right) = \frac{1}{12}$

Hence, required quadratic equation is $x^2 - \frac{7}{12} \cdot x + \frac{1}{12} = 0.$

$\Rightarrow \quad 12x^2 - 7x + 1 = 0$

SOLVING LIMIT OF 1^∞ INDETERMINATE FORM

Next indeterminate form is 1^∞

Let consider standard function $f(x) = (1+x)^{1/x}$ and its limit at zero

$$\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} e^{\ln(1+x)^{1/x}}$$

(using standard limit $\lim\limits_{x\to 0} \frac{\ln(1+x)}{x} = 1$)

$$= e^{\lim\limits_{x\to 0} \frac{\ln(1+x)}{x}} = e$$

Note

$$\lim_{x\to 0} (1+x)^{1/x} = e \text{ or } \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$$

Now, consider a bit more modified function having this form

$$y = \lim_{x\to\alpha} [f(x)]^{g(x)}$$

And as $x \to \alpha$, $\lim\limits_{x\to\alpha} f(x) = 1$, $\lim\limits_{x\to\alpha} g(x) \to \infty$

Clearly, it is 1^∞ form

$$y = \lim_{x\to\alpha} \{1+[f(x)-1]\}^{g(x)} = \lim_{x\to\alpha} e^{\ln(1+(f(x)-1))^{g(x)}}$$

$$= e^{\lim\limits_{x\to\alpha} g(x) \left(\frac{\ln(1+(f(x)-1))}{f(x)-1} \right)(f(x)-1)}$$

$$\left[\because \lim_{x\to\alpha} \frac{\ln(1+(f(x)-1))}{f(x)-1} = 1 \right]$$

Note

For 1^∞ form $\lim\limits_{x\to\alpha} [f(x)]^{g(x)} = e^{\lim\limits_{x\to\alpha} (f(x)-1)\cdot g(x)}$

as $x \to \alpha$, $\lim\limits_{x\to\alpha} f(x) = 1$, $\lim\limits_{x\to\alpha} g(x) \to \infty$

We can directly use this result

$$\lim_{x\to 0}(1+x)^{a/x} = \lim_{x\to 0} e^{x(a/x)} = e^{a}$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{x} = \lim_{x\to 0}(1+ax)^{1/x} = e^{a}$$

ILLUSTRATION 77

Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{3x^2+5x+3}{3x^2+x+2}\right)^{x}$.

SOLUTION

$$\lim_{x\to\infty}\left(\frac{3x^2+5x+3}{3x^2+x+2}\right)^{x} \text{ Base} = 1 \text{ as } x \to \infty\text{, It is } 1^{\infty} \text{ form}$$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{3x^2+5x+3}{3x^2+x+2}-1\right)x} = e^{\lim\limits_{x\to\infty}\frac{(4x+1)x}{3x^2+x+2}}$$

$$= e^{\lim\limits_{x\to\infty}\left(\frac{4+\frac{1}{x}}{3+\frac{1}{x}+\frac{2}{x^2}}\right)} = e^{4/3}$$

ILLUSTRATION 78

If $\lim\limits_{x\to 0}(1+ax+bx^2)^{3/x} = e^{7}$, then, find value of a and b.

SOLUTION

$$\lim_{x\to 0}\left[1+(ax+bx^2)\right]^{3/x} \text{ It is of } 1^{\infty} \text{ form}$$

$$= e^{\lim\limits_{x\to 0}(ax+bx^2)\frac{3}{x}} = e^{\lim\limits_{x\to 0}(a+bx)\,3} = e^{3a}$$

Since, $e^{3a} = e^{7} \Rightarrow a = 7/3$, b can take any finite value $a = 7/3, b \in R$.

ILLUSTRATION 79

Evaluate $\lim\limits_{m\to\infty}\left(\cos\dfrac{x}{m}\right)^{m}$ for fixed x.

SOLUTION

It is 1^{∞} form

$$\lim_{m\to\infty}\left(\cos\frac{x}{m}\right)^{m} = e^{\lim\limits_{m\to\infty} m\left(\cos\frac{x}{m}-1\right)} = e^{\lim\limits_{m\to\infty} -2m\sin^2\frac{x}{2m}}$$

$$= e^{\lim\limits_{m\to\infty} -2m\left(\frac{\sin\frac{x}{2m}}{\frac{x}{2m}}\right)^{2}\left(\frac{x}{2m}\right)^{2}} = e^{\lim\limits_{m\to\infty} -\frac{x^2}{2m}} = e^{-0} = 1$$

ILLUSTRATION 80

If $\lim\limits_{x\to 0}\dfrac{h(x)}{\sin^2 x} = 2$ and $\lim\limits_{x\to 0}\dfrac{k(x)}{e^{x}-x-1} = 5$ then, find $\lim\limits_{x\to 0}(1+h(x))^{\frac{1}{k(x)}}$.

SOLUTION

$$\lim_{x\to 0}\frac{h(x)}{\sin^2 x} = \lim_{x\to 0}\frac{h(x)}{x^2}\times\frac{x^2}{\sin^2 x} = 2 \Rightarrow \lim_{x\to 0}\frac{h(x)}{x^2} = 2$$

$$\lim_{x\to 0}\frac{k(x)}{e^{x}-x-1} = \lim_{x\to 0}\frac{k(x)}{x^2}\times\frac{x^2}{(e^{x}-x-1)} = 5$$

$$\Rightarrow \lim_{x\to 0}\frac{k(x)}{x^2}\,\frac{x^2}{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\ldots-x-1\right)} = 5$$

$$\Rightarrow \lim_{x\to 0}\frac{2k(x)}{x^2} = 5$$

Now, required limit

$$\lim_{x\to 0}\left(1+h(x)\right)^{1/k(x)} \quad (1^{\infty} \text{ form})$$

$$= \lim_{x\to 0} e^{\frac{h(x)}{k(x)}} = \lim_{x\to 0} e^{\frac{h(x)}{x^2}\times\frac{x^2}{k(x)}} = e^{\frac{2\times 2}{5}} = e^{\frac{4}{5}}$$

ILLUSTRATION 81

Find $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$.

SOLUTION

Given, limit $= \lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{\sin x}{x-\sin x}}$ $(1^{\infty}$ form)

$$= e^{l} \text{ where } l = \lim_{x\to 0}\left(\frac{\sin x}{x}-1\right)\left(\frac{\sin x}{x-\sin x}\right)$$

$$l = \lim_{x\to 0}\frac{\sin x}{x-\sin x}\left(\frac{\sin x - x}{x}\right) = -1$$

Hence, required Limit $= e^{-1} = \dfrac{1}{e}$.

ILLUSTRATION 82

Let $f(x, y) = \sin(2x-y)\cos y + \cos(2x-y)\sin y \ \forall\ x, y \in R$.

Find the value of $\lim\limits_{x\to 0}(1+\tan x - x)^{\frac{1}{x^2 f(x,y)}}$.

SOLUTION

$$f(x, y) = \sin(2x-y)\cos y + \cos(2x-y)\sin y$$

$$f(x, y) = \sin(2x-y+y) = \sin 2x$$

$$\therefore \quad \lim_{x\to 0}(1+\tan x - x)^{\frac{1}{x^2(\sin 2x)}} \qquad (1^{\infty} \text{ form})$$

$$= e^{\lim\limits_{x\to 0}\frac{\tan x - x}{x^2\sin 2x}} = e^{\lim\limits_{x\to 0}\left(\frac{x+\frac{x^3}{3}+\ldots-x}{2x^3\left(\frac{\sin 2x}{2x}\right)}\right)} = e^{1/6}.$$

ILLUSTRATION 83

Evaluate $\lim\limits_{n\to\infty}\left(\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right)\right)^{n}$.

SOLUTION

$$\lim_{n\to\infty}\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right)\right]^{n} \qquad \text{(It is } 1^{\infty} \text{ form)}$$

$$= e^{\lim\limits_{n\to\infty}\left[\tan\left(\frac{\pi}{4}+\frac{\pi}{n}\right)-1\right]n} = e^{\lim\limits_{n\to\infty}\left[\frac{1+\tan\frac{\pi}{n}}{1-\tan\frac{\pi}{n}}-1\right]n}$$

$$= e^{\lim\limits_{n\to\infty}\left[2\left(\frac{\tan\pi/n}{\pi/n}\right)\frac{\pi}{(1-\tan\pi/n)}\right]} \qquad \left(\because \lim_{n\to\infty}\frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}=1\right)$$

$$= e^{\lim\limits_{n\to\infty}\left[2\frac{\pi}{(1-\tan\pi/n)}\right]} = e^{2\pi}$$

ILLUSTRATION 84

If $\lim\limits_{x\to a}\left(2-\frac{a}{x}\right)^{a\tan\left(\frac{\pi x}{2a}\right)}=\frac{1}{e}$, then, find a.

SOLUTION

$$\lim_{x\to a}\left(2-\frac{a}{x}\right)^{a\tan\left(\frac{\pi x}{2a}\right)} \qquad \text{(It is } 1^{\infty} \text{ form)} = e^{-1} \qquad \text{(Given)}$$

Now, $\lim\limits_{x\to a}\left(1-\frac{a}{x}\right)a\cdot\tan\left(\frac{\pi x}{2a}\right)=-1$ Put $x=a+h$;

$$\Rightarrow \quad \lim_{h\to 0}\left(1-\frac{a}{a+h}\right)\cdot a\tan\left(\frac{\pi}{2a}(a+h)\right)=-1$$

$$\Rightarrow \quad \lim_{h\to 0}\left(\frac{a+h-a}{a+h}\right)a\tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right)$$

$$\Rightarrow \quad -\lim_{h\to 0}\left(\frac{ah}{a+h}\right)\cot\left(\frac{\pi h}{2a}\right)=-1$$

$$\Rightarrow \quad \lim_{h\to 0}\left(\frac{\pi h/2a}{\tan\frac{\pi h}{2a}}\right)\left(\frac{2a^{2}h}{\pi h}\right)\left(\frac{1}{a+h}\right)=1$$

$$\Rightarrow \quad \frac{2a}{\pi}=1,\text{ we get } a=\frac{\pi}{2}$$

ILLUSTRATION 85

Evaluate $\lim\limits_{x\to 1}\left(\frac{x^{2}-1}{2(x-1)}\right)^{\frac{2\sin^{-1}x}{\sin\pi x}}$.

SOLUTION

As $\sin^{-1}x$ is present in function, RHL is not defined (not part of domain). We can compute LHL here

$$\lim_{x\to 1^{-}}\left(\frac{x^{2}-1}{2(x-1)}\right)^{\frac{2\sin^{-1}x}{\sin\pi x}} = \lim_{x\to 1^{-}}\left(\frac{(x-1)(x+1)}{2(x-1)}\right)^{\frac{2\sin^{-1}x}{\sin\pi x}}$$

$$= \lim_{x\to 1^{-}}\left(\frac{x+1}{2}\right)^{\frac{2\sin^{-1}x}{\sin\pi x}} \qquad \text{(It is } 1^{\infty} \text{ Form)}$$

$$= e^{\lim\limits_{x\to 1^{-}}\left(\frac{x+1}{2}-1\right)\frac{2\sin^{-1}x}{\sin\pi x}}$$

$$= e^{\lim\limits_{x\to 1^{-}}\left(\frac{x-1}{2}\right)\frac{2\sin^{-1}x}{\sin\pi x}} \qquad (\text{put } x\to 1-h)$$

$$= e^{\lim\limits_{h\to 0}\frac{-h\sin^{-1}(1-h)}{\sin(\pi(1-h))}} \qquad (\because \sin(\pi-\pi h)=\sin\pi h)$$

$$= e^{\lim\limits_{h\to 0}\left(\frac{-\pi h}{\sin\pi h}\right)\frac{\sin^{-1}(1-h)}{\pi}} = e^{-\frac{1}{\pi}\times\frac{\pi}{2}} = e^{-1/2}$$

ILLUSTRATION 86

Evaluate $\lim\limits_{x\to\infty}(2^{x}+3^{x}+5^{x}+7^{x})^{\frac{1}{x}}$.

SOLUTION

$$\lim_{x\to\infty}(2^{x}+3^{x}+5^{x}+7^{x})^{\frac{1}{x}}$$

$$= \lim_{x\to\infty}7\left(\left(\frac{2}{7}\right)^{x}+\left(\frac{3}{7}\right)^{x}+\left(\frac{5}{7}\right)^{x}+1\right)^{\frac{1}{x}}$$

$$\left(\because \lim_{x\to\infty}a^{x}=0,\text{ when } -1<a<1\right)$$

$$= 7(0+0+0+1)^{0}=7$$

ILLUSTRATION 87

Let $a=n^{\left(n+\frac{1}{\ln n}\right)}$ and $b=\left(n+\frac{1}{\ln(n)}\right)^{n}$. Find the value of $\lim\limits_{n\to\infty}\left(\frac{a}{b}\right)$.

SOLUTION

$$a=n^{n}\cdot n^{\frac{1}{\ln n}}=n^{n}\;n^{\log_{n}e}=n^{n}\cdot e$$

$$b=\left(n+\frac{1}{\ln(n)}\right)^{n}$$

Now, $\dfrac{a}{b}=\dfrac{n^{n}.e}{\left(n+\frac{1}{\ln n}\right)^{n}}=\dfrac{n^{n}.e}{n^{n}\left(1+\frac{1}{n\ln n}\right)^{n}}$

$$\Rightarrow \quad \frac{a}{b}=\frac{e}{\left(1+\frac{1}{n\ln n}\right)^{n}}$$

Now, consider denominator of $\dfrac{a}{b}$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n\ln n}\right)^{n} \qquad (1^{\infty}\text{ form})$$

$$= e^{\lim\limits_{n\to\infty}n\left(1+\frac{1}{n\ln n}-1\right)} = e^{0}=1.$$

Hence, $\lim\limits_{n\to\infty}\left(\dfrac{a}{b}\right)=e$

ILLUSTRATION 88

Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{(x+1)^x}{x^x\cdot e}\right)^x$.

SOLUTION

$$L=\lim_{x\to\infty}\left(\frac{(x+1)^x}{x^x\cdot e}\right)^x=e^{\lim\limits_{x\to\infty}x\left[\left(1+\frac{1}{x}\right)^x\frac{1}{e}-1\right]} \qquad (1^\infty \text{ form})$$

$$L=e^{\lim\limits_{x\to\infty}x\left[\frac{e^{x\ln\left(1+\frac{1}{x}\right)}}{e}-1\right]}$$

$$=e^{\lim\limits_{x\to\infty}x\frac{\left[e^{x\ln\left(1+\frac{1}{x}\right)-1}-1\right]}{x\ln\left(1+\frac{1}{x}\right)-1}\left(x\ln\left(1+\frac{1}{x}\right)-1\right)}$$

$$L=e^{\lim\limits_{x\to\infty}x\left[x\ln\left(1+\frac{1}{x}\right)-1\right]}$$

Put $x=\dfrac{1}{y}$, as $x\to\infty$, $y\to 0^+$

$$L=e^{\lim\limits_{y\to0^+}\frac{\ln(1+y)-y}{y^2}}=e^{\lim\limits_{y\to0^+}\frac{\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\dots\right)-y}{y^2}}=e^{-\frac{1}{2}}$$

ILLUSTRATION 89

If $\lim\limits_{x\to\infty}\left(\dfrac{ax-b\cos\frac{1}{x}}{ax-b\sin\frac{1}{x}}\right)^x=e$, $a\neq 0$, then, find relation in a and b.

SOLUTION

$$\lim_{x\to\infty}\left(\frac{ax-b\cos\frac{1}{x}}{ax-b\sin\frac{1}{x}}\right)^x=e^{\lim\limits_{x\to\infty}\left(\frac{ax-b\cos\frac{1}{x}}{ax-b\sin\frac{1}{x}}-1\right)x}=e \qquad (\text{It is } 1^\infty \text{ from})$$

We get $\lim\limits_{x\to\infty}x\left(\dfrac{b\sin\frac{1}{x}-b\cos\frac{1}{x}}{ax-b\sin\frac{1}{x}}\right)=1$

$$\Rightarrow \lim_{x\to\infty}\left(\frac{b\sin\frac{1}{x}-b\cos\frac{1}{x}}{a-\frac{b}{x}\sin\frac{1}{x}}\right)=\frac{0-b}{a-0}=1$$

$$\Rightarrow -\frac{b}{a}=1$$

Hence, $a+b=0$.

Test Your Concepts-V

(Solutions on page H.53)

1. If the value of $\lim\limits_{x\to\infty}\left(\sqrt[3]{x^3+4x^2}-\sqrt{x^2+\dfrac{x}{3}}\right)$ is $\dfrac{a}{b}$ where a and b are co-prime then, find the value of $(a+b)$.
2. Evaluate $\lim\limits_{x\to2}\left(\dfrac{1}{x-2}-\dfrac{2(2x-3)}{x^3-3x^2+2x}\right)$.
3. Evaluate $\lim\limits_{x\to\infty}\dfrac{a+bx+cx^2+dx^3}{e+fx^2+gx^3}$.
4. Evaluate $\lim\limits_{x\to\infty}\dfrac{3x^2-4x+5}{4x^2+x^3-7}$.
5. Evaluate $\lim\limits_{x\to0^+}\left(\dfrac{3\ln^2x+5\ln x+6}{1+\ln^2x}\right)$.
6. If $f(x)=\sqrt{\dfrac{x-\sin x}{x+\cos^2x}}$; find $\lim\limits_{x\to\infty}f(x)$.
7. Evaluate $\lim\limits_{x\to\infty}\dfrac{x^2+x+1}{3x^2+2x-5}$.
8. If the value of $\lim\limits_{n\to\infty}\left(\dfrac{n^2+n+3}{n^2+3n+5}\right)^n$ is equal to $\dfrac{p}{e^2}$, then, find the value of p.
9. Evaluate $\lim\limits_{x\to-\infty}\dfrac{\sqrt{x^2+1}}{3x-6}$.
10. Solve $\lim\limits_{x\to\pi/2}(\sin x)^{\tan x}$.
11. Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{2x^2+3}{2x^2+5}\right)^{8x^2+3}$.
12. Evaluate $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x^2}\right)^{\frac{8x^3}{\pi}\sin\left(\frac{\pi}{2x}\right)}$.
13. Evaluate $\lim\limits_{x\to a}(\log_a x)^{\frac{1}{x-a}}$, where $0<a<1$.
14. Evaluate $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n^2+\cos n}\right)^{n^2+n}$.

Answer key

1. 13 **2.** $-\dfrac{1}{2}$ **3.** $\dfrac{d}{g}$ **4.** 0 **5.** 3 **6.** 1 **7.** $\dfrac{1}{3}$

8. 1 **9.** $-\dfrac{1}{3}$ **10.** 1 **11.** e^{-8} **12.** e^4 **13.** $\dfrac{1}{e^{a\ln a}}$ **14.** e

SOLVING LIMIT OF 0^0 AND ∞^0 FORM

To compute the limit of the form 0^0 and ∞^0, we generally let the limit be y and then, use logarithm. Or we can take log and antilog simultaneously, and we need to convert indeterminate form in either $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form.

It can be evaluated by standard limit/series expansion/rationalization/factorization/substitution/L-Hospital rule.

ILLUSTRATION 90

Evaluate $\lim\limits_{x\to 0^+} x^x$.

SOLUTION

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln x^x} = e^{\lim\limits_{x\to 0^+} x\ln x} \qquad (0^0 \text{ form })$$

$$= e^{\lim\limits_{x\to 0^+}\frac{\ln x}{1/x}} \qquad \left[\frac{\infty}{\infty}\text{ form, using L-H Rule}\right]$$

$$= e^{\lim\limits_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)} = e^{\lim\limits_{x\to 0^+}(-x)} = 1$$

ILLUSTRATION 91

Evaluate $\lim\limits_{x\to 0^+} (\sin x)^x$.

SOLUTION

$$\lim_{x\to 0^+}(\sin x)^x = e^{\lim\limits_{x\to 0^+}\ln(\sin x)^x} = e^{\lim\limits_{x\to 0^+}\frac{\ln\sin x}{1/x}} \quad \left(\text{It is }\left(\frac{\infty}{\infty}\right)\text{ form}\right)$$

Using L-H Rule

$$= e^{\lim\limits_{x\to 0^+}\frac{\frac{1}{\sin x}\cdot\cos x}{-1/x^2}} = e^{\lim\limits_{x\to 0^+} -\left(\frac{x}{\tan x}\right)x} = e^{-(1)0} = e^{o} = 1$$

ILLUSTRATION 92

Evaluate $\lim\limits_{x\to 0^+}\dfrac{e^{x^x-1}-x^x}{((x^2)^x-1)^2}$.

SOLUTION

We know $\lim\limits_{x\to 0^+} x^x = 1$

$$\text{Required limit} = \lim_{x\to 0^+}\frac{e^{(x^x-1)}-x^x}{((x^2)^x-1)^2}$$

$$= \lim_{x\to 0^+}\frac{\left(1+(x^x-1)+\frac{(x^x-1)^2}{2!}+\frac{(x^x-1)^3}{3!}+\ldots\right)-x^x}{(x^x-1)^2(x^x+1)^2}$$

$$= \lim_{x\to 0^+}\frac{\frac{(x^x-1)^2}{2!}+\frac{(x^x-1)^3}{3!}+\ldots}{(x^x-1)^2(x^x+1)^2} = \lim_{x\to 0^+}\frac{\frac{1}{2!}+\frac{(x^x-1)}{3!}+\ldots}{(x^x+1)^2}$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{8}$$

ILLUSTRATION 93

Evaluate $\lim\limits_{x\to\frac{\pi}{2}^-}(\tan x)^{\frac{\pi}{2}-x}$.

SOLUTION

$$\lim_{x\to\frac{\pi}{2}^-}(\tan x)^{\frac{\pi}{2}-x} \qquad (\text{It is }(\infty^o)\text{ form})$$

$$= e^{\lim\limits_{x\to\frac{\pi}{2}^-}\ln(\tan x)^{\pi/2-x}} = e^{\lim\limits_{x\to\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\ln\tan x}$$

Put $x = \dfrac{\pi}{2} - h$

$$= e^{\lim\limits_{h\to 0} h\ln\cot h} = e^{\lim\limits_{h\to 0}\frac{\ln\cot h}{1/h}} \qquad \left(\frac{\infty}{\infty}\text{form using L-H Rule}\right)$$

$$= e^{\lim\limits_{h\to 0}\frac{\frac{1}{\cot h}(-\text{cosec}^2 h)}{-1/h^2}} = e^{\lim\limits_{h\to 0}\left(\frac{h}{\sin h}\right)^2\tan h} = e^0 = 1$$

ILLUSTRATION 94

Evaluate $\lim\limits_{x\to\infty}\left(\dfrac{x^2+\sin x}{x+\sin x}\right)^{\frac{1}{x}}$.

SOLUTION

$$L = \lim_{x\to\infty}\left(\frac{x^2+\sin x}{x+\sin x}\right)^{\frac{1}{x}} \qquad (\infty^o\text{ form})$$

$$L = \lim_{x\to\infty}\left(\frac{x^2\left(1+\frac{\sin x}{x^2}\right)}{x\left(1+\frac{\sin x}{x}\right)}\right)^{\frac{1}{x}} = \lim_{x\to\infty}(x)^{\frac{1}{x}}\frac{\left(1+\frac{\sin x}{x^2}\right)^{1/x}}{\left(1+\frac{\sin x}{x}\right)^{1/x}}$$

$$L = \lim_{x\to\infty} x^{\frac{1}{x}} \qquad (\infty^o\text{ form})$$

Now, taking log,

$$\ln L = \lim_{x\to\infty}\frac{\ln x}{x} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

(Using L-H Rule)

$$= \lim_{x\to\infty}\frac{1}{x} = 0$$

Hence, required limit $L = 1$

ILLUSTRATION 95

If $L = \lim\limits_{x\to 0}\dfrac{(1+\tan x)^{\cot x}-e}{\sin\left(\frac{x}{2}\right)}$, then, compute $|[L]|$ is equal to (where [.] denotes greatest integer function).

SOLUTION

$$L = \lim_{x\to 0}\frac{(1+\tan x)^{\cot x}-e}{\sin\left(\frac{x}{2}\right)}$$

$$L=\lim_{x\to0}\frac{e^{\cot x\ln(1+\tan x)}-e}{\sin\frac{x}{2}}=\lim_{x\to0}\frac{e^{\cot x\left(\tan x-\frac{\tan^2x}{2}+\frac{\tan^3x}{3}-.....\right)}-e}{\sin\frac{x}{2}}$$

$$=e\lim_{x\to0}\frac{e^{-\frac{\tan x}{2}+\frac{\tan^2x}{3}-\dots}-1}{\left(-\frac{\tan x}{2}+\frac{\tan^2x}{3}-\dots\right)}\cdot\frac{\left(-\frac{\tan x}{2}+\frac{\tan^2x}{3}-\dots\right)}{\frac{x}{2}}\cdot\frac{\frac{x}{2}}{\sin\frac{x}{2}}$$

$$=e\lim_{x\to0}\left(-\frac{\tan x}{x}+\frac{2\tan^2x}{3x}-\dots\right)=-e\times1=-e$$

Hence, we have $|[L]|=3$

ILLUSTRATION 96

If $\lim\limits_{x\to0}\dfrac{\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}}-e^2}{x^2}$ is equal to $\left(\dfrac{p}{q}\right)e^2$, $(p,q\in N)$, then, the minimum value of $(p+q)$.

SOLUTION

$$\lim_{x\to0}\frac{\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{\frac{1}{x}}-e^2}{x^2}=\lim_{x\to0}\frac{e^{\frac{1}{x}\ln\left(\frac{1+\tan x}{1-\tan x}\right)}-e^2}{x^2}$$

$$=e^2\lim_{x\to0}\frac{\left(e^{\frac{1}{x}\ln\cdot\left(\frac{1+\tan x}{1-\tan x}\right)-2}-1\right)}{x^2}$$

$$=e^2\lim_{x\to0}\frac{e^{\frac{1}{x}\ln\left(\frac{1+\tan x}{1-\tan x}\right)-2}-1}{\left\{\frac{1}{x}\ln\left(\frac{1+\tan x}{1-\tan x}\right)-2\right\}}\cdot\frac{\ln\left(\frac{1+\tan x}{1-\tan x}\right)-2x}{x^3}$$

$$=e^2\lim_{x\to0}\frac{\ln\left(\frac{1+\tan x}{1-\tan x}\right)-2x}{x^3}$$

$$\left[\because\lim_{x\to0}\frac{1}{x}\ln\left(\frac{1+\tan x}{1-\tan x}\right)-2=0\right]$$

$$=e^2\lim_{x\to0}\frac{\ln(1+\tan x)-\ln(1-\tan x)-2x}{x^3}$$

$$=e^2\lim_{x\to0}\frac{\left[\left(\tan x-\frac{\tan^2x}{2}+\frac{\tan^3x}{3}+\dots\right)+\left(\tan x+\frac{\tan^2x}{2}+\frac{\tan^3x}{3}+\dots\right)-2x\right]}{x^3}$$

[Using series expansions]

$$=e^2\lim_{x\to0}\frac{2(\tan x-x)+\frac{2}{3}\tan^3x\dots}{x^3}$$

$$=e^2\lim_{x\to0}\frac{2\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\dots-x\right)+\frac{2}{3}\tan^3x+\dots}{x^3}$$

$$=e^2\left(\frac{2}{3}+\frac{2}{3}\right)=\frac{4e^2}{3}=\frac{p}{q}e^2$$

We get $p=4k$, $q=3k$, then minimum of $p+q=7$.

ILLUSTRATION 97

If $\alpha_n\in\left(0,\frac{\pi}{4}\right)$ is a root of the equation $\tan x+\cot x=n$, then, find the value of $\lim\limits_{n\to\infty}(\sin\alpha_n+\cos\alpha_n)^n$.

SOLUTION

We have,

$$\tan x+\cot x=n\Rightarrow\frac{1}{n}=\sin x\cos x \qquad \dots(1)$$

Given, α_n is the root of (1), so $\sin\alpha_n\cos\alpha_n=\dfrac{1}{n}$

Let $L=\lim\limits_{n\to\infty}(\sin^2\alpha_n+\cos^2\alpha_n+2\sin\alpha_n\cos\alpha_n)^{n/2}$

$$=\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{\frac{n}{2}}=e$$

LIMIT OF A COMPOSITE FUNCTION

When $f(x)$ is continuous at $x=\lim\limits_{x\to\alpha}g(x)$, then

$$\lim_{x\to\alpha}f(g(x))=f\left(\lim_{x\to\alpha}g(x)\right)$$

In particular,

(i) $\lim\limits_{x\to\alpha}\log f(x)=\log\left(\lim\limits_{x\to\alpha}f(x)\right)$

(ii) $\lim\limits_{x\to\alpha}e^{f(x)}=e^{\lim\limits_{x\to\alpha}f(x)}$

(iii) $\lim\limits_{x\to\alpha}|f(x)|=\left|\lim\limits_{x\to\alpha}f(x)\right|$

As we know that log, e^x, $|x|$ are continuous function so we can take limit inside the function, but in case of discontinuous function we should avoid taking limit inside the function like greatest integer function [], fractional part function {.}, signum function sgn(x).

In case of discontinuous function, we should use LHL and RHL computation to conclude about the existence of limit.

Useful Results

$\sin x < x < \tan x$ for $x \in (0, \pi/2)$

$\sin^{-1} x > x > \tan^{-1} x$ for $x \in (0, 1)$

Let consider $\lim\limits_{x\to 0}\left[\dfrac{\sin x}{x}\right] = \lim\limits_{x\to 0} 0 = 0$

$$\left(\because \sin x < x \Rightarrow \frac{\sin x}{x} < 1\right)$$

$$\left[\lim_{x\to 0}\frac{\sin x}{x}\right] = [1] = 1$$

$\therefore$ Hence, $\lim\limits_{x\to 0}\left[\dfrac{\sin x}{x}\right] \neq \left[\lim\limits_{x\to 0}\dfrac{\sin x}{x}\right]$

ILLUSTRATION 98

Evaluate $\lim\limits_{x\to 0}\left[\dfrac{\sin^{-1} x}{x}\right] + \left[\dfrac{-\tan^{-1} x}{\sin^{-1} x}\right] + \left[\dfrac{-3\tan^{-1} x}{x}\right]$

where [.] is GIF.

SOLUTION

As we know that $\sin^{-1} x > x > \tan^{-1} x$ for x varies from 0 to 1

$$\frac{\sin^{-1} x}{x} \approx 1.00000000001\ ,\ \frac{-\tan^{-1} x}{\sin^{-1} x} \approx -0.999999,$$

$$\frac{-3\tan^{-1} x}{x} \approx -2.999999$$

Now, $\lim\limits_{x\to 0}\left[\dfrac{\sin^{-1} x}{x}\right] + \left[\dfrac{-\tan^{-1} x}{\sin^{-1} x}\right] + \left[\dfrac{-3\tan^{-1} x}{x}\right]$

$$= \lim_{x\to 0} (1 - 1 - 3) = -3$$

ILLUSTRATION 99

Evaluate $\lim\limits_{x\to 0}\left[\dfrac{\tan x}{x}\right] + \left[\dfrac{\tan^{-1} x}{x}\right]$ where [.] is GIF.

SOLUTION

As we know that $x < \tan x$ and $\tan^{-1} x < x$ for x, varies from 0 to 1.

$$\frac{\tan x}{x} \approx 1.000000001 \text{ and } \frac{\tan^{-1} x}{x} \approx 0.9999999$$

$$\lim_{x\to 0}\left[\frac{\tan x}{x}\right] + \left[\frac{\tan^{-1} x}{x}\right] = \lim_{x\to 0} (1 + 0) = 1$$

ILLUSTRATION 100

Evaluate $\lim\limits_{x\to 0}\dfrac{\sin^3 x - x^3 \operatorname{sgn}\left(1 - \left[\dfrac{x}{\sin^{-1} x}\right]\right)}{x\tan^2(x).\sin(\pi\cos x)}$ where [.] is GIF.

SOLUTION

$$\frac{x}{\sin^{-1} x} < 1 \Rightarrow \left[\frac{x}{\sin^{-1} x}\right] = 0 \Rightarrow \operatorname{sgn}\left(1 - \left[\frac{x}{\sin^{-1} x}\right]\right) = 1$$

$$\lim_{x\to 0}\frac{\sin^3 x - x^3}{x.\tan^2 x.\sin(\pi\cos x)}$$

$$= \lim_{x\to 0}\frac{(\sin x - x)(\sin^2 x + x\sin x + x^2)}{x.\dfrac{\tan^2 x}{x^2}\cdot x^2.\sin(\pi - \pi\cos x)}$$

$$= \lim_{x\to 0}\frac{(\sin x - x)\left(\dfrac{\sin^2 x}{x^2} + \dfrac{x\sin x}{x^2} + \dfrac{x^2}{x^2}\right)}{x\left(\dfrac{\sin(\pi - \pi\cos x)}{\pi - \pi\cos x}\right)\dfrac{\pi(1-\cos x)}{x^2}\cdot x^2}$$

$$= \lim_{x\to 0}\frac{(\sin x - x)(3)}{x^3\,(1)\dfrac{\pi}{2}}$$

$$\left(\because \lim_{x\to 0}\frac{\sin x - x}{x^3} = \lim_{x\to 0}\left(\frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - .... - x}{x^3}\right) = -\frac{1}{6}\right)$$

Hence, required limit $= \dfrac{-1}{6} \times \dfrac{3}{\pi} \times 2 = \dfrac{-1}{\pi}$

ILLUSTRATION 101

If $f(x) = \begin{cases} 1 - x, & x < 1 \\ x - 1, & x \geq 1 \end{cases}$ and $g(x) = \begin{cases} 3 - x & x < 0 \\ 1 - x^2 & x \geq 0 \end{cases}$

Compute $\lim\limits_{x\to 1} g(f(x))$, $\lim\limits_{x\to 0} f(g(x))$, $\lim\limits_{x\to 1} f(f(x))$.

SOLUTION

As we are computing limit of composite function, There is no need to compute complete definition of composite function, we can discuss behaviour of function around the given point.

(a) $\lim\limits_{x\to 1} g(f(x))$

Let $x = 1 + h$ where $h > 0$

RHL $= \lim\limits_{x\to 1^+} g(f(x)) = \lim\limits_{h\to 0} g(1 + h - 1) = \lim\limits_{h\to 0} g(h)$

$= \lim\limits_{h\to 0} (1 - h^2) = 1 - 0 = 1$

Let $x = 1 - h$ where $h > 0$

LHL $= \lim\limits_{x\to 1^-} g(f(x)) = \lim\limits_{h\to 0} g(1 - (1 - h)) = \lim\limits_{h\to 0} g(h)$

$= \lim\limits_{h\to 0} 1 - h^2 = 1$

As we are getting both LHL and RHL both as 1, then, $\lim\limits_{x\to 1} g(f(x)) = 1$.

(b) $\lim\limits_{x\to 0} f(g(x))$

Let $x = 0 + h$

$h > 0$

RHL $= \lim\limits_{x\to 0^+} f(g(x)) = \lim\limits_{h\to 0} f(1 - h^2) = \lim\limits_{h\to 0} 1 - (1 - h^2)$

$= \lim\limits_{h\to 0} h^2 = 0$

Let $x = 0 - h$ where $h > 0$

$$\text{LHL} = \lim_{x\to 0^-} f(g(x)) = \lim_{h\to 0} f(g(-h)) = \lim_{x\to 0} f(3+h))$$

$$= \lim_{h\to 0} 3 + h - 1 = 2$$

Clearly, LHL and RHL are not equal hence, $\lim_{x\to 0} f(g(x))$ does not exist.

(c) $\lim_{x\to 1} f(f(x))$

Put $x = 1 + h$,

$$\text{RHL} = \lim_{x\to 1^+} f(f(x)) = \lim_{h\to 0} f(1+h-1) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} 1 - h = 1$$

Put $x = 1 - h$

$$\text{LHL} = \lim_{x\to 1^-} f(f(x)) = \lim_{h\to 0} f(1-(1-h)) = \lim_{h\to 0} f(h)$$

$$= \lim_{h\to 0} 1 - h = 1$$

Hence, $\lim_{x\to 1} f(f(x)) = 1$

ILLUSTRATION 102

If $f(x) = \begin{cases} x^2 - 2x + 3, & -2 < x < 0 \\ x^3 + x^2 + 2, & 0 \le x \le 2 \\ 3 - 2x, & 2 < x \le 5 \end{cases}$ and

$g(x) = \begin{cases} 2x - 3, & x \le -2 \\ x^3 + 1, & |x| < 2 \\ |x-3|, & 2 \le x < 4 \\ x + 2, & 4 \le x < 7 \end{cases}$ then, compute $\lim_{x\to 3^-} (gofog)(x)$.

SOLUTION

$$\lim_{x\to 3^-} (gofog)(x)$$

Put $x = 3 - h$

$$= \lim_{h\to 0} g(f(g(3-h)))$$

$$= \lim_{h\to 0} g(f(|3-h-3|)) = \lim_{h\to 0} g(f(h))$$

$$= \lim_{h\to 0} g(h^3 + h^2 + 2) = g(2^+) = 1$$

ILLUSTRATION 103

Find the value of limit $\lim_{\theta\to\pi}\left[\dfrac{11\cos^2\theta + 12\cos\theta + 1}{\cos^2\theta + \cos\theta}\right]$ (where $[k]$ denotes greatest integer functionless than or equal to k).

SOLUTION

$$\lim_{\theta\to\pi}\left[\frac{11\cos^2\theta + 12\cos\theta + 1}{\cos^2\theta + \cos\theta}\right]$$

$$= \lim_{\theta\to\pi}\left[\frac{(11\cos\theta + 1)(1 + \cos\theta)}{\cos\theta.(\cos\theta + 1)}\right] = \lim_{\theta\to\pi}\left[\frac{11\cos\theta + 1}{\cos\theta}\right]$$

$\lim_{\theta\to\pi} [11 + \sec\theta] = 9$ ($\because$ as $\theta \to \pi$, $\sec\theta \to -1^-$).

ILLUSTRATION 104

If $\lim_{x\to 0}\left(\left[\dfrac{\sin^{-1}x}{x}\right] + \left[\dfrac{2^2\sin^{-1}2x}{x}\right] + \left[\dfrac{3^2\sin^{-1}3x}{x}\right] + \ldots + \left[\dfrac{n^2\sin^{-1}nx}{x}\right]\right) = 100$, then, find n. Where [.] denoted GIF.

SOLUTION

In vicinity of $x = 0$, $|\sin^{-1}x| > |x|$

$\Rightarrow \dfrac{\sin^{-1}x}{x} > 1$, in vicinity of $x = 0$.

Also $\dfrac{\sin^{-1}2x}{2x} > 1, \dfrac{\sin^{-1}3x}{3x} > 1, \ldots, \dfrac{\sin^{-1}nx}{nx} > 1$

$\therefore$ Required limit $= 1^3 + 2^3 + 3^3 + \ldots + n^3$

$$= \left(\frac{n(n+1)}{2}\right)^2 = 100 \Rightarrow n\,(n+1) = 20$$

We get $n = 4$

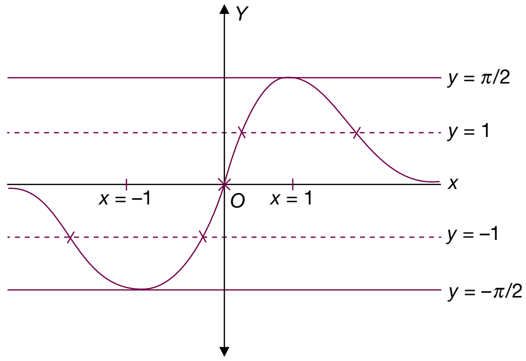
ILLUSTRATION 105

If $\lim_{x\to a}\left[\sin^{-1}\dfrac{2x}{1+x^2}\right]$ does not exist, then, find the number of possible values of a.

SOLUTION

This limit will not exist, when $\sin^{-1}\dfrac{2x}{1+x^2} = -1, 0, 1$

Also, $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ $= -1$(two values of x)
$= 0$(one values of x)
$= 1$(two values of x)



Graph of $f(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

There are 5 points where limit does not exist.

ILLUSTRATION 106

If $k=\lim\limits_{x\to 0^+}\left(\text{Min. }(t^2-6(t+x)+12)+\left[\left[\dfrac{\ln(1+2x)}{x}\right]\right]\right)$ where $t\in R$ and $[y]$ denotes greatest integer less than or equal to y, then, find k.

SOLUTION

$$k=\lim_{x\to 0^+}\left(\text{Min. }(t^2-6(t+x)+12)+\left[\left[\frac{\ln(1+2x)}{2x}\cdot 2\right]\right]\right)$$

$$\left\{\text{as } x\to 0^+,\ \frac{\ln(1+2x)}{2x}<1\right\}$$

$$=\text{Min. }(t^2-6t+12)+1$$

$$\therefore\quad k=\text{Min. }((t-3)^2+3)+1=4$$

ILLUSTRATION 107

Let $f(x)=\dfrac{\tan x}{x}$, then, the find value of $\lim\limits_{x\to 0}([f(x)]+x^2)^{\frac{1}{\{f(x)\}}}$ where [.] is GIF and {.} is FPF.

SOLUTION

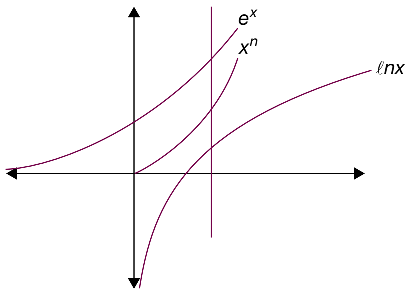
$$\lim_{x\to 0}\left(\left[\frac{\tan x}{x}\right]+x^2\right)^{\frac{1}{\left\{\frac{\tan x}{x}\right\}}}=\lim_{x\to 0}\left(1+x^2\right)^{\frac{1}{\frac{\tan x}{x}-1}}$$

$$=e^{\lim\limits_{x\to 0}\frac{x^2}{\frac{\tan x}{x}-1}}=e^{\lim\limits_{x\to 0}\frac{x^3}{\tan x-x}}$$

$$=e^{\lim\limits_{x\to 0}\frac{x^3}{x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots-x}}$$

$$=\lim_{x\to 0}e^{\lim\limits_{x\to 0}\frac{1}{\frac{1}{3}+\frac{2}{15}x^2+\ldots.}}=e^3$$

COMPARISON OF POLYNOMIALS, EXPONENTIAL, LOGARITHMIC FUNCTION AT INFINITY



As $x\to\infty$, $e^x >>>>> x^n >>>> \ln x$

e^x reaches infinity very fast compare to polynomials and log functions and log functions reaches infinity quite slowly compare to exponential and polynomial functions. We can use this information for calculating limits.

ILLUSTRATION 108

Evaluate $\lim\limits_{x\to\infty}\dfrac{\ln x}{e^x}$.

SOLUTION

As $x\to\infty$ then, $e^x >>>>>>> \ln x$

$$\lim_{x\to\infty}\frac{\ln x}{e^x}=0$$

Alternate:

$$\lim_{x\to\infty}\frac{\ln x}{e^x}\qquad\left(\frac{\infty}{\infty}\text{ form, using L-Hospital rule}\right)$$

$$=\lim_{x\to\infty}\frac{1/x}{e^x}=\lim_{x\to\infty}\frac{1}{xe^x}=0$$

ILLUSTRATION 109

Evaluate $\lim\limits_{x\to\infty}\dfrac{x^2-x+1}{\ln^2 x}$.

SOLUTION

$$\lim_{x\to\infty}\frac{x^2-x+1}{\ln^2 x}\to\infty\qquad(\text{as } x\to\infty \text{ then, } x^n >>>> \ln x)$$

We can conclude its limit directly.

Alternate:

$$\lim_{x\to\infty}\frac{x^2-x+1}{\ln^2 x}\qquad\left(\frac{\infty}{\infty}\text{ form using L-Hospital Rule}\right)$$

$$=\lim_{x\to\infty}\frac{2x-1}{2(\ln x)/x}=\lim_{x\to\infty}\frac{2x^2-x}{2\ln x}\qquad\left(\frac{\infty}{\infty}\text{ form}\right)$$

Again, using L-Hospital Rule

$$=\lim_{x\to\infty}\frac{4x-1}{2/x}=\lim_{x\to\infty}\frac{4x^2-x}{2}\to\infty$$

ILLUSTRATION 110

Evaluate $\lim\limits_{x\to\infty}\dfrac{\ln x-[x]}{[x]}$ where [.] is GIF.

SOLUTION

$$\lim_{x\to\infty}\frac{\ln x-[x]}{[x]}=\lim_{x\to\infty}\left(\frac{\ln x}{[x]}-1\right)=\lim_{x\to\infty}\left(\frac{\ln x}{x}-1\right)$$

$$(\text{As } x\to\infty \text{ then, } x >>>> \ln x)$$

$$=0-1=-1$$

SANDWICH THEOREM/SQUEEZE THEOREM

Sandwich theorem is used for evaluating the limit of a function.by assuming two suitable functions.

Let f, g and h be real functions such that $g(x)\le f(x)\le h(x)$ in $x\in(\alpha-\delta,\ \alpha+\delta)$ for all x in neighbourhood of $x=\alpha$.

For some real number α, if limits of both the extreme functions $g(x)$ and $h(x)$ converge to same finite limit l, then, function lying between these two function also converge to same limit.

$$g(x) \le f(x) \le h(x) \text{ for } x \in (\alpha - \delta, \alpha + \delta)$$

$$\lim_{x\to\alpha} g(x) = l = \lim_{x\to\alpha} h(x), \text{ then, } \lim_{x\to\alpha} f(x) = l.$$

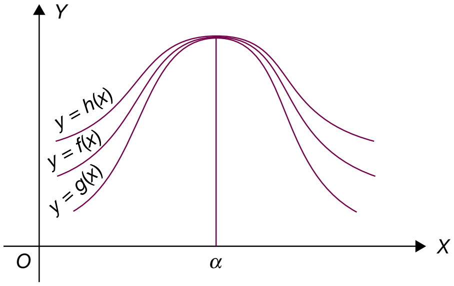


ILLUSTRATION 111

Evaluate $\lim\limits_{n\to\infty} \dfrac{[x]+[2x]+[3x]+....+[nx]}{n^2}$ where [.] is GIF.

SOLUTION

We know that $x - 1 \le [x] \le x$ for all x.
We are trying to get two functions where one function is lesser than given function and other function is greater than given function:

$$x - 1 \le [x] \le x$$

$$2x - 1 \le [2x] \le 2x$$

$$3x - 1 \le [3x] \le 3x$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$nx - 1 \le [nx] \le nx$$

Adding all of above terms, we get

$x(1 + 2 + 3 + \ldots\ldots + n) - n \le [x] + [2x] + [3x] + \ldots + [nx] \le x + 2x + 3x + \ldots + nx$

$$\Rightarrow \lim_{n\to\infty} \frac{x+2x+3x+\ldots+nx-n}{n^2} \le$$

$$\lim_{n\to\infty} \frac{[x]+[2x]+[3x]+\ldots+[nx]}{n^2} \le \lim_{n\to\infty} \frac{x+2x+3x+\ldots+nx}{n^2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{x(1+2+3+\ldots+n)-n}{n^2} \le$$

$$\lim_{n\to\infty} f(x) \le \lim_{n\to\infty} \frac{x(1+2+3+\ldots+n)}{n^2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{xn(n+1)}{2n^2} - \frac{1}{n} \le \lim_{n\to\infty} f(x) \le \lim_{n\to\infty} \frac{xn(n+1)}{2n^2}$$

$$\Rightarrow \frac{x}{2}(1+0) - 0 \le \lim_{n\to\infty} f(x) \le \frac{x}{2}(1+0)$$

$$\Rightarrow \frac{x}{2} \le \lim_{n\to\infty} f(x) \le \frac{x}{2}$$

∴ From Sandwich Theorem,

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} \frac{[x]+[2x]+[3x]+....+[nx]}{n^2} = \frac{x}{2}$$

Alternate:

$$\lim_{n\to\infty} \frac{[x]+[2x]+[3x]+....+[nx]}{n^2}$$

Where $x = [x] + \{x\}$ or $[x] = x - \{x\}$

$$= \lim_{n\to\infty} \frac{x+2x+3x+\ldots+nx-(\{x\}+\{2x\}+\{3x\}+\ldots+\{nx\})}{n^2}$$

$$= \lim_{n\to\infty} \frac{x(1+2+3+\ldots+n)}{n^2} - \lim_{x\to\infty} \frac{\{x\}+\{2x\}+\{3x\}+\ldots+\{nx\}}{n^2}$$

We know that,

$$0 \le \{x\} < 1, 0 \le \{2x\} < 1, 0 \le \{3x\} < 1, \ldots, 0 \le \{nx\} < 1$$

$$= \lim_{n\to\infty} \frac{xn(n+1)}{2n^2} - \lim_{x\to\infty} \frac{\text{Between 0 to } n}{n^2}$$

$$= \lim_{n\to\infty} \frac{x(1+1/n)}{2} - 0 = \frac{x}{2}$$

ILLUSTRATION 112

If $[x]$ denotes the greatest integer less than or equal to x, then, evaluate

$$\lim_{n\to\infty} \frac{1}{n^3}\left(\left[1^1 x\right]+\left[2^2 x\right]+\left[3^2 x\right]+\ldots+\left[n^2 x\right]\right).$$

SOLUTION

We have, $\lim\limits_{n\to\infty} \dfrac{\sum\limits_{r=1}^{n}[r^2x]}{n^3} = \lim\limits_{n\to\infty} \dfrac{\sum\limits_{r=1}^{n}\left(r^2x - \{r^2x\}\right)}{n^3}$,

Where $\{\cdot\}$ denotes the fractional part function,

$$= \lim_{n\to\infty} \left(\frac{x \cdot \dfrac{n(n+1)(2n+1)}{6}}{n^3} - \sum_{r=1}^{n} \frac{\{r^2x\}}{n^3} \right)$$

$$= \frac{x}{6} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\{r^2x\}}{n^3}$$

$$= \frac{x}{3} - 0$$

$$(\because 0 \le \{r^2x\} < 1, 0 \le \sum_{r=1}^{n} \{r^2x\} < n \Rightarrow 0 \le \sum_{r=1}^{n} \frac{\{r^2x\}}{n^3} < \frac{1}{n^2})$$

$$= \frac{x}{3}$$

ILLUSTRATION 113

Let $f(x) = x\cos x$, then, compute

$$\lim_{x\to 0}\left(\left[\frac{f(x)}{\sin x}\right]+\left[\frac{f(2x)}{\sin 2x}\right]+\left[\frac{f(3x)}{\sin 3x}\right]+\ldots+\left[\frac{f(2015x)}{\sin 2015x}\right]\right)$$

(where [.] represents greatest integer function).

SOLUTION

$$\lim_{x\to 0}\left(\left[\frac{f(x)}{\sin x}\right]+\left[\frac{f(2x)}{\sin 2x}\right]+\left[\frac{f(3x)}{\sin 3x}\right]+\ldots+\left[\frac{f(2015x)}{\sin 2015x}\right]\right)$$

$$= \lim_{x\to 0}\left(\left[\frac{x\cos x}{\sin x}\right]+\left[\frac{2x\cos 2x}{\sin 2x}\right]+\left[\frac{3x\cos 3x}{\sin 3x}\right]+ \ldots+\left[\frac{2015x\cos 2015x}{\sin 2015x}\right]\right)$$

$$= \lim_{x\to 0}\left(\left[\frac{x}{\tan x}\right]+\left[\frac{2x}{\tan 2x}\right]+\left[\frac{3x}{\tan 3x}\right]+\ldots+\left[\frac{2015x}{\tan 2015x}\right]\right),$$

$$\left(\because \frac{\tan x}{x} > 1 \Rightarrow \frac{x}{\tan x} < 1\right)$$

$$= 0 + 0 + \ldots 0 = 0$$

ILLUSTRATION 114

Find the value of $\lim\limits_{x\to 0}\sum\limits_{r=1}^{10}[r^2 x\cot x]$.

SOLUTION

Given, limit $\lim\limits_{x\to 0}\sum\limits_{r=1}^{10}[r^2 x\cot x]$

$$= \lim_{x\to 0}\left(\left[1^2\cdot\frac{x}{\tan x}\right]+\left[2^2\cdot\frac{x}{\tan x}\right]+\ldots.+\left[10^2\cdot\frac{x}{\tan x}\right]\right)$$

$$\because \quad \frac{x}{\tan x} < 1$$

As x moving to zero,

$$\left[1^2.\frac{x}{\tan x}\right] = 1^2 - 1, \left[2^2.\frac{x}{\tan x}\right] = 2^2 - 1, \left[3^2.\frac{x}{\tan x}\right]$$

$$= 3^2 - 1 \ldots, \left[10^2.\frac{x}{\tan x}\right] = 10^2 - 1$$

$$\text{Required limit} = (1^2 + 2^2 + \ldots + 10^2) - \left(\underbrace{1+1+\ldots+1}_{10\text{ times}}\right)$$

$$= \left(\frac{10\times 11\times 21}{6}\right) - 10 = (55\times 7) - 10$$

$$= 385 - 10 = 375.$$

ILLUSTRATION 115

Let $M(n)$ be the largest integer m such that $\binom{m}{n-1} > \binom{m-1}{n}$ (where $\binom{n}{r} = {}^nC_r$ and $n \in N$), then, find the value of $\lim\limits_{n\to\infty}\frac{M(n)}{n}$.

SOLUTION

Given ${}^mC_{n-1} > {}^{m-1}C_n$

$$\Rightarrow \quad \frac{m!}{(n-1)!(m-n+1)!} > \frac{(m-1)!}{n!(m-1-n)!}$$

$$\Rightarrow \quad \frac{m(m-1)!}{(n-1)!(m-n+1)(m-n)(m-n-1)!} > \frac{(m-1)!}{n(n-1)!(m-1-n)!}$$

$$\Rightarrow \quad \frac{m}{(m-n+1)(m-n)} > \frac{1}{n}$$

$$\therefore \quad mn > (m - n + 1)(m - n)$$

$$\Rightarrow \quad m^2 - 3mn + m + n^2 - n < 0$$

$$\Rightarrow \quad m^2 - (3n - 1)m + n^2 - n < 0$$

$$\Rightarrow \quad m \in \left(\frac{3n-1-\sqrt{5n^2-2n+1}}{2}, \frac{3n-1+\sqrt{5n^2-2n+1}}{2}\right)$$

Clearly, largest integer must be lying in this interval let it be $M(n)$.

$$\frac{3n-1+\sqrt{5n^2-2n+1}}{2} - 1 < M(n) < \frac{3n-1+\sqrt{5n^2-2n+1}}{2}$$

$\therefore$ According to sandwich theorem,

$$\text{Hence, } \lim_{n\to\infty}\frac{M(n)}{n} = \frac{3+\sqrt{5}}{2}$$

ILLUSTRATION 116

Evaluate $\lim\limits_{n\to\infty}\frac{1}{1+n^2}+\frac{2}{2+n^2}+\ldots+\frac{n}{n+n^2}$.

SOLUTION

For using sandwich theorem, we need two suitable function one is greater than above given and one lesser than given function.

$$f_n = \frac{1}{1+n^2}+\frac{2}{2+n^2}+\ldots+\frac{n}{n+n^2}$$

$$\text{Now, } f_n \le \frac{1}{1+n^2}+\frac{2}{1+n^2}+\ldots+\frac{n}{1+n^2}$$

$$\Rightarrow \quad f_n \le \frac{1}{1+n^2}(1+2+3+\ldots+n) = \frac{n(n+1)}{2(1+n^2)}$$

Also, $f_n \ge \frac{1}{n+n^2} + \frac{2}{n+n^2} + \frac{3}{n+n^2} + \ldots + \frac{n}{n+n^2} = \frac{n(n+1)}{2(n+n^2)}$

Thus, $\frac{n(n+1)}{2(n+n^2)} \le f_n \le \frac{n(n+1)}{2(1+n^2)}$

or, $\lim\limits_{n\to\infty} \frac{n(n+1)}{2(n+n^2)} \le \lim\limits_{n\to\infty} f_n \le \lim\limits_{n\to\infty} \frac{n(n+1)}{2(1+n^2)}$

or, $\lim\limits_{n\to\infty} \frac{1\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n}+1\right)} \le \lim\limits_{n\to\infty} f_n \le \lim\limits_{n\to\infty} \frac{1\left(1+\frac{1}{n}\right)}{2\left(\frac{1}{n^2}+1\right)}$

or, $\frac{1}{2} \le \lim\limits_{n\to\infty} f_n \le \frac{1}{2}$

From sandwich theorem, we can conclude that $\lim\limits_{n\to\infty} f_n = \frac{1}{2}$.

ILLUSTRATION 117

Find the value of $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \left(\frac{r}{n^2+r^2}\right)$.

SOLUTION

$$\lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{r}{n^2+r^2}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^2+1^2} + \frac{2}{n^2+2^2} + \frac{3}{n^2+3^2} + \ldots + \frac{n}{n^2+n^2}\right)$$

Let $f(n) = \frac{1}{n^2+1^2} + \frac{2}{n^2+2^2} + \ldots + \frac{n}{n^2+n^2}$

Now, lets consider two function $h(n) \le f(n) \le g(n)$

$$h(n) = \frac{1}{n^2+n^2} + \frac{2}{n^2+n^2} + \frac{3}{n^2+n^2} + \ldots \frac{n}{n^2+n^2}$$

$$= \frac{1+2+3+\ldots+n}{n^2+n^2}$$

$$\lim_{n\to\infty} h(n) = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \lim_{n\to\infty} \frac{n^2\left(1+\frac{1}{n}\right)}{2n^2} = \frac{1}{2}$$

$$g(n) = \frac{1}{n^2+1} + \frac{2}{n^2+1} + \ldots + \frac{n}{n^2+1} = \frac{n(n+1)}{2(n^2+1)}$$

$$\lim_{n\to\infty} h(n) = \lim_{n\to\infty} \frac{n^2\left(1+\left(\frac{1}{n}\right)\right)}{2n^2\left(1+\left(\frac{1}{n^2}\right)\right)} = \frac{1}{2}$$

Now, using sandwich theorem,

$$\lim_{n\to\infty} h(n) \le \lim_{n\to\infty} f(n) \le \lim_{n\to\infty} g(n)$$

Hence, $\lim\limits_{n\to\infty} f(n) = \frac{1}{2}$

ILLUSTRATION 118

Find the value of $\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \frac{r^2}{n^3+n^2+r}$.

SOLUTION

Let $f(n) = \sum\limits_{r=1}^{n} \frac{r^2}{n^3+n^2+r}$

$$= \frac{1^2}{n^3+n^2+1} + \frac{2^2}{n^3+n^2+2} + \ldots + \frac{n^2}{n^3+n^2+n}$$

Now, for using sandwich theorem, we have to choose two suitable functions.

$$\because \quad \frac{1^2+2^2+\ldots+n^2}{n^3+n^2+n} \le \sum_{r=1}^{n} \frac{r^2}{n^3+n^2+r} \le \frac{1^2+2^2+\ldots+n^2}{n^3+n^2+1}$$

$$\Rightarrow \quad \frac{n(n+1)(2n+1)}{6(n^3+n^2+n)} \le \sum_{r=1}^{n} \frac{r^2}{n^3+n^2+r} \le \frac{n(n+1)(2n+1)}{6(n^3+n^2+1)}$$

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6\left(1+\frac{1}{n}+\frac{1}{n^2}\right)} \le \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{n^3+n^2+r} \le$$

$$\lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6\left(1+\frac{1}{n}+\frac{n}{n^2}\right)}$$

$$\frac{1}{3} \le \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{n^3+n^2+r} \le \frac{1}{3}$$

$\therefore$ According to sandwich theorem, so given limit $= \frac{1}{3}$.

ILLUSTRATION 119

Let $f: R \to (0, \infty)$ be such that $f(x) + \frac{e^{x+x^2}}{f(x)} \le e^x + e^{x^2}\ \forall\ x > 0$, then, find $\lim\limits_{x\to 1} f(x)$.

SOLUTION

$$f(x) + \frac{e^{x+x^2}}{f(x)} \le e^x + e^{x^2} \text{ as given } f(x) > 0$$

$$\Rightarrow \quad (f(x))^2 - (e^x + e^{x^2})\,f(x) + e^x \cdot e^{x^2} \le 0$$

$$\Rightarrow \quad (f(x) - e^x)(f(x) - e^{x^2}) \le 0$$

$$\Rightarrow \quad e^{x^2} \le f(x) \le e^x;\ \forall\ x \in (0, 1)$$

$$e^x \le f(x) \le e^{x^2};\ \forall\ x \in (1, \infty)$$

$$\therefore \quad \lim_{x\to 1}(e^x) = \lim_{x\to 1}(e^{x^2}) = e$$

Hence, $\lim\limits_{x\to 1} f(x) = e$, by sandwich theorem.

LIMITS IN TWO VARIABLES

Considering *h* and *n* are two independent variables.

$\lim\limits_{h\to 0}\lim\limits_{n\to\infty}(1-h)^n=0$ (Base is less than 1 and having infinite power, becomes zero)

$\lim\limits_{h\to 0}\lim\limits_{n\to\infty}(1+h)^n\to\infty$ (Base is greater than 1 and having infinite power leads to infinity

If *h* and *n* are dependent on each other, let they are reciprocal of each other $n = 1/h$ then,

$$\lim_{h\to 0}\lim_{n\to\infty}(1+h)^n=\lim_{h\to 0}(1+h)^{1/h}=e.$$

ILLUSTRATION 120

Evaluate $f(x)=\lim\limits_{n\to\infty}\dfrac{\cos\pi x-x^2\sin(x-1)}{1+x^{2n+1}-x^{2n}}$ find $\lim\limits_{x\to 1}f(x)$.

SOLUTION

Put, $x = 1 - h$

$$\text{LHL}=\lim_{x\to 1^-}f(x)=\lim_{h\to 0}\lim_{n\to\infty}\frac{\cos\pi(1-h)-(1-h)^{2n}\sin(-h)}{1+(1-h)^{2n+1}-(1-h)^{2n}}$$

$$\left(\because \lim_{h\to 0}\lim_{n\to\infty}(1-h)^{2n}\to 0\right)$$

$$=\lim_{h\to 0}\frac{\cos(\pi(1-h))-0}{1+0-0}=\lim_{h\to 0}\frac{\cos(\pi(1-h))}{1}=\cos\pi=-1$$

$$\text{RHL}=\lim_{x\to 1^+}f(x)=\lim_{h\to 0}\lim_{n\to\infty}\frac{\cos(\pi+\pi h)-(1+h)^{2n}\sin h}{1+(1+h)^{2n+1}-(1+h)^{2n}}$$

$$\left(\frac{\infty}{\infty}\right)\text{form,}$$

Put, $x = 1 + h$

$$=\lim_{h\to 0}\lim_{n\to\infty}\frac{(1+h)^{2n}\left[\dfrac{-\cos\pi h}{(1+h)^{2n}}-\sin h\right]}{(1+h)^{2n}\left[\dfrac{1}{(1+h)^{2n}}+h\right]}=\lim_{h\to 0}\frac{0-\sinh}{0+h}=-1$$

LHL and RHL both are equal and unique.

Hence $\lim\limits_{x\to 1}f(x)=-1$

ILLUSTRATION 121

Evaluate $f(x)=\lim\limits_{n\to\infty}\dfrac{\tan\pi x^2+(x+1)^n\sin x}{x^2+(x+1)^n}$ find $\lim\limits_{x\to 0}f(x)$

SOLUTION

$$\text{LHL}=\lim_{x\to 0^-}f(x)=\lim_{h\to 0}\lim_{n\to\infty}\frac{\tan\pi h^2+(1-h)^n\sin(0-h)}{(0-h)^2+(1-h)^n},$$

Put $x = 0 - h$

$$=\lim_{h\to 0}\left(\frac{\tan\pi h^2}{\pi h^2}\right)\pi=\pi(1)=\pi$$

$$\text{RHL}=\lim_{x\to 0^+}f(x)=\lim_{h\to 0}\lim_{n\to\infty}\frac{\tan\pi h^2+(1+h)^n\sin h}{h^2+(1+h)^n}$$

$$\left(\frac{\infty}{\infty}\right)\text{form}$$

Put $x = 0 + h$

$$=\lim_{h\to 0}\lim_{n\to\infty}\left(\frac{\dfrac{\tan\pi h^2}{(1+h)^n}+\sin h}{\dfrac{h^2}{(1+h)^n}+1}\right)=\lim_{h\to 0}\frac{0+\sin h}{0+1}=\frac{0+0}{0+1}=0$$

LHL ≠ RHL

$\therefore\quad \lim\limits_{x\to 0}f(x)$ does not exist.

ILLUSTRATION 122

Form a set using the values $\lim\limits_{n\to\infty}\dfrac{x^{2n}-1}{x^{2n}+1}$ can take.

SOLUTION

Although it looks to be dependent on *n* but it also depends on values of *x*. Thats why we will have to consider all possible value of *x*.

(i) If $-1 < x < 1$ then, $x^2 < 1$; $x^{2n}\to 0$; when $n\to\infty$

$$\lim_{n\to\infty}\frac{(x^2)^n-1}{(x^2)^n+1}=\frac{0-1}{0+1}=-1$$

(ii) If $x = -1, 1$ then, $x^2 = 1$

$$\lim_{n\to\infty}\frac{(x^2)^n-1}{(x^2)^n+1}=\lim_{n\to\infty}\frac{(1)^n-1}{(1)^n+1}=\frac{1-1}{1+1}=0$$

(iii) if $x < -1, x > 1$ then, $x^2 > 1$: $x^{2n}\to\infty$; $\dfrac{1}{x^{2n}}\to 0$

$$\lim_{n\to\infty}\frac{(x^2)^n-1}{(x^2)^n+1}=\lim_{n\to\infty}\frac{1-\dfrac{1}{(x^2)^n}}{1+\dfrac{1}{(x^2)^n}}=\frac{1-0}{1+0}=1$$

Here, we will get three different answers depending on the values of *x*.

Hence, required set = {–1, 0, 1}

ILLUSTRATION 123

If *x* is a real number then, evaluate

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\cos^{2m}((n!\pi x))\right)$$

SOLUTION

(i) Let *x* is a rational number $x\in Q$, let $x=\dfrac{p}{q}$

n! contains product of all natural numbers starting from 1 to very large number so *q* will also be present in this product, which concludes $n!\,\pi x$ is integral multiple of π as $n\to\infty$

Hence, $\cos\ (n!\ \pi x) = \pm 1$

$$\Rightarrow \lim_{m\to\infty}\left(\lim_{n\to\infty}\cos^{2m}(n!\pi x)\right) = \lim_{m\to\infty}(\pm 1)^{2m} = 1$$

(ii) Let x is irrational number $x \notin Q$

$n!\ \pi x$ is not integral multiple of π so $-1 < \cos n!\pi x < 1$

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\ (\cos n!\pi x)^{2m}\right) = \lim_{m\to\infty}\left((-1 \text{ to } 1)^{2m}\right)$$

$$= \lim_{m\to\infty}\left((0 \text{ to } 1)^{m}\right) = 0$$

ILLUSTRATION 124

Let $f(n) = \lim\limits_{x\to 0}\left(1+\sum\limits_{k=1}^{n}\sin^2 kx\right)^{\frac{1}{n^3x^2}}$. If $\lim\limits_{n\to\infty} f(n)$ exist and is equal to $e^{\frac{p}{q}}$, where p and q are relatively prime, find $(p+q)$.

SOLUTION

$$\lim_{n\to\infty} f(n) = \lim_{n\to\infty}\left(\lim_{x\to 0}\left(1+\sum_{k=1}^{n}\sin^2 kx\right)^{\frac{1}{n^3x^2}}\right) = e^L \text{ (say)},$$

(1^∞ form) …(1)

$$\text{Now, } L = \lim_{n\to\infty}\lim_{x\to 0}\frac{1}{n^3x^2}\left(\sum_{k=1}^{n}\sin^2 kx\right)$$

$$L = \lim_{n\to\infty}\frac{1}{n^3}\lim_{x\to 0}\sum_{k=1}^{n}k^2\left(\frac{\sin kx}{kx}\right)^2; \quad \because \left(\lim_{x\to 0}\frac{\sin kx}{kx} = 1\right)$$

$$L = \lim_{n\to\infty}\frac{1}{n^3}\sum_{k=1}^{n}k^2 = \lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$$

$$L = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6} = \frac{1}{3}$$

So, required limit $= e^L = e^{\frac{1}{3}} = e^{\frac{p}{q}}$ (Given)

$\therefore\quad p = 1$ and $q = 3$

Hence, $p + q = 4$

ILLUSTRATION 125

$\lim\limits_{n\to\infty}\dfrac{n.3^n}{n(x-2)^n + n.3^{n+1} - 3^n} = \dfrac{1}{3}$, where $n \in N$, then, find the number of integers in the range of 'x'.

SOLUTION

$$\lim_{n\to\infty}\frac{n.3^n}{n(x-2)^n + n.3^{n+1} - 3^n} = \frac{1}{3}$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{\left(\frac{x-2}{3}\right)^n + 3 - \frac{1}{n}} = \frac{1}{3}$$

Which is possible iff $\left|\dfrac{x-2}{3}\right| < 1 \Rightarrow -3 < x-2 < 3$

Hence, $-1 < x < 5$

ILLUSTRATION 126

Let $f(x) = \lim\limits_{n\to\infty}\dfrac{\sin(\pi x^4) + (x+2)^n\cdot\dfrac{\tan(\pi x)}{x+1}}{1+(x+2)^n - x^4}$, then, find $\lim\limits_{x\to -1} f(x)$.

SOLUTION

$$\text{RHL} = \lim_{x\to -1^+} f(x) = \lim_{n\to\infty}\lim_{x\to -1^+}\frac{\sin(\pi x^4) + (x+2)^n\cdot\frac{\tan(\pi x)}{x+1}}{1+(x+2)^n - x^4}$$

Put $x = -1 + h$

$$= \lim_{n\to\infty}\lim_{h\to 0}\frac{\sin(\pi(-1+h)^4 + (1+h)^n\cdot\frac{\tan(\pi(-1+h)}{h}}{1+(1+h)^n - (-1+h)^4}$$

$$= \lim_{n\to\infty}\lim_{h\to 0}\frac{\frac{\sin(\pi(-1+h)^4}{(1+h)^n} + \frac{\tan\pi h}{\pi h}.\pi}{\frac{1}{(1+h)^n} + 1 - \frac{(-1+h)^4}{(1+h)^n}}$$

$$= \lim_{h\to 0}\frac{0+\frac{\tan\pi h}{\pi h}.\pi}{0+1-0} = \pi$$

$$\text{LHL} = \lim_{x\to -1^-} f(x) = \lim_{n\to\infty}\lim_{x\to -1^-}\frac{\sin(\pi x^4) + (x+2)^n\cdot\frac{\tan(\pi x)}{x+1}}{1+(x+2)^n - x^4}$$

Put $x = -1 - h$

$$= \lim_{n\to\infty}\lim_{h\to 0}\frac{\sin(\pi(1+h)^4) + (1-h)^n\cdot\frac{\tan(\pi(-1-h))}{-h}}{1+(1-h)^n - (-1-h)^4}$$

$$= \lim_{h\to 0}\frac{\sin(\pi(1+h)^4)+0}{1+0-(-1-h)^4} = \lim_{h\to 0}\frac{\sin(\pi(1-(1+h)^4)}{\pi(1-(1+h)^4)}.\pi = \pi$$

Hence, $\lim\limits_{x\to -1} f(x) = \pi$

ILLUSTRATION 127

Compute $\lim\limits_{n\to\infty}\sum\limits_{x=1}^{10}\dfrac{1-\cos^{2n}(x-11)}{(x-11)^2}$.

SOLUTION

$$\lim_{n\to\infty}\sum_{x=1}^{10}\frac{1-(\cos^2(x-11))^n}{(x-11)^2}$$

$\because\quad x - 11$ will not be an integral multiple of π

$\therefore\quad 0 < \cos^2(x - 11) < 1$

$\therefore\quad \lim\limits_{n\to\infty}(\cos^2(x - 11))^n = 0$

$\therefore\quad$ Given, limit $= \sum\limits_{x=1}^{10}\dfrac{1}{(x-11)^2} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots + \dfrac{1}{10^2}$

Test Your Concepts-VI

(Solutions on page H.54)

1. Let $g: (0, \infty) \to R$ be defined as $g(x) = 2x^{\sin 2x} \cdot \cos 2x$, then, find $\lim\limits_{x\to 0^+} g(x)$.

2. Evaluate $\lim\limits_{x\to \frac{\pi^-}{2}} (\sec x)^{\cos x}$.

3. Evaluate $\lim\limits_{x\to 0^+} \log_{\sin x} \sin 2x$.

4. Evaluate $\lim\limits_{x\to 0}\left[\dfrac{\sin x}{\tan x}\right]+\left[\dfrac{-\sin x}{x}\right]+\left[\dfrac{-\tan x}{x}\right]$ where [.] is GIF.

5. Evaluate $\lim\limits_{x\to 0}\left[\dfrac{5\sin x}{x}\right]+\left[\dfrac{-5\tan x}{x}\right]$ where [.] is GIF.

6. $\lim\limits_{x\to 0}\left[\dfrac{\sin^{-1}x}{x}\right]+\left[\dfrac{-\tan^{-1}x}{\sin^{-1}x}\right]+\left[\dfrac{-3\tan^{-1}x}{x}\right]$ where [.] is GIF.

7. Let $f: (1, \infty) \to (0, \infty)$ be a continuous decreasing function with $\lim\limits_{x\to\infty}\dfrac{f(4x)}{f(8x)} = 1$, then, compute $\lim\limits_{x\to\infty}\dfrac{f(6x)}{f(8x)}$.

8. $\lim\limits_{n\to\infty}\dfrac{\{x\}+\{2x\}+\ldots\{nx\}}{n^2}$ where {.} is fractional part function.

9. Evaluate $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2}}+\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\ldots+\dfrac{1}{\sqrt{n^2+2n}}\right)$.

Answer key

1. 2 **2.** 1 **3.** 1 **4.** –3 **5.** –2 **6.** –3 **7.** 1 **8.** 0 **9.** 2

SOLVED PROBLEMS

PROBLEM 1

If α and β are the roots of $ax^2 + bx + c = 0$, then, find,

$$\lim_{x\to\frac{1}{\alpha}} \sqrt{\frac{1-\cos(cx^2+bx+a)}{2(1-\alpha x)^2}}.$$

SOLUTION

$$\lim_{x\to\frac{1}{\alpha}} \sqrt{\frac{1-\cos(cx^2+bx+a)}{2(1-\alpha x)^2}}; \qquad \left(\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right)$$

$$= \lim_{x\to\frac{1}{\alpha}} \left|\frac{\sin\left(\dfrac{cx^2+bx+a}{2}\right)}{(1-\alpha x)}\right| \qquad (\because \sqrt{x^2} = |\ x\ |)$$

$\because$ α and β are roots of $ax^2+bx+c=0$

$\therefore$ $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$ are roots of $cx^2+bx+a=0$ then, $cx^2+bx+a=0=c\left(x-\dfrac{1}{\alpha}\right)\left(x-\dfrac{1}{\beta}\right)$.

$$\text{Required limit} = \lim_{x\to\frac{1}{\alpha}} \left|\frac{\sin\left\{\dfrac{c}{2}\left(x-\dfrac{1}{\alpha}\right)\left(x-\dfrac{1}{\beta}\right)\right\}}{-\alpha\left(x-\dfrac{1}{\alpha}\right)}\right|$$

$$= \lim_{x\to\frac{1}{\alpha}} \left|\frac{C\sin\left\{\dfrac{C}{2}\left(x-\dfrac{1}{\alpha}\right)\left(x-\dfrac{1}{\beta}\right)\right\}}{-\alpha\left(x-\dfrac{1}{\alpha}\right)\dfrac{C}{2}.2}\right|$$

$$= \lim_{x\to\frac{1}{\alpha}} \left|-\frac{c}{2\alpha}\left(x-\frac{1}{\beta}\right)\right| = \left|\frac{1}{2}\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\cdot\frac{c}{\alpha}\right|$$

PROBLEM 2

Find $\displaystyle\lim_{n\to\infty} \frac{\sqrt{1-x_0{}^2}}{x_1x_2x_3x_4\ldots x_n}$, where $x_{r+1} = \sqrt{\dfrac{1+x_r}{2}}$, $0 \le r \le (n-1)$, $r \in W, n \in N$.

SOLUTION

Let $x_0 = \cos\theta$ then, $x_1 = \sqrt{\dfrac{1+x_0}{2}} = \cos\dfrac{\theta}{2}$

$$x_2 = \sqrt{\frac{1+x_1}{2}} = \cos\frac{\theta}{2^2}, \ldots\ x_n = \cos\frac{\theta}{2^n}$$

$$\lim_{n\to\infty} \frac{\sqrt{1-x_0{}^2}}{x_1x_2x_3x_4\ldots x_n}$$

$$\therefore \quad = \lim_{n\to\infty} \frac{\sin\theta}{\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{4}\cos\dfrac{\theta}{8}\ldots\cos\dfrac{\theta}{2^n}}$$

$$= \lim_{n\to\infty} \frac{\sin\theta}{\dfrac{\sin\theta}{2^n\sin\dfrac{\theta}{2^n}}} = \lim_{n\to\infty} \frac{\sin\theta}{\sin\theta} 2^n\sin\frac{\theta}{2^n}$$

$$= \lim_{n\to\infty} \theta\cdot\frac{\sin\dfrac{\theta}{2^n}}{\dfrac{\theta}{2^n}} = \theta = \cos^{-1} x_0$$

PROBLEM 3

Let $f(x)$ be a polynomial function satisfying $\dfrac{2x-11}{3} \le f(x) \le \dfrac{2x+13}{3}$; $\forall x \in R$ and $\displaystyle\lim_{x\to1}(f(x))^{\frac{1}{x-1}} = e^{\frac{p}{q}}$, where $p, q \in N$, then, find the least value of $p+q$.

SOLUTION

$$\frac{2x-11}{3} \le f(x) \le \frac{2x+13}{3}; \forall x \in R \quad f(x) \text{ is polynomial}$$

$$\Rightarrow \quad f(x) = \frac{2x+c}{3}$$

As $\displaystyle\lim_{x\to1}(f(x))^{\frac{1}{x-1}}$ exists then, it should of form 1^∞

$$\lim_{x\to1} f(x) = 1 \Rightarrow \lim_{x\to1}\frac{2x+c}{3} = 1; \Rightarrow \frac{2+c}{3} = 1 \Rightarrow c = 3-2$$

$$\Rightarrow \quad c = 1$$

$$\lim_{x\to1}(f(x))^{\frac{1}{x-1}}$$

$$= \lim_{x\to1}\left(\frac{2x+1}{3}\right)^{\frac{1}{x-1}} = \lim_{x\to1} e^{\left(\frac{2x+1}{3}-1\right)\frac{1}{x-1}} = e^{\frac{2}{3}} = e^{\frac{p}{q}}$$

Hence, $p+q=5$

PROBLEM 4

If S be the sum of coefficients in the expansion of $(px + qy - rz)^n$ (where $p, q, r > 0$), then, compute the value of $\displaystyle\lim_{n\to\infty}\frac{S}{(S^{1/n}+1)^n}$.

SOLUTION

$$S = (p+q-r)^n \qquad \{\text{Putting } x = y = z = 1\}$$

$$\lim_{n\to\infty}\frac{S}{(S^{1/n}+1)^n}=\lim_{n\to\infty}\frac{(p+q-r)^n}{\{(p+q-r)+1\}^n}$$

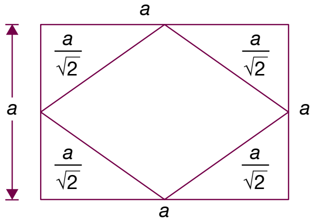
$$=\lim_{n\to\infty}\left(\frac{p+q-r}{(p+q-r+1)}\right)^n=0 \qquad \left\{as\left(\frac{p+q-r}{p+q-r+1}\right)<1\right\}$$

PROBLEM 5

Let S_1 be a square of side a. Another square S_2 is formed by joining the mid-points of the sides of S_1. The same process is applied to S_2 to form yet another square S_3, and so on. If $A_1, A_2, A_3 \ldots$ be the areas and $P_1, P_2, P_3, \ldots$ be the perimeters of $S_1, S_2, S_3, \ldots$ respectively then, compute $\lim_{n\to\infty}\frac{P_1+P_2+P_3+\ldots+P_n}{A_1+A_2+A_3+\ldots+A_n}$.

SOLUTION

$$A_1=a^2,\ P_1=4a$$



$$A_2=\frac{a^2}{2},\ P_2=\frac{4a}{\sqrt2},\ A_3=\frac{a^2}{4},\ P_3=\frac{4a}{(\sqrt2)^2},\text{ and so on}$$

$$\therefore\quad \lim_{n\to\infty}\frac{P_1+P_2+P_3+\ldots+P_n}{A_1+A_2+A_3+\ldots+A_n}$$

$$=\frac{4a+\frac{4a}{\sqrt2}+\frac{4a}{(\sqrt2)^2}+\frac{4a}{(\sqrt2)^3}+\ldots\infty}{a^2+\frac{a^2}{2}+\frac{a^2}{4}+\frac{a^2}{8}+\ldots\infty}=\frac{4a\left(\frac{1}{1-\frac{1}{\sqrt2}}\right)}{a^2\left(\frac{1}{1-\frac{1}{2}}\right)};$$

(Sum of infinite G.P. $=\frac{a}{1-r}$ where a is first term of G.P. and r is common ratio.)

$$=\frac{2}{a}\left(\frac{\sqrt2}{\sqrt2-1}\right)=\frac{2}{a}(2+\sqrt2)$$

PROBLEM 6

Let $a=\lim_{t\to0}\sin\left(\frac{\pi}{2}\cos(\tan t)\right)$;

$b=\lim_{y\to1}\sec(y\sec^2 y-\tan^2 y-1)$;

$c=\lim_{x\to0}\tan\left(\frac{\pi}{4}\cos\left(\sin x^{\frac{1}{3}}\right)\right)$ and

$d=\lim_{x\to\frac{\pi}{6}}\sqrt{\operatorname{cosec}^2(x)+5\sqrt3\tan(x)}$, then, find the value of $\frac{a+b+c}{d}$.

SOLUTION

$$a=\lim_{t\to0}\sin\left(\frac{\pi}{2}\cos(\tan t)\right)$$

$$=\sin\left(\frac{\pi}{2}(\cos(\tan 0))\right)=\sin\left(\frac{\pi}{2}\cos 0\right)=\sin\left(\frac{\pi}{2}\right)=1$$

$$b=\lim_{y\to1}\sec(y\sec^2 y-\tan^2 y-1)$$

$$=\sec(\sec^2 1-\tan^2 1-1)$$

$$=\sec(1-1)=\sec 0=1$$

$$c=\lim_{x\to0}\tan\left(\frac{\pi}{4}\cos\left(\sin x^{1/3}\right)\right)$$

$$=\tan\left(\frac{\pi}{4}(\cos(\sin 0))\right)=\tan\left(\frac{\pi}{4}\cos 0\right)=\tan\frac{\pi}{4}=1$$

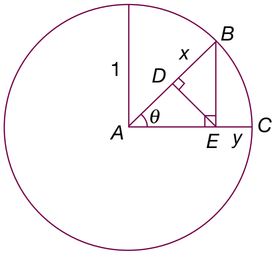
$$d=\lim_{x\to\frac{\pi}{6}}\sqrt{\operatorname{cosec}^2 x+5\sqrt3\tan x}=\sqrt{\operatorname{cosec}^2\frac{\pi}{6}+5\sqrt3\tan\frac{\pi}{6}}$$

$$=\sqrt{2^2+5\sqrt3\cdot\frac{1}{\sqrt3}}=\sqrt{4+5}=3$$

Hence, required term $=\frac{(a+b+c)}{d}=\frac{3}{3}=1$

PROBLEM 7

In the figure lengths of AB and AC are both equal to 1, the length of BD is denoted as x and the length of CE is denoted as y. BE is perpendicular to AC and ED is perpendicular to AB. Then, compute $\lim_{\theta\to0}\frac{x}{y}$



SOLUTION

From given figure,

In ΔAEB, $AE=AB\cos\theta$

$\Rightarrow\quad 1-y=\cos\theta\Rightarrow y=1-\cos\theta$

Also, in ΔADE, $AD=AE\cos\theta$

$\therefore\quad AD=\cos^2\theta$ [As, $AE=\cos\theta$]

So, $DB=x=AB-AD=(1-\cos^2\theta)$

$$\text{Now, }\lim_{\theta\to0}\frac{x}{y}=\lim_{\theta\to0}\left(\frac{1-\cos^2\theta}{1-\cos\theta}\right)=\lim_{\theta\to0}(1+\cos\theta)=1+1=2$$

PROBLEM 8

If the family of lines $(\sqrt{1+a}-1)\,px + (\sqrt[3]{1+a}-1)\,qy + (\sqrt[4]{1+a}-1)\,r = 0$, where p, q, r are in A.P., are concurrent at the point $(\alpha(a), \beta(a))$ then, find the value of $\lim\limits_{a\to 0^+}(\alpha(a)-\beta(a))$.

SOLUTION

$$\lim_{a\to 0^+}\left(\frac{\sqrt{1+a}-1}{a}\right)px + \lim_{a\to 0^+}\left(\frac{\sqrt[3]{1+a}-1}{a}\right)qy + \lim_{a\to 0^+}\left(\frac{\sqrt[4]{1+a}-1}{a}\right)r = 0$$

$$\lim_{a\to 0^+}\frac{\left(\sqrt{1+a}-1\right)\left(\sqrt{1+a}+1\right)}{a\left(\sqrt{1+a}+1\right)}px + \lim_{a\to 0^+}\frac{1+a-1}{a\left\{(1+a)^{2/3}+(1+a)^{1/3}+1\right\}}qy + \left(\frac{\frac{1}{4}(1+a)^{-3/4}}{1}\right)r = 0$$

$$\Rightarrow \quad \frac{px}{2}+\frac{qy}{3}+\frac{r}{4}=0 \qquad \ldots(1)$$

p, q, r are in A.P. $\Rightarrow p-2q+r=0$ $\qquad\ldots(2)$

On comparing (1) and (2), we get $\dfrac{\frac{x}{2}}{1}=\dfrac{\frac{y}{3}}{-2}=\dfrac{\frac{1}{4}}{1}$

$$\Rightarrow \quad x=\frac{2}{4} \text{ and } y=\frac{-6}{4}$$

Hence, $\lim\limits_{a\to 0^+}(\alpha(a)-\beta(a)) = \dfrac{1}{2}-\left(\dfrac{-3}{2}\right)=2$

PROBLEM 9

If α, β, γ are the roots equation $x^3+bx^2+cx+d=0$. Where b, c, d are non-zero distinct real numbers then

$$\lim_{x\to\alpha}\frac{e^{(x^3+bx^2+cx+d)}-1}{(x-\alpha)}+\lim_{x\to\beta}\frac{e^{(x^3+bx^2+cx+d)}-1}{(x-\beta)}+\lim_{x\to\gamma}\frac{e^{(x^3+bx^2+cx+d)}-1}{(x-\gamma)}$$

SOLUTION

$x^3+bx^2+cx+d=0=(x-\alpha)(x-\beta)(x-\gamma)$

Using relations between roots and coefficient, we get $\alpha+\beta+\gamma=-b$, $\alpha\beta+\beta\gamma+\alpha\gamma=c$

$$\lim_{x\to\alpha}\frac{e^{(x^3+bx^2+cx+d)}-1}{(x-\alpha)}+\lim_{x\to\beta}\frac{e^{(x^3+bx^2+cx+d)}-1}{(x-\beta)}+\lim_{x\to\gamma}\frac{e^{(x^3+bx^2+cx+d)}-1}{(x-\gamma)}$$

$$=\lim_{x\to\alpha}\frac{e^{(x-\alpha)(x-\beta)(x-\gamma)}-1}{(x-\alpha)(x-\beta)(x-\gamma)}\cdot(x-\beta)(x-\gamma)+\lim_{x\to\beta}\frac{e^{(x-\alpha)(x-\beta)(x-\gamma)}-1}{(x-\alpha)(x-\beta)(x-\gamma)}\cdot(x-\alpha)(x-\gamma)+\lim_{x\to\gamma}\frac{e^{(x-\alpha)(x-\beta)(x-\gamma)}-1}{(x-\alpha)(x-\beta)(x-\gamma)}\cdot(x-\alpha)(x-\beta)$$

$$\text{Limit}=(\alpha-\beta)(\alpha-\gamma)+(\beta-\alpha)(\beta-\gamma)+(\gamma-\alpha)(\gamma-\beta)$$

$$=\alpha^2+\beta^2+\gamma^2-\alpha\beta-\beta\gamma-\alpha\gamma$$

$$=(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\alpha\gamma)$$

(Using relations between roots and coefficients)

$$=b^2-3c$$

PROBLEM 10

Let f be a cubic polynomial such that $f(x)+f(-x)=0$ for all $x\in R$ and $\lim\limits_{x\to 0}\left(1+\dfrac{f(x)}{x}\right)^{\frac{1}{x^2}}=e^4$, then, compute $\sum\limits_{r=1}^{\infty}\tan^{-1}\left(\dfrac{2r}{f(r)}\right)$.

SOLUTION

$$\lim_{x\to 0}\left(1+\frac{f(x)}{x}\right)^{\frac{1}{x^2}}=e^4 \qquad (1^\infty \text{ form})$$

$$\Rightarrow \quad =e^{\lim\limits_{x\to 0}\frac{f(x)}{x}\cdot\frac{1}{x^2}}=e^4\Rightarrow f(x)=4x^3$$

Here, $f(x)=4x^3$

$$\therefore \quad \sum_{r=1}^{\infty}\tan^{-1}\left(\frac{2r}{f(r)}\right)$$

$$=\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{1}{2r^2}\right)=\sum_{r=1}^{\infty}\tan^{-1}\left(\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}\right)$$

$$=\sum_{r=1}^{\infty}\left[\tan^{-1}(2r+1)-\tan^{-1}(2r-1)\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left(\tan^{-1}(2r+1)-\tan^{-1}(2r-1)\right)$$

$$=\lim_{n\to\infty}\tan^{-1}(2n+1)-\tan^{-1}1=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

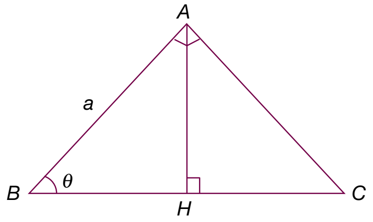
PROBLEM 11

Suppose ABC is a triangle such that $\angle BAC$ is a right angle. A point H is located at BC such that AH is perpendicular to BC. If $\angle ABC=\theta$, $AB=a$, and the value of $\lim\limits_{\theta\to 0}\dfrac{AC+AH}{\theta}=10$, then, find the value of a.

SOLUTION

$$AC=a\tan\theta;\ AH=a\sin\theta$$

Now, $\lim\limits_{\theta\to 0}\dfrac{AC+AH}{\theta}=\lim\limits_{\theta\to 0}\left(\dfrac{a\tan\theta+a\sin\theta}{\theta}\right)$



$$= a(1+1) = 2a = 10 \Rightarrow a = 5.$$

PROBLEM 12

Let $\lim\limits_{x\to 0^+} f(x) = 3$ $\lim\limits_{x\to 0^-} f(x) = 4$ and $f(0) = 5$ then, find length of perpendicular from point (0, 1) to the straight line $y \lim\limits_{x\to 0} f(\cos^3 x - \cos^2 x) - x \lim\limits_{x\to 0} f(\sin^2 x - \sin^3 x) = 0$.

SOLUTION

As $x \to 0$, $\cos^3 x - \cos^2 x = \cos^2 x\,(\cos x - 1) \to 0^-$

$\therefore \quad \lim\limits_{x\to 0} f(\cos^3 x - \cos^2 x) = f(0^-) = 4$

As $x \to 0$, $\sin^2 x - \sin^3 x = \sin x(1 - \sin x) \to 0^+$

$\lim\limits_{x\to 0} f(\sin^2 x - \sin^3 x) = f\,(0^+) = 3$

Given, line $y.\lim\limits_{x\to 0} f(\cos^3 x - \cos^2 x) - x \cdot \lim\limits_{x\to 0} f(\sin^2 x - \sin^3 x) = 0$

$\therefore \quad$ Equation is $4y - 3x = 0$

Length of perpendicular from point (0, 1) to above line $= \dfrac{|4-0|}{\sqrt{16+9}} = \dfrac{4}{5}$.

PROBLEM 13

The sequence $\langle a_{n-1} \rangle$, $n \in N$ is an arithmetical progression and d is its common difference.

If $\lim\limits_{n\to\infty}\left(\left(1-\dfrac{d^2}{a_1^2}\right)\left(1-\dfrac{d^2}{a_2^2}\right)\ldots\left(1-\dfrac{d^2}{a_n^2}\right)\right)$ converges to $\dfrac{1}{4}$ and $a_1 = 8$, then, find the value of d.

SOLUTION

$$\prod_{k=1}^{n}\left(1-\frac{d^2}{a_k^2}\right) = \prod_{k=1}^{n}\frac{a_k^2-d^2}{a_k^2} = \prod_{k=1}^{n}\frac{(a_k-d)}{a_k}\cdot\frac{(a_k+d)}{a_k}$$

$$= \prod_{k=1}^{n}\frac{a_{k-1}}{a_k}\cdot\frac{a_{k+1}}{a_k} = \left(\frac{a_0}{a_1}\cdot\frac{a_1}{a_2}\cdots\frac{a_{n-1}}{a_n}\right)\left(\frac{a_2}{a_1}\cdot\frac{a_3}{a_2}\cdots\frac{a_{n+1}}{a_n}\right)$$

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left(1-\frac{d^2}{a_k^2}\right) = \left(\frac{a_0}{a_1}\right)\left(\frac{a_{n+1}}{a_n}\right) \qquad [\text{but } a_0 = a_1 - d]$$

$$= \left(\frac{a_1-d}{a_1}\right)\left(\frac{a_n+d}{a_n}\right)$$

$$= \left(1-\frac{d}{a_1}\right)\left(1+\underbrace{\frac{d}{a_1+(n-1)d}}_{\text{zero as } n\to\infty}\right) = 1-\frac{d}{a_1}$$

But $1 - \dfrac{d}{a_1} = \dfrac{1}{4} \Rightarrow \dfrac{3}{4} = \dfrac{d}{a_1} \Rightarrow \dfrac{8.3}{4} = d$

Hence, $d = 6$

PROBLEM 14

Let $f(x) = \lim\limits_{n\to\infty} \tan^{-1}\left(4n^2\left(1-\cos\dfrac{x}{n}\right)\right)$ and $g(x) = \lim\limits_{n\to\infty}\dfrac{n^2}{2}\ln\cos\left(\dfrac{2x}{n}\right)$ then, compute $\lim\limits_{x\to 0}\left(\dfrac{e^{-2g(x)}-e^{f(x)}}{x^6}\right)$.

SOLUTION

$$f(x) = \lim_{n\to\infty}\tan^{-1}\left[4\left(\frac{1-\cos\frac{x}{n}}{x^2\cdot\frac{1}{n^2}}\right)\cdot x^2\right]$$

$$f(x) = \tan^{-1}\left(\frac{4x^2}{2}\right) = \tan^{-1}2x^2$$

$$g(x) = \lim_{n\to\infty}\frac{n^2}{2}\ln\cos\left(\frac{2x}{n}\right) = \lim_{n\to\infty}\frac{\ln\cos\left(\frac{2x}{n}\right)}{\frac{2}{n^2}}$$

$$= \lim_{n\to\infty}\frac{\sin\left(\frac{2x}{n}\right)2x}{\cos\left(\frac{2x}{n}\right)n^2 2\cdot\left(-\frac{2}{n^3}\right)}$$

$$= -\lim_{n\to\infty}\frac{x^2\sin\left(\frac{2x}{n}\right)}{\left(\frac{2x}{n}\right)} = -x^2$$

$$\lim_{x\to 0}\left(\frac{e^{-2g(x)}-e^{f(x)}}{x^6}\right) = \lim_{x\to 0}\frac{e^{2x^2}-e^{\tan^{-1}2x^2}}{x^6}$$

$$= \lim_{x\to 0}\frac{e^{\tan^{-1}2x^2}\left(e^{2x^2-\tan^{-1}2x^2}-1\right)}{\left(2x^2-\tan^{-1}2x^2\right)}\frac{\left(2x^2-\tan^{-1}2x^2\right)}{x^6}$$

$$= \lim_{x\to 0}1.\frac{\left(2x^2-(2x^2-\frac{(2x^2)^3}{3}+\ldots\right)}{x^6} = \frac{8}{3}$$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. The value of $\lim\limits_{x\to 1}\dfrac{x^{1/3}-x^{1/4}}{x^3-1}$ is _____

(A) $\dfrac{1}{36}$ (B) $-\dfrac{1}{36}$

(C) $-\dfrac{1}{12}$ (D) $\dfrac{1}{12}$

2. If $\lim\limits_{x\to 0}\dfrac{e^{ax}-e^x-x}{x^2}=b$ (finite) then _____

(A) $a=2, b=0$ (B) $a=0, b=\dfrac{3}{2}$

(C) $a=2, b=\dfrac{3}{2}$ (D) $a=0, b=2$

3. If $\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=a$ and $\lim\limits_{x\to 0}\dfrac{f(1-\cos x)}{g(x)\sin^2 x}=b$ (where $b\neq 0$), then $\lim\limits_{x\to 0}\dfrac{g(1-\cos 2x)}{x^4}$ is

(A) $\dfrac{4a}{b}$ (B) $\dfrac{a}{4b}$

(C) $\dfrac{a}{b}$ (D) None of these

4. If $\lim\limits_{x\to 0}\dfrac{\sin(\sin x)-\sin x}{ax^5+bx^3+c}=-\dfrac{1}{12}$, then

(A) $a=2,\ b=0, c=1$ (B) $a=0,\ b=2, c\in R$

(C) $a=2,\ b\in R, c=0$ (D) $a\in R,\ b=2, c=0$

5. $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{\frac{1}{x^2}}$

(A) $e^{\frac{1}{2}}$ (B) 1

(C) $e^{\frac{1}{6}}$ (D) $e^{\frac{1}{3}}$

6. $\lim\limits_{x\to\infty}\dfrac{\sum\limits_{r=1}^{10}(x+r)^{2010}}{\left(x^{1006}+1\right)\left(2x^{1004}+1\right)}=\ldots$

(A) 5 (B) 2010

(C) $\dfrac{1}{2}$ (D) 0

7. If α and β are roots of $x^2-\sqrt{1-\cos 2\theta}\,x+\theta=0$ where $0<\theta<\dfrac{\pi}{2}$ then $\lim\limits_{\theta\to 0^-}\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)=\ldots$

(A) $\dfrac{1}{\sqrt{2}}$ (B) $-\sqrt{2}$

(C) $\sqrt{2}$ (D) Doesn't exist

8. $\lim\limits_{\theta\to\frac{\pi}{2}}\left[\dfrac{\theta-\frac{\pi}{2}}{\cot\theta}\right]$, [where [.] denotes the greatest integer function] is

(A) –2 (B) 0

(C) –1 (D) 1

9. $\lim\limits_{x\to\infty}\dfrac{x^3\sin\left(\frac{1}{x}\right)-2x^2}{1+3x^2}=$

(A) 0 (B) $-\dfrac{1}{3}$

(C) –1 (D) $-\dfrac{2}{3}$

10. $\lim\limits_{x\to 0}\dfrac{\tan\left(\left[-\pi^2\right]x^2\right)-x^2\tan\left(\left[-\pi^2\right]\right)}{\sin^2 x}$ where $[.]$ denote GIF

(A) $\tan 10+10$ (B) $\tan 10-10$

(C) $10-\tan 10$ (D) None of these

11. If $\lim\limits_{x\to 0}\dfrac{((a-n)nx-\tan x)\sin nx}{x^2}=0$ where n is a non zero real number then a is equal to

(A) 0 (B) $\dfrac{n+1}{n}$

(C) n (D) $n+\dfrac{1}{n}$

12. If $\lim\limits_{x\to\infty}x\left(\tan^{-1}\left(\dfrac{x+\lambda}{x+\mu}\right)-\dfrac{\pi}{4}\right)=1$ then ordered pair(s) (λ,μ) can be

(A) (2000, 2011) (B) (0, 1)

(C) (5, 3) (D) (1, 0)

13. The value of $\lim\limits_{x\to 0}f(x)$ where $f(x)=\dfrac{\cos(\sin x)-\cos x}{x^4}$, is

(A) 2 (B) $1/6$

(C) $2/3$ (D) $-1/3$

14. $\lim\limits_{x\to 0}\dfrac{729^x-243^x-81^x+9^x+3^x-1}{x^3}=K(\log 3)^3$. Then $K=$ _____

(A) 4 (B) 5

(C) 6 (D) 7

15. $\lim\limits_{x\to 0}\dfrac{e^x-e^{\sin x}}{2(x-\sin x)}=$

(A) $-\dfrac{1}{2}$ (B) 1

(C) $\dfrac{3}{2}$ (D) $\dfrac{1}{2}$

16. $\lim_{h\to0}\left[\frac{1}{h\sqrt[3]{8+h}}-\frac{1}{2h}\right]=$

(A) $-\frac{1}{48}$ (B) $-\frac{4}{3}$

(C) $-\frac{16}{3}$ (D) $\frac{1}{12}$

17. $\lim_{x\to2^+}\{x\}\frac{\sin(x-2)}{(x-2)^2}$ where $\{\cdot\}$ is fractional part function

(A) 0 (B) ∞

(C) 1 (D) Does not exist

18. $\lim_{x\to\infty}x\left(\tan^{-1}\frac{x+1}{x+2}-\frac{\pi}{4}\right)=$ ______

(A) 0 (B) $-\frac{1}{2}$

(C) $\frac{1}{2}$ (D) Does not exist

19. $f(x)=\frac{1}{3}\left(f(x+1)+\frac{5}{f(x+2)}\right)$ and $f(x)>0,\forall x\in R$ then $\lim_{x\to\infty}f(x)$ is

(A) 0 (B) $\sqrt{\frac{2}{5}}$

(C) $\sqrt{\frac{5}{2}}$ (D) ∞

20. $\lim_{x\to0}\left(\left[\frac{100\sin x}{x}\right]+\left[\frac{100\tan x}{x}\right]\right)$, where [.] is GIF

(A) 199 (B) Does not exist

(C) 201 (D) 200

21. $\lim_{x\to0}\frac{5\sin x-7\sin2x+3\sin3x}{x^2\sin x}=$

(A) –5 (B) 5

(C) 0 (D) ∞

22. If α,β are the roots of the quadratic equation $ax^2+bx+c=0$, then $\lim_{x\to\alpha}\frac{1-\cos\left(ax^2+bx+c\right)}{(x-\alpha)^2}$ equals to

(A) 0 (B) $\frac{1}{2}(\alpha-\beta)^2$

(C) $\frac{a^2}{2}(\alpha-\beta)^2$ (D) $-\frac{a^2}{2}(\alpha-\beta)^2$

23. *ABC* is an isosceles triangle inscribed in a circle of radius r. If $AB=AC$ and h is the altitude from A to BC and P be the perimeter of ABC then $\lim_{h\to0}\frac{\Delta}{P^3}$ equals (where Δ is the area of the triangle)

(A) $\frac{1}{32r}$ (B) $\frac{1}{64r}$

(C) $\frac{1}{128r}$ (D) None

24. The value of the limit $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$ is

(A) 1 (B) $\frac{1}{4}$

(C) $\frac{1}{3}$ (D) $\frac{1}{2}$

25. $\lim_{x\to\frac{\pi}{2}}\frac{\sin x}{\cos^{-1}\left[\frac{1}{4}(3\sin x-\sin3x)\right]}$ where [ ] denotes greatest integer function, is

(A) $\frac{2}{\pi}$ (B) 1

(C) $\frac{4}{\pi}$ (D) Does not exist

26. If $\lim_{x\to0}\frac{\ln(3+x)-\ln(3-x)}{x}=k$, the value of k is

(A) $\frac{2}{3}$ (B) $-\frac{1}{3}$

(C) $-\frac{2}{3}$ (D) 0

27. The function $f(x)=\lim_{n\to\infty}\frac{x^{2n}-1}{x^{2n}+1}$ is identical with the function

(A) $g(x)=\text{sgn}(x-1)$ (B) $h(x)=\text{sgn}(\tan^{-1}x)$

(C) $u(x)=\text{sgn}(|x|-1)$ (D) $v(x)=\text{sgn}(\cot^{-1}x)$

28. The limiting value of the function $f(x)=\frac{2\sqrt{2}-(\cos x+\sin x)^3}{1-\sin2x}$ when $x\to\frac{\pi}{4}$ is

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$

(C) $3\sqrt{2}$ (D) $\frac{3}{\sqrt{2}}$

29. For $\lim_{x\to8}\frac{\sin\{x-10\}}{\{10-x\}}$ (where { } denotes fractional part function)

(A) LHL exist but RHL does not exist

(B) RHL exist but LHL does not exist.

(C) Neither LHL nor RHL does not exist

(D) Both RHL and LHL exist and equals to 1

30. Set of all values of x such that $\lim_{n\to\infty}\frac{1}{1+\left(\frac{4\tan^{-1}(2\pi x)}{\pi}\right)^{4n}}$ is non-zero and finite number when $n\in N$ is

(A) $\left(0,\frac{1}{2\pi}\right)$ (B) $\left(-\frac{1}{\pi},\frac{1}{\pi}\right)$

(C) $\left[-\frac{1}{2\pi},\frac{1}{2\pi}\right]$ (D) $\left(-\frac{1}{2\pi},\frac{1}{2\pi}\right)$

31. If $\lim_{x\to0}\left[1+x^2+\frac{f(x)}{x}\right]^{\frac{1}{x^2}}$ is equal to e^7, then $\lim_{x\to0}\frac{f(x)}{x^3}=\ldots$

(A) 7 (B) 4

(C) 5 (D) 6

32. The value of $\lim_{x\to0}\left((\sin x)^{1/x}+\left(\frac{1}{x}\right)^{\sin x}\right)$ where $x>0$ is

(A) 0 (B) –1

(C) 1 (D) 2

33. $\lim_{x\to 0}\left(\left[\frac{2009\sin x}{x}\right]+\left[\frac{2010\tan x}{x}\right]\right)$ is equal to _____ where [.] GIF

(A) 4020
(B) 4021
(C) 4010
(D) 4018

34. $\lim_{\substack{x\to 0\\ y\to 0}}\frac{y^2+\sin x}{x^2+\sin y^2}$, when $(x, y)\to(0, 0)$ along the curve $x = y^2$ is

(A) 0
(B) 1
(C) 2
(D) Does not exist

35. If $f(x)=\lim_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^x+\sin\frac{1}{n}\right)^n$ where $x\in Q$ then $f(1)=$

(A) 0
(B) –1
(C) 1
(D) $\frac{1}{2}$

36. If $\lim_{x\to 0}(x^{-3}\sin 3x + ax^{-2} + b)$ exists and is equal to zero then

(A) $a = -3$ and $b = 9/2$
(B) $a = 3$ and $b = 9/2$
(C) $a = -3$ and $b = -9/2$
(D) $a = 3$ and $b = -9/2$

37. The value of $\lim_{x\to\infty}\frac{\left(2^{x^n}\right)^{\frac{1}{e^x}}-\left(3^{x^n}\right)^{\frac{1}{e^x}}}{x^n}$ (where $n\in N$) is

(A) $\ln\left(\frac{2}{3}\right)$
(B) 0
(C) $n\ln\left(\frac{2}{3}\right)$
(D) Not Defined

38. For a certain value of c, $\lim_{x\to-\infty}[(x^5+7x^4+2)^C - x]$ is finite and non zero. The value of c and the value of the limit is

(A) $\frac{1}{5},\frac{7}{5}$
(B) 0, 1
(C) $1,\frac{7}{5}$
(D) None

39. $\lim_{n\to\infty}\frac{1^2\,n+2^2\,(n-1)+3^2\,(n-2)+\ldots+n^2.1}{1^3+2^3+3^3+\ldots+n^3}$ is equal to

(A) $\frac{1}{3}$
(B) $\frac{2}{3}$
(C) $\frac{1}{2}$
(D) $\frac{1}{6}$

40. The value of $\lim_{x\to\infty}\frac{\cot^{-1}\left(x^{-a}\log_a x\right)}{\sec^{-1}\left(a^x\log_x a\right)}$ $(a > 1)$ is equal to

(A) 1
(B) 0
(C) $\pi/2$
(D) Does not exist

41. Let $f:(1, 2)\to R$ satisfies the inequality

$$\frac{\cos(2x-4)-33}{2}<f(x)<\frac{x^2|4x-8|}{x-2},\ \forall\ x\in(1,2).$$

Then, $\lim_{x\to 2^-}f(x)$ is equal to

(A) 16
(B) –16
(C) Cannot be determined from the given information
(D) Does not exists

42. Let $a = \min[x^2+2x+3,\ x\in R]$ and $b=\lim_{x\to 0}\frac{\sin x\cos x}{e^x-e^{-x}}$. Then, the value of $\sum_{r=0}^{n}a^r b^{n-r}$ is

(A) $\frac{2^{n+1}+1}{3\cdot 2^n}$
(B) $\frac{2^{n+1}-1}{3\cdot 2^n}$
(C) $\frac{2^n-1}{3\cdot 2^n}$
(D) $\frac{4^{n+1}-1}{3\cdot 2^n}$

43. $\lim_{x\to 1/\sqrt{2}}\frac{x-\cos(\sin^{-1}x)}{1-\tan(\sin^{-1}x)}$ is

(A) $\frac{1}{\sqrt{2}}$
(B) $-\frac{1}{\sqrt{2}}$
(C) $\sqrt{2}$
(D) $-\sqrt{2}$

44. Range of the function $f(x)=\left[\frac{1}{\ln(x^2+e)}\right]+\frac{1}{\sqrt{1+x^2}}$ is, where [*] denotes the greatest integer function and $e=\lim_{\alpha\to 0}(1+\alpha)^{1/\alpha}$

(A) $\left(0,\frac{e+1}{e}\right)\cup\{2\}$
(B) (0, 1)
(C) $(0, 1]\cup\{2\}$
(D) $(0, 1)\cup\{2\}$

45. $\lim_{x\to\infty}\frac{\cot^{-1}\left(\sqrt{x+1}-\sqrt{x}\right)}{\sec^{-1}\left\{\left(\frac{2x+1}{x-1}\right)^x\right\}}$ is equal to

(A) 1
(B) 0
(C) $\pi/2$
(D) Non existent

46. Let $f(x)=\frac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}$. If $\lim_{x\to\infty}f(x)=l$ and $\lim_{x\to-\infty}f(x)=m$ then

(A) $l = m$
(B) $l = 2m$
(C) $2l = m$
(D) $l + m = 0$

47. $\lim_{n\to\infty}\cos\left(\pi\sqrt{n^2+n}\right)$ when n is an integer

(A) Is equal to 1
(B) Is equal to –1
(C) Is equal to zero
(D) Does not exist

48. $\lim_{x\to 0}\frac{(\sin x-\tan x)^2-(1-\cos 2x)^4+x^5}{7.(\tan^{-1}x)^7+(\sin^{-1}x)^6+3\sin^5 x}$ is equal to

(A) 0
(B) $\frac{1}{7}$
(C) $\frac{1}{3}$
(D) 1

49. Let $\lim_{x\to 0}\frac{[x]^2}{x^2}=l$ and $\lim_{x\to 0}\frac{[x^2]}{x^2}=m$, where [ ] denotes greatest integer, then

(A) l exists but m does not
(B) m exists but l does not
(C) l and m both exist
(D) Neither l nor m exists.

50. The value of $\lim_{x\to 0}\frac{(\tan(\{x\}-1))\ \sin\{x\}}{\{x\}\ (\{x\}-1)}$ where $\{x\}$ denotes the fractional part function, is

(A) 1 (B) tan 1
(C) sin 1 (D) Non-existent

51. $\lim\limits_{x \to \infty} \dfrac{2+2x+\sin 2x}{(2x+\sin 2x)e^{\sin x}}$ is

(A) Equal to zero (B) Equal to 1
(C) Equal to –1 (D) Non-existent

52. The value of $\lim\limits_{x\to 0}(\cos ax)^{\operatorname{cosec}^2 bx}$ is

(A) $e^{\left(-\frac{8b^2}{a^2}\right)}$ (B) $e^{\left(-\frac{8a^2}{b^2}\right)}$
(C) $e^{\left(-\frac{a^2}{2b^2}\right)}$ (D) $e^{\left(-\frac{b^2}{2a^2}\right)}$

53. If $f(3) = 6$ and $f'(3) = 2$, then $\lim\limits_{x \to 3} \dfrac{x\,f(3) - 3\,f(x)}{x-3}$ is given by

(A) 6 (B) 4
(C) 0 (D) None of these

54. $\lim\limits_{n \to \infty}\left[(1+x)\left(1+x^2\right)\left(1+x^4\right)\ldots\left(1+x^{2^n}\right)\right]$ if $|x| < 1$ has the value equal to

(A) 0 (B) 1
(C) $1 - x$ (D) $(1 - x)^{-1}$

55. $\lim\limits_{x\to 0} \dfrac{6x^2(\cot x)(\csc 2x)}{\sec\left(\cos x + \pi \tan\left(\dfrac{\pi}{4\sec x}\right) - 1\right)}$ has the value equal to

(A) 6 (B) –6
(C) 0 (D) –3

56. $\lim\limits_{x\to 1^-}\left(1+(arc\ \cos x)^{1-x}\right)^2$ has the value equal to

(A) 4 (B) 2
(C) 1 (D) 0

57. $\lim\limits_{x \to 0}\left(1 + \log^2_{\cos\frac{x}{2}} \cos x\right)^2$ is

(A) Equal to 4 (B) Equal to 9
(C) Equal to 289 (D) Non existent

58. $\lim\limits_{x\to 0^+} \dfrac{1}{x\sqrt{x}}\left(a\,arc\tan\dfrac{\sqrt{x}}{a} - b\ arc\tan\dfrac{\sqrt{x}}{b}\right)$ has the value equal to

(A) $\dfrac{a-b}{3}$ (B) 0
(C) $\dfrac{(a^2-b^2)}{6a^2b^2}$ (D) $\dfrac{a^2-b^2}{3a^2b^2}$

59. For $n \in N$, let $f_n(x) = \tan\dfrac{x}{2}(1+\sec x)(1+\sec 2x)(1+\sec 4x)\ldots\left(1+\sec 2^n x\right)$. Then $\lim\limits_{x\to 0}\dfrac{f_n(x)}{2x}$ is equal to

(A) 0 (B) 2^n
(C) 2^{n-1} (D) 2^{n+1}

60. If $\lim\limits_{n\to\infty} \dfrac{1}{\left(\dfrac{x^2+a}{x^2+1}\right)^n + 1} = 1$, then conditions on '$a$' is

(A) $0 < a < 1$ (B) $-1 < a < 1$
(C) $-1 < a < 2$ (D) None of these

61. If α is number of solution in $|x| = \log(x - [x])$ (where [.] denotes the greatest integer function). Then the value of $\lim\limits_{x\to\alpha}\dfrac{xe^{ax} - b\sin x}{x^3}$ is

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{3}$
(C) $\dfrac{1}{4}$ (D) $\dfrac{1}{6}$

62. Let $f : R \to R$ be a twice differentiable function such that $\lim\limits_{x\to\infty} f(x) = a\ (a \neq 0)$ then $\lim\limits_{x\to\infty} x^2 f''(x)$ is equal to

(A) a (B) Zero
(C) $a^2 + a$ (D) None of these

63. $\lim\limits_{x\to 1}\dfrac{(1-x)\left(1-x^2\right)\ldots\left(1-x^{2n}\right)}{\left[(1-x)\left(1-x^2\right)\ldots\ldots\left(1-x^n\right)\right]^2}$, is

(A) ${}^{2n}C_0$ (B) ${}^{2n}C_n$
(C) ${}^{2n}C_{n-1}$ (D) Limit does not exist

64. Let $x_1 = \tan^{-1}\dfrac{2}{\sqrt{3}} > x_2 > x_3\ldots$ are positive real numbers satisfying $\sin(x_{n+1} - x_n) + 3^{\frac{1-2n}{2}} \sin x_n \sin x_{n+1} = 0$ for $n \geq 1$. Then, $\lim\limits_{n\to\infty} x_n$ equals

(A) $\dfrac{\pi}{8}$ (B) $\dfrac{\pi}{6}$
(C) $\dfrac{\pi}{10}$ (D) $\dfrac{\pi}{12}$

65. If $f(2010) = \pi/2, f(2x) = f(x)\,\forall x \in R$ and $f(x)$ is continuous for $x \in R$, then the value of $\lim\limits_{x\to 0}\dfrac{\cos(f(x)) - \sqrt[3]{\cos(f(x))}}{\sin^2 x}$ is

(A) 1/4 (B) 1
(C) 0 (D) None of these

66. $\lim\limits_{n\to\infty}\prod\limits_{r=2}^{n}\left(\dfrac{r^2-r+1}{r^2+r+1}\right) =$

(A) 0 (B) 1
(C) $\dfrac{1}{2}$ (D) $\dfrac{3}{2}$

67. $\lim\limits_{n\to\infty}\prod\limits_{r=2}^{n}\left(\dfrac{r^3+1}{r^3-1}\right) =$

(A) 1 (B) $\dfrac{1}{2}$
(C) $\dfrac{3}{2}$ (D) 2

68. If [.] is greatest integer function and n is a positive integer, then $\lim\limits_{x\to 0}\left(\left[\dfrac{n\sin x}{x}\right] + \left[\dfrac{n\tan x}{x}\right]\right) =$

(A) 2^n (B) 2
(C) $2n - 1$ (D) $2n + 1$

69. The value of $\lim\limits_{x\to\infty}\left\{\sqrt[4]{(x+1)(x+2)(x+3)(x+4)}-x\right\}$ equals

(A) 1
(B) $\frac{5}{2}$
(C) $\frac{2}{3}$
(D) 0

70. If $\{x\}$ represents the fractional part of x. The value of $\lim\limits_{x\to 0}\left[\frac{1+\{x\}^{\frac{1}{\{x\}}}}{e}\right]^{\frac{1}{\{x\}}}$ is

(A) –1
(B) 0
(C) 1
(D) Does not exist

71. $\lim\limits_{x\to\infty} x.\ln\begin{vmatrix}\alpha/x & 1 & \gamma\\ 0 & 1/x & \beta\\ 1 & 0 & 1/x\end{vmatrix}=-5$, where α,β,γ are finite real numbers then

(A) $\alpha=2,\beta=1,\gamma\in R$
(B) $\alpha=2,\beta=2,\gamma=5$
(C) $\alpha\in R,\beta=1,\gamma\in R$
(D) $\alpha\in R,\beta=1,\gamma=5$

72. $P=\lim\limits_{n\to\infty}\left\{\lim\limits_{x\to 0}\frac{\log(1+nx)[\tan(2x)+\tan(4x)+...+\tan(2nx)]}{\sin^2 x+\sin^2 2x+...+\sin^2 nx}\right\}$.

If a circle touches x-axis at (P, 0) and having y-intercept l where $l=\lim\limits_{x\to 0}\frac{e^{\tan x}-e^{\sin x}}{\left(e^{\frac{x}{32}}-1\right)(1-\cos 2x)}$, then find the radius of the circle.

(A) 1
(B) 2
(C) 5
(D) 8

73. The value of $\lim\limits_{n\to\infty}\cos^n\left(\frac{x}{\sqrt{n}}\right)$ is equal to

(A) e^x
(B) $e^{-\frac{x}{2}}$
(C) $e^{\frac{x^2}{2}}$
(D) $e^{-\frac{x^2}{2}}$

74. $\lim\limits_{x\to\infty}\left\{\left(e^x+\pi^x\right)^{\frac{1}{x}}\right\}$,where {.} denotes the fractional part of x, is equal to

(A) $\pi-e$
(B) $\pi-3$
(C) $e-2$
(D) $3-e$

75. If α and β be the roots of the equation $px^2+qx+r=0$ and $1<\alpha<\beta$, if $\lim\limits_{x\to a}\frac{\left|px^2+qx+r\right|}{px^2+qx+r}=-1$ then

(A) $p>0, a<\alpha$ or $a<\beta$
(B) $p<0, \alpha<a<\beta$
(C) $p<0, a<\alpha$ or $a>\beta$
(D) $p>0, a>0$

76. $\lim\limits_{n\to\infty}\prod\limits_{r=3}^{n}\left(\frac{r^3-8}{r^3+8}\right)=$

(A) $\frac{2}{7}$
(B) $\frac{3}{7}$
(C) $\frac{4}{7}$
(D) $\frac{5}{7}$

77. If $f(x)=\lim\limits_{n\to\infty}\sum\limits_{r=0}^{n}\frac{\tan\frac{x}{2^{r+1}}+\tan^3\frac{x}{2^{r+1}}}{1-\tan^2\frac{x}{2^{r+1}}}$, then $\lim\limits_{x\to 0}\frac{f(x)-\sin x}{x^3}=$

(A) 0
(B) –1
(C) 1
(D) 1/2

78. If a_n and b_n are positive integers and $a_n+\sqrt{2}b_n=\left(2+\sqrt{2}\right)^n$, then $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right)=$

(A) $\sqrt{2}$
(B) 2
(C) $e^{\sqrt{2}}$
(D) e^2

79. $\lim\limits_{x\to\frac{\pi}{2}}\frac{(1-\sin x)\left(8x^3-\pi^3\right)\cos x}{(\pi-2x)^4}$

(A) $-\frac{\pi^2}{16}$
(B) $\frac{3\pi^2}{16}$
(C) $\frac{\pi^2}{16}$
(D) $-\frac{3\pi^2}{16}$

80. Let $f:R^+\to R^+$ be a function satisfying the relation $f(x.f(y))=f(xy)+x$ for all $x,y\in R^+$. Then $\lim\limits_{x\to 0}\left(\frac{(f(x))^{1/3}-1}{(f(x))^{1/2}-1}\right)=$

(A) 1
(B) $\frac{1}{2}$
(C) $\frac{2}{3}$
(D) $\frac{3}{2}$

81. Let $x_1=1$ and $x_{n+1}=\frac{4+3x_n}{3+2x_n}$ for $n\ge 1$. If $\lim\limits_{n\to\infty}x_n$ exists finitely, then the limit is equal to

(A) $\sqrt{2}$
(B) 1
(C) 2
(D) $\sqrt{2}+1$

82. Let $f(x)=x^3\left\{\sqrt{x^2+\sqrt{x^4+1}}-x\sqrt{2}\right\}$. Then, $\lim\limits_{x\to\infty}f(x)$ is equal to

(A) $\frac{1}{2\sqrt{2}}$
(B) $\frac{1}{4\sqrt{2}}$
(C) $\frac{3}{4\sqrt{2}}$
(D) Does not exist

83. If a_1 is the greatest value of $f(x)$ where $f(x)=\frac{1}{2+[\sin x]}$ where [.] is GIF and $a_{n+1}=\frac{(-1)^{n+2}}{n+1}+a_n$. Then, $\lim\limits_{n\to\infty}a_n=$ ______

(A) 0
(B) e
(C) 1
(D) $\log_e 2$

84. $\lim\limits_{x\to\frac{\pi}{2}}\left[\frac{[\sin x]-[\cos x]+1}{3}\right]=$ is where [.] denotes greatest integer function

(A) 0
(B) 1
(C) –1
(D) Does not exist

85. $\lim\limits_{x\to 0}\left(\dfrac{1+2x}{1+3x}\right)^{\frac{1}{x^2}}.e^{\frac{1}{x}} =$

(A) $e^{\frac{5}{2}}$

(B) e^2

(C) e^5

(D) 1

86. $\lim\limits_{x\to\infty}\sqrt[3]{x}\left(\sqrt[3]{(x+1)^2}-\sqrt[3]{(x-1)^2}\right)=$

(A) $\dfrac{1}{3}$

(B) $\dfrac{2}{3}$

(C) 1

(D) $\dfrac{4}{3}$

87. If $a>0$, $b>0$ then $\lim\limits_{n\to\infty}\left(\dfrac{a-1+b^{\frac{1}{n}}}{a}\right)^n =$

(A) $b^{\frac{1}{a}}$

(B) $a^{\frac{1}{b}}$

(C) a^b

(D) b^a

88. The value of $\lim\limits_{x\to 0}\dfrac{(1+x)^{1/x}-e}{x}$ is

(A) $\dfrac{e}{2}$

(B) $-\dfrac{e}{2}$

(C) $\dfrac{3e}{2}$

(D) $-\dfrac{2e}{3}$

89. $\lim\limits_{x\to 0}\dfrac{\sin[\cos x]}{1+[\cos x]}$, ([$x$] denotes the greatest integer less than or equal to)

(A) sin 1

(B) 0

(C) Does not exist

(D) $\dfrac{\sin 1}{2}$

90. The value of $\lim\limits_{x\to 0}\dfrac{\cos\left(\sin^2 x\right)-\cos\left(x^2\right)}{x^6}$ is

(A) 0

(B) $\dfrac{1}{2}$

(C) $\dfrac{1}{3}$

(D) $\dfrac{3}{4}$

91. If $\sum\limits_{r=1}^{k}\cos^{-1}\beta_r=\dfrac{k\pi}{2}$ for any $k\ge 1$ where $\beta_r\ge 0\,\forall r$ and $A=\sum\limits_{r=1}^{k}(\beta_r)^r$. Then $\lim\limits_{x\to A}\dfrac{(1+x^2)^{1/3}-(1-2x)^{1/4}}{x+x^2}=$

(A) $\dfrac{1}{2}$

(B) 0

(C) 3/2

(D) $\dfrac{\pi}{2}$

92. If [x] and {x} represent integral and fractional parts of x respectively and a is any real number, then $\lim\limits_{x\to[a]}\dfrac{e^{\{x\}}-\{x\}-1}{\{x\}^2}=$

(A) a

(B) $\{a\}$

(C) $\dfrac{1}{2}$

(D) Does not exist

93. $f(x)$ is a polynomial function and $\left(f(\alpha)\right)^2+\left(f'(\alpha)\right)^2=0$ then, the value of $\lim\limits_{x\to\alpha}\dfrac{f(x)}{f'(x)}\left[\dfrac{f'(x)}{f(x)}\right]$ (where $[.]$ denotes greatest integer function) is ___________

(A) 0

(B) 1

(C) –1

(D) 2

94. $\lim\limits_{x\to a^-}\left(\dfrac{|x|^3}{a}-\left[\dfrac{x}{a}\right]^3\right)(a>0)$, [.] GIF, is

(A) a^2-2

(B) a^2-1

(C) a^2

(D) a^2+1

95. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\sin x-(\sin x)^{\sin x}}{1-\sin x+\ln(\sin x)}=$

(A) 1

(B) 0

(C) 2

(D) –1

96. $\lim\limits_{x\to 1}\left(\tan^{-1}x.\dfrac{4}{\pi}\right)^{\frac{1}{x^2-1}}=$

(A) e^{π}

(B) $e^{\frac{1}{\pi}}$

(C) $\dfrac{1}{e^{\pi}}$

(D) $e^{-\frac{1}{\pi}}$

97. If a, b, c are positive real numbers then the least value of $\lim\limits_{x\to 0}\dfrac{\sqrt[3]{(x+a)(x+b)(x+c)}-\sqrt[3]{abc}}{x}$ is equal to

(A) 1

(B) 2

(C) 3

(D) 0

98. If $f(x)=8x^3+3x$, then $\lim\limits_{x\to\infty}\dfrac{f^{-1}(8x)-f^{-1}(x)}{x^{1/3}}$ is

(A) $\dfrac{1}{2}$

(B) $2\dfrac{1}{3}$

(C) $\dfrac{2^{1/3}-1}{2}$

(D) $\dfrac{1}{4}$

99. $\lim\limits_{x\to\infty}\dfrac{\cos^{-1}(\sqrt{x+1}-\sqrt{x})}{\sec^{-1}\left\{\left(\dfrac{2x+1}{x-1}\right)^x\right\}}$ is equal to

(A) 1

(B) 0

(C) $\pi/2$

(D) Non existent

100. $\lim\limits_{x\to 1}\dfrac{nx^{n+1}-(n+1)x^n+1}{(e^x-e)\sin\pi x}$ where $n=100$ is equal to

(A) $\dfrac{5050}{\pi e}$

(B) $\dfrac{100}{\pi e}$

(C) $-\dfrac{5050}{\pi e}$

(D) $-\dfrac{4950}{\pi e}$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. If $\lim\limits_{x\to0}\dfrac{x(1+a\cos x)-b\sin x}{(f(x))^3}=1$ and $\lim\limits_{x\to0}\dfrac{f(x)}{x}=1$ then

(A) $a=\dfrac{-5}{2}$ (B) $b=\dfrac{27}{2}$

(C) $a=\dfrac{-5}{3}$ (D) $b=\dfrac{-3}{2}$

2. $\lim\limits_{x\to\infty}\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)$ equal to

(A) $\lim\limits_{x\to0}\dfrac{\log(1+x)-x}{x^2}$ (B) $\lim\limits_{x\to0}\dfrac{1-\cos x}{x^2}$

(C) $\lim\limits_{x\to0}\dfrac{\sqrt{1+x}-1}{x}$ (D) $\lim\limits_{x\to0}\dfrac{\sqrt{x}}{\sqrt{x+\sqrt{x^2+2x}}}$

3. If $\lim\limits_{x\to0}g(x)$ exists and $\lim\limits_{x\to0}g(x).\left(\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right)$ also exists then $g(x)=$

(A) x (B) $x+4$

(C) x^2 (D) $4x^2+3x+2$

4. $\lim\limits_{x\to0}\dfrac{ae^x+b\cos x+ce^{-x}}{e^{2x}-2e^x+1}=4$. Then

(A) $a=2$ (B) $b=-4$

(C) $c=2$ (D) $a+b+c=-8$

5. α,β are the roots of $ax^2+bx+c=0$

$L_1=\lim\limits_{x\to\alpha}\dfrac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$, $L_2=\lim\limits_{x\to\beta}\dfrac{1-\cos(ax^2+bx+c)}{(x-\beta)^2}$,

$L_3=\lim\limits_{(x-\alpha)(x-\beta)\to0}\dfrac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2(x-\beta)^2}$. Then

(A) $L_1=L_2$ (B) $L_1=L_3\,(\alpha-\beta)^2$

(C) $L_2=L_3\,(\alpha-\beta)^2$ (D) $L_3=\dfrac{a^2}{2}$

6. If $\lim\limits_{x\to\infty}4x\left(\dfrac{\pi}{4}-\tan^{-1}\dfrac{x+1}{x+2}\right)=y^2+4y+5$, then y can be equal to

(A) 1 (B) –1

(C) –4 (D) –3

7. If $\alpha,\beta\in\left(-\dfrac{\pi}{2},0\right)$ such that $(\sin\alpha+\sin\beta)+\dfrac{\sin\alpha}{\sin\beta}=0$ and $(\sin\alpha+\sin\beta)\dfrac{\sin\alpha}{\sin\beta}=-1$ and $\lambda=\lim\limits_{n\to\infty}\dfrac{1+(2\sin\alpha)^{2n}}{(2\sin\beta)^{2n}}$ then

(A) $\alpha=-\dfrac{\pi}{6}$ (B) $\lambda=2$

(C) $\alpha=-\dfrac{\pi}{3}$ (D) $\lambda=1$

8. $\lim\limits_{n\to\infty}\left\{\left(\sqrt{2}+1\right)^n\right\}(-1)^{\left[\left(\sqrt{2}+1\right)^n\right]}$, where n is the a integer, (where [.] is greatest integer function, {.} fractional part of x), is

(A) 1 when n is even (B) 0 when n is odd

(C) 0 when n is even (D) –1 when n is odd

9. Let $f(x)=\dfrac{e^{\{x\}}-\{x\}-1}{\{x\}^2}$ where {.} fractional part of x and [.] is step function then

(A) $\lim\limits_{x\to[a]^-}f(x)=e-2$ (B) $\lim\limits_{x\to[a]^+}f(x)=\dfrac{1}{2}$

(C) $\lim\limits_{x\to[a]}f(x)=4$ (D) $\lim\limits_{x\to[a]}f(x-4)=0$

10. Let $L=\lim\limits_{x\to0}\dfrac{a-\sqrt{a^2-x^2}-\dfrac{x^2}{4}}{x^4}$, $a>0$. If L is finite then

(A) $a=2$ (B) $a=1$

(C) $L=\dfrac{1}{64}$ (D) $L=\dfrac{1}{32}$

11. If $f(x)=x\left(\dfrac{e^{|x|+[x]}-2}{|x|+[x]}\right)$ where $[\cdot]$ is GIF then

(A) $\lim\limits_{x\to0^+}f(x)=-1$ (B) $\lim\limits_{x\to0^-}f(x)=0$

(C) $\lim\limits_{x\to0}f(x)=-1$ (D) $\lim\limits_{x\to0}f(x)=0$

12. $\lim\limits_{x\to0}\left[\dfrac{\left|\sin^{-1}|x|\right|}{x}\right]$ (where [.] denotes greatest integer function) is

(A) Left hand limit is –2

(B) Left hand limit is –1

(C) Right hand limit is 1

(D) Limit exists and both are equal to 1

13. $\lim\limits_{x\to\infty}\dfrac{x^p+x^{p-1}+1}{x^q+x^{q-2}+2}=$ $(p>0, q>0)$

(A) 0 if $p<q$ (B) 1 if $p=q$

(C) Infinite if $p>q$ (D) 1 if $p>q$

14. α, β are roots of equation $ax^2+bx+c=0$ where $1<\alpha<\beta$ if $\lim\limits_{x\to m}\dfrac{|ax^2+bx+c|}{ax^2+bx+c}=1$, then which of the following are true

(A) $a<0$ and $\alpha<m<\beta$ (B) $a>0$ and $m<1$

(C) $a>0$ and $\alpha<m<\beta$ (B) $a>0$ and $m>1$

15. Let $f(x)=(e^x+2x)^{\frac{3}{x}}$, then

(A) $\lim\limits_{x\to0}f(x)=9$ (B) $\lim\limits_{x\to+\infty}f(x)=e^3$

(C) $\lim\limits_{x\to0}f(x)=e^9$ (D) $\lim\limits_{x\to+\infty}f(x)=3$

16. If $f(0)$, $f'(0)$, $f''(0)$ exists and non zero such that

$\lim\limits_{x\to0}\dfrac{af(2x)+2bf\left(\dfrac{x}{2}\right)-3f(x)}{\sin^{-1}(x^2)}=3$, then

(A) $a = b$ (B) $f''(0) = 4$

(C) $a^2 + b^2 = 2$ (D) $a^2 + b^2 = 8$

17. Let $f(x) = \dfrac{\cos^{-1}(1-\{x\})\sin^{-1}(1-\{x\})}{\sqrt{2\{x\}}(1-\{x\})}$ where $\{x\}$ denotes the fractional part of x, then

(A) $\lim\limits_{x\to 0^-} f(x) = \dfrac{\pi}{2\sqrt{2}}$ (B) $\lim\limits_{x\to 0^+} f(x) = \sqrt{2}\lim\limits_{x\to 0^-} f(x)$

(C) $\lim\limits_{x\to 0^-} f(x) = \dfrac{\pi}{4\sqrt{2}}$ (D) $\lim\limits_{x\to 0^+} f(x) = \dfrac{\pi}{2}$

18. If $\lim\limits_{n\to\infty}\left(\left(\cos\dfrac{k\pi}{4}\right)^n - \left(\cos\dfrac{k\pi}{6}\right)^n\right) = 0$ (where k is an integer) then which of the following statement(s) is (are) correct?

(A) k is neither divisible by 4 nor by 6.

(B) k is divisible by 12.

(C) k is divisible by 24.

(D) k is either divisible by 24 or k is neither divisible by 4 nor by 6.

19. Let $L = \lim\limits_{x\to 1}\dfrac{\sin(6\cos^{-1}x)}{\sqrt{1-x^2}}$ and $M = \lim\limits_{x\to 1}\dfrac{1-\cos(6\cos^{-1}x)}{1-x^2}$.

Which of the following is/are correct?

(A) $L + M = 24$ (B) $\dfrac{M}{L} = 3$

(C) $L - M = -12$ (D) $LM = 108$

20. If the function $f(x) = \dfrac{|2x-\pi|(\cos x)}{(\pi - 2x)}, x \neq \dfrac{\pi}{2}, f\left(\dfrac{\pi}{2}\right) = 0$ divides the circle $x^2 + y^2 = \pi^2$ in two parts having area A_1 and A_2 (where $A_1 < A_2$) then

(A) $\dfrac{A_1}{A_2} = \left(\dfrac{\pi-2}{\pi+2}\right)$ (B) $A_1 = \dfrac{\pi^3-4}{2}$

(C) $A_2 = \dfrac{\pi^3+4}{2}$ (D) $\dfrac{A_1}{A_2} = \dfrac{1}{2}$

21. Let $f(x) = \begin{cases} \dfrac{x}{\tan x}, & -\infty < x < 0 \\ 2, & x = 0 \\ \dfrac{\ln(1+2x)}{x} & 0 < x < \infty \end{cases}$ and

$g(x) = \begin{cases} x+4, & -\infty < x < 1 \\ x^2 - 5x + 11, & 1 \le x < 2 \\ x - 3, & 2 \le x < \infty \end{cases}$ then

which of the following is(are) correct?

(A) $\lim\limits_{x\to 0^-} g(f(x)) = 5$ (B) $\lim\limits_{x\to 0^-} g(f(x)) = 7$

(C) $\lim\limits_{x\to 0^+} g(f(x)) = 5$ (D) $\lim\limits_{x\to 0^+} g(f(x)) = -1$

22. Let $f(x) = \begin{cases} \dfrac{\tan^2\{x\}}{x^2 - [x]^2}, & x > 0 \\ 1, & x = 0 \\ \sqrt{\{x\}\cot\{x\}}, & x < 0 \end{cases}$ where $[x]$ is the step up function and $\{x\}$ is the fractional part function of x, then

(A) $\lim\limits_{x\to 0^+} f(x) = 1$ (B) $\lim\limits_{x\to 0^-} f(x) = 1$

(C) $\cot^{-1}\left(\lim\limits_{x\to 0^-} f(x)\right)^2 = 1$ (D) None of these

23. If variable line $x(3+\lambda) + 2y(2-\lambda) - (7-\lambda) = 0$ always passes through a fixed point (a, b) where λ is parameter and $\lambda = \lim\limits_{x\to(a-b)^-}\dfrac{[\sin x - 2] + \{\cos x\}}{x - [x] - 1}$ where $[y]$ and $\{y\}$ denotes greatest integer $\le$ and fractional part of y respectively, then

(A) $a + 2b = 3$ (B) $a - b + 2\ell = 2$

(C) $\ell = 1$ (D) ℓ does note exist

24. The possible value(s) of k for which

$$\lim_{x\to\infty}\frac{2x^3 - (\tan^{-1}x)^3}{\frac{8}{\pi}x^3\cot^{-1}(|kx|) + k^2x^6\sin\left(\frac{1}{x^3}\right) - 3kx^3} = \frac{1}{2}, \text{ is}$$

(A) 0 (B) –1

(C) 2 (D) 4

25. Consider, $f(x) = \begin{cases} 2-|x|, & -1 \le x \le 1 \\ |x-2| - x, & 1 < x \le 3 \end{cases}$

$g(x) = \begin{cases} \sin(x) - 1, & 0 \le x < \dfrac{\pi}{2} \\ [x] - \cos(x-2), & \dfrac{\pi}{2} \le x \le \pi \end{cases}$

where $[k]$ denotes greatest integer function of k. Identify the correct statement(s)?

(A) $\lim\limits_{x\to 1^+} g(f(x)) = -1$ (B) $\lim\limits_{x\to \frac{\pi}{2}^-} g(f(g(x))) = 0$

(C) $\lim\limits_{x\to 2^+}\dfrac{f(g(x))}{f(x)-2)} = \dfrac{1}{2}$ (D) $\lim\limits_{x\to 0^+}\dfrac{g(f(x))}{(f(x)-2)^2} = \dfrac{1}{2}$

26. Let $f(x) = \dfrac{1-\cos x(x)}{(x^4 + ax^3 + bx^2 + cx)^2}$. If $l = \lim\limits_{x\to 1^+} f(x)$, $m = \lim\limits_{x\to 2^+} f(x)$ and $n = \lim\limits_{x\to 3^+} f(x)$ where l, m and n are non-zero finite then

(A) $\lim\limits_{x\to 0^+} f(x) = \dfrac{1}{36}$ (B) $a + b + c = -1$

(C) $l + m + n = \dfrac{19}{72}$ (D) $\lim\limits_{x\to 4^+} f(x) = 0$

27. Consider $L = \lim\limits_{x\to 0}\dfrac{\sin(x^2) + 2\cos(bx) - ax^4}{e^{ax} - 1 - ax - 2x^2 - \dfrac{a^3x^3}{6}}$, $a, b \in I$. Identify the correct statement(s)?

(A) If $L = \dfrac{25}{8}$ then $a + b = 3$

(B) If $L = \dfrac{25}{8}$ then $a + b = -3$

(C) If $L = \dfrac{-23}{8}$ then $a + b = -1$

(D) If $L = \dfrac{-23}{8}$ then $a + b = 1$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

For $x > 0$ if $y = \sqrt{x + \sqrt{x + \sqrt{x + \ldots\infty}}}$ then $y = \sqrt{x + y}$ and hence $y^2 - y - x = 0$. So $y = \dfrac{1 + \sqrt{1 + 4x}}{2}$. Now, evalute the following limit values.

1. $\displaystyle\lim_{x \to 0^+} \frac{x^2\left[-1 + \sqrt{x + \sqrt{x + \sqrt{x + \ldots\infty}}}\right]}{\left[-1 + \sqrt{x^3 + \sqrt{x^3 + \sqrt{x^3 \ldots\infty}}}\right]} =$

(A) 0 (B) –1

(C) 1 (D) $\frac{1}{4}$

2. $\displaystyle\lim_{x \to 0^+} \frac{-1 + \sqrt{(\tan x - \sin x) + \sqrt{(\tan x - \sin x) + \sqrt{(\tan x - \sin x) + \ldots\infty}}}}{-1 + \sqrt{x^3 + \sqrt{x^3 + \sqrt{x^3 + \ldots\infty}}}} =$

(A) 0 (B) $\frac{1}{2}$

(C) $-\frac{1}{2}$ (D) 1

3. $\displaystyle\lim_{x \to \frac{\pi}{6}} -\frac{1}{2} + \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\ldots\infty}}} =$

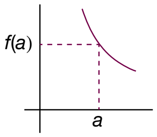
(A) 0 (B) 1

(C) $\sin\frac{\pi}{6}$ (D) $\cos\frac{\pi}{6}$

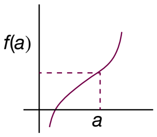
Comprehension 2

To evaluate $\lim\limits_{x \to a}[f(x)]$, we must analyse the $f(x)$ in a right hand neighborhood as well in left hand neighborhood of $x = a$. For example in case of continuous function we may come across following cases

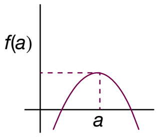
CASE 1:



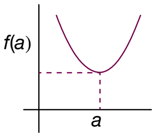
CASE 2:



CASE 3:



CASE 4:



If $f(a)$ is integer, the limit will exist in case of case 3 and case 4 but not in case 1 and case 2.
If $f(a)$ is not an integer, the limit will exist in all the cases.
Similar analysis may be required for Discontinuous functions at $x = a$

4. If $f'(1) = -3$, and $\lim\limits_{x \to 1}\left[f(x) - \frac{1}{2}\right]$ does not exist, (where [.] denotes the greatest integer function) then

(A) $f(1)$ may be integer

(B) $\{f(x)\} = \frac{1}{2}, \forall x \in R$ ({.} fractional part of x)

(C) $f(1)$ is not an integer

(D) None of these

5. $\lim\limits_{x \to 0}\left[\left(1 - e^x\right)\dfrac{\sin x}{|x|}\right]$ (where [.] denotes the greatest integer function)

(A) 0 (B) 1

(C) –1 (D) Does not exist

6. $\lim\limits_{x \to 1}\left[\operatorname{cosec}\dfrac{\pi x}{2}\right]^{-1/(1-x)}$ is equal to (where [.] denotes the greatest integer function)

(A) 0 (B) 1

(C) ∞ (D) Does not exist

Comprehension 3

Let $f(x) = \lim\limits_{n \to \infty}\left(\cos\sqrt{\dfrac{x}{n}}\right)^n$, $g(x) = \lim\limits_{n \to \infty}\left(1 - x + x\sqrt[n]{e}\right)^n$. Now, consider the function $y = h(x)$, where $h(x) = \tan^{-1}\left(g^{-1}f^{-1}(x)\right)$.

7. $\lim\limits_{x \to 0^+}\dfrac{\ln(f(x))}{\ln(g(x))}$ is equal to

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$

(C) 0 (D) 1

8. Domain of the function $y = h(x)$ is

(A) $(0, \infty)$ (B) R

(C) (0, 1) (D) [0, 1]

9. Range of the function $y = h(x)$ is

(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left(-\frac{\pi}{2}, 0\right)$

(C) R (D) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Comprehension 4

Let $f(x) = {}^nC_x p^x q^{n-x}; x = 0, 1, 2, \ldots\ldots n$ where $p + q = 1$, $0 < p < 1$. Here ${}^nC_x = \dfrac{n!}{x!(n-x)!}$ and $n! = 1 \times 2 \times 3 \times \ldots. \times n. np = \lambda$ (finite) when $p \to 0$ and $n \to \infty$. Now answer the following questions

10. $\lim\limits_{n\to\infty}\sum\limits_{x=0}^{n} x\,f(x) =$

(A) λ (B) $\dfrac{\lambda}{2}$

(C) 0 (D) 1

11. $\lim\limits_{n\to\infty}\sum\limits_{x=0}^{n} x^2 f(x) =$

(A) $1+\lambda$ (B) $\lambda+\lambda^2$

(C) λ^2 (D) 0

12. $\lim\limits_{n\to\infty}\sum\limits_{x=0}^{n} (x-\lambda)^2 f(x) =$

(A) λ (B) 0

(C) 2λ (D) $\lambda^2+\lambda$

Comprehension 5

Let $f(x)$ be a polynomial function satisfying $\lim\limits_{x\to\infty}\dfrac{x^4 f(x)}{x^8+1} = 3$, $f(2) = 5$, $f(3) = 10$, $f(-1) = 2$, $f(-6) = 37$, then

13. $f'(3) = \dots$

(A) 0 (B) -1

(C) 3 (D) 114

14. $\lim\limits_{x\to-6}\dfrac{f(x)-x^2-1}{3(x+6)} = \dots$

(A) $6!$ (B) $-6!$

(C) $\dfrac{6!}{2}$ (D) $-\dfrac{6!}{2}$

Comprehension 6

Consider a function $y = f(x)$. Let the functional rule for $y = f(x)$ is same as the functional rule for the height of a triangle ABC from the vertex A to the base BC and the vertical angle A. The triangle is inscribed in a circle of radius 6 and the area of the triangle is 12.

15. The least value of $f(x)$ is equal to

(A) $\dfrac{1}{2}$ (B) 2

(C) 4 (D) 6

16. If $g(x) = f(\sin^{-1}x)$ then $g'\left(\dfrac{4}{5}\right)$ is equal to

(A) $\dfrac{-25}{4}$ (B) $\dfrac{-25}{8}$

(C) $\dfrac{25}{8}$ (D) $\dfrac{25}{4}$

17. If $h(x) = \sec^{-1}\left(\dfrac{1}{2}f(x)\right)$ then $\lim\limits_{x\to\frac{\pi^-}{2}}\dfrac{e^{2h(x)}-2e^{\left(\frac{\pi}{2}-x\right)}+\sin x}{h(x)\cos x}$ is equal to

(A) $\dfrac{1}{2}$ (B) 2

(C) $\dfrac{3}{2}$ (D) 0

Comprehension 7

Let $f(x) = \max\{a, b, c\}$, where

$$a = \lim_{n\to\infty}\lim_{\alpha\to1^+}\frac{\alpha^n|\sin x|+\alpha^{-n}|\cos x|}{\alpha^n+\alpha^{-n}}$$

$$b = \lim_{n\to\infty}\lim_{\alpha\to1^-}\frac{\alpha^n|\sin x|+\alpha^{-n}|\cos x|}{\alpha^n+\alpha^{-n}}$$

$$c = \lim_{n\to\infty}\frac{\pi}{4n}\left[1+\cos\frac{\pi}{2n}+\cos\frac{2\pi}{2n}+\dots.+\cos\frac{(n-1)\pi}{2n}\right] \text{ Then}$$

18. The value of $b+c-\dfrac{1}{2}$ is

(A) $|\cos x|$ (B) $2|\cos x| - 1$

(C) $|\sin x| + 1$ (D) $|\sin x| + |\cos x|$

19. Range of $f(x)$ is

(A) $[0, 1]$ (B) $\left[\dfrac{1}{2}, 1\right]$

(C) $\left[\dfrac{1}{\sqrt{2}}, 1\right]$ (D) $\left[\dfrac{1}{2}, 2\right]$

Comprehension 8

Let f, g, h be real-valued function defined on R. Consider $f(x) = -1 + 4\sin^2 x$, $g(x) = 2x - 1$ and $h(x) = 2x + 1$.

20. The range of function $y = \dfrac{h(f(x))}{g(f(x))}$ is equal to

(A) $\left[\dfrac{1}{3}, \dfrac{7}{5}\right]$ (B) $(-\infty, 1) \cup \left[\dfrac{7}{5}, \infty\right)$

(C) $\left[1, \dfrac{7}{5}\right]$ (D) $\left(-\infty, \dfrac{1}{3}\right] \cup \left[\dfrac{7}{5}, \infty\right)$

21. The value of $\lim\limits_{x\to\frac{1}{2}}\dfrac{f(\sin^{-1}x)}{g(x)}$ is equal to

(A) 0 (B) 2

(C) 4 (D) 10

22. If $\lim\limits_{x\to\frac{-1}{2}}\dfrac{f(\sin^{-1}kx)-3}{h(x)}$ exists and has value equal to ℓ then $|k| + |l|$ is equal to

(A) 6 (B) 8

(C) 10 (D) 12

Comprehension 9

Let $P(x)$ be a polynomial of degree 6 with leading coefficient unity and $p(-x) = p(x)\ \forall\ x \in R$.

Also, $(P(1)+3)^2 + P^2(2) + (P(3)-5)^2 = 0$

23. The value of $\lim\limits_{x\to\frac{-1}{2}}\dfrac{\sin(P(x))}{(x-2)\tan(x+2)}$ is equal to

(A) 26 (B) 16

(C) -14 (D) -40

24. The value of $\lim\limits_{x\to\infty}\dfrac{\left(x^2-\sqrt{\dfrac{P(x)}{x^2-4}}\right)}{x\tan\dfrac{1}{x}}$ is equal to

(A) 10 (B) 5

(C) 3 (D) -4

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:
If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. Match the following:

COLUMN-I	COLUMN-II
(A) $\lim\limits_{x\to\frac{\pi}{2}}\sqrt{\dfrac{\tan x-\sin\left(\tan^{-1}\left(\tan x\right)\right)}{\tan x+\cos^2\left(\tan x\right)}}$ is equal to	(p) –1
(B) $\lim\limits_{x\to 0^+}\dfrac{1}{4}\dfrac{\log_{\sin x}\cos x}{\log_{\sin\frac{x}{2}}\cos\frac{x}{2}}=$	(q) 0
(C) $\lim\limits_{m\to\infty}\left(\cos\dfrac{x}{m}\right)^m$ is equal to	(r) 1
(D) $\lim\limits_{x\to\infty}\dfrac{\ln x-[x]}{[x]}$ is equal to	(s) 4

2. Match the following:

COLUMN-I	COLUMN-II
(A) $\lim\limits_{x\to 0}\dfrac{3e^x-x^3-3x-3}{\tan^2 x}$	(p) 0
(B) $\lim\limits_{x\to 0}\dfrac{2\sin x-\sin 2x}{x(\cos x-\cos 2x)}$	(q) $\dfrac{1}{4}$
(C) $\lim\limits_{x\to 0}\dfrac{e^x-e^{-x}-2x}{x-\sin x}$	(r) $\dfrac{2}{3}$
(D) $\lim\limits_{x\to 0}\dfrac{\pi-2\tan^{-1}x}{\log e^{\left(1+\frac{1}{x}\right)}}$	(s) $\dfrac{3}{2}$
	(t) 2

3. Match the following:

COLUMN-I	COLUMN-II
(A) $\lim\limits_{x\to\infty}\left(\dfrac{x}{1+x}\right)^x$	(p) e^2
(B) $\lim\limits_{x\to\infty}\left(\sin\dfrac{1}{x}+\cos\dfrac{1}{x}\right)^x$	(q) $e^{-1/2}$
(C) $\lim\limits_{x\to 0}\left(\cos x\right)^{\cot^2 x}$	(r) e
(D) $\lim\limits_{x\to 0}\left(\tan\left(\dfrac{\pi}{4}+x\right)\right)^{1/x}$	(s) e^{-1}

4. Match the following:

COLUMN-I	COLUMN-II
(A) Let $f(x)=[x-1]+[1-x]$, $[x]$ is the greatest integer function	(p) Continuous at a, a is an integer
(B) Let f be as in (A) a is not an integer	(q) $\lim\limits_{x\to 0}f(x)$ does not exist
(C) $f(x)=\dfrac{a^{[x]+x}-1}{[x]+x}, x\neq 0$	(r) $f(a)=0$
(D) $f(x)=\dfrac{a^{\cot x}-a^{\cos x}}{\cot x-\cos x}$	(s) $\lim\limits_{x\to\pi/2}f(x)=\log a$

5. Match the following

COLUMN-I	COLUMN-II
(A) $\lim\limits_{t\to 0}\left[\min\limits_{x\in R}(x^2+2x-2)\dfrac{\sin t}{t}\right]$	(p) –4
(B) $\lim\limits_{t\to\infty}\left[\min\limits_{x\in R}(x^2+2x-2)\dfrac{\sin^{-1}t}{t}\right]$	(q) –3
(C) $\lim\limits_{t\to\infty}\left[\min\limits_{x\in R}(-x^2+4x-7)\dfrac{t}{\sin^{-1}t}\right]$	(r) –2
(D) $\lim\limits_{t\to\infty}\left\{\min\limits_{x\in R}(-x^2+4x-7)\dfrac{\tan t}{t}\right\}$	(s) 1

6. Match the following

COLUMN-I	COLUMN-II
(A) $\lim\limits_{x\to0}\dfrac{\sqrt{1+x\sin x}-\sqrt{\cos 2x}}{\tan^2\frac{x}{2}}$	(p) −3
(B) $\lim\limits_{x\to1}\dfrac{\sin^2(x^3+x^2+x-3)}{1-\cos(x^2-4x+3)}$	(q) 6
(C) $\lim\limits_{x\to0}\dfrac{6x^2(\cot x)(\csc 2x)}{\sec\left(\cos x+\pi\tan\left(\frac{\pi}{4\sec x}\right)-1\right)}$ has the value equal to	(r) 8
(D) If $\lim\limits_{x\to-2^+}\dfrac{(2(a-3)(x+2)-6\sin^{-1}(x+2))\tan^{-1}(5x+10)}{(x+2)^2}$ then, the value of a is equal to	(s) 12
	(t) 18

7. Match the following

COLUMN-I	COLUMN-II
(A) $\lim\limits_{x\to\infty}x^2\left(\sqrt{\frac{x+2}{x}}-\sqrt[3]{\frac{x+3}{x}}\right)$ equals	(p) 0
(B) $\lim\limits_{x\to\infty}\dfrac{x^2(1+\sin^2 x)}{(x+\sin x)^2}$ equals	(q) $\frac{1}{2}$
(C) Let $a, b\in R$. If $\lim\limits_{x\to\infty}\left(\sqrt[3]{4x^2-x^3}+ax+b\right)=\frac{1}{3}$ then the sum $(a+b)$ equals	(r) 1
(D) $\lim\limits_{x\to\infty}\dfrac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x-x+1}}$ equals	(s) 2
	(t) Does not exist

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. If $\lim\limits_{x\to0}\dfrac{axe^x-b\log(1+x)+cxe^{-x}}{x^2\sin x}=2$ then $a-b+c=$ ________

2. $\lim\limits_{x\to0}|x|^{[\cos x]}$ ________ where [·] is GIF

3. If $f(x)=[[x]]-[x-1]$ and $a\in I$ where [·] is GIF, then $\lim\limits_{x\to a^-}f(x)+\lim\limits_{x\to a^+}f(x)$ ________.

4. $\lim\limits_{x\to0}\dfrac{3\log\left(\frac{3+x^2}{3-x^2}\right)}{x^2}$ is

5. If $\lim\limits_{x\to0}\dfrac{\cos 4x+a\cos 2x+b}{x^4}$ is finite, then $b-a$ ________.

6. Let $f(x)$ be a twice differentiable function and $f''(0)=2$, then $\lim\limits_{x\to0}\dfrac{2f(x)-3f(2x)+f(4x)}{x^2}$ ________.

7. If $f(x+y)=f(x)f(y)$ and $f(x)=1+xg(x)H(x)$ where $\lim\limits_{x\to0}g(x)=2$, $\lim\limits_{x\to0}H(x)=3$ then $f'(x)=Kf(x)$ if K ______.

8. $\lim\limits_{x\to0}\dfrac{e^{\tan x}-e^{\sin x}}{\left(e^{x/32}-1\right)(1-\cos 4x)}$ ________.

9. $\lim\limits_{x\to0}\dfrac{1-\cos^n(1-\cos x)}{\tan^m x}=1$. Then n/m is.

10. $\lim\limits_{x\to0}\dfrac{3\tan 3x-4\tan 2x-\tan x}{4x^2\tan x}$ ________.

11. $\lim\limits_{x\to0}\dfrac{ex}{e-(1+x)^{\frac{1}{x}}}$ ________.

12. $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\left(1-\tan\dfrac{x}{2}\right)}{\left(1+\tan\dfrac{x}{2}\right)} \cdot \dfrac{(1-\sin x)}{(\pi-2x)^3} = k$, then $96k$ ________.

13. If $p(x)$ is quadratic expression and $\lim\limits_{x \to 0}\left(1+\dfrac{p(x)}{x^2}\right) = 2$, then $p(3)$ _____.

14. If $\lim\limits_{x \to 0} \dfrac{\log_e(1+2x) - 2\log_e(1+x)}{x^2}$ is λ then $-\lambda$ is _______.

15. The integer 'n' for which $\lim\limits_{x \to 0} \dfrac{\cos^2 x - \cos x - e^x \cos x + e^x - \dfrac{x^3}{2}}{x^n}$ is finite non zero number is _______.

16. Let $a_1, a_2, \ldots, a_n$ be sequence of real numbers, with $a_{n+1} = a_n + \sqrt{1+a_n^2}$ and $a_0 = 0$ if $\lim\limits_{n \to \infty} \dfrac{a_n}{2^{n-1}}$ is $\dfrac{k}{\pi}$, then k is equal to ______________.

17. If $\lim\limits_{x \to \infty}\left(\sqrt{x^4+ax^3+3x^2+bx+2} - \sqrt{x^4+2x^3-cx^2+3x-d}\right) = 4$, then $a + c =$ _______.

18. The value of $\lim\limits_{x \to 0^+}\left(\dfrac{\pi^2 x}{2\sqrt{2}}\right)(\ln x)^n$ is equal to ________ $(n \geq 5$ and $n \in Z)$

19. $\lim\limits_{x \to 0} \dfrac{1}{f(x)} =$ ________ where $f(x) = \dfrac{\sin x \tan x}{x^2}$ [[.] denotes GIF]

20. If $\lim\limits_{x \to 0} \dfrac{\sin x^4 - x^4 \cos x^4 + x^{20}}{x^4\left(e^{2x^4} - 1 - 2x^4\right)} = k$, then $6k$ is equal to ________.

21. $\lim\limits_{\theta \to 0}\left\{\left[\dfrac{25\sin\theta}{\theta}\right] + \left[\dfrac{25\tan\theta}{\theta}\right]\right\}$, where $[x]$ denotes GIF $\leq x$, is equal to $7k$. then the value of 'k' is equal to ________.

22. If $f(x+y+z) = f(x) + f(y) + f(z)$ with $f(1) = 1$ and $f(2) = 2$ and $x, y, z \in R$, then evaluate $\lim\limits_{n \to \infty} \dfrac{\sum\limits_{r=1}^{n} (4r) f(3r)}{n^3}$ is equal to ______

23. If $f(x) = \begin{cases} \sin x & ; \quad x \neq n\pi \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \\ 2 & ; \quad \text{otherwise} \end{cases}$ and

$g(x) = \begin{cases} x^2+1 & ; \quad x \neq 0, 2 \\ 4 & ; \quad x = 0 \\ 5 & ; \quad x = 2 \end{cases}$ then $\lim\limits_{x \to 0} g[f(x)]$ is __________.

24. $\lim\limits_{x \to \infty}\left(\left((x-1)(x-2)(x+3)(x+10)(x+15)\right)^{\frac{1}{5}} - x\right) =$ ___

25. The value of $\lim\limits_{x \to 0} \dfrac{1 - \cos^5 x \cos^3 2x \cos^3 3x}{x^2}$ is 11 k where 'k' is _____.

26. $\lim\limits_{n \to \infty}\left\{\left(\sqrt{3}+1\right)^{2n}\right\} =$ ______ where { . } is fractional part of x

27. If $\{x\}, [x]$ are fractional part function and greatest integer functions of x respectively then for any real number a, the value of $\lim\limits_{x \to [a]} - \dfrac{e^{\{x\}} - \{x\} - 1}{\{x\}^2}$ is $e - K \Rightarrow K =$ _____

28. If $A = \lim\limits_{x \to -2} \dfrac{\tan \pi x}{x+2} + \lim\limits_{x \to \infty}\left(1+\dfrac{1}{x^2}\right)^x$ then $[A]$ is ______ where [.] denotes GIF

29. If $\lim\limits_{x \to 0} \dfrac{ae^x - b\cos x + ce^{-x}}{x \sin x} = 2$ then the value of $a+b+c =$ ____

30. If $\lim\limits_{x \to 0} \dfrac{\log(1+x)^{1+x}}{x^2} - \dfrac{1}{x} = k$, then value of 12 k is _____

31. The value of $\lim\limits_{x \to \frac{\pi}{2}} \sqrt{\dfrac{\tan x - \sin(\tan^{-1}(\tan x))}{\tan x + \cos^2(\tan x)}}$ is _____

32. Let $(\tan\alpha)x + (\sin\alpha)y = \alpha$ and $(\alpha \operatorname{cosec}\alpha)x + \cos\alpha y = 1$ be two variable straight lines, α being the parameter. Let P be the point of intersection of the lines. If the coordinates of P in the limiting position when $\alpha \to 0$ be (h, k), then is $h - k$ equal to ___.

33. Let $L = \lim\limits_{x \to \infty}\left(x \ln x + 2x \ln \sin\left(\dfrac{1}{\sqrt{x}}\right)\right)$. The value of $\left(-\dfrac{1}{L}\right)$ ________.

34. If the value of $\lim\limits_{x \to 2} \dfrac{\sqrt{x^2-2x+9} - \sqrt{9+2x-x^2}}{\sqrt{(x-1)^2+3} - \sqrt{5-(x-1)^2}}$ equal $\dfrac{p}{q}$ where p and q are relatively prime positive integers, then the value of $\sin^{-1}(\sin(p+q)) + \cos^{-1}(\cos(p+q))$ ______.

35. If L is the value of $\lim\limits_{x \to 0} \dfrac{(1+x)^{1/x} + e(x-1)}{e(\sin^{-1} x)}$ and M is the maximum value of the function __________.

$f(x) = \dfrac{2\tan^2 x}{\tan^2 x + (2\tan^2 x + 3)^2}$, where $x \neq (2n+1)\dfrac{\pi}{2}$, $n \in I$, then the value of $\left(L + \dfrac{1}{M}\right)$ is __________.

36. Find the sum of an infinite geometric series whose first term is $\lim\limits_{x \to 0} \sum\limits_{k=1}^{2013} \dfrac{\left\{\dfrac{x}{\tan x} + 2k\right\}}{2013}$ and whose common ratio is the value of $\lim\limits_{x \to 0} \dfrac{e^{\tan^3 x} - e^{x^3}}{2\ln(1+x^3 \sin^2 x)}$ _____.

37. The value of $\lim\limits_{x \to 0} \dfrac{\sin(x) + x^2 \cot(x) - x - \cos(x)}{(-x)^5}$ is $\dfrac{p}{q}$ where p and q are coprime, then the value of (p + q) _______.

38. Let $f(x) =$

$$\lim_{\theta \to 0} \frac{\cos(x+4\theta) - 4\cos(x+3\theta) + 6\cos(x+2\theta) - 4\cos(x+\theta) + \cos x}{\theta^4}$$

then the absolute value of difference between maximum and minimum and minimum value of $f(x)$ _______.

39. Let $f(x)$ be a quadratic function such that $f(0) = f(1) = 0$ and $f(2) = 1$. If $\lim\limits_{x \to 0} \dfrac{\cos\left(\dfrac{\pi}{2}\cos^2 x\right)}{f^2(x)}$ equals $k\pi$, $k \in N$, then the value of k ______.

40. The value of

$$\lim_{n \to \infty}\left(\sin\frac{\pi}{2r} - \sin\left(\frac{\pi}{2(r+2)}\right) - \cos\left(\frac{\pi}{2(r+2)}\right)\right)$$ is ______.

ARCHIVE: JEE MAIN

1. [JEE Mains 2021]

Let $f(x) = x^6 + 2x^4 + x^3 + 2x + 3$, $x \in R$. Then the natural number n for which $\lim\limits_{x\to1} \dfrac{x^n f(1) - f(x)}{x-1} = 44$ is ______.

2. [JEE Mains 2021]

If $\lim\limits_{x\to0} \dfrac{ax - (e^{4x} - 1)}{ax(e^{4x} - 1)}$ exists and is equal to b, then the value of $a - 2b$ is ______.

3. [JEE Mains 2021]

$\lim\limits_{x\to0} \dfrac{\sin^2\left(\pi \cos^4 x\right)}{x^4}$ is equal to:

(A) π^2
(B) $2\pi^2$
(C) $4\pi^2$
(D) 4π

4. [JEE Mains 2021]

If $\alpha = \lim\limits_{x\to\pi/4} \dfrac{\tan^3 x - \tan x}{\cos\left(x + \dfrac{\pi}{4}\right)}$ and $\beta = \lim\limits_{x\to0} (\cos x)^{\cot x}$ are the roots of the equation, $ax^2 + bx - 4 = 0$, then the ordered pair (a, b) is

(A) $(1, -3)$
(B) $(-1, 3)$
(C) $(-1, -3)$
(D) $(1, 3)$

5. [JEE Mains 2021]

Let $f : R \to R$ be a function such that $f(2) = 4$ and $f'(2) = 1$. Then, the value of $\lim\limits_{x\to2} \dfrac{x^2 f(2) - 4f(x)}{x-2}$ is equal to:

(A) 4
(B) 16
(C) 8
(D) 12

6. [JEE Mains 2021]

The value of $\lim\limits_{x\to0} \left(\dfrac{x}{\sqrt[8]{1-\sin x} - \sqrt[8]{1+\sin x}} \right)$ is equal to:

(A) -4
(B) 0
(C) -1
(D) 4

7. [JEE Mains 2021]

$\lim\limits_{n\to\infty} \left(1 + \dfrac{1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}}{n^2} \right)^n$ is equal to:

(A) $\dfrac{1}{2}$
(B) 0
(C) $\dfrac{1}{e}$
(D) 1

8. [JEE Mains 2021]

$\lim\limits_{n\to\infty} \left(1 + \dfrac{1 + \frac{1}{2} + \ldots\ldots + \frac{1}{n}}{n^2} \right)^n$ is equal to:

(A) $\dfrac{1}{2}$
(B) 0
(C) $\dfrac{1}{e}$
(D) 1

9. [JEE Mains 2020]

$\lim\limits_{x\to2} \dfrac{3^x + 3^{3-x} - 12}{3^{-x/2} - 3^{1-x}}$ is equal to ______.

10. [JEE Mains 2020]

If $\lim\limits_{x\to1} \dfrac{x + x^2 + x^3 + \ldots + x^n - n}{x-1} = 820.(n \in N)$ then the value of n is equal to ______.

11. [JEE Mains 2020]

If $\lim\limits_{x\to0} \left\{ \dfrac{1}{x^8} \left(1 - \cos\dfrac{x^2}{2} - \cos\dfrac{x^2}{4} + \cos\dfrac{x^2}{2}\cos\dfrac{x^2}{4} \right) \right\} = 2^{-k}$, then the value of k is ______

12. [JEE Mains 2020]

$\lim\limits_{x\to0} \left(\dfrac{3x^2+2}{7x^2+2} \right)^{1/x^2}$ is equal to

(A) $\dfrac{1}{e^2}$
(B) e
(C) e^2
(D) $\dfrac{1}{e}$

13. [JEE Mains 2020]

$\lim\limits_{x\to a} \dfrac{(a+2x)^{\frac{1}{3}} - (3x)^{\frac{1}{3}}}{(3a+x)^{\frac{1}{3}} - (4x)^{\frac{1}{3}}}$ $(a \neq 0)$ is equal to

(A) $\left(\dfrac{2}{3}\right)\left(\dfrac{2}{9}\right)^{\frac{1}{3}}$
(B) $\left(\dfrac{2}{3}\right)^{\frac{4}{3}}$
(C) $\left(\dfrac{2}{9}\right)^{\frac{4}{3}}$
(D) $\left(\dfrac{2}{9}\right)\left(\dfrac{2}{3}\right)^{\frac{1}{3}}$

14. [JEE Mains 2020]

Let $[t]$ denote the greatest integer $\leq t$. If for some $\lambda \in R - \{0, 1\}$. $\lim\limits_{x\to0} \left| \dfrac{1 - x + |x|}{\lambda - x + [x]} \right| = L$, then L is equal to

(A) 1
(B) 2
(C) $\dfrac{1}{2}$
(D) 0

15. [JEE Mains 2020]

$\lim\limits_{x\to0} \dfrac{x\left(e^{\left(\sqrt{1+x^2+x^4}-1\right)/x} - 1 \right)}{\sqrt{1+x^2+x^4} - 1}$

(A) Does not exist.
(B) Is equal to $\sqrt{e}$.
(C) Is equal to 0.
(D) Is equal to 1.

16. [JEE Mains 2020]

If α is the positive root of the equation, $p(x) = x^2 - x - 2 = 0$, then $\lim\limits_{x\to\alpha^+} \dfrac{\sqrt{1 - \cos p(x)}}{x + \alpha - 4}$ is equal to

(A) $\frac{3}{\sqrt{2}}$ (B) $\frac{3}{2}$

(C) $\frac{1}{\sqrt{2}}$ (D) $\frac{1}{2}$

17. [JEE Mains 2019]

$$\lim_{y\to 0}\frac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}$$

(A) Exists and equals $\frac{1}{2\sqrt{2}}$

(B) Exists and equals $\frac{1}{4\sqrt{2}}$

(C) Exists and equals $\frac{1}{2\sqrt{2}(\sqrt{2}+1)}$

(D) Does not exist

18. [JEE Mains 2019]

Let $[x]$ denote the greatest integer less than or equal to x. Then $\lim\limits_{x\to 0}\frac{\tan(\pi\sin^2 x)+(|x|-\sin(x[x]))^2}{x^2}$

(A) Equals $\pi+1$ (B) Equals 0

(C) Does not exist (D) Equals π

19. [JEE Mains 2019]

$\lim\limits_{x\to 0}\frac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}}$ equals

(A) 4 (B) $2\sqrt{2}$

(C) $4\sqrt{2}$ (D) $\sqrt{2}$

20. [JEE Mains 2019]

$\lim\limits_{x\to 0}\frac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x-x+1}}$ is

(A) 6 (B) 1

(C) 3 (D) 2

21. [JEE Mains 2018]

For each $t\in R$, let $[t]$ be the greatest integer less than or equal to t. Then $\lim\limits_{x\to 0^+} x\left(\left[\frac{1}{x}\right]+\left[\frac{2}{x}\right]+\ldots.+\left[\frac{15}{x}\right]\right)$

(A) Is equal to 15 (B) Is equal to 120

(C) Does not exist (in R) (D) Is equal to 0.

22. [JEE Mains 2017]

$\lim\limits_{x\to\frac{\pi}{2}}\frac{\cot x-\cos x}{(\pi-2x)^3}$ equals

(A) $\frac{1}{4}$ (B) $\frac{1}{24}$

(C) $\frac{1}{16}$ (D) $\frac{1}{8}$

23. [JEE Mains 2016]

Let $p=\lim\limits_{x\to 0+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ then $\log p$ is equal to

(A) $\frac{1}{4}$ (B) 2

(C) 1 (D) $\frac{1}{2}$

24. [JEE Mains 2014]

$\lim\limits_{x\to 0}\frac{\sin(\pi\cos^2 x)}{x^2}$ is equal to:

(A) $\frac{\pi}{2}$ (B) 1

(C) $-\pi$ (D) π

25. [JEE Main 2014]

If $\lim\limits_{x\to 2}\frac{\tan(x-2)\{x^2+(k-2)x-2k\}}{x^2-4x+4}=5$ then k is equal to

(A) 3 (B) 1

(C) 0 (D) 2

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. [JEE ADV 2018]

For any positive integer n, define $f_n:(0,\infty)\to R$ as

$$f_n(x)=\sum_{j=1}^{n}\tan^{-1}\left(\frac{1}{1+(x+j)(x+j-1)}\right)\text{ for all } x\in(0,\infty).$$

(Here, the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$)

Then, which of the following statement(s) is (are) True?

(A) $\sum\limits_{j=1}^{5}\tan^2(f_j(0))=55$

(B) $\sum\limits_{j=1}^{10}(1+f_j'(0))\sec^2(f_j(0))=10$

(C) For any fixed positive integer n, $\lim\limits_{x\to\infty}\tan(f_n(x))=\frac{1}{n}$

(D) For any fixed positive integer n, $\lim\limits_{x\to\infty}\sec^2(f_n(x))=1$

2. [JEE ADV 2012]

If $\lim\limits_{x\to\infty}\left(\frac{x^2+x+1}{x+1}-ax-b\right)=4$, then

(A) $a=1, b=4$

(B) $a=1, b=-4$

(C) $a=2, b=-3$

(D) $a=2, b=3$

3. [JEE ADV 2012]

Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $\left(\sqrt[3]{1+a}-1\right)x^2+\left(\sqrt{1+a}-1\right)x+\left(\sqrt[6]{1+a}-1\right)=0$ where $a>-1$. Then $\lim\limits_{a\to 0^+}\alpha(a)$ and $\lim\limits_{a\to 0^+}\beta(a)$ are

(A) $-\dfrac{5}{2}$ and 1 (B) $-\dfrac{1}{2}$ and –1

(C) $-\dfrac{7}{2}$ and 2 (D) $-\dfrac{9}{2}$ and 3

4. **[JEE ADV 2011]**

If $\lim\limits_{x\to 0}(1+x\ln(1+b^2))^{\frac{1}{x}} = 2\,b\sin^2\theta$, $b > 0$ and $\theta \in (-\pi, \pi]$, then the value of θ is

(A) $\pm\dfrac{\pi}{4}$ (B) $\pm\dfrac{\pi}{3}$

(C) $\pm\dfrac{\pi}{6}$ (D) $\pm\dfrac{\pi}{2}$

Multiple Correct Choice Type Problems

5. **[JEE ADV 2017]**

Let $f(x) = \dfrac{1-x(1+|1-x|)}{|1-x|}\cos\left(\dfrac{1}{1-x}\right)$ for $x \neq 1$. Then

(A) $\lim\limits_{x\to 1^-} f(x) = 0$ (B) $\lim\limits_{x\to 1^-} f(x)$ does not exist

(C) $\lim\limits_{x\to 1^+} f(x) = 0$ (D) $\lim\limits_{x\to 1^+} f(x)$ does not exist

6. **[JEE ADV 2015]**

Let $f(x) = \sin\left(\dfrac{\pi}{6}\sin\left(\dfrac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x) = \dfrac{\pi}{2}\sin x$ for all $x \in R$. Let $(f\,o\,g)\,(x)$ denote $f(g(x))$ and $(gof)(x)$ denote $g(f(x))$. Then which of the following is (*a*re) true?

(A) Range of f is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

(B) Range of $f\,o\,g$ is $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

(C) $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = \dfrac{\pi}{6}$

(D) There is an $x \in R$ such that $(gof)\,(x) = 1$

Integer/Numerical Answer Type Questions

7. **[JEE ADV 2020]**

Let e denote the base of the natural logarithm. The value of the real number a for which the right hand limit

$\lim\limits_{x\to 0^+}\dfrac{(1-x)^{1/x} - e^{-1}}{x^a}$ is equal to a non zero real number, is ________.

8. **[JEE ADV 2020]**

The value of the limit

$$\lim_{x\to\frac{\pi}{2}}\frac{4\sqrt{2}(\sin 3x + \sin x)}{\left(2\sin 2x\sin\dfrac{3x}{2} + \cos\dfrac{5x}{2}\right) - \left(\sqrt{2} + \sqrt{2}\cos 2x + \cos\dfrac{3x}{2}\right)} \text{ is}$$

____

9. **[JEE ADV 2016]**

Let $\alpha, \beta \in R$ be such that $\lim\limits_{x\to 0}\dfrac{x^2\sin(\beta x)}{\alpha x - \sin x} = 1$ then $6(\alpha + \beta)$ equals

10. **[JEE ADV 2015]**

Let m and n be two positive integers greater than 1. If $\lim\limits_{\alpha\to 0}\left(\dfrac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = -\left(\dfrac{e}{2}\right)$ then the value of $\dfrac{m}{n}$ is

11. **[JEE ADV 2014]**

The largest value of the non-negative integer a for which

$$\lim_{x\to 1}\left\{\frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$$

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A	**2.** C	**3.** C	**4.** D	**5.** C	**6.** A	**7.** B	**8.** C	**9.** B	**10.** B
11. D	**12.** C	**13.** B	**14.** C	**15.** D	**16.** A	**17.** C	**18.** B	**19.** C	**20.** A
21. A	**22.** C	**23.** C	**24.** D	**25.** A	**26.** A	**27.** C	**28.** D	**29.** B	**30.** C
31. D	**32.** C	**33.** D	**34.** C	**35.** C	**36.** A	**37.** B	**38.** A	**39.** A	**40.** A
41. B	**42.** D	**43.** B	**44.** D	**45.** A	**46.** A	**47.** C	**48.** C	**49.** B	**50.** D
51. D	**52.** C	**53.** C	**54.** D	**55.** D	**56.** A	**57.** C	**58.** D	**59.** C	**60.** B
61. D	**62.** B	**63.** B	**64.** B	**65.** C	**66.** A	**67.** C	**68.** C	**69.** B	**70.** D
71. D	**72.** C	**73.** D	**74.** B	**75.** C	**76.** A	**77.** D	**78.** A	**79.** D	**80.** C
81. A	**82.** B	**83.** D	**84.** A	**85.** A	**86.** D	**87.** A	**88.** B	**89.** B	**90.** C
91. A	**92.** D	**93.** B	**94.** C	**95.** C	**96.** B	**97.** A	**98.** A	**99.** A	**100.** C

Multiple Correct Choice Type Questions

1. A, D	**2.** B, C	**3.** A, C	**4.** A, B, C	**5.** A, B, C, D
6. B, D	**7.** A, B	**8.** C, D	**9.** A, B	**10.** A, C
11. A, B	**12.** A, C	**13.** A, B, C	**14.** A, B	**15.** B, C
16. A, B, C	**17.** A, B, D	**18.** A, C, D	**19.** A, B, C, D	**20.** B, C
21. A, C	**22.** A, C	**23.** A, B	**24.** A, B, D	**25.** B, D
26. B, C, D	**27.** B, D			

Linked Comprehension Type Questions

1. C	**2.** B	**3.** D	**4.** C	**5.** C	**6.** B	**7.** B	**8.** C	**9.** D	**10.** A
11. B	**12.** A	**13.** D	**14.** D	**15.** B	**16.** B	**17.** A	**18.** A	**19.** C	**20.** D
21. B	**22.** C	**23.** C	**24.** B						

Matrix Match/Column Match Type Questions

1. A → (r)	B → (s)	C → (r)	D → (p)
2. A → (s)	B → (r)	C → (t)	D → (t)
3. A → (s)	B → (r)	C → (q)	D → (p)
4. A → (r)	B → (p)	C → (q)	D → (s)
5. A → (q)	B → (p)	C → (p)	D → (s)
6. A → (q)	B → (t)	C → (p)	D → (q)
7. A → (q)	B → (t)	C → (p)	D → (r)

Integer/Numerical Answer Type Questions

1. 0	**2.** 1	**3.** 2	**4.** 2	**5.** 7	**6.** 6	**7.** 6	**8.** 2	**9.** 2	**10.** 4
11. 2	**12.** 3	**13.** 9	**14.** 1	**15.** 4	**16.** 4	**17.** 7	**18.** 0	**19.** 0	**20.** 1
21. 7	**22.** 4	**23.** 1	**24.** 5	**25.** 2	**26.** 1	**27.** 2	**28.** 4	**29.** 3	**30.** 6
31. 1	**32.** 3	**33.** 3	**34.** 0	**35.** 13	**36.** 2	**37.** 19	**38.** 2	**39.** 2	**40.** 3

ARCHIVE: JEE MAIN

1. 7 **2.** 5 **3.** C **4.** D **5.** D **6.** A **7.** D **8.** D **9.** 36 **10.** 40

11. 8 **12.** 1 **13.** A **14.** B **15.** D **16.** A **17.** B **18.** C **19.** C **20.** D

21. B **22.** C **23.** D **24.** D **25.** A

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. D **2.** B **3.** B **4.** D

Multiple Correct Choice Type Questions

5. A, D **6.** A, B, C

Integer/Numerical Answer Type Questions

7. 1 **8.** 8 **9.** 7 **10.** 2 **11.** 0

CHAPTER 4

Method of Differentiation

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Differentiation by First Principle
(b) Differentiation of Standard Functions
(c) Algebra of Differentiation
(d) Derivatives of Composite Function: [Chain Rule]
(e) Differentiation of Implicit Functions
(f) Differentiation Using Logarithm
(g) Differentiation of Infinite Series
(h) Differentiation of Parametric Functions
(i) Differentiation of a Function with Respect to Another Function
(j) Differentiation of Determinants
(k) Differentiation of Inverse Function
(l) Differentiation of Inverse Trigonometric Functions
(m) Higher Order Derivatives

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

DIFFERENTIATION BY FIRST PRINCIPLE

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}$$

If $y = f(x)$ is a function, then derivative of y with respect to x is given by,

$$f'(x) = \frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

This is also known as differential coefficient.

Let try to find out few standard differentiation using definition of derivatives.

(i) $\dfrac{d(x^n)}{dx} = nx^{n-1}$

PROOF:

$$\frac{d(x^n)}{dx} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = x^n\lim_{h\to 0}\frac{\left(1+\frac{h}{x}\right)^n - 1}{h}$$

$$= x^n\lim_{h\to 0}\frac{1+\frac{nh}{x}+\ldots.-1}{h} = nx^{n-1}$$

(ii) $\dfrac{d(a^x)}{dx} = a^x\log_e a,\ a>0,\ a\neq 1$

PROOF:

$$\frac{d(a^x)}{dx} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h}$$

$$= a^x\lim_{h\to 0}\frac{a^h-1}{h} = a^x\log_e a$$

Similarly, we can prove $\dfrac{d(e^x)}{dx} = e^x$

(iii) $\dfrac{d(\log_e|x|)}{dx} = \dfrac{1}{x},\ x\neq 0$

PROOF:

If $x > 0$,

$$\frac{d(\log_e x)}{dx} = \lim_{h\to 0}\frac{\log_e(x+h)-\log_e x}{h}$$

$$= \lim_{h\to 0}\frac{\log_e\left(1+\frac{h}{x}\right)}{\frac{h}{x}\cdot x} = \frac{1}{x} \quad \text{(Using standard limit)}$$

Also, it can be proved that $\dfrac{d(\log_e(-x))}{dx} = \dfrac{1}{x},\ x<0$

$$\therefore\ \frac{d(\log_e|x|)}{dx} = \frac{1}{x},\ x\neq 0$$

(iv) $\dfrac{d(\sin x)}{dx} = \cos x$

PROOF:

$$\frac{d(\sin x)}{dx} = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{2\sin\frac{h}{2}\cos\left(\frac{2x+h}{2}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right) = \cos x$$

Similarly, we can get $\frac{d(\cos x)}{dx} = -\sin x$

(v) $\frac{d(\tan x)}{dx} = \sec^2 x$

PROOF:

$$\frac{d(\tan x)}{dx} = \lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos x\cos(x+h)}$$

$$= \lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos x\cos(x+h)}$$

$$= \lim_{h\to 0}\frac{\sin h}{h}\lim_{h\to 0}\frac{1}{\cos x\cos(x+h)}$$

$$= (1)\frac{1}{\cos x\cdot\cos x} = \sec^2 x$$

Similarly, we can prove $\frac{d(\cot x)}{dx} = -\text{cosec}^2 x$,

$\frac{d(\sec x)}{dx} = \sec x\tan x$,

$\frac{d(\text{cosec}x)}{dx} = -\text{cosec}\, x\cot x$

DIFFERENTIATION OF STANDARD FUNCTIONS

(i) $\frac{d}{dx}k = 0$, where k is constant

(ii) $\frac{d}{dx}x^n = nx^{n-1}$

(iii) $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$

(iv) $\frac{d}{dx}(\log_e(ax+b)) = \frac{a}{ax+b}$

(v) $\frac{d}{dx}(\log_\alpha(ax+b)) = \frac{a}{(ax+b)\log_e\alpha}$

(vi) $\frac{d}{dx}a^{kx+\lambda} = k.\, a^{kx+\lambda}\log_e a$

(vii) $\frac{d}{dx}e^{ax+b} = a.\, e^{ax+b}$

(viii) $\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$,

(ix) $\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$

(x) $\frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)$

(xi) $\frac{d}{dx}\cot(ax+b) = -a\,\text{cosec}^2(ax+b)$

(xii) $\frac{d}{dx}\sec(ax+b) = a\sec(ax+b)\tan(ax+b)$

(xiii) $\frac{d}{dx}\text{cosec}(ax+b) = -a\,\text{cosec}(ax+b)\cot(ax+b)$

(xiv) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

(xv) $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

(xvi) $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$

(xvii) $\frac{d}{dx}\text{cosec}^{-1}x = -\frac{1}{|x|\sqrt{x^2-1}}$

(xviii) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

(xix) $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$

ALGEBRA OF DIFFERENTIATION

There are many rules of differentiation. Let us start with elementary rules.

(i) $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}f(x)$, where k is any constant.

(ii) $\frac{d}{dx}[f_1(x) \pm f_2(x) \pm f_3(x) \dots \pm f_n(x)]$

$$= \frac{d}{dx}f_1(x) \pm \frac{d}{dx}f_2(x) \pm \frac{d}{dx}f_3(x) \dots \pm \frac{d}{dx}f_n(x)]$$

i.e., derivative is distributive over addition and subtraction.

(iii) Product Rule:

$$\frac{d}{dx}(f(x)g(x)) = f(x)\frac{d}{dx}(g(x)) + g(x)\frac{d}{dx}(f(x))$$

In general,

$$\frac{d}{dx}(f_1(x).f_2(x).f_3(x)\dots f_n(x))$$

$$= \left(\frac{d}{dx}f_1(x)\right)(f_2(x)\cdot f_3(x)\cdot \dots \cdot f_n(x)) + f_1(x)\left(\frac{d}{dx}f_2(x)\right)\cdot f_3(x)\cdot \dots \cdot f_n(x) + f_1(x)\cdot f_2(x)\left(\frac{d}{dx}f_3(x)\right)(f_4(x)\cdot \dots \cdot f_n(x)) \dots + f_1(x))\cdot f_2(x)\cdot \dots \cdot f_{n-1}(x)\left(\frac{d}{dx}f_n(x)\right)$$

(iv) Division Rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{g(x)\frac{d(f(x))}{dx}-f(x)\frac{d(g(x))}{dx}}{(g(x))^2}$$

In particular case, if $f(x)=1$, then $\frac{d}{dx}\left(\frac{1}{g(x)}\right)=\frac{-g'(x)}{(g(x))^2}$

$$\frac{d}{dx}\left(\frac{N^r}{D^r}\right)=\frac{D^r\frac{d}{dx}(N^r)-N^r\frac{d}{dx}(D^r)}{(D^r)^2},$$

where N^r is numerator and D^r is denominator.

ILLUSTRATION 1

Find differential coefficient of

$$\left(x^{\frac{l+m}{m-n}}\right)^{\frac{1}{n-l}}\cdot\left(x^{\frac{m+n}{n-l}}\right)^{\frac{1}{l-m}}\cdot\left(x^{\frac{n+l}{l-m}}\right)^{\frac{1}{m-n}}\quad \text{w.r.t. } x.$$

SOLUTION

$$\text{Let } y=\left(x^{\frac{l+m}{m-n}}\right)^{\frac{1}{n-l}}\cdot\left(x^{\frac{m+n}{n-l}}\right)^{\frac{1}{l-m}}\cdot\left(x^{\frac{n+l}{l-m}}\right)^{\frac{1}{m-n}}$$

$$=x^{\frac{l+m}{(m-n)(n-l)}+\frac{m+n}{(n-l)(l-m)}+\frac{n+l}{(l-m)(m-n)}}$$

$$=x^{\frac{l^2-m^2+m^2-n^2+n^2-l^2}{(m-n)(n-l)(l-m)}}=x^0=1$$

$$\therefore\quad \frac{dy}{dx}=0$$

ILLUSTRATION 2

If $f(x)=e^x\cdot\sin x\cdot\ln x$, find $f'(x)$.

SOLUTION

$$f'(x)=\left(\frac{d}{dx}(e^x)\right)\sin x\ln x+e^x\left(\frac{d\sin x}{dx}\right)\ln x+e^x\sin x\left(\frac{d}{dx}\ln x\right)$$

$$=e^x\sin x\ln x+e^x\cos x\ln x+e^x\sin x\left(\frac{1}{x}\right)$$

$$=e^x[(\sin x+\cos x)\ln x+\frac{\sin x}{x}]$$

ILLUSTRATION 3

If $f(x)=\frac{(x^2-3x+2)\sin x}{x\ln x}$, find $f'(x)$.

SOLUTION

$$f'(x)=\frac{(x\ln x)\frac{d}{dx}((x^2-3x+2)\sin x)-\left((x^2-3x+2)\sin x\right)\frac{d}{dx}(x\ln x)}{(x\ln x)^2}$$

$$=\frac{x\ln x\left[(2x-3)\sin x+(x^2-3x+2)\cos x\right]-(x^2-3x+2)\sin x\left[x\left(\frac{1}{x}\right)+\ln x\right]}{(x\ln x)^2}$$

$$=\frac{x(2x-3)(\ln x)\sin x+(x^2-3x+2)(x\cos x.\ln x-\sin x(1+\ln x))}{(x\ln x)^2}$$

ILLUSTRATION 4

If $f(x)=\frac{(x^4-3x^2+2)\text{cosec}x\cdot\ln x}{x^2}$, find $f'(x)$.

SOLUTION

$$f(x)=\left(x^2-3+\frac{2}{x^2}\right)\text{cosec } x\ln x$$

$$f'(x)=\left(2x-0-\frac{4}{x^3}\right)\text{cosec } x\ln x+\left(x^2-3+\frac{2}{x^2}\right)(-\text{cosec } x\cot x)\ln x+\left(x^2-3+\frac{2}{x^2}\right)\text{cosec } x\left(\frac{1}{x}\right)$$

$$=2\left(x-\frac{2}{x^3}\right)\text{cosec } x\ln x+\left(x^2-3+\frac{2}{x^2}\right)\text{cosec } x\left[-\cot x\ln x+\frac{1}{x}\right]$$

ILLUSTRATION 5

Let $h(x)$ be a differentiable function for all $x\in R$ and let $f(x)=(kx+e^x)h(x)$, where k is constant. If $h(0)=5$, $h'(0)=-2$ and $f'(0)=18$, then find the value of k.

SOLUTION

$$f(x)=(kx+e^x)h(x)$$

$$f'(x)=(kx+e^x)\,h'(x)+h(x)\,(k+e^x)$$

$$\Rightarrow\quad f'(0)=h'(0)+h(0)\,(k+1)$$

$$\Rightarrow\quad 18=-2+5(k+1)\Rightarrow k=3.$$

ILLUSTRATION 6

Let f and g be differentiable function on R. If $f(6)=3$, $f'(6)=6$, $g(6)=1$ and $g'(6)=4$, then find $\left(\frac{fg}{f-g}\right)'(6)$.

SOLUTION

$$\left(\frac{fg}{f-g}\right)'=\frac{(f-g)(fg)'-fg(f-g)'}{(f-g)^2}$$

$$\Rightarrow\quad \left(\frac{fg}{f-g}\right)'=\frac{(f-g)(f'g+fg')-fg(f'-g')}{(f-g)^2}$$

$$\Rightarrow \quad \left(\frac{fg}{f-g}\right)'(6) = \frac{(f(6)-g(6))(f'(6)g(6)+f(6)g'(6)) - f(6)g(6)(f'(6)-g'(6))}{(f(6)-g(6))^2}$$

$$\therefore \quad \left(\frac{fg}{f-g}\right)'(6) = \frac{2(6+12)-3(2)}{2^2} = \frac{15}{2}$$

ILLUSTRATION 7

If $f(x) = 1 + \cos(x) + \cos^2(x) + \cos^3(x) + \ldots \infty$, then find the value of $f'\left(\frac{\pi}{3}\right)$.

SOLUTION

$$f(x) = 1 + \cos(x) + \cos^2(x) + \cos^3(x) + \ldots \infty$$

$$\because \quad a + ar + ar^2 + \ldots,\ |r| < 1, \text{ Sum of given series} = \frac{a}{1-r}$$

$$f(x) = \frac{1}{1-\cos x} \text{ and } f'(x) = \frac{-\sin x}{(1-\cos x)^2}$$

$$f'\left(\frac{\pi}{3}\right) = \frac{-\sin\frac{\pi}{3}}{\left(1-\cos\frac{\pi}{3}\right)^2} = \frac{\frac{-\sqrt{3}}{2}}{\left(1-\frac{1}{2}\right)^2} = -2\sqrt{3}$$

ILLUSTRATION 8

If the value of y is given as:

$$y = 1 + \frac{C_1}{x-C_1} + \frac{C_2 x}{(x-C_1)(x-C_2)} + \frac{C_3 x^2}{(x-C_1)(x-C_2)(x-C_3)},$$

Show that: $\dfrac{dy}{dx} = \dfrac{y}{x}\left[\dfrac{C_1}{C_1-x} + \dfrac{C_2}{C_2-x} + \dfrac{C_3}{C_3-x}\right]$

SOLUTION

$$y = \left(1 + \frac{C_1}{x-C_1}\right) + \frac{C_2 x}{(x-C_1)(x-C_2)} + \frac{C_3 x^2}{(x-C_1)(x-C_2)(x-C_3)}$$

$$= \left[\frac{x}{x-C_1} + \frac{C_2 x}{(x-C_1)(x-C_2)}\right] + \frac{C_3 x^2}{(x-C_1)(x-C_2)(x-C_3)}$$

$$= \frac{x^2}{(x-C_1)(x-C_2)} + \frac{C_3 x^2}{(x-C_1)(x-C_2)(x-C_3)}$$

$$= \frac{x^3}{(x-C_1)(x-C_2)(x-C_3)}$$

$$= \frac{x}{x-C_1}\cdot\frac{x}{x-C_2}\cdot\frac{x}{x-C_3}$$

$$\therefore \quad \log y = \log\left(\frac{x}{x-C_1}\right) + \log\left(\frac{x}{x-C_2}\right) + \log\left(\frac{x}{x-C_3}\right)$$

Differentiating w.r.t. x, we get,

$$\frac{1}{y}\frac{dy}{dx} = \frac{x-C_1}{x}\cdot\frac{(x-C_1-x)}{(x-C_1)^2} + \frac{x-C_2}{x}\cdot\frac{(x-C_2-x)}{(x-C_2)^2} + \frac{x-C_3}{x}\cdot\frac{(x-C_3-x)}{(x-C_3)^2}$$

$$= -\frac{1}{x}\left[\frac{C_1}{x-C_1} + \frac{C_2}{x-C_2} + \frac{C_3}{x-C_3}\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x}\left[\frac{C_1}{C_1-x} + \frac{C_2}{C_2-x} + \frac{C_3}{C_3-x}\right]$$

ILLUSTRATION 9

If $8f(x) + 6f\left(\frac{1}{x}\right) = x + 5$ and $y = x^2 f(x)$,

then find $\dfrac{dy}{dx}$ at $x = -1$.

SOLUTION

We have,

$$8f(x) + 6f\left(\frac{1}{x}\right) = x + 5 \qquad \ldots(1)$$

Replace x by $\frac{1}{x}$ in Eq. (1), we get

$$8f\left(\frac{1}{x}\right) + 6f(x) = \frac{1}{x} + 5 \qquad \ldots(2)$$

On solving eqs. (1) and (2), we get

$$f(x) = \frac{1}{28}\left(8x - \frac{6}{x} + 10\right)$$

Now,

$$y = x^2 f(x) = \frac{x^2}{28}\left(8x - \frac{6}{x} + 10\right) = \frac{1}{28}(8x^3 - 6x + 10x^2)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{28}(24x^2 - 6 + 20x)$$

$$\therefore \quad \left(\frac{dy}{dx}\right)_{x=-1} = \frac{1}{28}(24 - 6 - 20) = -\frac{1}{14}$$

DERIVATIVES OF COMPOSITE FUNCTION: [CHAIN RULE]

The Chain rule is used for differentiation of composite function of two or more functions.
The rule is as follows:

(i) If f and g are two real valued functions, then $(f(g(x))' = f'(g(x)) \times g'(x))$

(ii) If f, g and h are real valued functions, then $y' = (f(g(h(x))))' = f'(g(h(x))) \times g'(h(x)) \times h'(x)$

As we can see that there are three functions f, g and h are present, for differentiation outer layered function f is differentiated first then we differentiate second layer g and in last we differentiate last function h.
In other words, if y is a function of t and t is a function of x, then $\dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx}$.

If y be function of z, z be function of u, and u be function of v and v be function of x, then

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

E.g. $\frac{d(\cos e^x)}{dx}=\frac{d(\cos e^x)}{de^x}\frac{d(e^x)}{dx}$

$$=-\sin(e^x)\cdot\frac{d(e^x)}{dx}=(-\sin(e^x))e^x=-e^x\sin e^x$$

E.g. $\frac{d(\sqrt{\sin x})}{dx}=\frac{d\left((\sin x)^{\frac{1}{2}}\right)}{dx}$

$$=\frac{1}{2}\sin x^{\frac{1}{2}-1}.\frac{d(\sin x)}{dx}=\frac{1}{2\sqrt{\sin x}}.\cos x$$

ILLUSTRATION 10

Find derivative of $\log_e\left(\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}\right)$ w.r.t. x.

SOLUTION

$$f(x)=\log_e\left(\frac{x+\sqrt{x^2-a^2}}{x-\sqrt{x^2-a^2}}\right)$$

$$\Rightarrow\quad f(x)=\log_e(x+\sqrt{x^2-a^2})-\log(x-\sqrt{x^2-a^2})$$

$$\left[\because \log\frac{a}{b}=\log a-\log b\right]$$

$$\therefore\quad f'(x)=\left(\frac{1}{x+\sqrt{x^2-a^2}}\right)\left(1+\frac{2x}{2\sqrt{x^2-a^2}}\right)-\left[\left(\frac{1}{x-\sqrt{x^2-a^2}}\right)\left(1-\frac{2x}{2\sqrt{x^2-a^2}}\right)\right]$$

$$=\frac{1}{\sqrt{x^2-a^2}}+\frac{1}{\sqrt{x^2-a^2}}=\frac{2}{\sqrt{x^2-a^2}}$$

ILLUSTRATION 11

Find derivative of $f(x)=\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}$ w.r.t. x.

SOLUTION

$$\therefore\quad f'(x)=\frac{d\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}}{d\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}\;\frac{d\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}{d\sin\left(\frac{x^2}{3}-1\right)}\;\frac{d\sin\left(\frac{x^2}{3}-1\right)}{d\left(\frac{x^2}{3}-1\right)}\;\frac{d\left(\frac{x^2}{3}-1\right)}{dx}$$

$$=\frac{1}{2\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}}\times\frac{1}{\sin\left(\frac{x^2}{3}-1\right)}\times$$

$$\cos\left(\frac{x^2}{3}-1\right)\times\frac{2x}{3}=\frac{x\cot\left(\frac{x^2}{3}-1\right)}{3\sqrt{\log\left\{\sin\left(\frac{x^2}{3}-1\right)\right\}}}$$

ILLUSTRATION 12

If $y(x)=\sec(\ln x)+\ln(\sec x)$, then find $y'(1)$.

SOLUTION

$$y(x)=\sec(\ln x)+\ln(\sec x) \qquad \ldots(1)$$

$\therefore$ On differentiating $b.t.s$ of equation (1) w.r.t. x, we get

$$y'(x)=\frac{1}{x}\sec(\ln x)\,.\,\tan(\ln x)+\frac{\sec x.\tan x}{\sec x}$$

$$y'(x)=\frac{\sec(\ln x)\tan(\ln x)}{x}+\tan x$$

$$y'(1)=\frac{\sec(\ln 1)\tan(\ln 1)}{1}+\tan 1$$

$$y'(1)=\sec(0)\tan(0)+\tan 1$$

Hence, $y'(1)=\tan 1$

ILLUSTRATION 13

Let f and g be differentiable functions on r. Suppose $F(x)=f(g(x))$ and $g(2)=6$, $g'(2)=3$, $f'(2)=-1$, $f(2)=7$, $f'(6)=-4$. Find the value of $F'(2)$.

SOLUTION

$$F(x)=f(g(x))$$

$$F'(x)=f'(g(x))\cdot g'(x)$$

$$\therefore\quad F'(2)=f'(g(2))\cdot g'(2)=f'(6).(3)=(-4)(3)=-12$$

ILLUSTRATION 14

Let $f:(-2,2)\to R$ be a differentiable function such that $f(0)=-1$ and $f'(0)=1$. If $g(x)=(f(2f(x)+2))^2$ then find $g'(0)$.

SOLUTION

Given, $g(x)=(f(2f(x)+2))^2$

On differentiating w.r.t. x, we get

$$\therefore\quad g'(x)=2f(2f(x)+2)\times f'(2f(x)+2)\cdot 2f'(x)$$

So, $g'(0)=2f(2+2f(0))\times f'(2+2f(0))\times 2f'(0)\Rightarrow g'(0)=-4$

ILLUSTRATION 15

Let f and g be differentiable functions on R (the set of all real numbers) such that $g(1)=2=g'(1)$ and $f'(0)=4$.

If $h(x)=f(2x\,g(x)+\cos(\pi x)-3)$, then find $h'(1)$.

SOLUTION

We have, $h(x) = f(2x\,g(x) + \cos(\pi x) - 3)$
On differentiating w.r.t. x, we get

$$h'(x) = f'(2x \times g(x) + \cos(\pi x) - 3) \times (2x \times g'(x) + 2g(x) - \pi \sin \pi x)$$

$$\Rightarrow \quad h'(1) = f'(2g(1) - 4) \times (2g'(1) + 2g(1))$$

$$= f'(0) \times (2 \times 2 + 2 \times 2)$$

$$= 8f'(0) = 8 \times 4 = 32.$$

ILLUSTRATION 16

Let $g(x) = x^3 \ln(x^2 f(x))$, where $f(x)$ is a differentiable positive function on $(0, \infty)$ satisfying $f(2) = \dfrac{1}{4}$ and $f'(2) = -3$, then find $g'(2)$.

SOLUTION

Given, $g(x) = x^3 \ln(x^2 f(x))$
On differentiating both sides with respect to x, we get

$$g'(x) = \frac{x^3}{x^2 f(x)}(x^2 f'(x) + 2xf(x)) + 3x^2 \ln(x^2 f(x))$$

$$\therefore \quad g'(2) = \frac{2}{f(2)}(4f'(2) + 4f(2)) + 12\ln(4f(2))$$

$$= 8(-12 + 1) + 0 = -88$$

ILLUSTRATION 17

For a function $f(x) = \ln(1 + \sqrt{1-x^2}\,) - (\sqrt{1-x^2} + \ln x)$, where $0 < x < 1$, Find the value of $f'(x)$.

SOLUTION

$$f(x) = \ln(1 + \sqrt{1-x^2}\,) - (\sqrt{1-x^2} + \ln x)$$

On differentiating with respect to x, we get

$$f'(x) = \frac{\dfrac{-(2x)}{2\sqrt{1-x^2}}}{1+\sqrt{1-x^2}} + \frac{2x}{2\sqrt{1-x^2}} - \frac{1}{x}$$

$$= \frac{-x}{\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)} + \frac{x}{\sqrt{1-x^2}} - \frac{1}{x}$$

$$= \frac{x}{\sqrt{1-x^2}}\left(1 - \frac{1}{1+\sqrt{1-x^2}}\right) - \frac{1}{x}$$

$$= \frac{x\sqrt{1-x^2}}{\sqrt{1-x^2}\left(1+\sqrt{1-x^2}\right)} - \frac{1}{x}$$

$$= \frac{x}{1+\sqrt{1-x^2}} - \frac{1}{x} = \frac{x\left(1-\sqrt{1-x^2}\right)}{1-(1-x^2)} - \frac{1}{x}$$

$$= \frac{1-\sqrt{1-x^2}}{x} - \frac{1}{x} = -\frac{\sqrt{1-x^2}}{x}$$

ILLUSTRATION 18

Let $f(x) = \dfrac{x-1}{4} + \dfrac{(x-1)^3}{12} + \dfrac{(x-1)^5}{20} + \dfrac{(x-1)^7}{28} \ldots \infty$

for $x \in (0, 2)$, then find $f'\left(\dfrac{3}{2}\right)$.

SOLUTION

$$f(x) = \frac{x-1}{4} + \frac{(x-1)^3}{12} + \frac{(x-1)^5}{20} + \frac{(x-1)^7}{28} \ldots \infty$$

$$f(x) = \frac{1}{4}\left((x-1) + \frac{(x-1)^3}{3} + \frac{(x-1)^5}{5} + \frac{(x-1)^7}{7} \ldots\right)$$

$$f'(x) = \frac{1}{4}\left(1 + (x-1)^2 + (x-1)^4 + (x-1)^6 \ldots\right)$$

$$= \frac{1}{4(1-(x-1)^2)}$$

$$\therefore \quad f'(x) = \frac{1}{4x(2-x)}$$

Hence, $f'\left(\dfrac{3}{2}\right) = \dfrac{1}{3}$

ILLUSTRATION 19

If $f(x) = \sin^{-1}(\cos x)$, then find the value of $f(10) + f'(10)$.

SOLUTION

We have,

$$f(10) = \sin^{-1}(\cos 10) = \frac{\pi}{2} - \cos^{-1}\cos(10)$$

$$= \frac{\pi}{2} - (4\pi - 10) = 10 - \frac{7\pi}{2}$$

Also, $f'(x) = \dfrac{-\sin x}{\sqrt{1-\cos^2 x}} = \dfrac{-\sin x}{|\sin x|}$

$$\Rightarrow \quad f'(10) = \frac{-\sin 10}{|\sin 10|} = 1$$

Hence, $f(10) + f'(10) = \left(11 - \dfrac{7\pi}{2}\right)$

ILLUSTRATION 20

If $f(x) = \sin^{-1}\{[3x + 2] - \{3x + (x - \{2x\})\}\}$, $x \in \left(0, \dfrac{\pi}{2}\right)$ and $gof(x) = x \ \forall\ x \in \left(0, \dfrac{\pi}{12}\right)$. Then, find $g'\left(\dfrac{\pi}{6}\right)$. Where $[\cdot]$ and $\{\cdot\}$ denotes greatest integer function and fractional part function.

SOLUTION

$$f(x) = \sin^{-1}\{[3x + 2] - \{3x + (x - \{2x\})\}\}$$

$$\therefore \quad x \in \left(0, \frac{\pi}{12}\right) \Rightarrow 2x \in \left(0, \frac{\pi}{6}\right)$$

$$\Rightarrow \quad \{2x\} = 2x \text{ and } 3x \in \left(0, \frac{\pi}{4}\right)$$

$$f(x) = \sin^{-1}\{2 - \{3x - x\}\} = \sin^{-1}\{2 - 2x\}$$

(As $\{2 - 2x\} = 1 - 2x$)

$$\therefore \quad f(x) = \sin^{-1}(1 - 2x) = y \Rightarrow 1 - 2x = \sin y$$

$$\Rightarrow \quad x = \frac{1 - \sin y}{2} = f^{-1}(y)$$

$$\therefore \quad f^{-1}(x) = \frac{1 - \sin x}{2} = g(x) \Rightarrow g'(x) = \frac{-1}{2}\cos x$$

$$\text{Hence, } g'\left(\frac{\pi}{6}\right) = \frac{-1}{2}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{4}$$

ILLUSTRATION 21

Let $f(x)$ be a differentiable function such that $f(x) + 2f(-x) = \sin x$ for all $x \in R$. Find the value of $f'\left(\frac{\pi}{4}\right)$.

SOLUTION

$$f(x) + 2f(-x) = \sin x \qquad \ldots(1)$$

Put $x \to -x$

$$f(-x) + 2f(x) = -\sin x \qquad \ldots(2)$$

On solving equation (1) and (2)

$$f(x) = -\sin x \qquad \ldots(1)$$

On differentiation,

$$f'(x) = -\cos x \qquad \ldots(1)$$

$$\text{Hence, } f'\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$$

ILLUSTRATION 22

If $f(x) = \sin\left\{\frac{\pi}{3}[x] - x^2\right\}$ for $2 < x < 3$ and $[x]$ denotes the greatest integer less than or equal to x, then find $f'\left(\sqrt{\frac{\pi}{3}}\right)$.

SOLUTION

$$f(x) = \sin\left\{\frac{\pi}{3}[x] - x^2\right\} \text{ for } 2 < x < 3, \text{ we have } [x] = 2$$

$$\therefore \quad f(x) = \sin\left(\frac{2\pi}{3} - x^2\right) \Rightarrow f'(x) = -2x\cos\left(\frac{2\pi}{3} - x^2\right)$$

$$\Rightarrow \quad f'\left(\sqrt{\frac{\pi}{3}}\right) = -2\sqrt{\frac{\pi}{3}}\cos\frac{\pi}{3} = -\sqrt{\frac{\pi}{3}}$$

Test Your Concepts-I

(Solutions on page H.82)

Differentiate the following w.r.t. x.

1. $f(x) = (x^2 - 4x + 5)(x^3 - 2)$

2. $f(x) = (3x + 8)^{7/3} \cdot (x + 7)^{-3}$

3. $f(x) = \dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}$

4. $f(x) = \dfrac{(x^2 + 1)(x + 3)}{x}$

5. $f(x) = \sqrt{a + \sqrt{a + x}}$

6. $f(x) = \sqrt{4 + \sqrt{4 + \sqrt{4 + x^2}}}$

7. $f(x) = \dfrac{\sqrt{a + x} - \sqrt{a - x}}{\sqrt{a + x} + \sqrt{a - x}}$

8. $f(x) = \log(x + \sqrt{x^2 + a^2})$

9. $f(x) = 10^{10^x} + \log\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$

10. $f(x) = e^{ax} \cdot \cot(bx + c) + \log_{\sin x}\cos x$

11. $f(x) = \log_e(\sec x + \tan x) + \log_e(\sin\sqrt{x^2 + 1})$

Answer key

1. $5x^4 - 16x^3 + 15x^2 - 4x + 8$

2. $(25 - 2x)(3x + 8)^{4/3}(x + 7)^{-4}$

3. $\dfrac{-1}{\sqrt{x}(1 + \sqrt{x})^2}$

4. $\dfrac{2x^3 + 3x^2 - 3}{x^2}$

5. $\dfrac{1}{4(\sqrt{a + \sqrt{a + x}})(\sqrt{a + x})}$

6. $\dfrac{x}{4\sqrt{4 + \sqrt{4 + \sqrt{4 + x^2}}}\sqrt{4 + \sqrt{4 + x^2}}\sqrt{4 + x^2}}$

7. $\dfrac{2a}{(\sqrt{a + x} + \sqrt{a - x})^2\sqrt{a^2 - x^2}}$

8. $\dfrac{1}{\sqrt{x^2 + a^2}}$

9. $10^{10^x} \cdot 10^x(\log 10)^2 + \sec x$

10. $e^{ax}[a\cot(bx + c) - b\,\mathrm{cosec}^2(bx + c)] + \dfrac{(\log\sin x)(-\tan x) - (\log\cos x)(\cot x)}{(\log\sin x)^2}$

11. $\sec x + \dfrac{x\cot\sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}$

MOD MAR JAAEGA,PAR NEGATIVE NAHI HOGA

DIFFERENTIATION OF IMPLICIT FUNCTIONS

If the variable x and y are connected by a relation of the form $f(x, y) = 0$ and it is not possible or convenient to express y as a function of x in the form $y = f(x)$, then y is said to be an implicit function of x. To find $\frac{dy}{dx}$ in such case, we differentiate both sides of the given relation with respect to x using rules of differentiation then solve the expression for the term $\frac{dy}{dx}$.

ILLUSTRATION 23

If $y = y(x)$ follows the relation $x^2y + 3xy^3 = x - y + 3$, then find $y'(0)$.

SOLUTION

Given, $x^2y + 3xy^3 = x - y + 3$ …(1)

Put $x = 0$, $0 = -y + 3 \Rightarrow y = 3$

So, point (0, 3) lies on curve.

Now, differentiate both sides of equation (1) with respect to x, we get

$$x^2 \cdot \frac{dy}{dx} + 2xy + 3x(3y^2)\frac{dy}{dx} + 3y^3 \cdot 1 = 1 - \frac{dy}{dx}$$

Put $x = 0$, $y = 3$, we get

$$0 + 0 + 0 + 81 = 1 - \frac{dy}{dx}$$

$$\therefore \quad \left.\frac{dy}{dx}\right]_{P(0,3)} = y'(0) = 1 - 81 = -80.$$

ILLUSTRATION 24

If $y^3 + x^3 + 3xy = 15$, then find $\frac{dy}{dx}$ at (1, 2).

SOLUTION

$$y^3 + x^3 + 3xy = 15$$

On differentiation w.r. to x, we get

$$3y^2y' + 3x^2 + 3(xy' + y) = 0$$

$$\Rightarrow \quad y^2y' + x^2 + xy' + y = 0$$

At point (1, 2), put $x = 1$, $y = 2$, we get

$$4y' + 1 + y' + 2 = 0$$

$$\Rightarrow \quad 5y' = -3 \Rightarrow y'\left(\frac{dy}{dx}\right)_{(1,2)} = -\frac{3}{5}$$

ILLUSTRATION 25

If $\ln((e - 1)e^{xy} + x^2) = x^2 + y^2$, then find $\left(\frac{dy}{dx}\right)_{(1,0)}$.

SOLUTION

$$\ln((e - 1)e^{xy} + x^2) = x^2 + y^2$$

$$\Rightarrow \quad (e - 1)e^{xy} + x^2 = e^{x^2+y^2}$$

$$\Rightarrow \quad (e - 1)e^{xy}[xy' + y] + 2x = (2x + 2yy')\, e^{x^2+y^2}$$

Put $x = 1$, $y = 0$, we obtain

$$(e - 1)\, e^{0}(y' + 0) + 2 = (2 + 0)\, e$$

$$\Rightarrow \quad (e - 1)y' = 2(e - 1)$$

Hence, $y' = \left(\frac{dy}{dx}\right)_{(1,0)} = 2$

ILLUSTRATION 26

For the curve represented implicitly as $4^{2x} - 5^{2y} = 1$, then find the value of $\lim\limits_{x\to1}\left(\frac{dy}{dx}\right)$.

SOLUTION

Given, $16^x - 25^y = 1$

$$\Rightarrow \quad y\ln 25 = \ln(16^x - 1) \Rightarrow \frac{dy}{dx} \cdot \ln 25 = \frac{16^x \ln 16}{(16^x - 1)}$$

$$\therefore \quad \frac{dy}{dx} = \left(\frac{\ln 4}{\ln 5}\right)\left(\frac{16^x}{16^x - 1}\right)$$

$$\Rightarrow \quad \lim_{x\to1}\left(\frac{dy}{dx}\right) = \frac{\ln 4}{\ln 5} \cdot \left(\frac{16}{15}\right)$$

$$\Rightarrow \quad \lim_{x\to1}\left(\frac{dy}{dx}\right) = (\log_5 4)\left(\frac{16}{15}\right)$$

ILLUSTRATION 27

If $y = y(x)$ and its follows the relation $2e^{xy^2} + y\cos(x^2) = 4$, then find $y'(0)$.

SOLUTION

We have, $2e^{xy^2} + y\cos(x^2) = 4$ …(1)

Put $x = 0$, $2 + y = 4 \Rightarrow y = 2$

$\Rightarrow$ (0, 2) lies on the given curve.

Now, on differentiating (1) w.r.t. x, we get

$$2e^{xy^2}\left(2xy\frac{dy}{dx} + y^2\right) - 2xy\sin(x^2) + \cos(x^2)\frac{dy}{dx} = 0$$

As (0, 2) satisfy it, we get

$$2(0 + 4) - 0 + \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right]_{(0,2)} = y'(0) = -8$$

ILLUSTRATION 28

If $\ln(3\sin x - 4\cos x + 7 + 5y) = (\sin^2 x)y$, then find $y'(\pi)$.

SOLUTION

We have, $\ln(3\sin x - 4\cos x + 7 + 5y) = (\sin^2 x)y$, …(1)

Put $x = \pi$, we obtain

$$\ln(11 + 5y) = 0 \Rightarrow 11 + 5y = 1 \Rightarrow 5y = -10$$

$\therefore$ $y = -2$. So, $(\pi, -2)$ lies on the given curve.

Now, on differentiating equation (1) w.r.t. x, we get

$$\frac{1}{(3\sin x-4\cos x+7+5y)}\left(3\cos x+4\sin x+5\frac{dy}{dx}\right)$$

$$=(\sin^2 x)\frac{dy}{dx}+(\sin 2x)y$$

As $(\pi, -2)$ satisfy it, we get $1\cdot\left(-3+0+5\frac{dy}{dx}\right)=0$

Hence, $\left.\frac{dy}{dx}\right]_{(\pi,-2)}=y'(\pi)=\frac{3}{5}$

ILLUSTRATION 29

If $\ln(\ln x - \ln y) = e^{x^2 y}(1 - \ln x)$, then find $y'(e)$.

SOLUTION

$$\ln(\ln x-\ln y)=e^{x^2y}(1-\ln x) \qquad \ldots(1)$$

Put $x = e$, we get $\ln(\ln e - \ln y) = e^{e^2 y}(1 - \ln e) = 0$

$\Rightarrow \quad \ln(1 - \ln y) = 0 \Rightarrow 1 - \ln y = 1 \Rightarrow y = 1$

Here, Point $(e, 1)$ lies on curve.

Differentiate equation (1) and put $(e, 1)$, we get

$$\frac{\frac{1}{x}-\frac{1}{y}\frac{dy}{dx}}{\ln x-\ln y}=\frac{-e^{x^2y}}{x}+(1-\ln x)\ e^{x^2y}\ (x^2y'+y(2x))$$

Now, put $x = e$, $y = 1$, we obtain

$$\frac{1}{e}-\frac{dy}{dx}=-\frac{e^{e^2}}{e}+0$$

Hence, $\frac{dy}{dx}=y'(e)=\frac{1+e^{e^2}}{e}$

DIFFERENTIATION USING LOGARITHM

It is convenient to take the logarithm of the function first and then differentiate. This approach is known as derivative using the logarithmic function.

Profile (i) $y=[f(x)]^{g(x)}$

By taking log both side

$$\log_e y = g(x)\log_e(f(x))$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=(g(x))'\log_e(f(x))+g(x)\frac{(f(x))'}{f(x)}$$

$$\frac{dy}{dx}=[f(x)]^{g(x)}\ [(g(x))'\log_e(f(x))+g(x)\frac{(f(x))'}{f(x)}]$$

Note

$y=[f(x)]^{g(x)}$

$\frac{dy}{dx}$ = Differentiation of y, assuming $f(x)$ as constant + Differentiation of y, assuming $g(x)$ as constant

Profile (ii) $y=\frac{f_1(x)\cdot f_2(x)\cdot f_3(x)\ldots}{g_1(x)\cdot g_2(x)\cdot g_3(x)\ldots}$

By taking log both side

$$\log_e y = [\log_e(f_1(x)) + \log_e(f_2(x)) + \log_e(f_3(x)) + \ldots] - [\log_e(g_1(x)) + \log_e(g_2(x)) + \log_e(g_3(x)) + \ldots]$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx}=\left[\frac{(f_1(x))'}{f_1(x)}+\frac{(f_2(x))'}{f_2(x)}+\frac{(f_3(x))'}{f_3(x)}+\ldots\right]-\left[\frac{(g_1(x))'}{g_1(x)}+\frac{(g_2(x))'}{g_2(x)}+\frac{(g_3(x))'}{g_3(x)}+\ldots\right]$$

ILLUSTRATION 30

Find the derivative of y w.r.t. x, where $y=\log\left[(\cos x)^x+e^{-x^2}\right]$

SOLUTION

$$y=\log\left[(\cos x)^x+e^{-x^2}\right]$$

Let $u = (\cos x)^x$ and $v = e^{-x^2}$ Then $y = \log(u + v)$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx}=\frac{1}{u+v}\left[\frac{du}{dx}+\frac{dv}{dx}\right] \qquad \ldots(1)$$

We have, $u=(\cos x)^x=e^{\log(\cos x)^x}=e^{x\log\cos x}$

$$\Rightarrow \quad \frac{du}{dx}=e^{x\log\cos x}\left[\log\cos x+x\left(\frac{-\sin x}{\cos x}\right)\right]$$

$$\Rightarrow \quad \frac{du}{dx}=u\,[\log(\cos x)-x\tan x]$$

$$=(\cos x)^x\,[\log(\cos x)-x\tan x] \qquad \ldots(2)$$

For the second function, we have $v=e^{-x^2}$

On differentiating w.r.t. x, we get

$$\frac{dv}{dx}=e^{-x^2}(-2x)=-2x\,e^{-x^2} \qquad \ldots(3)$$

Substituting (2) and (3) in (1), we obtain the required result.

$$\frac{dy}{dx}=\frac{1}{u+v}\left[\frac{du}{dx}+\frac{dv}{dx}\right]$$

$$=\frac{1}{(\cos x)^x+e^{-x^2}}[(\cos x)^x[\log\ (\cos\ x)-x\tan x]-2xe^{-x^2}]$$

ILLUSTRATION 31

If $y=e^x\sin x^3.+(\tan x)^x$, find $\frac{dy}{dx}$.

SOLUTION

Let $u=e^x\sin x^3$ and $v=(\tan x)^x$

Now, $u=e^x\sin x^3$

On differentiating w.r.t. x, we get

$$\frac{du}{dx}=e^x.\frac{d\{\sin(x)^3\}}{dx}+\sin(x^3)\frac{d}{dx}(e^x)$$

$$=e^x.\cos x^3.3x^2+\sin x^3.e^x$$

Hence, $\frac{du}{dx} = 3x^2 e^x \cos x^3 + e^x \sin x^3$

And, $v = (\tan x)^x$

$\Rightarrow \log v = x \log(\tan x)$

On differentiating w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = 1.\log(\tan x) + x.\frac{1}{\tan x}\sec^2 x$$

$$\Rightarrow \quad \frac{dv}{dx} = v[\log(\tan x) + x\cot x.\sec^2 x]$$

$$= (\tan x)^x[\log(\tan x) + x\cot x \sec^2 x]$$

Now, $y = u + v$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= 3x^2 e^x \cos(x^3) + e^x \sin(x^3) + (\tan x)^x[\log(\tan x) + x\cot x \sec^2 x]$$

ILLUSTRATION 32

Let $f(x) = (x - 1)\ (x - 2)\ (x - 3)\ \ldots\ (x - n)$, $n \in N$ and $f'(n) = 720$, then find the value of n.

SOLUTION

$$f(x) = (x - 1)\ (x - 2)\ (x - 3)\ \ldots\ (x - n)$$

$$\Rightarrow \quad \ln f(x) = \ln\ (x - 1) + \ln\ (x - 2) + \ldots + \ln\ (x - n)$$

On differentiating both sides, we get

$$\frac{f'(x)}{f(x)} = \left(\frac{1}{x-1} + \frac{1}{x-2} + \ldots + \frac{1}{x-n}\right)$$

$$\Rightarrow \quad f'(x) = (x-1)(x-2)\ldots(x-n)\left(\frac{1}{x-1} + \frac{1}{x-2} + \ldots + \frac{1}{x-n}\right)$$

$$= (x - 2)\ (x - 3)\ \ldots\ (x - n) + (x - 1)\ (x - 3)\ (x - 4)\ \ldots\ (x - n) + \ldots + (x - 1)\ (x - 2)\ \ldots\ (x - (n - 1))$$

$$\Rightarrow \quad f'(n) = (n - 1)\ (n - 2)\ \ldots\ 2 \times 1 = (n - 1)!$$

$$\Rightarrow \quad (n - 1)! = 720 = 6! \Rightarrow n - 1 = 6 \Rightarrow n = 7.$$

ILLUSTRATION 33

Let $f(x) = \lim\limits_{h\to 0} \frac{(\sin(x+h))^{\ln(x+h)} - (\sin x)^{\ln x}}{h}$, then find $f\left(\frac{\pi}{2}\right)$.

SOLUTION

Let $g(x) = (\sin\ x)^{\ln x} = e^{\ln x \times \ln(\sin\ x)}$

$$g'(x) = e^{\ln x\ (\ln\ \sin\ x)}\left[\ln x\left(\frac{\cos x}{\sin x}\right) + \ln\sin x.\left(\frac{1}{x}\right)\right]$$

$$f(x) = \lim_{h\to 0}\frac{(\sin(x+h))^{\ln(x+h)} - (\sin x)^{\ln x}}{h}$$

$$= \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = g'(x)$$

$$= (\sin\ x)^{\ln x}\left[\ln x.(\cot x) + \frac{\ln(\sin x)}{x}\right]$$

$$f\left(\frac{\pi}{2}\right) = \left(\sin\frac{\pi}{2}\right)^{\ln\frac{\pi}{2}}\left[\ln\frac{\pi}{2}\cot\frac{\pi}{2} + \frac{\ln\sin\frac{\pi}{2}}{\frac{\pi}{2}}\right]$$

$$\Rightarrow \quad f\left(\frac{\pi}{2}\right) = 1[0 + 0] \Rightarrow f\left(\frac{\pi}{2}\right) = 0$$

ILLUSTRATION 34

If $f(x) = \frac{(x+1)^7\sqrt{1+x^2}}{(x^2-x+1)^6}$, then find the value of $f'(0)$.

SOLUTION

$$f(x) = \frac{(x+1)^7\sqrt{1+x^2}}{(x^2-x+1)^6}$$

Take log both side, we get

$$\log_e f(x) = 7\log_e(x + 1) + \frac{1}{2}\log_e(1 + x^2) - 6\log_e(x^2 - x + 1)$$

$$\Rightarrow \quad \frac{1}{f(x)} \times f'(x) = \frac{7}{x+1} + \frac{x}{x^2+1} - \frac{6(2x-1)}{x^2-x+1}$$

$$\Rightarrow \quad f'(x) = \frac{(x+1)^7\sqrt{1+x^2}}{(x^2-x+1)^6}\left[\frac{7}{x+1} + \frac{x}{x^2+1} - \frac{(12x-6)}{x^2-x+1}\right]$$

$$\Rightarrow \quad f'(0) = \left[7 + 0 - \left(\frac{-6}{1}\right)\right]$$

Hence, $f'(0) = 13$

ILLUSTRATION 35

If $f(x) = \prod\limits_{n=1}^{100}(x-n)^{n(101-n)}$, then find $\frac{f'(101)}{f(101)}$.

SOLUTION

We have,

$$f(x) = \prod_{n=1}^{100}(x-n)^{n(101-n)}$$

$$\Rightarrow \quad f(x) = (x - 1)^{100}\ .\ (x - 2)^{2\times 99}\ .(x - 3)^{3\times 98}\ \ldots\ (x - 100)^{100}$$

On taking log both sides, we get

$$\log f(x) = 100\log(x - 1) + 2 \times 99\log\ (x - 2) + 3 \times 98\log\ (x - 3) + \ldots + 100\log\ (x - 100)$$

On differentiating w.r.t. x, we get

$$\frac{1}{f(x)}f'(x) = \frac{100}{x-1} + \frac{2\times 99}{x-2} + \frac{3\times 98}{x-3} + \ldots + \frac{100}{x-100}$$

$$\frac{f'(101)}{f(101)} = \frac{100}{100} + \frac{2\times 99}{99} + \frac{3\times 98}{98} + \ldots + \frac{100}{1}$$

$$= 1 + 2 + 3 + 4 + \ldots. + 100$$

Hence, $\frac{f'(101)}{f(101)} = 5050$

Test Your Concepts-II

(Solutions on page H.83)

1. Find $\frac{dy}{dx}$, if $xy^3 - yx^3 = x$.

2. Find $\frac{dy}{dx}$, if $x^3 + y^3 = 3axy$.

3. Find $\frac{dy}{dx}$, if $x \log_e y + y \log_e x = 5$.

4. If $y = \log_y x$, then find $\frac{dy}{dx}$.

5. Find derivative of $f(x) = x^{x^x}$ w.r.t. x.

6. Find derivative of $f(x) = (\sin x)^{\cos^{-1} x}$ w.r.t. x.

7. Find derivative of $\frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$ w.r.t. x.

8. If $y = x^{x^2}$, then $\frac{dy}{dx} = (2 \log_e x + 1) \cdot x^{x^2+k}$, find k.

Answer key

1. $\frac{1+3x^2y-y^3}{3xy^2-x^3}$

2. $\frac{ay-x^2}{y^2-ax}$

3. $\frac{-y}{x}\left(\frac{x\log y+y}{x+y\log x}\right)$

4. $\frac{1}{x(1+\log_e y)}$

5. $x^{x^x} x^{x-1}[x \log x + x(\log x)^2 + 1]$

6. $(\sin x)^{\cos^{-1}x}\left(\cot x\cos^{-1}x - \frac{\log\sin x}{\sqrt{1-x^2}}\right)$

7. $\frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}\left[\frac{1}{2x}+\frac{3}{2x+8}-\frac{16}{12x-9}\right]$

8. 1

DIFFERENTIATION OF INFINITE SERIES

If taking out one or more than one terms from an infinite series, it remains unchanged, such that

If $y = \sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+\ldots\infty}}}$ then $y = \sqrt{f(x)+y}$

$$\Rightarrow \quad y^2 = f(x)+y$$

$$\Rightarrow \quad y^2 - y = f(x)$$

On differentiating w.r. to x, we get

$$(2y-1)\frac{dy}{dx} = f'(x) \Rightarrow \frac{dy}{dx} = \frac{f'(x)}{2y-1}$$

ILLUSTRATION 36

If $y = \cfrac{\sin x}{1+\cfrac{\cos x}{1+\cfrac{\sin x}{1+\cfrac{\cos x}{\ldots\ldots\ldots}}}}$, Prove that $\frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x-\sin x}$

SOLUTION

Given, function is $y = \cfrac{\sin x}{1+\cfrac{\cos x}{1+y}} = \frac{(1+y)\sin x}{1+y+\cos x}$

$$\Rightarrow \quad y + y^2 + y\cos x = (1+y)\sin x$$

On differentiating both sides with respect to x, we get

$$\frac{dy}{dx} + 2y\frac{dy}{dx} + \frac{dy}{dx}\cos x - y\sin x$$

$$= (1+y)\cos x + \left(\frac{dy}{dx}\right)\sin x$$

$$\Rightarrow \quad \frac{dy}{dx}[1+2y+\cos x-\sin x] = (1+y)\cos x + y\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1+y)\cos x + y\sin x}{1+2y+\cos x-\sin x}$$

ILLUSTRATION 37

If $y = (\log x)^{\log x^{\log x^{\log x^{\cdots\infty}}}}$ then find $\frac{dy}{dx}$.

SOLUTION

$y = (\log x)^{\log x^{\log x^{\log x^{\cdots\infty}}}}$ can also be written as $y = (\log x)^y$

Taking log both side, we get *log* $y = y(\log(\log x))$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{y}{x\log x} + \log(\log x)\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{1}{y} - \log(\log x)\right) = \frac{y}{x\log x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y/(x\log x)}{\frac{1}{y} - \log(\log x)}$$

$$= \frac{y^2}{(x\log x)(1 - y\log(\log x))}$$

ILLUSTRATION 38

Let $f(x) = x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + \ldots\infty}}}}$,

find $\dfrac{f(100)\cdot f'(100)}{100}$

SOLUTION

We have, $f(x) = x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + \ldots\infty}}}}$

$$\Rightarrow \quad f(x) = x + \frac{1}{x + f(x)} \Rightarrow f(x) - x = \frac{1}{x + f(x)}$$

$$\Rightarrow \quad [f(x)]^2 - x^2 = 1$$

On differentiating w.r.t. x, we get

$$2f(x)\cdot f'(x) = 2x \Rightarrow \frac{f(x)\cdot f'(x)}{x} = 1$$

Hence, $\dfrac{f(100)\cdot f'(100)}{100} = 1$

DIFFERENTIATION OF PARAMETRIC FUNCTIONS

In many cases, functions are not defined in terms of x explicitly, variable x and y are defined in some other parameter (variable) as t, θ, p, α.

Let x and y be the functions of parameter t, thus $x = f(t)$; $y = \phi(t)$ then $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\phi'(t)}{f'(t)}$.

ILLUSTRATION 39

Find dy/dx when $x = a\,(\theta + \sin\theta)$ and $y = a\,(1 + \sin\theta)$.

SOLUTION

$$x = a\,(\theta + \sin\theta) \text{ and } y = a\,(1 + \sin\theta)$$

On differentiating with respect to θ, we get

$$\frac{dx}{d\theta} = a\,(1 + \cos\,\theta) \text{ and } \frac{dy}{d\theta} = a\cos\,\theta.$$

Now, we get

$$\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta} = \frac{a\cos\theta}{a(1+\cos\theta)} = \frac{\cos\theta}{1+\cos\theta}$$

ILLUSTRATION 40

If $y = \tan u$ where $u = v - \dfrac{1}{v}$ and $v = \ln x$, then find the value of $\dfrac{dy}{dx}$ at $x = e$.

SOLUTION

$$y = \tan u\,,\ \frac{dy}{du} = \sec^2 u,$$

Given, $u = v - \dfrac{1}{v}$, $\dfrac{du}{dv} = 1 + \dfrac{1}{v^2}$,

$$\frac{dv}{dx} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

$$= \sec^2 u\left(1 + \frac{1}{v^2}\right)\left(\frac{1}{x}\right)$$

At $x = e$; $v = \ln e = 1$; $u = 1 - 1 = 0$; $y = \tan 0 = 0$

$$\therefore \quad \left.\frac{dy}{dx}\right|_{x=e} = 1 \times (1 + 1) \times \frac{1}{e} = \frac{2}{e}.$$

ILLUSTRATION 41

Let the parametric equations of a curve be $x = \dfrac{1-t^2}{1+t^2}$, $y = \dfrac{2t}{1+t^2}$. Find derivative of y relative to x at the point $t = 2$.

SOLUTION

$$x = \frac{1-t^2}{1+t^2},\ y = \frac{2t}{1+t^2} \text{ at } t = 2, \text{ we get } (x, y) = \left(\frac{-3}{5}, \frac{4}{5}\right)$$

Put $t = \tan\theta$, we get

$$x = \frac{1-\tan^2\theta}{1+\tan^2\theta},\ y = \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\Rightarrow \quad x = \cos 2\theta,\ y = \sin 2\theta$$

$$\therefore \quad x^2 + y^2 = 1;\ P\left(\frac{-3}{5}, \frac{4}{5}\right)$$

On differentiating, $2x + 2y\,dy/dx = 0$

Put $x = -3/5$, $y = 4/5$, we obtain

Hence, $\dfrac{dy}{dx} = \dfrac{3}{4}$

ILLUSTRATION 42

For $t \in (0, 1)$, let $x = \sqrt{2^{\sin^{-1}t}}$ and $y = \sqrt{2^{\cos^{-1}t}}$

then find $1 + \left(\dfrac{dy}{dx}\right)^2$.

SOLUTION

Given $x = \sqrt{2^{\sin^{-1}t}} \Rightarrow x^2 = 2^{\sin^{-1}t}$

$$\Rightarrow \quad \log_2(x^2) = \sin^{-1}(t)$$

$$y = \sqrt{2^{\cos^{-1}t}} \Rightarrow \log_2(y^2) = \cos^{-1}t$$

$$\Rightarrow \quad \log_2(x^2) + \log_2(y^2) = \frac{\pi}{2}$$

$$\Rightarrow \quad 2(\log x + \log y) = \frac{\pi}{2}\ln 2$$

On differentiation w.r.t. x, we get

$$\Rightarrow \quad \frac{2}{x} + \frac{2}{y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

Hence, $1 + \left(\frac{dy}{dx}\right)^2 = \frac{x^2 + y^2}{x^2}$

ILLUSTRATION 43

Find dy/dx at point (2, 3) for the curve $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 1$.

SOLUTION

Given curve $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 1$

At $y = 3$ and $x = 2$; we get $t = 2$

On differentiating both x and y with respect to t, we get

$$\frac{dx}{dt} = 2t + 3 \text{ and } \frac{dy}{dt} = 4t - 2 = 2(2t - 1)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2(2t-1)}{(2t+3)}$$

Hence, $\frac{dy}{dx} = \frac{6}{7}$

ILLUSTRATION 44

If $x = \sec\theta - \cos\theta$ and $y = \sec^3\theta - \cos^3\theta$, then find the value of $\left(\frac{dy}{dx}\right)^2$ at $x = 0$.

SOLUTION

We have, $x = \sec\theta - \cos\theta$

For $x = 0$; $\sec\theta - \cos\theta = 0$

$$\Rightarrow \quad \frac{1}{\cos\theta} = \cos\theta$$

$$\Rightarrow \quad \cos^2\theta = 1 \Rightarrow \cos\theta = 1 \text{ or } -1$$

On differentiating with respect to θ, we get

$$\Rightarrow \quad \frac{dx}{d\theta} = \sec\theta \cdot \tan\theta + \sin\theta = \tan\theta \cdot (\sec\theta + \cos\theta)$$

Also, $y = (\sec^3\theta - \cos^3\theta)$

$$\Rightarrow \quad \frac{dy}{d\theta} = 3\sec^2\theta\sec\theta \cdot \tan\theta + 3\cos^2\theta\sin\theta$$

$$= 3\tan\theta(\sec^3\theta + \cos^3\theta)$$

So, $\left(\frac{dy}{dx}\right)^2 = \left(\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}\right)^2 = \frac{9\tan^2\theta(\sec^3\theta + \cos^3\theta)^2}{\tan^2\theta(\sec\theta + \cos\theta)^2}$

At $x = 0$, $\cos\theta = \sec\theta = \pm 1$ we get $\left(\frac{dy}{dx}\right)^2 = \frac{9(1+1)^2}{(1+1)^2} = 9$

DIFFERENTIATION OF A FUNCTION WITH RESPECT TO ANOTHER FUNCTION

Let $y = f(x)$ and $z = g(x)$ be two functions of x, now the derivative of $f(x)$ w.r.t. $g(x)$ is the ratio of the derivative of $f(x)$ and $g(x)$ w.r.t. x.

$$\frac{df(x)}{dg(x)} = \frac{\frac{d}{dx}f(x)}{\frac{d}{dx}g(x)} = \frac{f'(x)}{g'(x)}$$

ILLUSTRATION 45

Find the derivative of y with respect to x, when $y = \frac{3+u}{2+u}$, where $u = \sin^{-1} x$.

SOLUTION

$y = \frac{3+u}{2+u}$ where $u = \sin^{-1} x$.

We find that,

$$\frac{dy}{du} = \left[\frac{(2+u)(1) - (3+u)(1)}{(2+u)^2}\right] = -\frac{1}{(2+u)^2}$$

And $\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

Hence, we obtain

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = -\frac{1}{(2+u)^2}\left[\frac{1}{\sqrt{1-x^2}}\right] \text{where } u = \sin^{-1} x.$$

ILLUSTRATION 46

Let $f(\alpha) = \cos\left(\cot^{-1}\left(\frac{\cos\alpha}{\sqrt{-\cos 2\alpha}}\right)\right)$ where $\frac{\pi}{4} < \alpha < \frac{\pi}{2}$, then find the value of $\frac{df(\alpha)}{d(\cot\alpha)}$.

SOLUTION

$$f(\alpha) = \cos\left(\cot^{-1}\left(\frac{\cos\alpha}{\sqrt{-\cos 2\alpha}}\right)\right)$$

$$\Rightarrow \quad f(\alpha) = \cos\left(\cos^{-1}\left(\frac{\cos\alpha}{\sqrt{\cos^2\alpha - \cos 2\alpha}}\right)\right)$$

$$\Rightarrow \quad f(\alpha) = \cos\cos^{-1}\left(\frac{\cos\alpha}{\sqrt{\cos^2\alpha - 2\cos^2\alpha + 1}}\right)$$

$$\Rightarrow \quad f(\alpha) = \frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}} = \frac{\cos\alpha}{|\sin\alpha|} = \cot\alpha \quad \left(\because \frac{\pi}{4} < \alpha < \frac{\pi}{2}\right)$$

Hence, $\frac{d\,f(\alpha)}{d(\cot\alpha)} = 1$.

Test Your Concepts-III

(Solutions on page H.83)

1. If $y = x - x^2$, then find the derivative of y^2 w.r.t. x^2.

2. If $x = e^{y+e^{y+e^{y+\ldots\ldots\infty}}}$, $x > 0$, then $\dfrac{dy}{dx}$.

3. Find $\dfrac{dy}{dx}$ for the following.

(i) $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\infty}}}$

(ii) $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots\infty}}}$

4. Find $\dfrac{dy}{dx}$, if $x = a\left(\cos t + \dfrac{1}{2}\log_e \tan^2 \dfrac{t}{2}\right)$, $y = a \sin t$.

5. If $x = \sin t\sqrt{\cos 2t}$ and $y = \cos t\sqrt{\cos 2t}$, find dy/dx.

6. Differentiate $\tan^{-1}\left(\dfrac{1+2x}{1-2x}\right)$ w.r.t. $\sqrt{1+4x^2}$.

7. Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ w.r.t. $\tan^{-1}x$.

8. The derivative of $\sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right)$ w.r.t. $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, $(x > 0)$ is:

Answer key

1. $2x^2 - 3x + 1$

2. $\dfrac{1-x}{x}$

3. (i) $\dfrac{\cos x}{2y-1}$; (ii) $\dfrac{\sin x}{1-2y}$

4. $\tan t$

5. $-\tan 3t$

6. $\dfrac{1}{2x\sqrt{1+4x^2}}$

7. $\dfrac{1}{2}$

8. 2

DIFFERENTIATION OF DETERMINANTS

To differentiate a determinant, differentiate one row (or column) at a time, keeping others rows (columns) unchanged.

If $\Delta(x) = \begin{vmatrix} A(x) & B(x) \\ C(x) & D(x) \end{vmatrix}$,

then $\dfrac{d}{dx}(\Delta(x)) = \begin{vmatrix} A'(x) & B'(x) \\ C(x) & D(x) \end{vmatrix} + \begin{vmatrix} A(x) & B(x) \\ C'(x) & D'(x) \end{vmatrix}$

ILLUSTRATION 47

If $y = \cos ax$, prove that $\begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix} = 0$, where $y_r = \dfrac{d^r y}{dx^r}$.

SOLUTION

Given, $y = \cos ax$

$\Rightarrow$ $y_1 = -a \sin ax$; $y_2 = -a^2 \cos ax$; $y_3 = a^3 \sin ax$

$y_4 = a^4 \cos ax$; $y_5 = -a^5 \sin ax$; $y_6 = -a^6 \cos ax$;

$y_7 = a^7 \sin ax$, $y_8 = a^8 \cos ax$

$$\therefore \quad \text{Determinant } \Delta(x) = \begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix}$$

$$= \begin{vmatrix} \cos ax & -a\sin ax & -a^2 \cos ax \\ a^3 \sin ax & a^4 \cos ax & -a^5 \sin ax \\ -a^6 \cos ax & a^7 \sin ax & a^8 \cos ax \end{vmatrix}$$

$$= a^3 \times a^6 \begin{vmatrix} \cos ax & -a \sin ax & -a^2 \cos ax \\ \sin ax & a \cos ax & -a^2 \sin ax \\ -\cos ax & a\sin ax & a^2 \cos ax \end{vmatrix}$$

Apply $R_1 \to R_1 + R_3$, we get

$$\Delta(x) = a^9 \begin{vmatrix} 0 & 0 & 0 \\ \sin ax & a\cos ax & -a^2 \sin ax \\ -\cos ax & a\sin ax & a^2 \cos ax \end{vmatrix}$$

Hence, $\Delta(x) = a^9 \times (0) = 0$.

ILLUSTRATION 48

If $f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2 and $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$, then find degree of $\Delta(x)$.

SOLUTION

$f(x)$, $g(x)$ and $h(x)$ are three polynomials of degree 2 as of type $ax^2 + bx + c$

$\because \quad f'''(x) = g'''(x) = h'''(x) = 0$, $\forall\, x \in R$

$$\Delta'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f''(x) & g''(x) & h''(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix}$$

$\Rightarrow \quad \Delta'(x) = 0 + 0 + 0 = 0$

$\Rightarrow \quad \Delta(x) = \text{Constant (differentiate row wise)}$

Hence, Degree of $\Delta(x) = 0$.

ILLUSTRATION 49

$$\text{If } \Delta(x) = x^2 \begin{vmatrix} \cos x & \sin x & 1 \\ 2\cos^2 x & \sin 2x & 2\cos(x)+1 \\ 4\cos^3 x & \sin 3x & (2\cos(x)+1)^2 \end{vmatrix} - 10\cos(x) - 2x^2,$$

then find the value of $\Delta'(0) + \Delta''(0)$.

SOLUTION

$$\Delta(x) = x^2 \sin x \begin{vmatrix} \cos x & 1 & 1 \\ 2\cos^2 x & 2\cos x & 2\cos x+1 \\ 4\cos^3 x & 3-4\sin^2 x & (2\cos x+1)^2 \end{vmatrix} - 10\cos x - 2x^2$$

$\Rightarrow \quad \Delta(x) = x^2 \sin x\phi(x) - 10\cos x - 2x^2$ (Form)

$$\Rightarrow \quad \Delta'(x) = 2x\sin x\phi(x) + x^2\cos x\phi(x) + x^2\sin x\phi'(x) + 10\sin x - 4x$$

$\Rightarrow \quad \Delta'(0) = 0$

$$\Rightarrow \quad \Delta''(x) = 2\sin x\phi(x) + 2x\cos x\phi(x) + 2x\sin x\phi'(x) + \frac{d}{dx}x^2(\cos x\phi(x) + \sin x\phi'(x)) + 10\cos x - 4$$

$\Rightarrow \quad \Delta''(0) = 0 + 0 + 0 + 10 - 4 = 6$

Hence, $\Delta'(0) + \Delta''(0) = 6$

DIFFERENTIATION OF INVERSE FUNCTION

Let f and g are inverse function of each other, then

$f(g(x)) = x$ or $f'(g(x))g'(x) = 1$ or $f'(g(x)) = \dfrac{1}{g'(x)}$.

Again, differentiating w.r.t. x, we obtain

$$f''(g(x))g'(x) = -\frac{1}{(g'(x))^2}g''(x)$$

ILLUSTRATION 50

Let $f: R \to R$ be defined as $f(x) = x^3 + 2x^2 + 4x + \sin\left(\dfrac{\pi x}{2}\right)$ and g be the inverse function of f, then find $g'(8)$.

SOLUTION

We have, $f(x) = x^3 + 2x^2 + 4x + \sin\left(\dfrac{\pi x}{2}\right)$

$$\therefore \quad f'(x) = 3x^3 + 4x + 4 + \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right) \Rightarrow f'(1) = 11$$

Also, $f(1) = 8$ and given that g be the inverse of f, then $g(f(x)) = x \quad \Rightarrow \quad g'(f(x))\,(f'(x)) = 1$

Put $x = 1$, we obtain

$$g'(f(1)) = \frac{1}{f'(1)}$$

Hence, $g'(8) = \dfrac{1}{f'(1)} = \dfrac{1}{11}$

ILLUSTRATION 51

If a differentiable function $f(x) = e^x + 2x$ is given, then find $\dfrac{d}{dx}(f^{-1}(x))$ at $x = f(\ln 3)$.

SOLUTION

$$f(x) = e^x + 2x \Rightarrow f'(x) = e^x + 2$$

$\Rightarrow \quad f'(\ln 3) = e^{\ln 3} + 2 = 3 + 2 = 5$

Let $f^{-1}(x) = g(x) \Rightarrow f(g(x)) = x$

$$\Rightarrow \quad g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(f^{-1}(x))}$$

$$\Rightarrow \quad \frac{d}{dx}(f^{-1}(x))\Big|_{x=f(\ln 3)} = (g'(f(\ln 3)) = \frac{1}{f'(f^{-1}f(\ln 3))}$$

Hence, $\dfrac{1}{f'(\ln 3)} = \dfrac{1}{5}$

ILLUSTRATION 52

If $f(x) = \dfrac{3x+2}{5x-3}$, and g is the inverse of f, then the value of $g'\left(\dfrac{5}{2}\right)$.

SOLUTION

$$y = f(x) = \frac{3x+2}{5x-3}$$

$\Rightarrow \quad 5xy - 3y = 3x + 2 \Rightarrow x(5y - 3) = 3y + 2$

$$\Rightarrow \quad x = \frac{3y+2}{5y-3}$$

$$\therefore \quad f^{-1}(y) = g(y) = \frac{3y+2}{5y-3}$$

$$g'(y) = \frac{(5y-3)3 - (3y+2)5}{(5y-3)^2} = \frac{-19}{(5y-3)^2}$$

$$\text{Hence, } g'\left(\frac{5}{2}\right) = \frac{-19}{\left(5\cdot\frac{5}{2} - 3\right)^2} = -\frac{4}{19}$$

ILLUSTRATION 53

Let $h(x) = xg(x)$ where g is the inverse of $f(x)$. Also the values of $f(x)$ and $f'(x)$ are given as

X	2	3	5
$f(x)$	4	5	1
$f'(x)$	–1	2	3

Then, find the value of $h'(5)$.

SOLUTION

Given, g is the inverse of $f(x)$ then $g(f(x)) = x$,

$\Rightarrow \quad g'(f(x))\, f'(x) = 1$

Put $x = 3$, we get

$$g'(f(3))\, f'(3) = 1 \Rightarrow g'(5) = \frac{1}{f'(3)}$$

Also, $g(f(3)) = 3 \Rightarrow g(5) = 3$

Given, $h(x) = xg(x)$ differentiation with respect to x, we get

$$h'(x) = xg'(x) + g(x) \text{ put } x = 5,$$

$$h'(5) = 5g'(5) + g(5)$$

$$= 5.\frac{1}{f'(3)} + g(5) = \frac{5}{2} + 3 = \frac{11}{2}.$$

ILLUSTRATION 54

Suppose f^{-1} is the inverse function of a differentiable function f and let $G(x) = \frac{1}{f^{-1}(x)}$. If $f(3) = 2$ and $f'(3) = \frac{1}{9}$, then find $G'(2)$.

SOLUTION

$$y = f(x) \Rightarrow f^{-1}(x) = g(x)$$

Given, $f(3) = 2$

$\because \quad g(f(x)) = x$

Put $x = 3$, $g(f(3)) = 3 \Rightarrow g(2) = 3$

Now, on differentiation $g'(f(x))\, f'(x) = 1$

Put $x = 3$, $g'(2) = \frac{1}{f'(3)} = 9$

Given, $G(x) = \frac{1}{f^{-1}(x)} = \frac{1}{g(x)}$

$$G'(x) = \frac{-1}{g^2(2)} \cdot g'(2) = \left(\frac{-1}{9}\right)(9) = -1$$

ILLUSTRATION 55

Suppose g is the inverse function of a differentiable function f and $G(x) = \frac{-4}{g^2(x)}$. If $f(5) = 3$, $f'(5) = \frac{1}{125}$, then find $G'(3)$.

SOLUTION

$$f(5) = 3 \Rightarrow f^{-1}(3) = g(3) = 5$$

So, $g'(f(x)) = \frac{1}{f'(x)} \Rightarrow g'(f(5)) = g'(3) = \frac{1}{f'(5)} = 125$

$$\therefore \quad G(x) = \frac{(-4)(-2)}{g^3(x)}\, g'(x)$$

$$\Rightarrow \quad G'(3) = \frac{8}{g^3(3)} \times g'(3) = \frac{8}{125} \times 125 = 8$$

ILLUSTRATION 56

If $f(x) = x^3 + 3x + 4$ and g is the inverse function of f, then find the value of $\frac{d}{dx}\left(\frac{g(x)}{g(g(x))}\right)$ at $x = 4$.

SOLUTION

Given, $f(0) = 4 \Rightarrow f^{-1}(4) = g(4) = 0$

$$g(f(x)) = x \Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

Put $x = 0$, we get $g'(f(0)) = \frac{1}{f'(0)}$

$$\Rightarrow \quad g'(4) = \frac{1}{f'(0)} = \frac{1}{3}$$

Also, $f(-1) = 0 \Rightarrow f^{-1}(0) = g(0) = -1$

At $x = 4$, $\frac{d}{dx}\left(\frac{g(x)}{g(g(x))}\right)$

$$= \left.\frac{g(g(x))g'(x) - g(x)\cdot g'(g(x))\cdot g'(x)}{(g(g(x)))^2}\right]_{x=4}$$

$$= \frac{g(g(4))g'(4) - g(4)\cdot g'(g(4))\cdot g'(4)}{(g(g(4)))^2}$$

$$= \frac{g(0)\left(\frac{1}{3}\right) - 0.g'(0)\left(\frac{1}{3}\right)}{\left(g(0)\right)^2} = \frac{-\frac{1}{3}}{(-1)^2} = -\frac{1}{3}$$

DIFFERENTIATION OF INVERSE TRIGONOMETRIC FUNCTIONS

Basic trigonometric functions have corresponding inverse functions when appropriate restrictions are placed on the domain of the original functions. All the inverse trigonometric functions have derivatives.

ILLUSTRATION 57

Find derivative of $\cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$ where $0 < x < \pi$.

SOLUTION

Let $y = \cot^{-1}\left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right)$ where $0 < x < \pi$

$$y=\cot^{-1}\left(\frac{|\cos x/2+\sin x/2|+|\cos x/2-\sin x/2|}{|\cos x/2+\sin x/2|-|\cos x/2-\sin x/2|}\right)$$

$$\left[\begin{array}{l}\because \sqrt{1+\sin x}=|\cos x/2+\sin x/2|\\ \text{and } \sqrt{1-\sin x}=|\cos x/2-\sin x/2|\end{array}\right]$$

(a) If $0<x<\pi/2$ or $0<x/2<\pi/4$ then $\cos x/2>\sin x/2$

$$y=\cot^{-1}\left(\frac{(\cos x/2+\sin x/2)+(\cos x/2-\sin x/2)}{(\cos x/2+\sin x/2)-(\cos x/2-\sin x/2)}\right)$$

$$=\cot^{-1}\left(\frac{\cos x/2}{\sin x/2}\right)=\cot^{-1}\cot x/2=x/2$$

$$\therefore\quad \frac{dy}{dx}=\frac{1}{2}$$

(b) If $\pi/2<x<\pi$ or $\pi/4<x/2<\pi/2$ then $\cos x/2<\sin x/2$

$$y=\cot^{-1}\left(\frac{(\cos x/2+\sin x/2)+(\sin x/2-\cos x/2)}{(\cos x/2+\sin x/2)-(\sin x/2-\cos x/2)}\right)$$

$$=\cot^{-1}\left(\frac{\sin x/2}{\cos x/2}\right)=\cot^{-1}(\tan x/2)$$

$$y=\pi/2-\tan^{-1}(\tan x/2)=\pi/2-x/2$$

$$\therefore\quad \frac{dy}{dx}=-\frac{1}{2}$$

ILLUSTRATION 58

If $f(x)=\tan^{-1}\left(\dfrac{3-x^3}{x+3x^2}\right)$, then find value of $f'(-1)$.

SOLUTION

$$f(x)=\tan^{-1}\left(\frac{3-x^3}{x+3x^2}\right)$$

$$\Rightarrow\quad f(x)=\tan^{-1}\left(\frac{3-x^3}{x(1+3x)}\right)=\tan^{-1}\left(\frac{3}{x}\right)-\tan^{-1}x^2$$

On differentiating with respect to x, we get

$$\Rightarrow\quad f'(x)=\frac{1}{1+\frac{9}{x^2}}\times\left(-\frac{3}{x^2}\right)-\frac{1}{1+x^4}(2x)$$

$$\therefore\quad f'(x)=\frac{-3}{x^2+9}-\frac{2x}{1+x^4}$$

Hence, $f'(-1)=\dfrac{-3}{10}+\dfrac{2}{2}=\dfrac{7}{10}$

ILLUSTRATION 59

Let $f(x)=\cos^{-1}(2x\sqrt{1-x^2})$ then find $f'\left(\dfrac{\sqrt{3}}{2}\right)$.

SOLUTION

$$f(x)=\cos^{-1}(2x\sqrt{1-x^2})$$

$$\Rightarrow\quad f(x)=\frac{\pi}{2}-\sin^{-1}(2x\sqrt{1-x^2})=\frac{\pi}{2}-(\pi-2\sin^{-1}x);$$

When $\dfrac{1}{\sqrt{2}}\le x\le 1$, we get $f(x)=-\dfrac{\pi}{2}+2\sin^{-1}x$

$$\Rightarrow\quad f'(x)=0+\frac{2}{\sqrt{1-x^2}}$$

$$\therefore\quad f'\left(\frac{\sqrt{3}}{2}\right)=\frac{2}{\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}}=\frac{2\times 2}{\sqrt{4-3}}=4$$

ILLUSTRATION 60

If $\phi(x)=2\cos^{-1}\left(\dfrac{2}{\sqrt{x}}\right)+\sin^{-1}\left(\dfrac{4\sqrt{x-4}}{x}\right)$, then find $\phi'(9)$.

SOLUTION

$$\phi(x)=2\cos^{-1}\left(\frac{2}{\sqrt{x}}\right)+\sin^{-1}\left(\frac{4\sqrt{x-4}}{x}\right)$$

Domain : $x\ge 4$

Put, $x=4\sec^2\theta$; $\theta\in\left[0,\dfrac{\pi}{2}\right)$

$$\phi(x)=2\cos^{-1}(\cos\theta)+\sin^{-1}(\sin 2\theta)$$

$$\phi(x)=\left[\begin{array}{ll}2\theta+2\theta, & \theta\in\left[0,\frac{\pi}{4}\right)\\ 2\theta+\pi-2\theta, & \theta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)\end{array}\right.=\left[\begin{array}{ll}4\theta, & \theta\in\left[0,\frac{\pi}{4}\right)\\ \pi, & \theta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)\end{array}\right.$$

$$\therefore\quad \phi(x)=\left[\begin{array}{ll}4\cos^{-1}\left(\frac{2}{\sqrt{x}}\right), & x\in[4,8)\\ \pi, & x\ge 8\end{array}\right.$$

$$\therefore\quad \phi'(9)=0$$

ILLUSTRATION 61

Let $f(x)=\cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$ and $g(x)=\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$, then find derivative of $f(x)$ with respect to $g(x)$ at $x=\dfrac{1}{2}$.

SOLUTION

Given, $f(x)=\cos^{-1}\left(\dfrac{2x}{1+x^2}\right)=\dfrac{\pi}{2}-\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$$=\frac{\pi}{2}-2\tan^{-1}x,\ -1\le x\le 1.$$

$$\therefore\quad f'(x)\Big]_{x=\frac{1}{2}}=\frac{-2}{1+x^2}=\frac{-2}{1+\frac{1}{4}}=\frac{-8}{5}$$

Also, $g(x)=\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)=\dfrac{\pi}{2}-\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$$= \frac{\pi}{2} - 2\tan^{-1}x,\ x \ge 0.$$

$$\therefore \quad g'(x)\Big]_{x=\frac{1}{2}} = \frac{-2}{1+x^2} = \frac{-2}{1+\frac{1}{4}} = -\frac{8}{5}.$$

So, derivative of $f(x)$ with respect to $g(x)$ at $x = \frac{1}{2}$

$$= \frac{f'\left(\frac{1}{2}\right)}{g'\left(\frac{1}{2}\right)} = 1$$

ILLUSTRATION 62

Find $\frac{dy}{dx}$ when $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left[2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right]$ where $0 < x < 1$.

SOLUTION

Put $x = \cos\theta$, we get

We get $y = \tan^{-1}\left(\frac{\cos\theta}{1+\sin\theta}\right) + \sin\left[2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right]$

$$= \tan^{-1}\left(\frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}\right) + \sin\theta$$

$$= \tan^{-1}\left(\frac{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}\right) + \sin\theta$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \sin\theta = \frac{\pi}{4} - \frac{\theta}{2} + \sin\theta$$

[As $0 < x < 1$ and $x = \cos\theta \Rightarrow 0 < \theta < \pi/2 \Rightarrow 0 < \frac{\theta}{2} < \frac{\pi}{4}$

Then, $\tan^{-1}\tan\frac{\theta}{2} = \frac{\theta}{2}$ and $\tan^{-1}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) = \frac{\pi}{4} - \frac{\theta}{2}$]

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \left(-\frac{1}{2} + \cos\theta\right)\left(-\frac{1}{\sin\theta}\right)$$

$$= \left(\frac{\frac{1}{2} - \cos\theta}{\sin\theta}\right) = \left(\frac{\frac{1}{2} - x}{\sqrt{1-x^2}}\right)$$

ILLUSTRATION 63

Obtain differential coefficient of $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$.

SOLUTION

Assume $u = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, $v = \cos^{-1}\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}$.

The function needs simplification before differentiation.

Let $x = \tan\theta$

$$\therefore \quad u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$v = \sqrt{\frac{1+\sqrt{1+\tan^2\theta}}{2\sqrt{1+\tan^2\theta}}} = \cos^{-1}\sqrt{\frac{1+\sec\theta}{2\sec\theta}}$$

$$= \cos^{-1}\sqrt{\frac{1+\cos\theta}{2}} = \cos^{-1}\left(\cos\frac{\theta}{2}\right) = \frac{\theta}{2}$$

$$\Rightarrow \quad u = v$$

Hence, $\frac{du}{dv} = 1$

Test Your Concepts-IV

(Solutions on page H.84)

1. Let $e^{f(x)} = \log_e x$. If $g(x)$ is the inverse function of $f(x)$, then $g'(x)$.
2. Let g be the inverse function of f and $f'(x) = \frac{x^{10}}{1+a^2}$. If $g(2) = a$, then find $g'(2)$.
3. Differentiate $f(x) = \tan^{-1}(\sqrt{1+x^2} + x)$ w.r.t. x.
4. Find derivative of $f(x) = \sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$.
5. Find derivative of $f(x) = \sin^{-1}(x^2\sqrt{1-x^2} + x\sqrt{1-x^4})$ w.r.t. x.
6. If $y = \sin^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$, $|x| > 1$, then find $\frac{dy}{dx}$.

7. If $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin 2\alpha & \sin 2\beta & \sin 2\gamma \end{vmatrix}$, where α and β are real constants, then find the value of

$$2f\left(\frac{\pi}{4}\right) - 3f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right).$$

Answer key

1. e^{x+e^x}

2. $\frac{1+a^2}{a^{10}}$

3. $\frac{1}{2(1+x^2)}$

4. $\frac{x}{\sqrt{1-x^4}}$

5. $\frac{2x+\sqrt{1+x^2}}{\sqrt{1-x^4}}$

6. 0

7. 0

HIGHER ORDER DERIVATIVES

If $y = f(x)$ be a differentiable function of x. Such that whose second, third … n^{th} derivative exist. the 1^{st}, 2^{nd}, 3^{rd} … n^{th} derivative of $y = f(x)$ are denoted respectively by,

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n} \quad \text{or} \quad y', y'', \ldots y_n$$

or, $\quad y_1, y_2, \ldots y_n \quad$ or $\quad Dy, D^2y, D^3y, \ldots D^ny$.

Second derivative of y w.r.t. $x = y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Third derivative of $f(x)$ with respect to x

$$= y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

n^{th} derivative of y w.r.t x is denoted by $y^n = \frac{d^ny}{dx^n}$

Conceptual Note(s)

(i) $\frac{d^2y}{dx^2} \neq \frac{d^2x}{dy^2}$ both have different meaning as $\frac{d^2y}{dx^2} =$ second derivative of y w.r.t. x; but $\frac{d^2x}{dy^2} =$ second derivative of x w.r.t. y

(ii) $\frac{dy}{dx} = \frac{1}{dx/dy}$ **(iii)** $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Let x and y be the functions of parameter t, thus $x = f(t)$; $y = \phi(t)$ then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)}$

Second derivative of y relative to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{\phi'(t)}{f'(t)}\right)\frac{dt}{dx} = \frac{d}{dt}\left(\frac{\phi'(t)}{f'(t)}\right)\frac{1}{f'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{f'(t)\phi''(t) - \phi'(t)f''(t)}{[f'(t)]^3}$$

Note

$$\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2} \quad \text{or} \quad \frac{d^2y}{dx^2} \neq \frac{\phi''(t)}{f''(t)}$$

ILLUSTRATION 64

If $y = \sin(\cos x)$, find second derivative of y.

SOLUTION

$$y = \sin(\cos x)$$

First derivative $= dy/dx = \cos(\cos x)(-\sin x)$

Second derivative $= d^2y/dx^2$

$$= \cos(\cos x)(-\cos x) - \sin(\cos x)(-\sin x)^2$$

$$= -\cos x.\cos(\cos x) - \sin^2 x \sin(\cos x)$$

ILLUSTRATION 65

If the functions are $u = e^x \sin x$; $v = e^x \cos x$ then show that,

(i) $v\frac{du}{dx} - u\frac{dv}{dx} = u^2 + v^2$ **(ii)** $\frac{d^2u}{dx^2} = 2v$

(iii) $\frac{d^2v}{dx^2} = -2u$

SOLUTION

$$u = e^x \sin x \Rightarrow \frac{du}{dx} = e^x \sin x + e^x \cos x = u + v$$

$$v = e^x \cos x \Rightarrow \frac{dv}{dx} = e^x \cos x - e^x \sin x = v - u$$

(i) $$v\frac{du}{dx} - \frac{u\,dv}{dx} = v\,(u+v) - u\,(v-u) = u^2 + v^2$$

(ii) $$\because \frac{du}{dx} = u + v$$

On differentiating with respect to x, we get

$$\Rightarrow \frac{d^2u}{dx^2} = \frac{du}{dx} + \frac{dv}{dx} = u + v + v - u = 2v$$

(iii) $$\because \frac{dv}{dx} = v - u$$

On differentiating with respect to x, we get

$$\frac{d^2v}{dx^2} = \frac{dv}{dx} - \frac{du}{dx} = (v-u) - (v+u) = -2u.$$

ILLUSTRATION 66

For the curve $xy + y^2 = 1$, show that $\dfrac{d^2y}{dx^2} = \dfrac{2}{(x+2y)^3}$.

SOLUTION

$$xy + y^2 = 1$$

On differentiating with respect to x, we get

$$\Rightarrow x\frac{dy}{dx} + y + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x+2y}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{(x+2y)\left(\frac{-dy}{dx}\right) + y\left(1 + \frac{2dy}{dx}\right)}{(x+2y)^2}$$

$$= \frac{1}{(x+2y)^2}\left\{\frac{(x+2y)y}{x+2y} + y + (2y)\left(\frac{-y}{x+2y}\right)\right\}$$

$$= \frac{1}{(x+2y)^2}\left\{\frac{2y(x+y)}{x+2y}\right\} = \frac{2y(x+y)}{(x+2y)^3} = \frac{2(xy+y^2)}{(x+2y)^3}$$

Since, $xy + y^2 = 1$

$$\frac{d^2y}{dx^2} = \frac{2}{(x+2y)^3}$$

ILLUSTRATION 67

Let $g(x) = f(x) \sin x$, where $f(x)$ is a twice differentiable function on $(-\infty, \infty)$ such that $f'(-\pi) = 1$. Find the value of $g''(-\pi)$.

SOLUTION

We have, $g(x) = f(x) \sin x$ …(1)

On differentiating equation (1) w.r.t. x, we get

$$g'(x) = f(x) \cos x + f'(x) \sin x \quad \text{…(2)}$$

Again, differentiating equation (2) w.r.t. x, we get

$$g''(x) = f(x)\,(-\sin x) + (f'(x) \cos x + f'(x) \cos x + f''(x) \sin x \quad \text{…(3)}$$

$$\Rightarrow \quad g''(-\pi) = 2f'(-\pi) \cos(-\pi) = 2 \times 1 \times (-1) = -2$$

Hence, $g''(-\pi) = -2$.

ILLUSTRATION 68

If $f(x) = (x-1)^4(x-2)^3(x-3)^2$, then

Find the value of $f'''(1) + f''(2) + f'(3)$.

SOLUTION

If $(x-1)$ is a factor of $f(x)$ repeated 4 times hence, $f'''(1) = 0$. Similarly, $(x-2)$ is a factor of $f(x)$ repeated 3 times hence, $f''(2)$ and $(x-3)$ is a factor of $f(x)$ repeated twice hence, $f'(3)$ will be zero.

Hence, $f'''(1) + f''(2) + f'(3) = 0 + 0 + 0 = 0$.

ILLUSTRATION 69

If $y^3 + y = 2\cos x$, then find the value of $\dfrac{d^2y}{dx^2}$ at the point with abscissa zero.

SOLUTION

$$y^3 + y = 2\cos x, \text{ when } x = 0,\ y = 1$$

On differentiating w.r.t. x, we get

$$3y^2\frac{dy}{dx} + \frac{dy}{dx} = -2\sin x$$

Put $x = 0$; $y = 1$, we obtain

$$3\frac{dy}{dx} + \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 3y^2\frac{d^2y}{dx^2} + \frac{dy}{dx}\,6y\frac{dy}{dx} + \frac{d^2y}{dx^2} = -2\cos x$$

Put $x = 0$, $y = 1$, we get

$$3\frac{d^2y}{dx^2} + 0 + \frac{d^2y}{dx^2} = -2$$

Hence, $y'' = \dfrac{d^2y}{dx^2} = -\dfrac{1}{2}$.

ILLUSTRATION 70

Let $f : R \to R$ be defined by $f(x) = x^3 + 3x + 1$ and g is the inverse of f, then find the value of $g''(5)$.

SOLUTION

$$f(1) = 5 \text{ then } g(5) = 1$$

$$\Rightarrow \quad f'(x) = 3x^2 + 3,\ f'(1) = 6$$

$$\Rightarrow \quad f''(x) = 6x,\ f''(1) = 6.$$

$$f(g(x)) = x \Rightarrow f'(g(x))g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}$$

Now, $g''(x) = -\dfrac{f''(g(x))g'(x)}{[f'(g(x))]^2} = \dfrac{f''(g(x))}{[f'(g(x))]^3}$

$$\Rightarrow \quad g''(5) = -\frac{f''(g(5))}{[f'(g(5))]^3} = -\frac{f''(1)}{[f'(1)]^3}$$

$$\therefore \quad g''(5) = \frac{-6}{216} = -\frac{1}{36}$$

ILLUSTRATION 71

If a curve is represented parametrically by the equations $x = f(t)$ and $y = g(t)$, then find $\left(\dfrac{d^2y}{dx^2}\right)\bigg/\left(\dfrac{d^2x}{dy^2}\right)$.

SOLUTION

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{g'(t)}{f'(t)}\right)\cdot\frac{dt}{dx}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^3} \qquad \ldots(1)$$

And, $\dfrac{dx}{dy} = \dfrac{\frac{dx}{dt}}{\frac{dy}{dt}} = \dfrac{f'(t)}{g'(t)}$

$$\Rightarrow \quad \frac{d^2x}{dy^2} = \frac{d}{dt}\left(\frac{f'(t)}{g'(t)}\right)\cdot\frac{dt}{dy} = \frac{g'(t)f''(t) - f'(t)g''(t)}{(g'(t))^3} \qquad \ldots(2)$$

Hence, $\left(\dfrac{d^2y}{dx^2}\right)\bigg/\left(\dfrac{d^2x}{dy^2}\right) = -\left(\dfrac{g'(t)}{f'(t)}\right)^3$

ILLUSTRATION 72

Let $f(x) = (\tan^{-1}x) + \ln^2(x + \sqrt{x^2+1}) + x^3$, then the value of $f'(0) + f''(0)$.

SOLUTION

$$f(x) = (\tan^{-1}x) + \ln^2(x + \sqrt{x^2+1}) + x^3$$

On differentiating w.r.t. x, we get

$$f'(x) = \frac{1}{1+x^2} + \frac{2\ln(x+\sqrt{x^2+1})}{\sqrt{x^2+1}} + 3x^2$$

$$\therefore \quad f'(0) = 1$$

$$f''(x) = -\frac{2x}{(1+x^2)^2} + \frac{\dfrac{2\sqrt{x^2+1}}{\left(x+\sqrt{x^2+1}\right)}\left[1+\dfrac{2x}{2\sqrt{x^2+1}}\right] - \dfrac{2\ln\left(x+\sqrt{x^2+1}\right)2x}{2\sqrt{x^2+1}}}{(x^2+1)} + 6x$$

$$f''(x) = -\frac{2x}{1+x^2} + \frac{2}{1+x^2} - \frac{2\ln\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}(x^2+1)} + 6x$$

$$\Rightarrow \quad f''(0) = 2$$

Hence, $f'(0) + f''(0) = 1 + 2 = 3$

ILLUSTRATION 73

Let $x = f(t)$ and $y = g(t)$, where x and y are twice differentiable. If $f'(0) = g'(0) = f''(0) = 2$, $g''(0) = 6$, then find the value of $\left(\dfrac{d^2y}{dx^2}\right)_{t=0}$.

SOLUTION

$$x = f(t) \text{ and } y = g(t)$$

$$\because \quad \frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(\frac{f'(t)g''(t) - g'(t)f''(t)}{(f'(t))^2}\right)\cdot\frac{dt}{dx}$$

$$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)_{t=0} = \frac{f'(0)g''(0) - g'(0)f''(0)}{(f'(0))^3} = \frac{2\cdot 6 - 2\cdot 2}{(2)^3} = 1.$$

ILLUSTRATION 74

Let $f(x) = g(x)\cos 2x + x^2 - 5$ where $g(x)$ is a twice differentiable function on R such that $g'\left(\dfrac{\pi}{4}\right) = -1$ then, find $f''\left(\dfrac{\pi}{4}\right)$.

SOLUTION

We have, $f(x) = g(x)\cos 2x + x^2 - 5 \qquad \ldots(1)$

$\therefore$ On differentiating both the sides of equation (1) with respect to x, we get

$$f'(x) = g'(x)\cos 2x - 2g(x)\sin 2x + 2x \qquad \ldots(2)$$

Again, differentiating both sides of equation (2) w.r.t. x, we get

$$f''(x) = g''(x)\cos 2x - 2\sin 2x\, g'(x) - 2\sin 2x g'(x) - 4g(x)\cos 2x + 2$$

$$\therefore \quad f''\left(\frac{\pi}{4}\right) = -2g'\left(\frac{\pi}{4}\right) - 2g'\left(\frac{\pi}{4}\right) + 2$$

$$= 2 + 2 + 2 = 6$$

ILLUSTRATION 75

If $x = \cos t$ and $y = \ln t$, then find the value of $\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2$ at $t = \dfrac{\pi}{2}$.

SOLUTION

$$x = \cos t \Rightarrow \frac{dx}{dt} = -\sin t$$

$$y = \ln t \Rightarrow \frac{dy}{dt} = \frac{1}{t}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-1}{t\sin t} \Rightarrow \left(\frac{dy}{dx}\right)^2_{t=\pi/2} = \frac{4}{\pi^2} \qquad \ldots(1)$$

Now, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{-1}{t\sin t}\right)$

$$= \frac{d}{dt}\left(\frac{-1}{t\sin t}\right)\frac{dt}{dx}$$

$$= \frac{1}{(t\sin t)^2}(t\cos t + \sin t)\left(\frac{-1}{\sin t}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{-t\cos t - \sin t}{t^2\sin^3 t}$$

$$\therefore \quad \left.\frac{d^2y}{dx^2}\right|_{t=\pi/2} = \frac{-4}{\pi^2} \qquad \ldots(2)$$

After adding (1) and (2), we get

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

ILLUSTRATION 76

If $y = x + e^x$, then show that $\dfrac{d^2x}{dy^2} = -\dfrac{e^x}{(1+e^x)^3}$.

SOLUTION

We have, $y = x + e^x$

$$\frac{dy}{dx} = 1 + e^x \qquad \left(\because \frac{dx}{dy} = \frac{1}{1+e^x}\right)$$

Now, on differentiating w.r.t. y, we get

$$\frac{d^2x}{dy^2} = -\frac{1}{(1+e^x)^2}\, e^x \cdot \frac{dx}{dy}$$

$$\therefore \quad \frac{d^2x}{dy^2} = -\frac{e^x}{(1+e^x)^3}$$

Test Your Concepts-V

(Solutions on page H.85)

1. If $\log_e y = \tan^{-1}x$, then show that $(1+x^2)y_2 + (2x-1)y_1 = 0$.
2. If $y = \tan x + \sec x$, then show that $y_2 = \dfrac{\cos x}{(1-\sin x)^2}$.
3. If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then show that $$\frac{d^2y}{dx^2} = \frac{1}{3a}\sec^4\theta \cdot \operatorname{cosec}\theta.$$
4. If $y = e^x \tan^{-1}x$, then find $(1+x^2)\,y_2 - 2(1-x+x^2)\,y_1 + (1-x)^2\,y$.
5. Let $f(x)$ be a polynomial in x, then find the second derivative of $f(e^x)$.

Answer key

4. **0** **5.** $f''(e^x)\cdot e^{2x} + f'(e^x)\cdot e^x$

SOLVED PROBLEMS

PROBLEM 1

If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f(x^2)$, then find $\dfrac{dy}{dx}$ at $x = 1$.

SOLUTION

We have, $y = f(x^2)$ and $f'(x) = \sqrt{2x^2 - 1}$

$$\Rightarrow \quad \frac{dy}{dx} = f'(x^2)\cdot 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{2x^4 - 1}\cdot 2x \qquad [\because f'(x) = \sqrt{2x^2 - 1}\,]$$

Hence, $\left(\dfrac{dy}{dx}\right)_{x=1} = 2$

PROBLEM 2

If, $f(x)$ is differentiable $\forall\, x \in R$ satisfying the relationship

$f(x) = \lim\limits_{n\to\infty} \dfrac{[1^2(f(x))^x] + [2^2(f(x))^x] + \ldots + [n^2(f(x))^x]}{n^3}$, find $\dfrac{d}{dx} f(x)$, where [.] denotes the greatest integer function.

SOLUTION

We have, $x - 1 < [x] \le x$,
(Note as an important fact)

$$\therefore \quad 1^2(f(x))^x - 1 < \left[1^2(f(x))^x\right] \le 1^2(f(x))^x$$

$$2^2(f(x))^x - 1 < \left[2^2(f(x))^x\right] \le 2^2(f(x))^x$$

....

$$n^2(f(x))^x - 1 < \left[n^2(f(x))^x\right] \le n^2(f(x))^x$$

On addition and taking the limit as $n \to \infty$

$$\lim_{n\to\infty} \frac{\left(1^2 + \ldots + n^2\right)((f(x)))^x - n}{n^3} \le$$

$$\lim_{n\to\infty} \frac{[1^2(f(x))^x] + \ldots + [n^2(f(x))^x]}{n^3}$$

$$\le \lim_{n\to\infty} \frac{\left(1^2 + \ldots + n^2\right)(f(x))^x}{n^3}$$

$$\because \quad \lim_{n\to\infty} \frac{(1^2 + 2^2 + \ldots + n^2)(f(x))^x - n}{n^3}$$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)(f(x))^x}{6n^3} - \frac{1}{n^2} = \frac{1}{3}(f(x))^x$$

Similarly, $\lim\limits_{n\to\infty} \dfrac{(1^2 + 2^2 + \ldots + n^2)(f(x))^x}{n^3}$

$$= \lim_{n\to\infty} \frac{n(n+1)(2n+1)(f(x))^x}{6n^3} = \frac{1}{3}(f(x))^x$$

$\therefore$ From Sandwich theorem, we get

$$f(x) = \lim_{n\to\infty} \frac{[1^2(f(x))^x] + \ldots + [n^2(f(x))^x]}{n^3} = \frac{1}{3}(f(x))^x$$

$$\therefore \quad y = \frac{1}{3} y^x \Rightarrow \log y = x \log y - \log 3$$

$$\Rightarrow \quad (x - 1)\log y = \log 3, \Rightarrow \log y^{(x-1)} = \log 3$$

$$\Rightarrow \quad y^{(x-1)} = 3 \Rightarrow y = 3^{\frac{1}{x-1}}$$

Hence, $\dfrac{dy}{dx} = 3^{\frac{1}{x-1}} \log 3 \,.\, \dfrac{-1}{(x-1)^2} = -\dfrac{3^{\frac{1}{x-1}} \log 3}{(x-1)^2}$

PROBLEM 3

Let $f(x)$ and $g(x)$ be two functions having finite 3rd order derivative $f'''(x)$ and $g'''(x)$ for all $x \in R$.

If $f(x).g(x) = 1$ for all x, show that $\dfrac{f'''}{f'} - \dfrac{3f''}{f} - \dfrac{g'''}{g'} + \dfrac{3g''}{g} = 0$, where $f' = f'(x)$, $g' = g'(x)$ etc.

SOLUTION

Given, $f \,.\, g = 1$, on differentiation $f'g + f g' = 0$
On dividing by $f.g$, we get

$$\therefore \quad \frac{f'}{f} + \frac{g'}{g} = 0 \Rightarrow \frac{f'}{f} = -\frac{g'}{g} \qquad \ldots(1)$$

On differentiating for 2nd time, we get

$$\frac{f''}{f} - \frac{f'^2}{f^2} + \frac{g''}{g} - \frac{g'^2}{g^2} = 0 \quad \text{(Using the product rule)}$$

On differentiating third time, we get

$$\frac{f'''}{f} - \frac{f'}{f^2} f'' + 2\frac{f'^3}{f^3} - \frac{1}{f^2} \,.\, 2f'f'' +$$

$$\frac{g'''}{g} - \frac{g''g'}{g^2} + 2\frac{g'^3}{g^3} - \frac{2g'g''}{g^2} = 0$$

$$\Rightarrow \quad \frac{f'''}{f'}\cdot\frac{f'}{f} - \frac{3f''}{f}\cdot\frac{f'}{f} + \frac{g'''}{g'}\cdot\frac{g'}{g}$$

$$- \frac{3g''}{g}\cdot\frac{g'}{g} + 2\left(\frac{f'^3}{f^3} + \frac{g'^3}{g^3}\right) = 0$$

Using relation (1), we get

$$\frac{f'''}{f'} - \frac{3f''}{f} - \frac{g'''}{g'} + \frac{3g''}{g} = 0.$$

Hence proved.

PROBLEM 4

Let f be a continuous and differentiable function on R satisfying $f(-x) = f(x)$ and $f(2 + x) = f(2 - x)$ $\forall\, x \in R$ and $f'(1) = -5$.

Then find the value of $\displaystyle\sum_{r=0}^{100} (-1)^r f'(r)$.

SOLUTION

$$f(-x) = f(x) \Rightarrow -f'(-x) = f'(x)$$

Put $x = 0 \Rightarrow -f'(0) = 0$

$$f(2 + x) = f(2 - x)$$

$$f'(2 + x) = -f'(2 - x)$$

Put $x = 0$, we get $f'(2) = 0$

Replace x by $2 - x$, we get $f(x) = f(4 - x)$

$\Rightarrow \quad f'(x) = -f'(4 - x) = -f'(-x)$

$\Rightarrow \quad f'(4 - x) = f'(-x)$

Now $f'(4 + t) = f'(t)$ periodic with 4

As $f'(0) = f'(2) = 0$

and $\quad f'(4) = f'(6) = f'(8) = \ldots = f'(100) = 0$

$\Rightarrow \quad f'(x) + f'(4 - x) = 0$

Put $x = 1$, we get

$f'(1) + f'(3) = 0,\ f'(5) + f'(7) = 0, \ldots,$
$f'(97) + f'(99) = 0$

$$\sum_{r=0}^{100} (-1)^r f'(r)$$

$$= f'(0) - f'(1) + f'(2) - f'(3) + \ldots - f'(99) + f'(100)$$

$$= [f'(0) + f'(2) + f'(4) + f'(6) + \ldots + f'(98) + f'(100)] -$$

$$[(f'(1) + f'(3)) + (f'(5) + f'(7)) + \ldots + (f'(97) + f'(99))]$$

$$= 0 - 0 = 0$$

PROBLEM 5

If $f(x) = \begin{cases} \ln(x + \sqrt{1 + x^2}); & x > 0 \\ 0\ ; & x = 0 \\ 10 \ln\left(\dfrac{1 - x}{1 + |x|}\right); & x < 0 \end{cases}$ and $g(x)$ is such that $h(x) = f(x) + g(x)$ is an odd function and identically zero for all $x \in R$, then find $f'(-\ln 2) + g'(\tan 1) + h'(\ln 2)$.

SOLUTION

$$f(x) = \begin{cases} \ln(x + \sqrt{x^2 + 1}); & x > 0 \\ 0\ ; & x \le 0 \end{cases}$$

Since, $h(x)$ is identically zero for all x,

$\therefore \quad g(x) = -f(x)$

$$g(x) = \begin{cases} -\ln(x + \sqrt{x^2 + 1}); & x > 0 \\ 0; & x \le 0 \end{cases}$$

$\therefore \quad f'(-\ln 2) = 0$

Since, $h(x) = 0$

$\therefore \quad h'(x) = 0 \Rightarrow h'(\ln 2) = 0$

Now, $g'(x) = \dfrac{-1}{\sqrt{x^2 + 1}} \Rightarrow g'(\tan 1) = -\cos 1$.

$$f'(-\ln 2) + g'(\tan 1) + h'(\ln 2) = -\cos 1$$

PROBLEM 6

Let F, f, g, h are differentiable functions such that

$$F(x) = \begin{cases} \left((f(0) - g(0))x \cdot \sin\left(\dfrac{1}{x}\right) + (g(0) - 1) \cdot \cos\dfrac{1}{x} + h(0)\right), & x \ne 0 \\ 0\ , & x = 0 \end{cases}$$

and $(f'(x) - g(x))\, t^2 + (g'(x) - h(x))\, t + (h'(x) - f(x)) = 0\ \forall\, t \in R$.

Find the value of $f^3(x) + g^3(x) + h^3(x) - 3f(x))\, g(x)\, h(x)$ at $x = 5$.

SOLUTION

As F is differentiable at $x = 0$. (Given)

$\therefore \quad f(0) = g(0) = 1,\ h(0) = 0 \qquad \ldots(1)$

Also, $(f'(x) - g(x))\, t^2 + (g'(x) - h(x))\, t + (h'(x) - f(x)) = 0\ \forall\, t \in R$

$\therefore \quad f'(x) = g(x),\ g'(x) = h(x),\ h'(x) = f(x) \qquad \ldots(2)$

$$g(x) = f'(x),\ h(x) = g'(x) = f''(x),\ h'(x) = f'''(x)$$

Let $\phi(x) = f^3(x) + g^3(x) + h^3(x) - 3f(x)\, g(x)\, h(x)$

$$= f^3(x) + (f'(x))^3 + (f''(x))^3 - 3f(x)\, f'(x)\, f''(x)$$

$\Rightarrow \quad \phi'(x)$

$$= 3f^2(x)\, f'(x) + 3(f'(x))^2 f''(x) + 3(f''(x))^2 f'''(x) -$$

$$3f'(x)\, f'(x)\, f''(x) - 3f(x)\, f''(x)\, f''(x) - 3f(x)\, f'(x)\, f'''(x)$$

We get $\phi'(x) = 0$ (Using equation (2))

$\therefore \quad \phi(x)$ is a constant function.

$\Rightarrow \quad \phi(5) = \phi(0) = 1 + 1 + 0 - 0 = 2 \qquad$ (Using equation (1))

PROBLEM 7

Let f and g be two real valued differentiable functions on R such that $\dfrac{f'(x)}{g'(x)} = \dfrac{g^2(x)}{f^2(x)}\ \forall\, x \in R$.

If $f(10) = 5$, $g(10) = 2$ and $g(2) = 1$, then find the value of $(f(2))^3$.

SOLUTION

$$\frac{f'(x)}{g'(x)} = \frac{g^2(x)}{f^2(x)}$$

$$\frac{d}{dx}(f^3(x) - g^3(x)) = 3f^2(x)\,f'(x) - 3g^2(x)\,g'(x) = 0$$

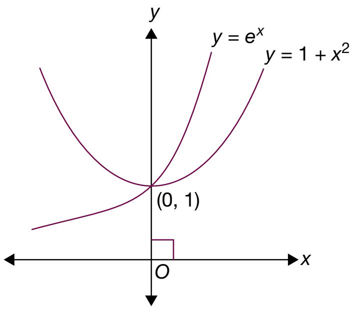
So, $f^3(x) - g^3(x) = \text{Constant.}$

$$\therefore \quad f^3(10) - g^3(10) = f^3(2) - g^3(2)$$

$$\Rightarrow \quad f^3(2) = (5)^3 - (2)^3 + (1)^3 = 125 - 8 + 1 = 126 - 8 = 118$$

PROBLEM 8

Let $f(x)$ and $g(x)$ be twice differentiable function satisfying $f(x) = xg(x)$ and $g'(x) = f(x)$, where $g(x) \neq 0\ \forall\ x \in R$. If $f'(x) = g(x) \times h(x)$, then find the number of roots of the equation $h(x) = e^x$.



SOLUTION

$$g'(x) = f(x) = xg(x)$$

$$\Rightarrow \quad \frac{g'(x)}{g(x)} = x \Rightarrow \ln(g(x)) = \frac{x^2}{2} + C \Rightarrow g(x) = k \cdot e^{\frac{x^2}{2}}$$

$$\therefore \quad f(x) = x\,g(x) = kxe^{\frac{x^2}{2}}$$

$$\therefore \quad f'(x) = ke^{\frac{x^2}{2}}(1 + x^2) = g(x)\,h(x)$$

$$\therefore \quad h(x) = 1 + x^2$$

Now, $1 + x^2 = e^x$
Hence, only one possible solution, $x = 0$

PROBLEM 9

If $f(x) = \sqrt{x + 2\sqrt{2x-4}} + \sqrt{x - 2\sqrt{2x-4}}$, then find the value of $100f'(3) + 10\,f'(102)$ is:

SOLUTION

$$f(x) = \sqrt{(\sqrt{x-2} + \sqrt{2})^2} + \sqrt{(\sqrt{x-2} - \sqrt{2})^2}$$

$$= |\sqrt{x-2} + \sqrt{2}| + |\sqrt{x-2} - \sqrt{2}|$$

For $\sqrt{x-2}$ to exist, $x \geq 2$. Also, $\sqrt{x-2} + \sqrt{2} > 0$

$$\sqrt{x-2} - \sqrt{2} \geq 0 \Rightarrow x \geq 4$$

$$\therefore \quad f(x) = \begin{cases} \sqrt{x-2} + \sqrt{2} + \sqrt{x-2} - \sqrt{2}, & x \geq 4 \\ \sqrt{x-2} + \sqrt{2} - \sqrt{x-2} + \sqrt{2} & 2 \leq x < 4 \end{cases}$$

$$f(x) = \begin{cases} 2\sqrt{x-2}, & x \geq 4 \\ 2\sqrt{2}, & 2 \leq x < 4 \end{cases}$$

$$\therefore \quad 100f'(3) + 10f'(102) = 100\,(0) + 10 \cdot \frac{1}{\sqrt{102-2}} = 1$$

PROBLEM 10

Suppose $f(x)$ and $g(x)$ are two functions such that $f, g : R \to R$, $f(x) = \log_e(1 + \sqrt{1+x^2})$ and $g(x) = \log_e(x + \sqrt{1+x^2})$.

Then find the value of $x \cdot e^{g(x)} \cdot \left[f\left(\frac{1}{x}\right)\right]' + g'(x)$ at $x = 1$

SOLUTION

$$f\left(\frac{1}{x}\right) = \log\left(\frac{x + \sqrt{x^2+1}}{x}\right);$$

$$f\left(\frac{1}{x}\right) = \log\left(x + \sqrt{1+x^2}\right) - \log x$$

On differentiating w.r.t. x, we get

$$\left[f\left(\frac{1}{x}\right)\right]' = \left(\frac{1}{x + \sqrt{1+x^2}}\right)\left(1 + \frac{2x}{2\sqrt{1+x^2}}\right) - \frac{1}{x}$$

$$= \frac{1}{\sqrt{1+x^2}} - \frac{1}{x}$$

Now, at $x = 1$, $\left[f\left(\frac{1}{x}\right)\right]' = \frac{1}{\sqrt{2}} - 1 = \frac{(1-\sqrt{2})}{\sqrt{2}}$ …(1)

Also, $g(x) = \log\left(x + \sqrt{1+x^2}\right)$

$$\therefore \quad g'(x) = \frac{1}{\sqrt{1+x^2}} \Rightarrow [g'(x)]_{x=1} = \frac{1}{\sqrt{2}} \qquad \text{…(2)}$$

Thus, $\left[x \cdot e^{g(x)} \cdot \left\{f\left(\frac{1}{x}\right)' + g'(x)\right\}\right]_{\text{at } x=1}$

$$= \left[1 \cdot (1 + \sqrt{2}) \cdot \frac{(1-\sqrt{2})}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right]$$

$$= \frac{-(2-1)}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 0$$

PROBLEM 11

If $f''(x) = -f(x)$ and $g(x) = f'(x)$ and $F(x) = \left[f\left(\frac{x}{2}\right)\right]^2 + \left[g\left(\frac{x}{2}\right)\right]^2$ and given that $F(5) = 5$, then find $F(10)$.

SOLUTION

We have, $F(x) = \left[f\left(\frac{x}{2}\right)\right]^2 + \left[g\left(\frac{x}{2}\right)\right]^2$

$$F'(x) = 2f\left(\frac{x}{2}\right) \times f'\left(\frac{x}{2}\right)\cdot\frac{1}{2} + 2g\left(\frac{x}{2}\right) \times g'\left(\frac{x}{2}\right)\cdot\frac{1}{2}$$

$$= f\left(\frac{x}{2}\right) g\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right)\left[-f\left(\frac{x}{2}\right)\right] = 0$$

$[\because f''(x) = -f(x)$ and $g(x) = f'(x)]$

$\because \quad F'(x) = 0$

This shows, $F(x)$ is a constant function. $[\because F(x) = C]$

We have, $F(5) = 5 \quad \therefore F(10) = 5$

PROBLEM 12

If $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + n^2y = 0$

SOLUTION

$$\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n = n\,(\log x - \log n)$$

On differentiating with respect to x, we get

$$-\frac{1}{\sqrt{1-\left(\frac{y}{b}\right)^2}}\,\frac{1}{b}\frac{dy}{dx} = \frac{n}{x}$$

or, $$\frac{-1}{\sqrt{b^2-y^2}}\frac{dy}{dx} = \frac{n}{x} \Rightarrow n^2(b^2-y^2) = \left(\frac{dy}{dx}\right)^2 x^2 \qquad \ldots(2)$$

On differentiating both sides of (2) with respect to x, we get

$$0 - 2n^2y\frac{dy}{dx} = 2\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}x^2 + 2x\left(\frac{dy}{dx}\right)^2$$

or, $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + n^2y = 0$$

PROBLEM 13

If α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be the polynomials of degree 3, 4 and 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where dash denotes the derivative.

SOLUTION

Let $g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$\therefore \quad g(\alpha) = 0$

And $g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$\because \quad g'(\alpha) = 0 \qquad \therefore \alpha$ is a repeated root of $g(x)$.

$\therefore \quad g(x)$ is divisible by $f(x)$. ($f(x)$ is a quadratic equation).

PROBLEM 14

Show that the function $y = f(x)$ defined by the parametric equations $x = e^t\sin t, y = e^t\cos t$ satisfies the relation $y''(x+y)^2 = 2(xy'-y)$.

SOLUTION

$$x = e^t\sin t \text{ and } y = e^t\cos t$$

$$\Rightarrow \quad x^2 + y^2 = e^{2t} \Rightarrow e^t = \sqrt{x^2+y^2} \qquad \ldots(1)$$

And, $\tan t = \frac{x}{y} \Rightarrow t = \tan^{-1}\left(\frac{x}{y}\right)$, put in (1)

$$\therefore \quad e^{\tan^{-1}\left(\frac{x}{y}\right)} = \sqrt{x^2+y^2} \qquad \ldots(2)$$

Taking log of both sides, we get

$$\tan^{-1}\left(\frac{x}{y}\right) = \frac{1}{2}\ln\left(x^2+y^2\right)$$

On differentiating both sides w.r.t. x, we get

$$\Rightarrow \quad \left(\frac{1}{1+\frac{x^2}{y^2}}\right)\left(\frac{y.1-x.y'}{y^2}\right) = \frac{1}{2}\cdot\frac{(2x+2yy')}{(x^2+y^2)}$$

$$\Rightarrow \quad y' = \frac{y-x}{x+y} \qquad \ldots(3)$$

Again, differentiating equation (3) w.r.t. x, we get

$$\Rightarrow \quad y'' = \frac{(x+y)(y'-1)-(y-x)(1+y')}{(x+y)^2}$$

$$\Rightarrow \quad y''(x+y)^2 = y'(2x) - 2y$$

$$\Rightarrow \quad y''(x+y)^2 = 2(xy'-y)$$

Hence proved.

PROBLEM 15

A function $f : R\ [1, \infty)$ satisfies the equation $f(xy) = f(x)\,f(y) - f(x) - f(y) + 2$. If f is differentiable on $R - \{0\}$ and $f(2) = 5$, $f'(x) = \dfrac{f(x)-1}{x}\,.\,\lambda$, then compute λ.

SOLUTION

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f\left\{x\left(1+\frac{h}{x}\right)\right\}-f(x)}{h} \qquad (x \neq 0 \text{ given})$$

$$= \lim_{x\to 0}\frac{f\left(1+\frac{h}{x}\right)-2}{\frac{h}{x}}\cdot\frac{f(x)-1}{x}$$

Putting $x = 1$, $y = 2$ in the given functional equation, $f(1) = 2$

$$\therefore \quad f'(x) = \frac{f(x)-1}{x}\cdot\lim_{h\to 0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{h/x} = \frac{f(x)-1}{x}.f'(1)$$

PROBLEM 16

Let $\dfrac{d(\sin^{-1}(2x\sqrt{1-x^2}))}{d(\sin^{-1}x)} = p$, where $x \in \left(\dfrac{-1}{\sqrt{2}},\dfrac{1}{\sqrt{2}}\right)$ and

$$\frac{d\left(\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)\right)}{d(\tan^{-1}x)} = q, \text{ where } x \in R - \{0\}.$$

Find the value of $2\{p+q\}+pq$. Where $\{\cdot\}$ denotes fractional part of x.

SOLUTION

$$p = \frac{d\left(\sin^{-1}2x\sqrt{1-x^2}\right)}{d(\sin^{-1}x)} \Rightarrow p = \frac{d(2\sin^{-1}x)}{d(\sin^{-1}x)} = 2$$

Also, $q = \dfrac{d\left(\frac{1}{2}\tan^{-1}x\right)}{d(\tan^{-1}x)} = \dfrac{1}{2}$

Hence, $2\{p+q\}+pq = 2\left(\dfrac{1}{2}\right)+1 = 1+1 = 2$

PROBLEM 17

Let f be a twice differentiable function, graph of which touches the x axis at $x = 1$ and $f''(1) = 4$.

If $\lim\limits_{n\to\infty}\left(\lim\limits_{n\to 0}\left(\dfrac{\sum\limits_{r=1}^{n}.f\left(1+\frac{x}{2^r}\right)}{\ln(1+\tan^2 x+\tan^4 x)}\right)\right) = \dfrac{p}{q}$, where $p, q \in N$,

then find the least value of $(p+q)$.

SOLUTION

$f(1) = 0$

$f'(1) = 0; f''(1) = 4$

$$\lim_{\substack{x\to 0\\ n\to\infty}}\frac{f\left(1+\frac{x}{2}\right)+f\left(1+\frac{x}{2^2}\right)+\ldots+f\left(1+\frac{x}{2^n}\right)}{x^2}$$

Applying L'Hopital rule 2 times, we get

$$= 2\lim_{n\to\infty}\left[\left(\frac{1}{2}\right)^2+\left(\frac{1}{2^2}\right)^2+\ldots+\left(\frac{1}{2^n}\right)^2\right] = \frac{2}{3}$$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. If $y=\frac{1+x^2+x^4}{1+x+x^2}$ and $\frac{dy}{dx}=ax+b$, then the values of a and b are

(A) $a=2, b=1$ (B) $a=-2, b=1$

(C) $a=2, b=-1$ (D) $a=-2, b=-1$

2. If $y=\frac{1}{1+x^{n-m}+x^{p-m}}+\frac{1}{1+x^{m-n}+x^{p-n}}+\frac{1}{1+x^{m-p}+x^{n-p}}$, then $\frac{dy}{dx}$ at $x=e^{m^{n^p}}$ is equal to

(A) e^{mnp} (B) $e^{mn/p}$

(C) $e^{np/m}$ (D) None of these

3. If $f(x)=\log_x(\log x)$, then $f'(x)$ at $x=e$ is

(A) $\frac{1}{e}$ (B) 1

(C) e (D) None of these

4. Let $f(x)$ be a polynomial function of second degree. If $f(1)=f(-1)$ and a, b, c are in an AP, then, $f'(a), f'(b), f'(c)$ are in

(A) AP (B) GP

(C) HP (D) None of these

5. If $y=\frac{x}{\sqrt{a^2-1}}-\frac{2}{\sqrt{a^2-1}}\tan^{-1}\left(\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)$, where $a\in(-\infty,-1)\cup(1,\infty)$ then $y'\left(\frac{\pi}{2}\right)$ equals

(A) $\frac{1}{a}$ (B) $\frac{2}{a}$

(C) $\frac{1}{2a}$ (D) a

6. The value of $\frac{dy}{dz}$ at $x=e$ where $y=e^x\ln^2 x$; $z=\ln x$ is

(A) $e(e+2)$ (B) $e^2(1+2e)$

(C) $e^e(e+2)$ (D) None of these

7. If $f(x)=(2015+x)^n$ where x is a real variable and n be a positive integer, then the value of $f(0)+f'(0)+\frac{f''(0)}{\lfloor\underline{2}}+\ldots\ldots\frac{f^{(n-1)}(0)}{\lfloor\underline{n-1}}$ is (where $f^{(n)}(0)$ denotes n^{th} derivative of f at $x=0$).

(A) $(2015)^n$ (B) $(2015)^n-1$

(C) $(2016)^n$ (D) $(2016)^n-1$

8. If $f''(x)=e^x\cos(e^x)$, $\lim\limits_{x\to-\infty}f'(x)=0$ and $y=f\left(\frac{x-1}{x+1}\right)$, then $\frac{dy}{dx}$ at $x=1$ is

(A) $\frac{\sin 1}{2}$ (B) $\sin 1$

(C) $e\sin 1$ (D) $\frac{e\sin 1}{2}$

9. If $f(x)=\log x$, then $f'(\log x)$ equals

(A) $\frac{x}{\log x}$ (B) $\frac{\log x}{x}$

(C) $\frac{1}{x\log x}$ (D) $\frac{1}{\log x}$

10. If $u=f(\tan x), v=g(\sec x)$ and $f'(x)=\tan^{-1}x$, $g'(x)=\operatorname{cosec}^{-1}x$, then du/dv at $x=\pi/4$ is

(A) $\sqrt{2}$ (B) $2\sqrt{2}$

(C) $3\sqrt{2}$ (D) $4\sqrt{2}$

11. If $f(x)=(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$, then at $x=0, f'(x)$ is

(A) 1 (B) –1

(C) 0 (D) None

12. $\frac{d}{dx}\left[\cos^{-1}\left(x\sqrt{x}-\sqrt{(1-x)(1-x^2)}\right)\right]$ is

(A) $\frac{1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x-x^2}}$ (B) $\frac{-1}{\sqrt{1-x^2}}-\frac{1}{2\sqrt{x-x^2}}$

(C) $\frac{1}{\sqrt{1-x^2}}+\frac{1}{2\sqrt{x-x^2}}$ (D) $\frac{1}{\sqrt{1-x^2}}$

13. If $y=\frac{1}{1+x+x^2+x^3}$, then $\left(\frac{dy}{dx}\right)_{x=0}$ and $\left(\frac{d^2y}{dx^2}\right)_{x=0}$ ...

(A) –1,0 (B) 1,0

(C) 0,2 (D) 0,1

14. If $5f(x)+3f\left(\frac{1}{x}\right)=x+2$ and $y=x\,f(x)$, then $\frac{dy}{dx}$ at $x=1$ is

(A) 14 (B) 7/8

(C) 1 (D) None

15. If $y(x)=f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x)=\sin x^2$, then $y'(0)$ is

(A) sin 1 (B) cos 1

(C) 2 sin 1 (D) $\frac{\cos 1}{2}$

16. If $y=\left(\frac{\alpha x+\beta}{\gamma x+\delta}\right)$, then $2\frac{dy}{dx}.\frac{d^3y}{dx^3}$ is

(A) $7\left(\frac{d^2y}{dx^2}\right)^2$ (B) $5\left(\frac{d^2y}{dx^2}\right)^2$

(C) $3\left(\frac{d^2y}{dx^2}\right)^2$ (D) $\left(\frac{d^2y}{dx^2}\right)^2$

17. If $f(x)=|\cos x|$, then $f'\left(\frac{3\pi}{4}\right)$ is

(A) $\sqrt{2}$

(B) $\frac{-1}{\sqrt{2}}$

(C) $\frac{1}{\sqrt{2}}$

(D) $-\sqrt{2}$

18. If $y=\frac{1+\sin x-\cos x}{1+\sin x+\cos x}$, then second derivative is

(A) $\cos^2\frac{x}{2}$

(B) $\sec^2\frac{x}{2}$

(C) $\frac{1}{3}\sec^2\left(\frac{x}{2}\right)$

(D) $\frac{1}{2}\sec^2(x/2)\tan(x/2)$

19. $\frac{d}{dx}\left[\frac{\sin x+\cos x}{\sqrt{1+\sin 2x}}\right],\left(0<x<\frac{\pi}{4}\right)$, is

(A) 1

(B) 0

(C) –1

(D) 2

20. Let $f_{n+1}(x)=e^{f_n(x)}$ for all $1\le n\le 9$ and $f_1(x)=e^x$. Then the first derivative of $f_n(x)$ for $n=10$ is

(A) $f_9(x)f_{10}(x)$

(B) $\prod_{r=1}^{9} f_r(x)e^x$

(C) $\prod_{r=1}^{10} e^x f_r(x)$

(D) $\prod_{r=1}^{10} f_r(x)$

[Where $\prod$ denotes continued product.]

21. The second derivative of a single valued function parametrically represented by $x=\phi(t)$ and $y=\psi(t), \alpha<1<\beta$, where $\phi(t)$ and $\psi(t)$ are different functions and is given by

(A) $\frac{d^2y}{dx^2}=\frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right)-\left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^3}$

(B) $\frac{d^2y}{dx^2}=\frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right)-\left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}$

(C) $\frac{d^2y}{dx^2}=\frac{\left(\frac{d^2x}{dt^2}\right)\frac{dy}{dt}-\frac{dx}{dt}\left(\frac{d^2y}{dt^2}\right)}{\left(\frac{dx}{dt}\right)^3}$

(D) $\frac{d^2y}{dx^2}=\frac{\left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)-\left(\frac{d^2y}{dt^2}\right)\frac{dx}{dt}}{\left(\frac{dy}{dt}\right)^3}$

22. $\frac{d}{dx}\left\{\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}=f(x)$ then $y=\left(f(x)\right)^2+x^2$ is

(A) Constant

(B) Line inclined to x-axis

(C) Parabola

(D) Circle

23. If $F(x)=f(x)\phi(x)$ and $f'(x)\phi'(x)=c$, then $\frac{f''(x)}{f(x)}+\frac{\phi''(x)}{\phi(x)}+\frac{2C}{f(x)\phi(x)}$ is

(A) $\frac{F''(x)}{F(x)}$

(B) $2\frac{F''(x)}{F(x)}$

(C) $-\frac{F''(x)}{F(x)}$

(D) $-2\frac{F''(x)}{F(x)}$

24. $y=f(x)$ be a real valued twice differentiable function defined on R, then $\frac{d^2y}{dx^2}\left(\frac{dx}{dy}\right)^3+\frac{d^2x}{dy^2}$ is

(A) $\frac{dy}{dx}$

(B) $\frac{dx}{dy}$

(C) $\frac{d^2y}{dx^2}$

(D) 0

25. If $y=(\sin^{-1}x)^2+(\cos^{-1}x)^2$, then $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}$ is

(A) –4

(B) 4

(C) 2

(D) –2

26. If $y=x\sin(\log x)+x\log x$, then $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+2y$ is

(A) $\ln(x)$

(B) $x\log x$

(C) $-x\ln(x)$

(D) $\frac{\ln x}{x}$

27. By introducing a new variable t, putting $x=\cos t$, the expression $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y$ is transformed into

(A) $\frac{d^2y}{dt^2}+y$

(B) $\frac{d^2y}{dt^2}-t\frac{dy}{dt}+y$

(C) $\frac{d^2y}{dt^2}-y$

(D) $\frac{d^2y}{dt^2}$

28. Let $f(x)$ and $g(x)$ be two functions having finite non-zero third order derivatives $f'''(x)$ and $g'''(x)$ for all $x\in R$. If $f(x)g(x)=1$ for all $x\in R$, then $\frac{f'''}{f'}-\frac{g'''}{g'}$ is equal to

(A) $3\left(\frac{f''}{g}-\frac{g''}{f}\right)$

(B) $3\left(\frac{f''}{f}-\frac{g''}{g}\right)$

(C) $3\left(\frac{g''}{g}-\frac{f''}{g}\right)$

(D) $3\left(\frac{f''}{g}-\frac{g''}{f}\right)$

29. If $y^{1/m}=\left[x+\sqrt{1+x^2}\right]$, then $(1+x^2)y_2+xy_1$ is equal to

(A) m^2y

(B) my^2

(C) m^2y^2

(D) my

30. If $x+\cos\theta=\sec\theta$, $y+\cos^8\theta=\sec^8\theta$, then $\left(\frac{x^2+4}{y^2+4}\right)\left(\frac{dy}{dx}\right)^2$ is

(A) 8

(B) 16

(C) 64

(D) 49

31. Let $y = f(x)$ and $f : R \to R$ be an odd function which is differentiable such that $f'''(x) > 0$ and $\varphi(\alpha, \beta) = \sin^8\alpha + \cos^8\beta + 2 - 4\sin^2\alpha\cos^2\beta$. If $f''(\varphi(\alpha, \beta)) = 0$, then $\sin^2\alpha + \sin^2\beta$ is

(A) 0
(B) 1
(C) 2
(D) 3

32. If $y = \sin(\sin x)$ and $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx}\tan x + f(x) = 0$, then $f(x)$ equals.

(A) $\sin^2 x \sin(\cos x)$
(B) $\sin^2 x \cos(\sin x)$
(C) $\cos^2 x \sin(\cos x)$
(D) $\cos^2 x \sin(\sin x)$

33. If $(a + bx)\, e^{\frac{y}{x}} = x$, then $x^3 \dfrac{d^2y}{dx^2}$ equals

(A) $\left(\dfrac{dy}{dx} + x\right)^2$
(B) $\left(x\dfrac{dy}{dx} - y\right)^2$
(C) $\left(\dfrac{dy}{dx} - y\right)^2$
(D) $\left(x\dfrac{dy}{dx} + y\right)^2$

34. If $y = \tan^{-1}\left(\dfrac{3 + 2\log_e x}{1 - 6\log_e x}\right) + \tan^{-1}\left(\dfrac{\log_e\left(\dfrac{e}{x^2}\right)}{\log_e(ex^2)}\right)$, then $\dfrac{d^2y}{dx^2}$ is

(A) 0
(B) 1
(C) –1
(D) 2

35. Let f be a function such that $f(x + y) = f(x) + f(y)$ for all x and y and $f(x) = (2x^2 + 3x)\, g(x)$ for all x where $g(x)$ is continuous and $g(0) = 9$, then $f'(x)$ is equals to

(A) 9
(B) 3
(C) 27
(D) 6

36. If $y = \tan x^{\tan x^{\tan x}}$, then $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{4}$ is

(A) 1
(B) 2
(C) 3
(D) 4

37. If $t\,(1 + x^2) = x$ and $x^2 + t^2 = y$, then at $x = 2$, the value of $\dfrac{dy}{dx}$ is

(A) $\dfrac{24}{5}$
(B) $\dfrac{101}{125}$
(C) $\dfrac{488}{125}$
(D) $\dfrac{358}{125}$

38. If $x = a \sin 2\theta(1 + \cos 2\theta)$ and $y = b \cos 2\theta(1 - \cos 2\theta)$, then $\left(\dfrac{dy}{dx}\right)_{\theta = \frac{\pi}{3}}$ is

(A) 0
(B) $\dfrac{ab}{\sqrt{3}}$
(C) $\dfrac{b}{a}\sqrt{3}$
(D) 1

39. If $x = \dfrac{1 + t}{t^3}$, $y = \dfrac{3}{2t^2} + \dfrac{2}{t}$ satisfies $f(x)\left(\dfrac{dy}{dx}\right)^3 = 1 + \dfrac{dy}{dx}$, then $f(x)$

(A) x
(B) $\dfrac{x^2}{1 + x^2}$
(C) $x + \dfrac{1}{x}$
(D) $x - \dfrac{1}{x}$

40. If $y = (1 + x)(1 + x^2)(1 + x^4)$, then $\dfrac{dy}{dx}$ at $x = 0$ is

(A) 1
(B) – 1
(C) 0
(D) 2

41. If $\cos y = x \cos(a + y)$ and $\dfrac{dy}{dx} = \dfrac{k}{1 + x^2 - 2x\cos a}$, then the value of k is

(A) $\sin a$
(B) $\cos a$
(C) 1
(D) $-\sin a$

42. Given that $f(x) = \begin{cases} \dfrac{x.g(x)}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, $g(0) = 0 = g'(0)$, then $f'(0)$ equals

(A) 1
(B) –1
(C) 2
(D) 0

43. If $f(x) = x + \tan x$, and f is inverse of g, then $g'(x)$ equal to

(A) $\dfrac{1}{1 + [g(x) - x]^2}$
(B) $\dfrac{1}{2 - [g(x) - x]^2}$
(C) $\dfrac{1}{2 + [g(x) - x]^2}$
(D) $\dfrac{1}{1 - [g(x) - x]^2}$

44. If $\sqrt{x^2 + y^2} = ae^{\tan^{-1}\left(\frac{y}{x}\right)}$ $(a > 0)$, $(y(0) > 0)$, then $y''(0)$ is

(A) $\dfrac{a}{2}\, e^{\pi/2}$
(B) $ae^{-\pi/2}$
(C) $\dfrac{-2}{a}\, e^{-\pi/2}$
(D) $\dfrac{a}{2} e^{-\pi/2}$

45. If $y = e^{-x}\cos x$ and $y_4 + ky = 0$, where $y_4 = \dfrac{d^4y}{dx^4}$, then k is

(A) 4
(B) – 4
(C) 2
(D) – 2

46. Let $y = e^{2x}$. Then $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{d^2x}{dy^2}\right)$ is

(A) 1
(B) e^{-2x}
(C) $2e^{-2x}$
(D) $-2e^{-2x}$

47. If $y^2 = P(x)$, is a polynomial of degree 3, then $2\left(\dfrac{d}{dx}\right)\left(y^3 . \dfrac{d^2y}{dx^2}\right)$ equals

(A) $P'''(x) + P'(x)$
(B) $P''(x).P'''(x)$
(C) $P(x).P'''(x)$
(D) a constant

48. If $f''(x) = -f(x)$ and $g(x) = f'(x)$ and $F(x) = \left(f\left(\dfrac{x}{2}\right)\right)^2 + \left(g\left(\dfrac{x}{2}\right)\right)^2$ and given that $F(5) = 5$, then $F(10)$ is equal to

(A) 5
(B) 10
(C) 0
(D) 15

49. If $f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$ (where p is constant), then at $x=0, \dfrac{d^3 f(x)}{dx^3}=0$

(A) p
(B) p^2
(C) $-p$
(D) 0

50. If $x^y = e^{x-y}$, then $\dfrac{dy}{dx}=$

(A) $(1+\log x)^{-1}$
(B) $(1+\log x)^{-2}$
(C) $\log x(1+\log x)^{-2}$
(D) $\log x(1+\log x)^{-1}$

51. If $y=\tan^{-1}\left(\dfrac{2^x}{1+2^{2x+1}}\right)$, then $\dfrac{dy}{dx}$ at $x=0$ is

(A) 1
(B) 2
(C) $\ln 2$
(D) $-\dfrac{1}{10}\ln 2$

52. A function g defined for all real $x > 0$ satisfies $g(1) = 1$, $g'(x^2)=x^3$ for all $x>0$, then $g(4)$ equals

(A) $\dfrac{13}{3}$
(B) 3
(C) $\dfrac{67}{5}$
(D) None of these

53. f is a strictly monotonic differentiable function with $f'(x)=\dfrac{1}{\sqrt{1+x^3}}$. If g is the inverse of f, then $g''(x)$ is

(A) $\dfrac{3x^2}{2\sqrt{1+x^3}}$
(B) $\dfrac{3g^2(x)}{2\sqrt{1+g^2(x)}}$
(C) $\dfrac{3}{2}g^2(x)$
(D) $\dfrac{x^2}{\sqrt{1+x^3}}$

54. Let $f(x)=\dfrac{g(x)}{x}$ when $x\neq 0$ and $f(0)=0$. If $g(0)=g'(0)=0$ and $g''(0)=17$, then $f'(0)$ is

(A) 3/4
(B) –1/2
(C) 17/3
(D) 17/2

55. Let $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$ then $265\left(\lim\limits_{x\to 0}\dfrac{h^4+3h^2}{(f(1-h)-f(1))\sin 5h}\right)$ is

(A) 1
(B) 2
(C) 3
(D) –3

56. $\lim\limits_{x\to 0}\dfrac{d}{dx}\left(\dfrac{e^{e^{x^2}}-e}{x}\right)$ is

(A) 0
(B) e
(C) e^2
(D) $-e$

57. Let $y(x) = e^{2\sin^{-1}x}$, $x \in [-1, 1]$ and $(1-x^2)\, y''(x) = xy'(x) + \lambda y(x)$, then λ equals

(A) 1
(B) 2
(C) 3
(D) 4

58. Let $f(x) = (2x - \pi)^3 + 2x - \cos x$. If the value of $\dfrac{d}{dx}\left(f^{-1}(x)\right)$ at $x = \pi$ can be expressed in the form of $\dfrac{p}{q}$ (where p and q are natural numbers in their lowest form), then the value of $(p + q)$ is

(A) 1
(B) 2
(C) 3
(D) 4

59. If $\ln(\ln x - \ln y) = e^{x^2 y}(1 - \ln x)$, then $y'(e)$ equals

(A) $\dfrac{1+e^{e^2}}{e}$
(B) $\dfrac{1-e^{e}}{e}$
(C) $\dfrac{e^{e^2}-1}{e}$
(D) $\dfrac{e^{e}+1}{e}$

60. If $y=\left(1+\dfrac{1}{x}\right)^x$, then $y'(2)$ is equal to

(A) $\dfrac{9}{4}\left(\dfrac{-1}{3}+\log\dfrac{3}{2}\right)$
(B) $\dfrac{4}{9}\left(\dfrac{-1}{3}+\log\dfrac{3}{2}\right)$
(C) $\dfrac{9}{4}\left(\dfrac{-1}{3}-\log\dfrac{3}{2}\right)$
(D) None of these

61. Let $f(x) = \tan^{-1}(\sin x + \cos x)$, then $f'(0)$ equals

(A) $\dfrac{1}{2}$
(B) 0
(C) 1
(D) Does not exist

62. $y = 6\tan x\,(e^x - x - 1) - 3x^3 - x^4 - x^5$, if n^{th} derivative at $x = 0$ is non zero, then least value of n is

(A) 3
(B) 4
(C) 5
(D) 6

63. If $f_n(x) = e^{f_{n-1}(x)}$ for all $n \in N$ and $f_0(x) = x$, then $\dfrac{d}{dx}\{f_n(x)\}$ is equal to

(A) $f_n(x).\dfrac{d}{dx}\{f_{n-1}(x)\}$
(B) $f_n(x).f_{n-1}(x)$
(C) $f_n(x).f_{n-1}(x)\ldots f_2(x).f_1(x)f_0(x)$
(D) None of these

64. Let $u(x)$ and $v(x)$ are differentiable function such that $\dfrac{u(x)}{v(x)}=7$. If $\dfrac{u'(x)}{v'(x)}=p$ and $\dfrac{d}{dx}\left(\dfrac{u'(x)}{v'(x)}\right)=q$ then $\dfrac{p+q}{p-q}$ has the value equal to

(A) 1
(B) 0
(C) 7
(D) –7

65. A function is represented parametrically by the equation $x=\dfrac{1+t}{t^3}, y=\dfrac{3}{2t^2}+\dfrac{2}{t}$, then $\dfrac{dy}{dx}-x\left(\dfrac{dy}{dx}\right)^3$ has the value equal to

(A) 2
(B) 0
(C) –1
(D) –2

66. Let g is the inverse function of f and $f'(x) = \dfrac{x^{10}}{(1+x^2)}$. If $g(2) = a$ then $g'(2)$ is equal to

(A) $\frac{5}{2^{10}}$ (B) $\frac{1+a^2}{a^{10}}$

(C) $\frac{a^{10}}{1+a^2}$ (D) $\frac{1+a^{10}}{a^2}$

67. If f is twice differentiable such that $f''(x) = -f(x), f'(x) = g(x)$; $h'(x) = [f(x)]^2 + [g(x)]^2$ and $h(0) = 2, h(1) = 4$, then the equation $y = h(x)$ represents

(A) A curve of degree 2

(B) A curve passing through the origin

(C) A straight line with slope 2

(D) A straight line with y intercept equal to –2

68. Suppose $f(x) = e^{ax} + e^{bx}$, where $a \neq b$, and that $f''(x) - 2f'(x) - 15f(x) = 0$ for all x. Then the product ab is equal to

(A) 25 (B) 9

(C) –15 (D) –9

69. If $x = t^3 + t + 5$ & $y = \sin t$, then $\left(\frac{d^2y}{dx^2}\right)$ is

(A) $-\frac{(3t^2+1)\sin t + 6t\cos t}{(3t^2+1)^3}$ (B) $\frac{(3t^2+1)\sin t + 6t\cos t}{(3t^2+1)^2}$

(C) $-\frac{(3t^2+1)\sin t + 6t\cos t}{(3t^2+1)^2}$ (D) $\frac{\cos t}{3t^2+1}$

70. The equation $y^2e^{xy} = 9e^{-3}.x^2$ defines y as a differentiable function of x. Then value of $\frac{dy}{dx}$ for $x = -1$ and $y = 3$ is

(A) $-\frac{15}{2}$ (B) $-\frac{9}{5}$

(C) 3 (D) 15

71. A non-zero polynomial with rela coefficients has the property that $f(x) = f'(x).f''(x)$, then the leading coefficient of $f(x)$ is

(A) $\frac{1}{6}$ (B) $\frac{1}{9}$

(C) $\frac{1}{12}$ (D) $\frac{1}{18}$

72. If $x^2 + y^2 = R^2$ $(R > 0)$ then $k = \frac{y''}{\sqrt{(1+y'^2)^3}}$, where k in terms of R alone is equal to

(A) $-\frac{1}{R^2}$ (B) $-\frac{1}{R}$

(C) $\frac{2}{R}$ (D) $-\frac{2}{R^2}$

73. Suppose the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivative 7 at $x = 2$. The derivative of the function $f(x) - f(4x)$ at $x = 1$, has the value equal to

(A) 19 (B) 9

(C) 17 (D) 14

74. The derivative of the function,

$$f(x) = \cos^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x - 3\sin x)\right\} + \sin^{-1}\left\{\frac{1}{\sqrt{13}}(2\cos x + 3\sin x)\right\}$$ w.r.t $\sqrt{1+x^2}$ at $x = \frac{3}{4}$ is

(A) $\frac{3}{2}$ (B) $\frac{5}{2}$

(C) $\frac{10}{3}$ (D) 0

75. If the function $f(x) = 2 - e^{-x}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1) + g''(1)$ is equal to

(A) –1 (B) 0

(C) 1 (D) 2

76. Let $f(x) = \begin{vmatrix} \cos x & \sin x & \cos x \\ \cos 2x & \sin 2x & 2\cos 2x \\ \cos 3x & \sin 3x & 3\cos 3x \end{vmatrix}$ then $f'\left(\frac{\pi}{2}\right)$

(A) 0 (B) –12

(C) 4 (D) 12

77. Let $y = \ln(1 + \cos x)^2$ then the value of $\frac{d^2y}{dx^2} + \frac{2}{e^{y/2}}$ equals

(A) 0 (B) $\frac{2}{1+\cos x}$

(C) $\frac{4}{(1+\cos x)}$ (D) $\frac{-4}{(1+\cos x)^2}$

78. Let $g : R \to R$ be a differentiable function such that $g(2) = -40$ and $g'(2) = -5$. Then $\lim\limits_{x\to 0}\left(\frac{g(2-x^2)}{g(2)}\right)^{4/x^2}$ is

(A) e^{32} (B) $\sqrt{e}$

(C) $\frac{1}{\sqrt{e}}$ (D) e^{-5}

79. Let P(x) be the polynomial $x^3 + ax^2 + bx + c$, where $a, b, c \in R$. If $P(-3) = P(+2) = 0$ and $P'(-3) < 0$, which of the following is a possible value of 'c' ?

(A) –27 (B) –18

(C) –6 (D) –3

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. $f(x) = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then

(A) $f'(2) = f'(3)$ (B) $f'(2) = 0$

(C) $f'(1/2) = \frac{16}{5}$ (D) $f'\left(\frac{1}{2}\right) = 0$

2. Let $y = \sqrt{x + \sqrt{x + \sqrt{x + ...\infty}}}$, then $\frac{dy}{dx}$ is

(A) $\frac{1}{2y-1}$ (B) $\frac{x}{x+2y}$

(C) $\frac{1}{\sqrt{1+4x}}$ (D) $\frac{y}{2x+y}$

3. A curve in the xy-plane is parametrically given by $x = t + t^3$ and $y = t^2$, where $t \in R$ is the parameter. If $\frac{dy}{dx} = \frac{1}{2}$, then t is/are

(A) $\frac{1}{3}$ (B) 2

(C) 3 (D) 1

4. If $\frac{d}{dx}\{f(x)\} = g(x)$, where $g(x) = e^{-x}.f(x) + e^{x}.f(-x)$, then $\left(\text{given } f(0) = 0\right)$.

(A) $g(x)$ is an even function (B) $f(x)$ is an odd function

(C) $g'(0) = f'(0) = g(0)$ (D) $g'(0) \neq f(0)$

5. Given $f(x) = -\frac{x^3}{3} + x^2 \sin 1.5\,a - x \sin a.\, \sin 2a - 5$ $\sin^{-1}\left(a^2 - 8a + 17\right)$ then

(A) $f'(x) = -x^2 + 2x\sin 6 - \sin 4 \sin 8$

(B) $f'(\sin 8) > 0$

(C) $f'(x)$ is not defined at $x = \sin 8$

(D) $f'(\sin 8) < 0$

6. Two functions f and g have first and second derivative at $x = 0$ and satisfy the relations $f(0) = \frac{2}{g(0)}$; $f'(0) = 2g'(0) = 4g(0)$; $g''(0) = 5f''(0) = 6f(0) = 3$, then

(A) If $h(x) = \frac{f(x)}{g(x)}$, then $h'(0) = \frac{15}{4}$

(B) If $k(x) = f(x)\, g(x) \sin x$, then $k'(0) = 2$

(C) $\lim\limits_{x \to 0} \frac{g'(x)}{f'(x)} = \frac{1}{2}$

(D) None of these

7. Let f, g and h are function differentiable on some open interval around 0 and satisfy the equations

$f' = 2f^2gh + \frac{1}{gh}$, $f(0) = 1$, $g' = fg^2h + \frac{4}{fh}$, $g(0) = 1$ and $h' = 3fgh^2 + \frac{1}{fg}$, $h(0) = 1$. The function f is given by

(A) $f(x) = 2^{-1/12}\left(\frac{\sin\left(6x + \frac{\pi}{4}\right)}{\cos^2\left(6x + \frac{\pi}{4}\right)}\right)^{1/6}$

(B) $f(x) = 2^{-1/6}\left(\frac{\sin\left(6x + \frac{\pi}{4}\right)}{\cos^2\left(6x + \frac{\pi}{4}\right)}\right)^{1/12}$

(C) $f(x) = (\sec 12x)^{1/12}\,(\sec 12x + \tan 12x)^{1/4}$

(D) $f(x) = (\sec 12x)^{1/4}\,(\sec 12x + \tan 12x)^{1/12}$

8. Let $f(x) = x^2 + xg'(1) + g''(2)$ and $g(x) = x^2 + xf'(2) + f''(3)$, then which of the following option/s is/are correct?

(A) $f'(1) = 4 + f'(2)$ (B) $g'(2) = 8 + g'(1)$

(C) $g''(2) + f''(3) = 4$ (D) $g'(2) = 6 + g'(1)$

9. Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ is an odd function such that $f'''(0) = 6$ and $f'(0) = 1$, then

(A) Degree of $f(f(x))$ is 9

(B) $f(x)$ is a one-one function

(C) $f(1) + f(2) + \ldots + f(n) = \frac{n(n+1)}{4}(n^2 + n + 2)$ $(n \in N)$

(D) $f(1) + f(2) + \ldots + f(n) = 0$ $(n \in N)$

10. Let $f(x)$ is differentiable $\forall\, x \in R$, satisfy $f(x + y) + f(x - y) = 2f(x)$ $\forall\, x, y \in R$, $f(0) = 0$, then

(A) f must be odd (B) f must be even

(C) $f'(1) = f'(2)$ (D) $f'(2) = f'(3) = f'(4)$

11. $\frac{d(x^{\ln x})}{d(\ln x)}$ is equal to

(A) $2(\ln x)\, e^{(\ln^2 x)}$ (B) $2(\ell n\, x)\,(x^{\ell n\, x})$

(C) $x.2(\ln^2 x)\,(x^{\ln x})$ (D) $2x^2(\ell n^2\, x)$

12. If $f(\theta) = \tan\left(\sin^{-1}\sqrt{\frac{2}{3 + \cos 2\theta}}\right)$, then

(A) $f'\left(\frac{\pi}{4}\right) = \sqrt{2}$ (B) $f\left(\frac{\pi}{4}\right) = \sqrt{2}$

(C) $\frac{d(f(\theta))}{d(\cos\theta)}$ at $\theta = \frac{\pi}{4}$ is $-\sqrt{2}$ (D) $\frac{d(f(\theta))}{d(\cos\theta)}$ at $\theta = \frac{\pi}{4}$ is -2

13. A student made the mistake in writing quotient rule of differentiation and writes $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f(x) \cdot g'(x) - g(x) f'(x)}{(g(x))^2}$ and still gets correct answer. Let $F(\text{n}) = \frac{f(n)}{g(n)}$

(A) $F(n) = F(n + 1)$

(B) $F(n) = F(n + 2)$

(C) $\sum\limits_{n=1}^{20} F(n) = \sum\limits_{n=1}^{20} F(2n - 1)$

(D) $F(1), F(2), F(3) \ldots$ form a G.P.

14. If $f(x - y)$, $(f(x).f(y))$ and $f(x + y)$ are in A.P. for all $x, y \in R$ and $f(0) \neq 0$ then

(A) $f'(x)$ is an even function (B) $f'(1) + f'(-1) = 0$

(C) $f'(2) - f'(-2) = 0$ (D) $f'(3) + f'(-3) = 0$

15. Let $f(x) = 2^x + 4^x + 8^x$ and $g(x)$ is inverse function of $f(x)$, then (where [.] represents greatest integer function) -

(A) $g'(14) \ln 2 < 1$ (B) $[g'(14) \ln 2] = 0$

(C) $g'(14) \ln 2 > 1$ (D) $[g'(14) \ln 2] = 1$

16. Derivative of $\sec^{-1}\left(\frac{1}{2x^2 - 1}\right)$ with respect to $\sqrt{1 - x^2}$ is

(A) 4 at $x = \frac{1}{2}$

(B) 2 at $x = 1$

(C) Does not exist at $x = \frac{1}{\sqrt{2}}$

(D) -4 at $x = \frac{1}{2}$

17. Let $y=\sin\left(\frac{x}{y}\right)$, then

(A) $\frac{dy}{dx}=\frac{y\cos\left(\frac{x}{y}\right)}{x\cos\left(\frac{x}{y}\right)+y^2}$

(B) $\frac{dy}{dx}=\frac{y\cos\left(\frac{x}{y}\right)}{y^2x\cos\left(\frac{x}{y}\right)+1}$

(C) $2\frac{dy}{dx}=\frac{\sin\left(\frac{2x}{y}\right)}{1+x\cos\left(\frac{x}{y}\right)-\cos^2\left(\frac{x}{y}\right)}$

(D) $\frac{dy}{dx}=\frac{\sin\left(\frac{2x}{y}\right)}{1+2x\cos\left(\frac{x}{y}\right)-\cos\left(\frac{2x}{y}\right)}$

18. Let a function $y=f(x)$ is defined as $y=e^{\theta}\cos\theta$ and $x=e^{-\theta}\sin\theta$ (where θ is a parameter), then

(A) $f(x)$ is one-one function

(B) $x^2\frac{dy}{dx}+y^2\frac{dx}{dy}=1$

(C) $y^{-2}\frac{dy}{dx}-\frac{x^2}{y^2}\left(\frac{dy}{dx}\right)=1$

(D) $\frac{dy}{dx}+2=0$ for infinite value of θ.

19. Let $f(x)=e^x$, then $f'(x)+f''(x)+\ldots+f''(x)\ldots$ n times is

(A) $nf(x)$ (B) $nf'(x)$

(C) $n\,f(f(x))$ (D) $(f(x))^2$

20. Let a function $y=f(x)$ is defined as $y=\theta^2\sin\theta$ and $x=\theta\sin\theta-\cos\theta$, then (where θ is a parameter)

(A) $\frac{dy}{dx}=\frac{y}{x-\cos\theta}$

(B) $\frac{dy}{dx}=\frac{y}{x+\cos\theta}$

(C) $f(x)$ is an injective function

(D) Tangent at $(-1, 0)$ to $f(x)$ is perpendicular to y-axis

21. Let $f:(0,\infty)\to(-\infty,\infty)$ be defined as $f(x)=e^x+\ln x$ and $g=f^{-1}$, then

(A) $g''(e)=\frac{1-e}{(1+e)^3}$ (B) $g''(e)=\frac{e-1}{(1+e)^3}$

(C) $g'(e)=e+1$ (D) $g'(e)=\frac{1}{e+1}$

22. Let $f(x)=\frac{x^{100}}{100}+\frac{x^{99}}{99}+\frac{x^{98}}{98}+\ldots+\frac{x^2}{2}+x+1$. Then which of the following is (are) correct?

(A) $f'(1)=100$ (B) $f'(0)=0$

(C) $f'(-1)=1$ (D) $f'(i)=0$, where $i^2=-1$.

23. If $f(x)=\frac{x^4-x^2+1}{x^2+\sqrt{3}x+1}$ then $f'(x)=2ax+b$ then a^2+b^2 is less, then

(A) 6 (B) 5

(C) 4 (D) 3

24. Let $f:R\to R$ be defined as $f(x)=(2x-3\pi)^3+\frac{4}{3}x+\cos x$ and $g=f^{-1}$, then

(A) $g'(2\pi)=\frac{3}{7}$ (B) $g'(2\pi)=\frac{7}{3}$

(C) $g''(2\pi)=0$ (D) $g''(2\pi)=\frac{-27}{343}$

25. Let $f:R\to(0,\infty)$ be defined as $f(x)=e^{x+\sin x}$ and $g(x)=f^{-1}(x)$, then

(A) $g'(1)=2$ (B) $g'(1)=\frac{1}{2}$

(C) $g''(1)=\frac{-1}{2}$ (D) $g''(1)=-2$

26. If $f(x)=\sin^{-1}\sqrt{\frac{\sqrt{1+x^2}-1}{2\sqrt{1+x^2}}}$, then which of the following is (are) correct?

(A) $f'(-1)=\frac{-1}{4}$

(B) $f'(1)=\frac{1}{4}$

(C) $f'(x)$ is an odd function

(D) $f'(x)$ is an even function

27. Let $P(x)=x^4-ax^3+bx^2-cx+480\ \forall\ x\in R$ and all its zeroes are integers which are greater than 1. Also, given that $|P(4)|+|P'(4)|=0$,

(A) Then least possible value of a is 19.

(B) Then greatest possible value of a is 25.

(C) Then number of polynomial $P(x)$ is 4.

(D) The least possible value of $P(2)$ is 0.

28. If the independent variable x is changed to y then the differential equation $x\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3-\frac{dy}{dx}=0$ is changed to $x\frac{d^2x}{dy^2}+\left(\frac{dx}{dy}\right)^2=k$ where k is equal to, when $[\cdot]$ is G.I.F.

(A) $\lim\limits_{x\to0}\left[\frac{2\tan x}{x}\right]$ (B) $\lim\limits_{x\to0}\left[\frac{2x}{\tan x}\right]$

(C) $\lim\limits_{x\to0}\left[\frac{\tan x}{x}\right]$ (D) 1

29. Let $f:R\to$ R be a function defined as

$f(x)=e^{x^2}\left(e^{x^3+x+1}-e^{-x^3-x-3}\right)+2x+5$ and g is the inverse function of f, then

(A) $\frac{d}{dx}(xf(g(x)))\big|_{x=5}=10$ (B) $\frac{d}{dx}(g(x))\big|_{x=3}=\frac{1}{10}$

(C) $\frac{d}{dx}\left(\frac{x}{g(x)}\right)\bigg|_{x=3}=-31$ (D) $\frac{d}{dx}\left(\frac{g(f(x))}{x^2}\right)\bigg|_{x=e}=1$

30. Which of the following function(s) has/have their derivatives as $2(1 + x^2)^{-1}$ in the specified interval?

(A) $f(x) = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), 0 < x < 1$

(B) $g(x) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), 0 < x < 1$

(C) $h(x) = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right), -1 < x < 1$

(D) $k(x) = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right), -1 < x < 1$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Let f be a differentiable function. Define

$D^* f(x) = \lim\limits_{h\to 0}\dfrac{f^2(x+h)-f^2(x)}{h}$, where $f^2(x) = (f(x))^2$.

Answer the following

1. If $f(x) = \tan x$, $g(x) = \ln x$ then $D^*(fg)(1) =$

(A) tan1
(B) 0
(C) $(\tan 1)^2$
(D) $2(\tan 1)^2$

2. If $f(x) = x^5$ and $g(x) = e^x$ then $D^*(fg)(1) =$

(A) e^2
(B) $4e^2$
(C) $8e^2$
(D) $12e^2$

3. If $f(x) = \sin x$, $g(x) = \cos x$, then $D^*(fog)(x) =$

(A) $-\sin(2\cos x)\sin x$
(B) $-\sin(\cos x)\sin x$
(C) $-\sin^2(\cos x)\sin x$
(D) $-\sin(\cos x)\sin^2 x$

Comprehension 2

If two function $f(x)$ and $F(x)$ are differentiable in $[a, b]$ and $F'(x) \neq 0$ for any $x \in [a, b]$, then there exists at least one $c \in (a, b)$ such that $\dfrac{f(b)-f(a)}{F(b)-F(a)} = \dfrac{f'(c)}{F'(c)}$.

4. If $f(x) = e^x$ and $F(x) = e^{-x}$, then $c \in (a, b)$ is such that a, c, b are in

(A) AP
(B) GP
(C) HP
(D) None of these

5. If $f(x) = \sin x$, $F(x) = \cos x$ and $c \in (a, b)$, where $\dfrac{f(b)-f(a)}{F(b)-F(a)} = \dfrac{f'(c)}{F'(c)}$, then c is

(A) AM of a, b
(B) GM of a, b
(C) HM of a, b
(D) None of these

6. If $f(x)$ is differentiable in $[a, b]$ and for $c \in (a, b)$, $(a^2 - b^2) f'(c) = k\{f(a) - f(b)\}$, then k is equal to

(A) 2
(B) $2c$
(C) c
(D) c^2

Comprehension 3

In certain problem the differentiation of $\{f(x).\, g(x)\}$ appears. One student commits mistake and differentiates as $\dfrac{df}{dx}.\dfrac{dg}{dx}$ but the gets correct result if $f(x) = x^3$ and $g(0) = \dfrac{1}{3}$.

7. The function $g(x)$ is

(A) $\dfrac{3}{|x-3|^3}$
(B) $\dfrac{4}{|x-3|^3}$
(C) $\dfrac{9}{|x-3|^3}$
(D) $\dfrac{27}{|x-3|^3}$

8. Derivative of $\{f(x - 3).\, g(x)\}$ with respect to x at $x = 8^{100}$ is

(A) 0
(B) – 1
(C) 1
(D) 2

9. $\lim\limits_{x\to 0}\dfrac{f(x).g(x)}{x(1+g(x))}$ will be

(A) 0
(B) – 1
(C) 1
(D) 2

Comprehension 4

$f, g : R \to R$ are functions such that $f(x) = 3x - \sin\left(\dfrac{\pi x}{2}\right)$, $g(x) = x^3 + 2x - \sin\left(\dfrac{\pi x}{2}\right)$

10. The value of $\dfrac{d}{dx}f^{-1}(g^{-1}(x))_{x=12}$ is equal to

(A) $\dfrac{2}{30+\pi}$
(B) $\dfrac{2}{30-\pi}$
(C) $\dfrac{2}{3(28-\pi)}$
(D) $\dfrac{2}{3(28+\pi)}$

11. Number of solution of equation $\dfrac{d}{dx}f(x) = \dfrac{d}{dx}g(x)$ is

(A) 0
(B) 1
(C) 2
(D) Infinite

Comprehension 5

A curve is represented parametrically by the equations $x = f(t) = a^{\ell n(b^t)}$ and $y = g(t) = a^{-\ell n(b^t)}$ $a, b > 0$ and $a \neq 1, b \neq 1$, where $t \in R$

12. Which of the following is not a correct expression for $\dfrac{dy}{dx}$?

(A) $\dfrac{-1}{f(t)^2}$
(B) $-(g(t))^2$
(C) $\dfrac{-g(t)}{f(t)}$
(D) $\dfrac{-f(t)}{g(t)}$

13. The value of $\frac{d^2y}{dx^2}$ at the point, where $f(t) = g(t)$ is

(A) 0 (B) $\frac{1}{2}$

(C) 1 (D) 2

14. The value of $\frac{f(t)}{f'(t)} \cdot \frac{f''(-t)}{f'(-t)} + \frac{f(-t)}{f'(-t)} \cdot \frac{f''(t)}{f'(t)} \forall t \in R,$ is equal to

(A) –2 (B) 2

(C) –4 (D) 4

Comprehension 6

Let 'r' represents the distance of the point ($a \cos\theta$, $b \sin\theta$) from origin, (a, b are fixed positive real numbers), then

15. The value of $\left(r\frac{d^2r}{d\theta^2} + \left(\frac{dr}{d\theta}\right)^2 + 2r^2\right)$ equals

(A) $2ab$ (B) $a^2 + b^2$

(C) $\sqrt{ab}(a+b)$ (D) $2(a^2 - ab + b^2)$

16. The value of $\left(r^3\frac{d^2r}{d\theta^2} + r^4\right)$ equals

(A) a^2b^2 (B) $\frac{1}{2}ab(a^2+b^2)$

(C) $\frac{1}{2}(a^4+b^4)$ (D) $\frac{1}{4}(a^2+b^2)^2$

Comprehension 7

Consider two real- valued functions defined on R (the set of all real numbers) as $f(x) = \tan^{-1}|x-1|$ and $g(x) = \cos^{-1}\left(\frac{x^2-1}{x^2+1}\right)$

17. The derivative of $f(x)$ with respect to $g(x)$ when $x = 1$, is equal to

(A) –1

(B) 0

(C) 1

(D) Non-existent

18. The value of $\frac{d^2g}{dx^2}$ when $x = -1$, is equal to

(A) –1

(B) 0

(C) 1

(D) Non-existent

Comprehension 8

Consider $f(x) = x^{\ln x}$ and $g(x) = e^2x$. Let α and β ($\alpha < \beta$) be two values of x satisfying the equation $f(x) = g(x)$

19. The product $\alpha\beta$ equals

(A) $\frac{1}{e^2}$ (B) $\frac{1}{e}$

(C) e (D) e^2

20. If $\lim\limits_{x\to\beta}\frac{f(x) - c\beta}{g(x) - \beta^2}$ exists and is equal to l, then the value of $(c - l)$ is equal to

(A) $4 - e^2$ (B) $e^2 - 4$

(C) $4 - e$ (D) $e - 4$

21. If $h(x) = \frac{f(x)}{g(x)}$, then $h'(\alpha)$ is equal to

(A) e (B) $-e$

(C) $3e$ (D) $-3e$

Comprehension 9

Let a polynomial f satisfies the relation $f(f(f(x))) + (1 - p)f(x) = 3$ $\forall x \in R$, where $p \in R$.

22. If leading coefficient of $f(x)$ is 2, then the value of $\frac{d}{dx}(f(f(x)))$ at $x = p$ is

(A) 2 (B) 4

(C) 9 (D) 17

23. If leading coefficient of $f(x)$ is negative and $f(0) = 4$, then $f^{-1}(4)$ is equal to

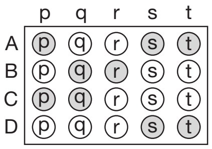
(A) 8 (B) 16

(C) 0 (D) 64

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. Match the following:

COLUMN-I	COLUMN-II
(A) $y = \tan^{-1}\left[x + \sqrt{1+x^2}\right]$	(p) $y'(0) = 2$
(B) $y = \dfrac{1 + x\tan^{-1}x}{\sqrt{1+x^2}}$	(q) $y'(0) = 0$
(C) $y = \min\left(x^2, x^3\right)$	(r) $y'(0) = 1$
(D) $y = \dfrac{2x+1}{x+1} - \dfrac{x+1}{2x+1}$	(s) $y'(0) = \dfrac{1}{2}$

2. Column-I contains the function and Column-II contains their derivatives at $x = 0$. Select the correct answer using the codes given below the list.

COLUMN-I	COLUMN-II
(A) $f(x) = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right)$	(p) 2
(B) $g(x) = \cos^{-1}(2x^2 - 1)$	(q) 3
(C) $h(x) = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$	(r) –2
(D) $k(x) = \tan^{-1}\left(\dfrac{3x - x^3}{1-3x^2}\right)$	(s) non-existent

3. Match the following:

COLUMN-I	COLUMN-II
(A) If $f(0) = 0$, $f'(0) = 2$, then the derivative of $y = f\left(f\left(f\left(f(x)\right)\right)\right)$, at $x = 0$, assuming it exists is	(p) 1
(B) For the function $f(x) = ln\tan\left(\dfrac{\pi}{4} + \dfrac{x}{2}\right)$ If $\dfrac{dy}{dx} = \sec x + p$, then p is equal to	(q) 16
(C) If $y = \cos^{-1}\left(\dfrac{5\cos x - 12\sin x}{13}\right)$, $x \in \left(0, \dfrac{\pi}{2}\right)$ then $\dfrac{dy}{dx}$ is equal to	(r) 0
(D) If $xe^{xy} - y = \sin^2 x$, then $\dfrac{dy}{dx}$ at $x = 0$ is	(s) –2

4. Match the following:

COLUMN-I	COLUMN-II
(A) Let f and g be differentiable functions satisfying $g'(a) = 2$, $g(a) = b$ and $fog = I$ (Identity function), then $f'(b)$ is equal to	(p) 2
(B) If $f(x) = \cos x.\cos 2x.\cos 4x.\cos 8x.\cos 16x$, then $f'\left(\dfrac{\pi}{4}\right)$ is equal to	(q) $\dfrac{1}{2}$
(C) If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f\left(x^2\right)$, then $\dfrac{dy}{dx}$ at $x = 1$ is	(r) $\sqrt{2}$
(D) If $y = \sin^2\cot^{-1}\left(\sqrt{\dfrac{1-x}{1+x}}\right)$, then $\dfrac{dy}{dx} =$	(s) $-\sqrt{2}$

(*Continued*)

5. Match the following :

COLUMN-I	COLUMN-II
(A) If $f'(x) = \sqrt{3x^2 + 6}$ & $y = f\left(x^3\right)$, then at $x = 1$, $\dfrac{dy}{dx}$	(p) –2
(B) If f be a differentiable function such that $f(xy) = f(x) + f(y)$ $x, y \in R$, then $f(e) + f(1/e)$	(q) 4
(C) If f be a twice differentiable function such that $f''(x) = -f(x)$ & $f'(x) = g(x)$, If $h(x) = \left(f(x)\right)^2 + \left(g(x)\right)^2$ & $h(5) = 9$, then $h(10)$	(r) 0
(D) $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x), \dfrac{\pi}{2} < x < \pi$, then $\dfrac{dy}{dx}$	(s) –1
	(t) 9

6. Match the following:

COLUMN-I	COLUMN-II
(A) If $y = \cos^{-1}(\cos x)$, then y' at $x = 5$ is equal to	(p) Does not exist
(B) For the function $f(x) = \ln\lvert\tan x\rvert$, then $f'\left(-\dfrac{\pi}{4}\right)$ is equal to	(q) –2
(C) The derivative of $\tan^{-1}\left(\dfrac{1+x}{1-x}\right)$ at $x = -1$ is	(r) $\dfrac{1}{2}$
(D) The derivative of $\dfrac{\log\lvert x\rvert}{x}$ at $x = -1$ is	(s) 1
	(t) –1

7. Match the following:

COLUMN-I	COLUMN-II
(A) If minimum and maximum values of $f(x) = 2\ \lvert x-1 \rvert + \lvert x+3 \rvert - 3\ \lvert x-4 \rvert$ are m and M respectively, then $(m + M)$ equals	(p) 0
(B) If $y - x + \sqrt{x^2+y^2} = 0$, then possible finite value of $\frac{dy}{dx}$ at its any point on y-axis is	(q) 1
(C) If $f(x) = (x^3 + bx^2 + cx + d) \times \lvert (x-1)(x-2)(x-3)^2(x-4)^3 \rvert$ is derivable for all $x \in R$ and $f'(3) = f'''(4) = 0$, then $\lvert b+c+d \rvert$ equals	(r) 2
(D) The number of integral value(s) in the range of $f(x) = \sin\left(\cos^{-1}\frac{x}{2}\right) + \sin\left(\tan^{-1}\frac{\sqrt{4+x^2}}{x}\right)$ is (are)	(s) 3
	(t) 4

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. If y is a function of x and $\ln(x + y) = 2xy$, then find the value of $y''(0)$.

2. Let f and g be two real-valued differentiable functions on R. If $f'(x) = g(x)$ and $g'(x) = f(x)\ \forall\ x \in R$ and $f(3) = 5$, $g(3) = 4$, then find the value of $(f^2(\pi) - g^2(\pi))$.

3. If $f'(2) = 6$ and $f'(1) = 4$, then $\lim\limits_{h \to 0} \dfrac{f(2h+2+h^2) - f(2)}{f(h-h^2+1) - f(1)}$ is ______.

4. If $y = \cos^{-1}(\cos x)$, then $1 + \left(\dfrac{dy}{dx}\right)_{x=\frac{13\pi}{4}}$ is _____.

5. If $x^2 + y^2 = t - \dfrac{1}{t}$ and $x^4 + y^4 = t^2 + \dfrac{1}{t^2}$, then $x^3 y \dfrac{dy}{dx}$ is ______.

6. If y is a function of x and $\log(x+y) = 2xy$, then $y'(0)$ is ______.

7. $xy = (x+y)^n$ and $\dfrac{dy}{dx} = \dfrac{y}{x}$, then n is ______.

8. Suppose the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivative 7 at $x = 2$. The derivative of the function $f(x) - f(4x) - 10x$ at $x = 1$, is equal to ______.

9. $y = \dfrac{2x^2}{(x-2)(x-3)(x-4)} + \dfrac{3x}{(x-3)(x-4)} + \dfrac{x}{x-4}$.

   If $\dfrac{x}{y}\dfrac{dy}{dx} = \dfrac{a}{2-x} + \dfrac{b}{3-x} + \dfrac{c}{4-x}$, then $a + b + c$ is ______.

10. If the function $f(x) = 4x^5 + 3x^3 + 2x^2 + e^{x/7}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is ______.

11. If $f(x) = x^3 + x^2.f'(1) + x.f''(2) - f'''(3)$, $\forall x \in R$, then $\lvert f(0) + f(3) + 16 \rvert$ is ______.

12. Let $f : R \to R$ and $g : R \to R$ be twice differentiable functions satisfying $f''(x) = g''(x), 2f'(1) = g'(1) = 4$ and $3f(2) = g(2) = 9$, then the value of $15 + f(4) - g(4)$ is equal to ______.

13. If $y = \tan^{-1}\dfrac{4x}{1+5x^2} + \tan^{-1}\dfrac{2+3x}{3-2x}$ and $\dfrac{dy}{dx} = \dfrac{k}{1+25x^2}$, then k

14. If $y = e^{\sin^{-1}x} + e^{-\cos^{-1}x}$, then find the value of $\left\lvert \dfrac{y''(x^2-1)+y}{xy'} \right\rvert$ (where y' and y'' denotes first order and 2nd order derivative of y respectively)

15. A function $f : R \to R$ satisfies $\sin x \cos y\ (f(2x+2y) - f(2x-2y)) = \cos x \sin y\ (f(2x+2y) + f(2x-2y))$. If $f'(0) = \dfrac{1}{2}$, then the value of $4f''(x) + f(x)$ is ______.

16. If $f(x) + f(y) = f\left(\dfrac{x-y}{1-xy}\right)$ for all, $x, y \in$ R and $xy \neq 1$ and $\lim\limits_{x \to 0}\dfrac{f(x)}{x} = 2$, then the value of $\dfrac{15}{\pi}\dfrac{f(\sqrt{3})}{f'(-2)}$ must be.

17. Let a differentiable function f satisfies the relation $f(xy) = 2f(x) + f(y) + x + 2y + 1\ \forall\ x, y \in R$ such that $f'(1) = 1$, then $f(e^3)$ is equal to ______.

18. If $y = e^{\sqrt{x}} + e^{-\sqrt{x}}$ satisfies $Axy'' + By' + Cy = 0$, then $A + B + C$ is equal to ______.

19. If $f(x) = \lambda x^2 + \mu x + 12$, $f'(4) = 15$ and $f'(2) = 11$, then $\lambda + \mu$ is ______.

20. If $f(x) = (ax + b)\sin x + (cx + d)\cos x$, then the values of a, b, c and d such that $f'(x) = x \cos x$ for all x, then $(a + b + c + d)$ is ______.

21. If $f'(x) = \sqrt{3x^2+6}$ and $y = f(x^3)$, then at $x = 1$, $\left(\dfrac{dy}{dx}\right)$ is equal to ______.

22. If $f(x) = \tan^{-1}\left(\dfrac{1}{x^2+x+1}\right) + \tan^{-1}\left(\dfrac{1}{x^2+3x+3}\right) + \tan^{-1}\left(\dfrac{1}{x^2+5x+7}\right) + \ldots \infty$ terms. Then, the value of $\lvert f'(0) \rvert$ is _____.

23. If $f(x) = x^2 - 4x - 3$, $x > 2$ and let $g(x)$ be the inverse of f, then the value of $8[g'(x)]$ at $f(x) = 9$ is ______.

24. Let $h(x)$ be differentiable for all x and let $f(x) = (kx + e^{x}) h(x)$, where k is some constant. If $h(0) = 5$, $h'(0) = -2$ and $f'(0) = 18$, then the value of k is ______.

25. Suppose the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivation 7 and $x = 2$, find derivative of $g(x) = f(x) - f(4x) - 16x$ at $x = 1$.

26. Let f: R to R be defined as $f(x) = x + \cos(x) + 2$ and $g(x)$ be the inverse function of $f(x)$, find $(g(3) + g''(3))$.

27. Let $P(x)$ is a polynomial satisfying $P(x) - P'(x) = x^2 + 2x + 1$, then $P(4)$ is equal to ______.

28. If $f(x) = \dfrac{10\cos(x) + 5\cos(3x) + \cos(5x)}{\cos(6x) + 6\cos(4x) + 15\cos(2x) + 10}$, then find $f(0) + f'(0) + f''(0)$.

29. Let $g(x) = f\left(\dfrac{x}{f(f(x))}\right)$ where $f(f(x))$ is differentiable positive function on $(0, \infty)$ such that $f(1) = 1$ and $f'(1) = 1$. Determine the valuie of $g'(1)$.

30. Let $f(x)$ and $g(x)$ be two polynomials of degree 2 defined as $f(x) = ax^2 + bx + c$ and $g(x) = px^2 + qx + r$. Given $a^2 + b^2 + c^2 - 2a + 4b - 2c + 6 = 0$, where $a, b, c \in R$. Also $g(0) = 2$, $g'(0) = -3$ and $g''(0) = 2$. Find the value of $(f(1) + g(1))$.

31. Let $p(x)$ be a polynomial of degree four such that $p(1) = p(2) = p(3) = p'(4) = 0$. If the real number $x \neq 1, 2, 3$ is such that $p(x) = 0$ can be expressed as $x = \dfrac{p}{q}$ where p and q are relatively prime, then find the value of $(p + q)$.

32. If a curve is represented parametrically by the equations

$$x = \sin\left(t + \frac{7\pi}{12}\right) + \sin\left(t - \frac{\pi}{12}\right) + \sin\left(t + \frac{3\pi}{12}\right),\ y = \cos\left(t + \frac{7\pi}{12}\right) + \cos\left(t - \frac{\pi}{12}\right) + \cos\left(t + \frac{3\pi}{12}\right)$$

then find the value of $\dfrac{d}{dt}\left(\dfrac{x}{y} - \dfrac{y}{x}\right)$ at $t = \dfrac{\pi}{8}$.

33. Let $F(x) = f(x)\, g(x)\, h(x)\ \forall\ x \in R$, where $f(x)\, g(x)$ and $h(x)$ are differentiable functions. At some point x_0, $F'(x_0) = 5\,F(x_0)$, $f'(x_0) = 2\,f(x_0)$, $g'(x_0) = -5g(x_0)$ and $h'(x_0) = kh(x_0)$, then k is equal to

34. Let f be a function defined implicitly by the equation $\dfrac{1 - e^{f(x)}}{1 + e^{f(x)}} = x$ and g be the inverse f. If $g''(\ln 3) - g'(\ln 3) = \dfrac{p}{q}$ where p and q relative prime numbers, then find the value of $(p + q)$.

35. Let $z = \cos^5 x$ and $y = \sin x$, then $\dfrac{d^2z}{dy^2}$ at $x = \dfrac{2\pi}{9}$ is equal to

36. Consider, $f : R \to R$, $f(x) = xe^{\tan^{-1}(x+2)} + 6e^{(x+2)}$ and g be the inverse of f. If $G(x) = \dfrac{x^2}{g(x)}$, then find the absolute value of $5G'(4)$.

37. Let $f(x) = \cos^{-1}(4x^3 - 3x)$. If $f'(0) = p$ and $f'\left(\dfrac{4}{5}\right) = q$, then find the value of $(2p - 3q)$.

38. If a curve is defined parametrically by the equation $x = \cos t$, $y = \sin^3 t$, where t is parameter, then find $\left|\dfrac{yd^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3\right|$ at $t = \dfrac{\pi}{2}$.

39. If $2x = \left(y^{\frac{1}{3}} + y^{\frac{-1}{3}}\right)$, then find the value of $\dfrac{(x^2 - 1)}{y} \cdot \dfrac{d^2y}{dx^2} + \dfrac{x}{y} \cdot \dfrac{dy}{dx}$

40. Let $e^y = \left(\dfrac{\sqrt{1+\alpha} + \sqrt{1-\alpha}}{\sqrt{1+\alpha} - \sqrt{1-\alpha}}\right)$ and $\tan\left(\dfrac{x}{2}\right) = \sqrt{\dfrac{1-\alpha}{1+\alpha}}$, $\alpha \in [-1, 0) \cup (0, 1]$, then find $\left.\dfrac{dy}{dx}\right|_{x=\frac{\pi}{3}}$

ARCHIVE: JEE MAIN

1. **[JEE Mains 2021]**
If $y = y(x)$ is an implicit function of x such that $\log_e(x + y) = 4xy$, then $\dfrac{d^2y}{dx^2}$ at $x = 0$ is equal to ______.

2. **[JEE Mains 2021]**
If $f(x) = \sin\left(\cos^{-1}\left(\dfrac{1 - 2^{2x}}{1 + 2^{2x}}\right)\right)$ and its first derivative with respect to x is $-\dfrac{b}{a}\log_c 2$ when $x = 1$, where a and b are integers, then the minimum value of $|a^2 - b^2|$ is ______.

3. **[JEE Mains 2021]**
Let $f : S \to S$ where $S = (0, \infty)$ be a twice differentiable function such that $f(x + 1) = xf(x)$. If $g : S \to R$ be defined as $g(x) = \log_e f(x)$, then the value of $|g''(5) - g''(1)|$ is equal to :

(A) $\dfrac{205}{144}$ (B) $\dfrac{197}{144}$

(C) $\dfrac{187}{144}$ (D) 1

4. **[JEE Mains 2021]**
Let $f(x) = \cos\left(2\tan^{-1}\sin\left(\cos^{-1}\sqrt{\dfrac{1-x}{x}}\right)\right)$, $0 < x < 1$. Then:

(A) $(1-x)^2 f'(x) - 2(f(x))^2 = 0$
(B) $(1+x)^2 f'(x) + 2(f(x))^2 = 0$
(C) $(1-x)^2 f'(x) + 2(f(x))^2 = 0$
(D) $(1+x)^2 f'(x) - 2(f(x))^2 = 0$

5. [JEE Mains 2021]

If $y(x) = \cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$, $x \in \left(\dfrac{\pi}{2}, \pi\right)$, then $\dfrac{dy}{dx}$ at $x = \dfrac{5\pi}{6}$ is :

(A) $-\dfrac{1}{2}$ (B) -1
(C) $\dfrac{1}{2}$ (D) 0

6. [JEE Mains 2021]

If $\dfrac{dy}{dx} = \dfrac{2^{x+y} - 2^x}{2^y}$, $y(0) = 1$, then $y(1)$ is equal to :

(A) $\log_2(2+e)$ (B) $\log_2(1+e)$
(C) $\log_2(2e)$ (D) $\log_2(1+e^2)$

7. [JEE Mains 2020]

If $y = \sum_{k=1}^{6} k\cos^{-1}\left\{\dfrac{3}{5}\cos kx - \dfrac{4}{5}\sin kx\right\}$, then $\dfrac{dy}{dx}$ at $x = 0$ is _____.

8. [JEE Mains 2020]

If $y(\alpha) = \sqrt{2\left(\dfrac{\tan\alpha + \cot\alpha}{1+\tan^2\alpha}\right) + \dfrac{1}{\sin^2\alpha}}$, $\alpha \in \left(\dfrac{3\pi}{4}, \pi\right)$, then $\dfrac{dy}{d\alpha}$ at $\alpha = \dfrac{5\pi}{6}$ is

(A) 4 (B) -4
(C) $\dfrac{4}{3}$ (D) $-\dfrac{1}{4}$

9. [JEE Mains 2020]

Let $x^k + y^k = a^k$, $(a, k > 0)$ and $\dfrac{dy}{dx} + \left(\dfrac{y}{x}\right)^{1/3} = 0$, then k is

(A) $\dfrac{3}{2}$ (B) $\dfrac{1}{3}$
(C) $\dfrac{2}{3}$ (D) $\dfrac{4}{3}$

10. [JEE Mains 2020]

Let $y = y(x)$ be a function of x satisfying $y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$ where k is a constant and $y\left(\dfrac{1}{2}\right) = -\dfrac{1}{4}$. Then $\dfrac{dy}{dx}$ at $x = \dfrac{1}{2}$, is equal to

(A) $\dfrac{\sqrt{5}}{2}$ (B) $-\dfrac{\sqrt{5}}{2}$
(C) $\dfrac{2}{\sqrt{5}}$ (D) $-\dfrac{\sqrt{5}}{4}$

11. [JEE Mains 2020]

Let $f(x) = (\sin(\tan^{-1}x) + \sin(\cot^{-1}x))^2 - 1$, $|x| > 1$. If $\dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{d}{dx}(\sin^{-1}(f(x)))$ and $y(\sqrt{3}) = \dfrac{\pi}{6}$, then $y(-\sqrt{3})$ is equal to

(A) $\dfrac{5\pi}{6}$ (B) $-\dfrac{\pi}{6}$
(C) $\dfrac{\pi}{3}$ (D) $\dfrac{2\pi}{3}$

12. [JEE Mains 2020]

If $y^2 + \log_e(\cos^2 x) = y$, $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, then

(A) $|y''(0)| = 2$ (B) $|y'(0)| + |y''(0)| = 3$
(C) $|y'(0)| + |y''(0)| = 1$ (D) $y''(0) = 0$

13. [JEE Mains 2020]

The derivative of $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = \dfrac{1}{2}$ is

(A) $\dfrac{\sqrt{3}}{12}$ (B) $\dfrac{\sqrt{3}}{10}$
(C) $\dfrac{2\sqrt{3}}{5}$ (D) $\dfrac{2\sqrt{3}}{3}$

14. [JEE Mains 2019]

Let f be a differentiable function such that $f(1) = 2$ and $f'(x) = f(x)$ for all $x \in R$. If $h(x) = f(f(x))$, then $h'(1)$ is equal to

(A) $4e$ (B) $2e^2$
(C) $4e^2$ (D) $2e$

15. [JEE Mains 2019]

If $2y = \left(\cot^{-1}\left(\dfrac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2$, $x \in \left(0, \dfrac{\pi}{2}\right)$, then $\dfrac{dy}{dx}$ is equal to

(A) $x - \dfrac{\pi}{6}$ (B) $\dfrac{\pi}{3} - x$
(C) $\dfrac{\pi}{6} - x$ (D) $2x - \dfrac{\pi}{3}$

16. [JEE Mains 2019]

If $f(1) = 1$, $f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at $x = 1$ is

(A) 9 (B) 12
(C) 15 (D) 33

17. [JEE Mains 2019]

Let $f(x) = \log_e(\sin x)$, $(0 < x < \pi)$ and $g(x) = \sin^{-1}(e^{-x})$, $(x \geq 0)$. If α is a positive real number such that $a = (fog)'(\alpha)$ and $b = (fog)(\alpha)$, then

(A) $a\alpha^2 + b\alpha - a = -2\alpha^2$ (B) $a\alpha^2 + b\alpha + a = 0$
(C) $a\alpha^2 + b\alpha + a = 0$ (D) $a\alpha^2 - b\alpha - a = 1$

18. [JEE Mains 2019]

The derivative of $\tan^{-1}\left(\dfrac{\sin x - \cos x}{\sin x + \cos x}\right)$, with respect to $\dfrac{x}{2}$, where $\left(x \in \left(0, \dfrac{\pi}{2}\right)\right)$ is

(A) $\frac{2}{3}$ (B) 1

(C) 2 (D) $\frac{1}{2}$

19. [JEE Mains 2017]

If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x}\,g(x)$, then $g(x)$ equals

(A) $\frac{3}{1+9x^3}$ (B) $\frac{9}{1+9x^3}$

(C) $\frac{3x\sqrt{x}}{1-9x^3}$ (D) $\frac{3x}{1-9x^3}$

20. [JEE-Mains 2014]

If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to

(A) $1+x^5$ (B) $5x^4$

(C) $\frac{1}{1+\{g(x)\}^5}$ (D) $1+\{g(x)\}^5$

21. [JEE-Mains 2013]

If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$

(C) 1 (D) $\sqrt{2}$

ARCHIVE: JEE ADVANCED

1. [JEE Adv 2019]

Let $f : R \to R$ be a function. We say that f has PROPERTY 1: if $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{\sqrt{|h|}}$ exists and is finite, and PROPERTY 2: if $\lim\limits_{h\to 0}\frac{f(h)-f(0)}{h^2}$ exists and is finite. Then which of the following option is/are correct?

(A) $f(x) = |x|$ has PROPERTY 1

(B) $f(x) = x^{2/3}$ has PROPERTY 1

(C) $f(x) = x|x|$ has PROPERTY 2

(D) $f(x) = \sin x$ has PROPERTY 2

2. [JEE Adv 2016]

Let $f : R \to R$, $g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$

(C) $h(0) = 16$ (D) $h(g(3)) = 36$

3. [JEE Adv 2011]

Let $f(\theta) = \sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

4. [JEE Adv 2009]

If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is

5. [JEE Adv 2008]

Let $g(x) = \ln f(x)$ where $f(x)$ is a twice differentiable positive function on $(0, \infty)$ such that $f(x + 1) = x\, f(x)$. Then for $N = 1, 2, 3$

$$g''\left(N+\frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(A) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1+\frac{1}{9}+\frac{1}{25}+\ldots+\frac{1}{(2N+1)^2}\right\}$

6. Let f and g be real valued functions defined on interval $(-1, 1)$ such that $g''(x)$ is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x) \sin x$.

Statement-1 : $\lim\limits_{x\to 0}\,[g(x)\cot x - g(0)\operatorname{cosec} x] = f''(0)$ and

Statement-2 : $f'(0) = g(0)$

(A) Statement-1 is True, Statement-2 is True ; statement-2 is a correct explanation for statement-1

(B) Statement-1 is True, Statement-2 is True ; statement-2 is NOT a correct explanation for statement1

(C) Statement-1 is True, Statement-2 is False

(D) Statement-1 is False, Statement-2 is True

7. [JEE Adv 2007]

$\frac{d^2x}{dy^2}$ equals

(A) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (B) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(C) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ (D) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

8. Let $f : R \to R$, $g : R \to R$ and $h : R \to R$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in R$. Then

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$

(C) $h(0) = 16$ (D) $h(g(3)) = 36$

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. C	**2.** D	**3.** A	**4.** A	**5.** A	**6.** C	**7.** D	**8.** A	**9.** D	**10.** A
11. A	**12.** B	**13.** A	**14.** B	**15.** C	**16.** C	**17.** C	**18.** D	**19.** B	**20.** D
21. A	**22.** A	**23.** A	**24.** D	**25.** B	**26.** B	**27.** A	**28.** B	**29.** A	**30.** C
31. B	**32.** D	**33.** B	**34.** A	**35.** C	**36.** B	**37.** C	**38.** C	**39.** A	**40.** A
41. A	**42.** D	**43.** C	**44.** C	**45.** A	**46.** D	**47.** C	**48.** A	**49.** D	**50.** C
51. D	**52.** C	**53.** C	**54.** D	**55.** C	**56.** B	**57.** D	**58.** D	**59.** A	**60.** A
61. A	**62.** C	**63.** A	**64.** A	**65.** C	**66.** B	**67.** C	**68.** C	**69.** A	**70.** D
71. D	**72.** B	**73.** A	**74.** C	**75.** D	**76.** C	**77.** A	**78.** C	**79.** A	

Multiple Correct Choice Type Questions

1. A, B, C	**2.** A, C, D	**3.** A, D	**4.** A, B, C	**5.** A, D
6. A, B, C	**7.** A, C	**8.** A, B, C	**9.** A, B, C	**10.** A, C, D
11. A, B	**12.** A, B, D	**13.** A, B, C, D	**14.** B, D	**15.** A, B
16. A, C	**17.** A, C, D	**18.** A, B, C	**19.** A, B	**20.** B, D
21. A, D	**22.** A, D	**23.** A, B	**24.** A, C	**25.** B, C
26. A, B, C	**27.** A, B, D	**28.** B, C, D	**29.** A, B	**30.** B, D

Linked Comprehension Type Questions

1. B	**2.** D	**3.** A	**4.** A	**5.** A	**6.** B	**7.** C	**8.** A	**9.** A	**10.** D
11. C	**12.** D	**13.** D	**14.** B	**15.** B	**16.** A	**17.** D	**18.** C	**19.** C	**20.** B
21. D	**22.** B	**23.** C							

Matrix Match/Column Match Type Questions

1. A → (s)	B → (q)	C → (q)	D → (p)
2. A → (r)	B → (s)	C → (s)	D → (q)
3. A → (q)	B → (r)	C → (p)	D → (p)
4. A → (q)	B → (r)	C → (p)	D → (q)
5. A → (t)	B → (r)	C → (t)	D → (p)
6. A → (t)	B → (q)	C → (r)	D → (s)
7. A → (p)	B → (p)	C → (q)	D → (r)

Integer/Numerical Answer Type Questions

1. 8	**2.** 9	**3.** 3	**4.** 0	**5.** 1	**6.** 1	**7.** 2	**8.** 9	**9.** 9	**10.** 7
11. 8	**12.** 5	**13.** 5	**14.** 1	**15.** 0	**16.** 25	**17.** 7	**18.** 5	**19.** 8	**20.** 2
21. 9	**22.** 1	**23.** 1	**24.** 3	**25.** 3	**26.** 2	**27.** 37	**28.** 1	**29.** 0	**30.** 0
31. 61	**32.** 8	**33.** 8	**34.** 25	**35.** 5/2	**36.** 24	**37.** 21	**38.** 3	**39.** 9	**40.** 2

ARCHIVE: JEE MAIN

1. 40	**2.** 481	**3.** A	**4.** C	**5.** A	**6.** B	**7.** 91	**8.** A	**9.** C	**10.** B
11. A	**12.** A	**13.** B	**14.** A	**15.** A	**16.** D	**17.** D	**18.** C	**19.** B	**20.** D
21. A									

ARCHIVE: JEE ADVANCED

1. A, B	**2.** B, C	**3.** 1	**4.** 2	**5.** A	**6.** B	**7.** D	**8.** B, C

CHAPTER 5

Continuity and Differentiability of Function

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Continuity of Function
(b) Types of Discontinuity
(c) Theorem on Continuous Function
(d) Continuity Test in an Interval
(e) Intermediate Mean Value Theorem
(f) Differentiability of function at Point $x = \alpha$
(g) Relation between Continuity and Differentiability
(h) Theorem on Differentiability
(i) Differentiability Test in an Interval
(j) Continuity and Differentiability of Composite Function
(k) Extended Formula of Derivative
(l) Functional Equation

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

CONTINUITY OF FUNCTION

Continuity of function tells that graph of function is not broken, it can be drawn in one go without lifting the pen.

A function $f(x)$ is said to be continuous at a point $x = \alpha$, if right hand limit at α = left hand limit at α = value of the function at α

$$\lim_{x\to\alpha^+} f(x) = \lim_{x\to\alpha^-} f(x) = f(\alpha) = \text{Finite}$$

or $$f(\alpha^+) = f(\alpha^-) = f(\alpha) = \text{Finite}$$

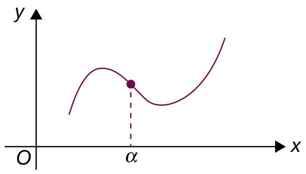
$$\text{RHL} = \text{LHL} = f(\alpha) = \text{Finite}$$

If above relation is not true, then function $f(x)$ is said to be discontinuous at $x = \alpha$.

Geometrical Meaning

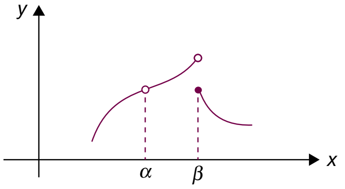
Function $f(x)$ will be continuous at $x = \alpha$, if graph of $f(x)$ does not break at point $(\alpha, f(\alpha))$

Figure (i) shows a continuous function



(i)

Figure (ii) shows a discontinuous function at $x = \alpha, \beta$



(ii)

Discontinuous functions are $f(x) = 1/x$ at $x = 0$, $f(x) = e^{1/x}$ at $x = 0$, $f(x) = \sin 1/x$, $f(x) = \cos 1/x$ at $x = 0$, $f(x) = [x]$ at every integer, $f(x) = \{x\} = x - [x]$ at every integer,

$$f(x) = \tan x, f(x) = \sec x \text{ when } x = (2n+1)\frac{\pi}{2}, n \in Z.$$

$$f(x) = \cot x, f(x) = \text{cosec } x \text{ when } x = n\pi, n \in Z.$$

ILLUSTRATION 1

Let $f(x) = \begin{cases} x^2, & x \in Z \\ -\dfrac{(x^2-4)}{2-x}, & x \notin Z \end{cases}$ where Z is the set of all integers, then check continuity of $f(x)$ at $x = 2$

SOLUTION

$$f(2) = 4,$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{(2-x)(2+x)}{(2-x)} = \lim_{x\to 2}(2+x) = 4$$

$$\therefore \quad f(2) = 4 = \lim_{x\to 2} f(x)$$

Hence, $f(x)$ is continuous at $x = 2$

ILLUSTRATION 2

The function $f(x) = \dfrac{1-\sin x+\cos x}{1+\sin x+\cos x}$, $x \neq \pi$, if $f(x)$ is continuous at $x = \pi$, find $f(\pi)$,

SOLUTION

$$\lim_{x\to\pi} f(x) = \lim_{x\to\pi} \frac{(1+\cos x)-\sin x}{(1+\cos x)+\sin x}$$

$$= \lim_{x\to\pi} \frac{2\cos^2\frac{x}{2} - 2\sin^2\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \lim_{x\to\pi} \frac{\cos\frac{x}{2} - \sin\frac{x}{2}}{\cos\frac{x}{2} + \sin\frac{x}{2}} = \lim_{x\to\pi} \frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}$$

$$= \lim_{x\to\pi} \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = -1$$

Hence, $f(x)$ is continuous at $x = \pi$, so $f(\pi) = \lim\limits_{x\to\pi} f(x) = -1$

TYPES OF DISCONTINUITY

Function is discontinuous at point $x = \alpha$, there are mainly two categories:

Discontinuity of Ist Kind

In this category $\lim\limits_{x\to\alpha} f(x)$ exists but is not equal to $f(\alpha)$ then: the function is said to have a **removable type of discontinuity of the first kind**. If LHL = RHL $\neq f(\alpha)$ provided, LHL and RHL both are unique and finite. In this case, we can redefine the function such that, $\lim\limits_{x\to\alpha} f(x) = f(\alpha)$ and make it continuous at $x = \alpha$. Types of removable discontinuity of first kind can be further classified as:

(a) Missing Point Discontinuity

When $\lim\limits_{x\to\alpha} f(x)$ exists finitely, but $f(\alpha)$ is not defined.

E.g. $f(x) = \dfrac{(2-x)(9-x^2)}{(2-x)}$ has a missing point discontinuity at $x = 2$.

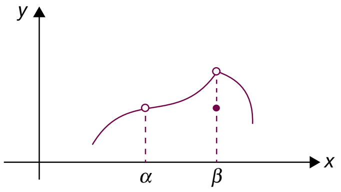
(b) Isolated point discontinuity

When $\lim\limits_{x\to\beta} f(x)$ exists finitely, $f(\beta)$ is defined, and $\lim\limits_{x\to\beta} f(x) \neq f(\beta)$

E.g. $f(x) = \dfrac{(x^2-16)}{(x-4)}$; $x \neq 4$ and $f(4) = 9$, function $f(x)$ breaks at $x = 4$.

$$\lim_{x\to 4} f(x) = 8 \neq f(4)$$

$f(x)$ has isolated point discontinuity at $x = 4$
Let consider a function shown in Figure (iii)



(iii)

Function has missing point discontinuity at $x = \alpha$ and isolated point discontinuity at $x = \beta$

Discontinuity of IInd Kind

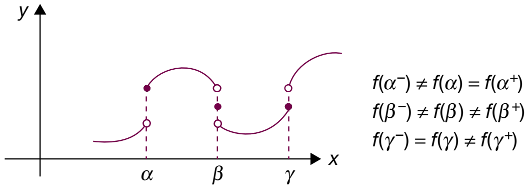
In this category, limit of function at point $x = \alpha$ does not exist so, redefine the function cannot make it continuous at $x = \alpha$, hence, it is known as **non-removable type of discontinuity of IInd kind.** Types of non – removable discontinuity of second kind can be further classified as:

(a) Finite Type Discontinuity

If LHL ≠ RHL provided, then LHL and RHL are both distinct and finite. Hence, limit does not exist. It is known as finite type, non-removable discontinuity.

Jump of discontinuity = | LHL – RHL |

Let consider a function having graph shown in figure (iv), it has finite type ofdiscontinuity at point $x = \alpha, \beta, \gamma$



(iv)

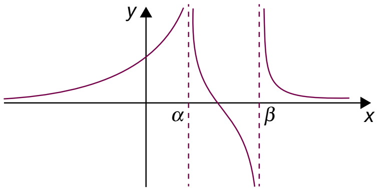
(b) Infinite Type Discontinuity

Either LHL or RHL or both may leading to ∞ or –∞, is having infinite type discontinuity.

E.g. $f(x) = \dfrac{1}{x-2}$ or $g(x) = \dfrac{1}{(x-2)^2}$ at $x = 2$,

$h(x) = \cot x$ has infinite discontinuity at $x = n\pi$, $n \in I$,
$\phi(x) = \tan x$, also has infinite type discontinuity at

$$x = \left(\frac{2n+1}{2}\right)\pi,\ n \in I$$



From above graph, we can conclude that at $x = \alpha$; LHL and RHL $\to \infty$ and at $x = \beta$; LHL $\to -\infty$, RHL $\to \infty$

(c) Oscillatory Discontinuity

LHL or RHL or both are not having finite value, they are oscillating between finite numbers.

E.g. $f(x) = \sin\frac{1}{x}$ at $x = 0$. In this case the $\lim\limits_{x\to 0} f(x)$ is not having finite value, it oscillating between –1 and 1.

We can summarize type of discontinuity using following chart.

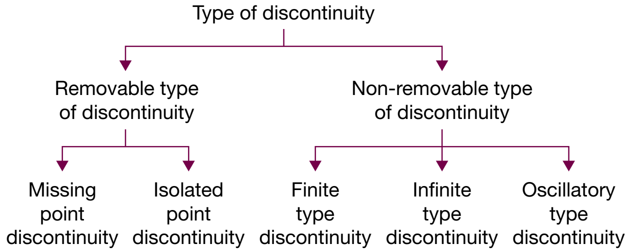

Type of discontinuity
Removable type of discontinuity
Non-removable type of discontinuity
Missing point discontinuity
Isolated point discontinuity
Finite type discontinuity
Infinite type discontinuity
Oscillatory type discontinuity

ILLUSTRATION 3

Let $f(x) = \begin{cases} \left(\dfrac{(1+2x)^{\frac{1}{x}}}{e^2}\right)^{\frac{1}{x}}, & x \neq 0 \\ e, & x = 0 \end{cases}$. Check continuity of $f(x)$ at $x = 0$, if discontinuous then specify type of discontinuity.

SOLUTION

$$\text{Let } L = \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{(1+2x)^{\frac{1}{x}}}{e^2}\right)^{\frac{1}{x}}$$

$$\Rightarrow \quad \ln L = \lim_{x\to 0}\frac{1}{x}\left[\ln(1+2x)^{1/x} - \ln e^2\right]$$

$$\Rightarrow \quad \ln L = \lim_{x\to 0}\frac{\left(\frac{1}{x}\ln(1+2x) - 2\right)}{x} = \lim_{x\to 0}\frac{\ln(1+2x) - 2x}{x^2}$$

$$\Rightarrow \quad \ln L = \lim_{x\to 0}\frac{2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \ldots - 2x}{x^2} = -2$$

$$\Rightarrow \quad L = e^{-2}$$

Therefore, $f(0) = e$

Hence, function is discontinuous at $x = 0$.

It has isolated point, removable type discontinuity of first kind.

ILLUSTRATION 4

A function f is defined as follows $f(x) = \begin{cases} \sin x, \text{ if } x \le 0 \\ 2x+3, \text{ if } x > 0 \end{cases}$ discuss continuity of $f(x)$ at $x = 0$, find jump of discontinuity.

SOLUTION

For continuity of $f(x)$ at $x = 0$

$$\text{LHL} = f(0^-) = \lim_{x\to 0^-} f(x) = f(0) = \sin 0 = 0$$

$$\text{RHL} = f(0^+) = 3$$

Clearly, $f(0^-) \neq f(0^+)$ so limit does not exist, we can conclude that it is non-removable finite type discontinuity of II$^{\text{nd}}$ type.

Hence, jump of discontinuity = $|\text{LHL} - \text{RHL}| = 3$

ILLUSTRATION 5

If $f(x) = \begin{cases} \dfrac{\sin\{\cos x\}}{x - \frac{\pi}{2}}, & x \neq \dfrac{\pi}{2} \\ 1, & x = \dfrac{\pi}{2} \end{cases}$ where $\{k\}$ represents the fractional part of k, discuss the continuity of $f(x)$ at $x = \dfrac{\pi}{2}$

SOLUTION

$$\text{LHL} = \lim_{x\to\frac{\pi}{2}^-} f(x) = \lim_{h\to 0}\frac{\sin\left\{\cos\left(\frac{\pi}{2} - h\right)\right\}}{\frac{\pi}{2} - h - \frac{\pi}{2}}$$

Put $x = \pi/2 - h$

$$= \lim_{h\to 0}\frac{\sin\{\sin h\}}{-h} = -\lim_{h\to 0}\left(\frac{\sin(\sin h)}{\sin h}\right)\left(\frac{\sin h}{h}\right) = -1$$

$$\text{RHL} = \lim_{x\to\frac{\pi}{2}^+} f(x) = \lim_{h\to 0}\frac{\sin\left\{\cos\left(\frac{\pi}{2} + h\right)\right\}}{\frac{\pi}{2} + h - \frac{\pi}{2}}$$

$$= \lim_{h\to 0}\frac{\sin\{-\sin h\}}{h} = \lim_{h\to 0}\frac{\sin(1 - \sin h)}{h} \to \infty$$

Hence, $\lim\limits_{x\to\frac{\pi}{2}} f(x)$ does not exist. So, $f(x)$ is not continuous at $x = \dfrac{\pi}{2}$, it has infinite type, non-removable type of discontinuity.

ILLUSTRATION 6

$f(x) = \begin{cases} \dfrac{(e^{x^2} - 1)\sin(\ln(1+2x))}{\tan^6(2\sqrt{x})}, & x \neq 0 \\ k, & x = 0 \end{cases}$ Find the value of k for which $f(x)$ is continuous at $x = 0$

SOLUTION

$\because$ $f(x)$ is continuous at $x = 0 \Rightarrow f(0) = \lim\limits_{x\to 0} f(x)$

$$\text{Given, } f(0) = k = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(e^{x^2}-1)\sin(\ln(1+2x))}{\tan^6(2\sqrt{x})}$$

$$= \lim_{x\to 0} \frac{\frac{(e^{x^2}-1)}{x^2}\cdot x^2 \cdot \frac{\sin[\ln(1+2x)]}{\ln(1+2x)}\ln(1+2x)}{\left(\frac{\tan(2\sqrt{x})}{2\sqrt{x}}\right)^6 (2\sqrt{x})^6}$$

$$k = \lim_{x\to 0} \frac{1\cdot x^2 1\cdot \frac{\ln(1+2x)}{2x}\cdot 2x}{1\cdot 2.^6 x^3} = \lim_{x\to 0} \frac{2x^3}{2^6 x^3} = \frac{1}{2^5}$$

Hence, $k = \frac{1}{2^5}$

ILLUSTRATION 7

Let $f(x) = \begin{cases} \dfrac{\sin 2x - 2\tan x}{\ln(1+x^3)}, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$. If $f(x)$ is continuous at $x = 0$, then find the value of k.

SOLUTION

Continuous at $x = 0$

$$\lim_{x\to 0} f(x) = f(0) = k = \lim_{x\to 0}\left(\frac{\sin 2x - 2\tan x}{\ln(1+x^3)}\right)$$

$$k = \lim_{x\to 0} \frac{2\sin x\cos x - 2\frac{\sin x}{\cos x}}{\frac{\ln(1+x^3)}{x^3}x^3}$$

$$= \lim_{x\to 0}\frac{\sin x}{x} \cdot \lim_{x\to 0}\frac{2\cos x - \frac{2}{\cos x}}{x^2}$$

$$k = \lim_{x\to 0} (-2)\cdot \frac{(1-\cos^2 x)}{(\cos x)x^2}$$

$$= -2\lim_{x\to 0}\frac{(1+\cos x)(1-\cos x)}{\cos x \; x^2}$$

$$k = -2 \lim_{x\to 0} \frac{2(1-\cos x)}{x^2} = (-2)\,(2)\left(\frac{1}{2}\right) = -2.$$

ILLUSTRATION 8

Let $g(x) = \begin{cases} \left(\dfrac{1}{1+[\sin x]}\right)^{\frac{2}{2x-\pi}}, & x \neq \dfrac{\pi}{2} \\ p^2 - 1, & x = \dfrac{\pi}{2} \end{cases}$ then find the value of p for which $f(x)$ is continuous at $x = \frac{\pi}{2}$, where $[k]$ denotes the largest integer less than or equal to k.

SOLUTION

As, $g(x)$ is continuous at $x = \frac{\pi}{2}$,

$$\text{So, } g\left(\frac{\pi}{2}\right) = \lim_{x\to\frac{\pi}{2}}\left(\frac{1}{1+[\sin x]}\right)^{\frac{2}{2x-\pi}}$$

Where we have exact 1 as base, (It is not 1^∞ indeterminate form): $g\left(\frac{\pi}{2}\right) = 1$

Hence, $p^2 - 1 = 1 \Rightarrow p^2 = 2 \Rightarrow p = \pm\sqrt{2}$

ILLUSTRATION 9

Given, $a, b \in R$ and $[x]$ denotes the largest integer less than or equal to x.

If $f(x) = \begin{cases} \sin\dfrac{\pi}{2}(x-[x]), & x < 5 \\ 5(b-1), & x = 5 \\ \dfrac{ab^2\,|x^2-11x+24|}{x-3}, & x > 5 \end{cases}$ is continuous at $x = 5$, find a and b.

SOLUTION

As, $f(x)$ is continuous at $x = 5$, so $f(5^+) = f(5) = f(5^-)$ …(1)

$$\text{Now, } f(5^-) = \lim_{x\to 5^-} \sin\frac{\pi}{2}(x-[x]) = \sin\frac{\pi}{2}(5-4) = 1.$$

$$\text{Also, } f(5^+) = \lim_{x\to 5^+}\frac{ab^2\,|x^2-11x+24|}{(x-3)}$$

$$= \lim_{x\to 5^+}\frac{-ab^2(x-3)(x-8)}{(x-3)} = 3ab^2$$

$\therefore$ From equation (1), we get $3ab^2 = 5(b-1) = 1$

$$\Rightarrow \quad b = \frac{6}{5} \text{ and } a = \frac{25}{108}$$

ILLUSTRATION 10

If $f(x) = \begin{cases} 3(1+|\tan x|)^{\frac{\alpha}{|\tan x|}}, & \dfrac{-1}{2} < x < 0 \\ \beta, & x = 0 \\ 3\left(1+\left|\dfrac{\sin x}{3}\right|\right)^{\frac{2}{|\tan x|}}, & 0 < x < \dfrac{2}{3} \end{cases}$ is a continuous function at $x = 0$, then find the ordered pair (α, β).

SOLUTION

$$\text{LHL} = f(0^-) = \lim_{x\to 0^-} 3\left(1+|\tan x|\right)^{\frac{\alpha}{|\tan x|}}, (1^\infty \text{ Form})$$

$$= 3e^{\lim\limits_{x\to 0^-}\frac{|\tan x|\alpha}{|\tan x|}} = 3e^{\alpha}$$

$$\text{Now, RHL} = f(0^+) = \lim_{x\to 0^+} f(x)$$

$$= \lim_{x\to 0^+} 3\left(1+\frac{|\sin x|}{3}\right)^{\frac{2}{|\tan x|}}, \ (1^\infty \text{ Form})$$

$$= 3e^{\lim\limits_{x\to 0^+}\frac{|\sin x|}{3}\frac{2}{|\tan x|}} = 3e^{2/3}$$

So, for the function $f(x)$ to be continuous at $x = 0$, we must have

$$3e^{\alpha} = 3e^{2/3} = \beta. \text{ so, } \alpha = 2/3,\ \beta = 3e^{2/3}$$

ILLUSTRATION 11

$$\text{Let } f(x) = \begin{cases} (\sin x + \cos x)^{\operatorname{cosec} x}, & \dfrac{-\pi}{2} < x < 0 \\ a & x = 0 \\ \dfrac{e^{\frac{1}{x}} + e^{\frac{2}{x}} + e^{\frac{3}{|x|}}}{ae^{\frac{2}{x}} + be^{\frac{3}{|x|}}}, & 0 < x < \dfrac{\pi}{2} \end{cases}.$$

If $f(x)$ is continuous at $x = 0$, then find $(a^2 + b^2)$

SOLUTION

Now, $\text{LHL} = f(0^-) = \lim\limits_{h\to 0} (\sin(0 - h) + \cos(0 - h))^{\operatorname{cosec}(0 - h)}$

$$= \lim_{h\to 0} (-\sin h + \cos h)^{-\operatorname{cosec} h} = e^L, \qquad (1^\infty \text{ Form})$$

Where $L = \lim\limits_{h\to 0} (-\sin h + \cos h - 1)(-\operatorname{cosec} h)$

$$= \lim_{h\to 0}\left(\frac{-\sin h + \cos h - 1}{-\sin h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\dfrac{\sin h}{h} + \dfrac{1-\cos h}{h}}{\dfrac{\sin h}{h}}\right) = \frac{1+0}{1} = 1$$

$$\text{LHL} = e^L = e \Rightarrow f(0) = a = e$$

$$\text{RHL} = f(0^+) = \lim_{h\to 0}\frac{e^{\frac{1}{h}} + e^{\frac{2}{h}} + e^{\frac{3}{h}}}{ae^{\frac{2}{h}} + be^{\frac{3}{h}}} = \lim_{h\to 0}\frac{e^{\frac{-2}{h}} + e^{\frac{-1}{h}} + 1}{ae^{\frac{-1}{h}} + b} = \frac{1}{b}$$

$\because$ $f(x)$ is continuous at $x = 0$,

$\Rightarrow$ $f(0) = f(0^-) = f(0^+)$ $\quad \therefore a = e = \dfrac{1}{b}$

Hence, $(a^2 + b^2) = e^2 + e^{-2}$

ILLUSTRATION 12

$$\text{Let } f(x) = \begin{cases} \dfrac{\alpha(1 - x\sin x) + \beta\cos x + 5}{x^2}, & x < 0 \\ 3, & x = 0\,. \\ \left(1 + \left(\dfrac{\gamma x + \delta x^3}{x^2}\right)\right)^{\frac{1}{x}}, & x > 0 \end{cases}$$

If $f(x)$ is continuous at $x = 0$, then find the value of

$$(\alpha^2 + \beta^2 + \gamma^2 + e^{2\delta})$$

SOLUTION

As, $f(x)$ is continuous at $x = 0$, so at $x = 0$, so at $x = 0$, both left and right limits must exist and both must be equal to 3.

Now, $\lim\limits_{x\to 0^-}\dfrac{\alpha(1 - x\sin x) + \beta\cos x + 5}{x^2}$

$$= \lim_{x\to 0^-}\frac{\alpha\left\{1 - x\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}\right)\right] + \beta\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}\ldots\right) + 5}{x^2}$$

$$= \lim_{x\to 0^-}\frac{(\alpha + \beta + 5) + \left(-\alpha - \dfrac{\beta}{2}\right)x^2 + \ldots}{x^2} = -\alpha - \frac{\beta}{2}$$

If $\lim\limits_{x\to 0^-} f(x)$ exist, then $\alpha + \beta + 5 = 0$ …(1)

and $\lim\limits_{x\to 0^-} f(x) = 3 \Rightarrow -\alpha - \dfrac{\beta}{2} = 3$ …(2)

Solving (1) and (2), we get

$$\alpha = -1,\ \beta = -4.$$

Since, RHL = $f(0)$

$$\Rightarrow \quad \left(1 + \left(\frac{\gamma x + \delta x^3}{x^2}\right)\right)^{\frac{1}{x}} = 3$$

$\Rightarrow$ $e^{\lim\limits_{x\to 0^+}\delta} = 3 \Rightarrow e^{\delta} = 3$ and $\gamma = 0$ for limit to exist

Hence, $\alpha^2 + \beta^2 + \gamma^2 + e^{2\delta} = (-1)^2 + (-4)^2 + (0)^2 + (3)^2$

$$= 1 + 16 + 0 + 9 = 26$$

ILLUSTRATION 13

Let $f(x)$ and $g(x)$ be continuous function for all $x \in R$.

If $\lim\limits_{x\to 3}\dfrac{f(x)}{x-3} = \lim\limits_{x\to f(3)}\dfrac{g(x)}{x^2} = 5$, then evaluate $\lim\limits_{x\to g(f(3))}\dfrac{\tan(10x)}{x}$

SOLUTION

Given, $\lim\limits_{x\to 3}\dfrac{f(x)}{x-3} = 5$ (exists)

So, $\lim\limits_{x\to 3} f(x) = 0 \Rightarrow f(3) = 0$ (As, f is continuous on R)

Also, $\lim\limits_{x\to f(3)}\dfrac{g(x)}{x^2} = 5$(exists) $\Rightarrow \lim\limits_{x\to 0}\dfrac{g(x)}{x^2} = 5$ (exists)

So, $\lim\limits_{x\to 0} g(x) = 0 \Rightarrow g(0) = 0$ (As, g is continuous on R)

Hence, $\lim\limits_{x\to g(f(3))}\dfrac{\tan(10x)}{x} = 10\lim\limits_{x\to 0}\dfrac{\tan(10x)}{10x} = 10$

Test Your Concepts-I

(Solutions on page H.109)

1. $f(x) = \begin{cases} x-1, & x<0 \\ \frac{1}{4}, & x=0 \\ x^2, & x>0 \end{cases}$

With the help of graph, discuss discontinuity at $x = 0$.

2. Redefine the function $f(x) = x \sin 1/x$ in such a way that it becomes continuous in R, then find $f(0)$.

3. Let $f(x) = \begin{cases} (2-\tan x \cdot \sqrt{\cot x})^{\frac{1}{\sin x - \cos x}}, & \text{if } x \neq \frac{\pi}{4} \\ k, & \text{if } x = \frac{\pi}{4} \end{cases}$.

If $f(x)$ continuous at $x = \frac{\pi}{4}$, then find k.

4. Let $f(x) = \begin{cases} \dfrac{e^{x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{1-x}}{\ln(1+\sqrt{x})}, & x \in (0,1) \\ k, & x \leq 0 \end{cases}$.

If $f(x)$ is continuous at $x = 0$, then find the value of k.

5. Let $f(x) = \begin{cases} \left(\dfrac{x\sin x + 2\cos 2x}{2}\right)^{\frac{1}{x^2}}, & x \neq 0 \\ e^{\frac{-k}{2}}, & x = 0 \end{cases}$.

If $f(x)$ is continuous at $x = 0$, then find the value of k.

6. Find the value of k, so that the function

$f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$ is continuous at $x = \pi$.

7. Show that the function $f(x) = \begin{cases} \dfrac{\sin x}{x} + \cos x, & x \neq 0 \\ 2, & x = 0 \end{cases}$ is continuous at $x = 0$.

8. Examine the continuity of the function

$f(x) = \begin{cases} \dfrac{|\sin x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ at $x = 0$.

9. If the function $f(x) = \begin{cases} 3ax+b, & \text{if } x>1 \\ 11, & \text{if } x=1 \\ 5ax-2b, & \text{if } x<1 \end{cases}$ is continuous at $x = 1$, find the values of a and b.

10. Examine the continuity of the function

$f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at $x = 0$.

11. If $f(x)$ is continuous at $x = 0$, find the values of a, b and c,

where $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{if } x<0 \\ c, & \text{if } x=0 \\ \dfrac{\sqrt{x+bx^2}-\sqrt{x}}{bx^{3/2}}, & \text{if } x>0 \end{cases}$.

12. For what value of k, is the following function continuous at $x = 0$?

$f(x) = \begin{cases} \dfrac{1-\cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$

Answer key

1. Non-removable finite type discontinuity **2.** 0 **3.** $e^{\frac{-1}{\sqrt{2}}}$ **4.** $k = \frac{2}{\pi}$ **5.** $k = 3$ **6.** $-\frac{2}{\pi}$

8. Discontinuous **9.** $a = 3, b = 2$ **10.** Continuous **11.** $a = -3/2$, $b \in R$ and $c = 1/2$ **12.** 1

THEOREM ON CONTINUOUS FUNCTION

A function is said to be continuous function if it is continuous at every point in its domain.

All polynomial function, exponential function, logarithm function, trigonometric functions are continuous function in their domain.

$$f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_nx^n \text{ (Polynomial)}$$

$$f(x) = |x|, x + |x|, x - |x|, x|x|$$

$$f(x) = \sin x, f(x) = \cos x, f(x) = e^x, f(x) = a^x, a > 0,$$
$$f(x) = \log x, f(x) = \log a^x, a > 0$$

If $f(x)$ and $g(x)$ are continuous functions then following are also continuous functions:

$$f(x) + g(x), f(x) - g(x), f(x) \cdot g(x), \lambda f(x), |f(x)|, (f(x))^n,$$
$$f(g(x)), f(x)/g(x), \text{ if } g(x) \neq 0.$$

If a function is consisting of two or more functions and we have knowledge of continuity of sub-functions at some point, then using following results we talk about continuity of resultant function. It certainly saves time and efforts.

Table 5.1 **At point $x = \alpha$**

$f(x)$	$g(x)$	$f(x) \pm g(x)$	$f(x) \cdot g(x)$	$f(x)/g(x)$ where $g(\alpha) \neq 0$
Continuous	Continuous	Continuous	Continuous	Continuous
Discontinuous	Continuous	Discontinuous	Can't Say (?)	Can't Say (?)
Discontinuous	Discontinuous	Can't Say (?)	Can't Say (?)	Can't Say (?)

(A) A function consisting of sum or difference of continuous and discontinuous sub-functions at $x = \alpha$ then resultant function must be discontinuous function at $x = \alpha$.

Let consider $f(x) = [x]$ where [.] is gif, discontinuous at $x = 0$.

$$g(x) = |x| \text{ continuous at } x = 0$$

$$h(x) = f(x) + g(x)$$

$$h(x) = [x] + |x| = \begin{cases} -1-x & -1 < x < 0 \\ 0-x & a < x < 1 \end{cases}$$

$h(0^-) = -1$, $h(0^+) = 1$ at $x = 0$ discontinuous.

We can conclude that sum or difference of continuous and discontinuous function results in giving us discontinuous function.

(B) A function consisting of sum or difference of two discontinuous sub-functions at $x = \alpha$, then resultant function may or may not be continuous function.

CASE 1: Let consider $f(x) = [x]$ discontinuous at $x = 0$ where [.] is gif.

$$g(x) = \{x\} \text{ discontinuous at } x = 0 \text{ where } \{.\} \text{ is fpf}$$

$$h(x) = f(x) + g(x) = [x] + \{x\} = x$$

It is continuous at $x = 0$. we can see that two discontinuous function, on addition, are giving us continuous function

CASE 2: Let consider $f(x) = [x]$ discontinuous at $x = 0$ where [.] is gif.

If we consider $k(x) = -\{x\}$, discontinuous at $x = 0$, where {.} is fpf

$$h(x) = f(x) + k(x) = [x] - \{x\}$$

$$= [x] - (x - [x]) = 2[x] - x$$

$$h(0^+) = 2(0) - 0 = 0$$

$$h(0) = 0 - 0 = 0$$

$$h(0^-) = 2(-1) - 0 = -2 \text{ discontinuous at } x = 0$$

We can see that two discontinuous function, on addition, are giving us discontinuous function clearly. We can conclude that at one point sum or difference of two discontinuous function may be continuous or discontinuous.

(C) A function consisting of product of two discontinuous sub-functions at $x = \alpha$, then resultant function may or may not be continuous function.

Lets consider the product of two discontinuous function.

Let consider $f(x) = [x]$, where [.] is gif.

$$g(x) = \{x\} \text{ where } \{.\} \text{ is fpf}$$

$$T(x) = f(x)\, g(x) = [x]\, \{x\}$$

CASE 1: At $x = 0$, $f(x)$ and $g(x)$ are discontinuous.

$T(0) = 0 = T(0^+) = T(0^-)$

$T(x)$ is continuous function at $x = 0$.

CASE 2: At $x = 2$, $f(x)$ and $g(x)$ are discontinuous.

At $x = 2$, function $f(x)$ and $g(x)$ are discontinuous.

$T(2) = 0 = T(2^+)$; $T(2^-) = (1)\,(1) = 1$

$T(x)$ is discontinuous function at $x = 2$.

CONTINUITY TEST IN AN INTERVAL

A function $f(x)$ is said to be continuous in the closed interval $[a, b]$ if it is

(i) Continuous at every point of the open interval (a, b)

$$f(\alpha^+) = f(\alpha^-) = f(\alpha) = \text{Finite, for every } \alpha \in (a, b)$$

(ii) Right continuous at $x = a$, $f(a) = f(a^+) = \text{Finite}$

(iii) Left continuous at $x = b$, $f(b) = f(b^-) = \text{Finite}$

Function $f(x)$ will be continuous in an interval, if graph of function does not break in the entire interval.

Working Approach

In an interval there are infinite points, so it becomes tedious work to check function's continuity. Therefore, we should first identify few suspect points where function can be discontinuous

Possible point of discontinuity of function $f(x)$ where $x \in [a, b]$

(i) Boundary points of an closed interval.

(ii) If function is having different definitions for different intervals of x, then all definition changing points.

(iii) Critical point related to specific function like in case of $[g(x)]$, $\{g(x)\}$ all the points where $g(x)$ is an integer where [.] and {.} are greatest integer function and fractional part function respectively.

For $\text{Sgn}(g(x))$ all the points where $g(x) = 0$.

Note

These are possible point of discontinuity, for them to be points of discontinuity we have to perform continuity test at each point, then only we should conclude them as continuous or discontinuous points.

ILLUSTRATION 14

Discuss the continuity of $f(x) = \begin{cases} |x+1|, & x < -2 \\ 2x+3, & -2 \le x < 0 \\ x^2+3, & 0 \le x < 3 \\ x^3-15, & x \ge 3 \end{cases}$

SOLUTION

Since, $f(x)$ is a polynomial function in each of the given intervals. Hence, $f(x)$ is continuous in each of the four intervals $(-\infty, -2)$, $(-2, 0)$, $(0, 3)$ and $(3, \infty)$. Hence, we check the continuity at definition changing points $x = -2, 0, 3$.

At point $x = -2$

$$\lim_{x\to -2^-} f(x) = \lim_{h\to 0} f(-2-h) = \lim_{h\to 0}(-(-2-h)-1) = 1$$

$$\lim_{x\to -2^+} f(x) = \lim_{h\to 0} f(-2+h) = \lim_{h\to 0} (2(-2+h)+3) = -1$$

Since, $\lim\limits_{x\to -2} f(x)$ does not exist and hence, $f(x)$ is not continuous at $x = -2$

At point $x = 0$

$$\text{LHL} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} (2(0-h)+3) = 3$$

$$\text{RHL} = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} ((0+h)^2+3) = 3$$

$$f(0) = (0)2 + 3 = 3$$

Since, LHL = RHL = $f(0)$ Hence, $f(x)$ is continuous at $x = 0$

At point $x = 3$

$$\text{LHL} = \lim_{h\to 0} f(3-h) = \lim_{h\to 0} ((3-h)^2+3) = 12$$

$$\text{RHL} = \lim_{h\to 0} f(3+h) = \lim_{h\to 0} ((3+h)^3-15) = 12$$

$$f(3) = 3^3 - 15 = 12$$

Since, LHL = RHL = $f(3)$. Therefore, $f(x)$ is continuous at $x = 3$. Hence, $f(x)$ is continuous in $R - \{-2\}$.

ILLUSTRATION 15

Let $f(x) = \begin{cases} \dfrac{1}{|x-1|}, & -\infty < x < 0 \\ |x|^2+1, & 0 \le x < 1 \\ 2x, & 1 \le x < 2 \\ \dfrac{x^2-1}{x-1}+1, & 2 \le x < \infty \end{cases}$

then discuss continuity of $f(x)$.

SOLUTION

We need to check continuity where $f(x)$ change the definition i.e. $x = 0, 1, 2$.

(i) Continuity check at $x = 0$

$$\text{LHL} = \lim_{x\to 0^-} f(x) = \frac{1}{|0-1|} = 1$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = |0|^2 + 1 = 1$$

$$\text{LHL} = \text{RHL} = f(0) = 1$$

Hence, it is continuous at $x = 0$

(ii) Continuity check at $x = 1$

$$\text{LHL} = \lim_{x\to 1^-} f(x)$$

$$= \lim_{x\to 1^-} (|x|^2+1) = |1|^2+1 = 2$$

$$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2$$

Also, $f(1) = 2$

$\Rightarrow$ LHL = RHL = $f(1)$

Hence, $f(x)$ is continuous at $x = 1$

(iii) Continuity check at $x = 2$

$$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (2x) = 4$$

$$\text{RHL} = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \left(\frac{x^2-1}{x-1}+1\right)$$

$$= \lim_{x\to 2^+} (x+1+1) = 4$$

$$f(2) = \frac{2^2-1}{2-1} + 1 = 4$$

$\Rightarrow$ LHL = RHL = $f(2)$

$\Rightarrow$ $f(x)$ is continuous at $x = 2$.

Hence, $f(x)$ is always continuous in $(-\infty, \infty)$.

ILLUSTRATION 16

$f(x) = [x^2]$ for $3 \le x \le 5$ where $[\cdot]$ is the greatest integer function. Find the number of point of discontinuity.

SOLUTION

If $3 \le x \le 5$ then $9 \le x^2 \le 25$

Integral values of x^2 are 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25

Possible point of discontinuity $x = 3, \sqrt{10}, \sqrt{11}, \ldots, \sqrt{24}, 5$

$f(3) = f(3^+) = 9$ it means $f(x)$ is continuous at $x = 3$

$$f(\sqrt{10}) = 10; f(\sqrt{10}^-) = 9$$

$$f(\sqrt{11}) = 11; f(\sqrt{11}^-) = 10$$

……………………………

$$f(\sqrt{24}) = 24; f(\sqrt{24}^-) = 23$$

$$f(5) = 25; f(5^-) = 24$$

$f(x)$ is discontinuous at $x = \sqrt{10}, \sqrt{11}, \ldots, \sqrt{24}, 5$ (16 points)

ILLUSTRATION 17

$f(x) = [\sqrt{x}\,]$ for $x \in [1, 81]$ where $[\cdot]$ is the greatest integer function.

SOLUTION

Given $1 \le x \le 81$ then $1 \le \sqrt{x} \le 9$

Integral values of $\sqrt{x}$ are 1, 2, 3, 4, 5, 6, 7, 8, 9

Possible point of discontinuity $x = 1, 4, 9, 16, 25, 36, 49, 64, 81$

$f(1) = f(1^+) = 1$ hence $f(x)$ is continuous at $x = 1$

$$f(4^-) = 1; f(4) = 2; f(4^+) = 2$$

$$f(9^-) = 2; f(9) = 3; f(9^+) = 3$$

……………………………..

$$f(64^-) = 7; f(64) = 8; f(64^+) = 8$$

$$f(81^-) = 8; f(81) = 9$$

$f(x)$ is discontinuous at $x = 4, 9, 16, 25, 36, 49, 64, 81$ (8 points)

ILLUSTRATION 18

$f(x) = [\ln x]\,\mathrm{sgn}\left(\{x\} - \frac{1}{2}\right)$ where $1 \le x \le 3$ where $[\cdot]$ is gif and $\{\cdot\}$ is fpf

SOLUTION

For $1 \le x < e \Rightarrow 0 \le \ln x < 1 \Rightarrow [\ln x] = 0$

For $e \le x < 3 \Rightarrow 1 \le \ln x < 2 \Rightarrow [\ln x] = 1$

$$e \le x < 3 \Rightarrow [x] = 2$$

$$f(x) = \begin{cases} 0 & 1 \le x < e \\ 1\,\mathrm{sgn}\left(x - 2 - \frac{1}{2}\right) = 1 & e \le x < 3 \\ -1 & x = 3 \end{cases}$$

$$f(e^-) = 0;\ f(e) = 1;\ f(e^+) = 1$$

$$f(3^-) = 1;\ f(3) = -1$$

$f(x)$ is discontinuous at $x = e, 3$

ILLUSTRATION 19

$$f(x) \begin{cases} [\sin \pi x], & 0 \le x < 1 \\ \left\{x - \frac{2}{3}\right\} \mathrm{sgn}\left(x - \frac{5}{4}\right), & 1 \le x \le 2 \end{cases}$$

where $[\cdot]$ is the greatest integer function and $\{\cdot\}$ is fractional part function. Discuss continuity of $f(x)$ when $x \in [0, 2]$

SOLUTION

$$f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2} \\ 1, & x = \frac{1}{2} \\ 0, & \frac{1}{2} < x < 1 \\ -\left\{x - \frac{2}{3}\right\}, & 1 \le x < \frac{5}{4} \\ 0 & x = \frac{5}{4} \\ \left\{x - \frac{2}{3}\right\}, & \frac{5}{4} < x \le 2 \end{cases}$$

When $1 \le x < \frac{5}{4}$; We have $x = [x] + \{x\}$

$$\left\{x - \frac{2}{3}\right\} = x - \frac{2}{3} - \left[x - \frac{2}{3}\right]$$

$$\frac{1}{3} < x - \frac{2}{3} < \frac{7}{12} \text{ so } \left[x - \frac{2}{3}\right] = 0 \quad \therefore \quad \left\{x - \frac{2}{3}\right\} = x - \frac{2}{3}$$

When $\frac{5}{4} < x \le 2$, we have $\left\{x - \frac{2}{3}\right\} = \begin{cases} x - \frac{2}{3}, & \frac{5}{4} < x < \frac{5}{3} \\ 0, & x = \frac{5}{3} \\ \frac{x-2}{3-1}, & \frac{5}{3} < x \le 2 \end{cases}$

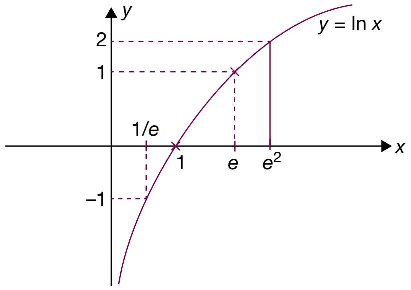
$$f(x) = \begin{cases} 0, & 0 \le x < \frac{1}{2}, \frac{1}{2} < x < 1 \\ 1, & x = \frac{1}{2} \\ -\left(x - \frac{2}{3}\right), & 1 \le x < \frac{5}{4} \\ 0, & x = \frac{5}{4}, \frac{5}{3} \\ x - \frac{2}{3}, & \frac{5}{4} < x < \frac{5}{3} \\ \frac{x-2}{3-1}, & \frac{5}{3} < x \le 2 \end{cases}$$

Now we have simplified function $f(x)$ and we can conclude that $f(x)$ is discontinuous at $x = \frac{1}{2}, 1, \frac{5}{4}, \frac{5}{3}$

ILLUSTRATION 20

$f(x) = [\ln x]$ for $x \in \left[\frac{1}{e}, e^2\right]$ where $[\cdot]$ the is greatest integer function.

SOLUTION



As given $\frac{1}{e} \le x \le e^2$

$\Rightarrow \quad -1 \le \ln x \le 2$

From the graph of $f(x)$, we can conclude that $f(x)$ is discontinuous at $x = 1, e, e^2$

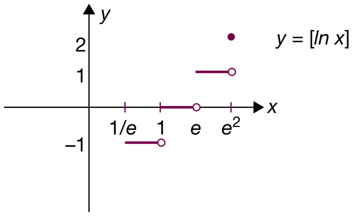
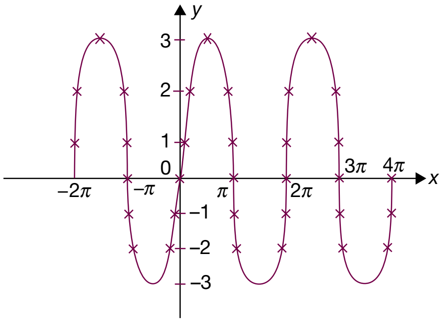


ILLUSTRATION 21

$f(x) = [3 \sin x]$ for $x \in [-2\pi, 4\pi]$ where $[\cdot]$ is GIF.
Find number of point discontinuity.

SOLUTION

$f(x) = [3 \sin x]$ for $x \in [-2\pi, 4\pi]$ is drawn as shown in diagram.
$[3 \sin x]$ becomes discontinuous whenever it crosses an integer.



For $x \in [-2\pi, 4\pi]$ $f(x)$ has 33 points of discontinuity.

ILLUSTRATION 22

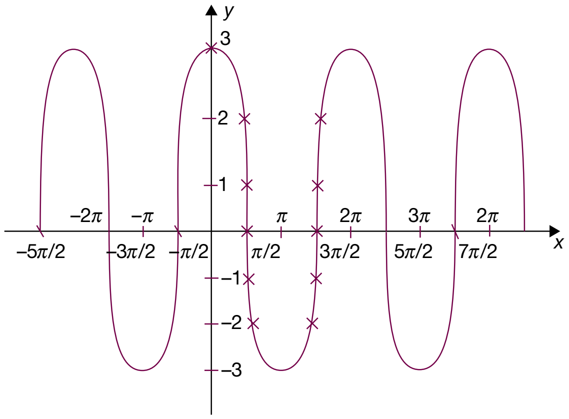
$f(x) = \{3 \cos x\}$, find number of point of discontinuity where $\{\cdot\}$ is fractional part function.
If **(i)** $x \in [0, 2\pi]$ **(ii)** $x \in [-2\pi, 4\pi]$

SOLUTION

$$f(x) = \{3 \cos x\} = 3 \cos x - [3 \cos x] \qquad (\because\ x = [x] + \{x\})$$

Number of points of discontinuity of $\{3 \cos x\}$ and $[3 \cos x]$ are same.

From graph of $3 \cos x$, we can say that it is discontinuous at 11 points when $x \in [0, 2\pi]$
For length of one time period $T = 2\pi$, continuous function



If $x \in [0, 2\pi)$, then $f(x)$ has 11 points of discontinuity.
If $x \in [0, 2\pi]$, then $f(x)$ has 12 points of discontinuity.
If $x \in [-2\pi, 4\pi]$, then $f(x)$ has $3 \times 11 + 1 = 34$ points of discontinuity.

ILLUSTRATION 23

Let function $f(x) = \begin{cases} x^2 - 5x + 6, & -2 \le x \le 4 \\ \alpha - \tan\left(\frac{\pi x}{4}\right), & 4 < x \le 5 \\ \log_{10}(\beta x), & 5 < x \le 10 \end{cases}$

is continuous in $[-2, 10]$, then find $(\alpha + \beta)$

SOLUTION

$\because$ $f(x)$ is continuous in $[-2, 10]$ then also continuous at $x = 4$ and 5.

$$\lim_{x \to 4^+} f(x) = f(4)$$

$$\Rightarrow \quad \lim_{x \to 4^+} \left(\alpha - \tan\frac{\pi x}{4}\right) = 4^2 - 5 \times 4 + 6$$

$$\Rightarrow \quad \alpha - 0 = 2 \Rightarrow \alpha = 2$$

Also, $\lim_{x \to 5^+} f(x) = f(5)$

$$\Rightarrow \quad \lim_{x \to 5^+} \log_{10} \beta x = 2 - \tan\left(\frac{5\pi}{4}\right)$$

$$\Rightarrow \quad \log_{10} 5\beta = 2 - 1$$

$$\Rightarrow \quad 5\beta = 10 \Rightarrow \beta = 2$$

Hence, $\alpha + \beta = 2 + 2 = 4$

ILLUSTRATION 24

Let $f(x) = \dfrac{\sqrt{\text{sgn}(ax^2+ax+1)}}{\cot^{-1}(x^2-a)}$. If $f(x)$ is continuous for all $x \in R$, then find number of integers in the range of a.

SOLUTION

As, domain is given as R then $ax^2 + ax + 1 \geq 0 \ \forall \ x \in R$

$$D \leq 0$$

$$a^2 - 4a \leq 0 \Rightarrow a\,(a-4) \leq 0 \Rightarrow a \in [0, 4].$$

Check at boundary points.

(i) At $a = 0$

$$\text{sgn}\,(ax^2 + ax + 1) = \text{sgn}\,(1) = 1$$

$\Rightarrow$ For $a = 0$ function is always continuous.

(ii) At $a = 4$

$$\text{sgn}\,(ax^2 + ax + 1) = \text{sgn}\,(4x^2 + 4x + 1)$$

$$= \text{sgn}\,(2x+1)^2 = \begin{cases} 1, & x \neq -1/2 \\ 0, & x = -1/2 \end{cases}$$

$f(x)$ becomes discontinuous at $x = -1/2$

$\therefore \ a \in [0, 4),$

$a = 0, 1, 2, 3$

Hence, no. of acceptable integers are 4.

ILLUSTRATION 25

If the function $f(x) = \begin{cases} x^2 + px + 1, & x \text{ is rational} \\ px^2 + 2x + q, & x \text{ is irrational} \end{cases}$ is continuous at $x = 1$ and $x = 2$, then find $(p + q)$

SOLUTION

Here, $x = 1$ and $x = 2$ are roots of the equation $x^2 + px + 1 = px^2 + 2x + q$

Here $(p-1)x^2 + (2-p)x + (q-1) = 0$

$$\therefore \quad \frac{p-2}{p-1} = 3 \Rightarrow p = \frac{1}{2} \text{ and } \frac{q-1}{p-1} = 2 \Rightarrow q = 0$$

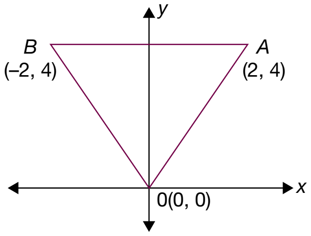
Hence, $(p+q) = \dfrac{1}{2} + 0 = \dfrac{1}{2}$

ILLUSTRATION 26

Let $f(x) = \begin{cases} x^2, & x \text{ is rational} \\ 8 - x^2, & x \text{ is irrational} \end{cases}$. If $f(x)$ is continuous at two points A and B, then find area of ΔOAB where O is origin.

SOLUTION

$f(x)$ will be continuous at two points where $x^2 = 8 - x^2$



$\Rightarrow \quad 2x^2 = 8 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$\therefore \quad$ Points A and B are (2, 4) and (–2, 4).

$\therefore \quad$ Area of $\Delta OAB = \dfrac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times 4 \times 4 = 8 \text{ sq. unit.}$$

ILLUSTRATION 27

Let $f(x) = \begin{cases} ax + b, & x \in Q \\ 2x^3 + x^2 + 3, & x \notin Q \end{cases}$ is continuous at points $x = -1, 1$ and α, then find the value of α.

SOLUTION

Points of continuity of $f(x)$ are given by equation,

$$ax + b = 2x^3 + x^2 + 3$$

$\Rightarrow \quad 2x^3 + x^2 - ax + 3 - b = 0$ whose roots are ± 1 and α

Put $x = 1$, $a + b = 6$ and put $x = -1$, $a - b = -2$

$\Rightarrow \quad a = 2$ and $b = 4$.

Now, cubic equation is $2x^3 + x^2 - 2x - 1 = 0$

Product of roots $(-1)\,(1)\,\alpha = \dfrac{1}{2} \Rightarrow \alpha = -\dfrac{1}{2}$

ILLUSTRATION 28

$$\text{Let } f(x) = \begin{cases} a\tan^{-1}\left(\dfrac{1}{x-4}\right), & 0 \leq x < 4 \\ \dfrac{\pi}{2}, & x = 4 \\ b\tan^{-1}\left(\dfrac{2}{x-4}\right), & 4 < x < 6 \\ \sin^{-1}(7-x) + \dfrac{a\pi}{4}, & 6 \leq x \leq 8 \end{cases}.$$

if $f(x)$ is continuous in [0, 8], then find the value of $b - a$.

SOLUTION

Clearly, $f(x)$ must also be continuous at $x = 4$ and $x = 6$.

So, $f(4^-) = f(4) = f(4^+)$

$$\Rightarrow \quad \lim_{x \to 4^-} f(x) = \frac{\pi}{2} = \lim_{x \to 4^+} f(x)$$

$$\Rightarrow \quad \lim_{x\to 4^-} a\tan^{-1}\left(\frac{1}{x-4}\right) = \frac{\pi}{2} = \lim_{x\to 4^+} b\tan^{-1}\frac{2}{x-4}$$

$$\Rightarrow \quad \frac{-a\pi}{2} = \frac{\pi}{2} = \frac{b\pi}{2}$$

Hence, $a = -1, b = 1 \Rightarrow b - a = 2$

ILLUSTRATION 29

$$\text{Let } f(x) = \begin{cases} 5 - cx, & x \le -1 \\ b - 2x, & -1 < x < 0 \\ 2, & x = 0 \\ \ln\left((\cos x + a\sin bx)^{\frac{1}{x}}\right), & 0 < x < \frac{\pi}{2} \end{cases}$$

be a continuous function in its domain.
If $\cos^{-1}(\cos(a + c + 3b)) = m\pi + n$ (where m and n are integers), then find the value of $(m - n)$.

SOLUTION

$f(x)$ is continuous at $x = 0$

$$f(0) = f(0^-)$$

$$\Rightarrow \quad 2 = \lim_{x\to 0^-}(b - 2x) = b \Rightarrow b = 2$$

$$\text{RHL} = f(0^+) = f(0)$$

$$\Rightarrow \quad \lim_{x\to 0^+} \ln(\cos x + a\sin bx)^{1/x} = 2$$

$$\Rightarrow \quad \lim_{x\to 0^+} \frac{\ln(\cos x + a\sin bx)}{x} \quad \text{using L-H rule}$$

$$\Rightarrow \quad \lim_{x\to 0^+} \frac{-\sin x + ab\cos bx}{\cos x + a\sin bx} = ab = 2$$

We get $a = 1$

$f(x)$ is continuous at $x = -1$

$$f(-1) = f(-1^+)$$

$$5 + c = \lim_{x\to -1^+}(b - 2x) = b - 2(-1) \Rightarrow c = -1$$

Now, $\cos^{-1}\cos(a + c + 3b) = \cos^{-1}(\cos 6) = 2\pi - 6 = m\pi + n$
We get $m = 2, n = -6$

Hence, $(m - n) = 8$

ILLUSTRATION 30

Let $f(x)$ be an odd continuous function on R and takes only rational values, then find the number of distinct real roots of the equation $x^3 - (2f(x) + 1)x^2 + (f^2(x) + 2f(x))x + 2f(x) = 0$.

SOLUTION

$$x^3 - (2f(x) + 1)x^2 + (f^2(x) + 2f(x))x + 2f(x) = 0$$

Since, $f(x)$ is odd continuous function.

$$\Rightarrow \quad f(0) = 0$$

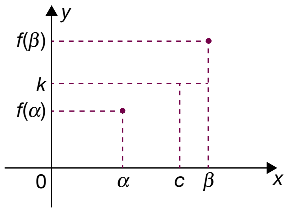
It takes only rational values $\Rightarrow f(x) = 0 \;\forall\; x \in R$

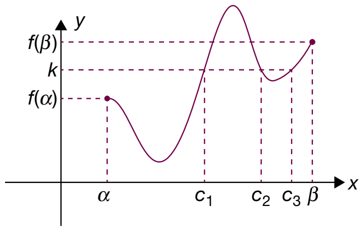
Then, cubic equation becomes $x^3 - x^2 = 0 \Rightarrow x = 0, 1$.

Hence, number of distinct real roots = 2.

INTERMEDIATE MEAN VALUE THEOREM

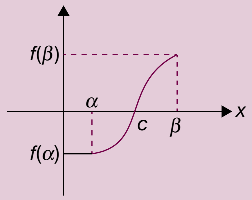
IMVT is application of continuity of function. Suppose $f(x)$ is continuous on an interval I. α and β are any two points lying in given interval I and k is a number between $f(\alpha)$ and $f(\beta)$, then according to Intermediate Mean Value Theorem their exists atleast one number c between α and β such that $f(c) = k$.

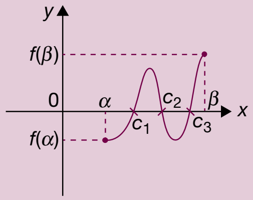




In first figure equation $f(c) = k$ has one solution but in second figure equation has three solutions $x = c_1, c_2, c_3$

Let $f(x)$ is continuous in $[\alpha, \beta]$ and $f(\alpha) < 0$, $f(\beta) > 0$, then $f(x) = 0$ for atleast one value of $x \in (\alpha, \beta)$





In first figure equation $f(c) = 0$ has one solution but in second figure equation has three solutions $x = c_1, c_2, c_3$

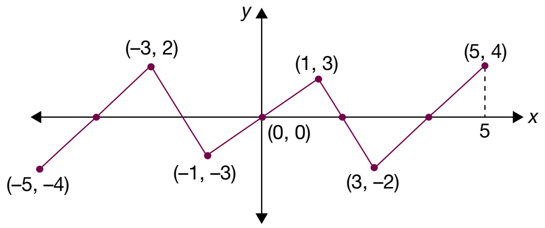
ILLUSTRATION 31

Let $f : R \to R$ be a continuous onto function satisfying $f(x) = -f(-x) \;\forall\; x \in R$. If $f(-3) = 2$, $f(-1) = -3$ and $f(5) = 4$ in $x \in [-5, 5]$ then find minimum number of solutions of equation $f(x) = 0$.

SOLUTION

Given, $f(x) = -f(-x)$ in $[-5, 5]$

$\Rightarrow \quad f(x)$ is an odd function.



Since, points (–3, 2), (–1, –3), and (5, 4) lies on the curve. So, (3, –2), (1, 3) and (–5, –4) will also lies on the curve. For minimum number of roots, one of the possible graph of continuous function $f(x)$ is as follows.

Hence, $f(x) = 0$ has atleast 5 solutions.

ILLUSTRATION 32

If $a > b > c > d$, prove that $(x - a)(x - c) + 2(x - b)(x - d) = 0$ has real and distinct roots.

SOLUTION

Given $a > b > c > d$

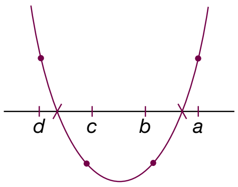
Let $f(x) = (x - a)(x - c) + 2(x - b)(x - d)$

$$f(a) = 0 + 2(a - b)(a - d) > 0$$

$$f(b) = (b - a)(b - c) + 0 < 0$$

$$f(c) = 0 + 2(c - b)(c - d) < 0$$

$$f(d) = (d - a)(d - c) + 0 < 0$$



From Intermediate Mean Value Theorem, we can say that $f(x)$ has one zero between a and b and other zero between c and d. As $f(x) = 0$ is quadratic in x, we can conclude that roots are real and distinct.

ILLUSTRATION 33

Show that $e^{2x} + e^{x} + 2^{\sin^{-1}x} + x - \pi = 0$ has atleast one $x \in (0, 1)$

SOLUTION

$f(x) = e^{2x} + e^{x} + 2^{\sin^{-1}x} + x - \pi$ is continuous for $x \in [0, 1]$

$$f(0) = 1 + 1 + 2^{\circ} + 0 - \pi = 3 - \pi < 0$$

$$f(1) = e^{2} + e + 2^{\pi/2} + 1 - \pi > 0$$

As $f(x)$ is continuous function, then from IMVT we can claim that $f(x) = 0$ for atleast one $x \in (0, 1)$.

ILLUSTRATION 34

Show that the function $f(x) = (x - a)^2 (x - b)^2 + x$ takes the value $\frac{a+b}{2}$ for some value of $x \in [a, b]$.

SOLUTION

$f(a) = a; f(b) = b$. Also, f is continuous in $[a, b]$ and $\frac{a+b}{2} \in [a, b]$.

Let $g(x) = f(x) - \frac{a+b}{2}$ which is continuous in $x \in [a, b]$

$$g(a) = f(a) - \frac{a+b}{2} = a - \frac{a+b}{2} = \frac{a-b}{2} < 0$$

$$g(b) = f(b) - \frac{a+b}{2} = b - \frac{a+b}{2} = \frac{b-a}{2} > 0$$

Hence, using IMVT, there exists at least one $c \in [a, b]$ such that $g(x) = 0$

$$\Rightarrow \quad f(c) = \frac{a+b}{2}.$$

ILLUSTRATION 35

Let $f(x)$ be continuous in $x \in [0, 1]$ and $f(0) = 0 = f(1)$. Prove that $f(c) = 1 - 2c^2$ is true for some value of c where $c \in (0, 1)$.

SOLUTION

Let $g(x) = \underset{\text{continuous}}{f(x)} + \underset{\text{continuous}}{2x^2 - 1}$ where $x \in [0, 1]$

We can say that $g(x)$ is also continuous function

$$\left.\begin{matrix} g(0) = f(0) + 0 - 1 = -1 < 0 \\ g(1) = f(1) + 2 - 1 = 1 > 0 \end{matrix}\right]$$

From IMVT, $g(x) = 0$ for some $x \in (0, 1)$

$$\Rightarrow \quad f(x) + 2x^2 - 1 = 0$$

$$\Rightarrow \quad f(x) = 1 - 2x^2 \text{ for some } x, \text{ where } x \in (0, 1)$$

Test Your Concepts-II

(Solutions on page H.110)

1. Let $f : R \to R$ is given by $f(x) = \left[\begin{matrix} x^2 + 6x + 2, x \in Q \\ x^2 + 5x - 4, \text{otherwise} \end{matrix}\right.$.

   Then, find values of x for which function f is continuous.

2. Let $f : \left[0, \frac{3}{2}\right) \to R$ be a function defined as

   $f(x) = [3x] - \{2x\}$. Find the number of points of discontinuity of $f(x)$where [.] and {.} are GIF and FPF.

3. If the function $f(x) = \begin{cases} \dfrac{\sin(p+1)x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \dfrac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases}$

   is continuous for all x on R, then find $(2p + 8q)$.

4. Find the value of the constant c for which the function $f(x) = \begin{cases} cx^2 + 2x, & x < 2 \\ x^3 - cx, & x \ge 2 \end{cases}$ is continuous on $(-\infty, \infty)$

5. Let $f(x) = \begin{cases} \cos^2 x, & x \in Q \\ -\cos^2 x, & x \notin Q \end{cases}$. Then find the set of points, where $f(x)$ is continuous.

6. Find the number of points of discontinuity of $f(x) = [2x^3 - 5]$ in $[1, 2)$ where $[x]$ denotes greatest integer less than or equal to x.

7. If $f(x) = [x]\,(\sin kx)^p$ is continuous for real x for $k \in \{n\pi, n \in I\}, p > 0$ (True/false)

8. $f(x) = \begin{cases} x + 2, & x < 0 \\ -x^2 - 2, & 0 \le x < 1 \\ x, & x \ge 1 \end{cases}$, then find the number of points of discontinuity of $|f(x)|$.

9. The function $f(x) = \begin{cases} \dfrac{1}{2} & \text{if } x \in Q \\ \dfrac{1}{3} & \text{if } x \notin Q \end{cases}$ and $g(x) = \begin{cases} x, & x \in Q \\ -x, & x \in R \sim Q \end{cases}$ are continuous at number of points p and q respectively, then find $p + q$.

10. If $f(x)$ is continuous function on R which attains only irrational values, then find $\sum_{r=1}^{20} f(r)$ where $f(10) = \sqrt{5}$.

11. If $f(x) = |[x]x|$ and $[.]$ denotes G.I.F and $-1 \le x \le 2$ then find the number of discontinuous points.

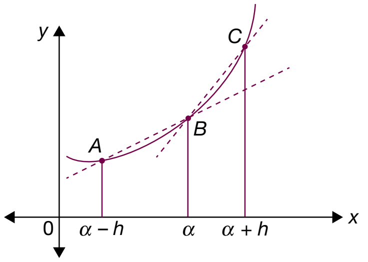
Answer key

1. $x = -6$ **2.** 5 **3.** 1 **4.** 2/3 **5.** $x = (2k+1)\dfrac{\pi}{2}, k \in I.$ **6.** 13

7. True **8.** 1 **9.** 1 **10.** $20\sqrt{5}$ **11.** 2

DIFFERENTIABILITY OF FUNCTION AT POINT $x = \alpha$

Geometrically, $f(x)$ is differentiable or derivable at a point iff there exists a unique non-vertical tangent at the point.



Let consider a function $= f(x)$,

Assuming points on the curve $A(\alpha - h, f(\alpha - h))$, $B(\alpha, f(\alpha))$, $C(\alpha + h, f(\alpha + h))$

$$\text{Slope of } AB = m_{AB} = \frac{f(\alpha - h) - f(\alpha)}{-h}$$

$$\text{Slope of } BC = m_{BC} = \frac{f(\alpha + h) - f(\alpha)}{h}$$

For $f(x)$ to be differentiable at point $x = \alpha$, it should have non-vertical tangent of finite slope then $\lim\limits_{h \to 0} m_{AB} = \lim\limits_{h \to 0} m_{BC}$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(\alpha - h) - f(\alpha)}{-h} = \lim_{h \to 0} \frac{f(\alpha + h) - f(\alpha)}{h}$$

$$\Rightarrow \quad f'(\alpha^-) = f'(\alpha^+) = \text{Finite}$$

Let $y = f(x)$ be continuous at point $x = \alpha$, then $f(x)$ is said to be differentiable at $x = \alpha$,

If LHD = RHD = finite

Where Left Hand Derivative (LHD)

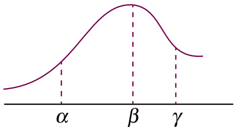
$$= \lim_{h \to 0} \frac{f(\alpha - h) - f(\alpha)}{-h} = f'(\alpha^-)$$

Right Hand Derivative (RHD)

$$= \lim_{h \to 0} \frac{f(\alpha + h) - f(\alpha)}{h} = f'(\alpha^+)$$

For a function to be differentiable at point $x = \alpha$, then $f(x)$ must be continuous at $x = \alpha$.

If a function is discontinuous at a point, it is not differentiable at that point irrespective of LHD and RHD whether they are equal or not.

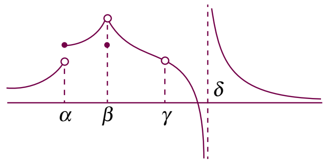


(i)

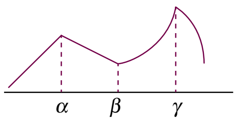
Function shown in Figure (i) is continuous and differentiable at $x = \alpha, \beta, \gamma$.

Function shown in Figure (ii) is discontinuous and not differentiable at $x = \alpha, \beta, \gamma, \delta$.

Function shown in Figure (iii) is always continuous but not differentiable at $x = \alpha, \beta, \gamma$.



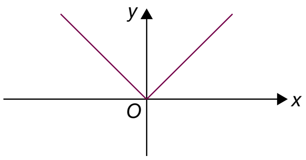
(ii)



(iii)

Let consider function $f(x) = |x|$

$$f(x) = \begin{cases} +x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

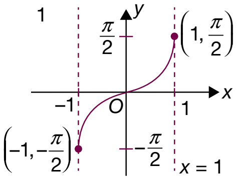


It is continuous at $x = 0$
but $f'(0^+) = 1$ and $f'(0^-) = -1$
Hence, it is not differentiable at $x = 0$, from graph we can clearly observe a corner at $x = 0$

Let consider inverse trigonometric function

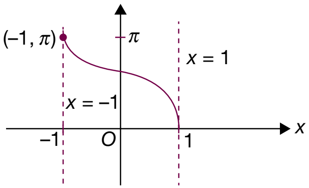
$$f(x) = \sin^{-1}x \text{ where } x \in [-1, 1]$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \text{ for } -1 < x < 1$$



$\sin^{-1}x$ has vertical tangents $x = \pm 1$
Hence, not differentiable at $x = \pm 1$.
Similarly, $g(x) = \cos^{-1}x$, where $x \in [-1, 1]$

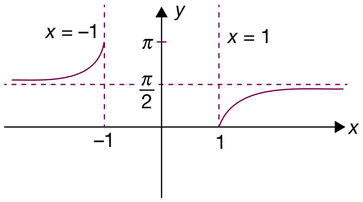
$$g'(x) = -\frac{1}{\sqrt{1-x^2}}, \text{ for } -1 < x < 1$$



$\cos^{-1}x$ has vertical tangents $x = \pm 1$ Hence, not differentiable at $x = \pm 1$.

Let function $f(x) = \sec^{-1}x$, where $|x| \geq 1$

$$f'(x) = \frac{1}{|x|\sqrt{x^2-1}}; \text{ for } |x| > 1$$



$f(x) = \sec^{-1}x$ has vertical tangents so it does not have derivatives at $x = \pm 1$

Similarly, $f(x) = \operatorname{cosec}^{-1}x$ also has vertical tangents so, it does not have derivatives at $x = \pm 1$

Relation between Continuity and Differentiability

(i) If a function is continuous at $x = \alpha$, then it may or may not be differentiable at $x = \alpha$.

(ii) If a function is discontinuous at $x = \alpha$, then surely it is not differentiable at $x = \alpha$.
As, graph is broken due to discontinuity, there cannot be tangent possible in this case so we can directly conclude that function is not differentiable at that point.

(iii) If a function is differentiable at $x = \alpha$, then function is surely continuous at $x = \alpha$.

(iv) If a function is not - differentiable at $x = \alpha$, then function may or may not be continuous at $x = \alpha$.

(v) If a function f is not differentiable but is continuous at $x = \alpha$, it geometrically implies a sharp corner at $x = \alpha$.

Working Rule

Check differentiability of $f(x)$ at $x = \alpha$

- If discontinuous, then we can directly conclude not-differentiable at $x = \alpha$
- If continuous, then we need to check if RHD = LHD = finite, then differentiable else not-differentiable

Note

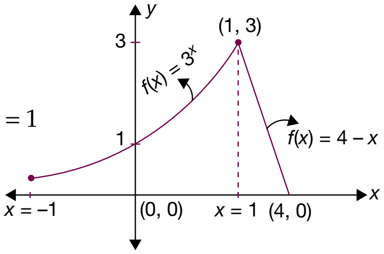
A function will fail to have a derivative at $x = \alpha$, then it may have following reasons.

(i) Discontinuity at $x = \alpha$ or
(ii) Vertical tangent, $x = \alpha$ or
(iii) A corner/cusp/sharp point at $x = \alpha$

ILLUSTRATION 36

Let $f(x) = \begin{cases} 3^x, & -1 \leq x \leq 1 \\ 4-x, & 1 < x \leq 4 \end{cases}$, then check differentiability of $f(x)$ at $x = 1$

SOLUTION



As, $f(1^+) = 3 = f(1^-) = f(1)$

$\Rightarrow$ f is continuous at $x = 1$

$$f'(x) = \begin{cases} 3^x \ln 3, & x < 1 \\ -1, & x > 1 \end{cases}$$

Also, LHD $= f'(1^-) = 3\ln 3$

and RHD $= f'(1^+) = -1$

As, $f'(1^-) \neq f'(1^+) \Rightarrow f$ is non-derivable at $x = 1$

Hence, graph of $f(x)$ also clearly tells that at $x = 1$, there is a sharp corner so it is not differentiable.

ILLUSTRATION 37

If the right hand derivative of $f(x) = [x] \tan \pi x$ at $x = 7$ is $k\pi$, then find k where [.] is GIF.

SOLUTION

$$f(x) = [x] \tan \pi x$$

$$f'(7^+) = \lim_{h \to 0} \frac{f(7+h) - f(7)}{h}$$

$$= \lim_{h \to 0} \frac{[7+h]\tan \pi(7+h) - [7]\tan 7\pi}{h}$$

$$= \lim_{h \to 0} \frac{7\tan \pi(7+h) - 0}{h}$$

$$= 7\pi \lim_{h \to 0} \frac{\tan \pi h}{\pi h} = 7\pi = k\pi \quad \text{(Given)}$$

Hence, $k = 7$

ILLUSTRATION 38

Let $f(x) = \begin{cases} x\left(\dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$,

then check differentiability of $f(x)$ at $x = 0$

SOLUTION

We have continuous $f(x)$ at $x = 0$, $f(0^+) = f(0) = f(0^-) = 0$

LHD at $x = 0 = f'(0^-) = \lim\limits_{h \to 0} \dfrac{f(0-h) - f(0)}{-h}$

$$= \lim_{h \to 0} \frac{-h\left(\dfrac{e^{-1/h} - e^{1/h}}{e^{-1/h} + e^{1/h}}\right) - 0}{-h} = \lim_{h \to 0} \left(\frac{e^{-2/h} - 1}{e^{-2/h} + 1}\right) = -1$$

RHD at $x = 0 = f'(0^+) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{h\left(\dfrac{e^{1/h} - e^{-1/h}}{e^{1/h} + e^{-1/h}}\right) - 0}{h} = \lim_{h \to 0} \left(\frac{1 - e^{-2/h}}{1 + e^{-2/h}}\right)$$

$$= \frac{1-0}{1+0} = 1$$

$\therefore$ (LHD at $x = 0$) $\neq$ (RHD at $x = 0$)

Hence, $f(x)$ is not differentiable at $x = 0$

ILLUSTRATION 39

Discuss differentiability of function

$$f(x) = \begin{cases} 2x - 2, & x < 1 \\ x^2, & 1 \leq x \leq 2 \quad \text{at } x = 1 \text{ and } x = 2 \\ -2 + 8x - x^2, & x > 2 \end{cases}$$

SOLUTION

$$f'(x) = \begin{cases} 2, & x < 1 \\ 2x, & 1 < x < 2 \\ 8 - 2x, & x > 2 \end{cases}$$

$$f'(1^-) = f'(1^+) = 2,$$

$$f'(2^-) = f'(2^+) = 4$$

Clearly, we can see that LHD = RHD, but still this function is not differentiable because it is not continuous

$$f(1^-) = 0,\ f(1^+) = 1$$

$$f(2^-) = 4,\ f(2^+) = -2 + 16 - 4 = 8$$

So, we must check continuity first, then we should check differentiability.

Hence, $f(x)$ is discontinuous and non-differentiable at $x = 1, 2$

ILLUSTRATION 40

Let $f(x) = \begin{cases} x^3, & x \leq 1 \\ ax^2 + bx + c, & x > 1 \end{cases}$. If $f''(x)$ is continuous everywhere, then find a, b, c

SOLUTION

Continuity of $f(x)$ at $x = 1$,

$$f(1^-) = f(1) = f(1^+) \Rightarrow a + b + c = 1$$

Continuity of $f'(x)$ at $x = 1$

$$f'(1^-) = f'(1^+) \Rightarrow 3 = 2a + b$$

Continuity of $f''(x)$ at $x = 1$

$$f''(1^-) = f''(1^+) \Rightarrow 6 = 2a \Rightarrow a = 3, b = -3, c = 1$$

ILLUSTRATION 41

If $f(x) = \begin{cases} x^3 + 2x^2 + ax + 6, & x < 1 \\ 2x + b, & x \geq 1 \end{cases}$ is differentiable for all $x \in R$, then find the value of $(a + b)$

SOLUTION

$\because$ $f(x)$ is differentiable for all x.

So, f is also continuous for $x \in R$

$$f(1^-) = f(1) \Rightarrow a + 9 = 2 + b \Rightarrow b - a = 7 \qquad \ldots(1)$$

For L.H.D at $x = 1 = f'(1^-) = 3x^2 + 4x + a\big|_{x=1^-} = 3 + 4 + a$

$$\text{R.H.D.} = f'(1^{+}) = 2$$

$$\text{LHD} = \text{RHD} \Rightarrow 3 + 4 + a = 2$$

We get, $a = -5$, $b = 2$.
Hence, $(a + b) = -3$

ILLUSTRATION 42

Let $f(x) = |x - 2|e^{x}g(x) + \sin x$, where $g(x)$ is a continuous function. If $f(x)$ is differentiable at $x = 2$, then find $g(2)$

SOLUTION

Left hand derivative of $f(x)$

$$f'(2^{-}) = \lim_{h\to 0} \frac{e^{2-h}|(2-h)-2|g(2-h)+\sin(2-h)-0-\sin 2}{-h}$$

$$= \lim_{h\to 0} \frac{e^{2-h}hg(2-h)}{-h} + \lim_{h\to 0} \frac{2\cos\left(\frac{4-h}{2}\right)\sin\left(\frac{-h}{2}\right)}{-h}$$

$$= -\lim_{h\to 0} e^{2-h}g(2-h) + \lim_{h\to 0} \frac{2\cos\left(\frac{4-h}{2}\right)\sin\left(\frac{h}{2}\right)}{2\left(\frac{h}{2}\right)}$$

$$\text{LHD} = f'(2^{-}) = -e^{2}g(2) + \cos 2$$

Right Hand derivative of $f(x)$

$$f'(2^{+}) = \lim_{h\to 0} \frac{e^{2+h}|(2+h)-2|g(2+h)+\sin(2+h)-0-\sin 2}{h}$$

$$= \lim_{h\to 0} \frac{e^{2+h}hg(2+h)}{h} + \lim_{h\to 0} \frac{2\cos\left(\frac{4+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}$$

$$\text{RHD} = f'(2^{+}) = e^{2}g(2) + \cos 2$$

$f(x)$ is differentiable at $x = 2$, so

$$f'(2^{+}) = f'(2^{-}) \Rightarrow -e^{2}g(2) + \cos 2 = e^{2}g(2) + \cos 2$$

Hence, $g(2) = 0$

ILLUSTRATION 43

Let $f(x) = \begin{cases} x^{2}\left|\cos\frac{\pi}{2x}\right|, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Find $f'\left(\frac{1^{+}}{3}\right) + f'\left(\frac{1^{-}}{3}\right)$.

SOLUTION

For region where x is just smaller than $1/3$,

we have $f(x) = x^{2}\cos\frac{\pi}{2x}$

For LHD $= f'(x) = 2x\cos\frac{\pi}{2x} + \frac{\pi}{2}\sin\frac{\pi}{2x}$

$$f'\left(\frac{1^{-}}{3}\right) = -\frac{\pi}{2}$$

When x is just greater than $1/3$, we have $f(x) = -x^{2}\cos\frac{\pi}{2x}$

For RHD $= f'(x) = -2x\cos\frac{\pi}{2x} - \frac{\pi}{2}\sin\frac{\pi}{2x}$

$$f'\left(\frac{1^{+}}{3}\right) = \frac{\pi}{2}$$

Hence, $f'\left(\frac{1^{-}}{3}\right) + f'\left(\frac{1^{+}}{3}\right) = 0$

ILLUSTRATION 44

$f(x) = \lim\limits_{x\to\infty} \frac{x^{n} - \sin x^{n}}{x^{n} + \sin x^{n}}$ $x > 0$; $x \neq 1$ $f(1) = 0$ discuss differentiability at $x = 1$

SOLUTION

$$f(1) = 0$$

$$\text{LHL} = f(1^{-}) = \lim_{x\to\infty}\lim_{h\to 0} \frac{(1-h)^{n} - \sin(1-h)^{n}}{(1-h)^{n} + \sin(1-h)^{n}} \text{ let } x = 1 - h$$

$$= \lim_{x\to\infty}\lim_{h\to 0} \frac{(1-h)^{n}\left[1 - \frac{\sin(1-h)^{n}}{(1-h)^{n}}\right]}{(1-h)^{n}\left[1 + \frac{\sin(1-h)^{n}}{(1-h)^{n}}\right]} = \frac{1-1}{1+1} = 0$$

$$\text{RHL} = f(1^{+}) = \lim_{h\to 0}\lim_{x\to\infty} \frac{(1+h)^{n} - \sin(1+h)^{n}}{(1+h)^{n} + \sin(1+h)^{n}}$$

$$= \lim_{h\to 0}\lim_{x\to\infty} \frac{(1+h)^{n}\left[1 - \frac{\sin(1+h)^{n}}{(1+h)^{n}}\right]}{(1+h)^{n}\left[1 + \frac{\sin(1+h)^{n}}{(1+h)^{n}}\right]}$$

$$= \frac{1 - \frac{-1 to 1}{\infty}}{1 + \frac{-1 to 1}{\infty}} = \frac{1-0}{1+0} = 1$$

$$f(1) = f(1^{-}) = 0;\ f(1^{+}) = 1$$

$f(x)$ is discontinuous at $x = 1$

ILLUSTRATION 45

$f(x) = \begin{cases} \left\{x + \frac{1}{3}\right\}[\sin \pi x] & 0 \le x < 1 \\ [2x]\operatorname{sgn}(x - 4/3) & 1 \le x \le 2 \end{cases}$ check differentiability at $x = 1$ where $[\cdot]$ is GIF $\{\cdot\}$ is fpf

SOLUTION

Let $x = 1 - h$; LHL $= f(1^{-}) = \lim\limits_{h\to 0} \left\{1 - h + \frac{1}{3}\right\} [\sin(\pi - \pi h)]$

$$= \lim_{h\to 0} \left(\frac{1}{3} - h\right) \underset{0.00000001}{[\sin \pi h]} = \lim_{h\to 0} \left(\frac{1}{3} - h\right) 0 = 0$$

Let $x = 1 + h$;

$$\text{RHL} = f(1^+) = \lim_{h\to 0} [2+2h]\,\text{sgn}(1+h-4/3) = \lim_{h\to 0} 2(-1) = -2$$

LHL ≠ RHL, so $f(x)$ is discontinuous and non-differentiable at $x = 1$

ILLUSTRATION 46

$$f(x) = \begin{cases} x^2\,\text{sgn}[x] + \{x\} & 0 \le x < 2 \\ \sin x + |x-3| & 2 \le x \le 4 \end{cases}$$

Check differentiability at $x = 2$ where $[\cdot]$ is GIF : $\{\cdot\}$ is fpf

SOLUTION

$$f(2) = \sin 2 + 1$$

Let $x = 2 - h$;

$$\text{LHL} = f(2^-) = \lim_{h\to 0} (2-h)^2\,\text{sgn}[\underbrace{2-h}_{1.999}] + \{2-h\}$$

$$= \lim_{h\to 0} (2-h)^2\,(1) + 1 - h = 4 + 1 = 5$$

$f(x)$ is discontinuous and non-differentiable at $x = 2$

ILLUSTRATION 47

$$\text{Let } f(x) = \begin{cases} \dfrac{5e^{\frac{1}{x}}+2}{3-e^{\frac{1}{x}}}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

Check (i) $f(x)$ (ii) $x^2 f(x)$ for continuity and differentiability at $x = 0$.

SOLUTION

(i) For $f(x)$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{5e^{\frac{1}{x}}+2}{3-e^{\frac{1}{x}}} = \lim_{x\to 0^+} \frac{5+2e^{-\frac{1}{x}}}{3e^{-\frac{1}{x}}-1} = \frac{5+0}{0-1} = -5$$

But $f(0) = 0$

Since, f is not continuous at $x = 0$, hence, not differentiable at $x = 0$.

(ii) For $x^2 f(x)$, let $F(x) = x^2 f(x)$

$$\text{LHD at } x = 0 = \lim_{h\to 0} \frac{F(0-h)-F(0)}{-h} = \lim_{h\to 0} \frac{h^2 f(-h) - 0}{-h}$$

$$= \lim_{h\to 0} h\, f(-h) = 0$$

$$\text{RHD at } x = 0 = \lim_{h\to 0} \frac{F(0+h)-F(0)}{h} = \lim_{h\to 0} \frac{h^2 f(h) - 0}{h}$$

$$= \lim_{h\to 0} h\, f(h) = 0$$

Since, LHD = RHD

Therefore, $x^2 f(x)$ is differentiable at $x = 0$ and hence, it will always be continuous at $x = 0$

ILLUSTRATION 48

Let a_1 and a_2 be two values of a for which

$$f(x) = \begin{cases} x\cdot\left(\dfrac{\ln(1+x)+\ln(1-x)}{\sec x - \cos x}\right), & x \in (-1, 0) \\ (a^2-3a-1)\sin x + x^2, & x \in [0, \infty) \end{cases}$$

is differentiable at $x = 0$, then find the value of $(a_1^2 + a_2^2)$.

SOLUTION

$$f'(0^-) = \lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{\dfrac{-h(\ln(1-h)+\ln(1+h))}{\sec h - \cos h} - 0}{-h}$$

$$= \lim_{h\to 0} \frac{\ln(1-h^2)}{\dfrac{(1-\cos^2 h)}{\cos h}} = \lim_{h\to 0} \frac{\dfrac{\ln(1-h^2)}{-h^2}\times(-h^2)}{\left(\dfrac{1-\cos h}{h^2}\right)h^2\,\dfrac{(1+\cos h)}{\cos h}}$$

$$= \lim_{h\to 0} \frac{(-h^2)}{\dfrac{1}{2}\times 2h^2} = -1$$

$$f'(0^+) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{(a^2-3a-1)\sin h + h^2 - 0}{h}$$

$$= \lim_{h\to 0} \left\{(a^2-3a-1)\frac{\sin h}{h} + h\right\} = a^2 - 3a - 1$$

∵ As, given function is differentiable at $x = 0$.

We have $f'(0^+) = f'(0^-) \Rightarrow a^2 - 3a - 1 = -1$

$\Rightarrow \quad a^2 - 3a = 0$

$\Rightarrow \quad a = 0$ or 3, let $a_1 = 0$, $a_2 = 3$

Hence, $a_1^2 + a_2^2 = 9$.

Test Your Concepts-III

(Solutions on page H.111)

1. Examine differentiability of $f(x)$ at $x = 0$ for

$$f(x) = \begin{cases} \dfrac{1-\cos x}{x\sin x}, & x \ne 0 \\ \dfrac{1}{2}, & x = 0 \end{cases}$$

2. Check the differentiability of

$$f(x) = \begin{cases} x+2, & x > 3 \\ 5, & x = 3 \\ 8-x, & x < 3 \end{cases} \text{ at } x = 3$$

3. If $f(x)=\begin{cases}|x-1|([x]-x) & , \quad x\neq 1\\ 0 & , \quad x=1\end{cases}$, test the differentiability at $x = 1$, where $[\cdot]$ denotes the greatest integer function.

4. Find the values of a and b such that the function

$f(x)=\begin{cases}ax+3, & x\geq 1\\ x^2+b, & x<1\end{cases}$ is continuous and differentiable at $x=1$.

5. Let $f(x) = \begin{cases} x\tan^{-1}x+\sec^{-1}\dfrac{1}{x}, & x\in(-1,1)-\{0\}\\ \dfrac{\pi}{2}, & x=0\end{cases}$

If $l=f'(0^+)$ and $m=f'(0^-)$, then find the value of l^2+m^2.

6. Let $f(x) = \begin{bmatrix} \dfrac{\sin x^2}{x}, & x\neq 0\\ 0, \text{at} & x=0\end{bmatrix}$, then find the value of $f'(0^+)+f'(0^-)$.

7. Let $g(x) = \begin{cases}\dfrac{x^2+x\tan x-x\tan 2x}{ax+\tan x-\tan 3x}, & x\neq 0\\ 0, & x=0\end{cases}$. If $g'(0)$ exists and is equal to non zero value b, then find $\dfrac{b}{a}$.

8. Let $f(x) = \begin{cases}\sqrt{10-x^2}, & -3\leq x<3\\ 2-e^{x-3}, & x\geq 3\end{cases}$. Prove that $f(x)$ is continuous but not differentiable at $x=3$.

9. Find the number of values of a for the which the function $f(x) = (x + 1)\,|x - a|$ is differentiable in R

10. Let $g(x) = \begin{cases}2x+\tan^{-1}x+a, & -\infty<x\leq 0\\ x^3+x^2+bx, & 0<x<\infty\end{cases}$. If $g(x)$ is differentiable for all $x \in (-\infty, \infty)$, then find $(a^2 + b^2)$

11. Let $f(x) = \dfrac{x^3}{\sqrt{a+x}\,(bx-\sin x)}$ $(a, b \in R)$ and $g(x) = |2 - x|$.

If p is the right hand derivative of $g(x)$ at $x = 2$ and $\lim\limits_{x\to 0} f(x) = p$, then find the value of $(a + 2b)$.

12. If $f(x) = \lim\limits_{t\to x}\left(\dfrac{\sec t-\sec x}{t-x}\right)$, then find

$$\left(\frac{f'\left(\frac{\pi}{3}\right)+f'\left(\frac{-\pi}{3}\right)}{4}\right)$$

13. $f(x)=\text{Max}\left\{x, x^3\right\}$, then at $x = 0$ discuss continuity and differentiability

14. $f(x)=\begin{cases}\left(\dfrac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}\right), & x\neq 0\\ 0, & x=0\end{cases}$ then at $x = 0$ discuss continuity and differentiability

Answer key

1. differentiable **2.** not differentiable **3.** not differentiable **4.** $a=2$, $b=4$
5. 2 **6.** 2 **7.** 7/52 **8.** not differentiable **9.** 1 **10.** 9 **11.** 38
12. 7 **13.** continuous but not differentiable **14.** neither continuous nor differentiable

THEOREM ON DIFFERENTIABILITY

If a function is consisting of two or more functions and we have *k*nowledge of continuity and differentiability of these sub functions at some point then using following results we talk about differentiability of resultant function. It certainly saves time and efforts.

Table 5.2 At point $x = \alpha$

f(x)	g(x)	f(x) ± g(x)	f(x) g(x)	f(x)/g(x) where g(α) ≠ 0
Differentiable	Differentiable	Differentiable	Differentiable	Differentiable
Not-Differentiable	Differentiable	Not- Differentiable	Can't Say (?)	Can't Say (?)
Not- Differentiable	Not-Differentiable	Can't Say (?)	Can't Say (?)	Can't Say (?)

(A) Function which is sum or difference of differentiable and non differentiable function at $x = \alpha$ must be non-differentiable at $x = \alpha$.

Let consider function $f(x) = x^2$, it is differentiable at $x = 0$.

Another function $g(x) = |x|$, it is not differentiable at $x = 0$.

$$h(x) = f(x) + g(x) = x^2 + |x| = \begin{cases} x^2 + x & x \ge 0 \\ x^2 - x & x < 0 \end{cases}$$

$$h'(0^+) = 2x + 1\big|_{x=0^+} = 1$$

$$h'(0^-) = 2x - 1\big|_{x=0^-} = -1$$

Hence, at $x = 0$, $h(x)$ is not differentiable.

(B) Function which is sum or difference of two non-differentiable functions may or may not be differentiable at $x = \alpha$.

CASE 1: Let consider function $f(x) = |x|$, it is not differentiable at $x = 0$.

Another function $g(x) = -|x|$, it is not differentiable at $x = 0$.

Now, $h(x) = f(x) + g(x) = 0$, it is always differentiable.

CASE 2: Let consider function $f(x) = |x|$, it is not differentiable at $x = 0$.

Another function $g(x) = 2|x|$, it is not differentiable at $x = 0$.

Now, $h(x) = f(x) + g(x) = 3|x|$, it is not differentiable at $x = 0$.

(C) Function which is product of two non-differentiable functions may or may not be differentiable at $x = \alpha$.

CASE 1: Let consider function $f(x) = x^2 + |x|$, it is not differentiable at $x = 0$.

Another function $g(x) = |x|$, it is not differentiable at $x = 0$

$$h(x) = f(x)\, g(x) = (x^2 + |x|)\, |x|$$

$$h(x) = \begin{cases} \left(x^2 + x\right)x & x \ge 0 \\ \left(x^2 - x\right)(-x) & x < 0 \end{cases} = \begin{cases} x^3 + x^2 & x \ge 0 \\ -x^3 + x^2 & x < 0 \end{cases}$$

$$h'(0^+) = 3x^2 + 2x\big|_{x=0^+} = 0;\ h'(0^-) = -3x^2 + 2x\big|_{x=0^-} = 0$$

$h(x)$ is differentiable at $x = 0$. We can see that two non-differentiable functions, on addition, are giving us differentiable function.

CASE 2: Let consider function $h(x) = |x| + 10$, it is not differentiable at $x = 0$

Another function $g(x) = |x|$, it is not differentiable at $x = 0$

$$k(x) = h(x)\, g(x) = |x|^2 + 10|x| = \begin{cases} x^2 + 10x & x \ge 0 \\ x^2 - 10x & x < 0 \end{cases}$$

$$k'(0^+) = 10;\ k'(0^-) = -10$$

$k(x)$ is not differentiable at $x = 0$. We can see that two non-differentiable functions, on addition, are giving us non-differentiable function.

Now, we can conclude that product of two non-differentiable functions may or may not be differentiable or non-differentiable.

ILLUSTRATION 49

Discuss differentiability for function

$f(x) = |x - 1.5| + |x - 0.7| + \tan x$ in the interval (0, 2).

SOLUTION

We have, $y = |x - a|$ is always continuous but not differentiable at $x = a$

$\therefore$ $f(x)$ is not differentiable at $x = 0.7$ and $x = 1.5$

Also, $\tan x$ is discontinuous and not differentiable at $x = \dfrac{\pi}{2}$

Now, using the theorem of continuity and differentiability, we can conclude that $f(x)$ is discontinuous at one point and non -differentiable at three points in (0, 2).

ILLUSTRATION 50

Find the number of points, where $f(x) = |(x - 3)^7| + |(x - 7)^3| + |x - 5|$ is non-differentiable.

SOLUTION

$\because$ $|(x - 3)^7|$ and $|(x - 7)^3|$ both are differentiable every where but $|x - 5|$ is not differentiable at $x = 5$.

Now, using the theorem of differentiability, we can conclude that $f(x)$ is non-differentiable at $x = 5$ only.

DIFFERENTIABILITY TEST IN AN INTERVAL

A function $f(x)$ is said to be differentiable in the closed interval $[a, b]$

(i) It must be continuous at every point of (a, b)

$$f(\alpha^+) = f(\alpha^-) = f(\alpha) = \text{Finite}$$

and $f'(\alpha^-) = f'(\alpha^+) = \text{Finite}$ for every $\alpha \in (a, b)$

(ii) $f(a) = f(a^+) = \text{Finite}$ and RHD $= f'(a^+) = \text{Finite}$

(iii) $f(b) = f(b^-) = \text{Finite}$ and LHD $= f'(b^-) = \text{Finite}$

Conceptual Note(s)

To check differentiability of function $f(x)$ for $x \in [a, b]$, as in an interval there are infinite points, so it becomes tedious work to check function's differentiability at infinite points so we should first identify few suspect points where function can be non-differentiable

- (i) All the points where continuity need to be checked
  - (a) Boundary points
  - (b) Definition of changing points
  - (c) Critical points specific to certain functions like GIF $[g(x)]$, FPF $\{g(x)\}$, sgn(x) if present
- (ii) Critical points of modulus function and inverse trigonometric function if present

Note

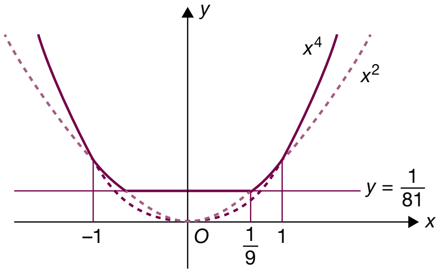
These are **possible** point of non-differentiability, we have to perform differentiability test at each of these points, then only we should conclude them as differentiable or non-differentiable points.

ILLUSTRATION 51

If $f(x) = \max.\left\{x^4, x^2, \frac{1}{81}\right\}\ \forall\ x \in R$, then find the sum of the square of reciprocal of all the values of x where $f(x)$ is non-differentiable

SOLUTION

By observation from graph



$$\text{We can conclude } f(x) = \begin{cases} \frac{1}{81}, & -\frac{1}{9} \le x \le \frac{1}{9} \\ x^2, & \frac{1}{9} < x \le 1 \text{ or } -1 \le x \le -\frac{1}{9} \\ x^4, & x > 1 \text{ or } x < -1 \end{cases}$$

Clearly, $f(x)$ has sharp points at two points in given interval.

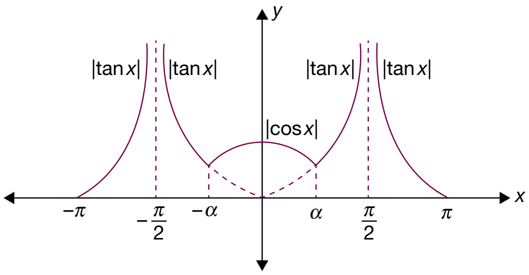
Hence, $f(x)$ is non-differentiable at $x = \frac{1}{9}, 1, -1, -\frac{1}{9}$

Hence, sum of squares of reciprocals $= 9^2 + 1 + 1 + 9^2 = 164$

ILLUSTRATION 52

Find the number of points where the function $f(x) = \max(|\tan x|, \cos|x|)$ is non-differentiable in the interval $(-\pi, \pi)$.

SOLUTION



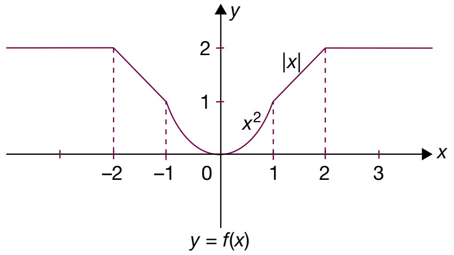
Discontinuous at $\pm\frac{\pi}{2}$ and sharp corner at $x = \alpha$ and $-\alpha$.

Hence, non-derivable at 4 points.

ILLUSTRATION 53

Let $f(x) = \min.(|x|, x^2, 2), x \in [-5, 5]$. Then, find the number of points where $f(x)$ is non-derivable.

SOLUTION



$$f(x) = \begin{cases} 2 & 2 \le |x| \le 5 \\ |x| & 1 < |x| < 2 \\ x^2 & -1 \le x \le 1 \end{cases}$$

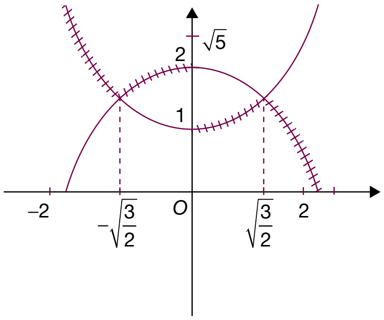
From the graph we can get corner points
Hence, function not derivable at 4 points $x = -2, -1, 1, 2$

ILLUSTRATION 54

$$f(x) = \begin{cases} \max\{\sqrt{1+x^2}, \sqrt{4-x^2}\}, -2 \le x \le 0 \\ \min\{\sqrt{1+x^2}, \sqrt{4-x^2}\}, 0 < x \le 2 \end{cases}$$

Discuss continuity and differentiability of $f(x)$.

SOLUTION



From the graph we can conclude that

$$f(x) = \begin{cases} \sqrt{1+x^2}, & -2 \le x < -\sqrt{\frac{3}{2}} \\ \sqrt{4-x^2}, & -\sqrt{\frac{3}{2}} \le x \le 0 \\ \sqrt{1+x^2}, & 0 < x < \sqrt{\frac{3}{2}} \\ \sqrt{4-x^2}, & \sqrt{\frac{3}{2}} \le x \le 2 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{x}{\sqrt{1+x^2}}, & -2 \le x < -\sqrt{\dfrac{3}{2}} \\ \dfrac{-x}{\sqrt{4-x^2}}, & -\sqrt{\dfrac{3}{2}} \le x \le 0 \\ \dfrac{x}{\sqrt{1+x^2}}, & 0 < x < \sqrt{\dfrac{3}{2}} \\ \dfrac{-x}{\sqrt{4-x^2}}, & \sqrt{\dfrac{3}{2}} \le x \le 2 \end{cases}$$

$$f(0) = 2 : f(0^+) = 1$$

$f(x)$ is discontinuous at $x = 0$

$$f'\left(-\sqrt{\frac{3}{2}}^{-}\right) \ne f'\left(-\sqrt{\frac{3}{2}}^{+}\right);\ f'\left(\sqrt{\frac{3}{2}}^{-}\right) \ne f'\left(\sqrt{\frac{3}{2}}^{+}\right)$$

$$f'(2^-) \to \infty$$

Hence, $f(x)$ is not differentiable at $x = \pm\sqrt{\dfrac{3}{2}}, 0, 2$

ILLUSTRATION 55

If $f(x) = (x^5 + 1)\,|x^2 - 4x - 5| + \sin|x| + \cos(|x - 1|)$, then find number of points such that $f(x)$ is not differentiable.

SOLUTION

$$f(x) = (x^5 + 1)\,|x^2 - 4x - 5| + \sin|x| + \cos(|x - 1|)$$

$$\because \quad \cos(|x - 1|) = \cos(x - 1)$$

$$f(x) = (x^5 + 1)\,|(x - 5)(x + 1)| + \sin|x| + \cos(x - 1)$$

$(x^5 + 1)\,|(x - 5)(x + 1)|$ contains term $(x + 1)^2$ so at $x = -1$ it is differentiable

Now, using the theorem of differentiability we can conclude that

Hence, $f(x)$ is non-differentiable at two points. $x = 0, 5$

ILLUSTRATION 56

Find the number of values of x in $[-4, 4]$ where

$f(x) = [3x + 14] + |4x^2 - 1|(2x^2 + 3x - 2) + \sin\left(\dfrac{\pi x}{2}\right)$ is non-derivable where [.] is GIF.

SOLUTION

We have

$$f(x) = [3x] + 14 + |(2x + 1)(2x - 1)|\,(2x - 1)(x + 2) + \sin\left(\frac{\pi x}{2}\right)$$

$\sin\left(\dfrac{\pi x}{2}\right)$ is always continuous and differentiable.

For $x \in [-4, 4]$, the interval will be $-12 \le 3x \le 12$

Points to be considered for testing discontinuity of $[3x]$ are

$$\therefore \quad x = -\frac{12}{3}, -\frac{11}{3}, -\frac{10}{3}, \ldots, -\frac{1}{3}, 0, \frac{1}{3}, \ldots, \frac{11}{3}, \frac{12}{3} \text{ [25 points]}$$

But $f(-4) = f(-4^+)$, Hence, $f(x)$ is continuous at $x = -4$.

$|4x^2 - 1|(2x^2 + 3x - 2) + \sin\left(\dfrac{\pi x}{2}\right)$ are differentiable at these points. So, we have concluded that $f(x)$ has 24 points of discontinuity.

$|4x^2 - 1|(2x^2 + 3x - 2) = |(2x + 1)(2x - 1)|\,(2x - 1)(x + 2)$ is always continuous for all x.

It has term $(2x - 1)^2$ so it is differentiable at $x = 1/2$ but $f(x)$ is non-derivable at $x = -\dfrac{1}{2}$

As, discontinuous functions are also non-differentiable,

Now, using the theorem of differentiability we can conclude that $f(x)$ is non-derivable at 25 points in $[-4, 4]$

Hence, $x = -\dfrac{11}{3}, -\dfrac{10}{3}, \ldots, -\dfrac{1}{3}, 0, \dfrac{1}{3}, \ldots, \dfrac{11}{3}, 4$ and $-\dfrac{1}{2}$

ILLUSTRATION 57

Let $f(x) = (x^2 - 3x + 2)\,|(x^3 - 6x^2 + 11x - 6)| + \left|\sin\left(x + \dfrac{\pi}{4}\right)\right|$.

Find number of points at which the function $f(x)$ is non-differentiable in $[0, 2\pi]$.

SOLUTION

We have

$$f(x) = (x - 1)(x - 2)\,|(x - 1)(x - 2)(x - 3)| + \left|\sin\left(x + \frac{\pi}{4}\right)\right|$$
$$= g(x) + h(x)$$

$$h(x) = \sin\left(x + \frac{\pi}{4}\right) = 0$$

When $x + \dfrac{\pi}{4} = \pi$ or $x + \dfrac{\pi}{4} = 2\pi \Rightarrow x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$ in $[0, 2\pi]$

$h(x)$ is always continuous but not differentiable at 2 points $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

Now, $g(x) = (x - 1)(x - 2)\,|(x - 1)(x - 2)(x - 3)|$

$$g(x) = \begin{cases} (x - 1)^2(x - 2)^2(x - 3) & x \in [1, 2] \cup [3, \infty) \\ -(x - 1)^2(x - 2)^2(x - 3) & x \in (-\infty, 1) \cup (2, 3) \end{cases}$$

$g(x)$ is always differentiable except at $x = 3$, $g'(3^-) = -4$, $g'(3^+) = 4$

Hence, $f(x)$ is not differentiable at $x = 3, \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

ILLUSTRATION 58

$f(x) = x[x]$ for x $[-1, 3]$ where $[\cdot]$ is GIF

Discuss continuity and differentiability.

SOLUTION

$$f(x) = \begin{cases} -x & -1 \le x < 0 \\ 0 & 0 \le x < 1 \\ x & 1 \le x < 2 \\ 2x & 2 \le x < 3 \\ 9 & x = 3 \end{cases}$$

$f(x)$ is discontinuous at $x = 1, 2, 3$

$f(x)$ is not differentiable at $x = 0, 1, 2, 3$

ILLUSTRATION 59

$f(x) = \max\{\sin x, \cos x\}$ for $x \in [0, 2\pi]$

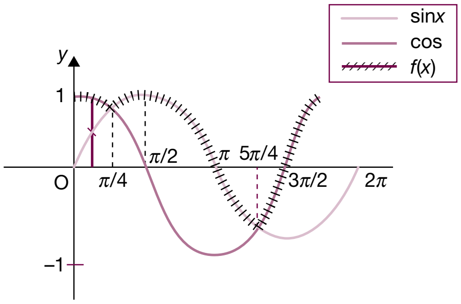
Discuss continuity and differentiability.

SOLUTION

$$f(x) = \begin{cases} \cos x & 0 \le x \le \dfrac{\pi}{4} \\ \sin x & \dfrac{\pi}{4} < x < \dfrac{5\pi}{4} \\ \cos x & \dfrac{5\pi}{4} \le x \le 2\pi \end{cases}$$

$$f'(x) = \begin{cases} -\sin x & 0 \le x \le \dfrac{\pi}{4} \\ \text{DNE} & x = \dfrac{\pi}{4} \\ \cos x & \dfrac{\pi}{4} < x < \dfrac{5\pi}{4} \\ \text{DNE} & x = \dfrac{5\pi}{4} \\ -\sin x & \dfrac{5\pi}{4} < x \le 2\pi \end{cases}$$

$$f(x) \to \text{cont. not diff. at } x = \frac{\pi}{4}, \frac{5\pi}{4}$$



$$\left.\begin{aligned} f'\left(\frac{\pi}{4}^-\right) = f'\left(\frac{5\pi}{4}^-\right) = -\frac{1}{\sqrt{2}} \\ f'\left(\frac{\pi}{4}^+\right) = f'\left(\frac{5\pi}{4}^+\right) = \frac{1}{\sqrt{2}} \end{aligned}\right]$$

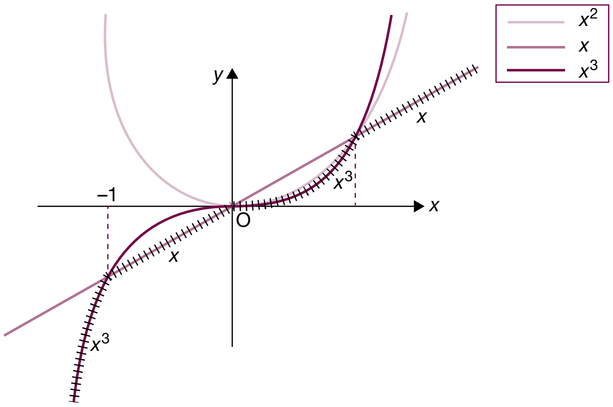
$f(x)$ is not differentiable at $x = \dfrac{5\pi}{4}, \dfrac{\pi}{4}$

ILLUSTRATION 60

$f(x) = \min\{x, x^2, x^3\}$

Discuss continuity and differentiability

SOLUTION



$$f(x) = \begin{cases} x^3 & x < -1 \\ x & -1 \le x \le 0 \\ x^3 & 0 < x \le 1 \\ x & x > 1 \end{cases}$$

$$f'(x) = \begin{cases} 3x^2 & x < -1 \\ 1 & -1 < x < 0 \\ 3x^2 & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

$$f'(-1^-) = 3,\ f'(-1^+) = 1,\ f'(0^-) = 1,$$
$$f'(0^+) = 0,\ f'(1^-) = 3,\ f'(1^+) = 1$$

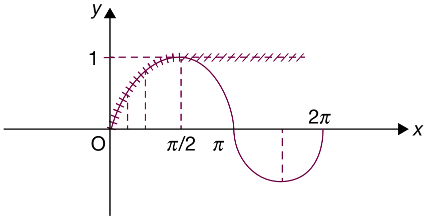
not differentiable at $x = -1, 0, 1$

ILLUSTRATION 61

$f(x) = \max\{\sin t : 0 \le t \le x\}$ where $x\ [0, 2\pi]$.

Discuss continuity and differentiability of $f(x)$.

SOLUTION



$$f\left(\frac{\pi}{3}\right) = \max\left\{\sin t, 0 \le t \le \frac{\pi}{3}\right\}$$

$$f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$f\left(\frac{2\pi}{3}\right) = \max\left\{\sin t, 0 \le t \le \frac{2\pi}{3}\right\}$$

$$f\left(\frac{2\pi}{3}\right) = \sin\frac{\pi}{2} = 1$$

$$f(x)=\begin{cases}\sin x, & 0\le x\le \dfrac{\pi}{2}\\ 1, & \dfrac{\pi}{2}<x\le 2\pi\end{cases}$$

$$f'(x)=\begin{cases}\cos x, & 0\le x<\dfrac{\pi}{2}\\ 0, & \dfrac{\pi}{2}<x\le 2\pi\end{cases}$$

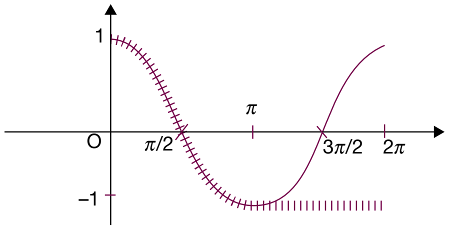
$f(x)$ is continuous and differentiable.

ILLUSTRATION 62

$g(x)=\min\{\cos t : 0\le t\le x\}\ x\in[0,2\pi]$

Discuss continuity and differentiability of $g(x)$.

SOLUTION



$$g(x)=\begin{cases}\cos x & 0\le x\le \pi\\ -1 & \pi<x\le 2\pi\end{cases}$$

$$g\left(\frac{3\pi}{2}\right)=\min\left\{\cos t:0\le t\le\frac{3\pi}{2}\right\}$$

$$=\cos\pi=-1$$

$g(x)$ is continuous and differentiable.

ILLUSTRATION 63

$$f(x)=x^3-x^2+x+1 \text{ and } g(x)=\begin{cases}\max\{f(t),0\le t\le x\}, & 0\le x\le 1\\ \min\{f(t),0\le t\le x\}, & 1<x\le 2\\ 3-x, & 2<x\le 3\end{cases}$$

SOLUTION

$$f'(x)=3x^2-2x+1$$

$$D=(-2)^2-4(3)(1)=4-12=-8<0$$

$f'(x)=3x^2-2x+1>0$, for all real x

It means $f(x)$ is strictly increasing function.

$$g(x)=\begin{cases}f(x), & 0\le x\le 1\\ f(0), & 1<x\le 2\\ 3-x, & 2<x\le 3\end{cases}=\begin{cases}x^3-x^2+x+1, & 0\le x\le 1\\ 1, & 1<x\le 2\\ 3-x, & 2<x\le 3\end{cases}$$

$$g(1^-)=2,\ g(1^+)=1$$

$g(x)$ at $x=1$ discontinuous and not differentiable.

$$g(2^-)=g(2)=g(2^+)=1$$

At $x=2$ $g(x)$ is continuous but not differentiable as $g'(2^+)=-1, g'(2^-)=0$

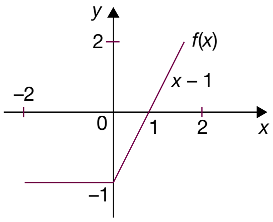
ILLUSTRATION 64

$$f(x)=\begin{cases}-1, & -2\le x\le 0\\ x-1, & 0<x\le 2\end{cases}$$

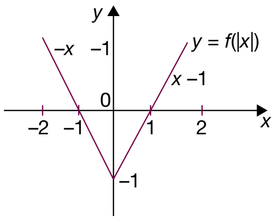
$g(x)=f(|x|)+|f(x)|$ Discuss continuity and differentiability of $g(x)$ where $x\in[-2,2]$

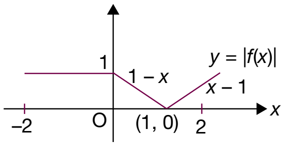
SOLUTION

Graph of $f(x)$ is as shown



Based of $f(x)$, graph of $f(|x|)$ and $|f(x)|$ are also drawn





$$g(x)=f(|x|)+|f(x)|=\begin{cases}-x-1+1, & -2\le x\le 0\\ x-1+1-x, & 0<x<1\\ x-1+x-1, & 1\le x\le 2\end{cases}$$

$$g(x)=\begin{cases}-x & -2\le x\le 0\\ 0 & 0<x<1\\ 2x-2 & 1\le x\le 2\end{cases}$$

$$\left.\begin{matrix}g'(0^-)=-1\\ g'(0^+)=0\end{matrix}\right]\ \left.\begin{matrix}g'(1^-)=0\\ g'(1^+)=2\end{matrix}\right]$$

$g(x)$ is continuous in x [–2, 2] not diff. at $x=0, 1$

ILLUSTRATION 65

Let $f(t)=|t|+|t-1|\ \forall\ t\in R$ and

$$g(x)=\begin{cases}\max.(f(t)), & x-1\le t\le x, 0\le x\le 1\\ 3-x, & 1<x\le 2\end{cases}$$

Find number of points where $g(x)$ is non-derivable in [0, 2].

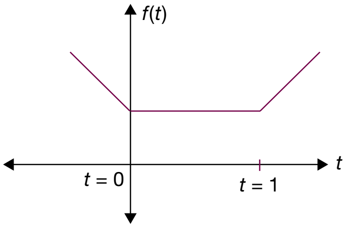
SOLUTION

By making function $f(x)$ modulus free, we get

$$f(t)=\begin{cases}1-2t, & t<0\\ 1, & 0\le t\le 1\\ 2t-1, & t>0\end{cases}$$

For $0 \le x \le 1$,

$$x-1 \le t \le x,\ \max f(t) = f(x-1)$$



$$g(x) = \begin{cases} \max.(f(t)), & x-1 \le t \le x, 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}$$

$$g(x) = \begin{cases} f(x-1), & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases} = \begin{cases} 1-2(x-1), & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}$$

$$\Rightarrow \quad g(x) = \begin{cases} 3-2x, & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}$$

In [0,2] $g(x)$ has polynomial definition which is always continuous and differentiable, but we need to test continuity and differentiability at definition changing point $x = 1$

$$g(1^-) = 1, g(1^+) = 2$$

Hence, function $g(x)$ is discontinuous and non differentiable at $x = 1$

ILLUSTRATION 66

If $f(x)$ = max, $(1 - |x|, \text{Min.}(x^2, 1))$ then find number of points where $f(x)$ is non-differentiable.

SOLUTION

We have plotted the given function

Hence, there are 5 points where $f(x)$ is non-differentiable.

$$x = 0, \pm 1, \pm\left(\frac{\sqrt{5}-1}{2}\right)$$

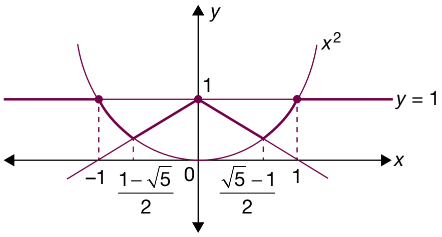


ILLUSTRATION 67

If $f(x) = \begin{cases} |x^5 - 4x^3 + 2|\cos \pi x & x \ge 0 \\ x^3 - \sin(x) - |(x^2+8x+12)x^2| & x < 0 \end{cases}$, then find number of points where $f(-|x|)$ is not differentiable.

SOLUTION

$f(x)$ is not differentiable at $x = -2, -6$ for $x < 0$.

$$f(x) = \begin{cases} |x^5 - 4x^3 + 2|\cos \pi x, & x \ge 0 \\ x^3 - \sin x - |(x+2)(x+6)||x|^2, & x < 0 \end{cases}$$

$\therefore \quad f(-|x|)$ is not differentiable at $x = \pm 2$ and ± 6.

Also, $f(0)$ is not defined for $x < 0$ and $f(x)$ is discontinuous at $x = 0$

$\therefore \quad f(-|x|)$ is not differentiable at $x = 0$.

Hence, $f(-|x|)$ is not differentiable at 5 points.

ILLUSTRATION 68

If n_1 = number of points of non-derivability of

$$f(x) = \begin{cases} \max.\{|x|, \sin|x|\}, & x \in \left(0, \frac{\pi}{2}\right) \\ \max.\{|x|, \tan|x|\}, & x \in \left(\frac{-\pi}{2}, 0\right) \end{cases} \quad \text{in} \quad \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$$

and n_2 = number of points of discontinuity of $g(x) = [\sin x] + [\cos x]$ where $[\cdot]$ is GIF in $[0, 2\pi]$ then find $(n_1 + n_2)$

SOLUTION

For $x \in \left(0, \frac{\pi}{2}\right)$, $\sin x < x < \tan x$ and

For $x \in \left(-\frac{\pi}{2}, 0\right)$, $\tan x < x < \sin x$

$$f(x) = \begin{cases} \max.\{|x|, \sin|x|\}, & x \in \left(0, \frac{\pi}{2}\right) \\ \max.\{|x|, \tan|x|\}, & x \in \left(\frac{-\pi}{2}, 0\right) \end{cases}$$

$$= \begin{cases} x & x \in \left(0, \frac{\pi}{2}\right) \\ \tan(-x) & x \in \left(\frac{-\pi}{2}, 0\right) \end{cases}$$

It is continuous but not differentiable at $x = 0$ so $n_1 = 1$

$$g(x) = [\sin x] + [\cos x]$$

$$g(x) = \begin{cases} 1, & x = 0, \frac{\pi}{2}, 2\pi \\ 0, & 0 < x < \frac{\pi}{2} \\ -1, & \frac{\pi}{2} < x \le \pi, \frac{3\pi}{2} \le x < 2\pi \\ -2, & \pi < x < \frac{3\pi}{2} \end{cases}$$

We can clearly count points of discontinuity as $x = 0, \pi/2, \pi, 3\pi/2, 2\pi$ we have $n_2 = 5$

Hence, $n_1 + n_2 = 6$.

ILLUSTRATION 69

Discuss the continuity and differentiability of the function $y = f(x)$ defined parametrically.

$$x = 2t - |t-1| \text{ and } y = 2t^2 + t\,|t|$$

SOLUTION

$$x = 2t - |t-1| \text{ and } y = 2t^2 + t\,|t|$$

CASE 1: When $t < 0$,

$$x = 2t + (t-1) = 3t - 1 \Rightarrow t = \frac{x+1}{3} < 0 \Rightarrow x < -1$$

$$y = 2t^2 - t^2 = t^2 \Rightarrow y = \frac{1}{9}(x+1)^2 \text{ for } x < -1$$

CASE 2: When $0 \le t < 1$

$$x = 2t + (t-1) = 3t - 1 \Rightarrow t = \frac{x+1}{3}$$

$$y = 2t^2 + t^2 = 3t^2 \Rightarrow y = \frac{1}{3}(x+1)^2$$

Here, $0 \le t < 1 \Rightarrow 0 \le \frac{x+1}{3} < 1 \Rightarrow -1 \le x < 2$

CASE 3: When $t \ge 1$

$$x = 2t - (t-1) = t + 1 \Rightarrow t = x - 1$$

$$y = 2t^2 + t^2 = 3t^2 \Rightarrow y = 3(x-1)^2$$

Here, $t \ge 1 \Rightarrow x - 1 \ge 1 \Rightarrow x \ge 2$

Thus,

$$y = f(x) = \begin{cases} \frac{1}{9}(x+1)^2, & x < -1 \\ \frac{1}{3}(x+1)^2, & -1 \le x < 2 \\ 3(x-1)^2, & x \ge 2 \end{cases}$$

Continuity at $x = -1$

$$\text{LHL} = \lim_{h\to 0} f(-1-h) = \lim_{h\to 0} \frac{1}{9}(-1-h+1)^2 = 0$$

$$\text{RHL} = \lim_{h\to 0} f(-1+h) = \lim_{h\to 0} \frac{1}{3}(-1+h+1)^2 = 0$$

$$f(-1) = \frac{1}{3}(-1+1)^2 = 0$$

Since, LHL = RHL = $f(-1)$ hence, continuous at $x = -1$

continuity at $x = 2$

$$\text{LHL} = \lim_{h\to 0} f(2-h) = \lim_{h\to 0} \frac{1}{3}(2-h+1)^2 = 3$$

$$\text{RHL} = \lim_{h\to 0} f(2+h) = \lim_{h\to 0} 3(2+h-1)^2 = 3$$

$$f(2) = 3(2-1)^2 = 3$$

Since, LHL = RHL = $f(2)$ hence, continuous at $x = 2$

Differentiability at $x = -1$

$$\text{LHD} = \lim_{h\to 0} \frac{f(-1-h) - f(-1)}{-h} = \lim_{h\to 0} \frac{\frac{1}{9}(-1-h+1)^2 - 0}{-h} = 0$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\frac{1}{3}(-1+h+1)^2 - 0}{h} = 0$$

Since, LHD = RHD Hence, f is differentiable at $x = -1$

Differentiability at $x = 2$

$$\text{LHD} = \lim_{h\to 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h\to 0} \frac{\frac{1}{3}(2-h+1)^2 - 3}{-h} = 2$$

$$\text{RHD} = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{3(2+h-1)^2 - 3}{h} = 6$$

Since, LHD ≠ RHD Hence, f is not differentiable at $x = 2$. Hence, f is continuous for all x and differentiable for all x except $x = 2$.

ILLUSTRATION 70

Let $f(x) = 1 + 4x - x^2 \quad x \in R$,

$$g(x) = \begin{cases} \max\{f(t),\ x \le t \le x+1, 0 \le x \le 3\} \\ \min\{(x+3), 3 \le x \le 5\} \end{cases}$$

Check continuity and differentiability of $g(x)$ for all $x \in [0, 5]$

SOLUTION

Since, $f(t) = 1 + 4t - t^2$

$$f'(t) = 4 - 2t = 0 \Rightarrow t = 2$$

$f''(t) = -2 < 0$ Hence, $f(t)$ has maximum at $t = 2$

$\max\{f(t);\ x \le t \le (x+1), 0 \le x \le 3\}$

$$= \begin{cases} f(x+1) & \text{if } [x, x+1] < 2 \\ f(2) & \text{if } x \le 2 \le x+1 \\ f(x) & \text{if } 2 < [x, x+1] \end{cases}$$

$$= \begin{cases} 1 + 4(x+1) - (x+1)^2, & 0 \le x < 1 \\ 5, & 1 \le x \le 2 \\ 1 + 4x - x^2, & 2 < x < 3 \end{cases}$$

$$= \begin{cases} 4 + 2x - x^2, & 0 \le x < 1 \\ 5, & 1 \le x \le 2 \\ 1 + 4x - x^2, & 2 < x < 3 \end{cases}$$

And $\min\{(x+3);\ 3 \le x \le 5\} = 6$

$$\therefore \quad g(x) = \begin{cases} 4 + 2x - x^2, & 0 \le x < 1 \\ 5, & 1 \le x \le 2 \\ 1 + 4x - x^2, & 2 < x < 3 \\ 6, & 3 \le x \le 5 \end{cases}$$

Continuity at $x = 1$

$$\text{LHL} = \lim_{h\to 0} g(1-h) = \lim_{h\to 0} [4 + 2(1-h) - (1-h)^2] = 5$$

$$\text{RHL} = \lim_{h\to 0} g(1+h) = 5$$

$f(1) = 5$ hence, $g(x)$ is continuous at $x = 1$

At $x = 2$

$$\text{LHL} = \lim_{h\to 0} g(2-h) = 5$$

$$\text{RHL} = \lim_{h\to 0} g(2+h) = \lim_{h\to 0}[1+4(2+h)-(2+h)^2] = 5$$

$$f(2) = 5$$

Since, LHL = RHL = $f(2)$ hence, $g(x)$ is continuous at $x = 2$

At $x = 3$

$$\text{LHL} = \lim_{h\to 0} g(3-h) = \lim_{h\to 0}[1+4(3-h)-(3-h)^2] = 4$$

$$\text{RHL} = \lim_{h\to 0} g(3+h) = 6$$

Since, LHL ≠ RHL hence, $g(x)$ is not continuous at $x = 3$

Hence, $g(x)$ is continuous in $x \in [0, 5] - \{3\}$

$$g'(x) = \begin{cases} 2-2x & 0 \le x < 1 \\ 0 & 1 \le x \le 2 \\ 4-2x & 2 < x < 3 \\ 6 & 3 < x \le 5 \end{cases}$$

$$g'(1^-) = 0 = g'(1^+) \text{ and } g'(2^-) = 0 = g'(2^+)$$

Hence, $g(x)$ is continuous in $x \in [0, 5] - \{3\}$, differentiable in $x \in [0, 5] - \{3\}$

Test Your Concepts-IV

(Solutions on page H.113)

1. Let $f(x) = \{|x| - |x - 1|\}^2$. Draw the graph of $f(x)$, discuss continuity and differentiability of $f(x)$.

2. Discuss the differentiability of $f(x) = \tan \pi [x] + \tan x$ where [.] is GIF.

3. Let $f : R \to R$ be a function defined by $f(x) = \min.(x + 1, |x| + 1)$, then prove $f(x)$ is differentiable everywhere

4. If $f(x) = \begin{cases} px + q, & x \le -1 \\ px^2 + x + 2q, & x > -1 \end{cases}$ is differentiable for all $x \in R$, then find the ordered pair (p, q).

5. Let $f(x) = \begin{cases} 2x+1, & x \le 1 \\ x^2+2, & 1 < x \le 2 \\ 4x-2, & x > 2 \end{cases}$. Then find the number of points where $f(x)$ is non-differentiable.

6. Let $g(x) = \begin{cases} 2x^2 - 3x + \dfrac{3}{2}, & \text{for } x < 1 \\ \dfrac{x^2}{2}, & \text{for } x \ge 1 \end{cases}$.

If L and M denotes the point of discontinuity and non-derivability of $g(x)$, then find the value of $(L + M)$.

7. If $f(x) = \max\{3 - x, 3 + x, 6\}$ then no. of points where $f(x)$ is not-derivable.

8. If $f(x) = \dfrac{\sin 2\pi[\pi^2 x]}{5+[x^2]}$ where [.] denotes the greatest integer function, then show that $f(x)$ is always differentiable.

9. The function defined by

$f(x) = \max\left\{x^2, (x-1)^2, 2x(1-x)\right\}, 0 \le x < 1$

Find number of point where $f(x)$ is not differentiable.

10. Find total number of points belonging to $(0, 2\pi)$ where $f(x) = \min\{\sin x, \cos x, 1 - \sin x\}$ is not differentiable.

11. If $f(x) = x^3 - 6x^2 + 11x - 6$, $x \in R$ then find the number of points where $|f(|x|)|$ is non-derivable.

12. Find number of points where $f(x) = \sqrt[3]{x^2\,|x|} - |x| - 1$ is not differentiable.

Answer key

1. Always continuous and not differentiable at $x = 0, 1$

2. Discontinuous and non-differentiable at $x = \left(\dfrac{2n+1}{2}\right)\pi, n \in I$.

4. (1/3, 1/3) **5.** 0 **6.** 0 **7.** 2 **9.** 2 **10.** 3 **11.** 7 **12.** 0

CONTINUITY AND DIFFERENTIABILITY OF COMPOSITE FUNCTION

ILLUSTRATION 71

If $f(x) = \begin{cases} 1+x^3 & x<0 \\ x^2-1 & x \ge 0 \end{cases}$ and $g(x) = \begin{cases} (x-1)^{1/3} & x<0 \\ \sqrt{x+1} & x \ge 0 \end{cases}$

Check continuity and differentiability of *gof*(x) at $x = 0$

SOLUTION

Check for continuity of *gof*(x) at $x = 0$

Let $x = 0 - h$; LHL $= \lim\limits_{h\to 0} g(f(0-h))$

$$= \lim_{h\to 0} g(1-h^3) = \lim_{h\to 0} \sqrt{1-h^3+1} = \sqrt{2}$$

$$g(f(0)) = g(-1) = (-1-1)^{1/3} = -2^{1/3}$$

Let $x = 0 + h$; RHL $= \lim\limits_{h\to 0} g(f(0+h))$

$$= \lim_{h\to 0} g(h^2-1) = \lim_{h\to 0} (h^2-1-1)^{1/3} = -2^{1/3}$$

Clearly, we can see that LHL ≠ RHL, so *gof*(x) is discontinuous and non-differentiable at $x = 0$.

ILLUSTRATION 72

$f(x) = \begin{cases} 1+x & x<0 \\ 2x-1 & x \ge 0 \end{cases}$ $\quad g(x) = \begin{cases} 4-x & x<-1 \\ 2x+7 & x \ge -1 \end{cases}$

Check continuity and differentiability of *gof*(x) at $x = 0$

SOLUTION

Let $x = 0 - h$; LHL $= \lim\limits_{h\to 0} g(f(0-h)$

$$= \lim_{h\to 0} g(1-h) = \lim_{h\to 0} 2(1-h)+7 = 9$$

$$gof(0) = g(-1) = -2+7 = 5$$

$$\text{RHL} = \lim_{h\to 0} g(f(0+4)) = \lim_{h\to 0} g(2h-1)$$

$$= \lim_{h\to 0} 2(2h-1)+7 = 5$$

Clearly, we can see that LHL ≠ RHL so *gof*(x) is discontinuous and non-differentiable at $x = 0$.

ILLUSTRATION 73

$f(x) = \dfrac{x+1}{x-1}$; $g(x) = \dfrac{1}{x-2}$ discuss continuity of (i) $f(x)$; (ii) $g(x)$; (iii) *fog*(x); (iv) *gof*(x)

SOLUTION

$f(x) = \dfrac{x+1}{x-1}$; $f(x)$ is discontinuous at $x = 1$

$g(x) = \dfrac{1}{x-2}$; $g(x)$ is discontinuous at $x = 2$

$$fog(x) = f\left(\frac{1}{x-2}\right) = \frac{\frac{1}{x-2}+1}{\frac{1}{x-2}-1} = \frac{1+x-2}{1-(x-2)} = \frac{x-1}{3-x}$$

fog(x) is discontinuous at $x = 2, 3$.

$$gof(x) = g\left(\frac{x+1}{x-1}\right) = \frac{1}{\left(\frac{x+1}{x-1}\right)-2} = \frac{x-1}{x+1-2(x-1)} = \frac{x-1}{3-x}$$

fog(x) is discontinuous at $x = 1, 3$

(v) $fof(x) = f\left(\dfrac{x+1}{x-1}\right) = \dfrac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \dfrac{x+1+x-1}{x+1-(x-1)} = x$

fog(x) is discontinuous at $x = 1$

(vi) $gog(x) = g\left(\dfrac{1}{x-2}\right) = \dfrac{1}{\frac{1}{x-2}-2} = \dfrac{x-2}{1-2(x-2)} = \dfrac{x-2}{5-2x}$

gog(x) is discontinuous at $x = 2, \dfrac{5}{2}$

ILLUSTRATION 74

Let f be a composite function of x defined by $f(u) = \dfrac{1}{u^3-6u^2+11u-6}$, where $u(x) = \dfrac{1}{x}$. Then find the number of points x where f is discontinuous.

SOLUTION

$$f(u) = \frac{1}{u^3-6u^2+11u-6} = \frac{1}{(u-1)(u-2)(u-3)},$$

As, $u(x) = \dfrac{1}{x}$

$u(x)$ is discontinuous at $x = 0$

⇒ $f(x)$ is also discontinuous at $x = 0$

So, $f = \dfrac{1}{\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)\left(\frac{1}{x}-3\right)}$

Hence, f is discontinuous at $x = 1, \dfrac{1}{2}, \dfrac{1}{3} . 0$

ILLUSTRATION 75

Let $f(x) = -1 + |x-2|$ and $g(x) = 1 - |x|$, then find the set of all points where (fog) (x) is discontinuous.

SOLUTION

$$f(x) = -1 + |x-2|,\ g(x) = 1 - |x|$$

both are continuous function.

$fog(x) = -1 + |1 - |x| - 2| = -1 + |1 + |x|| = -1 + 1 + |x|$

Hence, $fog(x) = |x|$ which is continuous always.

ILLUSTRATION 76

Let $f : R \to R$ be a continuous function such that $f(x) = 19$, $x \in Q$ and $g(x) = \sqrt{39 + x^2}$. Then find the value of $g(f(x))$

SOLUTION

$$f(x) = 19 \ \forall \ x \in Q$$

As, $f(x)$ is continuous

$$f(x) = 19 \ \forall \ x \in R \ (f(x) \text{ is constant})$$

Hence, $g(f(x)) = \sqrt{39 + f^2(x)} = \sqrt{39 + 361} = 20 \ \forall \ x \in R$

ILLUSTRATION 77

Examine the continuity of $f(x) = \lim\limits_{n\to\infty} \dfrac{x}{(2\sin x)^{2n} + 1} \ \forall \ x \in R$.

SOLUTION

If $|2 \sin x| < 1 \Rightarrow |\sin x| < \dfrac{1}{2}$

$$\Rightarrow \quad -\frac{1}{2} < \sin x < \frac{1}{2} \Rightarrow n\pi - \frac{\pi}{6} < x < n\pi + \frac{\pi}{6}$$

$$f(x) = \lim_{n\to\infty} \frac{x}{(2\sin x)^{2n} + 1} = \frac{x}{0+1} = x$$

If $|2 \sin x| = 1 \Rightarrow |\sin x| = \dfrac{1}{2} \Rightarrow \sin x = \pm\dfrac{1}{2} \Rightarrow x = n\pi \pm \dfrac{\pi}{6}$

$$f(x) = \lim_{n\to\infty} \frac{x}{(2\sin x)^{2n} + 1} = \frac{x}{1+1} = \frac{x}{2}$$

If $|2 \sin x| > 1 \Rightarrow |\sin x| > \dfrac{1}{2}$

$$\Rightarrow \quad \sin x < -\frac{1}{2} \text{ or } \sin x > \frac{1}{2}$$

$$\Rightarrow \quad n\pi + \frac{\pi}{6} < x < n\pi + \frac{5\pi}{6}$$

$$f(x) = \lim_{n\to\infty} \frac{x}{(2\sin x)^{2n} + 1} = \frac{x}{\infty + 1} = 0$$

$$\text{Hence, } f(x) = \begin{cases} x, & n\pi - \dfrac{\pi}{6} < x < n\pi + \dfrac{\pi}{6} \\ \dfrac{x}{2}, & x = n\pi \pm \dfrac{\pi}{6} \\ 0, & n\pi + \dfrac{\pi}{6} < x < n\pi + \dfrac{5\pi}{6} \end{cases}$$

Hence, $f(x)$ is discontinuous at $x = n\pi \pm \dfrac{\pi}{6}$

ILLUSTRATION 78

Let $f(x) = \lim\limits_{n\to\infty} \dfrac{(x^2 + 4x + 5 + e^x + \text{sgn}(e^{-x}))^n - 5}{2(x^2 + 4x + 5 + e^x - \text{sgn}(e^{-x}))^n + 9}$ then discuss continuity and differentiability.

SOLUTION

As, let $g(x) = x^2 + 4x + 5 + e^x + \text{sgn}(e^{-x}) = (x + 2)^2 + 2 + e^x > 2.$

$$f(x) = \lim_{n\to\infty} \frac{(g(x))^n - 5}{2(g(x))^n + 9} = \lim_{n\to\infty} \frac{1 - \dfrac{5}{(g(x))^n}}{2 + \dfrac{9}{(g(x))^n}} = \frac{1-0}{2+0} = \frac{1}{2}$$

So, $f(x) = \dfrac{1}{2} \ \forall \ x \in R.$

Hence, $f(x)$ is always continuous and differentiable.

ILLUSTRATION 79

Let a and c be real numbers such that $c > 0$ and $f(x)$ is defined on [0, 1] as $f(x) = \begin{cases} x^a \sin(x^{-c}), & \text{if} \quad x \neq 0 \\ 0, & \text{if} \quad x = 0 \end{cases}$

Show that

(i) $f(x)$ is continuous if and only if $a > 0$
(ii) $f'(0)$ exists if and only if $a > 1$
(iii) $f'(x)$ is continuous if and only if $a > 1 + c$
(iv) $f''(0)$ exists if and only if $a > 2 + c$

SOLUTION

(i) $f(x) = \begin{cases} x^a \sin\left(\dfrac{1}{x^c}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$f(0) = 0$$

$$\lim_{x\to 0} x^a \sin\left(\frac{1}{x^c}\right) = (0)(-1 \text{ to } 1) = 0 \ \text{ if } a > 0.$$

For continuous function $a > 0$

(ii) $\text{RHD} = f'(0^+) = \lim\limits_{h\to 0} \dfrac{(0+h)^a \sin\left(\dfrac{1}{(0+h)^c}\right) - 0}{h}$

$$= \lim_{h\to 0} h^{a-1} \sin\frac{1}{h^c} = 0 \qquad (\text{for } a > 1)$$

$$\text{LHD} = f'(0^-) = \lim_{h\to 0} \frac{(0-h)^a \sin\left(\dfrac{1}{(0-h)^c}\right) - 0}{-h}$$

$$= \lim_{h\to 0} (-h)^{a-1} \sin\frac{1}{(-h)^c} = 0 \qquad \text{for } a > 1$$

Hence, $f'(0)$ exists if and only if $a > 1$

(iii) To discuss about continuity of $f'(x)$, we should get derivative

$$f'(x) = ax^{a-1} \sin\frac{1}{x^c} + x^a \cos\left(\frac{1}{x^c}\right)\left(\frac{-c}{x^{c+1}}\right)$$

$$= ac^{a-1} \sin\frac{1}{x^c} - cx^{a-c-1} \cos\frac{1}{x^c};$$

$$f'(x) = x^{a-c-1}\left(ax^c \sin\frac{1}{x^c} - c\cos\frac{1}{x^c}\right)$$

Hence, $f'(x)$ is continuous if and only if $a > 1 + c.$

(iv) $f''(0)$, we should talk about LHD and RHD of $f'(x)$ at $x = 0$

$$\text{RHD} = f'(0^+) = \lim_{h\to 0}\frac{h^{a-c-1}\left(ah^c sm\frac{1}{h^c} - c\cos\frac{1}{h^c}\right) - 0}{h}$$

$$= \lim_{h\to 0} h^{a-c-2}\left(-1 \text{ to } 1\right) = 0$$

If $a - c - 2 > 0 \Rightarrow a > 2 + c$

LHD =

$$f'(0^-) = \lim_{h\to 0}\frac{(-h)^{a-c-1}\left(a(-h)^c \sin\frac{1}{(-h)^c} - c\cos\frac{1}{(-h)^c}\right) - 0}{-h}$$

$$= \lim_{h\to 0}(-h)^{a-c-2}\left(-1 \text{ to } 1\right) = 0$$

If $a - c - 2 > 0 \Rightarrow a > 2 + c$

Hence, $f''(0)$ exists if and only if $a > 2 + c$.

CONTINUITY OF DERIVATIVE

If a function is differentiable, it means that its derivative exists but if we say that a function has a continuous derivative, then it implies that it is differentiable and its derivative is a continuous function.

It may be possible that function $y = f(x)$ is differentiable at $x = a$, but it is not necessary that its derivative $f'(x)$ is continuous at $x = a$.

For example, consider function $f(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

At $x = 0$, RHD $= f'(0^+)$

$$= \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0(-1\,to\,1) = 0$$

LHD $= f'(0^-)$

$$= \lim_{h\to 0}\frac{(-h)^2\sin\frac{1}{-h} - 0}{-h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0(-1\,to\,1) = 0$$

As LHD = RHD, $f(x)$ is differentiable at $x = 0$

Now lets find its derivative

For $x \neq 0$,

$$f'(x) = 2x\sin\frac{1}{x} + x^2\left(-\frac{1}{x^2}\right)\cos\frac{1}{x} = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right) \text{ does not exist}$$

Clearly, we can conclude that $f(x)$ is differentiable at $x = 0$ but its derivative $f'(x)$ is not continuous at $x = 0$.

ILLUSTRATION 80

Let $f(x) = \begin{cases} x^2\cos\frac{1}{x}, & x < 0 \\ 0, & x = 0 \\ x^2\sin\frac{1}{x}, & x > 0 \end{cases}$, then discuss continuity and differentiability of $f(x)$ and $f'(x)$ at $x = 0$.

SOLUTION

$$\left.\begin{aligned} f(0^+) &= \lim_{x\to 0^+} x^2\sin\frac{1}{x} = 0 \\ f(0^-) &= \lim_{x\to 0^-} x^2\cos\frac{1}{x} = 0 \end{aligned}\right\};\quad f(0) = 0 \text{ it means } f(x) \text{ is}$$

continuous at $x = 0$

$$f'(0^+) = \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h} = \lim_{h\to 0} h\sin\frac{1}{h} = 0(-1+0+1) = 0$$

$$f'(0^-) = \lim_{h\to 0}\frac{h^2\cos\frac{1}{h} - 0}{-h} = \lim_{h\to 0} -h\cos\frac{1}{h} = 0(-1+0+1) = 0$$

$f(x)$ is differentiable at $x = 0$

$$f'(x) = \begin{cases} 2x\cos\frac{1}{x} + x^2\left(-\sin\frac{1}{x}\left(\frac{-1}{x^2}\right)\right) & x < 0 \\ 0 & x = 0 \\ 2x\sin\frac{1}{x} + x^2\left(\cos\frac{1}{x}\left(\frac{-1}{x^2}\right)\right) & x > 0 \end{cases}$$

$$= \begin{cases} 2x\cos\frac{1}{x} + \sin\frac{1}{x} & x < 0 \\ & x = 0 \\ 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x > 0 \end{cases}$$

$\lim\limits_{x\to 0} f'(x)$ does not exist.

$f(x)$ is continuous, differentiable but $f'(x)$ is not continuous and not differentiable.

EXTENDED FORMULA OF DERIVATIVE

Let $p(h)$ and $q(h)$ are two functions of h and as $h \to 0$, $p(h) \to 0$ and $q(h) \to 0$ then

$$\lim_{h\to 0}\frac{f(\alpha + p(h) - f(\alpha + q(h))}{p(h) - q(h)} = f'(\alpha),$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

ILLUSTRATION 81

If $f(x)$ is a differentiable function such that find $f'(2) = 6$, $f'(1) = 4$ and $\lim\limits_{h\to 0} \dfrac{f(h^3+3h+2)-f(2)}{f(2h-2h^2+1)-f(1)}$.

SOLUTION

Given,

$$\lim_{h\to 0} \frac{f(h^3+3h+2)-f(2)}{f(2h-2h^2+1)-f(1)} \quad \text{(Using L–Hospital Rule)}$$

$$= \lim_{h\to 0} \frac{f'(h^3+3h+2)\cdot(3h^2+3)-0}{f'(2h-2h^2+1)(2-4h)-0}$$

$$= \frac{3f'(2)}{2f'(1)} = \frac{6\cdot 3}{4\cdot 2} = \frac{9}{4}$$

ILLUSTRATION 82

Let f be a differentiable function on R and $f'(3) = 2$, then find $\lim\limits_{h\to 0}\left(\dfrac{f(3+\sin^3 h)-f(3-\sin^2 h)}{h^2}\right)$

SOLUTION

$$\lim_{h\to 0}\left(\frac{f(3+\sin^3 h)-f(3-\sin^2 h)}{h^2}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(3+\sin^3 h)-f(3-\sin^2 h)}{\sin^3 h+\sin^2 h}\right)\cdot\left(\frac{\sin^3 h+\sin^2 h}{h^2}\right)$$

$$= f'(3)\times(0+1) = 2.$$

ILLUSTRATION 83

If $f'(a) = \dfrac{1}{4}$, then evaluate $\lim\limits_{h\to 0} \dfrac{f(a+2h^2)-f(a-2h^2)}{f(a+h^3-h^2)-f(a-h^3+h^2)}$

SOLUTION

$$\lim_{h\to 0} \frac{f(a+2h^2)-f(a-2h^2)}{f(a+h^3-h^2)-f(a-h^3+h^2)}$$

$$= \lim_{h\to 0} \frac{f(a+2h^2)-f(a-2h^2)}{2h^2-(-2h^2)}.$$

$$\frac{4h^2}{\dfrac{f(a+h^3-h^2)-f(a-h^3+h^2)}{(h^3-h^2)-(-h^3+h^2)}.(2h^3-2h^2)}$$

$$= \lim_{h\to 0} \frac{f'(a)\,(4h^2)}{f'(a)(2h^3-2h^2)} = \lim_{h\to 0} \frac{4}{2h-2} = -2$$

SOLVING FUNCTIONAL EQUATION

ILLUSTRATION 84

If $f(x+y) = f(x) + f(y) + |x|\, y + xy^2, \forall\, x, y \in R$ and $f'(2) = 0$, then discuss continuity and differentiability of $f'(x)$ in $x \in R$

SOLUTION

$$f(x+y) = f(x) + f(y) + |x|y + xy^2 (f(0) = 0)$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)+|x|h+xh^2}{h}$$

$$f'(x) = \lim_{h\to 0}\left[\frac{f(0+h)-f(0)}{h}+|x|+xh\right]$$

$$f'(x) = f'(0) + |x|$$

$$f'(2) = 0 = f'(0) + 2 \Rightarrow f'(0) = -2$$

$$\Rightarrow \quad f'(x) = |x| - 2$$

Hence, $f'(x)$ is always continuous but not diff. at $x = 0$.

ILLUSTRATION 85

If a twice differentiable function satisfying a relation $f(x^2y) = x^2f(y) + y\,f(x^2)\ \forall\ x, y > 0$ and $f'(1) = 1$, then find the value of $f''\left(\dfrac{1}{7}\right)$.

SOLUTION

$$f(x^2y) = x^2f(y) + yf(x^2)\ \forall\ x, y > 0$$

Differentiate w.r.t. x keeping y constant,

$$2xyf'(x^2y) = f(y)\ 2x + y2xf'(x^2)$$

$$\therefore \quad yf'(x^2y) = f(y) + yf'(x^2)$$

Put $x = 1$

$$yf'(y) = f(y) + yf'(1)$$

$$yf'(y) = f(y) + y \qquad \ldots(1)$$

Now, again differentiate w.r.t. y

$$yf''(y) + f'(y) = f'(y) + 1$$

Hence, $f''(y) = \dfrac{1}{y} \Rightarrow f''\left(\dfrac{1}{7}\right) = 7$

ILLUSTRATION 86

Let R be the set of real numbers and $f : R \to R$, be a differentiable function such that $|f(y) - f(y)| \le |x - y|^3\ \forall\ x, y \in R$. If $f(10) = 100$, then find the value of $f(20)$.

SOLUTION

$$|f(y) - f(y)| \le |x - y|^3$$

Put $x = x + h$

$$\therefore \quad |f(x+h) - f(x)| \le |h|^3$$

$$\Rightarrow \quad \lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| \le \lim_{h\to 0} h^2$$

$\Rightarrow \quad |f'(x)| \le 0$, modulus function can not be negative

$\Rightarrow \quad |f'(x)| = 0 \Rightarrow f'(x) = 0$

Therefore, $f(x)$ = constant, $(\because f(10) = 100)$

$\therefore \quad f(x) = 100$

Hence, $f(20) = 100$

ILLUSTRATION 87

If $(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2-y^2)$ for all x, y and $f(1) = 2$ find $f(x)$

SOLUTION

$$(x-y)\frac{f(x+y)}{(x^2-y^2)} - \frac{(x+y)f(x-y)}{(x^2-y^2)} = 4xy$$

$$\Rightarrow \quad \frac{f(x+y)}{x+y} - \frac{f(x-y)}{x-y} = (x+y)^2 - (x-y)^2$$

$$\Rightarrow \quad \frac{f(x+y)}{x+y} - (x+y)^2 = \frac{f(x-y)}{x-y} - (x-y)^2$$

We can say that $\dfrac{f(t)}{t} - t^2 = k$

$f(t) - t^3 = kt$

$f(t) = kt + t^3$

Given $f(1) = 2$, we get $k = 1$

$f(x) = x + x^3$

ILLUSTRATION 88

Let $f(x)$ be a polynomial function satisfying

$f(x) + f(y) = f(x)f(y) - f(xy) + f(1)\ \forall\ x, y \in R$ and $f'(1) = 5$.

Find number of divisors of $f'(1) \,.\, f''(1) \,.\, f'''(1)$

SOLUTION

Putting $y = \dfrac{1}{x}$, we get

$$f(x) + f\left(\frac{1}{x}\right) = f(x)\,f\left(\frac{1}{x}\right)$$

$\Rightarrow \quad f(x) = 1 \pm x^n$

$\Rightarrow \quad f'(x) = \pm\, nx^{n-1} \Rightarrow f'(1) = \pm\, n = 5$

$\Rightarrow \quad n = 5$ (–ve sign rejected)

$\therefore \quad f(x) = 1 + x^5$

$f'(1)\,.\,f''(1)\,.\,f'''(1) = 5 \times (5 \times 4) \times (5 \times 4 \times 3) = 5^3 \times 2^4 \times 3^1$

$\therefore$ Total number of divisors $N = (3+1)(4+1)(1+1)$

$= 4 \times 5 \times 2 = 40.$

ILLUSTRATION 89

Let $f(x+y) = f(x) + f(y) - 2xy - 1$ for all x and y. If $f'(0)$ exists and $f'(0) = -\sin\alpha$, then find

(i) $f(x)$ **(ii)** $f(f'(0))$

SOLUTION

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\{f(x) + f(h) - 2xh - 1\} - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{-2xh + f(h) - 1}{h}$$

$$f'(x) = -2x + \lim_{h\to 0}\frac{f(h) - 1}{h} \qquad \ldots(1)$$

Since, $f(x+y) = f(x) + f(y) - 2xy - 1$

Putting $x = y = 0$

$f(0) = f(0) + f(0) - 0 - 1 \Rightarrow f(0) = 1$

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{f(h) - 1}{h}$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{f(h) - 1}{h} = f'(0) = -\sin\alpha$$

$\therefore \quad$ From (1), $f'(x) = -2x - \sin\alpha$

$\Rightarrow \quad f(x) = -x^2 - x\sin\alpha + c$

Where c is constant of integration.
Putting $x = 0$

$f(0) = 0 - 0 + c$

$\Rightarrow \quad c = 1$

Therefore, $f(x) = -x^2 - x\sin\alpha + 1$

Hence, $f\{f'(0)\} = f(-\sin\alpha) = -\sin^2\alpha + \sin^2\alpha + 1 = 1$

ILLUSTRATION 90

Let $f : R \to R$ be a real-valued differentiable function satisfying $f\left(\dfrac{2x+3y}{5}\right) = \dfrac{2f(x)+3f(y)}{5}$ for all $x, y \in R$ and $f(0) = 1, f'(0) = 2$. Find the value of $f(3) + f'(4)$

SOLUTION

$$f\left(\frac{2x+3y}{5}\right) = \frac{2f(x)+3f(y)}{5}$$

Differentiate w.r.t. x, keeping y constant

$$f'\left(\frac{2x+3y}{5}\right) = f'(x) \qquad \ldots(1)$$

Similarly, differentiate w.r.t. y, keeping x constant

$$f'\left(\frac{2x+3y}{5}\right) = f'(y) \qquad \ldots(2)$$

From (1) and (2), we get

$f'(x) = f'(y)$

Hence, $f'(x) = c = 2$ (given $f'(0) = 2$)

On integrating, we get

$$f(x) = 2x + C$$

Given, $f(0) = 1 \Rightarrow C = 1$

Hence, $f(x) = 2x + 1$

$$f(3) + f'(4) = 7 + 2 = 9$$

ILLUSTRATION 91

Let $f(x) = 2\tan^{-1}x$ and $g(x)$ be a differentiable function satisfying $g\left(\frac{x+2y}{3}\right) = \frac{g(x)+2g(y)}{3}$ $\forall\ x, y \in R$ and $g'(0) = 1$, $g(0) = 2$. Find the number of integers satisfying $f^2(g(x)) - 5f(g(x)) + 4 > 0$ where $x \in (-10, 10)$.

SOLUTION

$$g\left(\frac{x+2y}{3}\right) = \frac{g(x)+2g(y)}{3}$$

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$$

$$= \lim_{h\to 0}\frac{g\left(\frac{3x+2(3h/2)}{3}\right)-g\left(\frac{3x+2(0)}{3}\right)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{g(3x)+2g(3h/2)}{3}-\frac{g(3x)+2g(0)}{3}}{h}$$

$$= \lim_{h\to 0}\left(\frac{g(0+3h/2)-g(0)}{3h/2}\right) = g'(0) = 1$$

$\Rightarrow\quad g'(x) = 1$

$\Rightarrow\quad g(x) = x + 2 \qquad$ as $g(0) = 2$

$f(x) = 2\tan^{-1}x$ and $g(x) = x + 2 \Rightarrow f(g(x)) = 2\tan^{-1}(x + 2)$

Solution of inequality $f^2(g(x)) - 5f(g(x)) + 4 > 0$ is

$$f(g(x)) < 1 \text{ or } f(g(x)) > 4$$

$$\Rightarrow\quad \tan^{-1}(x+2) < \frac{1}{2} \text{ or } \tan^{-1}(x+2) > 2$$

$$\left[\text{As, } \tan^{-1}(x+2) < \frac{\pi}{2}\right]$$

We get $\tan^{-1}(x+2) < \frac{1}{2}$

$$\Rightarrow\quad x + 2 < \tan\left(\frac{1}{2}\right) \Rightarrow x \in \left(-10, \tan\left(\frac{1}{2}\right)-2\right)$$

As, $\frac{1}{2} < \frac{\pi}{6} \Rightarrow \tan\frac{1}{2} < \frac{1}{\sqrt{3}} \Rightarrow \tan\frac{1}{2} - 2 < \frac{1}{\sqrt{3}} - 2$

In given interval $x \in (-10, 10)$

Hence, total integer in the range are $\{-9, -8, -7, -6, -5, -4, -3, -2\}$, 8 integers.

Test Your Concepts-V

(Solutions on page H.114)

1. Let f be a differentiable function such that $\lim_{h\to 0}\frac{f((\pi+h)^3)-f(\pi^3)}{h} = \pi$, then find the value of $f'(\pi^3)$.

2. If $f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3} \forall x, y \in R$ and $f'(0) = 1$, then show that $f(x)$ is a linear function in x.

3. Let $f: R \to R$ be a differentiable function at $x = 0$ satisfying $f(0) = 0$ and $f'(0) = 1$, then find the value of $\lim_{x\to 0}\frac{1}{x}\sum_{n=1}^{\infty}(-1)^n f\left(\frac{x}{n}\right)$.

4. If $f\left(\frac{x+y}{3}\right) = \frac{2+f(x)+f(y)}{3}$ for all real x and y and $f(x)$ is a differentiable function and $f'(2) = 2$, then find $f(1/2) + f(3) + f(5)$.

5. Find the point of discontinuity of $y = f(u)$, where $f(u) = \frac{3}{2u^2+5u-3}$ and $u = \frac{1}{x+2}$.

6. Let $f(x)$ be a continuous function which satisfies $f(x^2 + 1) = \frac{x^{h2}}{f(2^x)-1}$ and $f(x) > 0$ for all $x \in R$, then find $\lim_{x\to 1} f(x)$.

7. If $f'(2) = 6$ and $f'(1) = 4$, then find $\lim_{h\to 0}\frac{f(2h+2+h^2)-f(2)}{f(h-h^2+1)-f(1)}$

8. Let $f(x) = \sin x$, $g(x) = [x + 1]$ and $h(x) = gof(x)$, where $[\cdot]$ is greatest integer function, then find $h'(\pi/2)$.

9. Let f be a differentiable function satisfying the condition: $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ for all $x, y\ (\neq 0) \in R$ and $f(y) \neq 0$. If $f'(1) = 2$, then show that $f'(x) = \frac{2f(x)}{x}$.

10. If $f(x + y) = 2f(x) \cdot f(y)$ for all x, y, where $f'(0) = 3$ and $f(4) = 2$, then find $f'(4)$.

11. Let $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for real values of x and y. If $f'(0)$ exists and equals to -1 and $f(0) = 1$, then find $f'(2)$.

Answer key

1. $1/3\pi$ **3.** $-\ln 2$ **4.** 23 **5.** $x = 0, -2, -7/3$ **6.** 2

7. 3 **8.** Does not exist **10.** 12 **11.** -1

SOLVED PROBLEMS

PROBLEM 1

Let the equation $x^3 + 2x^2 + px + b = 0$ and $x^3 + x^2 + px + r = 0$ have two roots in common and the third root of each equation are represented by α and β respectively.

If $f(x) = \begin{cases} e^{x\log_{1+x}|\alpha+\beta|}, & -1 < x < 0 \\ a, & x = 0 \\ b\dfrac{\ln(e^{x^2} + \alpha\beta\sqrt{x})}{\tan\sqrt{x}}, & 0 < x < 1 \end{cases}$ is continuous at $x = 0$, then find the value of $2(a + b)$.

SOLUTION

Given, $x^3 + 2x^2 + px + b = 0$ ...(1)

And $x^3 + x^2 + px + r = 0$ ...(2)

Subtracting equation (2) from (1), common roots are the roots of $x^2 + b - r = 0$ which has two roots say x_1 and x_2 such that $x_1 + x_2 = 0$

$\Rightarrow \quad x_1 + x_2 + \alpha = -2$ and $x_1 + x_2 + \beta = -1$

$\therefore \quad \alpha = -2, \beta = -1$

$$\text{LHL} = \lim_{x\to 0^-} e^{x\log_{1+x}|\alpha+\beta|} = \lim_{h\to 0} e^{-h\frac{\ln 3}{\ln(1-h)}} = e^{\ln 3} = 3$$

$\Rightarrow \quad f(0^-) = f(0) = a = 3$

$$\text{RHL} = \lim_{h\to 0} b\frac{\ln(e^{h^2} + \alpha\beta\sqrt{h})}{\tan\sqrt{h}}$$

$$= b\lim_{h\to 0}\frac{\ln(1 + e^{h^2} - 1 + 2\sqrt{h})}{(e^{h^2} - 1 + 2\sqrt{h})}\cdot\frac{(e^{h^2} - 1 + 2\sqrt{h})}{\sqrt{h}}\,\frac{\sqrt{h}}{\tan\sqrt{h}}$$

$$\text{RHL} = b\lim_{h\to 0}\left(\frac{1 + h^2 + \frac{h^4}{2!} - 1 + 2\sqrt{h}}{\sqrt{h}}\right) = 2b = f(0) = 3$$

$\Rightarrow \quad b = \dfrac{3}{2}$ Hence, $2(a + b) = 9$

PROBLEM 2

Let $f(x) = \lim\limits_{n\to\infty}\dfrac{(x^2 + 2x + 3 + \sin\pi x)^n - 1}{(x^2 + 2x + 3 + \sin\pi x)^n + 1}$, then discuss continuity and differentiability of $f(x)$.

SOLUTION

$$x^2 + 2x + 3 + \sin\pi x = (x + 1)^2 + 2 + \sin\pi x > 1$$

$$\lim_{n\to\infty}\left((x+1)^2 + 2 + \sin\pi x\right)^n \to \infty$$

$$f(x) = \lim_{n\to\infty}\frac{1 - \dfrac{1}{(x^2 + 2x + 3 + \sin\pi x)^n}}{1 + \dfrac{1}{(x^2 + 2x + 3 + \sin\pi x)^n}} = \frac{1 - 0}{1 + 0} = 1$$

Hence, $f(x) = 1\ \forall\ x \in R$, always continuous and derivable.

PROBLEM 3

If $f(x) = \lim\limits_{n\to\infty}\dfrac{(1 + \sin x)^n + e^x}{1 + (1 + \sin x)^n}$, then the find the total number of point of discontinuity of $f(x)$.

SOLUTION

For $x \in (2n\pi, (2n + 1)\pi)$, we have $\lim\limits_{n\to\infty}(1 + \sin x)^n \to \infty$

$$f(x) = \lim_{n\to\infty}\left(\frac{1 + \dfrac{e^x}{(1 + \sin x)^n}}{\dfrac{1}{(1 + \sin x)^n} + 1}\right) = \lim_{n\to\infty}\left(\frac{1 + 0}{0 + 1}\right) = 1$$

For $x = n\pi$, $\lim\limits_{n\to\infty}(1 + \sin x)^n = 1$

$$f(x) = \lim_{n\to\infty}\left(\frac{(1 + \sin x)^n + e^x}{1 + (1 + \sin x)^n}\right) = \frac{1 + e^x}{1 + 1} = \frac{e^x + 1}{2}$$

For $x \in ((2n - 1)\pi, 2n\pi)$, $\sin x < 0$ then $\lim\limits_{n\to\infty}(1 + \sin x)^n = 0$

$$f(x) = \lim_{n\to\infty}\left(\frac{(1 + \sin x)^n + e^x}{1 + (1 + \sin x)^n}\right) = \frac{0 + e^x}{1 + 0} = e^x$$

$$f(x) = \begin{cases} 1 & x \in (2n\pi, (2n + 1)\pi) \\ \dfrac{1 + e^x}{2} & x = n\pi \\ e^x & x \in ((2n - 1)\pi, 2n\pi) \end{cases}$$

Hence, we see that function discontinuous at all integral multiple of π.

PROBLEM 4

If $f(x) = \begin{cases} |x + 1 - a^2| + |a + 1 - 2x^2| + 2, & a \le x < 0 \\ \dfrac{\ln(b^2 - 1 + \sin 2x)}{e^{\sin x} - 1}, & 0 < x \le b \end{cases}$ is continuous at $x = 0$ with $f(0) = 2$ then find the value of $a + b$

SOLUTION

$f(x)$ is continuous at $x = 0$

$$\text{LHL} = \lim_{x\to 0^-}(|x + 1 - a^2| + |a + 1 - 2x^2| + 2) = f(0)$$

$(\because f(0) = 2)$

$\Rightarrow \quad |1 - a^2| + |a + 1| = 0 \Rightarrow a = -1$ for both modulus terms to be zero

$$\text{RHL} = \lim_{x\to 0^+} \frac{\ln(b^2 - 1 + \sin 2x)}{e^{\sin x} - 1} = f(0)$$

For limit to be the exist $\ln(b^2 - 1) = 0 \Rightarrow b^2 - 1 = 1$

$\Rightarrow \quad b^2 = 2 \Rightarrow b = \pm\sqrt{2}$ but $b > 0$

$\therefore \quad b = \sqrt{2}$

$$\text{RHL} = \lim_{x\to 0^+} \frac{\ln(1+\sin 2x)}{\dfrac{e^{\sin x}-1}{\sin x}\sin x}$$

$$= \lim_{x\to 0^+} \frac{\ln(1+\sin 2x)}{\sin 2x}\,\frac{\sin 2x}{\sin x} = \lim_{x\to 0^+} \frac{2\sin x\cos x}{\sin x} = 2$$

Hence, $a + b = \sqrt{2} - 1$

PROBLEM 5

If $\displaystyle\sum_{r=1}^{n} \sin^{-1}(\alpha_r) = \frac{n\pi}{2}$ for any $n \in N$ and $p = \displaystyle\prod_{r=1}^{n} (\alpha_r)^r$.

If $f(x) = \begin{cases} \dfrac{x^{1/3} - (3-2x)^{1/4}}{x^2 - x}, & x \neq p \\ k, & x = p \end{cases}$ is continuous at $x = p$, then find the value of $6(k + p)$.

SOLUTION

As, $\sin^{-1}\alpha_1 + \sin^{-1}\alpha_2 \ldots + \sin^{-1}\alpha_n = \dfrac{n\pi}{2}$ $\left(\because \dfrac{-\pi}{2} \leq \sin^{-1}\alpha \leq \dfrac{\pi}{2}\right)$

$\Rightarrow \quad \sin^{-1}\alpha_1 = \sin^{-1}\alpha_2 = \ldots = \sin^{-1}\alpha_n = \dfrac{\pi}{2}$

$\Rightarrow \quad \alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 1$

So, $p = \displaystyle\prod_{r=1}^{n} (\alpha_r)^r = 1.1.1.1\ldots\ldots.1 = 1$

Hence, $f(x) = \begin{cases} \dfrac{x^{1/3} - (3-2x)^{1/4}}{x^2 - x}, & x \neq 1 \\ k, & x = 1 \end{cases}$

Clearly, $f(x)$ is continuous at $x = 1$.

$$\therefore \quad k = \lim_{x\to 1} \frac{x^{1/3} - (3-2x)^{1/4}}{x^2 - x}$$

$$= \lim_{h\to 0} \frac{(1+h)^{1/3} - (1-2h)^{1/4}}{(1+h)^2 - (1+h)} \quad (\text{Putting } x = 1 + h)$$

$$= \lim_{h\to 0} \frac{(1+h)^{1/3} - (1-2h)^{1/4}}{h + h^2}$$

$$= \lim_{h\to 0} \frac{1/3(1+h)^{-2/3} - 1/4(1-2h)^{-3/4}(-2)}{1+2h}$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

Hence, $6(k + p) = 6\left(\dfrac{5}{6} + 1\right) = 11$

PROBLEM 6

Let $P(x)$ be a polynomial satisfying

$$P(x) - 2P'(x) = 3x^3 - 27x^2 + 38x + 1.$$

If the function $f(x) = \begin{cases} \dfrac{P''(x) + 18}{6}, & x \neq \dfrac{\pi}{2} \\ \sin^{-1}(ab) + \cos^{-1}(a + b - 3ab), & x = \dfrac{\pi}{2} \end{cases}$

is continuous at $x = \dfrac{\pi}{2}$, then find $(a + b)$.

SOLUTION

Let $P(x) = Ax^3 + Bx^2 + Cx + D$

So, $P'(x) = 3Ax^2 + 2Bx + C$

Now, putting above result in $P(x) - 2P'(x) = 3x^3 - 27x^2 + 38x + 1$.

We have, $A = 3$, $B = -9$, $C = 2$, $D = 5$.

$\therefore \quad P(x) = (3x^3 - 9x^2 + 2x + 5)$

$P'(x) = 9x^2 - 18x + 2$

$P''(x) = 18x - 18$

So, $f(x) = \begin{cases} 3x, & x \neq \dfrac{\pi}{2} \\ \sin^{-1}(ab) + \cos^{-1}(a + b - 3ab), & x = \dfrac{\pi}{2} \end{cases}$

As, $f(x)$ is continuous at $x = \dfrac{\pi}{2}$, so

$$\frac{3\pi}{2} = (\sin^{-1}(ab) + \cos^{-1}(a + b - 3ab))$$

It exists when $\sin^{-1}(ab) = \pi/2$, $\cos^{-1}(a + b - 3ab) = \pi$

$\therefore \quad ab = 1$ and $a + b - 3ab = -1$

Hence, we get $a + b = 2$.

PROBLEM 7

Let, $f(x + y) = f(x) + f(y) + 2xy - 1$, $\forall\ x, y \in R$. If $f(x)$ is differentiable and $f'(0) = \sin\phi$, then prove that $f(x) > 0$ for $x \in R$

SOLUTION

Put $y = 0$ in given relation,

We get, $f(x) = f(x) + f(0) - 1 \Rightarrow f(0) = 1$

We have $f'(x) = \displaystyle\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$$\Rightarrow \quad f'(x) = \lim_{h\to 0} \frac{f(x) + f(h) + 2hx - 1 - f(x)}{h}$$

$$f'(x) = 2x + \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = 2x + f'(0)$$

$$\Rightarrow \quad f'(x) = \frac{dy}{dx} = 2x + \sin\phi$$

$\Rightarrow \quad y = x^2 + x\sin\phi + k$

$\Rightarrow \quad y = f(x) = x^2 + x\sin\phi + 1 \qquad \{\because \text{ for } x = 0, y = 1\}$

$$\Rightarrow \quad f(x)=\left(x+\frac{\sin\phi}{2}\right)^2+\frac{\cos^2\phi}{4}+\frac{3}{4}>0$$

Always for all real x. Hence, proved.

PROBLEM 8

If, $f(x+y)=f(x)\,.\,f(y)$ for all real x and y, $f(x)>0$ and $f(x)$ is continuous. If $f(1)=\frac{1}{2}$, then compute

$$\lim_{n\to\infty}\{f(x)+2f(x+1)+3f(x+2)+\ldots+(n+1)f(x+n)\}=4f(x).$$

SOLUTION

$f(x+y)=f(x)\,f(y)\Rightarrow f(x)=a^x$, for $a>0$, obviously a^x is +ve and continuous.

Now, if $f(1)=\frac{1}{2}\Rightarrow a=\frac{1}{2}$

$$\therefore \quad f(x)=2^{-x}$$

$$f(x+1)=2^{-(x+1)}=\frac{1}{2}2^{-x}$$

$$f(x+2)=2^{-(x+2)}=\frac{1}{4}2^{-x}$$

$$f(x+3)=2^{-(x+3)}=\frac{1}{8}2^{-x}$$

……………………………………..

$$f(x+n)=2^{-(x+n)}=\frac{1}{2^n}2^{-x}$$

$$g(x)=\lim_{n\to\infty}[f(x)+2f(x+1)+3f(x+2)+4f(x+3)+\ldots+(n+1)f(x+n)]$$

$$g(x)=\left(1+2.\frac{1}{2}+3.\frac{1}{4}+4.\frac{1}{8}+\ldots\right)2^{-x} \qquad \text{[It is an AGP]}$$

$$\frac{g(x)}{2}=\left(0+\frac{1}{2}+2.\frac{1}{4}+3.\frac{1}{8}+4.\frac{1}{16}+\ldots\right)2^{-x}$$

By taking difference

We get $\frac{g(x)}{2}=\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\right)2^{-x}$

$$\frac{g(x)}{2}=\left(\frac{1}{1-1/2}\right)2^{-x}=2.2^{-x}$$

Required term $=g(x)=4\;2^{-x}=4\,f(x)$ (Hence proved)

PROBLEM 9

Examine differentiability of the function $f(x)=\sin^{-1}(\cos x)$ at $x=n\pi+\frac{\pi}{2}$ where $n\in I$.

SOLUTION

Given function is always continuous function

$$\text{LHD}=f'\left(n\pi+\frac{\pi}{2}\right)^-$$

$$=\lim_{h\to0}\left(\frac{f\left(n\pi+\frac{\pi}{2}-h\right)-f\left(n\pi+\frac{\pi}{2}\right)}{-h}\right)$$

$$=\lim_{h\to0}\left[\frac{\sin^{-1}\left\{\cos\left(n\pi+\frac{\pi}{2}-h\right)\right\}-\sin^{-1}\left\{\cos\left(n\pi+\frac{\pi}{2}\right)\right\}}{-h}\right]$$

$$=\lim_{h\to0}\left[\frac{\sin^{-1}\left\{(-1)^n\cos\left(\frac{\pi}{2}-h\right)\right\}-\sin^{-1}\left\{(-1)^n\cos\frac{\pi}{2}\right\}}{-h}\right]$$

$$=\lim_{h\to0}\left[\frac{\sin^{-1}\left\{\sin(-1)^n h\right\}-\sin^{-1}0}{-h}\right]$$

$$=\lim_{h\to0}\frac{(-1)^n h}{-h}=(-1)^{n-1}$$

$$\text{RHD}=f'\left(n\pi+\frac{\pi}{2}\right)^+=\lim_{h\to0}\left\{\frac{f\left(n\pi+\frac{\pi}{2}+h\right)-f\left(n\pi+\frac{\pi}{2}\right)}{h}\right\}$$

$$=\lim_{h\to0}\left\{\frac{\sin^{-1}\left[\left\{\cos\left(n\pi+\frac{\pi}{2}+h\right)\right\}-\sin^{-1}\left\{\cos\left(n\pi+\frac{\pi}{2}\right)\right\}\right]}{h}\right\}$$

$$=\lim_{h\to0}\left\{\frac{\sin^{-1}\left\{(-1)^n\cos\left(\frac{\pi}{2}+h\right)\right\}-\sin^{-1}\left\{(-1)^n\cos\frac{\pi}{2}\right\}}{h}\right\}$$

$$=\lim_{h\to0}\left[\frac{\sin^{-1}\left\{(-1)^{n+1}\sin h\right\}}{h}\right]=\lim_{h\to0}\left[\frac{\sin^{-1}\left\{\sin(-1)^{n+1}h\right\}}{h}\right]$$

$$=\lim_{h\to0}\frac{(-1)^{n+1}h}{h}=(-1)^{n+1}=(-1)^{n-1}(-1)^2=(-1)^{n-1}$$

Thus, we get LHD = RHD

Hence, $f(x)$ is differentiable at $x=n\pi+\frac{\pi}{2}$

PROBLEM 10

Find possible triplet $\{a,b,c\}$ for which the function

$$f(x)=\begin{cases}\dfrac{\sin\left(x-\frac{1}{2}\right)}{2x+a}, & x<\dfrac{1}{2}\\ b, & x=\dfrac{1}{2}\\ \dfrac{2x^2-3x+1}{c-2x}, & x>\dfrac{1}{2}\end{cases} \text{ is continuous at } x=\frac{1}{2}.$$

SOLUTION

Given, $\lim\limits_{x\to\frac{1^-}{2}} f(x) = \lim\limits_{x\to\frac{1^-}{2}} \dfrac{\sin\left(x-\frac{1}{2}\right)}{2x+a} = \dfrac{0}{1+a}$

CASE 1: When $a \neq -1$, then $\lim\limits_{x\to\frac{1^+}{2}} f(x) = 0$ ...(1)

CASE 2: When $a = -1$, then

$$\lim_{x\to\frac{1^-}{2}} f(x) = \lim_{x\to\frac{1^-}{2}} \frac{\sin\left(x-\frac{1}{2}\right)}{2x-1}, \qquad \left(\frac{0}{0}\text{ Form}\right)$$

$$= \lim_{x\to\frac{1^-}{2}} \frac{\sin\left(x-\frac{1}{2}\right)}{2\left(x-\frac{1}{2}\right)} = \frac{1}{2} \qquad \text{...(2)}$$

Also, $\lim\limits_{x\to\frac{1^+}{2}} f(x) = \lim\limits_{x\to\frac{1^+}{2}} \dfrac{2x^2-3x+1}{c-2x}$

$$= \lim_{x\to\frac{1^+}{2}} \frac{(2x-1)(x-1)}{c-2x} = \frac{0}{c-1}$$

CASE 1: When $c \neq 1$, then $b = \lim\limits_{x\to\frac{1^+}{2}} f(x) = 0$ ...(3)

CASE 2: When $c = 1$, then

$$b = \lim_{x\to\frac{1^+}{2}} f(x) = \lim_{x\to\frac{1^+}{2}} \frac{(2x-1)(x-1)}{(1-2x)} = \frac{1}{2} \qquad \text{...(4)}$$

Now, for the function $f(x)$ to be continuous at $x = \frac{1}{2}$
From (1) and (3), we have

$$a \neq -1, b = 0, c \neq 1$$

And from (2) and (4), we have

$a = -1$, $b = \frac{1}{2}$, $c = 1$ are the possible values, so there are 2 possible different values of triplet $\{a, b, c\}$.

PROBLEM 11

Let $f(x) = x^2 + ax + 3$ and $g(x) = x + b$, where $F(x) = \lim\limits_{n\to\infty} \dfrac{f(x)+x^{2n}g(x)}{1+x^{2n}}$, $n \in N$. If $F(x)$ is continuous at $x = 1$ and $x = -1$, then find the value of $(a^2 + b^2)$.

SOLUTION

$$F(x) = \begin{cases} f(x), & -1 < x < 1 \\ g(x), & x \in (-\infty, -1) \cup (1, \infty) \\ \dfrac{f(1)+g(1)}{2}, & x = 1 \\ \dfrac{f(-1)+g(-1)}{2}, & x = -1 \end{cases}$$

Continuous at $x = 1$, $F(1^-) = F(1^+) = F(1)$

$$\Rightarrow \quad f(1^-) = g(1^+) = \frac{f(1)+g(1)}{2}$$

$$\Rightarrow \quad f(1) = g(1) \Rightarrow 4 + a = 1 + b$$

$$\therefore \quad b - a = 3 \qquad \text{...(1)}$$

Continuous at $x = -1$, $F(-1^-) = F(-1^+) = F(-1)$

$$\Rightarrow \quad g(-1^-) = f(-1^+) = \frac{f(-1)+g(-1)}{2}$$

$$\Rightarrow \quad g(-1) = f(-1) \Rightarrow b - 1 = 4 - a$$

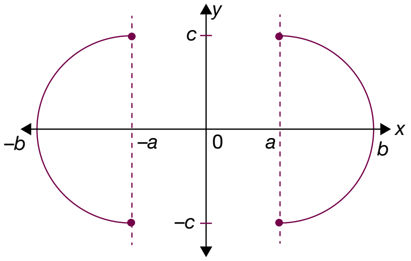
$$\therefore \quad a + b = 5 \qquad \text{...(2)}$$

(1) and (2)

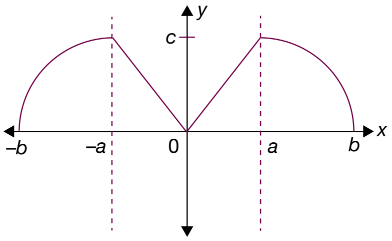
Hence, $a = 1$, $b = 4 \Rightarrow a^2 + b^2 = 17$

PROBLEM 12

If graphs of $|y| = f(x)$ and $y = |f(x)|$ are given as below $(a, b > 0)$.



|y| = f(x)

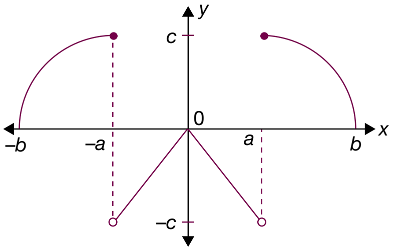


|y| = f(x)

Given that $f(0) = 0$. Find points of discontinuity and non-differentiability of $f(x)$

SOLUTION

From the Ist graph if $x \in (-a, a)$ then no value of y and hence, $f(x)$ must be negative in $(-a, a)$.
Also, from the above two graph if $x \in [-b, -a] \cup [a, b]$, then $f(x)$ is positive and hence, from the given graphs, graph of $f(x)$ is as follows.



Hence, $f(x)$ is discontinuous at 2 points $x = \pm a$ and non-differentiable at 3 points $x = 0, +a, -a$.

PROBLEM 13

If $f(x) = x|x^2 - 3x|\, e^{|x-3|} + \left[\dfrac{3+\operatorname{sgn}(x^2-1-5x)}{4+x^2}\right] + \left\{\dfrac{1}{3}+\left[\dfrac{x^2-2x}{|x|+1}\right]\right\}$ is non-derivable at $x = x_1, x_2, x_3, \ldots, x_n$, then find $\sum\limits_{i=1}^{n} x_i$ where [.] and {.} are greatest integer function and fractional part function respectively.

SOLUTION

($\because$ $3+\operatorname{sgn}(x^2-1-5x) \le 4$ and $x^2+4 \ge 4$, then

$$0 < \frac{3+\operatorname{sgn}(x^2-1-5x)}{x^2+4} < 1 \Rightarrow \left[\frac{3+\operatorname{sgn}(x^2-1-5x)}{x^2+4}\right] = 0$$

$\{I+f\} = f$ where I is an integer

Then, $\left\{\dfrac{1}{3}+\left[\dfrac{x^2-2x}{|x|+1}\right]\right\} = \dfrac{1}{3}$

Now, function $f(x) = x\,|x(x-3)|\, e^{|x-3|} + 0 + \dfrac{1}{3}$ is always continuous.

Now, check points at $x = 0, 3$.

$\therefore$ Derivable at $x = 0$ but non-derivable at $x = 3$

Hence, Sum $= \sum\limits_{i=1}^{n} x_i = 3$

PROBLEM 14

A function $f : R \to R$ where R is a set of real numbers satisfies the equation $f\left(\dfrac{x+y}{3}\right) = \dfrac{f(x)+f(y)+f(0)}{3}$ for all $x, y \in R$. If the function is differentiable at $x = 0$, then show that it is differentiable for all x in R.

SOLUTION

$$f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)+f(0)}{3}$$

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \text{exist}.$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f\left(\dfrac{3x+3h}{3}\right)-f\left(\dfrac{3x+0}{3}\right)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{1}{h}\left[\frac{f(3x)+f(3h)+f(0)}{3} - \frac{f(3x)+f(0)+f(0)}{3}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{f(3h)-f(0)}{3}\right]$$

$$= \lim_{h\to 0}\frac{f(3h)-f(0)}{3h} = f'(0)$$

Hence, $f(x)$ is differentiable for all x in R.

PROBLEM 15

If $f\left(\dfrac{x^2+y^2}{2}\right) = \dfrac{\{f(x)\}^2+\{f(y)\}^2}{2}$ for all $x, y \in R$, find $f(x)$.

SOLUTION

Differentiating both sides partially with respect to x, (keeping y as constant), we get

$$f'\left(\frac{x^2+y^2}{2}\right) \times \frac{1}{2}(2x) = \frac{1}{2}\left[2f(x)f'(x)+0\right]$$

$$\Rightarrow \quad xf'\left(\frac{x^2+y^2}{2}\right) = f(x)f'(x) \qquad \ldots(1)$$

Similarly, differentiating both sides partially with respect to y (keeping x as constant) or interchanging x and y, we get

$$yf'\left(\frac{x^2+y^2}{2}\right) = f(y)f'(y) \qquad \ldots(2)$$

Now, dividing (1) by (2), we have $\dfrac{x}{y} = \dfrac{f(x)f'(x)}{f(y)f'(y)}$

$$\Rightarrow \quad \frac{f(x)f'(x)}{x} = \frac{f(y)f'(y)}{y} = k\left(= f(1)f'(1)\right) \text{ say}$$

$$\Rightarrow \quad f(x)f'(x) = kx \Rightarrow \int f(x)f'(x)\,dx = k\int x\,dx$$

$$\Rightarrow \quad \frac{\{f(x)\}^2}{2} = k\frac{x^2}{2} + c \Rightarrow \{f(x)\}^2 = kx^2 + 2c.$$

When $x = 0, y = 0, f(0) = 0$ or 1.

$\Rightarrow \quad \{f(0)\}^2 = 0 + 2c \Rightarrow$ Either $0 = 2c$ or $1 = 2c$.

$\therefore \quad c = 0$ or $c = \dfrac{1}{2}$.

Hence, $f(x) = \pm kx^2$ or $\pm\sqrt{kx^2+1}$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. A function $f(x)$ is defined by,

$$f(x)=\begin{cases}\dfrac{[x^2]-1}{x^2-1}, \text{ for } x^2\neq 1\\ 0 \qquad\quad , \text{ for } x^2=1\end{cases} \qquad \text{Where [.] denotes GIF}$$

(A) Continuous at $x = -1$
(B) Discontinuous at $x = 1$
(C) Differentiable at $x = 1$
(D) None of these

2. If $f(x) = \begin{cases}\dfrac{\cos x(a\sin x-b)+b}{x^2} & x\neq 0\\ 2, & x=0\end{cases}$, the values of '$a$' and '$b$' so that $f(x)$ is continuous at $x = 0$ is

(A) $a = 0, b = 4$ (B) $a = 0, b = -4$
(C) $a = -1, b = 2$ (D) $a = 2, b = 2$

3. If $f(x) = \begin{cases}1, & x<0\\ 0, & x=0\\ -1, & x>0\end{cases}$ and $g(x) = \cos\left(x-\dfrac{\pi}{4}\right) - \sin\left(x-\dfrac{\pi}{4}\right)$, then points, where $f(g(x))$ is discontinuous in $0 \le x \le 2\pi$ are

(A) $\dfrac{\pi}{2}, \dfrac{3\pi}{2}$ (B) $\dfrac{3\pi}{4}, \dfrac{7\pi}{4}$
(C) $\dfrac{\pi}{4}, \dfrac{5\pi}{4}$ (D) $0, \pi, 2\pi$

4. Let $f(x) = \begin{cases}\dfrac{1}{x}-\dfrac{2}{e^{2x}-1} & ; x\neq 0\\ \lambda & ; x=0\end{cases}$. The value of λ for which $f(x)$ is continuous at $x = 0$

(A) 0 (B) 1
(C) -1 (D) No value

5. Consider $f(x) = \begin{cases}0 & x<-1\\ [x^3]-[x]^3 & -1\le x<2\\ 1 & 2\le x\end{cases}$ where [.] denotes greatest integer function.

(A) $f(x)$ is discontinuous at 7 points and not differentiable at 8 points.
(B) $f(x)$ is not differentiable at 7 points and continuous everywhere.
(C) $f(x)$ is discontinuous at 7 points and not differentiable at 7 points only.
(D) $f(x)$ is discontinuous at 8 points and not differentiable at 8 points only.

6. If $f(x)=\operatorname{sgn}(x)$ and $g(x)=x\left(1-x^2\right)$ then $(fog)(x)$ is discontinuous at

(A) exactly one point (B) exactly two points
(C) exactly three points (D) no point

7. Let $f(x) = \lim\limits_{n\to-\infty} x(x+1)\,(x^2-1)^2\,(\cos^2 \pi x)^n$, then $f'(x)$ is discontinuous at:

(A) All integers
(B) All integers except one integer
(C) All integers except two integers
(D) All integers except three integers

8. Number of points at which the function $f(x) = \max.\{a-x, a+x, b\}, -\infty<x<\infty, 0<a<b$ cannot be differentiable is,

(A) 2 (B) 3
(C) 1 (D) 0

9. The function $f(x)=\left(x^2-1\right)\left|x^2-3x+2\right|+\cos\left(|x|\right)$ is not differentiable at

(A) -1 (B) 0
(C) 1 (D) 2

10. Let $f(x) = x\,|x|$, $g(x) = \sin x$ and $h(x) = (gof)\,(x)$. then

(A) $h(x)$ is not differentiable at $x = 0$.
(B) $h(x)$ is differentiable at $x = 0$ but $h'(x)$ is discontinuous at $x = 0$.
(C) $h'(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$.
(D) $h'(x)$ is differentiable at $x = 0$.

11. If $f(x)$ is a function such that $f(0) = a$, $f'(0) = ab$, $f''(0) = ab^2$, $f'''(0)=ab^3$, and so on and $b > 0$, where dash denotes the derivatives, then $\lim\limits_{x\to-\infty} f(x)=$

(A) ∞ (B) $-\infty$
(C) 0 (D) none of these

12. If $f(x)=p|\sin x|+qe^{|x|}+r|x|^3$ and $f(x)$ is differentiable at $x = 0$, then

(A) $p=q=r=0$
(B) $p = 0, q = 0, r$ is any real number
(C) $q = 0, r = 0, p$ is any real number
(D) $r = 0, p = 0, q$ is any real number

13. Let $f(x)=\begin{cases}\dfrac{\tan 7x-\tan 4x-\tan 3x}{x^2} & x\neq 0\\ 0 & x=0\end{cases}$ then the value of $f'(0)$ is

(A) 14 (B) 0
(C) 84 (D) -84

14. If $y=f(x)$ represents a straight line passing through origin and not passing through any of the points with integral Co-ordinates in the co-ordinate plane. Then the number of such continuous functions on 'R' is _____ (it is known that straight line represents a function)

(A) 0 (B) finite
(C) infinite (D) at most one

15. If both $f(x)$ and $g(x)$ are differentiable functions at $x = x_0$, then the function defined as $h(x) = \text{maximum } \{f(x), g(x)\}$:

(A) Is always differentiable at $x = x_0$
(B) Is never differentiable at $x = x_0$
(C) Is differentiable at $x = x_0$ provided $f(x_0) \neq g(x_0)$
(D) Cannot be differentiable at $x = x_0$ if $f(x_0) \neq g(x_0)$

16. The no. of points of discontinuous of $g(x) = f(f(x))$, where $f(x)$ is defined as, $f(x) = \begin{cases} 1+x, & 0 \le x \le 2 \\ 3-x, & 2 < x \le 3 \end{cases}$

(A) 0 (B) 1
(C) 2 (D) >2

17. If $f(x)$ has isolated point of discontinuity at $x = a$, such that $|f(x)|$ is continuous at $x = a$, then

(A) $\lim_{x \to a} f(x)$ does not exist (B) $\lim_{x \to a} f(x) + f(a) = 0$
(C) $f(a) = 0$ (D) None of these

18. Number of points of discontinuity of $f(x) = \left\{\frac{x}{5}\right\} + \left[\frac{x}{2}\right]$ in $x \in [0, 100]$ is/are (where [.] denotes greatest integer function and {.} denotes fractional part function)

(A) 50 (B) 51
(C) 52 (D) 55

19. If $f(x) = \begin{cases} \alpha + \dfrac{\sin[x]}{x} & x > 0 \\ 2 & x = 0 \\ \beta + \left[\dfrac{\sin x - x}{x^3}\right] & x < 0 \end{cases}$; where [.] is G.I.F. If $f(x)$ is continuous at $x = 0$, then $\beta - \alpha$ equal to

(A) 1 (B) –1
(C) 2 (D) –2

20. Let $f(x) = \dfrac{\sqrt{x^2 + px + 1}}{x^2 - p}$. If $f(x)$ is discontinuous at exactly 2 values of x, then number of integers in the range of p is

(A) 1 (B) 2
(C) 3 (D) 4

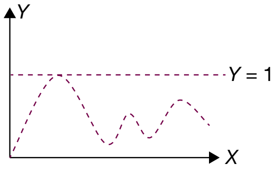
21. The number of point of discontinuity of $f(x) = [2x]^2 - \{2x\}^2$ (where [.] denotes the greatest integer function and { } is fractional part of x) in the interval (–2, 2) are:

(A) 6 (B) 8
(C) 7 (D) 3

22. The number of points of discontinuity of the function $f : (-2, 2) \to R, f(x) = \left[2^{-x^2}\left[2^{x^2}\right]\right]$ (where [.] represent (G.I.F) is

(A) 30 (B) 29
(C) 31 (D) None of these

23. Graph of $y = f(x)$ is shown in the figure; which of the following is true (where [ ] stands for greatest integer x)



(A) $y = [f(x)]$ is discontinuous for two values of x.
(B) $y = [-f(x)]$ is discontinuous for two values of x.
(C) $f(x) = [f(x)]$ has one solution.
(D) $f(x) = [-f(x)]$ has only one solution.

24. $f(x) = \left[\dfrac{x^2 + 1}{x^2 + [|x|] + 1}\right]$ is discontinuous at; (where [ ] stands for greatest integer function)

(A) One point
(B) Two points
(C) No point
(D) Infinite many points

25. $f(x) = \begin{cases} \sin\left(\dfrac{a-x}{2}\right)\tan\left[\dfrac{\pi x}{2a}\right] & \text{for } x > a \\ \dfrac{\left[\cos\left(\dfrac{\pi x}{2a}\right)\right]}{a-x} & \text{for } x < a \end{cases}$ where $[x]$ is the greatest integer function of x, and $a > 0$, then

(A) $f(a^-) < 0$
(B) f has a removable discontinuity at $x = a$
(C) f has an irremovable discontinuity at $x = a$
(D) $f(a^+) < 0$

26. If $f(x) = \dfrac{e^{2x} - (1 + 4x)^{1/2}}{\ln(1 - x^2)}$ for $x \neq 0$, then f has

(A) An irremovable discontinuity at $x = 0$
(B) A removable discontinuity at $x = 0$ and if $f(0) = -4$ then continuous
(C) A removable discontinuity at $x = 0$ and if $f(0) = -\frac{1}{4}$ then continuous
(D) A removable discontinuity at $x = 0$ and if $f(0) = 4$ then continuous

27. Let $f(x) = \dfrac{e^x \cos x - x \log_e(1 + x) - x}{x^2}$, $x \neq 0$. If $f(x)$ is continuous at $x = 0$, then $f(0)$ is equal to

(A) 0 (B) 1
(C) –1 (D) 2

28. Consider $f(x) = \begin{cases} x[x]^2 \log_{1+x} 2 & \text{for } -1 < x < 0 \\ \dfrac{\ln\left(e^{x^2} + 2\sqrt{\{x\}}\right)}{\tan\sqrt{x}} & \text{for } 0 < x < 1 \end{cases}$ where [*] and {*} are the greatest integer function and fractional part function respectively, then

(A) $f(0) = \ln 2 \Rightarrow f$ is continuous at $x = 0$
(B) $f(0) = 2 \Rightarrow$ f is continuous at $x = 0$
(C) $f(0) = e^2 \Rightarrow f$ is continuous at $x = 0$
(D) f has a irremovable discontinuity at $x = 0$

29. Let $f(x) = \dfrac{e^{\tan x} - e^x + \ln(\sec x + \tan x) - x}{\tan x - x}$ be a continuous function at $x = 0$. The value of $f(0)$ equals.

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$
(C) $\frac{3}{2}$ (D) 2

30. For the function $f(x) = \dfrac{1}{x + 2^{\frac{1}{x-2}}}$, $x \neq 2$, which of the following holds?

(A) $f(2) = \dfrac{1}{2}$ and f is continuous at $x = 2$

(B) $f(2) \neq 0$, $\dfrac{1}{2}$ and f is continuous at $x = 2$

(C) f can not be continuous at $x = 2$

(D) $f(2) = 0$ and f is continuous at $x = 2$

31. Given $f(x) = \begin{cases} x^2 e^{2(x-1)}, & 0 \leq x \leq 1 \\ a\cos(2x-2) + bx^2, & 1 < x \leq 2 \end{cases}$; $f(x)$ is differentiable at $x = 1$ provided

(A) $a = -1, b = 2$ (B) $a = 1, b = -2$

(C) $a = -3, b = 4$ (D) $a = 3, b = -4$

32. The domain of the derivative of the function

$$f(x) = \begin{cases} \tan^{-1} x, & \text{if } |x| \leq 1 \\ \dfrac{1}{2}(|x| - 1), & \text{if } |x| > 1 \end{cases} \text{ is}$$

(A) $R - \{0\}$ (B) $R - \{1\}$

(C) $R - \{-1\}$ (D) $R - \{-1, 1\}$

33. Given, $f(x) = \begin{cases} \dfrac{[\{|x|\}]e^{x^2}\{[x + \{x\}]\}}{\left(e^{1/x^2} - 1\right)\text{sgn}(\sin x)} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$, where $\{x\}$ is the fractional part function; $[x]$ is the step up function and sgn(x) is the signum function of x then, $f(x)$

(A) Is continuous at $x = 0$

(B) Is discontinuous at $x = 0$

(C) Has a removable discontinuity at $x = 0$

(D) Has an irremovable discontinuity at $x = 0$

34. Define $f : [0, \pi] \to R$ by $f(x) =$

$$\begin{cases} \tan^2 x\left[\sqrt{2\sin^2 x + 3\sin x + 4} - \sqrt{\sin^2 x + 6\sin x + 2}\right], & x \neq \pi/2 \\ k & , x = \pi/2 \end{cases}$$

is continuous at $x = \dfrac{\pi}{2}$, then k

(A) $-\dfrac{1}{4}$ (B) $\dfrac{1}{12}$

(C) $\dfrac{1}{24}$ (D) $\dfrac{1}{32}$

35. Let $|a_1 \sin x + a_2 \sin 2x + \ldots + a_8 \sin 8x| \leq |\sin x|$ for $x \in R$, define $P = a_1 + 2a_2 + 3a_3 + \ldots + 8a_8$. Then P satisfies

(A) $|P| \leq 1$ (B) $|P| < 1$

(C) $|P| > 1$ (D) $|P| \geq 1$

36. $f(x) = [x^2] - [x]^2$, then (where [.] denotes greatest integer function)

(A) f is not continuous $x = 0$ and $x = 1$

(B) f is continuous at $x = 0$ but not at $x = 1$

(C) f is not continuous at $x = 0$ but continuous at $x = 1$

(D) f is continuous at $x = 0$ and $x = 1$

37. Let $f(x) = \sec^{-1}([1 + \sin^2 x])$; where [.] denotes greatest integer function. Then the set of points where $f(x)$ is not continuous is

(A) $\left\{\dfrac{n\pi}{2}, n \in I\right\}$ (B) $\left\{(2n-1)\dfrac{\pi}{2}, n \in I\right\}$

(C) $\left\{(n-1)\dfrac{\pi}{2}, n \in I\right\}$ (D) $\{n\pi / n \in I\}$

38. The number of points in (1, 3), where $f(x) = a^{[x^2]}$, $a > 1$, is not differentiable where [x] denotes the integral part of x is

(A) 0 (B) 3

(C) 5 (D) 7

39. In $x \in \left[0, \dfrac{\pi}{2}\right]$, let $f(x) = \lim\limits_{n \to \infty} \dfrac{2^x - x^n \sin x}{1 + x^n}$, then

(A) $f(x)$ is a constant function

(B) $f(x)$ is continuous at $x = 1$

(C) $f(x)$ is discontinuous at $x = 1$

(D) None of these

40. Let $f(x) = [\cos x + \sin x]$, $0 < x < 2\pi$, where [x] denotes the integral part of x, then the number of points of discontinuity of $f(x)$ is

(A) 3 (B) 4

(C) 5 (D) 6

41. If [x] denotes the integral part of x and in $(0, \pi)$, we define

$$f(x) = \left[\frac{2(\sin x - \sin^n x) + |\sin x - \sin^n x|}{2(\sin x - \sin^n x) - |\sin x - \sin^n x|}\right]$$

for $x \in (0, \pi) - \left\{\dfrac{\pi}{2}\right\}$ for $n > 1$ and $f\left(\dfrac{\pi}{2}\right) = 3$. Then

(A) $f(x)$ is continuous but not differentiable at $x = \dfrac{\pi}{2}$

(B) Both continuous and differentiable at $x = \dfrac{\pi}{2}$

(C) Neither continuous nor differentiable at $x = \dfrac{\pi}{2}$

(D) $\lim\limits_{x \to \frac{\pi}{2}} f(x)$ exists but $\lim\limits_{x \to \frac{\pi}{2}} f(x) \neq f\left(\dfrac{\pi}{2}\right)$

42. If [x] denotes the integral part of x and $f(x) = [n + p \sin x]$, $0 < x < \pi$, $n \in I$ and p is a prime number, then the number of points where $f(x)$ is not differentiable is

(A) $p - 1$ (B) p

(C) $2p - 1$ (D) $2p + 1$

43. If $f(0) = 0$ and that 'f' is differentiable at $x = 0$, and 'k' is a positive integer.

Then $\lim\limits_{x \to 0} \dfrac{1}{x}\left[f(x) + f\left(\dfrac{x}{2}\right) + f\left(\dfrac{x}{3}\right) + \text{----} + f\left(\dfrac{x}{k}\right)\right]$

(A) $K.f'(0)$ (B) $\left(\displaystyle\sum_{r=1}^{K} \dfrac{1}{r}\right) f'(0)$

(C) $\displaystyle\sum_{r=1}^{K} \dfrac{1}{r}$ (D) Does not exist

44. Let $f(x)$ be a function which is differentiable any number of times $f\left(2x^2-1\right)=2x^3 f(x), \forall x \in R$ then $f^{(2010)}(0)=$ ____________ (Here $f^{(n)}(x)=n^{th}$ order derivative of ‘f’ at x)

(A) –1

(B) 1

(C) 0

(D) Data insufficient

45. Let $f(x)=\lim\limits_{n\to\infty}\sum\limits_{r=0}^{n-1}\dfrac{x}{(rx+1)\left((r+1)x+1\right)}$, then

(A) $f(x)$ is continuous but not differentiable at $x = 0$

(B) $f(x)$ is both continuous and differentiable at $x = 0$

(C) $f(x)$ is neither continuous not differentiable at $x = 0$

(D) $f(x)$ is a periodic function.

46. If $[x]$ denotes the integral part of x and

$$f(x)=[x]\left\{\frac{\sin\dfrac{\pi}{[x+1]}+\sin\pi[x+1]}{1+[x]}\right\}; \text{ then}$$

(A) $f(x)$ is continuous in R.

(B) $f(x)$ is continuous but not differentiable in R.

(C) $f''(x)$ exists for all x in R.

(D) $f(x)$ is discontinuous at all integral points in R.

47. A function $f: R \to R$ is such that $f(0)=4, f'(x)=1$ in $-1<x<1$ and $f'(x)=3$ in $1<x<3$. Also, f is continuous every where. Then $f(2)$ is

(A) 5

(B) 7

(C) 8

(D) Can not be determined

48. The value of $f(0)$, so that the function $f(x)=\dfrac{1-\cos(1-\cos x)}{x^4}$ is continuous everywhere, is

(A) 1/8

(B) 1/2

(C) 1/4

(D) 1/16

49. The function $f(x)$ = maximum $\left\{\sqrt{x(2-x)}, 2-x\right\}$ is non-differentiable at x equal to

(A) 1

(B) 0.2

(C) 0, 1

(D) 1, 2

50. $$f(x)=\begin{cases}\dfrac{e^{e/x}-e^{-e/x}}{e^{1/x}+e^{-1/x}} &, \quad x\neq 0\\ \quad k &, \quad x=0\end{cases}$$

(A) f is continuous at x, when $k = 0$.

(B) f is not continuous at $x = 0$ for any real k.

(C) $\lim\limits_{x\to 0} f(x)$ exist infinitely.

(D) None of these

51. If $f(x)$ = maximum $\left(\cos x, \dfrac{1}{2}, \{\sin x\}\right), 0 \le x \le 2\pi$, where { . } represents fractional part function, then number of points at which $f(x)$ is continuous but not differentiable, is

(A) 1

(B) 2

(C) 3

(D) 4

52. If $f(x)=\begin{cases}x^2\left\{e^{1/x}\right\}, & x\neq 0\\ k, & x=0\end{cases}$ is continuous at x = 0, then where {.} denotes fractional part function

(A) It is differentiable at $x = 0$.

(B) $k = 1$

(C) Continuous but not differentiable at $x = 0$.

(D) Continuous everywhere in its domain.

53. The function $f(x)=\min\{|x|-1, |x-2|-1, |x-1|-1\}$ is not differentiable at

(A) 2 points

(B) 5 points

(C) 4 points

(D) 3 points

54. Let $f(x)=\cos x$,

$$g(x)=\begin{cases}\text{minimum}\{f(t): 0\le t\le x\}, & x\in[0,\pi]\\ \sin x-1, & x>\pi\end{cases} \text{ then}$$

(A) $g(x)$ is discontinuous at $x=\pi$

(B) $g(x)$ is continuous for $x\in[0,\infty)$

(C) $g(x)$ is differentiable at $x=\pi$

(D) $g(x)$ is differentiable for $x\in[0,\infty)$

55. If $x \in R^+$ and $n \in N$, we can uniquely write $x = mn + r$ where $m \in W$ and $0 \le r < n$. We define x mod $n = r$. For example 10.3 mod 3 = 1.3. The number of points of discontinuity of the function $f(x) = (x \bmod 2)^2 + (x \bmod 4)$ in the interval $0 < x < 9$ is

(A) 0

(B) 2

(C) 4

(D) None of these

56. Let $f(x)$ be a continuous and differentiable function satisfying $f(x + y) = f(x) f(y)$ for all $x, y, \in R$.
If $f(x)$ can be expressed as $f(x) = 1 + x\,p(x) + x^2\,q(x)$ where $\lim\limits_{x\to 0} p(x) = a$ and $\lim\limits_{x\to 0} q(x) = b$, then $f'(x)$ is:

(A) $a\,f(x)$

(B) $b\,f(x)$

(C) $(a + b)\,f(x)$

(D) $(a + 2b)\,f(x)$

57. If $f(x)=\begin{cases}[x]+\sqrt{\{x\}} & x<1\\ \dfrac{1}{[x]+\{x\}^2} & x\ge 1\end{cases}$, then where [.] and { . } represent greatest integer and fractional part functions respectively

(A) $f(x)$ is continuous at $x = 1$ but not differentiable

(B) $f(x)$ is not continuous at $x = 1$

(C) $f(x)$ is differentiable at $x = 1$

(D) $\lim\limits_{x\to 1} f(x)$ does not exist.

58. If $f(x)$ is a differentiable function and $f'(2) = 6, f'(1) = 4, f'(x)$ represents the differentiation of $f(x)$ with respect to x, then

$$\lim_{h\to 0}\frac{f(2+2h+h^2)-f(2)}{f(1+h^2+h)-f(1)}=$$

(A) 3

(B) 4

(C) 6

(D) 14

59. Let $f(x)=\begin{cases}g(x)\,.\,\cos\dfrac{1}{x}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ where $g(x)$ is an even function differentiable at x = 0, passing through the origin then $f'(0)$

(A) is equal to 1

(B) is equal to 0

(C) is equal to 2

(D) does not exist

60. Let $f(x) = \dfrac{g(x)}{h(x)}$, where g and h are continuous functions on the open interval (a, b). Which of the following statements is true for $a < x < b$?

(A) f is continuous at all x for which x is not zero.
(B) f is continuous at all x for which $g(x) = 0$
(C) f is continuous at all x for which $g(x)$ is not equal to zero.
(D) f is continuous at all x for which $h(x)$ is not equal to zero.

61. If $f(x) = \dfrac{x - e^x + \cos 2x}{x^2}$, $x \neq 0$ is continuous at $x = 0$, then

(A) $f(0) = \dfrac{5}{2}$ (B) $[f(0)] = -2$
(C) $\{f(0)\} = -0.5$ (D) $[f(0)] \cdot \{f(0)\} = -1.5$

where $[x]$ and $\{x\}$ denotes greatest integer and fractional part function

62. Given a function $g(x)$ which has derivative $g'(x)$ for all x satisfying $g'(0) = 2$ and $g(x + y) = e^y \cdot g(x) + e^x \cdot g(y)$ for all x, $y \in R$. $g(5) = 32$ then the value of $g'(5) - 2e^5$ is equal to
(A) 5 (B) 25
(C) 32 (D) None of these

63. Let $f(x) = \begin{cases} \dfrac{2^x + 2^{3-x} - 6}{\sqrt{2^{-x}} - 2^{1-x}}, \text{ if } x > 2 \\ \dfrac{x^2 - 4}{x - \sqrt{3x - 2}}, \text{ if } x < 2 \end{cases}$ then

(A) $f(2) = 8 \Rightarrow f$ is continuous at $x = 2$
(B) $f(2) = 16 \Rightarrow f$ is continuous at $x = 2$
(C) $f(2^-) \neq f(2^+) \Rightarrow f$ is discontinuous
(D) f has a removable discontinuity at $x = 2$

64. If f is twice differentiable such that $f''(x) = -f(x)$, $f'(x) = g(x)$, $h'(x) = [f(x)]^2 + [g(x)]^2$ and $h(0) = 2$, $h(1) = 4$, then the equation $y = h(x)$ represents:
(A) A curve of degree 2
(B) A curve passing through the origin
(C) A straight line with slope 2
(D) A straight line with y intercept equal to –2.

65. Let f be a differentiable function on the open interval (a, b). Which of the following statements must be true?
I. f is continuous on the closed interval $[a, b]$
II. f is bounded on the open interval (a, b)
III. If $a < a_1 < b_1 < b$, and $f(a_1) < 0 < f(b_1)$, then there is a number c such that $a_1 < c < b_1$ and $f(c) = 0$
(A) I and II only (B) I and III only
(C) II and III only (D) only III

66. For the function $f(x) = \lim\limits_{n \to \infty} \dfrac{1}{1 + n\sin^2(\pi x)}$, which of the following holds?
(A) The range of f is a singleton set.
(B) f is continuous on R
(C) f is discontinuous for all $x \in I$
(D) f is discontinuous for some $x \in R$

67. Let $f(x) = \begin{cases} \lim\limits_{n \to \infty} \sum\limits_{n=1}^{n} \dfrac{1}{2^n} \tan\left(\dfrac{x}{2^n}\right), & x \in (0, \pi) - \left\{\dfrac{\pi}{2}\right\} \\ \dfrac{2}{\pi} & x = \dfrac{\pi}{2} \end{cases}$

Then which one of the following alternative is true?

(A) $f(x)$ has non-removable discontinuity of finite type at $x = \dfrac{\pi}{2}$
(B) $f(x)$ has missing point discontinuity at $x = \dfrac{\pi}{2}$
(C) $f(x)$ is continuous at $x = \dfrac{\pi}{2}$
(D) $f(x)$ has non-removable discontinuity of infinite type at $x = \dfrac{\pi}{2}$

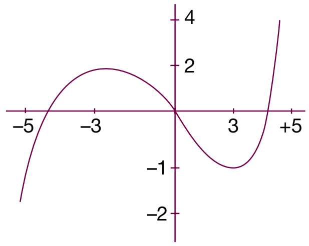
68. The graph of function f contains the point $P(1, 2)$ and $Q(s, r)$. The equation of the secant line through P and Q is $y = \left(\dfrac{s^2 + 2s - 3}{s - 1}\right) x - 1 - s$. The value of $f'(1)$, is
(A) 2 (B) 3
(C) 4 (D) Non existent

69. Let $f(x)$ be a real valued function such that $f(a) = 0$. If $g(x) = (x - a) f(x)$ is continuous but not differentiable at $x = a$ and $h(x) = (x - a)^2 f(x)$ is continuous and differentiable at $x = a$. Then $f(x)$
(A) Must be continuous and differentiable at $x = a$
(B) Must be continuous but not differentiable at $x = a$
(C) May or may not be continuous at $x = a$
(D) Must be discontinuous at $x = a$

70. Number of points of non-differentiability of the function $g(x) = [x^2]\{\cos^2 4x\} + \{x^2\}[\cos^2 4x] + x^2 \sin^2 4x + [x^2][\cos^2 4x] + \{x^2\}\{\cos^2 4x\}$ in $(-50, 50)$ where $[x]$ and $\{x\}$ denote the greatest integer function and fractional part function of x respectively, is equal to
(A) 98 (B) 99
(C) 100 (D) 0

71. Let $f(x) = \max.\{|x^2 - 2|x||, |x|\}$, then number of points where $f(x)$ is non derivable, is
(A) 2 (B) 3
(C) 4 (D) 5

72. Let $f : R \to R$ be a continuous and surjective function satisfying $f(x) + f(-x) = 0$, $\forall\ x \in R$. If $f(-3) = 2$ and $f(5) = 4$ in $[-5, 5]$, then the equation $f(x) = 0$ has



(A) Exactly three real roots (B) Exactly two real roots
(C) Atleast five real roots (D) Atleast three real roots

73. Let

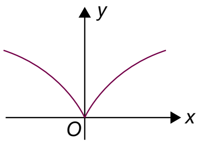
$$f(x) = \begin{cases} \lim\limits_{n \to \infty} \dfrac{ax(x-1)\left(\cot\dfrac{\pi x}{4}\right)^n + (px^2 + 2)}{\left(\cot\dfrac{\pi x}{4}\right)^n + 1}, & x \in (0,1) \cup (1,2) \\ 0 \quad , & x = 1 \end{cases}.$$

If $f(x)$ is differentiable for all $x \in (0, 2)$ then $(a^2 + p^2)$ equals

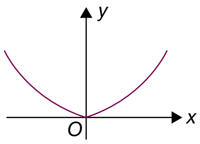
(A) 18 (B) 20
(C) 22 (D) 24

74. Let $y = f(x) = \begin{cases} e^{-1/x^2} \text{if } x \neq 0 \\ 0 \quad \text{if } x = 0 \end{cases}$ then which of the following can best represent the graph of $y = f(x)$?

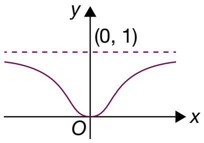
(A)



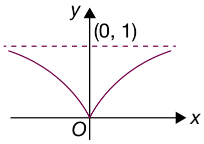
(B)



(C)



(D)



75. If $f(x) = \; f(x) = \left|\left|\sin\left(|x|-1\right)\right|-2\right|$

(A) $f(x)$ is continuous but not differentiable at $x = 2$
(B) $f(x)$ is discontinuous at $x = 2$
(C) $f(x)$ is differentiable at $x = 2$ and $f'(2) = \cos 1$
(D) $f(x)$ is differentiable at $x = 2$ and $f'(2) = -\cos 1$

76. Let $f''(x)$ be continuous at $x = 0$ and $f''(0) = 4$, the value of $\lim\limits_{x \to 0} \dfrac{2f(x) - 3f(2x) + f(4x)}{\sin(x^2)}$ is

(A) 6 (B) 10
(C) 11 (D) 12

77. If $f(x) = (1 + x^2)\operatorname{sgn}(x)$ (where sgn(x) is signum function), then $f^{-1}(x)$ is

(A) Discontinuous as it is inverse of a discontinuous function
(B) Discontinuous only at $x = 0$
(C) Continuous on its domain
(D) Discontinuous at some points of its domain

78. Let the functions defined as $g(x) = \tan^{-1}|x| - \cot^{-1}|x|$, $f(x) = \dfrac{[x]}{[x+1]}\{x\}$ and $h(x) = |g(f(x))|$, (where [.] denote the integral part, and {.} be the fractional part), which of the following holds good?

(A) h is continuous at $x = 0$ (B) h is discontinuous at $x = 0$
(C) $h'(0)$ does not exist (D) $h\left(0^+\right) = -\pi/2$

79. Let $f(x) = -1 + |x - 1|$, $g(x) = 2 - |x + 1|$, then $gof(x)$ is not continuous at x is equal to

(A) –1 (B) 0
(C) 1 (D) null set

80. If $f(x) = \begin{cases} \sin\left(\cos^{-1}x\right) + \cos\left(\sin^{-1}x\right), & x \le 0 \\ \sin\left(\cos^{-1}x\right) - \cos\left(\sin^{-1}x\right), & x > 0 \end{cases}$ then at $x = 0$

(A) $f(x)$ is continuous and differentiable.
(B) $f(x)$ is continuous but not differentiable.
(C) $f(x)$ not continuous but differentiable.
(D) $f(x)$ is neither continuous nor differentiable.

81. If $x = |t| + 2t$ and $y = t^2 - 2|t| + 1$, then the curve $y = f(x)$ is

(A) Continuous every where except at $x = 0$
(B) Differentiable every where except at $x = 0$
(C) Continuous every where except at $x = 0, -1, 3$
(D) Differentiable every where

82. If graph of the function $y = f(x)$ is continuous and passes through point (3, 1), then $\lim\limits_{x \to 3} \dfrac{\ln(3f(x) - 2)}{2(1 - f(x))}$ is equal

(A) $\dfrac{3}{2}$ (B) $\dfrac{1}{2}$
(C) $-\dfrac{3}{2}$ (D) $-\dfrac{1}{2}$

83. If $f(x) = \begin{cases} [x] + \sqrt{\{x\}} & x < 1 \\ \dfrac{1}{[x] + \{x\}^2} & x \ge 1 \end{cases}$, where [.] and { . } represent greatest integer and fractional part functions respectively

(A) $f(x)$ is continuous at $x = 1$ but not differentiable
(B) $f(x)$ is not continuous at $x = 1$
(C) $f(x)$ is differentiable at $x = 1$
(D) $\lim\limits_{x \to 1} f(x)$ does not exist

84. If $f(x) = \operatorname{sgn}(\sin^2 x - \sin x - 1)$ has exactly four points of discontinuity for $x \in (0, n\pi)$, $n \in N$, then

(A) Minimum value of n is 5.
(B) Maximum value of n is 6.
(C) There are exactly two possible values of n.
(D) There are exactly only one value of n.

85. If $f(x) = \left(\sum\limits_{r=0}^{n} a_r|x-1|^r\right)$, where a_r's are positive real constants, then $f(x)$ is

(A) Not continuous at $x = 1$
(B) Continuous everywhere but not differentiable at $x = 0$
(C) Continuous everywhere but not differentiable at $x = 1$
(D) Differentiable everywhere

86. If $f(x) = \begin{cases} \sin\left(\dfrac{\pi}{2}\left(x - [x]\right)\right), & x < 5 \\ 5(b-1), & x = 5 \\ ab^2 \dfrac{|x^2 - 11x + 24|}{x - 3}, & x > 5 \end{cases}$ is continuous at $x = 5$, $(a, b \in R)$ then ([.] denotes greatest integer function)

(A) $a = \dfrac{25}{108}, b = \dfrac{6}{5}$ (B) $a = \dfrac{6}{13}, b = \dfrac{17}{29}$
(C) $a = \dfrac{1}{2}, b = \dfrac{25}{36}$ (D) $a = \dfrac{23}{100}, b = \dfrac{6}{5}$

87. If $f(x) = \lim\limits_{n \to \infty} \dfrac{\left(x^2 + ax + 1\right) + x^{2n}\left(2x^2 + x + b\right)}{1 + x^{2n}}$ is a continuous function on R, then $(a, b) =$

(A) $(1, 0)$ (B) $(0, -1)$
(C) $(1, -1)$ (D) $\left(\dfrac{1}{2}, -\dfrac{1}{2}\right)$

88. Let $f(x)=\begin{cases} x^p\sin\left(\dfrac{1}{x}\right)+x|x|; & x\neq 0 \\ 0, & x=0 \end{cases}$ then

(A) For $P=1$, $f(x)$ is differentiable everywhere
(B) For $P=2$, $f(x)$ is not differentiable everywhere
(C) For $P=1$, $f(x)$ is not differentiable at $x=0$
(D) For any value of $Pf(x)$ is differentiable everywhere

89. Let $f(x)=x-[x]$ and $g(x)=\lim\limits_{n\to\infty}\dfrac{(f(x))^{2n}-1}{(f(x))^{2n}+1}$, then where $[.]$ denotes the greatest integer function

(A) g is continuous and differentiable on R
(B) g is not differentiable at $x=0, 1, 2$
(C) g is periodic with period 1
(D) g is not continuous at $x=0, 1$

90. If $f(x)=[\log_e x]+\sqrt{\{\log_e x\}}, x>1$, where [.] and {.} denote the greatest integer function and the fractional part function respectively, then

(A) $f(x)$ is continuous but non-differentiable at $x=e$
(B) $f(x)$ is differentiable at $x=e$
(C) $f(x)$ is discontinuous at $x=e$
(D) None of these

91. Suppose the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivate 7 at $x = 2$. The derivative of the function $f(x)-f(4x)$ at $x=1$ has the value equal to

(A) 19 (B) 9
(C) 17 (D) 14

92. If $f(x)=\begin{cases} \left|1-4x^2\right|, & 0\le x<1 \\ \left[x^2-2x\right], & 1\le x<2 \end{cases}$, where [.] denotes the greatest integer function, then

(A) $f(x)$ is differentiable at $x=\dfrac{1}{2}$
(B) $f(x)$ is differentiable at $x=1$
(C) $f(x)$ is differentiable in (0, 2) except at $x=\dfrac{1}{2},1$
(D) $f(x)$ is differentiable in (0, 2)

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. Let $f(x)=\begin{cases} \sin\left(\dfrac{\pi}{2}x\right) & ; \ -2\le x<0 \\ \{2x\} & ; \ 0\le x\le 2 \end{cases}$ where {.} is fpf, then

(A) Range of $f(x)$ is $[-1, 1]$
(B) $f(x)$ is discontinuous at 4 points in $[-2, 2]$
(C) $f(x)$ is discontinuous at 5 points in $[-2, 2]$
(D) $f(x)$ is continuous but not differentiable at a unique point

2. Consider the function $g(x)=\begin{cases} \dfrac{\max\left\{x,\dfrac{1}{x}\right\}}{\min\left\{x,\dfrac{1}{x}\right\}} & \text{when } x\neq 0 \\ 1 & ;\text{ when } x=0 \end{cases}$,

then

(A) $\lim\limits_{x\to 0^+} g(x)=0$
(B) $\lim\limits_{x\to 0^-} g(x)=0$
(C) $g(x)$ is continuous for all x except at $x=0$
(D) $g(x)$ is differentiable for all x except at $x=0, -1, 1$

3. If $f(x)=\dfrac{\sin 3\pi x+A\sin 5\pi x+B\sin \pi x}{(x-1)^5}$ $(x\neq 1)$ is continuous at $x=1$; then

(A) $A=-\dfrac{1}{5}$ (B) $B=2$
(C) $f(1)=\dfrac{16}{5}\pi^5$ (D) $f(1)=16\pi^5$

4. Which of the statement(*s*) is/are INCORRECT?

(A) If $f+g$ is continuous at $x=a$, then f and g are continuous at $x=a$.
(B) If $\lim\limits_{x\to a}(fg)$ exists, then $\lim\limits_{x\to a} f$ and $\lim\limits_{x\to a} g$ both exist.
(C) Discontinuity at $x=a \Rightarrow$ non existence of limit.
(D) All functions defined on a closed interval attain a maximum and a minimum value on that interval.

5. Which of the following statement(*s*) is(are) correct?

(A) Every bounded function in $[a, b]$ is also continuous.
(B) f $f(x)$ is continuous in $[a, b]$ and $f(a)\neq f(b)$ then there some $c\in(a, b)$ such that $2f(a)+f(b)=3f(c)$.
(C) Let $f: R\to R$ be a function defined as $f(x)=2x+3\cos x$ then $f(c)=10$ for some $c\in R$.
(D) Let $f(x)=5x^8+ax^5+bx^2-cx-10$ where $a, b, c, \in R$ and $f(2)=5$, $f(4)=-5$, then the equation $f(x)=0$ has atleast four real root.

6. Let $f(x)=\begin{cases} x^2+3x+a & x\le 0 \\ bx+2, & x>0 \end{cases}$, a and b constants, is differentiable function, then

(A) $b=3$
(B) $a=2$
(C) Range of $f(x)$ is $\left[-\dfrac{1}{4},\infty\right)$
(D) $f(x)$ is many-one function.

7. If $f(x)=\lim\limits_{n\to\infty}(2\sin x)^n$ for all $x\in\left[0,\dfrac{\pi}{6}\right]$. Then which of the following statements is true

(A) $f\left(\dfrac{\pi}{6}\right)$ not equal to 1
(B) $f(x)$ has irremovable discontinuity at $x=\dfrac{\pi}{6}$
(C) $f(x)$ has removable discontinuity at $x=\dfrac{\pi}{6}$
(D) $f(x)$ is continuous in $\left(0,\dfrac{\pi}{6}\right)$

8. Consider the function $f(x) = \sqrt{x^2} + [x]^2$ in $[-2, 2]$ which of the following is correct?
where $[y]$ denotes greatest integers less than or equal to y.
(A) f is many one but not an even function in $[-2, 2]$
(B) Range of f contains 3 integers
(C) f is periodic in $[-2, 2]$
(D) f is discontinuous at all integers in $(-2, 2)$

9. A function $f(x)$ defined as $f(x) = \lim\limits_{n\to\infty}\begin{cases}\cos^{2n}x; & x<0\\ \sqrt[n]{1+x^n}; & 0\le x\le 1\\ \dfrac{1}{1+x^n}; & x>1\end{cases}$.
Which of the following does not hold good?
(A) Continuous at $x = 0$ but discontinuous at $x = 1$
(B) Continuous at $x = 1$ but discontinuous at $x = 0$
(C) Continuous both $x = 1$ and $x = 0$
(D) Discontinuous both at $x = 1$ and $x = 0$

10. f is a continuous function in $[a, b]$; g is a continuous function in $[b, c]$. A function $h(x)$ is defined as $h(x) = \begin{cases} f(x); & x\in[a,b)\\ g(x); & x\in(b,c]\end{cases}$, if $f(b) = g(b)$, then
(A) $h(x)$ has a removable discontinuity at $x = b$
(B) $h(x)$ may or may not be continuous in $[a, c]$
(C) $h(b^-) = g(b^+)$ and $h(b^+) = f(b^-)$
(D) $h(b^+) = g(b^-)$ and $h(b^-) = f(b^+)$

11. Let $f(x) = |x-1| + |x-2|$ and
$g(x) = \begin{cases}\min\{f(t)\};\ 0\le t\le x,\ 0\le x\le 3\\ x-2,\ x>3\end{cases}$, then $g(x)$ is not differentiable at
(A) $x = 1$ (B) $x = 2$
(C) $x = 3$ (D) $x = 3/2$

12. If $f(x) = \begin{cases} a+b\sqrt{1-x^2}, & |x|\le 1\\ ce^{(1-x)^2}+d, & |x|>1\end{cases}$ is derivable on R (set of all real numbers) and $f\left(e^{-e^{100}}\right) = 4$, then
(A) $a = b = c = d = 4$ (B) $a + b + c + d = 8$
(C) $a = d = 4$ and $b = c = 0$ (D) $b = c = 0, a = 4$ and $d \in R$

13. Let $f(x)$ = minimum $\left(\cos x, \dfrac{1}{2}, \{\sin x\}\right)$, $0 < x < 2\pi$. Let $x = x_i$, $i = 1, 2, 3, \ldots n$ be the point where $f(x)$ is non differentiable. If $\sum\limits_{i=1}^{n} x_i = \dfrac{\lambda\pi}{6}$, then value of λ is greater than (where $\{k\}$ denotes the fractional part function of k)
(A) 11 (B) 18
(C) 19 (D) 23

14. Let 'f' be a differentiable function satisfying $f(x + y) = f(x) + f(y) + (e^x - 1)(e^y - 1)$ ∀ $x, y \in R$ and $f'(0) = 2$. Identify the correct statement(s)
(A) $\lim\limits_{x\to 0}\dfrac{f(f(x))}{f(x)-x} = 4$
(B) $\lim\limits_{x\to 0}(f(x)+\cos x)^{\frac{1}{e^x-1}} = e^2$
(C) Number of solutions of the equation $f(x) = 0$ is 2.
(D) Range of the function $y = f(x)$ is $(-\infty, \infty)$.

15. Let f and g be real valued functions defined on $(-1, 1)$ such that $g''(x)$ is continuous also $g(0)\ne 0$, $g'(0)=0$, $g''(0)\ne 0$ and $f(x) = g(x)\sin x$. The value of $\lim\limits_{x\to 0}(g(x)\cot x - g(x)\operatorname{cosec} x)$ is
(A) $g''(0)$ (B) $g'(0)$
(C) $f''(0)$ (D) 0

16. Given a real valued function f such that
$f(x)=\begin{cases}\dfrac{\tan^2 x}{\left(x^2-[x]^2\right)} & \text{for } x>0\\ 1 & x=0\\ \sqrt{\{x\}\cot\{x\}} & \text{for } x<0\end{cases}$, where $[x]$ is the integral part and $\{x\}$ is the fractional part of x, then
(A) $\lim\limits_{x\to 0} f(x) = 1$ (B) $\lim\limits_{x\to 0^-} f(x) = \sqrt{\cot 1}$
(C) $\cot^{-1}\left(\lim\limits_{x\to 0^-} f(x)\right)^2 = 1$ (D) f is continuous at $x = 0$

17. Select the correct statements.
(A) The function f defined by $f(x) = \begin{cases}2x^2+3; & x\le 1\\ 3x+2; & x>1\end{cases}$ is neither differentiable not continuous at $x = 1$.
(B) The function $f(x) = x^2|x|$ is twice differentiable at $x = 0$.
(C) If f is continuous at $x = 5$ and $f(5) = 2$ then, $\lim\limits_{x\to 2} f(4x^2-11)$ exists.
(D) If $\lim\limits_{x\to a}(f(x)+g(x)) = 2$ and $\lim\limits_{x\to a}(f(x)-g(x)) = 1$, then $\lim\limits_{x\to a} f(x).g(x)$ need not exist.

18. Let $f(x) = \min\{x^3, x^2\}$ and $g(x) = [x]^2 + \sqrt{\{x\}^2}$, where $[x]$ denotes the greatest integer and $\{x\}$ denotes the fractional part function. Then which of the following holds?
(A) f is continuous for all x
(B) g is discontinuous for all $x \in I$
(C) f is differentiable for all $x \in (1, \infty)$
(D) g is not differentiable for all $x \in I$

19. Which of the following statement(s) is/are correct?
(A) If f is differentiable at $x = 1$, then $\lim\limits_{x\to 1}\dfrac{f(x^3)-f(1)}{x-1} = 3f'(1)$
(B) If f is differentiable at $x = 0$, then $\lim\limits_{x\to 0}\dfrac{f(ax)-f(bx)}{cx} = \dfrac{a-b}{c}f'(0)$
(C) If f is differentiable at $x = x_0$, $\lim\limits_{x\to x_0}\dfrac{xf(x_0)-x_0 f(x)}{x-x_0} = f(x_0) - x_0 f'(x_0)$
(D) $f : R \to R$ is differentiable and $f(x) \le f(0)$ ∀ $x \in [-1, 1]$ then $f'(0) = 0$

20. Let $f(x) = \begin{cases}-4, & -4\le x<0\\ x^2-4, & 0\le x\le 4\end{cases}$ and $g(x) = f(|x|) + |f(x)|$, then which of the following statements is(are) correct?

(A) $f(x)$ is continuous but differentiable at $x = \frac{\pi}{2}$
(B) $f(x)$ is continuous but non-differentiable at $x = 0$
(C) $g(x)$ is continuous as well as differentiable at $x = 0$
(D) $g(x)$ is continuous but not differentiable at $x = 2$

21. Let $f(x) = \begin{cases} \left(\cos\frac{1}{x}\right)(\ln^2(1+x)), & x > 0 \\ 0, & x \le 0 \end{cases}$. Then which of the following is (are) correct?
(A) $f(x)$ is continuous at $x = 0$
(B) $f(x)$ is differentiable at $x = 0$
(C) $f'(x)$ is continuous at $x = 0$
(D) $f'(x)$ is non-derivable at $x = 0$

22. Given $f(x) = \begin{cases} 3 - \left[\cot^{-1}\left(\frac{2x^3 - 3}{x^2}\right)\right] & \text{for } x > 0 \\ \{x^2\} \cos e^{1/x} & \text{for } x < 0 \end{cases}$ where { } and [ ] denotes the fractional part and the integral part functions respectively, then which of the following statement does not hold good.
(A) $f(0^-) = 0$
(B) $f(0^+) = 3$
(C) $f(0) = 0 \Rightarrow$ continuity of f at $x = 0$
(D) Irremovable discontinuity of f at $x = 0$

23. Let $f(x) = \begin{cases} \frac{a(1 - x\sin x) + b\cos x + 5}{x^2} & x < 0 \\ 3 & x = 0 \\ \left(1 + \left(\frac{cx + dx^3}{x^2}\right)\right)^{\frac{1}{x}} & x > 0 \end{cases}$. If f is continuous at $x = 0$, then
(A) $a = -1$
(B) $b = -4$
(C) $c = 0$
(D) $d = \log_e 5$

24. $f(x) = \text{Min}\{1, x^2, x^3\}$ then
(A) $f(x)$ is continuous
(B) $f(x) > 1 \;\; \forall x > 1$
(C) $f(x)$ is not differentiable but continuous at $x = 1$
(D) $f(x)$ is not differentiable for two value of x

25. If $f(x)$ be such that $f(x) = \max\left(|3 - x|, 3 - x^3\right)$, then
(A) $f(x)$ is continuous $\forall x \in R$.
(B) $f(x)$ is differentiable $\forall x \in R$.
(C) $f(x)$ is non-differentiable at three points only.
(D) $f(x)$ is non-differentiable at four points only.

26. $f(x) = \left[x^{-2}\left[x^2\right]\right]$, (where [.] denotes the greatest integer function) $x \ne 0$, then
(A) $f(x)$ is continuous everywhere.
(B) $f(x)$ is discontinuous at $x = \sqrt{2}$.
(C) $f(x)$ is non-differentiable at $x = 1$.
(D) $f(x)$ is discontinuous at infinitely many points.

27. Let $f(x) = \begin{cases} \int_0^x \{5 + |1 - y|\}dy & \text{if } x > 2 \\ 5x + 1 & \text{if } x \le 2 \end{cases}$ then
(A) $f(x)$ is continuous at $x = 2$
(B) $f(x)$ is not continuous at $x = 2$
(C) $f(x)$ is not differentiable at $x = 2$
(D) the right derivative of $f(x)$ at $x = 2$ is 1

28. If $f(x) = \lim_{n\to\infty} \frac{2x^{2n} + x + 2}{x^{2n} - x^2 + 3}$, then
(A) $f(x)$ is continuous at $x = 1$
(B) $f(x)$ is discontinuous at $x = 1$
(C) $\lim_{x\to 2} f(x)$ exist finitely
(D) $\lim_{x\to -2} f(x)$ exists finitely

29. For the function $f : (0, 1) \to R$, $f(x) = [2^n x] + \{2^m x\}$, $(n, m \in N)$, the number of points of discontinuity of the function can be (where [.] { } represent G.I.F. and fractional part of x respectively)
(A) 24
(B) 28
(C) 26
(D) 496

30. On the interval $I = [-2, 2]$, the function $f(x) =$

$$\begin{cases} (x+1)e^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(A) Is continuous for all values of $x \in I$
(B) Is continuous for $x \in I - \{0\}$
(C) Assume all intermediate values from $f(-2)$ and $f(2)$
(D) Has a maximum value equal to $\frac{3}{e}$

31. A function $f : R \to R$ satisfies the equation $f(x+y) = f(x).f(y)$ for all x, y in R and $f(x) \ne 0$ for any $x \in R$. Let the function be differentiable at $x = 0$ and $f'(0) = 2$ then.
(A) $f'(x) = 2f(x) \forall x \in R$
(B) $f(x) = e^{2x}$
(C) $f(x)$ is every where continuous
(D) $f\left(\frac{1}{2}\right)$ is an irrational number

32. Let $f(x) = \begin{cases} \frac{e^x - 1 + ax}{x^2} & x > 0 \\ b, & x = 0 \\ \frac{\sin(x/2)}{x}, & x < 0 \end{cases}$ then
(A) $f(x)$ is continuous at $x = 0$ if $a = -1$, $b = \frac{1}{2}$
(B) $f(x)$ is discontinuous at $x = 0$, if $b \ne \frac{1}{2}$
(C) $f(x)$ has non-removable discontinuity at $x = 0$ if $a \ne -1$
(D) $f(x)$ has removable discontinuity at $x = 0$ if $a = -1$, $b \ne \frac{1}{2}$

33. Let $\phi\left(\frac{x + 2y}{3}\right) = \frac{\phi(x) + 2\phi(y)}{3} \forall x, y \in R$ and $\phi'(0) = 1$ and $\phi(0) = 2$ then

(A) $\phi(x)$ is continuous $\forall x \in R$
(B) $\phi(x)$ is differentiable $\forall x \in R$
(C) $\phi(x)$ is both continuous and differentiable
(D) $\phi(x)$ is discontinuous at $x = 0$

34. If $f(x) = \text{maximum } \{4, 1 + x^2, x^2 - 1\} \forall x \in R.$
Then the total number of points where $f(x)$ is not-differentiable at
(A) $\sqrt{3}$ (B) $-\sqrt{3}$
(C) Two irrational points (D) None

35. Suppose that $f : R \to R$ is continuous and satisfying the equation $f(x).f(f(x)) = 1$, for all real x.
Let $f(1000) = 999$, then which of the following is true?
(A) $f(500) = \dfrac{1}{500}$ (B) $f(199) = \dfrac{1}{199}$
(C) $f(x) = \dfrac{1}{x} \forall\, x \in R - \{0\}$ (D) $f(1999) = \dfrac{1}{1999}$

36. The function $f(x) = \begin{cases} \dfrac{x^2}{a} & ,0 \le x < 1 \\ a & ,1 \le x < \sqrt{2} \\ \dfrac{2b^2 - 4b}{x^2} & ,\sqrt{2} \le x < \infty \end{cases}$ is continuous for $0 \le x < \infty$. Then which of the following statements is correct?
(A) The number of all possible ordered pairs (a, b) is 3.
(B) The number of all possible order pairs (a, b) is 4.
(C) The product of all possible values of b is –1.
(D) The product of all possible values of b is 1.

37. $f(x) = \begin{cases} x^2 \left| \cos \dfrac{\pi}{x} \right|, & \text{when } x \ne 0 \\ 0, & \text{when } x = 0 \end{cases}$
(A) $f(x)$ is not differentiable at $x = 0$
(B) $f(x)$ is differentiable at $x = 0$
(C) $f(x)$ is not differentiable at $x = \dfrac{2}{2n+1} : n \in Z$
(D) $f(x)$ is differentiable at $x \ne \dfrac{2}{2n+1} : n \in Z$

38. A function $f(x)$ satisfies the relation $f(x + y) = f(x) + f(y) + xy(x + y)\ \forall\ x, y \in R$. If $f'(0) = -1$, then
(A) $f(x)$ is a polynomial function
(B) $f(x)$ is an exponential function
(C) $f(x)$ is twice differentiable for all $x \in R$
(D) $f'(3) = 8$

39. $f(x) = \dfrac{[x]+1}{\{x\}+1}$ for $f : \left[0, \dfrac{5}{2}\right) \to \left(\dfrac{1}{2}, 3\right]$, where [.] represents greatest integer function and { . } represents fractional part of x, then which of the following is true.
(A) $f(x)$ is injective and discontinuous function.
(B) $f(x)$ is surjective and non differentiable function.
(C) $\min\left(\min\limits_{x \to 1-} f(x), \lim\limits_{x \to 1+} f(x)\right) = f(1)$
(D) max (x values of point of discontinuity) = $f(1)$

40. Let $f(x) = [x]$ and $g(x) = \begin{cases} 0 & x \in I \\ x^2 & \text{otherwise} \end{cases}$. Then (where [.] represent greatest integer function)
(A) gof is differentiable function.
(B) fog is non-differentiable at infinite points.
(C) gof is not continuous at each integer.
(D) fog (1) = 0.

41. $f(x) = \lim\limits_{n \to \infty} \dfrac{(a + \sin \pi x)^n - 1}{(a + \sin \pi x)^n + 1}$, $x \in (0, 6)$, then
(A) If $a = 1$ then $f(x)$ has 5 points of discontinuity
(B) If $a = 3$ then $f(x)$ has no points of discontinuity
(C) If $a = \dfrac{1}{2}$ then $f(x)$ has 6 points of discontinuity
(D) If $a = 0$ then $f(x)$ has 3 points of discontinuity

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Consider two function $y = f(x)$ and $y = g(x)$ defined

as $f(x) = \begin{cases} ax^2 + b & , \quad 0 \le x \le 1 \\ 2bx + 2b & , \quad 1 < x \le 3 \\ (a-1)x + 2a - 3 & , \quad 3 < x \le 4 \end{cases}$ and

$$g(x) = \begin{cases} cx^2 + d & , \quad 0 \le x \le 2 \\ dx + 3 - c & , \quad 2 < x < 3 \\ x^2 + b + 1 & , \quad 3 \le x \le 4 \end{cases}$$

1. $f(x)$ is continuous at $x = 1$ but not differentiable at $x = 1$, if
(A) $a = 1, b = 0$ (B) $a = 1, b = 2$
(C) $a = 3, b = 1$ (D) a and b are integers

2. $g(x)$ is continuous at $x = 2$, if
(A) $c = 1, d = 2$ (B) $c = 2, d = 3$
(C) $c = 1, d = -1$ (D) $c = 1, d = 4$

3. If f is continuous and differentiable at $x = 3$, then
(A) $a = -\dfrac{1}{3}, b = \dfrac{2}{3}$ (B) $a = \dfrac{2}{3}, b = -\dfrac{1}{3}$
(C) $a = \dfrac{1}{3}, b = -\dfrac{2}{3}$ (D) $a = 2, b = \dfrac{1}{2}$

Comprehension 2

f be a polynomial function such that $f(x)\,f(y) + 2 = f(x) + f(y) + f(xy)$ for all $x, y \in [0, \infty)$ and $f(x)$ is one-one function on $[0, \infty)$ with $f(0) = 1$ and $f'(1) = 2$ then

4. $f(x)$ is given by

(A) $1+\dfrac{2x^3}{3}$ (B) x^2+1

(C) $1+\dfrac{3}{2}x^2$ (D) $1+\dfrac{3x^4}{4}$

5. The number of discontinuities of $g(x)=\left[\dfrac{1}{f(x)}\right]$ on $[0,10]$ is (where $[x]$ is greatest integer $\le x$)

(A) 1 (B) 0

(C) 10 (D) 11

6. If $h(x)=\min\left\{\dfrac{2}{f(x)}, x^2, |1-|x||\right\}$, then the number of points of non-differentiability of $h(x)$ is

(A) 3 (B) 4

(C) 5 (D) 6

Comprehension 3

Let $f : R \to R$ be a differential function satisfying $f\left(\dfrac{x+y}{3}\right)=\dfrac{2+f(x)+f(y)}{3}$ for all real x and y and $f'(2)=2$.

7. The range of $f(|x|)$ contains

(A) $[4,\infty)$ (B) $[1,\infty)$

(C) $[2,\infty)$ (D) None of these

8. If $g(x) = |f(|x|)-3|$ for all $x \in R$, then for $g(x)$

(A) One non-differentiable point

(B) Two non-differentiable points

(C) Three non-differentiable points

(D) Four non-differentiable points

9. The number of solutions of the equation $x^2+(f(|x|))^2=9$ are

(A) 0 (B) 2

(C) 3 (D) 5

Comprehension 4

Let $f(x) = |\,||x|-1|-1|$ and $g(x) = \{\max f(t) : x \le t \le x+1, x \in R\}$

10. The graph of $g(x)$ is symmetrical about

(A) $x=\dfrac{1}{2}$ (B) $x=\dfrac{-1}{2}$

(C) $x=0$ (D) $x=-1$

11. The number of points of non-differentiability of $g(x)$ is

(A) 7 (B) 8

(C) 9 (D) 5

Comprehension 5

Consider a function, $f(x) = \dfrac{2}{3}\ln^{3/2}(\sin x+\cos x)$ for $x \in \left(0,\dfrac{\pi}{2}\right)$

12. The value of $\displaystyle\lim_{x\to\frac{\pi}{2}}\frac{f(x)}{\left(\frac{\pi}{2}-x\right)^{3/2}}$ is

(A) $\dfrac{2}{3}$ (B) 1

(C) $\dfrac{2}{3}(2)^{3/4}$ (D) Non existent

13. The function $g(x)$ is defined as

$$g(x)=\begin{cases}\left(\dfrac{3}{2}f(x)\right)^{2/3}; & x\in\left(0,\dfrac{\pi}{2}\right)\\[2ex] \dfrac{e^{\left(\frac{3}{2}f\left(x-\frac{\pi}{2}\right)\right)^{2/3}}}{\sqrt{2}}; & x\in\left(\dfrac{\pi}{2},\pi\right)\end{cases}$$

. Then $g(x)$

(A) Is continuous at $x=\dfrac{\pi}{2}$ for $g\left(\dfrac{\pi}{2}\right)=0$

(B) Has a removable discontinuity at $x=\dfrac{\pi}{2}$

(C) $g(x)$ is discontinuous at $x=\dfrac{\pi}{2}$ and jump of discontinuity is equal to $\sqrt{2}$

(D) Has a non-removable discontinuity at $x=\dfrac{\pi}{2}$

Comprehension 6

Let $f(x) = f_1(x)-2f_2(x)$

$$f_1(x)=\begin{cases}\min\{x^2,|x|\} \text{ for } x\in[-1,1]\\ \max\{x^2,|x|\} \text{ for } |x|>1\end{cases};$$

$$f_2(x)=\begin{cases}\max\{x^2,|x|\} \text{ for } x\in[-1,1]\\ \min\{x^2,|x|\} \text{ for } |x|>1\end{cases}$$

14. The number of points where $f'(x)$ vanishes or does not exist is

(A) 1 (B) 2

(C) 3 (D) 4

15. If $g(x) = |f(x)|$, then the number of points where the function is non-differentiable is

(A) 0 (B) 1

(C) 3 (D) 4

Comprehension 7

Let $f(x)=\max\{f_1(x), f_2(x), f_3(x)\}$

Where $\displaystyle f_1(x)=\lim_{n\to\infty}\lim_{a\to1^+}\frac{a^n|\sin x|+|\cos x|a^{-n}}{a^n+a^{-n}}$

$\displaystyle f_2(x)=\lim_{n\to\infty}\lim_{a\to1^-}\frac{a^n|\sin x|+|\cos x|a^{-n}}{a^n+a^{-n}}$ and

$\displaystyle f_3(x)=\lim_{x\to0}\frac{g\left(\frac{x}{2}\right)-g(x^2)}{g(x)-g(0)}$

Where $g(x)$ is a differentiable function and $g^1(x)>0$.

16. The functions representing (f_1, f_2, f_3) are

(A) $\left(|\sin x|, |\cos x|, \dfrac{1}{2}\right)$ (B) $(|\sin x|, |\cos x|, -1)$

(C) $(|\cos x|, |\sin x|, -1)$ (D) $\left(|\cos x|, |\sin x|, \dfrac{1}{2}\right)$

17. Number of values of x for which $f(x)$ is non differentiable in $[0,2\pi]$ are

(A) 2 (B) 4

(C) 5 (D) 6

18. Number of values of x satisfying

$$[f_1(x)] \neq [f_2(x)]+[f_3(x)] \text{ in } [0,2\pi] \text{ are}$$

(where [.] is greatest integer function)

(A) 0 (B) 4

(C) 5 (D) infinite

Comprehension 8

Let $f(x)=\begin{cases}[x], -2\le x\le -\dfrac{1}{2}\\ 2x^2-1, -\dfrac{1}{2}<x\le 2\end{cases}$ and $g(x)=f(|x|)+|f(x)|$ where [ ] represents greatest integer function.

19. The number of points where $|f(x)|$ is non-differentiable is

(A) 3 (B) 4

(C) 2 (D) 5

20. The number of points where $g(x)$ is non-differentiable is

(A) 4 (B) 5

(C) 2 (D) 3

21. The number of points where $g(x)$ is discontinuous is

(A) 1

(B) 2

(C) 3

(D) none of these

Comprehension 9

For $x>0$; let $f(x)=\lim\limits_{n\to\infty}\dfrac{\log(2+x)+x^{2^n}\sin x}{1+x^{2^n}}$

Answer the following questions.

22. $\lim\limits_{x\to 0^+} f(x)$ is equal to

(A) 0

(B) $\log_e 3$

(C) $\log_e 2$

(D) does not exist

23. At $x=1$, $'f'$ _______

(A) continuous

(B) discontinuous

(C) both continuous and differentiable

(D) continuous but not differentiable.

24. In $[0,\pi/2]$, the number of points at which $'f'$ vanishes is

(A) 0 (B) 1

(C) 2 (D) 3

Comprehension 10

Let a function $f(x)$ be defined by

$$f(x)=\begin{cases} b\sin^{-1}\dfrac{x+c}{2}, \text{ if } -\dfrac{1}{2}<x<0\\ \dfrac{1}{2} \quad, \text{if } x=0\\ \dfrac{e^{\frac{ax}{2}}-1}{x} \quad, \text{ if } 0<x<\dfrac{1}{2}\end{cases}, \text{ it is given that } |c|<\frac{1}{2}.$$

Answer the following

25. $f(x)$ may be continuous for

(A) $a=1$ (B) $a=-\dfrac{1}{2}$

(C) $a=\dfrac{1}{2}$ (D) $a=2$

26. If $f(x)$ is differentiable at $x=0$, if

(A) $16b^2=4-c^2$ (B) $64b^2=4-c^2$

(C) $4b^2=4-c^2$ (D) $16b^2=c^2+4$

27. If $f(x)$ is differentiable at zero, then in addition to the conditions imposed on a,b,c described in a,b,c described in (58) and (59) above, we must have

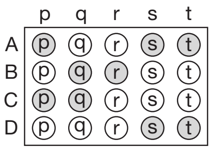
(A) $c=\dfrac{1}{2}\sin^{-1}\dfrac{1}{2b}$ (B) $c=2\sin\dfrac{1}{2b}$

(C) $c=\sin^{-1}\dfrac{1}{2b}$ (D) $c=\sin^{-1}(2b)$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. Match the following:

COLUMN-I	COLUMN-II
(A) Let $f: R \to R$ is defined by the equation $f(x+y) = f(x).f(y) \forall x, y \in R$, $f(0) \neq 0$, and $f'(0) = 2$, then $\frac{f'(x)}{f(x)}$ is	(p) 1
(B) Let $f(x) = \begin{cases} \frac{x^2-1}{4}, \lvert x \rvert < 1 \\ \tan^{-1} x, \lvert x \rvert \geq 1 \end{cases}$ then $f(x)$ is not differentiable at 'x' is equal to	(q) -1
(C) Let $f(x) = \begin{cases} x^2 + a, 0 \leq x < 1 \\ 2x + b, 1 \leq x \leq 2 \end{cases}$ and $g(x) = \begin{cases} 3x + b, 0 \leq x < 1 \\ x^3 \quad , \quad 1 \leq x \leq 2 \end{cases}$. If $\frac{df}{dg}$ exists, then $a =$	(r) 2
(D) If $y = \tan(x+y)$ then $\frac{d^3y}{dx^3} = -\left(\frac{6y^4 + 16y^2 + \lambda}{y^8}\right)$, then the value of $\left[\frac{\lambda}{3}\right]$, $[.]$ is greatest integer function.	(s) 3

2. For the quadratic polynomial $f(x) = ax^2 + bx + c$ where $a, b, c \in R$ and a > 0. Match the entries of column-I with their corresponding entries in column-II (Discriminant $D = b^2 - 4ac$).

COLUMN-I	COLUMN-II
(A) If $D > 0$, $b < 0$ and $c > 0$ then the number of points where the graph of $y = \lvert f(\lvert x \rvert) \rvert$ is not derivable, is	(p) 1
(B) If $D > 0$, $b < 0$ and $c < 0$ then the number of points where the graph of $y = \lvert f(\lvert x \rvert)$ is not derivable, is	(q) 2
(C) If $D < 0$, $b \neq 0$ and $c > 0$ then the number of points where the graph of $y = \lvert f(\lvert x \rvert) \rvert$ is not derivable, is	(r) 3
(D) If $D > 0$, $b = 0$ and $c < 0$ then the number of points where the graph of $y = \lvert f(\lvert x \rvert) \rvert$ is not derivable, is	(s) 4
	(t) 5

3. Match the following:

COLUMN-I	COLUMN-II
(A) Number of points where the function $f(x) = \begin{cases} 1 + \left[\cos\frac{\pi x}{2}\right], & 1 < x < 2 \\ 1 - \{x\}, & 0 \leq x < 1 \\ \lvert \sin \pi x \rvert, & -1 \leq x < 0 \end{cases}$ and $f(1) = 0$ is continuous but non-differentiable	(p) 0
(B) $f(x) = \begin{cases} x^2 e^{1/x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$, then $f'(0^-) =$	(q) 1
(C) The number of points at which $g(x) = \frac{1}{1 + \frac{2}{f(x)}}$ is not differentiable, where $f(x) = \frac{1}{1 + \frac{1}{x}}$, is	(r) 2
(D) Number of points where tangent does not exist for the curve $y = \text{sgn}\left(x^2 - 1\right)$	(s) 3
	(t) 4

4. Match the following:

COLUMN-I	COLUMN-II
(A) Consider the following functions defined from R to R (i) $f(x) = \lvert x \rvert + \lvert x + 1 \rvert$ (ii) $g(x) = 2x + \lvert x \rvert$ (iii) $h(x) = x \lvert x \rvert$ (iv) $k(x) = \lvert \sin x \rvert$ Number of functions which are not differentiable, is	(p) 1
(B) For $a, b \in R$, if $\lim_{x \to 0} \frac{(2 + x + 3x^2)a + (1 - x + 4x^2)b + (10x^3 - 2x^2 - x) - 5}{x^2}$ exists then $(a^2 + b^2)$ is equal to	(q) 2

(*Continued*)

COLUMN-I	COLUMN-II
(C) If $f(x)$ is derivable at $x = 2$ such that $f(2) = 2$ and $f'(2) = 4$, then the value of $\lim\limits_{x\to 0}\frac{1}{h^2}(\ln f(2+h^2) - \ln f(2-h^2))$ is equal to	(r) 3
(D) Let $f(x) = (x^2 - 3x + 2)\,\lvert(x^3 - 6x^2 + 11x - 6)\rvert + \left\lvert\sin\left(x+\frac{\pi}{4}\right)\right\rvert$. Number of points at which the function $f(x)$ is non-differentiable in $[0, 2\pi]$, is	(s) 4
	(t) 5

5. Match Column-I with Column-II and select the correct answer using the code given below the list.

COLUMN-I	COLUMN-II
(A) Jump of discontinuity of $f(x) = \frac{(e^{\tan x}-1)}{(e^{\tan x}+1)}$ at $x = \frac{\pi}{2}$ is	(p) 4
(B) Let $f(x) = x^2 - 4x + 1$, then the number of points where $\lvert f(\lvert x\rvert)\rvert$ is non differentiable is	(q) 5
(C) If $f(x) = \begin{cases} \max\{[x], x\} & ; \quad 0 < x < \frac{\pi}{4} \\ b & ; \quad x = 0 \\ \max\{\sin x, \cot x\} & ; \quad -\frac{\pi}{4} < x < 0 \end{cases}$ is continuous at $x = 0$, then $f'(0)$ is (where [.] represents greatest integer function)	(r) 1
(D) Let $\ell = \lim\limits_{x\to 0}\frac{1-\cos([\pi^2]x)}{(e^x-1)^2}$, the $\left[\frac{\ell}{9}\right]$ is (where [.] represents greatest integer function)	(s) 2

6. Column-I lists some function and Column-II lists its properties. Match the items of Column-I with those of Column-II

COLUMN-I	COLUMN-II
(A) $f(x) = \lim\limits_{n\to\infty}\frac{n^x - n^{-x}}{n^x + n^{-x}}, x \in R$	(p) Continuous at all points.
(B) $f(x) = \begin{cases} \lim\limits_{n\to\infty}\sqrt[n]{4^n + x^{2n} + \frac{1}{x^{2n}}}, & x \in R - \{0\} \\ 0 & , x \in \{0\} \end{cases}$	(q) Discontinuous at finite number of points.
(C) $f(x) = \lim\limits_{n\to\infty}\frac{\ln(e^n + \lvert x\rvert^n)}{n}, x \in R$	(r) Differentiable at all points.
(D) $f(x) = \lim\limits_{n\to\infty}\sqrt[2n]{\cos^{2n}x + \sin^{2n}x}, x \in R$	(s) Discontinuous at finitely many points but not differentiable at infinitely many points.

(*Continued*)

7. Match the following:

COLUMN-I	COLUMN-II
(A) $f(x) = \begin{cases} \frac{1}{\lvert x\rvert}, & \lvert x\rvert \ge 1 \\ ax^2 + b, & \lvert x\rvert < 1 \end{cases}$ is differentiable everywhere and $\lvert k\rvert = a + b$, then the value of k is	(p) 2
(B) If $f(x) = \operatorname{sgn}(x^2 - ax + 1)$ has exactly one point of discontinuity, then the value of a can be	(q) –2
(C) $f(x) = [2 + 3\lvert n\rvert\sin x], n \in N, x \in (0, \pi)$ has exactly 11 points of discontinuity, then the value of n is	(r) 1
(D) $f(x) = \lvert\lvert\lvert x\rvert - 2\rvert + a\rvert$ has exactly three points of non-differentiability, then the value of a is	(s) –1

8. Match the following:

COLUMN-I	COLUMN-II
(A) If the function $f(x) = \begin{cases} \frac{\sin 3x}{x}, & x \ne 0 \\ \frac{k}{2} & x = 0 \end{cases}$ is continuous at $x = 0$, then $k =$	(p) 6
(B) If $f(x) = \frac{x^2 - 10x + 25}{x^2 - 7x + 10}$ for $x \ne 5$ and it is continuous at $x = 5$, then $f(5) =$	(q) 0
(C) If $f : R \to R$ defined by $f(x) = \begin{cases} a^2\cos^2 x + b^2\sin^2 x\ (x \le 0) \\ e^{ax+b}\ (x > 0) \end{cases}$ is continuous function, then $b =$	(r) $2\log\lvert a\rvert$
(D) Let $f(x) = \begin{cases} \cos[x] & x \ge 0 \\ \lvert x\rvert + a & x < 0 \end{cases}$ if $\lim\limits_{x\to 0} f(x)$ exists then $a =$	(s) 1

9. Match the Following:

COLUMN-I	COLUMN-II
(A) If $f(x) = \begin{cases} \dfrac{a+3\cos x}{x^2}, & x<0 \\ b\tan\left(\dfrac{\pi}{x+3}\right), & x \ge 0 \end{cases}$ is continuous at $x = 0$, then (where [.] denotes the greatest integer function)	(p) $[a-2b] = -2$ (where [.] denotes G.I.F)
(B) If $f(x) = \begin{cases} -2\sin x, & -\pi \le x \le -\dfrac{\pi}{2} \\ a\sin x + b, & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ \cos x, & \dfrac{\pi}{2} \le x \le \pi \end{cases}$ is continuous in $[-\pi, \pi]$, then	(q) $\lvert a-b \rvert = 2$
(C) If $f(x) = \begin{cases} \left(\dfrac{3}{2}\right)^{(\cos 3x)/(\cot 2x)}, & -\pi \le x \le -\dfrac{\pi}{2} \\ b+3, & -\dfrac{\pi}{2} < x < \dfrac{\pi}{2} \\ (1+\lvert\cos x\rvert)\left(\dfrac{a\lvert\tan x\rvert}{b}\right), & \dfrac{\pi}{2} \le x \le \pi \end{cases}$ is continuous at $x = \pm\dfrac{\pi}{2}$, then	(r) $\lvert a+2b \rvert = 1$
(D) If $f(x) = \dfrac{a\sin x + b}{x}$, $f(0) = 1$ and $f(x)$ is continuous then	(s) $\lvert a+2b \rvert = 4$

(*Continued*)

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. Let $f(x) = [x] + \left[x+\frac{1}{4}\right] + \left[x+\frac{1}{2}\right] + \left[x+\frac{3}{4}\right]$. Then, no. of points of discontinuity of $f(x)$ in [0, 1] is [.] denotes G.I.F]

2. Number of point(s) in [1, 2] where $f(x) = (-2)^{[x^3]}$ is non differentiable (where [.] denotes greatest integer function)

3. Let $f : [1, 2] \to [1, 4]$, and $g : [1, 2] \to [2, 7]$ are two continuous and bijective function such that $f(1) = 4$ and $g(2) = 7$. Then, find the number of solutions of the equation $f(x) = g(x)\ \forall\, x \in [1, 2]$.

4. Let a and b are real numbers, $b > 0$ and f is defined on [0, 1] by $f(x) = \begin{cases} x^a \sin(x^{-b}) & x > 0 \\ 0 & x = 0 \end{cases}$. If f'' is continuous, then number of integral values of $a - 2b$ in interval [0, 50] is.

5. Find the number of points of discontinuity of the function

$$f(x) = |x| + \left|\frac{x}{3} - 1\right| + \left||x| - \left|\frac{x}{3} - 1\right|\right| \text{ in } x \in (-\infty, \infty)$$

6. Given $f(x) = \begin{cases} x^2 - 5x + \alpha & -\infty < x \le 0 \\ |7-3x| + \sin[x] & 0 < x < 1 \\ \beta x + 2, & 1 \le x < \infty \end{cases}$, where $\alpha, \beta \in R$ and $[x]$ denotes the greatest integer less than or equal to x. If $f(x)$ is continuous for all $x \in R$, then find $(\alpha - \beta)$.

7. Let S denote the sum of an infinite geometric progression whose first term is the value of the function

$$f(x) = \frac{\sin\left(x - \frac{\pi}{6}\right)}{\sqrt{3} - 2\cos x} \text{ at } x = \frac{\pi}{6}, \text{ if } f(x) \text{ is continuous at } x = \frac{\pi}{6} \text{ and}$$

whose common ratio is the limiting value of the function

$$g(x) = \frac{\sin(x)^{1/3} \ln(1+3x)}{(arc\ \tan\sqrt{x})^2 (e^{5.x^{1/3}} - 1)} \text{ as } x \to 0. \text{ Find the value of } 2S.$$

8. Let $f : R \to R$ and $g : R \to R$ be two functions given by $f(x) = -1 + |x|$ and $g(x) = x^3 - 1$ respectively.

Define $h : R \to R$ given by $h(x) = \begin{cases} \min(f(x), g(x)), \text{ if } x \le 1 \\ \max.(f(x), g(x)), \text{ if } x > 1 \end{cases}$.

Then the number of points at which $h(x)$ is non-differentiable is equal to

9. If $f(x)$ is *a* continuous function $\forall\, x \in R$ and the range of $f(x)$ is $(2, \sqrt{21})$ and $g(x) = \left[\frac{f(x)}{c}\right]$ is continuous $\forall\, x \in R$, then the least positive integral value of c is __________ (here $[x]$ is greatest integer $\le x$)

10. The value of p for which the function

$$f(x) = \begin{cases} \dfrac{(4^x - 1)^3}{\sin(x/p)\ \ln(1 + x^2/4)}, & x \ne 0 \\ 12(\ln 4)^3, & x = 0 \end{cases}$$

may be continuous at $x = 0$ is __________

11. The only integer lying in the interval (0, 4), where the function $f(x) = \lim\limits_{n\to\infty} (\cos \pi x / 2)^{2n}$, is discontinuous, is __________

12. The function given by $f(x) = \begin{cases} \dfrac{\sqrt{\pi} - \sqrt{\cos^{-1} x}}{\sqrt{x+1}}, & x \ne -1 \\ \dfrac{1}{\sqrt{\lambda\pi}}, & x = -1 \end{cases}$ the value of λ for which the function $f(x)$ is continuous at $x = -1$ from the right, must be __________

13. Let $f: R^+ \rightarrow R^+$ be *a* differentiable function satisfying $f(xy) = \dfrac{f(x)}{y} + \dfrac{f(y)}{x}$, also $\forall x, y \in R^+$ $f(1) = 0$, $f'(1) = 1$. If M be the greatest value of $f(x)$, then $[M + e]$, [.] gif is __________

14. Let $f(x) = \begin{cases} \dfrac{1}{|x|} & : |x| \geq 1 \\ ax^2 + b & : |x| < 1 \end{cases}$ be continuous and differentiable everywhere, then find the value of $\left|\dfrac{b}{a}\right|$.

15. If $\lim\limits_{x \to 0} \dfrac{1 - \cos\left(1 - \cos\dfrac{x}{2}\right)}{2^m x^n}$ is equal to the left hand derivative of $e^{-|x|}$ at $x = 0$, then find the value of $\dfrac{n - 10m}{37}$.

16. Let $f(x)$ be *a* differentiable function such that $2f(x + y) + f(x - y) = 3f(x) + 3f(y)$ $\forall x, y \in R$ and $f'(0) = 0$, then find the value of $f(5) + f'(5)$.

17. Suppose $f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax^2 + bx + c & \text{if } x \geq 1 \end{cases}$ and $f''(1)$ exist, then find the value of $(a^3 + b^3 + c^3)$.

18. Let $f(x) = \begin{cases} a\cot^{-1}\left(\dfrac{b + x}{4}\right), & \dfrac{-2}{3} < x < 0 \\ 2, & x = 0 \\ \dfrac{\ln(1 - cx)}{x}, & 0 < x < \dfrac{2}{3} \end{cases}$.

If the function $f(x)$ is differentiable at $x = 0$, then find the value of $(b^2 - 2a + c^6)$

19. Consider the function $f(x) = \begin{cases} a^2 + e^x, & -\infty < x < 0 \\ x + 2, & 0 \leq x \leq 3 \\ c - \dfrac{b^2}{x}, & 3 < x < \infty \end{cases}$.

If $f(x)$ is differentiable for every $x \in R$, then find the number of ordered triplets (a, b, c) of real numbers.

20. If 'f' is *a* polynomial function satisfying the condition $f(\tan x) + f(\cot x) = f(\tan x).f(\cot x)$ $\forall x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right) - \{0\}$ and $f(2) = 9$ then the value of $\dfrac{f'(2)}{6}$ is.

21. Number of solutions of $e^{\frac{2x-1}{2}} = \lim\limits_{n \to \infty} \dfrac{1 - \left(\sin^2 \pi x\right)^n}{1 + \left(\sin^2 \pi x\right)^n}$ are

22. Consider the function f defined as $f(x) = ||x| - 1|$ $\forall x \in R$ and another function $g(x)$ such that $g(x) = fof(x)$. Then, number of points where $g(x)$ is not differentiable is

23. If $f(x) = \min.(x - [x], -x - [-x])$, $-2 \leq x \leq 2$. Then, find the number of solutions of the equation $x^2 + f(x) = 1$ (where [.] denotes G.I.F.)

24. Let $f(x)$ be *a* continuous function defined on R such that range of $f(x)$ is $\left[\dfrac{1}{17}, \dfrac{1}{9}\right]$ and if $g(x) = [k f(x)]$, where $k \in N$ and [.] is the greatest integer $\leq x$. Then, the largest value of k for which $g(x)$ is continuous $\forall x \in R$ is

25. *A* function $f(x)$ is defined for all $x \in R$ such that $f(x + y) = f(x) + 2y^2 + kxy$ $\forall x, y \in R$, k is a constant and if $f(1) = 2$ and $f(2) = 8$ then the value of $f'(1)$ is

26. The function $f(x) = \left|x^2 - 3x + 2\right| + \cos|x|$ is not differentiable at how many values of x.

27. Let $f(x) = [\tan x\,[\cot x]]$, $x \in \left[\dfrac{\pi}{12}, \dfrac{\pi}{2}\right)$ (where [.] represents the greatest integer function), then the number of points where $f(x)$ is discontinuous is/are_____

28. Over the interval $\left(\dfrac{1}{2013\pi}, \dfrac{1}{2007\pi}\right)$ the function $\dfrac{\cos x}{\sin\left(\dfrac{1}{x}\right)}$ is discontinuous at K points, then K must be equal to __________

29. The number of points of discontinuity of the function $[x^2] + [x^3]$ ([.] greatest integer) on (1, 2) is ________

30. Number of points of discontinuity of $f(x) = \left[\dfrac{\sin^2 x + \cos^4 x}{\cos^2 x + \sin^4 x}\right]$ where [ ] is G.I.F in the interval $x \in (0, \pi)$ is

31. The number of two digits numbers 'a' whose sum of digits is 9 such that $f(x) = \left[\left(\dfrac{x - 2}{a}\right)^3\right]\sin(x - 2) + a\cos(x - 2)$ is continuous in $[4, 6]$ where [·] is GIF is

32. Number of points of discontinuity of $f(x) = \left[\sin^{-1} x\right] - [x]$ in its domain, where [.] is GIF is equal to

33. If $f(x) = \operatorname{sgn}\left(x^2 - (a - 1)x + 4\right)$ is *a* continuous on R, then the number of integral values of a is

34. The number of points of discontinuity of $f(x) = [2x]^2 - \{2x\}^2$ (where [ ] denotes the greatest integer function and { } is fractional part of x) in the interval (–2, 2), are

35. If $f(x) = \begin{cases} \operatorname{sgn}(x - 2).\left[\log_e x\right], & 1 \leq x \leq 3 \\ \left\{x^2\right\}, & 3 < x \leq 3.5 \end{cases}$ where [.] denotes the greatest integer function and {.} represents the fractional part function, then the number of points of discontinuity is

36. If $f(x) = \begin{cases} \dfrac{1 - \sin^3 x}{3\cos^2 x}, & x < \dfrac{\pi}{2} \\ a, & x = \dfrac{\pi}{2} \\ \dfrac{b(1 - \sin x)}{(\pi - 2x)^2}, & x > \dfrac{\pi}{2} \end{cases}$. If $f(x)$ is continuous $x = \dfrac{\pi}{2}$ then find $\dfrac{b}{a}$

37. The domain of the derivative of the function $f(x) = \begin{cases} \tan^{-1} x, & |x| \leq 1 \\ \dfrac{1}{2}(|x| - 1), & |x| > 1 \end{cases}$ is $R - \{K_1\}$, then $K_1 + 8$ is

38. The number of points in (–5, 5) at which the function $f(x)=x\sin\left([x]^4-5[x]^2+4\right)$ is discontinuous where [.] is GIF, is __________

39. The number of points at which $f(x)=[x]+|1-x|,-1\le x\le 3$ is not differentiable is where [.] denotes G.I.F

40. The number of points where $f(x)=\left|x-[x]-\frac{1}{2}\right|$ is not differentiable in $\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ where [.] is GIF

41. If $f(x)=2|x|-3|x-2|+5|3-4x|$ for $\frac{3}{4}<x<2$ then $f'(1)=\lambda$, then $\frac{\lambda}{5}$ is equal to

42. If $f(x)=\max\{\sin x,\cos x,0\}$, then number of non differentiable points of $f(x)$ in $(0,2n\pi)$ are Kn, then K is

43. If $f(x)$ is continuous function from $R\to R$ and attains only irrational values and $\lambda\sum_{r=1}^{10}f(r)=\sum_{11}^{50}f(r)$, then λ is equal to

44. If $f(x)=\begin{cases}\sin x, & x<0\\ \cos x-|x-1|, & x\ge 0\end{cases}$, then $g(x)=f(|x|)$ is non-differentiable for exactly ________ number of values of x

45. If $\alpha\in(-\infty,-1)\cup(-1,0)$, then the number of points where the function $f(x)=\left|x^2+(\alpha-1)|x|-\alpha\right|$ is not differentiable is.

46. If the function f defined by $f(x)=\dfrac{x(1+a\cos x)-b\sin x}{x^3}$ if $x\ne 0$ and $f(0)=1$ is continuous at $x=0$, then $2a-8b=$

47. Let $f(x)=\begin{cases}x^2\displaystyle\sum_{r=0}^{\left[\frac{1}{|x|}\right]}r & ;x\ne 0\\ \dfrac{k}{2}; & \text{otherwise}\end{cases}$ ([.]denotes the greatest integer function). The value of k such that f become continuous at $x=0$ is

48. Let $f:(-\infty,\infty)\to[0,\infty)$ be a continuous function such that $f(x+y)=f(x)+f(y)+f(x)f(y),\forall x,y\in R$. Also, $f'(0)=1$. Then $\left[\dfrac{f(4)}{f(2)}\right]$ equals ([•] represents greatest integer function)

49. If the function f defined by $f(x)=\dfrac{\log(1+x)^{1+x}}{x^2}-\dfrac{1}{x}$ if $f(x)$ is continuous at $x=0$, then 6 $(f(0))=$

50. If $f(x)=\begin{cases}\dfrac{\tan\left[x^2\right]\pi}{ax^2}+ax^3+b & ,\ 0\le x\le 1\\ 2\cos\pi x+\tan^{-1}x & ,1<x\le 2\end{cases}$ is differentiable in [0, 2], then $b=\frac{\pi}{4}-\frac{26}{k_2}$. Find $k_1^2+k_2^2$ {where [ ] denotes greatest integer function}.

51. Let $f(x)=\begin{cases}|x|^p\sin\frac{1}{x}+x|\tan x|^q & ,\ x\ne 0\\ 0 & ,\ x=0\end{cases}$ be differentiable at $x=0$, then find the least possible value of $[p+q]$, (where[.] represents greatest integer function)

52. Let function f be defined as $f:R^+\to R^+$ and function g is defined as $g:R^+\to R^+$. Functions f and g are continuous in their domain. Suppose the function $h(x)=\lim\limits_{n\to\infty}\dfrac{x^nf(x)+x^2}{x^n+g(x)}$, $x>0$. If $h(x)$ is continuous in its domain, then find $f(1)\cdot g(1)$

ARCHIVE: JEE MAIN

1. **[JEE Mains 2021]**
Let $[t]$ denote the greatest integer $\le t$. The number of points where the function
$f(x)=[x]\,|x^2-1|+\sin\left(\dfrac{\pi}{[x]+3}\right)-[x+1]$, $x\in(-2,2)$ is not continuous is _________.

2. **[JEE Mains 2021]**
The number of points, at which the function
$f(x)=|2x+1|-3|x+2|+|x^2+x-2|$, $x\in R$ is not differentiable, is ______.

3. **[JEE Mains 2021]**
A function f is defined on [–3, 3] as
$f(x)=\begin{cases}\min\{|x|,2-x^2\}, & -2\le x\le 2\\ [|x|], & 2<|x|\le 3\end{cases}$ where $[x]$ denotes the greatest integer $\le x$. Then number of point, where f is not differentiable in (–3, 3) is _______.

4. **[JEE Mains 2021]**
Let $f:R\to R$ and $g:R\to R$ be defined as $f(x)=\begin{cases}x+a, & x<0\\ |x-1|, & x\ge 0\end{cases}$ and $g(x)=\begin{cases}x+1, & x<0\\ (x-1)^2+b, & x\ge 0\end{cases}$ where a, b are non-negative real numbers. If $(gof)(x)$ is continuous for all $x\in R$, then $a+b$ is equal to ____.

5. **[JEE Mains 2021]**
Let a function $g:[0,4]\to R$ be defined as
$g(x)=\begin{cases}\max\limits_{0\le t\le x}\{t^3-6t^2+9t-3\}, & 0\le x\le 3\\ 4-x, & 3<x\le 4\end{cases}$, then the number of points in the interval (0, 4) where $g(x)$ is NOT differentiable, is ______.

6. **[JEE Mains 2021]**
Let $f:[0,3]\to R$ be defined by $f(x)=\min\{x-[x],1+[x]-x\}$ where $[x]$ is the greatest integer less than or equal to x.

Let P denote the set containing all $x \in [0, 3]$ where f is discontinuous, and Q denote the set containing all $x \in (0, 3)$ where f is not differentiable. Then the sum of number of elements in P and Q is equal to _________.

7. **[JEE Mains 2021]**

Let $f : R \to R$ be defined as $f(x) = \begin{cases} 2\sin\left(-\dfrac{\pi x}{2}\right); & x < -1 \\ |ax^2 + x + b|; & -1 \le x \le 1 \\ \sin(\pi x); & x > 1 \end{cases}$.

If $f(x)$ is continuous on R, then $a + b$ equals :

(A) 1

(B) –1

(C) –3

(D) 3

8. **[JEE Mains 2021]**

Let the functions $f : R \to R$ and $g : R \to R$ be defined as:

$f(x) = \begin{cases} x+2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$ and $g(x) = \begin{cases} x^3, & x < 1 \\ 3x - 2, & x \ge 1 \end{cases}$

Then, the number of points in R where $(fog)(x)$ is NOT differentiable is equal to:

(A) 3

(B) 1

(C) 0

(D) 2

9. **[JEE Mains 2021]**

Let $f : [0, \infty) \to [0, 3]$ be a function defined by

$$f(x) = \begin{cases} \max\{\sin t : 0 \le t \le x\}, & 0 \le x \le \pi \\ 2 + \cos x, & x > \pi \end{cases}$$

Then which of the following is true?

(A) f is not continuous exactly at two points in $(0, \infty)$

(B) f is continuous everywhere but not differentiable exactly at one point in $(0, \infty)$

(C) f is differentiable everywhere in $(0, \infty)$

(D) f is continuous everywhere but not differentiable exactly at two points in $(0, \infty)$

10. **[JEE Mains 2021]**

If the function $f(x) = \begin{cases} \dfrac{1}{x}\log_e\left(\dfrac{1 + \frac{x}{a}}{1 - \frac{x}{b}}\right) & , \ x < 0 \\ k & , \ x = 0 \\ \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1} - 1} & , \ x > 0 \end{cases}$

is continuous at $x = 0$, then $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{4}{k}$ is equal to:

(A) –5

(B) 5

(C) –4

(D) 4

11. **[JEE Mains 2020]**

If the function f defined on $\left(-\dfrac{1}{3}, \dfrac{1}{3}\right)$ by

$f(x) = \begin{cases} \dfrac{1}{x}\log_e\left(\dfrac{1+3x}{1-2x}\right), & \text{when } x \ne 0 \\ k, & \text{when } x = 0 \end{cases}$ is continuous, then k is equal to _________.

12. **[JEE Mains 2020]**

Let $f(x) = x \cdot \left[\dfrac{x}{2}\right]$, for $-10 < x < 10$, where $[t]$ denotes the greatest integer function. Then the number of points of discontinuity of f is equal to ______.

13. **[JEE Mains 2020]**

Let S be the set of points where the function, $f(x) = |2 - |x - 3||$, $x \in R$, is not differentiable. Then $\displaystyle\sum_{x \in S} f(f(x))$ is equal to _______.

14. **[JEE Mains 2020]**

Suppose a differentiable function $f(x)$ satisfies the identity $f(x + y) = f(x) + f(y) + xy^2 + x^2y$, for all real x and y. If $\displaystyle\lim_{x \to 0} \frac{f(x)}{x} = 1$, then $f'(3)$ is equal to _________.

15. **[JEE Mains 2020]**

Let $f : R \to R$ be defined as $f(x) = \begin{cases} x^5 \sin\left(\dfrac{1}{x}\right) + 5x^2 & , \ x < 0 \\ 0 & , \ x = 0 \\ x^5 \cos\left(\dfrac{1}{x}\right) + \lambda x^2 & , \ x > 0 \end{cases}$.

The value of λ for which $f''(0)$ exists, is

16. **[JEE Mains 2020]**

The function $f(x) = \begin{cases} \dfrac{\pi}{4} + \tan^{-1} x, & |x| \le 1 \\ \dfrac{1}{2}(|x| - 1), & |x| > 1 \end{cases}$ is:

(A) Continuous on $R - \{1\}$ and differentiable on $R - \{-1, 1\}$.

(B) Both continuous and differentiable on $R - \{-1\}$.

(C) Continuous on $R - \{-1\}$ and differentiable on $R - \{-i, 1\}$.

(D) Both continuous and differentiable on $R - \{1\}$.

17. **[JEE Mains 2019]**

Let $f(x) = 15 - |x - 10|$; $x \in R$. Then the set of all values of x, at which the function, $g(x) = f(f(x))$ is not differentiable, is:

(A) $\{10, 15\}$

(B) $\{5, 10, 15\}$

(C) $\{10\}$

(D) $\{5, 10, 15, 20\}$

18. **[JEE Mains 2019]**

Let $f : R \to R$ be differentiable at $c \in R$ and $f(c) = 0$. If $g(x) = |f(x)|$, then at $x = c$, g is:

(A) Differentiable if $f'(c) = 0$

(B) Differentiable if $f'(c) \ne 0$

(C) Not differentiable

(D) Not differentiable if $f'(c) = 0$

19. **[JEE Mains 2019]**

Let $f : R \to R$ be a function defined as

$f(x) = \begin{cases} 5, & \text{if} & x \le 1 \\ a + bx, & \text{if} & 1 < x < 3 \\ b + 5x, & \text{if} & 3 \le x < 5 \\ 30, & \text{if} & x \ge 5 \end{cases}$ then, f is

(A) Continuous if $a = 0$ and $b = 5$

(B) Continuous if $a = -5$ and $b = 10$

(C) Continuous if $a = 5$ and $b = 5$

(D) Not continuous for any values of a and b

20. [JEE Mains 2019]

Let $f : [-1, 3] \to R$ be defined as $f(x) = \begin{cases} |x| + [x], & -1 \le x < 1 \\ x + |x|, & 1 \le x < 2 \\ x + [x], & 2 \le x \le 3 \end{cases}$

Where $[t]$ denotes the greatest integer less than or equal to t. Then, f is discontinuous at

(A) Four or more points
(B) Only three points
(C) Only two points
(D) Only one point

21. [JEE Mains 2019]

If the function f defined on $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ by

$f(x) = \begin{cases} \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1} & x \ne \dfrac{\pi}{4} \\ k, & x = \dfrac{\pi}{4} \end{cases}$ is continuous, then k is equal to:

(A) $\frac{1}{\sqrt{2}}$
(B) 1
(C) $\frac{1}{2}$
(D) 2

22. [JEE Mains 2019]

If the function $f(x) = \begin{cases} a|\pi - x| + 1, & x \le 5 \\ b|x - \pi| + 3, & x > 5 \end{cases}$ is continuous at $x = 5$, then value of $a - b$ is

(A) $\frac{2}{\pi + 5}$
(B) $\frac{-2}{\pi + 5}$
(C) $\frac{2}{\pi - 5}$
(D) $\frac{2}{5 - \pi}$

23. [JEE Mains 2019]

If $f(x) = [x] - \left[\frac{x}{4}\right]$, $x \in R$, where $[x]$ denotes the greatest integer function, then:

(A) $\lim_{x \to 4^-} f(x)$ exists but $\lim_{x \to 4^+} f(x)$ does not exist.
(B) Both $\lim_{x \to 4^-} f(x)$ and $\lim_{x \to 4^+} f(x)$ exist but are not equal.
(C) $\lim_{x \to 4^+} f(x)$ exists but $\lim_{x \to 4^-} f(x)$ does not exist.
(D) f is continuous at $x = 4$.

24. [JEE Mains 2019 AIEEE 2011]

If $f(x) = \begin{cases} \dfrac{\sin(p+1)x + \sin x}{x} & x < 0 \\ q & x = 0 \\ \dfrac{\sqrt{x + x^2} - \sqrt{x}}{x^{3/2}} & x > 0 \end{cases}$ is continuous at $x = 0$, then the ordered pair (p, q) is equal to:

(A) $\left(-\frac{3}{2}, -\frac{1}{2}\right)$
(B) $\left(-\frac{1}{2}, \frac{3}{2}\right)$
(C) $\left(-\frac{3}{2}, \frac{1}{2}\right)$
(D) $\left(\frac{5}{2}, \frac{1}{2}\right)$

25. [JEE Mains 2018]

Let $S = \{t \in R : f(x) = |x - \pi|(e^{|x|} - 1)\sin|x|$ is not differentiable at $t\}$. Then the set S is equal to:

(A) $\{0\}$
(B) $\{\pi\}$
(C) $\{0, \pi\}$
(D) ϕ(an empty set)

26. [JEE Mains 2016]

For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then:

(A) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$
(B) g is not differentiable at $x = 0$
(C) $g'(0) = \cos(\log 2)$
(D) $g'(0) = -\cos(\log 2)$

27. [JEE Mains 2014]

If $f(x)$ is continuous and $f(9/2) = 2/9$, then $\lim_{x \to 0} f\left(\frac{1 - \cos 3x}{x^2}\right)$ is equal to:

(A) 9/2
(B) 0
(C) 2/9
(D) 8/9

ARCHIVE: JEE ADVANCED

1. [JEE Adv 2020]

Let the function $f : R \to R$ be defined by $f(x) = x^3 - x^2 + (x - 1)\sin x$ and let $g: R \to R$ be an arbitrary function. Let $f\,g: R \to R$ be the product function defined by $(fg)(x) = f(x)g(x)$. Then which of the following statements is/are TRUE?

(A) If g is continuous at $x = 1$, then fg is differentiable at $x = 1$
(B) If fg is differentiable at $x = 1$, then g is continuous at $x = 1$
(C) If g is differentiable at $x = 1$, then fg is differentiable at $x = 1$
(D) If fg is differentiable at $x = 1$, then g is differentiable at $x = 1$

2. [JEE Adv 2020]

Let the functions $f : (-1, 1) \to R$ and $g: (-1, 1) \to (-1, 1)$ be defined by $f(x) = |2x - 1| + |2x + 1|$ and $g(x) = x - [x]$ where $[x]$ denotes the greatest integer less than or equal to x. Let $fog: (-1, 1) \to R$ be the composite function defined by $(fog)(x) = f(g(x))$. Suppose c is the number of points in the interval $(-1, 1)$ at which fog is NOT continuous, and suppose d is the number of points in the interval $(-1, 1)$ at which $f\,o\,g$ is NOT differentiable. Then the value of $c + d$ is ____

3. [JEE Adv 2020]

Let $f : R \to R$ and $g: R \to R$ be functions satisfying $f(x + y) = f(x) + f(y) + f(x)f(y)$ and $f(x) = xg(x)$ for all $x, y \in R$. If $\lim_{x \to 0} g(x) = 1$, then which of the following statements is/are TRUE?

(A) f is differentiable at every $x \in R$
(B) If $g(0) = 1$, then g is differentiable at every $x \in R$
(C) The derivative $f'(1)$ is equal to 1
(D) The derivative $f'(0)$ is equal to 1

4. [JEE Adv 2018]

Let $f : R \to R$ be a differentiable function with $f(0) = 1$ and satisfying the equation $f(x + y) = f(x)f'(y) + f'(x)f(y)$ for all $x, y \in R$ Then, the value $\log_e(f(4))$ is _________.

5. **[JEE Adv 2018]**

Let $f_1 : R \to R, f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to R, f_3 : \left(-1, e^{\pi/2} - 2\right) \to R$ and $f_4 : R \to R$ be be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$,

(ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(iii) $f_3(x) = [\sin(\log_e(x + 2))]$, where, for $t \in R$, $[t]$ denotes the greatest integer less than or equal to t.

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

List-I	List-II
P. The function f_1 is	1. **NOT** continuous at $x = 0$
Q. The function f_2 is	2. Continuous at $x = 0$ and **NOT** differentiable at $x = 0$
R. The function f_3 is	3. Differentiable at $x = 0$ and its derivative is **NOT** continuous at $x = 0$
S. The function f_4 is	4. Differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is:
(A) P → 2; Q → 3; R → 1; S → 4
(B) P → 4; Q → 1; R → 2; S → 3
(C) P → 4; Q → 2; R → 1; S → 3
(D) P → 2; Q → 1; R → 4; S → 3

6. **[JEE Adv 2017]**

Let $[x]$ be the greatest integer less then or equals to x. Then, at which of the following point (s) the functions $f(x) = x\cos(\pi(x + [x]))$ is discontinuous?

(A) $x = -1$ (B) $x = 0$
(C) $x = 1$ (D) $x = 2$

7. **[JEE Adv 2016]**

Let $a, b \in R$ and $R \to R$ be defined by $f(x) = a\cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$. Then f is

(A) Differentiable at $x = 0$ if $a = 0$ and $b = 1$
(B) Differentiable at $x = 1$ if $a = 1$ and $b = 0$
(C) NOT differentiable at $x = 0$ if $a = 1$ and $b = 0$
(D) NOT differentiable at $x = 1$ if $a = 1$ and $b = 1$

8. **[JEE Adv 2016]**

Let $f : \left[-\frac{1}{2}, 2\right] \to R$ and $g : \left[-\frac{1}{2}, 2\right] \to R$ be functions defined by $f(x) = [x^2 - 3]$ and $g(x) = |x| f(x) + |4x - 7| f(x)$, where $[y]$ denotes the greatest less than or equal to y for $y \in R$. Then

(A) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$

(B) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$

(C) g is NOT differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$

(D) g is NOT differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$

9. **[JEE Adv 2015]**

Let $g: R \to R$ be a differentiable function with $g(0) = 0$, $g'(0) = 0$ and $g'(1) \neq 0$.

Let $f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$ and $h(x) = e^{|x|}$ for all $x \in R$. Let $(foh)(x)$ denote $f(h(x))$ and $(hof)(x)$ denote $h(f(x))$. Then which of the following is (are) true?

(A) f is differentiable at $x = 0$
(B) h is differential at $x = 0$
(C) foh is differentiable at $x = 0$
(D) hof is differentiable at $x = 0$

10. **[JEE Adv 2014]**

Let $f : R \to R$ and $g: R \to R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$.

Define $h: R \to R$ by

$$h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \le 0, \\ \min\{f(x), g(x)\} & \text{if } x > 0 \end{cases}$$

The number of points at which $h(x)$ is not differentiable is

11. **[JEE Adv 2014]**

Let $f_1: R \to R$, $f_2: [0, \infty) \to R$, $f_3: R \to R$ and $f_4: R \to [0, \infty)$ be defined by

$$f_1(x) = \begin{cases} |x| & \text{if } x < 0 \\ e^x & \text{if } x \ge 0; \end{cases} \quad f_2(x) = x^2;$$

$$f_3(x) = \begin{cases} \sin x & \text{if } x < 0 \\ x & \text{if } x \ge 0; \end{cases} \text{ and } f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if } x < 0 \\ f_2(f_1(x)) - 1 & \text{if } x \ge 0; \end{cases}$$

List-I	List-II
P. f_4 is	1. onto but not one-one
Q. f_3 is	2. neither continuous nor one-one
R. $f_2 o f_1$ is	3. differentiable but not one-one
S. f_2 is	4. continuous and one-one

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

12. **[JEE Adv 2014]**

For every pair of continuous functions $f, g: [0, 1] \to \mathbb{R}$ such that $\max\{f(x): x \in [0, 1]\} = \max\{g(x): x \in [0, 1]\}$, the correct statement(s) is(are):

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0, 1]$
(D) $(f(c))^2 = (g(c))^2$ for some $c \in [0, 1]$

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. B	**2.** A	**3.** A	**4.** B	**5.** A	**6.** C	**7.** C	**8.** A	**9.** D	**10.** C
11. C	**12.** B	**13.** C	**14.** C	**15.** C	**16.** C	**17.** B	**18.** A	**19.** A	**20.** B
21. A	**22.** B	**23.** D	**24.** B	**25.** B	**26.** B	**27.** A	**28.** D	**29.** C	**30.** C
31. A	**32.** D	**33.** A	**34.** B	**35.** A	**36.** C	**37.** B	**38.** D	**39.** C	**40.** C
41. B	**42.** C	**43.** B	**44.** C	**45.** C	**46.** D	**47.** C	**48.** A	**49.** D	**50.** B
51. D	**52.** A	**53.** B	**54.** B	**55.** C	**56.** A	**57.** A	**58.** A	**59.** B	**60.** D
61. D	**62.** C	**63.** C	**64.** C	**65.** D	**66.** C	**67.** C	**68.** C	**69.** D	**70.** D
71. D	**72.** D	**73.** B	**74.** C	**75.** D	**76.** D	**77.** C	**78.** A	**79.** D	**80.** D
81. B	**82.** C	**83.** A	**84.** A	**85.** C	**86.** A	**87.** A	**88.** C	**89.** A	**90.** A
91. A	**92.** C								

Multiple Correct Choice Type Questions

1. B, D	**2.** B, C, D	**3.** A, C	**4.** A, B, C, D	**5.** B, C, D
6. A, B, C, D	**7.** C, D	**8.** A, B, D	**9.** A, B, C	**10.** A, C
11. A, C	**12.** B, C	**13.** A, B, C	**14.** A, B, D	**15.** B, C, D
16. B, C	**17.** B, C	**18.** A, C	**19.** A, B, C, D	**20.** A, C, D
21. A, B, D	**22.** B, D	**23.** A, B, C	**24.** A, C	**25.** A, C
26. B, C, D	**27.** A, C	**28.** B, C, D	**29.** A, B, D	**30.** B, C, D
31. A, B, C, D	**32.** A, B, C, D	**33.** A, B, C	**34.** A, B, C	**35.** A, B
36. A, C	**37.** B, C, D	**38.** A, C, D	**39.** A, B, D	**40.** A, B, D
41. A, B, C, D				

Linked Comprehension Type Questions

1. C	**2.** A	**3.** D	**4.** B	**5.** A	**6.** D	**7.** C	**8.** C	**9.** B	**10.** B
11. C	**12.** A	**13.** D	**14.** C	**15.** C	**16.** A	**17.** B	**18.** C	**19.** A	**20.** D
21. B	**22.** C	**23.** B	**24.** A	**25.** A	**26.** B	**27.** B			

Matrix Match/Column Match Type Questions

1. A → (r)	B → (p, q)	C → (q)	D → (s)
2. A → (t)	B → (r)	C → (p)	D → (q)
3. A → (q)	B → (p)	C → (s)	D → (p)
4. A → (r)	B → (t)	C → (s)	D → (r)
5. A → (s)	B → (q)	C → (r)	D → (p)
6. A → (q)	B → (q)	C → (p)	D → (p)
7. A → (r, s)	B → (p, q)	C → (p, q)	D → (p, r)
8. A → (p)	B → (q)	C → (r)	D → (s)
9. A → (p)	B → (q, r)	C → (q, s)	D → (r)

Integer/Numerical Answer Type Questions

1. 4	**2.** 7	**3.** 1	**4.** 48	**5.** 0	**6.** 5	**7.** 5	**8.** 0	**9.** 5	**10.** 3
11. 2	**12.** 2	**13.** 3	**14.** 3	**15.** 2	**16.** 0	**17.** 1	**18.** 48	**19.** 4	**20.** 2
21. 0	**22.** 5	**23.** 2	**24.** 8	**25.** 4	**26.** 2	**27.** 3	**28.** 5	**29.** 8	**30.** 0
31. 3	**32.** 3	**33.** 7	**34.** 6	**35.** 5	**36.** 8	**37.** 7	**38.** 6	**39.** 4	**40.** 6
41. 5	**42.** 3	**43.** 4	**44.** 3	**45.** 5	**46.** 7	**47.** 1	**48.** 8	**49.** 3	**50.** 180
51. 1	**52.** 1								

ARCHIVE: JEE MAIN

1. 2	**2.** 2	**3.** 5	**4.** 1	**5.** 1	**6.** 5	**7.** B	**8.** B	**9.** C	**10.** A
11. 5	**12.** 8	**13.** 3	**14.** 10	**15.** 5	**16.** A	**17.** B	**18.** A	**19.** D	**20.** B
21. C	**22.** D	**23.** D	**24.** C	**25.** 4	**26.** C	**27.** C			

ARCHIVE: JEE ADVANCED

1. A, C	**2.** 4	**3.** A, B, D	**4.** 2	**5.** D	**6.** A, C, D	**7.** A, B	**8.** B, C	**9.** A, D	**10.** 3
11. D	**12.** A, D								

CHAPTER

6

Application of Derivatives

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

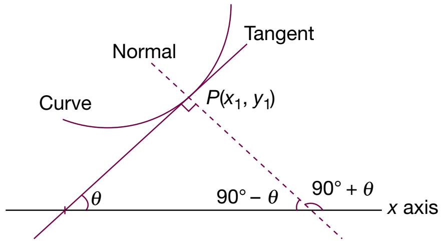
- (a) Tangent and Normal of a Curve
- (b) Angle between Curves, Orthogonal Curves
- (c) Length of Tangent/Normal/Sub-tangent/Sub-normal
- (d) Shortest Distance between Two Curves
- (e) Approximation Using Derivative
- (f) Physical Application of Derivative: Rate of Change
- (g) Rolle's Theorem
- (h) Lagrange's Mean Value Theorem and Cauchy's Mean Value Theorem

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

GEOMETRICAL APPLICATION OF DERIVATIVE: TANGENT AND NORMAL

Tangent and normal at a point on curve:
Curve $y = f(x)$ and point $P(x_1, y_1)$ lies on curve $y_1 = f(x_1)$.
Therefore,
Slope of tangent on curve $y = f(x)$ at point (x_1, y_1)

$$m = \tan\theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = f'(x)\Big|_{(x_1, y_1)}$$



Normal is always perpendicular to tangent

$$\text{Slope of normal} = \left(\frac{-1}{f'(x)}\right)_{(x_1, y_1)} = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

Conceptual Note(s)

Equation of tangent $y - y_1 = f'(x)\Big|_{(x_1, y_1)}(x - x_1)$

Equation of normal $y - y_1 = \left(\frac{-1}{f'(x)}\right)_{(x_1, y_1)}(x - x_1)$

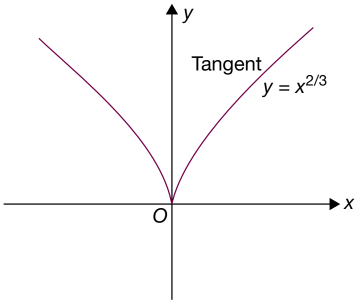
Following points for better understanding:

(i) When $\theta = 0$, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$

Equation of tangent is $y = y_1$;
Equation of normal $x = x_1$

(ii) When $\theta = \pi/2$, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \to \pm\infty$ or $\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$

Equation of tangent is $x = x_1$;
Equation of normal $y = y_1$

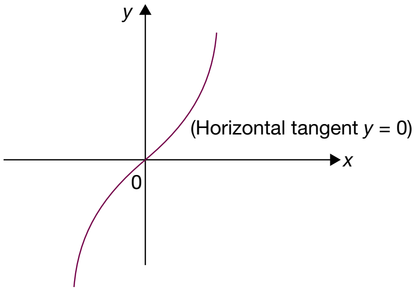


At point P, dy/dx does not exist then also tangent to a curve at point P can be drawn.
E.g., $x = 0$ is a tangent to $y = x^{2/3}$ at (0, 0).
$x = 1$ and $x = -1$ are tangent for curve $y = \sin^{-1}x$ and $y = \cos^{-1}x$, as dy/dx is infinity.

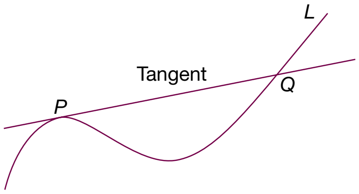
(iii) If the tangent at point P on the curve is equally inclined to both the axes, then at point P,
slope of tangent $= dy/dx = \pm 1$.

(iv) If the tangent at point P makes equal intercepts on the coordinate axes, then at point P,
slope of tangent $= dy/dx = -1$.

(v) A line which is tangent to the curve at point P. It may cross the curve at the same point also.
E.g., $f(x) = x^3$, $y = 0$ (x-axis) is a tangent. It crosses curve.



(vi) A tangent drawn at a point on the curve may intersect the curve again at other point.



In the above figure, line L is tangent to curve at P and it intersects the curve again at Q.

(vii) If a curve passing through the origin be given by a rational integral algebraic equation, the equation of the tangent (or tangents) at the origin is obtained by equating to zero the terms of the lowest degree in the equation, e.g., if the equation of a curve be $x^2 - y^2 + x^3 + 3x^2y - y^3 = 0$, the tangents at the origin are given by $x^2 - y^2 = 0$ i.e. $x + y = 0$ and $x - y = 0$.

ILLUSTRATION 1

Find the points on the curve $y = x^3 + x^2 + x$ at which the tangent to the curve is

(i) Parallel to the line $y = 2x + 3$,
(ii) Perpendicular to the line $x + y = 1$.

SOLUTION

Differentiating $y = x^3 + x^2 + x$,

We get $\dfrac{dy}{dx} = 3x^2 + 2x + 1$

(i) Let the tangent at the point (a, b) on the curve be parallel to the line $y = 2x + 3$. Since, parallel lines have the same slope, we have

$$\left(\frac{dy}{dx}\right)_{(a,b)} = 3a^2 + 2a + 1 = 2$$

Therefore, we get $3a^2 + 2a - 1 = 0$
or, $a = -1$ and $a = 1/3$
Since, the point (a, b) lies on the curve, we get
$b = a^3 + a^2 + a$
For $a = -1$, we obtain $b = -1$ and for $a = 1/3$, we obtain $b = 13/27$
Hence, at the points $(-1, -1)$ and $(1/3, 13/27)$, the tangents to the given curve are parallel to the line $y = 2x + 3$.

(ii) The slope of the given line $y = 1 - x$ is -1. Hence, the slope of the tangent, which is perpendicular to the given line is =1,

that is $\left(\dfrac{dy}{dx}\right)_{(a,b)} = 3a^2 + 2a + 1 = 1$, which gives $a = 0$ and $a = -2/3$.
Since, the point (a, b) lies on the curve, we get for $a = 0$, $b = 0$ and for $a = -2/3$, $b = -14/27$.
Therefore, at the points $(0, 0)$ and $(-2/3, -14/27)$, the tangents to the curve are perpendicular to the line $x + y = 1$.

ILLUSTRATION 2

Determine the parameters a, b, c in the curve $y = ax^2 + bx + c$ so that it passes through $(-1, 0)$ and the line $y = x$ becomes a tangent to it at the point $x = 1$.

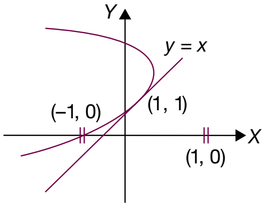
SOLUTION

Given curve is $y = ax^2 + bx + c$ …(1)
Differentiating with respect to x,

$$\frac{dy}{dx} = 2ax + b \quad \ldots(2)$$

According to given condition,

$$\left(\frac{dy}{dx}\right)_{x=1} = 1 = 2a + b \quad \ldots(3)$$



Since, (–1, 0) is a point upon curve.

$\Rightarrow \quad a - b + c = 0 \qquad \ldots(4)$

Given that point (1, 1) must lie upon curve.

$\Rightarrow \quad a + b + c = 1 \qquad \ldots(5)$

Solving (3), (4), (5), we get $2b = 1$ or $b = \frac{1}{2}$

Hence, $a = \frac{1}{4} = c$.

ILLUSTRATION 3

Find the points on the curve $y = x^3 - 2x^2 - x$ at which the tangent lines are parallel to the line $y = 3x - 2$.

SOLUTION

Let $P(x_1, y_1)$ be the required point. The given curve is

$$y = x^3 - 2x^2 - x$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 4x - 1 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\, y_1)} = 3{x_1}^2 - 4x_1 - 1$$

Since, the tangent at (x_1, y_1) is parallel to the line $y = 3x - 2$.

$\therefore$ Slope of the tangent at (x_1, y_1) = Slope of the line $y = 3x - 2$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 3 \Rightarrow 3{x_1}^2 - 4x_1 - 1 = 3$$

$$\Rightarrow \quad 3{x_1}^2 - 4x_1 - 4 = 0 \Rightarrow (x_1 - 2)\,(3x_1 + 2) = 0$$

$$\Rightarrow \quad x_1 = 2, \frac{-2}{3}.$$

Since, (x_1, y_1) lies on (1), therefore $y_1 = {x_1}^3 - 2{x_1}^2 - x_1$

When $x_1 = 2$; $y_1 = 2^3 - 2(2)^2 - 2 = -2$.

When $x_1 = -\frac{2}{3}$; $y_1 = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 + \frac{2}{3} = -\frac{14}{27}$.

Hence, the required points are (2, –2) and $\left(-\frac{2}{3}, -\frac{14}{27}\right)$.

ILLUSTRATION 4

Write equation of tangent and normal at point $x = -1/2$, on curve $y = |\ln|x||$.

SOLUTION

When $x = -\frac{1}{2}$; $y = \left|\ln\left|-\frac{1}{2}\right|\right| = |-\ln 2| = \ln 2$

Therefore, the point is $\left(-\frac{1}{2}, \ln 2\right)$

First we make, equation of curve free from modulus in neighbourhood of given point.

$$y = -\ln(-x) \text{ at } x = -\frac{1}{2}$$

Slope of tangent = $\frac{dy}{dx} = -\frac{1}{(-x)}(-1) = \left.\frac{-1}{x}\right|_{x=-\frac{1}{2}} = 2$

At point $\left(-\frac{1}{2}, \ln 2\right)$, slpoe of tangent $m_T = 2$

Equation of tangent $y - \ln 2 = 2\left(x + \frac{1}{2}\right) = 2x + 1$

$\Rightarrow \quad y = 2x + 1 + \ln 2$

Slope of normal $m_N = -\frac{1}{2}$

Equation of normal

$$y - \ln 2 = -\frac{1}{2}\left(x + \frac{1}{2}\right) \Rightarrow 2y - 2\ln 2 + x + \frac{1}{2} = 0$$

Hence, $4y - 4\ln 2 + 2x + 1 = 0$

ILLUSTRATION 5

Write equation of normal on curve $y = |x^2 - |x||$ at point $x = -2$.

SOLUTION

Given, $x = -2$ so, $y = |4 - |-2|| = |4 - 2| = |2| = 2$.

Therefore, point (–2, 2).

First we make, equation of curve free from modulus in neighbourhood of given point.

$$y = +(x^2 - (-x)) = x^2 + x$$

Slope of tangent = $m_T = \frac{dy}{dx} = 2x + 1\Big|_{x=-2} = -4 + 1 = -3$

Slope of normal = 1/3

Equation of normal $y - 2 = \frac{1}{3}(x + 2) \Rightarrow 3y - 6 = x + 2$

Hence, $x - 3y + 8 = 0$

ILLUSTRATION 6

If straight line $x\cos\alpha + y\sin\alpha = p$, touches the curve $x^m\,y^n = a^{m+n}$ then show that $p^{m+n}\,m^m\,n^n = (m + n)^{m+n}\,a^{m+n}$. $\sin^n\alpha.\cos^m\alpha$.

SOLUTION

Given curve is $x^m\,y^n = a^{m+n}$. $\qquad \ldots(1)$

Taking log, we get

$$m\log x + n\log y = (m + n)\log a$$

Differentiating, $\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = 0$

or, $\quad \frac{dy}{dx} = -\frac{m}{n}\,\frac{y}{x}$

∴ Equation of the tangent line at point (x_1, y_1) is,

$$y - y_1 = -\frac{m}{n}\frac{y_1}{x_1}(x - x_1)$$

or, $my_1x + nx_1y = (m + n)\, x_1y_1$ …(2)

According to given condition, equation (2) and the equation $x \cos \alpha + y \sin \alpha = p$ represent same line.

By taking ratios of coefficients

$$\frac{my_1}{\cos\alpha} = \frac{nx_1}{\sin\alpha} = \frac{(m+n)x_1y_1}{p}$$

$$\Rightarrow \quad x_1 = \frac{mp}{(m+n)\cos\alpha},\ y_1 = \frac{np}{(m+n)\sin\alpha}$$

The point on curve i.e., (x_1, y_1)

$$= \left(\frac{mp}{(m+n)\cos\alpha}, \frac{np}{(m+n)\sin\alpha}\right)$$

Will satisfy the equation of curve (1) .

$$\Rightarrow \quad \left(\frac{mp}{(m+n)\cos\alpha}\right)^m \left(\frac{np}{(m+n)\sin\alpha}\right)^n = a^{m+n}$$

Hence, $p^{m+n}\, m^m\, n^n = (m + n)^{m+n}\, a^{m+n} \sin^n\alpha \cos^m\alpha.$

ILLUSTRATION 7

Write equation of tangents on curve $y = \cos(x + y)$ where $-2\pi \le x \le 2\pi$ that is parallel to line $x + 2y = 0$.

SOLUTION

Given curve $y = \cos(x + y)$.

On differentiation,

$$\frac{dy}{dx} = -\sin(x+y)\left[1 + \frac{dy}{dx}\right]$$

$$\Rightarrow \quad \frac{dy}{dx}(1 + \sin(x + y)) = -\sin(x + y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-\sin(x+y)}{1+\sin(x+y)}$$

Tangent is parallel to given line $x + 2y = 0$.

$$\frac{-\sin(x+y)}{1+\sin(x+y)} = -\frac{1}{2} \qquad (\because \text{ parallel lines } m_1 = m_2)$$

$\Rightarrow \quad \sin(x + y) = 1$ or $\cos(x + y) = 0$

Curve $y = \cos(x + y) = 0$; then $\sin x = 1$ where $x \in [-2\pi, 2\pi]$

Acceptable solutions are $x = \dfrac{-3\pi}{2}, \dfrac{\pi}{2}$.

At point $A\left(\dfrac{-3\pi}{2}, 0\right)$, equation of tangent

$$y - 0 = -\frac{1}{2}\left(x + \frac{3\pi}{2}\right) \Rightarrow 4y + 2x + 3\pi = 0$$

At point $B\left(\dfrac{\pi}{2}, 0\right)$, equation of tangent

$$y - 0 = \frac{-1}{2}\left(x - \frac{\pi}{2}\right) \Rightarrow 4y + 2x - \pi = 0$$

ILLUSTRATION 8

Find area of triangle whose three sides are x-axis and tangent and normal to curve $y(2a - x) = x^2$ at point (a, a).

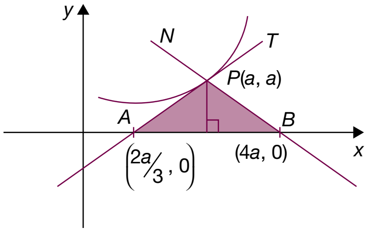
SOLUTION

Given curve $y(2a - x) = x^2$

On differentiation $\dfrac{dy}{dx} \cdot (2a - x) + y(0 - 1) = 2x$

At point $P(a, a)$, put $x = y = a$,

$$\frac{dy}{dx}(a) - a = 2a \Rightarrow \frac{dy}{dx} = 3 = m_T = \text{Slope of tangent}$$



Slope of normal $= m_N = -\dfrac{1}{3}$

Tangent $y - a = 3(x - a) \Rightarrow y = 3x - 2a$

Normal $y - a = -\dfrac{1}{3}(x - a) \Rightarrow x + 3y - 4a = 0$

Required area of $\Delta PAB = \dfrac{1}{2}$(base) height

$$= \frac{1}{2}\left(4a - \frac{2a}{3}\right)a = \frac{5a^2}{3} \text{ sq. units}$$

ILLUSTRATION 9

Line joining $A(0, 3)$ and $B(5, -2)$ is tangent to curve $y = \dfrac{c}{x+1}$. Find c.

SOLUTION

Calculus Method

Let (x_1, y_1) on the curve $y = \dfrac{c}{x+1}$ then $y_1 = \dfrac{c}{x_1+1}$ …(1)

Equation of AB $y = 3 - x$ is tangent to given curve, as point (x_1, y_1) also lies on it so $y_1 = 3 - x_1$.

From (1), we get $c = (3 - x_1)(x_1 + 1)$

$$\text{Slope of tangent} = \frac{dy}{dx} = \left.\frac{-c}{(x+1)^2}\right|_{x=x_1} = -1 = \text{Slope of } AB \qquad \text{…(2)}$$

$\Rightarrow \quad c = (x_1 + 1)^2 = (3 - x_1)(x_1 + 1)$

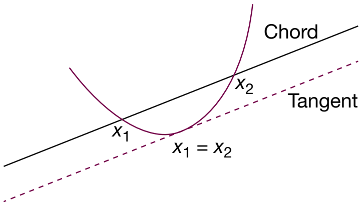
Hence, $x_1 \ne -1$; $x_1 + 1 = 3 - x_1$; $x_1 = 1$

From (2), we get $c = (1 + 1)^2 = 4$.

Analytical Method

Equation of AB $y-3=-1\,(x-0) \Rightarrow y=3-x$

Solving tangent AB with curve



$$y=\frac{c}{x+1}=3-x$$

$$\Rightarrow \quad c=(x+1)(3-x)$$

$$\Rightarrow \quad c=3x+3-x^2-x$$

$$\Rightarrow \quad x^2-2x+c-3=0$$

It is a quadratic equation in x, so tangency $x_1 = x_2$ which means discriminant $D=0 \Rightarrow B^2-4AC=0$

$$\Rightarrow \quad (-2)^2-4(1)(c-3)=0 \Rightarrow 4-4c+12=0$$

Hence, $c=4$

ILLUSTRATION 10

In interval [1, 2] and the tangent to the curve $y=f(x)=x^3-6ax^2+5x$ at $x=\frac{7}{4}$ is parallel to the chord joining the points of intersection of the curve with the ordinates $x=1$ and $x=2$, then find the value of a.

SOLUTION

$$f(x)=x^3-6ax^2+5x$$

$$f'(x)=3x^2-12ax+5$$

$$f'\left(\frac{7}{4}\right)=\frac{147}{16}-21a+5=\frac{227}{16}-21a$$

$$f(1)=6-6a,\ f(2)=18-24a$$

$$\because \quad \text{Slope of tangent at } x=\frac{7}{4}=f'\left(\frac{7}{4}\right)=\frac{f(2)-f(1)}{2-1}$$

(Slope of chord joining point $x=1$ and $x=2$)

$$\therefore \quad \frac{227}{16}-21a=\frac{(18-24a)-(6-6a)}{2-1}$$

$$\Rightarrow \quad \frac{227}{16}-21a=12-18 \Rightarrow a=\frac{35}{48}.$$

ILLUSTRATION 11

Find the equation of the normal to the curve $y=(1+x)^y+\sin^{-1}(\sin^2 x)$ at $x=0$.

SOLUTION

The given curve is $y=(1+x)^y+\sin^{-1}(\sin^2 x)$

Here, at $x=0$, $y=(1+0)^y+\sin^{-1}(0) \Rightarrow y=1$

$\therefore$ Point at which normal has been drawn is (0, 1).

Let $u=(1+x)^y$ …(1)

By taking log on both sides of equation (1), we get

$$\log u = y\log(1+x)$$

$$\Rightarrow \quad \frac{1}{u}\frac{du}{dx}=\frac{y}{1+x}+\log(1+x)\cdot\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{du}{dx}=(1+x)^y\left[\frac{y}{1+x}+\log(1+x)\frac{dy}{dx}\right]$$

Also, let $v=\sin^{-1}(\sin^2 x) \Rightarrow \frac{dv}{dx}=\frac{2\sin x\cos x}{\sqrt{1-\sin^4 x}}$

So, curve is $y=(1+x)^y+\sin^{-1}(\sin^2 x)$

$$y=u+v \Rightarrow \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}=(1+x)^y\left[\frac{y}{1+x}+\log(1+x)\frac{dy}{dx}\right]+\frac{2\sin x\cos x}{\sqrt{1-\sin^4 x}}$$

$$\Rightarrow \quad [1-(1+x)^y\log(1+x)]\frac{dy}{dx}=y(1+x)^{y-1}+\frac{2\sin x\cos x}{\sqrt{1-\sin^4 x}}$$

$$\text{Slope of tangent}=\frac{dy}{dx}=\left[\frac{y(1+x)^{y-1}+\dfrac{2\sin x\cos x}{\sqrt{1-\sin^4 x}}}{1-(1+x)^y\log(1+x)}\right]_{(0,1)}=1$$

$\therefore$ Slope of normal $=-1$

Hence, equation of normal to given curve at (0, 1) is

$$y-1=-1\,(x-0) \Rightarrow x+y=1.$$

ILLUSTRATION 12

Find the point(s) on the curve $y^3+3x^2=12y$, where the tangent is vertical.

SOLUTION

Differentiating the given curve about x, we get

$$3y^2\frac{dy}{dx}+6x=12\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=-\frac{2x}{y^2-4}$$

At point where the tangent(s) is (are) vertical, $\frac{dy}{dx}$ is not defined, i.e. at those points,

$$y^2-4=0 \Rightarrow y=\pm 2$$

When, $y=2, 8+3x^2=12(2) \Rightarrow 3x^2=16 \Rightarrow x=\pm\frac{4}{\sqrt{3}}$

When, $y=-2, -8+3x^2=-24 \Rightarrow 3x^2=-16$.

This is not possible.

Hence, the required points are $\left(\pm\frac{4}{\sqrt{3}}, 2\right)$.

TANGENT/NORMAL TO CURVE WITH PARAMETRIC CO-ORDINATES

Let curve $y = \psi(t)$ and $x = \phi(t)$,

So, the equation of tangent at any point $(t = t_0)$ is,

$$y - \psi(t_0) = \left(\frac{dy/dt}{dx/dt}\right)_{t=t_0} [x - \phi(t_0)]$$

or, $$y - \psi(t_0) = \left(\frac{dy}{dx}\right)_{t=t_0} (x - \phi(t_0))$$

Similarly, we can write equation of normal.

$$y - \psi(t_0) = -\frac{1}{(dy/dx)_{t=t_0}}(x - \phi(t_0))$$

ILLUSTRATION 13

Curve $x = a(\theta + \sin\theta)$; $y = a(1 + \cos\theta)$ at point $\theta = \pi/2$. Write equation of tangent/normal.

SOLUTION

At $\theta = \frac{\pi}{2}$, we get $x = a\left(\frac{\pi}{2}+1\right)$ and $y = a(1+0) = a$

Thus, point is $\left(a\left(\frac{\pi}{2}+1\right), a\right)$.

Slope of tangent, $m_T = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \left.\frac{a(0-\sin\theta)}{a(1+\cos\theta)}\right|_{\theta=\frac{\pi}{2}} = -1$

Equation of tangent

$$y - a = -1\left(x - a\left(\frac{\pi}{2}+1\right)\right) \Rightarrow x + y = a\left(\frac{\pi}{2}+2\right)$$

Slope of normal, $m_N = 1$

Equation of normal, $y - a = 1\left(x - a\left(\frac{\pi}{2}+1\right)\right) \Rightarrow y = x - a\frac{\pi}{2}$

ILLUSTRATION 14

Show that normal at any point on curve $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$ is at constant distance from origin.

SOLUTION

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a(\cos t - \cos t + t\sin t)}{a(-\sin t + \sin t + t\cos t)} = \tan t$$

Slope of tangent $= m_T = \tan t$

Slope of normal $= m_N = -\cot t$

Equation of Normal

$$y - a(\sin t - t\cos t) = -\cot t\,(x - a(\cos t + t\sin t))$$

$\Rightarrow \quad y + \cot t\; x = a\cot t\,(\cos t + t\sin t) + a\,(\sin t - t\cos t)$

$\Rightarrow \quad x\cot t + y = \dfrac{a(\cos^2 t + \sin^2 t)}{\sin t}$

$\Rightarrow \quad x\cos t + y\sin t = a$

Distance from origin to normal $x\cos t + y\sin t - a = 0$

$$= \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right| = \left|\frac{0+0-a}{\sqrt{\cos^2 + \sin^2 t}}\right| = a = \text{constant}$$

ILLUSTRATION 15

Consider $f(x) = \int_1^x \left(t + \frac{1}{t}\right)dt$ and $g(x) = f'(x)$ for $x \in \left[\frac{1}{2}, 3\right]$.

If P is a point on the curve $y = g(x)$ such that the tangent to this curve at P is parallel to a chord joining the points $\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)$ and $(3, g(3))$ of the curve, then find the coordinates of the point P.

SOLUTION

$$f(x) = \int_1^x \left(t + \frac{1}{t}\right)dt$$

$$g(x) = f'(x) = x + \frac{1}{x} \text{ for } x \in \left[\frac{1}{2}, 3\right]$$

$\Rightarrow \quad g(3) = 3 + \frac{1}{3} = \frac{10}{3};\ g\left(\frac{1}{2}\right) = \frac{1}{2} + 2 = \frac{5}{2}$

Slope of chord joining points $\left(\frac{1}{2}, g\left(\frac{1}{2}\right)\right)$ and $(3, g(3))$

$$= \frac{g(3) - g\left(\frac{1}{2}\right)}{3 - \frac{1}{2}} = \frac{\frac{10}{3} - \frac{5}{2}}{3 - \frac{1}{2}} = \frac{1}{3}$$

Slope of tangent $g'(x) = 1 - \frac{1}{x_1^2} = \frac{1}{3} \Rightarrow x_1 = \pm\sqrt{\frac{3}{2}}$

$$y = \sqrt{\frac{3}{2}} + \sqrt{\frac{2}{3}} = \frac{5}{\sqrt{6}}$$

Hence, required point $\left(\sqrt{\frac{3}{2}}, \frac{5}{\sqrt{6}}\right)$.

ILLUSTRATION 16

Let $f(x) = x^3 + x + 1$ and let $g(x)$ be its inverse function, then find equation of the tangent to $y = g(x)$ at $x = 3$.

SOLUTION

Let $(3, \alpha)$ be the point on $y = g(x)$ then $(\alpha, 3)$ lies on $y = f(x) = \alpha^2 + \alpha + 1 = 3 \Rightarrow \alpha = 1$.

Point of tangency is (3, 1) on $y = g(x)$.

Also, $x = f(g(x))$

$\Rightarrow \quad 1 = f'(g(x))g'(x)$

$$\Rightarrow \quad g'(3)=\frac{1}{f'(g(3))}=\frac{1}{f'(1)}=\frac{1}{3x^2+1}\bigg|_{x=1}=\frac{1}{4}$$

Hence, tangent is $y-1=\frac{1}{4}(x-3)$ or $x-4y+1=0$

ILLUSTRATION 17

If $f(x)+f(y)=f(x)\,f(y)+f(xy)$, $f(1)=0$, $f'(1)=-2$, then find equation to the tangent, drawn to the curve $y=f(x)$ at $x=\sqrt{2}$.

SOLUTION

$$f(x)+f(y)=f(x)\,f(y)+f(xy)$$

Put $y=\dfrac{1}{x}\Rightarrow f(x)+f\left(\dfrac{1}{x}\right)=f(x)\,f\left(\dfrac{1}{x}\right)$ $\quad(\because f(1)=0)$

$\Rightarrow \quad f(x)=1\pm x^n$ $\quad(\because f'(1)=-2)$

Clearly, $f(x)=1-x^2;\ f'(x)=-2x$

Now, we want tangent at point $x=\sqrt{2}$, $y=-1$

Hence, required tangent at $\left(\sqrt{2},-1\right)$ is, $y+1=-2\sqrt{2}\left(x-\sqrt{2}\right)$

ILLUSTRATION 18

Find the slope of the straight line which is both tangent and normal to the curve $4x^3=27y^2$.

SOLUTION

Given curve $4x^3=27y^2$,

We can assume point $x=3t^2$, $y=2t^3$, then

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{6t^2}{6t}=t$$

The tangent at point t, $y-2t^3=t(x-3t^2)$ or $y=tx-t^3$ …(1)

The normal at point t_1, $y-2t_1^3=-1/t_1(x-3t_1^2)$

or, $\quad y=-x/t_1+2t_1^3+3t_1$ …(2)

Eq. (1) and (2) are identical. By comparing, we get

$$-t^3=2t_1^3+3t_1,\ t_1=-\frac{1}{t}$$

Eliminating t_1, we get $t^6=2+3t^2$.

$\Rightarrow \quad t^2=2,\ t=\pm\sqrt{2}$

Hence, the slope of the straight line which is both tangent and normal to the curve $=\pm\sqrt{2}$.

Test Your Concepts-I

(Solutions on page H.148)

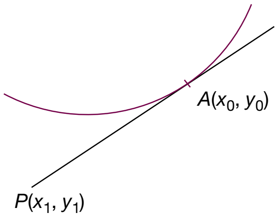
1. Curve; $y=\dfrac{x^3-x}{x^2+1}$ at point $x=1$. Write equation of tangent/normal.
2. Find equation of tangent to the curve $y=be^{-x/a}$ at the point where curve crosses y-axis.
3. Find The slope of the tangent and the normal to the curve $x^2+3y+y^2=5$ at $(1, 1)$.
4. Find the point on the curve $y=x^3-11x+5$ at which the tangent has the equation $y=x-11$.
5. The graph of the function $y=f(x)$ has a unique tangent at $\left(e^a,0\right)$ through which the graph passes, then find $\lim\limits_{x\to e^a}\dfrac{\log(1+7f(x))-\sin(f(x))}{3f(x)}$.
6. If $x+4y=14$ is normal to the curve $y^2=\alpha x^3-\beta$ at $(2,3)$, then find the value of $\alpha+\beta$.
7. Find the slope of the tangent to the curve represented by $x=t^2+3t-8$ and $y=2t^2-2t-5$ at the point $M(2,-1)$.
8. The curve $x+y-\ln(x+y)=2x+5$ has a horizontal normal at the point (α,β), then find $\alpha+\beta$.
9. The equation of normal to $y=x\log x$ is parallel to $2x-2y+3=0$, then find the equation of the normal.
10. Find the area bounded by the axes of reference and the normal to $y=\log x$ at the point $(1,0)$.
11. If the normal to the curve $y=f(x)$ at the point (3, 4) makes an angle $\dfrac{3\pi}{4}$ with the positive x-axis, then find $f'(3)$.
12. The tangent to the curve $y=\dfrac{1+3x^2}{3+x^2}$ drawn at the points for which $y=1$, intersect at point P. Find point P.
13. Find the value of parameter t so that the line $(4-t)x+y+(a^3-1)=0$ is normal to the curve $xy=1$.
14. The parametric equations of a curve are $x=t^2$ and $y=t^3$. $A(t_1)$, $B(t_2)$ are points on the curve. If $t_1=1$, $t_2=3$, then find the abscissa of the point P on the curve the tangent at which is parallel to chord AB.

Answer key

1. $x - y - 1 = 0; x + y - 1 = 0$
2. $\frac{x}{a} + \frac{y}{b} = 1$
3. $\frac{-2}{5}, \frac{5}{2}$
4. (2, –9)
5. 2
6. 9
7. 6/7
8. 1
9. $x - y = 3e^{-2}$
10. $\frac{1}{2}$
11. 1
12. (0, 0)
13. $(-\infty, 0) \cup (4, \infty)$
14. $\frac{169}{36}$

TANGENT AND NORMAL ON CURVE FROM EXTERNAL POINT $P(x_1, y_1)$

We need to find tangent and normal drawn from point $P(x_1, y_1)$ which is not on curve $y = f(x)$.



(i) Let assume a point $A(x_0, y_0)$ on given curve $y = f(x)$, as point A lies on it, we have $y_0 = f(x_0)$ ...(1)

(ii) Slope of tangent $= m_T = \left(\frac{dy}{dx}\right)_{(x_0, y_0)}$ and

Slope of normal $= m_N = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_0, y_0)}}$

Equation of tangent $y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$,

Equation of normal $y - y_0 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_0, y_0)}} (x - x_0)$

(iii) Given point (x_1, y_1) lies on tangent/normal, hence, satisfies it.

$$y_l - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x_1 - x_0) \text{ and}$$

$$y_1 - y_0 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_0, y_0)}} (x_1 - x_0) \qquad \text{...(2)}$$

Now, solve (1) and (2), we get point $(x_0\ y_0)$ and equation of tangent and normal.

ILLUSTRATION 19

Write equation of tangent drawn from origin to curve $y = (x + 1)^3$.

SOLUTION

Let $A(x_0, y_0)$ on curve $y_0 = (x_0 + 1)^3$ ...(1)

$$m_T = \frac{dy}{dx} = 3(x+1)^2 \Big|_{x = x_0} = 3(x_0 + 1)^2$$

Tangent $y - y_0 = 3(x_0 + 1)^2 (x - x_0)$

Origin (0, 0) lies on it, we get

$$0 - y_0 = 3(x_0 + 1)^2 (0 - x_0) \Rightarrow y_0 = 3x_0(x_0 + 1)^2 \qquad \text{...(2)}$$

Solving equation (1) and (2), we get

$$y_0 = (x_0 + 1)^3 = 3x_0(x_0 + 1)^2$$

$$\Rightarrow \quad (x_0 + 1)^2(x_0 + 1 - 3x_0) = 0 \Rightarrow x_0 = -1; x_0 = \frac{1}{2}$$

As $x_0 = -1$; $y_0 = 0$ point $A(-1, 0)$

$$m_T = 3(x_0 + 1)^2 = 0$$

Tangent $y - 0 = 0(x + 1) \Rightarrow y = 0$

As $x_0 = \frac{1}{2}$; $y_0 = \frac{27}{8}$, we get point $B\left(\frac{1}{2}, \frac{27}{8}\right)$

$$m_T = 3(x_0 + 1)^2 = 3\left(\frac{1}{2} + 1\right)^2 = \frac{27}{4}$$

So, tangent $y = \frac{27x}{4}$

Hence, from origin possible tangents are $y = 0$, $y = \frac{27x}{4}$

ILLUSTRATION 20

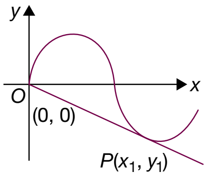
Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on curve $x^2y^2 = (x^2 - y^2)$.

SOLUTION

Given curve is $y = \sin x$; $\frac{dy}{dx} = \cos x$

If $P(x_1, y_1)$ be the point of contact, then the equation of the tangent will be

$$y - y_1 = \cos x_1 (x - x_1) \qquad \text{...(2)}$$



This line will pass through the origin (0, 0)

$\Rightarrow \quad y_1 = x_1 \cos x_1$ …(3)

Since, $P(x_1, y_1)$ is a point on curve, therefore $y_1 = \sin x_1$ …(4)

From eq. (3) and (4), we get

$$\left(\frac{y_1}{x_1}\right)^2 + y_1^2 = \cos^2 x_1 + \sin^2 x_1.$$

or, $\quad y_1^2 + x_1^2 y_1^2 = x_1^2.$

Hence, locus of point (x_1, y_1) is $x^2y^2 = x^2 - y^2$.

Assuming a point on curves in single parameter θ, as it always helps in calculation.

Table 6.1 : Equations and their points of contact

Equation	Point of contact	Equation	Point of contact
$x^2 + y^2 = a^2$	$P(a\cos\theta, a\sin\theta)$	$x^2 - y^2 = a^2$	$P(a\sec\theta, a\tan\theta)$
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$P(a\cos\theta, b\sin\theta)$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$P(a\sec\theta, b\tan\theta)$
$x^{2/3} + y^{2/3} = a^{2/3}$	$P(a\cos^3\theta, a\sin^3\theta)$	$x^{2/3} - y^{2/3} = a^{2/3}$	$P(a\sec^3\theta, a\tan^3\theta)$
$x^4 + y^4 = a^4$	$P(a\sqrt{\cos\theta}, a\sqrt{\sin\theta})$	$x^4 - y^4 = a^4$	$P(a\sqrt{\sec\theta}, a\sqrt{\tan\theta})$
$\sqrt{x} + \sqrt{y} = \sqrt{a}$	$P(a\cos^4\theta, a\sin^4\theta)$	$\sqrt{x} - \sqrt{y} = \sqrt{a}$	$P(a\sec^4\theta, a\tan^4\theta)$
$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{a^2}$	$P(a\sec\theta, a\operatorname{cosec}\theta)$	$\frac{1}{x^2} - \frac{1}{y^2} = \frac{1}{a^2}$	$P(a\cos\theta, a\cot\theta)$

ILLUSTRATION 21

Show that segment of tangent to the curve $x^{2/3} + y^{2/3} = a^{2/3}$, intercepted between coordinate axes is constant.

SOLUTION

Let $P(a\cos^3\theta, a\sin^3\theta)$ on given curve $x^{2/3} + y^{2/3} = a^{2/3}$.

On differentiation $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0.$

$$\Rightarrow \quad \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{1/3}$$

Slope of tangent at point $P = \frac{dy}{dx} = -\left(\frac{a\sin^3\theta}{a\cos^3\theta}\right)^{\frac{1}{3}} = -\tan\theta$

Equation of Tangent $y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$

$\Rightarrow \quad y + \tan\theta\, x = a\sin^3\theta + a\tan\theta\cos^3\theta = a\sin\theta(\sin^2\theta + \cos^2\theta)$

$$\Rightarrow \quad \frac{x}{a\cos\theta} + \frac{y}{a\sin\theta} = 1$$

x-intercept of tangent $= OA = a\cos\theta$;

y-intercept of tangent $= OB = a\sin\theta$

Length of segment of tangent which is intercepted between Coordinate axes $= \ell = AB = \sqrt{OA^2 + OB^2}$

Hence, $\ell = \sqrt{a^2\cos^2\theta + a^2\sin^2\theta} = a =$ constant

Hence proved.

ILLUSTRATION 22

Prove that sum of intercepts on coordinates axes cut by tangents drawn at any point on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is constant.

SOLUTION

Let point $P(a\cos^4\theta, a\sin^4\theta)$ on curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

On differentiation $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0.$

$$\Rightarrow \quad \frac{dy}{dx} = -\sqrt{\frac{y}{x}}\Bigg|_P = -\sqrt{\frac{a\sin^4\theta}{a\cos^4\theta}} = -\tan^2\theta$$

Equation of tangent $y - a\sin^4\theta = -\tan^2\theta\,(x - a\cos^4\theta)$

$\Rightarrow \quad \tan^2\theta\, x + y = a\sin^2\theta\,(\sin^2\theta + \cos^2\theta)$

$$\Rightarrow \quad \tan^2\theta\, x + y = a\sin^2\theta \Rightarrow \frac{x}{a\cos^2\theta} + \frac{y}{a\sin^2\theta} = 1$$

x-intercept of tangent $= OA = a\cos^2\theta$;

y-intercept of tangent $= OB = a\sin^2\theta$

Hence, sum of intercepts on coordinate axes

$$= OA + OB = a\cos^2\theta + a\sin^2\theta = a = \text{constant}$$

Hence proved.

ILLUSTRATION 23

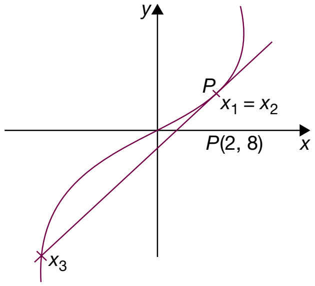
Tangent drawn to the curve $y = x^3$ at point $P(2, 8)$ meets the curve again at point Q. Find Q.

SOLUTION

Point $P(2, 8)$ lies on curve $y = x^3$

$$\frac{dy}{dx} = 3x^2\Big|_{x=2} = 12 = m_T$$

Equation of tangent $y - 8 = 12(x - 2) \Rightarrow y = 12x - 16$



As given tangent at P cuts curve again at point Q

Now, solving tangent with curve $y = x^3$;

$$y = x^3 = 12x - 16$$

$$\Rightarrow \quad x^3 - 12x + 16 = 0 \qquad \ldots(1)$$

$$\Rightarrow \quad x^3 + 0.x^2 - 12x + 16 = 0 \text{ has rots } x_1, x_2, x_3$$

Since, $x_1 + x_2 + x_3 = 0$

At point of tangency $x_1 = x_2 = 2$

$$\therefore \quad 2 + 2 + x_3 = 0 \Rightarrow x_3 = -4;\ y_3 = x_3^{\,3} = -64$$

Point $Q(-4, -64)$

We can also find point P and Q by factor.
From equation (1), we have

$$x^3 - 12x + 16 = 0 \Rightarrow (x - 2)^2\,(x + 4) = 0$$

Hence, roots are $x = 2, 2, 4$. It confirms that at point of tangency roots get repeated.

ILLUSTRATION 24

Curve $x = \sec^2 t$, $y = \cot t$. If tangent at $P\left(t = \dfrac{\pi}{4}\right)$ meets curve again at point Q. Find $|PQ|$.

SOLUTION

When $t = \dfrac{\pi}{4}$, we have $x = \sec^2\dfrac{\pi}{4} = (\sqrt{2})^2 = 2;\ y = \cot\dfrac{\pi}{4} = 1$

Thus, point $P\ (2, 1)$.

$$\text{Slope of tangent} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{-\operatorname{cosec}^2 t}{2\sec t \cdot \sec t \tan t}\Bigg|_{t=\frac{\pi}{4}} = -\frac{1}{2}$$

$$\text{Tangent } y - 1 = -\frac{1}{2}(x - 2) \Rightarrow x + 2y = 4 \qquad \ldots(1)$$

Curve $x = \sec^2 t$ and $y = \cot t$

By eliminating t, we get $x = 1 + \dfrac{1}{y^2} \Rightarrow (x - 1)y^2 = 1$

Solving tangent with curve, we have $(x - 1)\left(\dfrac{4 - x}{2}\right)^2 = 1$

$$\Rightarrow \quad (x - 1)\,(x^2 - 8x + 16) = 4$$

$$\Rightarrow \quad x^3 - 9x^2 + 24x - 20 = 0 \text{ has roots } x_1, x_2, x_3$$

Point of tangency is $P(2, 1)$ then $x_1 = x_2 = 2$

$$x_1 + x_2 + x_3 = 9 \Rightarrow 2 + 2 + x_3 = 9 \Rightarrow x_3 = 5$$

From (1), we have $x_3 + 2y_3 = 4$, we get $y_3 = -\dfrac{1}{2}$

Required point $Q\left(5, -\dfrac{1}{2}\right)$ and $P(2, 1)$

Hence, distance $PQ = \dfrac{3\sqrt{5}}{2}$

ILLUSTRATION 25

If tangent to curve $y^2 = x(2 - x)^2$ at point $M(1, 1)$ meets the curve again at point N. Find N.

SOLUTION

Curve $y^2 = x(2 - x)^2$

On differentiation,

$$2y\frac{dy}{dx} = (2 - x)^2 - x \cdot 2(2 - x)$$

At point $x = y = 1$, $2(1)\ m_T = 1 - 2(1) = -1$

Slope of tangent, $m_T = -\dfrac{1}{2}$

$$\text{Tangent } y - 1 = -\frac{1}{2}(x - 1) \Rightarrow x + 2y - 3 = 0 \qquad \ldots(1)$$

or, $\quad y = \dfrac{3 - x}{2}$ solving it with curve $y^2 = x(2 - x)^2$

$$\Rightarrow \quad \left(\frac{3 - x}{2}\right)^2 = x(x^2 + 4 - 4x)$$

$$\Rightarrow \quad x^2 + 9 - 6x = 4(x^3 + 4x - 4x^2)$$

$$\Rightarrow \quad 4x^3 - 17x^2 + 22x - 9 = 0 \text{ has roots } x_1, x_2, x_3$$

$P(1, 1)$ is point of tangency so $x_1 = x_2 = 1$

Sum of the roots $= x_1 + x_2 + x_3 = \frac{17}{4} \Rightarrow 1 + 1 + x_3 = \frac{17}{4}$

$\Rightarrow \quad x_3 = \frac{9}{4}$

From (1) we have $y_3 = \frac{3 - x_3}{2} = \frac{3}{8}$

Hence, required point $Q\left(\frac{9}{4}, \frac{3}{8}\right)$.

ILLUSTRATION 26

Tangent at a point P_1 {other than (0, 0)} on the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3, and so on. Show that the abscissa of P_1, P_2, P_3, …, P_n, form a GP. Also, find the ratio $\frac{Ar(\Delta P_1P_2P_3)}{Ar(\Delta P_2P_3P_4)}$.

SOLUTION

The given curve is $y = x^3$ …(1)

Let the point P_1 be (t, t^3), $t \neq 0$

Then, slope of tangent at P_1 is $\frac{dy}{dx} = (3x^2)_{x=t} = 3t^2$

$\therefore$ Equation of tangent at P_1 is $y - t^3 = 3t^2 (x - t)$

$\Rightarrow \quad 3t^2x - y - 2t^3 = 0$ …(2)

Now, this tangent meets the curve again at P_2 which can be obtained by solving (1) and (2)

i.e., $3t^2x - x^3 - 2t^3 = 0$ or $x^3 - 3t^2x + 2t^3 = 0$

$(x - t)^2 (x + 2t) = 0 \Rightarrow x = -2t$ is for point P_2 as $x = t$ is for P_1

$\therefore \quad y = -8t^3$

We get point P_2 is $(-2t, -8t^2) = (t_1, t_1^3)$ say.

Similarly, we can find that tangent at P_2 meets the curve again at P_3 $(2t_1, 8t_1^3)$ i.e., $(4t, 64t^3)$.

Similarly, $P_4 \equiv (-8t, -512t^3)$ and so on.

We observe that abscissa of pts. P_1, P_2, P_3, … are $t, -2t, 4t$, … which form a GP with common ratio –2. Also ordinates of these pts. $t^3, -8t^3, 64t^3$, …. Also form a GP with common ratio –8.

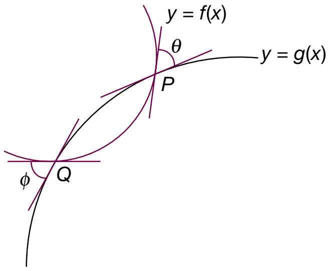
$$\text{Now, } \frac{Ar(\Delta P_1P_2P_3)}{Ar(\Delta P_2P_3P_4)} = \frac{\begin{vmatrix} 1 & t & t^3 \\ 1 & -2t & -8t^3 \\ 1 & 4t & 64t^3 \end{vmatrix}}{\begin{vmatrix} 1 & -2t & -8t^3 \\ 1 & 4t & 64t^3 \\ 1 & -8t & -512t^3 \end{vmatrix}}$$

$$= \frac{t^4\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -8 \\ 1 & 4 & 64 \end{vmatrix}}{(-2)(-8)t^4\begin{vmatrix} 1 & 1 & 1 \\ 1 & -2 & -8 \\ 1 & 4 & 64 \end{vmatrix}} = \frac{1}{16} \text{ sq. units}$$

ANGLE BETWEEN INTERSECTING CURVES

Let consider two curves $y = f(x)$ and $y = g(x)$ and they intersect at points $P(x_1, y_1)$ and $Q(x_2, y_2)$

Angle between two intersecting curves = Angle between tangents of given curves at point of intersection = Angle between normal of given curves at point of intersection



Curve $y = f(x)$ and $y = g(x)$

Point $P(x_1, y_1)$ is point of intersection of given curves

$$m_1 = \left.\frac{df(x)}{dx}\right|_{(x_1, y_1)}, \; m_2 = \left.\frac{dg(x)}{dx}\right|_{(x_1, y_1)},$$

Angle between intersecting curves $= \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

Note

Curves can intersect at different angles at different point of intersections P and Q.

ORTHOGONAL CURVES

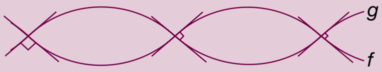
When two curves are intersecting at right angle at each point of intersection

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right| \text{ where } \theta = 90°$$

(i) For finite slopes, $m_1m_2 = -1$

(ii) If $m_1 = 0$ and $m_2 \to \pm\infty$ then also curves are orthogonal.

Two curves intersecting at right angles at each point of intersection then only they are considered as orthogonal curves.



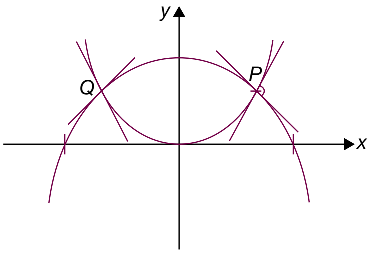
Orthogonal

ILLUSTRATION 27

Curves $y = x^2$ and $y = 8 - x^2$. Find angle between curves.

SOLUTION

First we should find point of intersection of given curves.



$$y = x^2 = 8 - x^2$$

$$\Rightarrow \quad x^2 = 4 \Rightarrow x = \pm 2 \Rightarrow y = x^2 = 4$$

Point of intersection are $P(2, 4)$ and $Q(-2, 4)$.
Both curves are symmetric about y axis so angle between curves is same for either of point.
Curve $y = x^2$

At point $P(2, 4)$, slope of tangent $m_1 = \dfrac{dy}{dx} = 2x\Big|_{x=2} = 4$

Curve $y = 8 - x^2$

At point $P(2, 4)$, slope of tangent $m_2 = \dfrac{dy}{dx} = -2x\Big|_{x=2} = -4$

Angle between curves at point P and Q, we get

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{4 - (-4)}{1 - 16}\right| = \frac{8}{15}$$

Hence, given curves are intersecting each other at an angle $\theta = \tan^{-1}\left(\dfrac{8}{15}\right)$ at points P and Q.

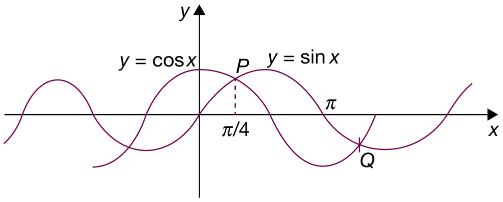
ILLUSTRATION 28

Curves $y = \sin x$, $y = \cos x$. Find angle between curves.

SOLUTION

Both given curves are periodic and they intersect at infinite points. Angle between curves remains same at each point of intersection due to symmetry.

Point $P\left(\dfrac{\pi}{4}, \dfrac{1}{\sqrt{2}}\right)$ is point of intersection of given curves in first quadrant.



We are going to use Point P for computation of angle between curves.
Curve $y = \sin x$

Slope of tangent at $P = m_1 = \dfrac{dy}{dx} = \cos x\Big|_{x=\frac{\pi}{4}} = \dfrac{1}{\sqrt{2}}$

Curve $y = \cos x$

Slope of tangent at $P = m_2 = \dfrac{dy}{dx} = -\sin x\Big|_{x=\frac{\pi}{4}} = -\dfrac{1}{\sqrt{2}}$

Angle between curves, $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$

$$= \left|\frac{\frac{1}{\sqrt{2}} - \left(-\frac{1}{\sqrt{2}}\right)}{1 - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}}\right| = \left|\frac{\sqrt{2}}{1 - \frac{1}{2}}\right| = 2\sqrt{2}$$

Hence, required angle between given curves is $\theta = \tan^{-1} 2\sqrt{2}$

ILLUSTRATION 29

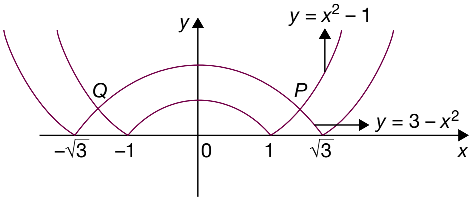
Find angle between curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$.

SOLUTION

For point of intersection, solving both given curves,

$$y = x^2 - 1 = 3 - x^2$$

$$\Rightarrow \quad x^2 = 2 \Rightarrow x = \pm\sqrt{2}$$



Point of intersections are $P\left(\sqrt{2}, 1\right)$ and $Q\left(-\sqrt{2}, 1\right)$.

As we can see from graph that given curves are symmetric about y-axis and intersect at point P and Q.
By symmetry angle of intersection should be same at both point P and Q.

Curve $y = x^2 - 1$

$\dfrac{dy}{dx} = 2x\Big|_{x=\sqrt{2}} = 2\sqrt{2} = m_1$

Curve $y = 3 - x^2$

$\dfrac{dy}{dx} = m_2 = -2x\Big|_{x=\sqrt{2}} = -2\sqrt{2}$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{2\sqrt{2} - \left(-2\sqrt{2}\right)}{1 + 2\sqrt{2}\left(-2\sqrt{2}\right)}\right| = \frac{4\sqrt{2}}{7}$$

Hence, given curves are intersecting each other at an angle $\theta = \tan^{-1}\left(\dfrac{4\sqrt{2}}{7}\right)$ at points P and Q.

ILLUSTRATION 30

If the angle of intersection of the curves

$y = [\,|\sin x| + |\cos x|\,]$ and $x^2 + y^2 = 5$, where [.] denotes the greatest integer function is θ, then find the value of $\tan\theta$.

SOLUTION

$$y = [\,|\sin x| + |\cos x|\,] \text{ has period } T = \frac{\pi}{2}$$

$$y = [\sin x + \cos x]$$

As $x \in \left[0, \frac{\pi}{2}\right]$; $1 \le \sin x + \cos x \le \sqrt{2}$

$\therefore \quad [\sin x + \cos x] = 1$

Given curves $y = 1$ and $x^2 + y^2 = 5 \Rightarrow x = \pm 2$

At point (±2, 1)

$$m_1 = \frac{dy}{dx} = 0 \text{ and } 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -2 = m_2$$

Hence, angle between given curves = $\tan^{-1} 2$.

ILLUSTRATION 31

Compute angle between curves $y = \dfrac{4e^x + 2e^{-x}}{3}$ and $y = e^{-x}$.

SOLUTION

Point of intersection of given curves $y = \dfrac{4e^x + 2e^{-x}}{3} = e^{-x}$

$$\Rightarrow \quad 4e^x + 2e^{-x} = 3e^{-x} \Rightarrow e^{2x} = \frac{1}{4}$$

$$x = -\ln 2;\ y = e^{\ln 2} = 2$$

Point of intersection (–ln 2, 2)

Curves $y = \dfrac{4e^x + 2e^{-x}}{3}$

Slope of tangent $m_1 = \dfrac{dy}{dx} = \dfrac{4e^x - 2e^{-x}}{3}\Bigg|_{x=-\ln 2} = \dfrac{4e^{-\ln 2} - 2e^{\ln 2}}{3}$

$$= \frac{4\left(\frac{1}{2}\right) - 2(2)}{3} = -\frac{2}{3}$$

Slope of tangent for second curve

$$m_2 = \frac{dy}{dx} = -e^{-x}\Big|_{x=-\ln 2} = -e^{\ln 2} = -2$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{2}{3} + 2}{1 + \frac{4}{3}}\right| = \frac{4}{7}$$

Hence, angle between given curves is $\tan^{-1}\left(\dfrac{4}{7}\right)$.

ILLUSTRATION 32

Curves $x^2 = 4y$, $y^2 = 4x$, are they orthogonal?

SOLUTION

For point of intersection, solving given curves $y = \dfrac{x^2}{4}$ and $y^2 = 4x$

$$\left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow \frac{x^4}{16} = 4x \Rightarrow x^4 = 64x \text{ we get } x = 0, 4$$

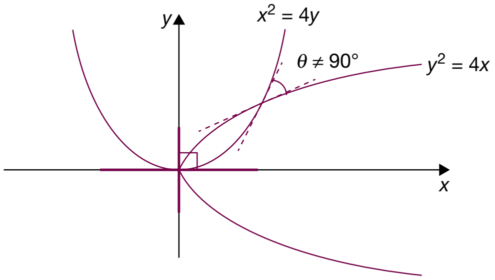
Point of intersection are $P(0, 0)$ and $Q(4, 4)$.

$$x^2 = 4y \Rightarrow \frac{dy}{dx} = \frac{x}{2}\Bigg|_{(0,0)} = 0, \text{ It is } x\text{-axis,}$$

$$y^2 = 4x \Rightarrow 2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}\Bigg|_{(0,0)} \to \infty$$

It is y axis,

Clearly, angle between curves at origin is $\theta = 90°$.



Lets compute angle at point $Q(4, 4)$

$$m_1 = \frac{dy}{dx} = \frac{x}{2}\Bigg|_{(4,4)} = \frac{4}{2} = 2$$

$$m_2 = \frac{dy}{dx} = \frac{2}{y}\Bigg|_{(4,4)} = \frac{4}{2} = \frac{1}{2}$$

At point Q, $m_1 m_2 = 2\left(\dfrac{1}{2}\right) = 1 \ne -1$.

So, angle between curves at point Q is $\theta \ne 90°$.
We can conclude that at points, curves are intersecting at right angle but at point Q they are not intersecting at 90°, it means these curves are not orthogonal.

ILLUSTRATION 33

If curves $2x = y^2$ and $2xy = k$ are orthogonal. Find k^2.

SOLUTION

Curve $2x = y^2$

On differentiation $2 = 2y\dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{y} = m_1$

Second curve $2xy = 1$, on differentiation

$$2y + 2x\frac{dy}{dx} = 0, \text{ we get } m_2 = \frac{dy}{dx} = -\frac{y}{x}$$

Let (x, y) is point of intersection of given curve,

For orthogonal curves, we have $m_1 m_2 = \left(\frac{1}{y}\right)\left(\frac{-y}{x}\right) = -1$

$\Rightarrow \quad x = 1$ and $y^2 = 2$

Hence, from second curve, $k^2 = 4x^2y^2 = 4(1)(2) = 8$

ILLUSTRATION 34

Curves $y^2 = bx$ and $ax^2 + by^2 = 1$ are orthogonal. Find relation in a and b?

SOLUTION

First curve $y^2 = bx$;

On differentiation $2yy' = b$

$$m_1 = \frac{dy}{dx} = \frac{b}{2y};$$

Second curve $ax^2 + by^2 = 1$

On differentiation $2ax + 2by\, y' = 0$

$$m_2 = \frac{dy}{dx} = -\frac{ax}{by}$$

As both curves are orthogonal, we get

$$m_1 m_2 = \left(\frac{b}{2y}\right)\left(\frac{-ax}{by}\right) = -1$$

$\Rightarrow \quad ax = 2y^2 \Rightarrow ax = 2(bx)$ using $y^2 = bx$

Hence, $a = 2b$

ILLUSTRATION 35

Find the angle of intersection between the curves $y = x^3$ and $6y = 7 - x^2$.

SOLUTION

To find the point of intersection, eliminate y.

Thus $6x^3 = 7 - x^2$, which is satisfied by $x = 1$. Then $y = 1$.

The other two values of x are imaginary. Hence, there is only one real point of intersection which is (1, 1).

Now $m_1 = \left(\frac{dy}{dx}\right) = 3x^2 = 3$ at $x = 1$

And $6\left(\frac{dy}{dx}\right) = -2x$ or $m_2 = \left(\frac{dy}{dx}\right) = -\frac{1}{3}$, at $x = 1$.

Since, $m_1\, m_2 = 3 \times \left(\frac{-1}{3}\right) = -1.$

Hence, the two curves cut orthogonally at (1, 1).

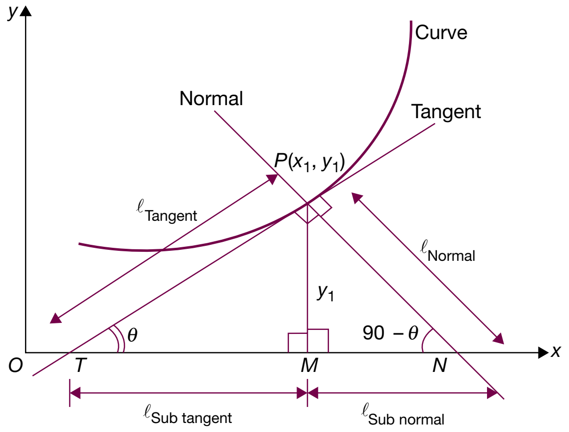
LENGTH OF TANGENT/NORMAL/SUB-TANGENT/SUB-NORMAL

Lets consider curve $y = f(x)$.

At point $P(x_1, y_1)$ tangent and normal are drawn to curve which cuts x-axis at point T and N respectively.

Length of tangent, normal, sub-tangent, sub-normal are shown in figure.

$$\text{Slope of tangent} = m = \frac{dy}{dx} = \left.\frac{df(x)}{dx}\right|_{P(x_1, y_1)} = \tan\theta$$



From the figure, $P(x_1, y_1)$ and $PM = y_1$

In ΔPTM

$$\sin\theta = \frac{PM}{PT} = \frac{|y_1|}{PT} \Rightarrow PT = |y_1|\operatorname{cosec}\theta$$

$$\tan\theta = \frac{PM}{TM} = \frac{y_1}{TM} \Rightarrow TM = |y_1|\cot\theta$$

In ΔPMN

$$\sin(90^\circ - \theta) = \frac{PM}{PN} = \cos\theta \Rightarrow PN = PM\sec\theta$$

$$\tan(90^\circ - \theta) = \frac{PM}{MN} = \cot\theta \Rightarrow MN = PM\tan\theta$$

Conceptual Note(s)

Tangent and normal are drawn at point (x_1, y_1) on curve then

$$l_{\text{Tangent}} = |y_1 \operatorname{cosec}\theta| = \left|y_1\sqrt{1 + \frac{1}{m^2}}\right|$$

$$l_{\text{Sub tangent}} = |y_1 \cot\theta| = \left|\frac{y_1}{m}\right|$$

$$l_{\text{Normal}} = |y_1 \sec\theta| = |y_1\sqrt{1 + m^2}|$$

$$l_{\text{Sub normal}} = |y_1 \tan\theta| = |y_1 m|$$

SHORTEST DISTANCE BETWEEN TWO CURVES

Shortest distance between two non-intersecting curves lie along their common normal wherever defined.

ILLUSTRATION 36

Curve $y = x^3 - x^2 + x$ at $x = -1$. Find length of tangent, sub tangent, normal, subnormal.

SOLUTION

At $x = -1$, $y = -1 - 1 - 1 = -3$ we get point $(-1, -3)$

Curve $y = x^3 - x^2 + x$

On differentiation,

$$\frac{dy}{dx} = 3x^2 - 2x + 1\,|_{x=-1} = 3 + 2 + 1 = 6 = m = \tan\theta$$

Length of tangent, $\ell_T = |y_1 \operatorname{cosec}\theta|$

$$= \left|y_1\sqrt{1+\frac{1}{m^2}}\right| = \left|3\sqrt{1+\frac{1}{36}}\right| = \frac{\sqrt{37}}{2}$$

Length of sub-tangent,

$$\ell_{ST} = |y_1 \cot\theta| = \left|\frac{y_1}{m}\right| = \left|\frac{3}{6}\right| = \frac{1}{2}$$

Length of normal,

$$\ell_N = |y_1 \sec\theta| = \left|y_1\sqrt{1+m^2}\right| = \left|3\sqrt{1+36}\right| = 3\sqrt{37}$$

Length of sub-normal, $\ell_{SN} = |y_1 \tan\theta| = |y_1 \times m| = 18$

ILLUSTRATION 37

Prove that for curve $x^a y^b = c^{a+b}$ at any point subtangent is proportional to abscissa of point of contact.

SOLUTION

Given curve $x^a y^a = c^{a+b}$

By taking log both side, we get

$$\ln(x^a y^b) = \ln c^{a+b} \Rightarrow a \ln x + b \ln y = (a+b)\ln c$$

$$\Rightarrow \quad a\left(\frac{1}{x}\right) + b\left(\frac{1}{y}\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{a}{b}\left(\frac{y}{x}\right) = m_T = \text{Slope of tangent}$$

Length of sub-tangent, $\ell_{ST} = y_1 \cot\theta = \dfrac{y}{m}$

$$= \frac{y}{\left(-\frac{a}{b}\frac{y}{x}\right)} = -\frac{b}{a}x$$

Cleary, we can see that $\ell_{ST} \propto x$

ILLUSTRATION 38

For curves $y = be^{x/a}$ show that subtangent is constant and sub normal is proportional to square of ordinate of point of contact.

SOLUTION

Curves $y = be^{x/a}$

On differentiation

$$\frac{dy}{dx} = \frac{b}{a}e^{\frac{x}{a}} = \frac{y}{a} = m = \text{Slope of tangent}$$

Length of sub-tangent, $\ell_{ST} = \dfrac{y}{m} = \dfrac{y}{\frac{y}{a}} = a = \text{constant}$

Length of sub-normal, $\ell_{SN} = ym = y\left(\dfrac{y}{a}\right) = \dfrac{y^2}{a} \propto y^2$

ILLUSTRATION 39

Find a point on parabola $y = x^2 + x + 1$ which is nearest form the line $y = x - 1$. Also find the shortest distance between them.

SOLUTION

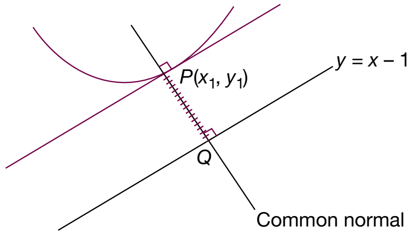
As we know that shortest distance lies along their common normal.

Let point $P(x_1, y_1)$ lies on given curve $y = x^2 + x + 1$.

So $\quad y_1 = x_1^2 + x_1 + 1 \qquad \ldots(1)$

For minimum distance tangent at P becomes parallel to given line $y = x - 1$.

$$y = x^2 + x + 1$$



Slope of tangent at $P = \dfrac{dy}{dx} = 2x + 1\big|_{x=x_1} = 1$

(slope of given line)

We get, $x_1 = 0$ and $y_1 = 0 + 0 + 1 = 1$

We get required point $P(0, 1)$ on curve now required shortest distance PQ = distance of point $P(0,1)$ from line

$$x - y - 1 = 0 = \left|\frac{0-1-1}{\sqrt{2}}\right| = \sqrt{2}.$$

ILLUSTRATION 40

If $(x-2)^2 + (y+1)^2 \le 4$, then find minimum of $(x+2)^2 + (y-1)^2$.

SOLUTION

$(x-2)^2 + (y+1)^2 \le 4$ it is an interior region of circle including circumference whose centre $C(2, -1)$ and radius $r = 2$.

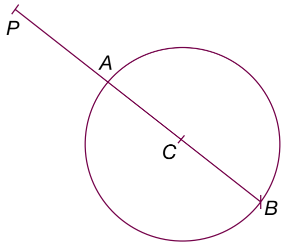
Let point $P(-2, 1)$ and $PC = \sqrt{16+4} = 2\sqrt{5}$.

Every normal of circle passes through centre.

Minimum of $(x+2)^2 + (y-1)^2$

$= $ Min of square of distance between (x, y) and $P(-2, 1)$

$$= PA^2 = (PC - r)^2 = \left(2\sqrt{5} - 2\right)^2$$



Maximum of $(x+2)^2 + (y-1)^2$

$= $ Max of square of distance between (x, y) and $P(-2, 1)$

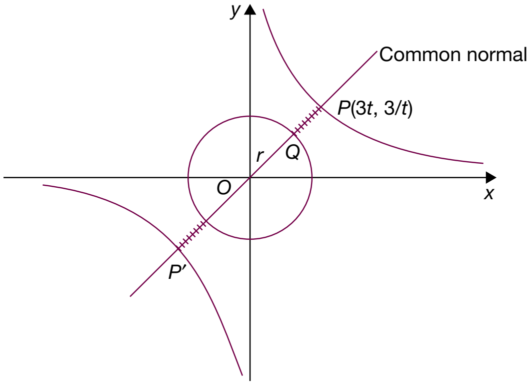
$$= PB^2 = (PC + r)^2 = \left(2\sqrt{5} + 2\right)^2$$

ILLUSTRATION 41

Find minimum distance between curve $xy = 9$ and circle $x^2 + y^2 = 1$.

SOLUTION

As we know that shortest distance lies along their common normal



Computing normal for curve $xy = 9$, $y = \frac{9}{x}$

Slope of tangent $m_T = \frac{dy}{dx} = -\frac{9}{x^2}\Big|_{x=3t} = -\frac{1}{t^2}$

Slope of normal, $m_N = t^2$

Normal at $P\left(3t, \frac{3}{t}\right)$, we get

$$y - \frac{3}{t} = t^2(x - 3t)$$

As it is a common normal for curve and circle.
Every normal of circle passes through centre C(0, 0)

$$0 - \frac{3}{t} = t^2(0 - 3t) \Rightarrow t^4 = 1 \Rightarrow t = \pm 1$$

We get, Point $P(3, 3)$ $P'(-3, -3)$.

Hence, minimum distance between given curves

$$= PQ = OP - r = 3\sqrt{2} - 1.$$

ILLUSTRATION 42

Find minimum value of $(x_1 - x_2)^2 + \left(\sqrt{8 - x_1^2} - 16/x_2\right)^2$.

SOLUTION

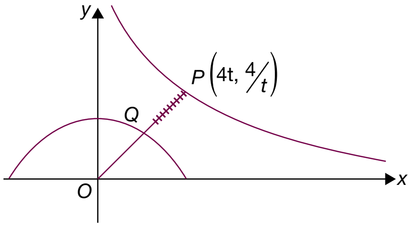
Let $y_1 = \sqrt{8 - x_1^2}$ and $y_2 = \frac{16}{x_2}$

Square of distance between points (x_1, y_1) and

$$(x_2, y_2) = (x_1 - x_2)^2 + \left(\sqrt{8 - x_1^2} - 16/x_2\right)^2$$

Further we can say that minimum value of

$(x_1 - x_2)^2 + \left(\sqrt{8 - x_1^2} - 16/x_2\right)^2$ = minimum distance between curves $x^2 + y^2 + = 8$ and $xy = 16$



Let point $P\left(4t, \frac{4}{t}\right)$ on curve $xy = 16$

$y = \frac{16}{x} \Rightarrow \frac{dy}{dx} = \frac{-16}{x^2}\Big|_{x=4t} = \frac{-1}{t^2} = m_T = $ Slope of tangent

Equation of Normal is $y - \frac{4}{t} = t^2(x - 4t)$ as it is a common normal for curve and circle.

Every normal of circle passes through centre C(0, 0).

$$0 - \frac{4}{t} = t^2(0 - 4t) \Rightarrow t^4 = 1 \Rightarrow t = 1 \text{ (acceptable)}$$

We get $P(4, 4)$, $OP = 4\sqrt{2}$, $OQ = R = 2\sqrt{2}$.

Minimum value of $(x_1 - x_2)^2 + \left(\sqrt{8 - x_1^2} - 16/x_2\right)^2$

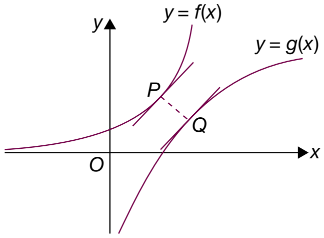
$$= PQ^2 = (OP - R)^2 = \left(4\sqrt{2} - 2\sqrt{2}\right)^2 = 8$$

ILLUSTRATION 43

Find the minimum distance between the curves $f(x) = e^x$ and $g(x) = \ln x$.

SOLUTION

$g(x) = \ln x$; Point $Q(x', \ln x')$ on curve $g'(x) = \frac{1}{x}$



Equation of normal $y - \ln x' = -x'(x - x')$

$$\Rightarrow \quad x'x + y = \ln x' + x'^2 \qquad \ldots(1)$$

$$f(x) = e^x;\ P(t, e^t) \text{ on curve } \frac{dy}{dx} = e^x$$

Equation of normal $y - e^t = -\frac{1}{e^t}(x - t)$

$$\Rightarrow \quad x + e^t y = t + e^{2t} \qquad \ldots(2)$$

Shortest distance lies along common normal.

Comparing eq. (1) and (2), we get

$$\frac{x'}{1} = \frac{1}{e^t} = \frac{\ln x' + x'^2}{t + e^{2t}}$$

$$\frac{1}{e^t} = \frac{\ln\left(\frac{1}{e^t}\right) + \frac{1}{e^{2t}}}{t + e^{2t}} = \frac{-t + \frac{1}{e^{2t}}}{t + e^{2t}} \Rightarrow t + e^{2t} = -te^t + e^{-t}$$

$\Rightarrow \quad t(1 + e^t) = e^{-t} - e^{2t}$, $t = 0$ is solution.

$$x' = \frac{1}{e^t} = 1$$

$$P(t, e^t) \equiv (0, 1);\ Q(x', \ln x') \equiv (1, 0)$$

Hence, shortest distance $PQ_{\min} = \sqrt{1+1} = \sqrt{2}$.

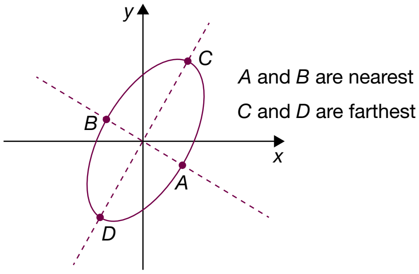
ILLUSTRATION 44

Find the coordinate of the points on the curve, $5x^2 - 6xy + 5y^2 = 4$ which are the nearest to the origin.

SOLUTION

$P(x_1, y_1)$ lies on curve $5x^2 - 6xy + 5y^2 = 4$

$$5x_1^2 - 6x_1 y_1 + 5y_1^2 = 4 \qquad \ldots(1)$$



Given curve is an ellipse with origin as centre, nearest and farthest both points lie on normal passes through centre.

On differentiating curve, $10x - 6y - 6xy' + 10y\,y' = 0$

$$y' = \frac{10x - 6y}{6x - 10y} = \frac{5x - 3y}{3x - 5y}$$

Normal, $y - y_1 = -\frac{(3x_1 - 5y_1)}{(5x_1 - 3y_1)}(x - x_1)$

It passes through origin.

$$0 - y_1 = -\frac{(3x_1 - 5y_1)}{(5x_1 - 3y_1)}(0 - x_1)$$

$$\Rightarrow \quad 5x_1y_1 - 3y_1^2 = -3x_1^2 + 5x_1y_1 \Rightarrow y_1 = \pm x_1$$

Solving with (1), we get

$$5x_1^2 - 6x_1y_1 + 5y_1^2 = 4 \Rightarrow 10x_1^2 - 6x_1(\pm x_1) = 4$$

We get $x_1 = \pm 1$, $y_1 = \pm 1$ and $x_1 = \pm\frac{1}{2}$, $y_1 = \pm\frac{1}{2}$

Hence, nearest point $\left(\frac{1}{2}, -\frac{1}{2}\right)\left(-\frac{1}{2}, \frac{1}{2}\right)$.

Alternate:

Let $P(r\cos\theta, r\sin\theta)$ any point whose distance from origin is r.

$P(r\cos\theta, r\sin\theta)$ lies on given curve

$$5x^2 - 6xy + 5y^2 = 4$$

$$\Rightarrow \quad 5r^2\cos^2\theta - 6r^2\cos\theta\sin\theta + 5r^2\sin^2\theta = 4$$

$$\Rightarrow \quad 5r^2 - 3r^2\sin 2\theta = 4$$

$$\Rightarrow \quad r^2 = \frac{4}{5 - 3\sin 2\theta}$$

$$\Rightarrow \quad r^2_{\min} = \frac{4}{5+3} = \frac{1}{2} \Rightarrow r_{\min} = \frac{1}{\sqrt{2}}$$

For $\sin 2\theta = -1$;

For $2\theta = -\frac{\pi}{2}, \frac{3\pi}{2}$ or $\theta = -\frac{\pi}{4}, \frac{3\pi}{4}$

Hence, points are $\left(\frac{1}{2}, -\frac{1}{2}\right)$ for $\theta = \frac{\pi}{4}$ and $\left(-\frac{1}{2}, \frac{1}{2}\right)$ for $\theta = \frac{3\pi}{4}$.

Test Your Concepts-II

(Solutions on page H.149)

1. Write equation of normal drawn from (1, 2) to curve $x^2 = 4y$.

2. If the tangent at (1, 1) on $y^2 = x(2-x)^2$ meets the curve again at P, then find point P.

3. Let the equation of a curve in parametric form be $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$. Find the angle between the tangent drawn at the point $\theta = \frac{\pi}{3}$ and normal drawn at the point $\theta = \frac{2\pi}{3}$.

4. Prove that If the tangent at any point of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ meets the axes of coordinates in A and B, then locus of mid point of AB is a circle.

5. The tangent at any point P on the curve $x^{2/3} + y^{2/3} = 4$ meets the coordinate axes at A and B, then find AB.

6. Find the minimum distance between the curves $y^2 = x - 1$ and $x^2 = y - 1$.

7. Find the lengths of the (i) tangent, (ii) normal, (iii) sub-tangent and (iv) sub-normal at any point $P(a, b)$ on the parabola $y^2 = 2px$ ($p > 0$).

8. Find the number of points of intersection of two curves $y = 2\sin x$ and $y = 5x^2 + 2x + 3$.

9. If the relation between the sub-normal and sub-tangent at any point on the curve $y^2 = (x + a)^3$ is $p(\text{S.N}) = q(\text{S.T})^2$, then compute $\frac{p}{q}$.

10. Find the sum of the lengths of subtangent and tangent to the curve $x = c[2\cos\theta - \log(\operatorname{cosec}\theta + \cot\theta)]$, $y = c\sin 2\theta$ at $\theta = \frac{\pi}{3}$.

11. Find the cosine of the angle of intersection of curves $f(x) = 2^x \log_e x$ and $g(x) = x^{2x} - 1$.

12. If tangents are drawn from origin to the curve $y = \cos x$, show that their points of contact lie on the curve $x^2 - y^2 = x^2y^2$.

13. Find the angle between tangents at the point of intersection of two curves $x^3 - 3xy^2 + 2 = 0$, $3x^2y - y^3 = 2$.

14. Find the maximum value of the sum of the intercepts made by any tangent to the curve $\left(a\sin^2\theta, 2a\sin\theta\right)$ with the axes.

15. Prove that the tangents to the curve $y = \frac{(1+3x^2)}{(3+x^2)}$ drawn at the points for which $y = 1$, intersect at the origin.

16. Find the angle of intersection of the curves,

(i) $x^2 - y^2 = 5$ and $\frac{x^2}{18} + \frac{y^2}{8} = 1$.

(ii) $y^2 = \frac{2x}{\pi}$ and $y = \sin x$.

(iii) $y^2 = 4x$ and $x^2 + y^2 = 5$.

Answer key

1. $x + y = 3$ **2.** $\left(\frac{9}{4}, \frac{3}{8}\right)$ **3.** $\frac{\pi}{3}$ **5.** 8 **6.** $\left(\frac{3\sqrt{2}}{4}\right)$

7. (i) $\left|\frac{b}{p}\right|\sqrt{b^2+p^2}$ (ii) $\sqrt{b^2+p^2}$ (iii) $\frac{b^2}{p}$ (iv) p **8.** 0 **9.** $\frac{8}{27}$ **10.** $\frac{3c}{2}$

11. 1 **13.** $\frac{\pi}{2}$ **14.** $2a$ **16.** (i) $\theta = 90°$ (ii) $\frac{\pi}{4}$ *at origin*, $\cot^{-1}(\pi)$ (iii) $\tan^{-1}\left(\pm\frac{5}{11}\right)$

APROXIMATION USING DERIVATIVE

Consider the function $y = f(x)$.
From the definition of derivative, we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

For small variation in x, Δx tends to zero then dy is good approximation of Δy.
Change in y according to change in x,

$$\Delta y = \left(\frac{dy}{dx}\right)\Delta x.$$

ILLUSTRATION 45

(i) Circular plate is expanded by heat from radius 5 cm to 5.06 cm, then compute approximate increase in area.

(ii) If an edge of cube is increased by 1%, then find percentage increase in volume.

(iii) If rate of decrease of $y = \frac{x^2}{2} - 2x + 5$ is twice the rate of decrease of x, then find x (rate of decrease is non zero).

SOLUTION

(i) $r = 5$ cm, $\Delta r = 0.06$ cm

$$A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r.\frac{dr}{dt}$$

Approximate increase in area $\Delta A = \left(\frac{dA}{dr}\right)\Delta r = 2\pi r\Delta r$
$= 2\pi.5.(5.06-5) = 0.6\pi$

(ii) $V = x^3;\ \frac{\Delta x}{x}\times 100 = 1$

$$\frac{dV}{dx} = 3x^2$$

$$\Delta V = \left(\frac{dV}{dx}\right)\Delta x = 3x^2\Delta x$$

$$\%\text{ change in volume} = \frac{\Delta V}{V}\times 100 = \frac{3x^2\Delta x}{x^3}\times 100$$

$$= 3\left(\frac{\Delta x}{x}\times 100\right) = 3\%$$

(iii) $y = \frac{x^2}{2} - 2x + 5$

$$\frac{dy}{dx} = x - 2$$

$$\frac{dy}{dt} = 2\frac{dx}{dt} \Rightarrow \frac{dy}{dx} = 2 = x - 2 \Rightarrow x = 4$$

ILLUSTRATION 46

(i) If radius ($r = 5$ cm) of circular balloon is increased by 0.5%. Find % change in volume using derivatives.

(ii) Find rate of increase in area of an equilateral triangle of side 15 cm, when each side is increasing at the rate 0.1 cm/s.

SOLUTION

(i) Given $\frac{\Delta r}{r}\times 100 = 0.5$

$$\Delta r = 0.5\% \times 5\text{ cm} = \frac{0.5}{100}\times 5 = \frac{2.5}{100}$$

$$\Delta r = 0.025\text{ cm}$$

$$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2$$

Change in volume $\Delta V = \left(\frac{dV}{dr}\right)\Delta r = 4\pi r^2.\Delta r$

$$\%\text{ change in volume} = \frac{\Delta V}{V}\times 100 = \frac{4\pi r^2\cdot\Delta r}{\frac{4}{3}\pi r^3}\times 100$$

$$= 3\left(\frac{\Delta r}{r}\right)100 = 3\times 0.5 = 1.5\%$$

(ii) Let area of equilateral triangle with side x

$$A = \frac{\sqrt{3}}{4}x^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2}x\frac{dx}{dt}$$

Given $\frac{dx}{dt} = 0.1\text{ cm/sec}$

$$x = 15\text{ cm}$$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2}\times 15\times 0.1 = \frac{3\sqrt{3}}{4}\text{ cm}^2/\text{sec}$$

PHYSICAL APPLICATION OF DERIVATIVE: RATE OF CHANGE

$\frac{dx}{dt}$ = Instantaneous rate of change of x about t

$\frac{dy}{dx}$ = Instantaneous rate of change of y about x

The value of $\frac{dy}{dx}$ at $x = x_0$ i.e. $\left(\frac{dy}{dx}\right)_{x=x_0}$ represents the rate of change (*instantaneous rate of change*) of y with respect to x at point $x = x_0$.

If $x = \phi(t)$ and $y = \psi(t)$, then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided that $\frac{dx}{dt} \neq 0$

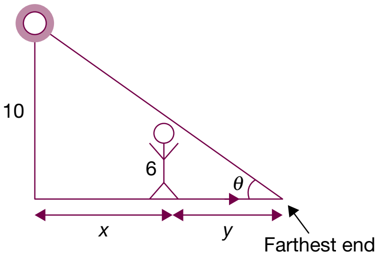
Thus, the rate of change of y with respect to x can be calculated by using the rate of change of y and that of x each with respect to t.

ILLUSTRATION 47

6 feet tall Rohit moves away from a light pole of 10 feet high at a rate 10 feet/sec, what is the rate of change in length of its shadow and how fast the farthest end of shadow is moving away.

SOLUTION

Rohit is at x distance from light pole and shadow is of length y as shown in diagram.



$$\tan\theta = \frac{6}{y} = \frac{10}{x+y}$$

$$\Rightarrow \quad 3(x+y) = 5y$$

$$\Rightarrow \quad 3x = 2y \Rightarrow \frac{3dx}{dt} = \frac{3dy}{dt} \qquad \left(\because \frac{dx}{dt} = 10\text{ feet/sec}\right)$$

$$\Rightarrow \quad 3\times 10 = \frac{2dy}{dt} = \frac{dy}{dt} = 15\text{ feet/sec}$$

Rate of change in shadow $= \frac{dy}{dt} = 15\text{ feet/sec}$

Rate by which farthest end of shadow is moving

$$= \frac{d}{dt}(x+y) = 10 + 15 = 25\text{ feet/sec}$$

ILLUSTRATION 48

Area of circle is increasing at the rate of 2 cm^2/sec.
Find the rate of change in area of square if square is

(i) Inscribed in circle.

(ii) Circumscribed the given circle.

SOLUTION

Let radius of circle as r and side of square as a

Area of circle $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} = 2 \Rightarrow \frac{rdr}{dt} = \frac{1}{\pi}$$

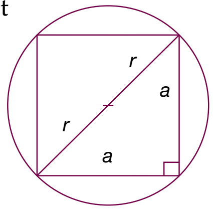
(i) Square is inscribed in circle

Using Pythagoras theorem, we get

$$2a^2 = 4r^2 \Rightarrow a^2 = 2r^2$$

$$S = a^2 = 2r^2$$

$$\Rightarrow \frac{dS}{dt} = 4r\frac{dr}{dt} = \frac{4}{\pi}\ \text{cm}^2/\text{sec}$$



(ii) Square is circumscribed the given circle

We have $a = 2r$

$$S = a^2 = 4r^2$$

$$\frac{dS}{dt} = 8\frac{rdr}{dt} = \frac{8}{\pi}\ \text{cm}^2/\text{sec}$$

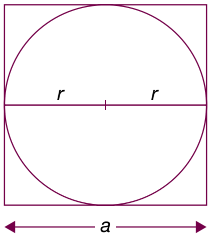


ILLUSTRATION 49

Water is poured into inverted conical vessel of which radius of base is 2 m and height 4m at a rate of 77 litre/min. Find the rate at which water level is rising at the instant when depth is 70 cm $\left(\text{Use } \pi \text{ as } \frac{22}{7}\right)$.

SOLUTION

As time t, water cone of radius r and height h.

We know that, 1 litre = 1000 cm^3

Volume of cone formed by water $V = \frac{1}{3}\pi r^2 h$

Given $\frac{dV}{dt} = 77$ litre/min.

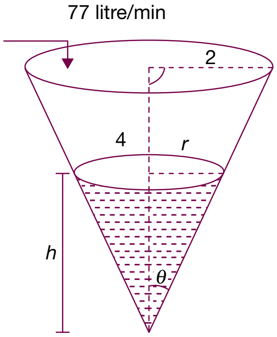
$$\tan\theta = \frac{r}{h} = \frac{2}{4} \Rightarrow h = 2r$$

$$\Rightarrow \quad r = \frac{h}{2}$$

$$\Rightarrow \quad V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$$

$$\Rightarrow \quad 77 \times 1000 = \frac{22}{7(4)} \times 70 \times 70\frac{dh}{dt}$$



Hence, rate by which water level is rising $\frac{dh}{dt} = 20$ cm/min

ILLUSTRATION 50

Water is dripping out from a conical funnel of semi-vertical angle $\frac{\pi}{4}$ at uniform rate of 2 cm^3/sec through a ring hold at the vertex in bottom. When the slant height of water is 4cm. Find the rate of decreases of slant height.

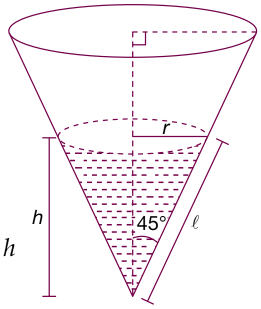
SOLUTION

At time t, water cone of height h, radius r and slant height l.

Given, $\theta = 45^\circ \Rightarrow \tan\theta = \frac{r}{h} = 1$

$$\Rightarrow \quad h = r$$

$$\ell = \sqrt{h^7 + r^2} = \sqrt{2}\,r \Rightarrow \frac{\ell}{\sqrt{2}} = h$$



Volume of cone, $V = \frac{1}{3}\pi r^2 h$ and $h = r$

$$V = \frac{\pi h^3}{3} \Rightarrow \frac{dV}{dt} = 2 = \pi h^2\frac{dh}{dt} \qquad \left(\because h = \frac{\ell}{\sqrt{2}}\right)$$

$$\Rightarrow \quad 2 = \frac{\pi\ell^2}{2}\frac{1}{\sqrt{2}}\frac{d\ell}{dt} = \frac{\pi\ell^2}{2\sqrt{2}}\frac{d\ell}{dt}$$

When $\ell = 4$ cm

Hence, $2 = \frac{\pi}{2\sqrt{2}}16\frac{d\ell}{dt} \Rightarrow \frac{d\ell}{dt} = \frac{1}{2\pi\sqrt{2}}$ cm/sec

ILLUSTRATION 51

The radius of a spherical soap bubble is increasing at the rate of 0.1 cm/sec.

Find the rate of increase in its (i) volume and (ii) surface area, when radius is 4 cm.

SOLUTION

If V denotes the volume and S denotes the surface area of the soap bubble with radius r, then

$$V = \frac{4}{3}\pi r^3 \text{ and } S = 4\pi r^2 \qquad \ldots(1)$$

We are given that $\frac{dr}{dt} = 0.1$ cm/sec.

Differentiating (1), we get

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \text{ and } \frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

Using $\frac{dr}{dt} = 0.1$ cm/sec and $r = 4$ cm

We have $\frac{dV}{dt} = 6.4\pi$ cm^3/sec and $\frac{dS}{dt} = 3.2\,\pi$ cm^2/sec

Test Your Concepts-III

(Solutions on page H.151)

1. A particle is moving along $x^2 + y^2 = 25$, with it is at point (3, 4), its abscissa change at the rate 2 cm/sec. Find the rate by which its ordinate changes at this point.

2. A point is moving along the curve $y^3 = 27x$, find interval of values of x in which ordinate changes faster than abscissa.

3. Find the rate of change of $\sqrt{x^2+16}$ with respect to $\frac{x}{x-1}$ at $x = 3$.

4. A spherical balloon is increasing at a rate 0.1 cm/sec. Find the rate of change in volume when radius is 5 cm.

5. Water is poured into an inverted conical vessel of which the radius of the base is 6 cm and height 12 cm at the rate of 5.5 cm cube per minute. At what rate is water level rising at the instant when the depth is 3.5 cm?

6. Air is being fed in a spherical balloon at a uniform rate 1 cm^3/sec. Find the rate at which its surface area is changing after 2 seconds.

7. A 13 feet ladder is leaning against a wall such that when base starts slipping away. At the instant its base is 12 feet away from the wall, the base is moving away from the wall at the rate of 5 feet/sec. Find the rate at which angle between ladder and ground is changing.

8. A cube of ice melts without its changing shape at the uniform rate of 4 cm^3/min. Find the rate of change of the surface area of the cube, in cm^2/min, when the volume of the cube is 125 cm^3.

9. A light is hung 6 m vertically above, a man is walking away from the light at the rate of 54 m/min. If the height of the man is 2 m, then find the rate of which the shadow increases.

10. The radius and height of a cylinder are equal to the radius of sphere. Find the ratio of the rates of change of the volume of the sphere and cylinder.

11. A spherical balloon is pumped at the rate of 10 inch^3/min, find the rate of increase of its radius if its radius is 15 inch.

12. Side of variable cube is increasing at the rate of 3 cm/s. How fast is the volume of the cube increasing when the edge is 10 cm long?

13. The top of a ladder 13 m long is resting against a vertical wall, when a ladder begins to slide. When the foot of the ladder is 5 m from the wall, it is sliding at the rate of 2 m/s. How fast then the top sliding downwards?

14. At what point of (1st quadrant) the ellipse $16x^2 + 9y^2 = 400$ does the ordinate decrease as the same rate at which the abscissa increases?

Answer key

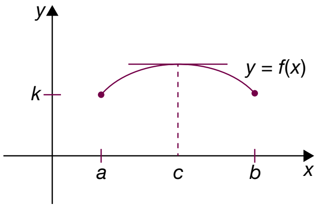
1. –1.5 cm/sec
2. $x \in (-1, 1)$
3. $-\frac{12}{5}$
4. 10π cm^3/sec
5. 0.571 cm/sec
6. $\left(\frac{3}{2\pi}\right)^{\frac{1}{3}}$
7. –1
8. $\frac{16}{5}$
9. 27 m/min
10. 4:3
11. $\frac{1}{90\pi}$ inch/min
12. 900 cc/sec.
13. $-\frac{5}{6}$ m/sec
14. $\left(3, \frac{16}{3}\right)$

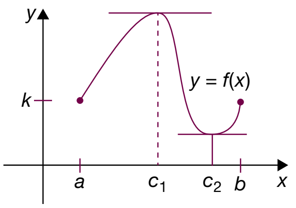
ROLLE'S THEOREM

If function $y = f(x)$ is

(i) Continuous in $x \in [a, b]$
(ii) Differentiable in $x \in (a, b)$
(iii) $f(a) = f(b) = k$,

then there exists atleast one c for which $f'(c) = 0$, where $c \in (a, b)$.





Geometrically Rolle's theorem claims that once all three conditions are satisfied then there atleast one horizontal tangent exist.

NOTE:

(i) We also say that between two roots of an equation $f(x) = 0$ where $f(x)$ is continuous and differential there exists atleast one root of the equation $f'(x) = 0$.

(ii) Let $f(x)$ is continuous and thrice differentiable between any two consecutive roots of $f(x) = K$. From Rolle's theorem we can say that $f'(x) = 0$ for atleast one x. If $f(x) = K$ has n distinct roots then by Rolle's theorem $f'(x) = 0$ and $f''(x) = 0$ have atleast $n - 1$ and $n - 2$ zeroes respectively.

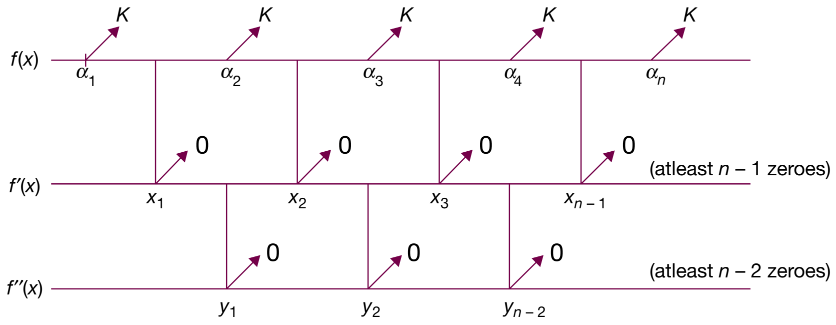


ILLUSTRATION 52

Verify Rolle's theorem

(i) $f(x) = x^2 - 3x + 2$ where $x \in [1, 2]$.

(ii) $f(x) = (x - a)^m (x - b)^n$ where $x \in [a, b]$, $m, n \in N$.

(iii) $f(x) = \dfrac{1}{|x|}$ for $x \in [-1, 1]$.

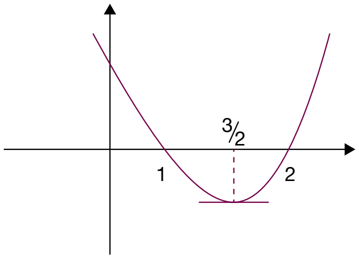
(iv) $f(x) = x(x + 3)e^{-x/2}$ for $x \in [-3, 0]$.

SOLUTION

(i) $f(x) = x^2 - 3x + 2$ where $x \in [1, 2]$

$f(x)$ is a quadratic equation so continuous and differentiable in $x \in [1, 2]$.

Thus, $f(1) = f(2) = 0$.



Now, from Rolle's Theorem

$f'(c) = 0$ for atleast one $c \in (1, 2)$

$\Rightarrow 2c - 3 = 0 \Rightarrow c = \dfrac{3}{2} \in (1, 2)$

Rolle's theorem is verified for given function.

(ii) $f(x) = (x - a)^m (x - b)^n$ where $x \in [a, b]$, $m, n \in N$

$f(x)$ is continuous and differentiable.

Thus, $f(a) = f(b) = 0$.

Now, from Rolle's theorem

$f'(c) = 0$ for atleast one $c \in (a, b)$

$\Rightarrow m(x - a)^{m-1} (x - b)^n + (x - a)^m n(x - b)^{n-1}\big|_{x=c} = 0$

$\Rightarrow (c - a)^{m-1} (c - b)^{n-1}[m(c - b) + n(c - a)] = 0$

$\Rightarrow c(m + n) = mb + na$

$\Rightarrow c = \dfrac{mb + na}{m + n} \in (a, b)$ as $m, n \in N$

Clearly, we can get c which divides a and b in the ratio $m : n$ internally.

Rolle's theorem is verified for given function.

(iii) $f(x) = \dfrac{1}{|x|}$ $x \in [-1, 1]$

$f(x)$ is discontinuous at $x = 0$ and not differentiable at $x = 0$.

Thus, $f(-1) = 1 = f(1)$.

But we can not apply Rolle's theorem here as two conditions are not satisfied.

(iv) $f(x) = x(x + 3)e^{-x/2}$

$x \in [-3, 0]$

$f(x)$ is continuous and differentiable.

Thus, $f(-3) = 0 = f(0)$.

From Rolle's theorem

$$f'(x) = (2x + 3)e^{-x/2} + (x^2 + 3x)e^{-x/2}\left(-\frac{1}{2}\right)$$

$$= e^{-x/2}\left[2x + 3 - \frac{x^2}{2} - \frac{3x}{2}\right] = \frac{e^{-x/2}}{2}(-x^2 + x + 6) = 0$$

$$\Rightarrow e^{-x/2}\left[\frac{-x^2}{2} + \frac{x}{2} + 3\right] = 0$$

$\Rightarrow x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = 3, x = -2$

$f'(c) = 0$ for atleast one $c \in (-3, 0)$.

Hence, $c = -2 \in (-3, 0)$.

Rolle's theorem is verified for given function.

ILLUSTRATION 53

Verify Rolle's theorem for $f(x) = (x^2 - 4x + 3)e^{2x}$ in $[1, 3]$.

SOLUTION

$(x^2 - 4x + 3)$ and e^{2x} are continuous and differential functions in given interval.

$f(x)$ is product of above two functions,therefore, it is also continuous and differentiable.

$$f(1) = (1 - 4 + 3)e^2 = 0;\ f(3) = (9 - 12 + 3)e^6 = 0$$

$\therefore \quad f(1) = f(3)$

All the conditions of the theorem are satisfied.
As such there exists one or more values of c, such that $f'(c) = 0$.

$$f'(x) = (x^2 - 4x + 3)\, 2e^{2x} + e^{2x}.\, (2x - 4)$$

$$= 2(x^2 - 3x + 1)\, e^{2x}.$$

Now, $f'(c) = 0 \Rightarrow c^2 - 3c + 1 = 0$.

$$\therefore \quad c = \frac{3 \pm \sqrt{5}}{2}$$

The value of $c = \dfrac{3 + \sqrt{5}}{2}$ lie in the interval (1, 3).

Hence, the Rolle's theorem is verified.

ILLUSTRATION 54

Verify Rolle's theorem for $f(x) = \log\left(\dfrac{x^2 + ab}{x(a+b)}\right)$ where $x \in [a, b]$, $b > a > 0$.

SOLUTION

We have

$$f(x) = \log (x^2 + ab) - \log(a + b) - \log x.$$

$f(x)$ is continuous in $a \le x \le b$.

Further, $f'(x) = \dfrac{x^2 - ab}{x(x^2 + ab)}$, $f(x)$ is differentiable also.

Now, $f(a) = f(b) = 0$.
All the three conditions of Rolle's theorem are satisfied.

Now, $f'(c) = 0 \Rightarrow \dfrac{c^2 - ab}{c(c^2 + ab)} = 0$

$\therefore \quad c^2 = ab$ or $c = \sqrt{ab}$ as it lies in the interval (a, b), being geometric mean a and b.

Hence, this verifies the Rolle's theorem.

ILLUSTRATION 55

By considering the function $\phi(x) = (x - \lambda) \log x$ where $\lambda > 1$, show that the equation $x \log x + x = \lambda$, is satisfied by atleast one value of x lying between 1 and λ.

SOLUTION

We have $\phi(x) = (x - \lambda) \log x$ for $1 \le x \le \lambda$.

The given function $\phi(x)$ is continuous in the close interval $1 \le x \le \lambda$.

Further, $\phi'(x) = \log x + \dfrac{(x - \lambda)}{x}$ exists in the open interval $1 < x < \lambda$.

Also, $\phi(1) = \phi(\lambda) = 0$

Thus, by Rolle's theorem there lies atleast one value of x in $(1, \lambda)$, where $\phi'(x) = 0$. That is the equation,

$$\log x + \frac{(x - \lambda)}{x} = 0 \text{ or } x \log x + x = \lambda$$

Hence, the equation will have atleast one root lying between 1 and λ.

ILLUSTRATION 56

Show that between two roots of equation $e^{-x} - \cos x = 0$ atleast one root of equation $\sin x - e^{-x} = 0$ exists.

SOLUTION

Let $f(x) = e^{-x} - \cos x$ where $x \in [\alpha, \beta]$.
Assuming that $f(x) = 0$ has roots $x = \alpha, \beta$
$f(x)$ is continuous and differentiable function.

$$f(\alpha) = f(\beta) = 0$$

Now, from Rolle's Theorem $f'(x) = 0$

$$\Rightarrow \quad e^{-x}(-1) - (-\sin x) = 0$$

$$\Rightarrow \quad \sin x - e^{-x} = 0 \text{ for atleast one } x \in (\alpha, \beta)$$

Hence proved.

ILLUSTRATION 57

If $2a + 3b + 6c = 0$, show that $ax^2 + bx + c = 0$ has atleast one root in between 0 to 1.

SOLUTION

Let $f(x) = \dfrac{ax^3}{3} + \dfrac{bx^2}{2} + cx + d$

For $x \in [0, 1]$, $f(x)$ is cubic polynomial so it is continuous and differentiable.

Thus, $f(0) = d$

$$f(1) = \frac{a}{3} + \frac{b}{2} + c + d = \frac{2a + 3b + 6c}{6} + d = 0 + d = d$$

$$f(0) = f(1)$$

From Rolle's theorem $f'(x) = 0$ for atleast one $x \in (0, 1)$.

Hence, $ax^2 + bx + c = 0$ has atleast one root in between 0 and 1. Hence proved.

ILLUSTRATION 58

Prove that equation $\cos x + 2\cos 2x + 4\cos 3x = 0$ has atleast one solution in $x \in (0, \pi)$.

SOLUTION

Let $f(x) = \sin x + \sin 2x + \dfrac{4}{3} \sin 3x$ for $x \in [0, \pi]$

$f(x)$ is continuous and differentiable.

Thus, $f(0) = f(\pi) = 0$.

From Rolle's theorem $f'(x) = 0$

$\Rightarrow \quad \cos x + 2\cos 2x + 4\cos 3x = 0$ for atleast one $x \in (0, \pi)$

Hence Proved.

ILLUSTRATION 59

For what value of a, m and b does the function $f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{cases}$ satisfy the hypothesis of the mean value theorem for the interval [0, 2].

SOLUTION

For mean value theorem function should be continuous and differentiable in the given interval for continuity at $x = 0$ at $x = 1$.

$$f(0) = f(0^+) \Rightarrow a = 3,$$

$$f(1^-) = a + 2 = f(1^+) = b + m$$

$$\Rightarrow \quad b + m = 5 \qquad \ldots(1)$$

As $f(x)$ is differentiable at $x = 1$.

$$f'(1^-) = -2 + 3 = 1 \text{ and } f'(1^+) = m$$

$$f'(1^-) = f'(1^+) \Rightarrow m = 1$$

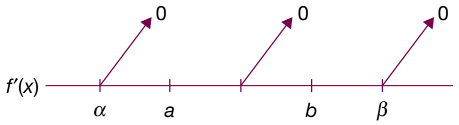
Hence, $a = 3$, $m = 1$, $b = 4$.

ILLUSTRATION 60

If $f(x)$ satisfies all necessary conditions of Rolle's theorem, then prove that between any two consecutive zeroes of $f'(x)$ there can be atmost one zero of $f(x)$.

SOLUTION

Let $x = \alpha, \beta$ are consecutive zeroes of $f'(x)$.
We are assuming that $f(x)$ has two zeroes in between α, β



$$f(a) = 0 = f(b) \text{ where } a, b \in (\alpha, \beta)$$

As $f(x)$ is continuous and differentiable. Now from Rolle's theorem, $f'(c) = 0$ for alteast one $c \in (a, b)$
But if this is true then α, β one not consecutive which concludes that our assumption is wrong.
It means $f(x)$ can not have two zeroes in between α, β.
Hence, $f(x)$ can have atmost one zero between any two consecutive zeroes of $f'(x)$.

ILLUSTRATION 61

If $f(x)$ and $g(x)$ are differentiable function such that $f'(x)\, g(x) \ne g'(x)\, f(x)$, then prove that between any two real solution of $f(x) = 0$, $g(x)$ has a zero.

SOLUTION

Let $f(\alpha) = f(\beta) = 0$ where $x \in [\alpha, \beta]$.
We are assuming that $g(x) \ne 0$ for any $x \in [\alpha, \beta]$. …(1)

Let $h(x) = \dfrac{f(x)}{g(x)}$ where $x \in [\alpha, \beta]$.

As given $f(x)$ and $g(x)$ are continuous and differentiable function then $h(x)$ is also continuous and differentiable.

$$h(\alpha) = \frac{f(\alpha)}{g(\alpha)} = 0; h(\beta) = \frac{f(\beta)}{g(\beta)} = 0$$

Rolle's theorem is applicable on $h(x)$ then

$$h'(x) = 0 \Rightarrow \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2} = 0$$

$\Rightarrow \quad g(x)\, f'(x) = f(x)\, g'(x)$ for atleast one $x \in (\alpha, \beta)$

But it is not true as given $f'(x)\, g(x) \ne g'(x)\, f(x)$
It means our assumption is wrong.
$g(x)$ contradiction but not possible.
Hence, $g(x) = 0$ for atleast one $x \in (\alpha, \beta)$

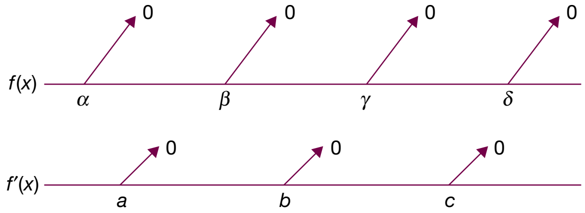
ILLUSTRATION 62

If $f(x) = 0$ has 4 zeroes and $f(x)$ is thrice differentiable function. Find minimum no. of zeroes of

$$\frac{d}{dx}((f'(x))^2 + f(x)\, f''(x)).$$

SOLUTION

$f(x)$ is continuous and differentiable functions and $f(x) = 0$ for four values $\alpha, \beta, \gamma, \delta$
By Rolle's theorem we can say that $f'(x) = 0$ for atleast three values.
Let consider a function $g(x) = f(x)\, f'(x)$. It has alteast 7 zeroes.



$$\frac{d}{dx}((f'(x))^2 + f(x)f''(x)) = \frac{d}{dx}\left[\frac{d}{dx}(f(x)f'(x))\right]$$

$$= \frac{d^2}{dx^2}(f(x)f'(x)) = g''(x)$$

$g(x)$ satisfies all conditions of Rolle's theorem.

$g(x)$ has minimum 7 zeroes then $g'(x)$ has atleast 6 zeroes and $g''(x)$ has atleast 5 zeroes.

Hence, minimum no. of zeroes of $\dfrac{d}{dx}((f'(x))^2 + f(x)f''(x)) = 5$

LAGRANGE'S MEAN VALUE THEOREM (LMVT)

If function $y = f(x)$ defined in $x \in [a, b]$ is

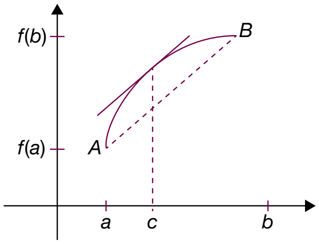
(i) Continuous $x \in [a, b]$
(ii) Differentiable in (a, b)

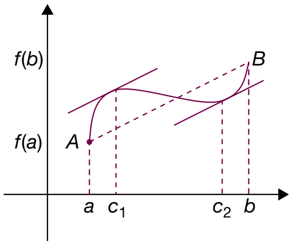
then there exists atleast one c for which $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Point $A(a, f(a))$ and $B(b, f(b))$ lies on curve.

Slope of chord joining, $AB = m_{AB} = \dfrac{f(b) - f(a)}{b - a} = f'(c)$

Geometrical Meaning of LMVT is, there exists atleast one tangent which is parallel to chord AB.





If $f(a) = f(b)$ then $f'(c) = 0$ so we are getting Rolle's theorem condition from LMVT.
Now we try to get LMVT using Rolle's Theorem.
Let $F(x) = kx + f(x)$ where k is a constant.
Now, we choose some k such that $F(a) = F(b)$.

$$\therefore \quad ka + f(a) = kb + f(b)$$

$$\therefore \quad k = -\frac{f(b) - f(a)}{b - a}$$

So, $F(x) = -\dfrac{f(b) - f(a)}{b - a}x + f(x)$

$F(x)$ satisfies all the conditions of Rolle's theorem as $F(x)$ is continuous and differentiable and $F(a) = F(b)$, so, there exists at least one $c \in (a, b)$ such that $F'(c) = 0$.

$$\therefore \quad -\frac{f(b) - f(a)}{b - a} + f'(c) = 0$$

$\therefore \quad f'(c) = \dfrac{f(b) - f(a)}{b - a}$ is true for atleast one $c \in (a, b)$

Which is the condition for Lagrange's mean value theorem.

CAUCHY'S MEAN VALUE THEOREM

Cauchy's mean value theorem, also known as the extended mean value theorem, is the more general form of the mean value theorem. It states that if functions $f(x)$ and $g(x)$ are both continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and $g'(x)$ is not zero on that open interval, then there exists some c in (a, b), such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

PROOF:

Let consider a function $h(x)$ such that it satisfies the conditions of Rolle's theorem.

$$h(x) = f(x) - g(x)\left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)$$

Now, $h(a) = h(b) = \dfrac{f(a)g(b) - g(a)f(b)}{g(b) - g(a)}$

So, according to the Rolle's theorem there exists at least one $c \in (a, b)$ such that

or, $\quad f'(c) - g'(c)\dfrac{f(b) - f(a)}{g(b) - g(a)} = 0$

or, $\quad \dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

ILLUSTRATION 63

Let $f(x)$ and $g(x)$ be two differentiable functions in R and $f(2) = 8$, $g(2) = 0$, $f(4) = 10$, and $g(4) = 8$. Then prove that $g'(x) = 4f'(x)$ for at least one $x \in (2, 4)$.

SOLUTION

Consider $h(x) = g(x) - 4f(x)$ in $[2, 4]$.

Now, $h(2) = g(2) - 4f(2) = -32$,

And $h(4) = g(4) - 4f(4) = 8 - 4 \times 10 = -32$

Therefore, $h'(x) = 0$ for at least one $x \in (2, 4)$ using Rolle's theorem.
Hence proved.

Alternate:

Using Cauchy's mean value theorem, there exists at least one $c \in (2, 4)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(4) - f(2)}{g(4) - g(2)} = \frac{10 - 8}{8 - 0} = \frac{1}{4}$$

or, $\quad 4f'(c) = g'(c)$

Hence, we can say that $g'(x) = 4f'(x)$ for at least one $x \in (2, 4)$.
Hence proved.

ILLUSTRATION 64

Suppose α, β are angles satisfying $0 < \alpha < \beta < \dfrac{\pi}{2}$.

Then prove that $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = -\cot\theta$. For atleast one $\theta \in (\alpha, \beta)$

SOLUTION

Let $f(x) = \sin x$ and $g(x) = \cos x$. Then f and g are continuous and derivable.

Also, $\sin x \neq 0$ for any $x \in \left(0, \frac{\pi}{2}\right)$.

So, by Cauchy's mean value theorem,

$$\frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}$$

or, $$\frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha} = \frac{\cos\theta}{-\sin\theta} = -\cot\theta \text{ is true}$$

For atleast one $\theta \in (\alpha, \beta)$

ILLUSTRATION 65

Let f be continuous on $[a, b]$, $a > 0$, and differentiable on (a, b). Prove that there exists $c \in (a, b)$ such that

$$\frac{bf(a)-af(b)}{b-a} = f(c) - cf'(c).$$

SOLUTION

We consider the functions

$$g(x) = \frac{f(x)}{x} \text{ and } h(x) = \frac{1}{x}$$

Both the functions $g(x)$ and $h(x)$ are continuous and differentiable in (a, b) as $a > 0$.

Therefore, using Cauchy mean value theorem, there exists at least one $c \in (a, b)$ for which

$$\frac{g'(c)}{f'(c)} = \frac{g(a)-g(b)}{h(a)-h(b)}$$

or, $$\frac{\frac{f(c)-cf'(c)}{c^2}}{\frac{1}{c^2}} = \frac{\frac{f(a)}{a}-\frac{f(b)}{b}}{\frac{1}{a}-\frac{1}{b}}$$

or, $f(c) - cf'(c) = \dfrac{bf(a)-af(b)}{b-a}$ is true for atleast one $c \in (a, b)$

Hence proved.

ILLUSTRATION 66

$A(x_1, y_1)$ and $B(x_2, y_2)$ are the two points on the curve $f(x) = ax^2 + bx + c$. Find a point $P(x_3, y_3)$ on the curve such that slope of tangent at point P becomes equal to slope of chord AB.

SOLUTION

Let's apply Lagrange's mean value theorem for the function. $f(x) = ax^2 + bx + c$ in $[x_1, x_2]$ The function $f(x) = ax^2 + bx + c$ is a polynomial function of x and is therefore continuous in $[x_1, x_2]$. $f'(x) = 2ax + b$; therefore, $f'(x)$ exists in the open interval (x_1, x_2).

Thus, both the conditions of Lagrange's mean value theorem are satisfied, so their must exists a point $x_3 \in (x_1, x_2)$ such that

$$f'(x_3) = \frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{a(x_2^2-x_1^2)+b(x_2-x_1)}{(x_2-x_1)}$$
$$= a(x_2+x_1)+b$$

$$\Rightarrow \quad 2ax_3 + b = ax_2 + b \Rightarrow x_3 = \frac{(x_2+x_1)}{2}$$

and $$y_3 = a\left(\frac{x_2+x_1}{2}\right)^2 + b\left(\frac{x_2+x_1}{2}\right) + c$$

ILLUSTRATION 67

Let $\int_x^{4x} f(t)dt = \int_2^x f(t)dt \;\; \forall\, x > 0$ and $f(x)$ is a derivable function such that $f(1) = 8$. According to Largange's mean value theorem, there exists some $c \in (4, 16)$ such that $|6f'(c)|$ is equal to

SOLUTION

$$\int_x^{4x} f(t)\,dt = \int_2^x f(t)\,dt \quad \forall\, x > 0$$

On differentiation

$$\Rightarrow \quad f(4x)\cdot 4 - f(x) = f(x) - 0 \Rightarrow f(x) = 2f(4x)$$

Put $x = 1$, we get $f(4) = \dfrac{f(1)}{2} = 4$

Put $x = 4$, $f(16) = \dfrac{f(4)}{2} = 2$

From LMVT

$$f'(c) = \frac{f(16)-f(4)}{16-4} = \frac{2-4}{12} = -\frac{1}{6}$$

Now, $|6f'(c)| = 1$

ILLUSTRATION 68

Let $f(x) = \text{Max.}\,\{x^2, (1-x)^2, 2x(1-x)\}$ where $0 \le x \le 1$. If Rolle's theorem applicable for $f(x)$ on $[\alpha, \beta]$, then find $\alpha + \beta + \gamma$ where $\gamma \in (\alpha, \beta)$ such that $f'(\gamma) = 0$.

SOLUTION

Point of intersection of curves $y = (1-x)^2$ and $y = 2x(1-x)$

$$y = (1-x)^2 = 2x(1-x)$$

$$\Rightarrow \quad (3x-1)(x-1) = 0 \Rightarrow x = \frac{1}{3} \text{ and } 1$$

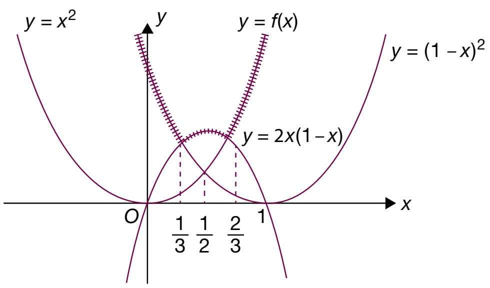
Point of intersection of curve $y = 2x(1-x)$ and $y = x^2$

$$y = 2x(1-x) = x^2 \Rightarrow 2x - 2x^2 = x^2 \Rightarrow 3x^2 - 2x = 0$$

$$\Rightarrow \quad x = 0 \text{ and } \frac{2}{3}$$

$f(x)$ is not differentiable at $x = \frac{1}{3}, \frac{2}{3}$

$f(x)$ is shown in figure.



$$f(x) = 2x - 2x^2;\ x \in \left[\frac{1}{3}, \frac{2}{3}\right]$$

$f(x)$ is continuous and differentiable.

$$f\left(\frac{1}{3}\right) = 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9}$$

$$f\left(\frac{2}{3}\right) = 2\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{4}{9}$$

We have $\alpha = \frac{1}{3}, \beta = \frac{2}{3}$

For Rolle's theorem $f'(x) = 2 - 4x = 0 \Rightarrow x = \frac{1}{2} = \gamma$

Hence, $\alpha + \beta + \gamma = \frac{1}{3} + \frac{2}{3} + \frac{1}{2} = \frac{3}{2}$

Test Your Concepts-IV

(Solutions on page H.152)

1. State why Rolle's theorem is not applicable to the function
$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \end{cases}.$$
2. $A(1, 0)$, $B(e, 1)$ are two points on the curve $y = \log_e x$. If P is a point on the curve at which the tangent to the curve is parallel to the chord AB, then find abscissa of P.
3. If a, b are two real number with $a < b$, show that a real number ca can be found between a and b such that $3c^2 = b^2 + ab + a^2$.
4. $f(x)$ and $g(x)$ are differentiable functions for $0 \le x \le 2$ such that $f(0) = 5$, $g(0) = 0$, $f(2) = 8$, $g(2) = 1$. Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3g'(c)$.
5. Assume that f is continuous on $[a, b]$, $a > b$ and differentiable on an open interval (a, b). Show that if $\frac{f(a)}{a} = \frac{f(b)}{b}$, then there exist $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f'(x_0)$.
6. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = a$ and $f(b) = b$, show that there exist distinct c_1, c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.
7. Let $a > 0$ and f be continuous in $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, find $f(0)$.
8. Let $f(x) = (x-a)(x-b)(x-c)(x-d);\ a < b < c < d$, then find minimum number of roots of the equation $f''(x) = 0$.
9. Let $f : [2,7] \to [0,\infty)$ be a continuous and differentiable function. Then show that
$$\frac{(f(7)-f(2))\left((f(7))^2 + (f(2))^2 + f(2)f(7)\right)}{3}$$
$= 5f^2(c)f'(c)$ where $c \in (2,7)$.
10. Let $f(x)$ is differentiable function in $[2, 5]$ such that $f(2) = \frac{1}{5}$ and $f(5) = \frac{1}{2}$, then Show that the exists a number c, $2 < c < 5$ for which $f'(c) = \frac{1}{10}$.

Answer key

1. not differentiable **2.** $e - 1$ **7.** 0 **8.** 2

SOLVED PROBLEMS

PROBLEM 1

The curve $y = ax^3 + bx^2 + cx + 5$, touches the x-axis at $P(-2, 0)$ and cuts the y-axis at a point Q, where its gradient is 3. Find a, b, c.

SOLUTION

Given that $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$

$$\left(\frac{dy}{dx}\right)_{x=-2} = 0 \Rightarrow 3ax^2 + 2bx + c]_{x=-2} = 0$$

$$\Rightarrow \quad 12a - 4b + c = 0 \qquad \ldots(1)$$

Point $P(-2, 0)$ lies on curve $y = ax^3 + bx^2 + cx + 5$

$$-8a + 4b - 2c + 5 = 0 \qquad \ldots(2)$$

Also, the curve cuts the y-axis at Q

for $x = 0$, $y = 5$; $Q(0, 5)$

At Q gradient of the curve is 3

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=0} = 3 \Rightarrow 3ax^2 + 2bx + c]_{x=0} = 3 \Rightarrow c = 3 \qquad \ldots(3)$$

Solving (1), (2) and (3), we get $a = -\frac{1}{2}$, $b = -\frac{3}{4}$ and $c = 3$.

PROBLEM 2

Consider the polynomial function,

$$f(x) = \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x$$

Show that Rolle's theorem is not applicable for $y = f(x)$ on any interval $[a, b]$ where $a, b \in R$.

SOLUTION

$$f(x) = \frac{x^7}{7} - \frac{x^6}{6} + \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x$$

$$f'(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$$

For $x < 0, f'(x) = x^6 + x^4 + x^2 + 1 - x^5 - x^3 - x > 0$

For $x > 1, f'(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 > 0$

$$= x^5 (x - 1) + x^3 (x - 1) + x (x - 1) + 1 > 0$$

If $0 \le x \le 1 \Rightarrow 1 - x \ge 0$

$$\Rightarrow \quad f'(x) = x^6 + x^4 - x^5 + x^2 - x^3 + 1 - x$$

$$= x^6 + x^4 (1 - x) + x^2 (1 - x) \ge 0$$

So, $\forall\ x \in R$, $f'(x) > 0$ then $f(x)$ is Strictly Monotonically Increasing. $f(x) = 0$, at exactly one point. Range of $f(x) \equiv R$. Hence, for Rolle's theorem $f(a) = f(b)$ is not possible here.

PROBLEM 3

Let 'A' be the point where the curve $5a^2x^3 + 10ax^2 + x + 2y - 4 = 0$ $\left(a \in R - \{0\}\right)$ meets y-axis. Then, find the equation of tangent to the curve at the point B where the normal at 'A' meets the curve again.

SOLUTION

Curve $5a^2x^3 + 10ax^2 + x + 2y - 4 = 0$ meets y-axis (i.e.) $x = 0$

$\Rightarrow \quad y = 2$, $A(0, 2)$

On differentiation, we obtain

$$5a^2 3x^2 + 10a.2x + 1 + 2\frac{dy}{dx} = 0 \qquad \ldots(1)$$

Slope of tangent $= \frac{dy}{dx}_{(0,2)} = -\frac{1}{2}$

Equation of normal at point A, $y - 2 = 2(x - 0)$ or $y = 2x + 2$

Now, normal meets curve $5a^2x^3 + 10ax^2 + x + 2y - 4 = 0$

$$5a^2x^3 + 10ax^2 + x + 4x + 4 - 4 = 0$$

$$\Rightarrow \quad x(a^2x^2 + 2ax + 1) = x(ax + 1)^2 = 0$$

$$\Rightarrow \quad x = -\frac{1}{a}, y = 2\left(1 - \frac{1}{a}\right)$$

From (1), put $x = -\frac{1}{a}$, $5a^2 3x^2 + 10a.2x + 1 + 2\frac{dy}{dx} = 0$

$$\Rightarrow \quad 15 - 20 + 1 + 2\frac{dy}{dx} = 0$$

Slope of tangent $= \frac{dy}{dx}_{(0,-1/a)} = 2$

Equation of tangent $y - 2\left(1 - \frac{1}{a}\right) = 2\left(x + \frac{1}{a}\right)$

$\Rightarrow \quad y = 2x + 2$

PROBLEM 4

Let C be the curve $y^3 - 3xy + 2 = 0$. If H is the set of points on the curve C where the tangent is horizontal and V is the set of the point on the curve C where the tangent is vertical, then $H = \ldots$ and $V = \ldots$

SOLUTION

The given curve is C: $y^3 - 3xy + 2 = 0$.

Differentiating it with respect to x, we get

$$3y^2 \frac{dy}{dx} - 3x\frac{dy}{dx} - 3y = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{y^2 - x}$$

$\therefore \quad$ Slope of tangent to C at pt (x_1, y_1) is $\frac{dy}{dx} = \frac{y_1}{-x_1 + y_1^2}$

For horizontal tangent $\frac{dy}{dx} = 0 \Rightarrow y_1 = 0$.

For $y_1 = 0$ in C, we get no value of x_1

∴ There is no pt. on C at which tangent is horizontal.

∴ $H = \phi$

For vertical tangent $-x_1 + y_1^2 = 0 \Rightarrow x_1 = y_1^2$

From C, $y_1^3 - 3y_1^3 + 2 = 0 \Rightarrow y_1^3 = 1$

$\Rightarrow \quad y_1 = 1 \Rightarrow x_1 = 1$

There is only one point (1, 1) at which vertical can be drawn ∴ $V = \{(1, 1)\}$

PROBLEM 5

Find the sum of the coordinates of the point on the graph of $f(x) = x^3 + 4x$ the tangent at which is parallel to the chord joining the points (–2, –16) and (1, 5).

SOLUTION

Slope of chord $= \dfrac{5-(-16)}{1-(-2)} = 7$

$$f'(x) = 3x^2 + 4$$

By L.M.V.T $\exists c \in (-2, 1)$ such that $f'(c) = 7$

$$3c^2 + 4 = 7 \Rightarrow c = \pm 1$$

∴ $c = -1$ point $(c, f(c)) = (-1, -5)$.

Hence, sum of co-ordinates of point = –1 – 5 = –6

PROBLEM 6

Show that the normal at the point $\left(3t, \dfrac{4}{t}\right)$ of the curve $xy = 12$ cuts the curve again at the point whose parameter t_1 is given by $t_1 = -\dfrac{16}{9t^3}$.

SOLUTION

Let point $\left(3t, \dfrac{4}{t}\right)$ on curve $xy = 12$.

$$x \cdot \frac{dy}{dx} + y \cdot 1 = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

Slope of tangent $= \left.\dfrac{dy}{dx}\right|_{(3t, 4/t)} = \dfrac{-4/t}{3t} = -\dfrac{4}{3t^2}$

Slope of normal $= \dfrac{3t^2}{4}$

Equation of normal $y - \dfrac{4}{t} = \dfrac{3t^2}{4}(x - 3t)$

$$\Rightarrow \quad y = \left(\frac{3t^2}{4}\right)x + \left(\frac{4}{t} - \frac{9t^3}{4}\right) \quad \ldots(1)$$

Solving equation (1) and $xy = 12$, we get

$$\frac{12}{x} = \left(\frac{3t^2}{4}\right)x + \left(\frac{4}{t} - \frac{9t^3}{4}\right)$$

$\Rightarrow \quad 3t^3x^2 + (16 - 9t^4)x - 48t = 0$ roots are x_1 and x_2

As $x_1 = 3t$; $x_2 = 3t_1$

$$x_1x_2 = -\frac{16}{t^2} \Rightarrow 3t \cdot 3t_1 = -\frac{16}{t^2} \Rightarrow t_1 = -\frac{16}{9t^3}$$

Hence proved.

PROBLEM 7

If p and q be the intercepts on the axes by the tangent at any point on the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$.

Show that $\dfrac{p^2}{a^2} + \dfrac{q^2}{b^2} = 1$.

SOLUTION

Curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$

Differentiating above curve w.r.t. x, we get

$$\frac{1}{a^{2/3}} \cdot \frac{2}{3} x^{-1/3} + \frac{1}{b^{2/3}} \cdot \frac{2}{3} y^{-1/3} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{b^{2/3}} \frac{1}{y^{1/3}} \frac{dy}{dx} = -\frac{1}{a^{2/3}} \frac{1}{x^{1/3}}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(x_1, y_1)} = -\left(\frac{b}{a}\right)^{2/3} \left(\frac{y_1}{x_1}\right)^{1/3}$$

Hence, equation of tangent at point (x_1, y_1).

$$y - y_1 = -\left(\frac{b}{a}\right)^{2/3} \left(\frac{y_1}{x_1}\right)^{1/3} (x - x_1)$$

$$\Rightarrow \quad a^{2/3} x_1^{1/3} y - a^{2/3} y_1 x_1^{1/3} = -b^{2/3} x y_1^{1/3} + b^{2/3} x_1 y_1^{1/3}$$

$$\Rightarrow \quad b^{2/3} y_1^{1/3} x + a^{2/3} x_1^{1/3} y = a^{2/3} y_1 x_1^{1/3} + b^{2/3} x_1 y_1^{1/3}$$

$$= x_1^{1/3} y_1^{1/3} (a^{2/3} y_1^{2/3} + b^{2/3} x_1^{2/3})$$

$$\frac{x}{\dfrac{x_1^{1/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})}{b^{2/3}}} + \frac{y}{\dfrac{y_1^{1/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})}{a^{2/3}}} = 1$$

Hence, $p = \dfrac{x_1^{1/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})}{b^{2/3}}$;

$$q = \frac{y_1^{1/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})}{a^{2/3}}$$

$$\frac{p^2}{a^2} = \frac{x_1^{2/3}}{a^2 b^{4/3}} (a^{2/3} y_1^{2/3} + b^{2/3} x_1^{2/3})^2;$$

$$\frac{q^2}{b^2} = \frac{y_1^{2/3}(a^{2/3}y_1^{2/3} + b^{2/3}x_1^{2/3})^2}{b^2 a^{4/3}}$$

$$\frac{p^2}{a^2}+\frac{q^2}{b^2}=(a^{2/3}\,{y_1}^{2/3}+b^{2/3}\,{x_1}^{2/3})^2\left[\frac{x_1^{2/3}}{a^2b^{4/3}}+\frac{y_1^{2/3}}{b^2a^{4/3}}\right]$$

$$=(a^{2/3}\,{y_1}^{2/3}+b^{2/3}\,{x_1}^{2/3})^2\,\frac{[b^{2/3}x_1^{2/3}+a^{2/3}y_1^{2/3}]}{a^2b^2}$$

$$=\frac{(a^{2/3}b^{2/3})^3}{a^2b^2}=\frac{a^2b^2}{a^2b^2}=1$$

Hence, $\dfrac{p^2}{a^2}+\dfrac{q^2}{b^2}=1$

PROBLEM 8

If the tangent to the curve $x = 1 - 3t^2$, $y = t - 3t^3$ at the point $P(-2, 2)$ meets the curve again at Q, show that tangents at P and Q are mutually perpendicular.

SOLUTION

Given curve $x = 1 - 3t^2$, $y = t - 3t^3$

For point P, $x = -2 = 1 - 3t^2$, then $t = -1, 1$

$y = 2 = t - 3t^3 \Rightarrow t = -1$ satisfies it

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\left.\frac{9t^2-1}{6t}\right|_{t=-1}\Rightarrow\frac{dy}{dx}=-\frac{4}{3}\qquad\ldots(1)$$

The tangent at P, $y - 2 = -\dfrac{4}{3}(x+2) \Rightarrow 4x + 3y = -2.$

Solving with given curve, $x = 1 - 3t^2$, $y = t - 3t^3$

$$4(1-3t^2)+3(t-3t^3)=-2$$

$$\Rightarrow\quad (t+1)^2\,(3t-2)=0$$

$t = -1$, point of tangency, slope of tangent at $P = m_1 = -\dfrac{4}{3}$

$t=\dfrac{2}{3}$ is for point Q

Slope of tangent at $Q = m_2 = \dfrac{dy}{dx}=\dfrac{9t^2-1}{6t}=\dfrac{9\left(\frac{4}{9}\right)-1}{6\left(\frac{2}{3}\right)}=\dfrac{3}{4}$

Clearly, $m_1m_2 = -1$

Hence, the tangents at Point P and Q are perpendicular.

PROBLEM 9

Show that in the curve $y = a\ln(x^2 - a^2)$, sum of the length of tangent and sub-tangent varies as the product of coordinates of the point of contact.

SOLUTION

$$y=a\ln(x^2-a^2)$$

Differentiate above curve w.r.t. x, we get

$$\frac{dy}{dx}=\frac{a}{(x^2-a^2)}2x=\frac{2ax}{(x^2-a^2)}=y'$$

$$\Rightarrow\quad 1+y^2=1+\left(\frac{2ax}{x^2-a^2}\right)^2=\frac{x^4+a^4-2a^2x^2+4a^2x^2}{(x^2-a^2)^2}$$

$$\Rightarrow\quad 1+y^2=\left(\frac{x^2+a^2}{x^2-a^2}\right)^2\Rightarrow\sqrt{1+y'^2}=\frac{x^2+a^2}{x^2-a^2}$$

Sum of tangent and sub-tangent $=\left|\dfrac{y}{y'}\times\sqrt{1+y'^2}\right|+\left|\dfrac{y}{y'}\right|$

$$=\left|\frac{y}{y'}\right|\left|\sqrt{1+y'^2}+1\right|=\left|\frac{y}{\frac{2ax}{x^2-a^2}}\right|\left|\frac{x^2+a^2}{x^2-a^2}+1\right|$$

$$=\left|\frac{y\times(x^2-a^2)}{2ax}\right|\left|\frac{x^2+a^2+x^2-a^2)}{x^2-a^2}\right|$$

$$=\left|\frac{y}{2ax}\times 2x^2\right|=\left|\frac{xy}{a}\right|$$

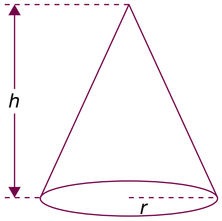
Hence, length of tangent and sub-tangent ∝ $|xy|$, proved.

PROBLEM 10

Sand is pouring from a pipe at the rate of 12 cc/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always 1/6 th of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

SOLUTION

Let V is volume of cone formed by sand. Cone has height h and radius r.



Given $\dfrac{dV}{dt}=12$ cc/sec; $h=\dfrac{1}{6}r\Rightarrow r=6h$

$$V=\text{volume of cone}=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi(6h)^2\times h=12\pi h^3$$

Differentiating equation of $V = 12\pi h^3$ w.r.t. time, we get

$$\frac{dV}{dt}=12\pi\times 3h^2\times\frac{dh}{dt}$$

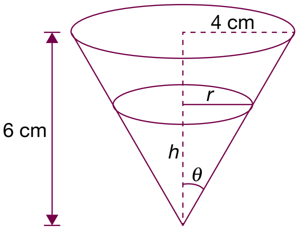
$$\Rightarrow\quad\frac{dh}{dt}=\frac{dV}{dt}\times\frac{1}{36\pi\times h^2}$$

$$\frac{dh}{dt}=\frac{12}{36\pi\times4\times4}\quad\frac{dh}{dt}=\frac{1}{48\pi}\text{ cm/sec.}$$

PROBLEM 11

A filter paper in the form of a right circular cone has diameter of its base equal to 8 cm. and depth 6 cm. Water is flowing out through the bottom of the cone at the rate of 50 cc/min. Find the rate at which the level of water is falling when the height of the water is 3 cm.

SOLUTION



Given,

$$\left.\begin{aligned} r &= 4\text{ cm}\\ h &= 6\text{ cm}\end{aligned}\right\} \frac{r}{h} = \frac{4}{6} = \frac{2}{3}$$

$$r = \frac{2}{3} \times h;\ \frac{dv}{dt} = 50\text{ cc/min.}$$

Volume of cone, $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi\left(\frac{2}{3}h\right)^2 h = \frac{4\pi}{27}h^3$$

On differentiating, we get,

$$\Rightarrow \quad \frac{dV}{dt} = \frac{4\pi}{27} \times 3h^2 \times \frac{dh}{dt}$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{dV}{dt}\left(\frac{27}{4\pi \times 3h^2}\right);$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{50 \times 27}{4\pi \times 3 \times (3)^2} = \frac{25}{2\pi}\text{ cm/min.}$$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. A function f is defined by $f(x) = \int_0^x \frac{dt}{1+t^2}$. The normal line to $y = f(x)$ at $x = 1$, has x-intercept equal to x and y-intercept equal to Y, then

(A) $X = 2Y$ (B) $Y = 2X$

(C) $Y + 2X = 0$ (D) $X + Y = 0$

2. Let a, b be real numbers and C be the graph of function $f(x) = e^{a+bx^2}$. If slope of the tangent line on the curve C at point $P(1, 1)$ is 2, then the value of $(a^2 + b^2)$ is equal to

(A) 1 (B) 2

(C) 4 (D) 8

3. The equation of tangents to the curve $C_1 : (1 + x^2)\, y = 1$, at the points of its intersection with the curve $C_2 : (x + 1)\, y = 1$ are given by

(A) $x + 2y = 1; y = 1$ (B) $x + 2y = 3; y = 1$

(C) $x + 2y = 2; y = 1$ (D) $x + 2y = 1; x = 1$

4. A tangent is drawn at the point M with abscissa unity on the curve $y = \left(5 - x^{\frac{2}{3}}\right)^{\frac{3}{2}}$. If length of segment of tangent intercepted between the co-ordinate axes is $\sqrt{N}$, then N is equal to

(A) 25 (B) 50

(C) 75 (D) 125

5. The equation of normal to the curve $x + y = x^y$, where it cuts the x-axis is equal to

(A) $y = -2x + 2$ (B) $y = 3x - 3$

(C) $y = x - 1$ (D) $2y = x - 1$

6. If $f(x) = \lim\limits_{h \to 0} \frac{e^{\sin^{-1}(x+h)} - e^{\sin^{-1} x}}{h}$, then equation of tangent drawn to the graph of the function $y = f(x)$ at $x = 0$, is

(A) $y = 2x + 1$ (B) $x + y = 1$

(C) $y = x + 1$ (D) $2x + y = 1$

7. If the line passing through the points $(-1, 8)$ and $(0, 6)$ is tangent to the curve $y = \frac{a^2}{x-1}$, then a^2 equals

(A) 2 (B) 4

(C) 1 (D) 8

8. Let $g(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$ then the equation of normal drawn to $g(x)$ at $x = 0$, is

(A) $x = 0$ (B) $y = 0$

(C) $x = 1$ (D) $y = 1$

9. The values of parameter such that the line $(\log_2(1 + 5a - a^2))x - 5y - (a^2 - 5) = 0$ is a normal to the curve $xy = 1$, may lie in the interval

(A) $(-\infty, 0)$ (B) $[0, 5]$

(C) $(0, 5)$ (D) $(-\infty, 0) \cup (5, \infty)$

10. Line l is tangent to the curve $y = e^x$ and is parallel to the line $x - 4y = 1$. The x-intercept of the line is equal to

(A) $-\ln 2e$ (B) $-\ln (4e)$

(C) $\ln\left(\frac{e}{4}\right)$ (D) $\ln\left(\frac{4}{e}\right)$

11. C is a curve defined by the parametric equation $x = t^2$ and $y = t^3 - 3t$. The equation of the tangent line to this curve at $(4, 2)$ is

(A) $y = 15x - 58$ (B) $y = 2$

(C) $y = x - 2$ (D) $y = \frac{9x}{4} - 7$

12. If the surface area of a sphere of radius r is increasing uniformly at the rate 8 cm^2/sec, then the rate of change of its volume is

(A) Proportional to $\sqrt{r}$ (B) Constant

(C) Proportional to r (D) Proportional to r^2

13. If curves $x^2 = 9a\,(9 - y)$ and $x^2 = a(y + 1)$ intersect orthogonally then value of 'a' is

(A) 1 (B) 2

(C) 3 (D) 4

14. Let a line l through origin is tangent to the curve $y = x^3 + x + 16$. The slope of the line l is

(A) 2 (B) 8

(C) 13 (D) 26

15. Water is dropped at the rate of 2 m^3/sec. into a cone of semi-vertical angle 45°. The rate at which periphery of water surface changes when height of the water in the cone is 2 meter, is

(A) 1 m/sec. (B) 2 m/sec.

(C) 3 m/sec. (D) 4 m/sec.

16. If the curves $x^{2/3} + y^{2/3} = c^{2/3}$ and $(x^2/a^2) + (y^2/b^2) = 1$ touches each other, then

(A) $a + b = c$ (B) $a - b = c$

(C) $a + 2b = c$ (D) $2a - b = c$

17. If $\sqrt{x-5} = mx + 2$ has two distinct real solutions, then range of m is equal to

(A) $\left(\frac{-1}{2}, \frac{1}{10}\right)$ (B) $\left(0, \frac{1}{10}\right)$

(C) $\left(\frac{-1}{10}, \frac{1}{2}\right)$ (D) $\left(0, \frac{1}{2}\right)$

18. If curves $C_1 : y^2 = 2ax(a > 0)$ and $C_2 : xy = 4\sqrt{2}$ intersect orthogonally, then a equals

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$

(C) 2 (D) $\frac{3}{2}$

19. If the graph of $f(x) = \frac{3}{x}$ and $g(x) = ax + 1$ intersect at two distinct points, then the range of a is equal to

(A) $(0, \infty)$ (B) $\left(-\infty, \frac{-1}{12}\right) \cup (0, \infty)$

(C) $\left(\frac{-1}{12}, 0\right) \cup (0, \infty)$ (D) $\left(-\infty, \frac{-1}{12}\right) \cup \left(\frac{-1}{12}, \infty\right)$

20. The value of x which satisfies the conclusion of the mean value theorem for the function $y = x^2 - 3x - 1$ over the interval [0, 4], is

(A) 0 (B) 1

(C) 2 (D) 3

21. Suppose that $f(0) = -3$ and $f'(x) \le 5$ for all values of x. Then the largest value which $f(2)$ can attain is

(A) 7 (B) –7

(C) 13 (D) 8

22. In which one of the following functions, Rolle's theorem is applicable?

(A) $f(x) = |x|$, in $-2 \le x \le 2$

(B) $f(x) = \tan x$, in $0 \le x \le \pi$

(C) $f(x) = 1 + (x - 2)^{2/3}$, in $1 \le x \le 3$

(D) $f(x) = x(x - 2)^2$, in $0 \le x \le 2$

23. If $2x^3 - 6x + a = 0$ has three real and distinct roots, then number of distinct integral values of 'a' is

(A) 5 (B) 6

(C) 7 (D) ∞

24. A spherical balloon is being inflated at the rate of 35 cc/min. The rate of increase in surface area (in cm^2/min.) of the balloon when its diameter is 14 cm, is

(A) 100 (B) $10\sqrt{10}$

(C) $\sqrt{10}$ (D) 10

25. If the curves $\frac{x^2}{c} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ intersect at right angles, then c is equal to

(A) $\frac{3}{4}$ (B) $\frac{4}{3}$

(C) $\frac{1}{3}$ (D) 2

26. The equation of tangent to the graph of $y = \ln(x^2 + 4) - x$ arc $\tan\left(\frac{x}{2}\right)$ at $x = 2$, is equal to

(A) $\frac{\pi x}{4} + y = \ln 8$ (B) $\frac{\pi x}{4} + y = \pi - \ln 8$

(C) $\frac{\pi x}{4} + y = \ln 4$ (D) $\frac{\pi x}{4} - y = \pi - \ln 4$

27. If tangent drawn to the curve $y = |x^2 - 6x + 8|$ at its point $P(h, k)$ where $h \in (2, 4)$, which also touches the curve $y = x^2 - 6x + \lambda$, then number of integral values of λ is

(A) 0 (B) 1

(C) 2 (D) 3

28. The angle between the tangent at any point P and the line joining P to the origin 'O' is the same at all points of the curve $\ln(x^2 + y^2) = c \tan^{-1}\left(\frac{y}{x}\right)$ and it is $\frac{\pi}{4}$. Find c. (where c is constant)

(A) 1 (B) 2

(C) 3 (D) 4

29. The triangle formed by the normal to the curve $f(x) = x^2 - ax + 2a$ at the point (2, 4) and the coordinate axes lie in second quadrant. If its area is 2 sq. units, then a can be

(A) 2 (B) $\frac{17}{4}$

(C) $\frac{19}{4}$ (D) 5

30. A lamp of negligible height is placed on the ground ℓ_1m. away from a wall. A man $\ell_2 m$ tall is walking at a speed of $\frac{\ell_1}{10}$ m/s from the lamp to the nearest point on the wall. When he is midway between the lamp and the wall, the rate of change in the length of this shadow on the wall is

(A) $-\frac{5\ell_2}{2}$m/s (B) $-\frac{2\ell_2}{5}$m/s

(C) $-\frac{\ell_2}{2}$m/s (D) $-\frac{\ell_2}{5}$m/s

31. Let $f(x)$ satisfy the requirements of LMVT in [0, 2] if $f(0) = 0$ and $|f'(x)| \le \frac{1}{2}$ for all x in (0, 2). Then

(A) $f(x) \le 2$ (B) $|f(x)| \le 1$

(C) $f(x) = 2x$ (D) $f(x) = 3$ for at least one x in [0, 2]

32. A man is moving away from a tower 41.6 m high at a rate of 2 m/s. If the eye level of the man is 1.6 m above the ground, then the rate at which the angle of elevation of the top of the tower changes, when he is at a distance of 30 m from the foot of the tower, is

(A) $-\frac{4}{125}$ radian/s (B) $-\frac{2}{25}$ radian/s

(C) $-\frac{1}{625}$ radian/s (D) None of these

33. If $3(a + 2c) = 4(b + 3d)$, then the equation $ax^3 + bx^2 + cx + d = 0$ will have

(A) No real solution

(B) At least one real root in (–1, 0)

(C) At least one real root in (0, 1)

(D) None of these

34. A line L is perpendicular to the curve $y = \frac{x^2}{4} - 2$ at its point P and passes through (10, –1). The coordinates of the point P are

(A) (2, –1) (B) (6, 7)

(C) (0, –2) (D) (4, 2)

35. The angle between the tangent to the graph of the function $f(x) = \int_2^x (2t - 5)dt$ at the points where the graph cuts the x-axis is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

36. If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b on x and y axis respectively, then the value of a^2b is

(A) $27\,c^3$ (B) $\frac{4}{27}c^3$

(C) $\frac{27}{4}c^3$ (D) $\frac{4}{9}c^3$

37. Let C be the curve $y = x^3$. The tangent at A meets the curve again at B. If the gradient at B is K times the gradient at A then K is equal to

(A) 4 (B) 2

(C) –2 (D) $\frac{1}{4}$

38. Let $f : [-1, 2] \to R$ be differentiable such that $0 \le f'(t) \le 1$ for $t \in [-1, 0]$ and $-1 \le f'(t) \le 0$ for $t \in [0, 2]$. Then

(A) $-2 \le f(2) - f(-1) \le 1$ (B) $1 \le f(2) - f(-1) \le 2$

(C) $-3 \le f(2) - f(-1) \le 0$ (D) $-2 \le f(2) - f(-1) \le 0$

39. A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. The value of $\frac{dx}{dt}$ when $x = 3$ is

(A) 4 (B) $\frac{9}{2}$

(C) $\frac{3\sqrt{3}}{2}$ (D) None of these

40. Number of solution of the equation $3\tan x + x^3 = 2$ in $\left(0, \frac{\pi}{4}\right)$ is

(A) 0 (B) 1

(C) 2 (D) 3

41. At the point $P(a, a^n)$ on the graph of $y = x^n$ $(n \in N)$ in the first quadrant a normal is drawn. The normal intersects the y-axis at the point $(0, b)$. If $\lim_{a\to 0} b = \frac{1}{2}$, then n equals

(A) 1 (B) 3

(C) 2 (D) 4

42. The curve $y = x^3 + x^2 - x$ has two horizontal tangents. The distance between these two horizontal lines, is

(A) $\frac{13}{9}$ (B) $\frac{11}{9}$

(C) $\frac{22}{27}$ (D) $\frac{32}{27}$

43. In which one of the following intervals Rolles theorem hold(s) good for $y = x^2 \sin\frac{1}{x} + x^3 \cos\frac{1}{2x}$:

(A) $\left[\frac{1}{\pi}, \frac{2}{\pi}\right]$ (B) $\left[\frac{1}{3\pi}, \frac{1}{\pi}\right]$

(C) $\left[\frac{1}{2\pi}, \frac{1}{\pi}\right]$ (D) $\left[\frac{1}{\pi}, \frac{3}{\pi}\right]$

44. Let $f(t) = t^2$ for 0 to 1. And $g(t) = t^3$ for 0 to 1. The value of c with $0 < c < 1$ at which $\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(c)}{g'(c)}$ is

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$

(C) $\frac{1}{2}$ (D) $\frac{1}{6}$

45. Given $f(x) = 4 - \left(\frac{1}{2} - x\right)^{2/3}$, $g(x) = \begin{cases} \frac{\tan[x]}{x} & x \ne 0 \\ 1, & x = 0 \end{cases}$; $h(x) = \{x\}$; $k(x) = 5^{\log_2(x+3)}$ then in $[0, 1]$,

Lagrange's Mean Value Theorem is not applicable to

(A) f, g, h (B) h, k

(C) f, g (D) g, h, k

Where $[x]$ and $\{x\}$ denotes the greatest integer and fractional part function.

46. Let a function f be defined as $f(x) = \begin{cases} \frac{|x-1|}{x^2+1} & \text{if} \quad x > -1 \\ x^2 & \text{if} \quad x \le -1 \end{cases}$.

Then the number of critical point(s) on the graph of this function is/are

(A) 4 (B) 3

(C) 2 (D) 1

47. The ordinate of all points on the curve $y = \frac{1}{2\sin^2 x + 3\cos^2 x}$ where the tangent is horizontal, may be

(A) Always equal to $\frac{1}{2}$

(B) Always equal to $\frac{1}{3}$

(C) $\frac{1}{2}$ or $\frac{1}{3}$ according as n is an even or an odd integer if abscissa is $\frac{n\pi}{2}$

(D) $\frac{1}{2}$ or $\frac{1}{3}$ according as n is an odd or an even integer if abscissa is $\frac{n\pi}{2}$

48. A function $y = f(x)$ is defined on $[0, 4]$ as

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 \le x \le 1 \\ (x-2)^2 & \text{if} \quad 1 < x \le 3 \\ 1 & \text{if} \quad 3 < x \le 4 \end{cases}$$

. For the function $y = f(x)$

All the three conditions of Rolles theorem are violated on $[0, 4]$ then

(A) $f'(x)$ vanishes at some point in $(0, 4)$

(B) $f'(x)$ does not vanish in $(0, 4)$

(C) $f'(x)$ may or may not vanish at some point in $(0, 4)$

(D) None

49. If $P(x) = 2013x^{2012} - 2012x^{2011} - 16x + 8$, then $P(x) = 0$ for $x \in [0, 8^{1/2011}]$ has

(A) Exactly one real root

(B) No real root

(C) Atleast one and at most two real roots

(D) Atleast two real roots

50. If $M(x_0, y_0)$ is the point on the curve $3x^2 - 4y^2 = 72$, which is nearest to the line $3x + 2y + 1 = 0$, then the value of $(x_0 + y_0)$ is equal to

(A) 3 (B) –3

(C) 9 (D) –9

51. The temperature of a hot body decreases at a rate proportional to the excess of the temperature of the body over the temperature of the surrounding medium. A body heated to 100° is kept in a room with temperature 30° at time $t = 0$. After 5 minutes the temperature T drops to 80°. Its temperature will be 60° when t is

(A) 10 (B) $10\ln\left(\frac{7}{5}\right)$

(C) 6 (D) $5\log_{\left(\frac{7}{5}\right)}\left(\frac{7}{3}\right)$

52. The curve with the property that the projection of ordinate on the normal is constant and has the length equal to 'a' is

(A) $-x + a\log(\sqrt{y^2 - a^2} + y) = c$ (B) $x + \sqrt{a^2 - y^2} = c$

(C) $(y - a)^2 = cx$ (D) $ay = \tan^{-1}(x + y)$

53. The curve $y = ax^3 + bx^2 + cx + 5$ touches the x-axis at $P(-2, 0)$ and cuts the y-axis at a point $Q(0, 5)$ where its gradient is 3. Then $a + 2b + c$ is

(A) $-\frac{1}{2}$ (B) $-\frac{3}{4}$

(C) –3 (D) 1

54. If the function $f(x) = x^3 - ax^2 + 2x$ satisfies the conditions of LMVT over the interval [0, 2] and the tangent to the curve $y = f(x)$ at $x = \frac{1}{2}$ is parallel to the chord that joins the points of intersection of the curve with ordinates at $x = 0$ and $x = 2$, then the value of a is

(A) $\frac{9}{4}$ (B) $\frac{11}{4}$

(C) $\frac{13}{4}$ (D) $\frac{15}{4}$

55. If function $y = f(x)$ satisfy condition of Rolle's theorem in [1, 3] then the value of $\int_1^3 f'(x)dx$ is equal to

(A) 0 (B) 1

(C) 2 (D) 3

56. The Mean value theorem guarantees the existence of a special point on the graph of $y = \sqrt{x}$ between (0, 0) and (4, 2). The coordinates of this point are

(A) $(3, \sqrt{3})$ (B) (1, 1)

(C) $(2, \sqrt{2})$ (D) $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

57. The tangent at point of local minima say P of the curve $y = x^2 - x^3$ meets it again at Q, then mid point of PQ satisfy the curve with equation

(A) $y = 1 - 9x + 28x^2 - 28x^3$ (B) $y = 1 + 9x - 28x^2 - 28x^3$

(C) $y = 1 + 9x - 28x^2 + 14x^3$ (D) $y = 2 + 9x + 28x^2 - 28x^3$

58. The distance of the point on $y = x^4 + 3x^2 + 2x$ which is nearest to the line $y = 2x - 1$ is

(A) $\frac{4}{\sqrt{5}}$ (B) $\frac{3}{\sqrt{5}}$

(C) $\frac{2}{\sqrt{5}}$ (D) $\frac{1}{\sqrt{5}}$

59. The value of the parameter a so that the line $(3 - a)x + ay + \left(a^2 - 1\right) = 0$ is normal to the curve $xy = 1$ may lie in the interval

(A) $(-\infty, 0) \cup (3, \infty)$ (B) $(1, 3)$

(C) $(-3, 3)$ (D) $(0, 3)$

60. The number of continuous and derivable function(s) $f(x)$ such that $f(1) = -1$, $f(4) = 7$ and $f'(x) > 3$ for all $x \in R$ is are

(A) 0 (B) 1

(C) 3 (D) infinite

61. The lines tangent to the curves $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

62. If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies the Rolle's theorem in [1, 3] and $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$, then the values of a and b are respectively

(A) –6, 1 (B) –2, 1

(C) 1, –2 (D) 1, –6

63. A man of height 2 m walks directly away from a lamp of height 5 m, on a level road at 3 m/s. The rate at which the length of his shadow is increasing is

(A) 1 m/s (B) 2 m/s

(C) 3 m/s (D) 4 m/s

64. The orthogonal trajectory family of the family of curves $x^{2/3} + y^{2/3} = a^{2/3}$ is (a is arbitrary)

(A) $x^{2/3} - y^{2/3} = c$ (B) $x^{4/3} + y^{4/3} = c$

(C) $x^{4/3} - y^{4/3} = c$ (D) $xy = c$

65. The number of points in the rectangle $\{(x, y) \mid -12 \le x \le 12 \text{ and } -3 \le y \le 3\}$ which lie on the curve $y = x + \sin x$ and at which the tangent to the curve is parallel to the x-axis is

(A) 0 (B) 2

(C) 4 (D) 8

66. If $3x + 2y = 1$ acts as a tangent to $y = f(x)$ at $x = \frac{1}{2}$ and if $p = \lim_{x \to 0} \frac{x(x-1)}{f\left(\frac{e^{2x}}{2}\right) - f\left(\frac{e^{-2x}}{2}\right)}$ then, $\sum_{r=1}^{\infty} p^r$ is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$
(C) $\frac{1}{6}$ (D) $\frac{1}{7}$

67. If the curves $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{l^2}-\frac{y^2}{m^2}=1$ cut each other orthogonally, then
(A) $a^2+b^2=l^2+m^2$ (B) $a^2-b^2=l^2-m^2$
(C) $a^2-b^2=l^2+m^2$ (D) $a^2+b^2=l^2-m^2$

68. Two runners A and B start at the origin and run along positive x-axis, with B running three times as fast as A. An observer, standing one unit above the origin, keeps A and B in view. Then the maximum angle of sight 'θ' between the observes view of A and B is
(A) $\frac{\pi}{8}$ (B) $\frac{\pi}{6}$
(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

69. Let $f(x)=x^2+xg'(1)+g''(2)$ and $g(x)=f(1)x^2+xf'(x)+f''(x)$, then $f(3)+g(3)=$
(A) 7 (B) –7
(C) 0 (D) 6

70. Let the equation of a curve be $\frac{x^2}{4}+\frac{y^2}{3}=1$ where ($2\cos\theta$, $\sqrt{3}\sin\theta$) is a general point on the curve. If the tangent to the given curve intersects the co-ordinate axes at points A, B, then the locus of midpoint of AB is
(A) $2x^2+\sqrt{3}y^2=4$ (B) $3x^2+4y^2=4x^2y^2$
(C) $3x^2+4y^2=x^2y^2$ (D) $4x^2+3y^2=4x^2y^2$

71. A curve represented parametrically by the equation $x=t^3-4t^2-3t$ and $y=2t^2+3t-5$ where $t\in R$. If H denotes the number of point(s) on the curve where the tangent is horizontal and V is the number of point(s) where the tangent is vertical then
(A) $H=2, V=1$ (B) $H=1, V=2$
(C) $H=2, V=2$ (D) $H=1, V=1$

72. The abscissa of two points on $y=(2010)x^2+(2011)x-2011$ are 2010 and 2012. if the chord joining those two points is parallel to tangent at P on the curve then the ordinate of P is equal to
(A) $(2009)\times(2010)\times(2011)$
(B) $(2010)\times(2011)\times(2012)$
(C) $(2011)\times(2012)\times(2013)$
(D) None

73. Tangent at P_1 other than origin on the curve $y=x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve again at P_3 and so on … then $\frac{\text{area of }\Delta P_1P_2P_3}{\text{area of }\Delta P_2P_3P_4}$ equals
(A) 1 : 20 (B) 1 : 16
(C) 1 : 8 (D) 1 : 2

74. Tangents are drawn from origin to the curve $y=\sin x+\cos x$. Then their points of contact lie on the curve
(A) $\frac{1}{x^2}+\frac{2}{x^2}=1$ (B) $\frac{2}{x^2}-\frac{1}{y^2}=1$
(C) $\frac{2}{x^2}+\frac{1}{y^2}=1$ (D) $\frac{2}{y^2}-\frac{1}{x^2}=1$

75. Let $f(x)=\begin{cases}(1+x)^{\frac{2}{3}}, & -2\le x<0\\ (3x-5)^2, & 0\le x\le 2\end{cases}$. Which one of the following statements is correct?
(A) $f(x)$ is continuous at each point on [–2, 2]
(B) $f(x)$ is differentiable for all 'x' in (–2, 2)
(C) Rolle's Theorem is applicable for $f(x)$ in [–2, 2]
(D) There exists at least one c in (–2, 2) for which $f'(c)=0$

76. For what values of a, m and b, Lagrange's mean value theorem is applicable to the function $f(x)$ for $x\in[0, 2]$

$$f(x)=\begin{cases}3 & x=0\\ -x^2+a & 0<x<1\\ mx+b & 1\le x\le 2\end{cases}$$

(A) $a=3, m=-2, b=0$ (B) $a=3, m=-2, b=4$
(C) $a=3, m=2, b=0$ (D) No such a, m, b exist

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. If the tangent to the curve $2y^3=ax^2+x^3$ at the point (a, a) cuts off intercepts α and β on the coordinate axes, (where $\alpha^2+\beta^2=61$) then the value of a is
(A) -30 (B) 10
(C) 20 (D) 30

2. If L_T, L_N, L_{ST} and L_{SN} denote the lengths of tangent, normal. Sub-tangent and sub-normal respectively of a curve $y=f(x)$ at a point $P(2009, 2010)$ on it. Then
(A) $\frac{L_{ST}}{2010}=\frac{2010}{L_{SN}}$ (B) $\left|\frac{L_T}{L_N}\sqrt{\frac{L_{SN}}{L_{ST}}}\right|=$ constant
(C) $1-L_{ST}L_{SN}=\frac{2009}{2010}$ (D) $\left(\frac{L_T+L_N}{L_T-L_N}\right)^2=\frac{L_{ST}}{L_{SN}}$

3. Consider the curve $x^3+xy+y^3=3$ and line $x+y=k$ ($k\in R$). Which of the following statement(s) is/are correct?
(A) Line is tangent to the curve if $k=2$
(B) Line is tangent to the curve if $k=3$
(C) Line meets the curve in two points if $2<k<3$
(D) Line meets the curve in two points if $\frac{1}{3}<k<2$

4. Let $f(x) = \begin{cases} |2x-1|^{\alpha}\sin(2\pi x), & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$. If Rolle's theorem is applicable to $f(x)$ for $x \in [0, 1]$, then α can be

(A) 0 (B) 1

(C) 2 (D) 3

5. If the equation $x^5 - 10a^3x^2 + b^4x + c^5 = 0$ has three equal roots, then

(A) $2b^2 - 10a^3b^2 + c^5 = 0$ (B) $6a^5 + c^5 = 0$

(C) $2c^5 - 10a^3a^2 + b^4c^5 = 0$ (D) $b^4 = 15a^4$

6. The coordinate of the point on the graph of the function, $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$ where the tangent drawn cut off intercepts from the coordinate axis which are equal in magnitude but opposite in sign, is

(A) $\left(2, \frac{8}{3}\right)$ (B) $\left(3, \frac{7}{2}\right)$

(C) $\left(1, \frac{5}{6}\right)$ (D) None of these

7. The curve $y = ax^3 + bx^2 + cx$ is inclined at 45° to x-axis at (0, 0) and it touches x-axis at point (1, 0) then

(A) $y'(1) = 0$ (B) $y''(1) = 2$

(C) $y''(2) = 12$ (D) $y(1) = 0$

8. Given $f'(1) = 1$ and $f(2x) = f(x)$, $\forall\, x > 0$. If $f'(x)$ is differentiable, then there exists a number $c \in (2, 4)$ such that $f''(c)$ is greater than

(A) $\frac{1}{4}$ (B) $\frac{-1}{2}$

(C) $-1\frac{1}{4}$ (D) $-\frac{1}{8}$

9. If the tangent at any point $P\left(4m^2, 8m^3\right)$ of $x^3 - y^2 = 0$ is a normal also to the curve $x^3 - y^2 = 0$, then the value of m is

(A) $m = \frac{\sqrt{2}}{3}$ (B) $m = -\frac{\sqrt{2}}{3}$

(C) $m = \frac{3}{\sqrt{2}}$ (D) $m = -\frac{3}{\sqrt{2}}$

10. The coordinates of the point on the curve $(x^2+1)(y-3) = x$ where a tangent to the curve has the greatest slope are given by

(A) $\left(\sqrt{3}, 3 + \frac{\sqrt{3}}{4}\right)$ (B) $\left(-\sqrt{3}, 3 - \frac{\sqrt{3}}{4}\right)$

(C) $(0, 3)$ (D) $(3, 0)$

11. Consider the function g defined by

$g(x) = \begin{cases} x^2 \sin\frac{\pi}{x} + (x-1)^2 \sin\left(\frac{\pi}{x-1}\right); & x \ne 0, 1 \\ 0, & x = 0, 1 \end{cases}$. Which of the following statement(s) is/are correct?

(A) $g(x)$ is differentiable $\forall\, x \in R$

(B) $g'(x)$ is discontinuous at $x = 0$ but continuous at $x = 1$

(C) $g'(x)$ is discontinuous at both $x = 0$ and $x = 1$

(D) Rolle's theorem is applicable for $g(x)$ in [0, 1]

12. Let $f(x) = \begin{cases} (1-x)^{\alpha} x.(1-\cos(2\pi x)); & 0 < x \le 1 \\ 0; & x = 0 \end{cases}$. If Rolle's theorem is applicable to $f(x)$ for $x \in [0, 1]$, then α can be

(A) -2 (B) -1

(C) $\frac{1}{2}$ (D) 1

13. The values of the parameter 'a' so that the line $(3-a)x + ay + a^2 - 1 = 0$ is a normal to the curve $xy = 1$ may belong to the interval

(A) $(3, \infty)$ (B) $(-\infty, 0)$

(C) $(0, 1)$ (D) $(1, 2)$

14. Tangent at a point P_1 [other than (0, 0)] or the curve $y = x^3$ meets the curve again at P_2. The tangent at P_2 meets the curve at P_3 and so on. Then

(A) Abscissae of $P_1, P_2, P_3, \ldots P_n$ are in A. P.

(B) Abscissae of $P_1, P_2, P_3, \ldots P_n$ form a G. P.

(C) Area $\left(\Delta P_2P_3P_4\right) =$ 16 area $\left(\Delta P_1P_2P_3\right)$

(D) Area $\left(\Delta P_1P_2P_3\right) = 16$ area $\left(\Delta P_2P_3P_4\right)$

15. If $f(x)$ is continuous and derivable $\forall\, x \in R$ and $f'(c) = 0$ for exactly 2 real values of 'c', then the number of real and distinct value of 'd' for which $f(d) = 0$ can be?

(A) 1 (B) 2

(C) 3 (D) 4

16. Which of the following is/are correct?

(A) Between any two roots of $e^x \cos x = 1$, then exists atleast one root of $\tan x = 1$

(B) Between any two roots of $e^x \sin x = 1$, then exists atleast one root of $\tan x = -1$

(C) Between any two roots of $e^x \cos x = 1$, then exists atleast one root of $e^x \sin x = 1$

(D) Between any two roots of $e^x \sin x = 1$, then exists atleast one root of $e^x \cos x = 1$

17. If f is differentiable on $[a, b]$ and if $f(a) = f(b) = 0$, then for some $x \in (a, b)$.

(A) $x f(x) + f'(x) = 0$ (B) $x f(x) - f'(x) = 0$

(C) $50 f(x) + f'(x) = 0$ (D) $f'(x) + (\ln 50) f(x) = 0$

18. Which of the following functions fail to satisfy the condition of Rolle's theorem on the interval [–1, 1], (where $[x]$ denotes the greatest integer less than or equal to x and $\{x\}$ denotes the fractional part of x).

(A) $f(x) = |x| + |\sin x|$

(B) $f(x) = \begin{cases} \frac{\tan(x^2-1)}{1-x^2} + \frac{\ln|x|}{|x|-1} & x \ne \pm 1 \\ f(x) = \{x\} + \{-x\} & x = \pm 1, 0 \end{cases}$

(C) $f(x) = \{x\} + \{-x\}$

(D) $f(x) = |x| - |\sin x|$

19. Let $f : [0, 8] \to R$ be differentiable function such that $f(0) = 0$, $f(4) = 1$, $f(8) = 1$, then which of the following hold(s) good?

(A) There exist some $c_1 \in (0, 8)$ where $f'(c_1) = \frac{1}{4}$.

(B) There exist some $c \in (0, 8)$ where $f'(c) = \frac{1}{12}$.

(C) There exist $c_1, c_2 \in [0, 8]$ where $8f'(c_1) f(c_2) = 1$.

(D) There exist some $\alpha, \beta \in (0, 2)$ where

$$\int_0^8 f(t)dt = 3(\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3))$$

20. The equation of normal(s) to the curve represented parametrically by $x = t^5 - t^3$ and $y = t^2$ at the point (0, 1) is/are

(A) $x + y + 1 = 0$ (B) $x + y - 1 = 0$
(C) $x - y + 1 = 0$ (D) $x - y - 1 = 0$

21. The normal to the curve $x = a(\cos\theta + \theta \sin\theta)$, $y = a(\sin\theta - \theta \cos\theta)$ at any θ is such that

(A) It makes constant angle with x-axis
(B) It is a constant distance from the origin
(C) It touches circle $(x - \sin\theta)^2 + (y - \cos\theta)^2 = a^2$
(D) It passes through origin

22. Which of the following statement(s) is (are) incorrect about $f(x) = \frac{e^x}{x}$?

(A) $f(x)$ is increasing in $x \in (0, \infty)$
(B) Tangent on the curve $f(x)$ at $x = 1$ is perpendicular to x-axis.
(C) Tangent on the curve $f(x)$ at $x = 1$ is parallel to x-axis.
(D) Range of $f(x)$ is $[e, \infty)$

23. If Rolle's Theorem is applicable to the function f defined by $f(x) = \begin{cases} ax^2 + b; & |x| < 1 \\ 1 \quad ; & |x| = 1 \\ \frac{c}{|x|} \quad ; & |x| > 1 \end{cases}$ in the interval [–3, 3] then which of the following alternative(s) is/are correct?

(A) $a + b + c = 2$ (B) $|a| + |b| + |c| = 3$
(C) $2a + 4b + 3c = 8$ (D) $4a^2 + 4b^2 + 5c^2 = 15$

24. Let r be a positive constant. If two curves $C_1 : y = \frac{2x^2}{x^2 + 1}$ and $C_2 : y = \sqrt{r^2 - x^2}$ intersect orthogonally, then r cannot be

(A) $\frac{2}{3}$ (B) $\sqrt{3}$

(C) $\sqrt{\frac{2}{3}}$ (D) $\sqrt{2}$

25. Let $f : R \to R$ be defined as $f(x) = \sin x + ax + b$ $(a, b \in R)$. Then the equation $f(x) = 0$ has

(A) Only one real root which is positive if $a > 1, b < 0$
(B) Only one real root which is negative if $a > 1, b > 0$
(C) Only one real root which is negative if $a < -1, b < 0$
(D) Only one real root which is negative if $a < -1, b < 0$

26. Let $f(x)$ be a twice differentiable function such that $f''(x) > 0$ in $[0, 1]$. Then,

(A) $f(0) + f(1) = 2f(c)$ for some $c \in (0, 1)$

(B) $f(0) + f(1) = 2f\left(\frac{1}{2}\right)$

(C) $f(0) + f(1) > 2f\left(\frac{1}{2}\right)$

(D) $f(0) + f(1) < 2f\left(\frac{1}{2}\right)$

27. For which of the following function, Rolle's hypothesis is not satisfied?

(A) $f(x) = x^{\frac{4}{3}}, x \in [-1, 1]$

(B) $k(x) = |x| + |x - 2|, x \in [-1, 2]$

(C) $h(x) = x + \frac{1}{x}, x + \in \left[\frac{1}{5}, 5\right]$

(D) $g(x) = \begin{cases} \cos\left(\frac{1}{x}\right), & x \in [-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases}$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Consider $f(x) = ax^3 + bx^2 + cx + 4$ where $a, b, c \in R$ and $f''\left(\frac{-2}{3}\right) = 0$ and tangent drawn to the graph of the function $y = f(x)$ at $x = \frac{-2}{3}$ is $y = \frac{5x}{3} + \frac{100}{27}$.

1. The value of $(a + b + c)$ is equal to

(A) 4 (B) 6
(C) 7 (D) 10

2. If g is the inverse of f, then $\frac{d}{dx}(g(x) \cdot f(g(x))$ at $x = 4$ is equal to

(A) $\frac{1}{4}$

(B) $\frac{4}{7}$

(C) $\frac{3}{4}$

(D) $\frac{4}{3}$

Comprehension 2

Consider the curve $x = 1 - 3t^2$, $y = t - 3t^3$. If a tangent at a point $\left(1-3t^2, t-3t^3\right)$ inclined at an angle θ to positive x-axis and another tangent at point $P(-2, 2)$ cuts the curve again at Q. Then

3. The value of $\tan\theta + \sec\theta$ is equal to

(A) $3t$
(B) t
(C) $t - t^2$
(D) $t^2 - 2t$

4. The point Q will be

(A) $(1, -2)$
(B) $\left(-\frac{1}{3}, -\frac{2}{3}\right)$
(C) $(-2, 1)$
(D) None of these

Comprehension 3

Let the Tangent to the cubic curve $x^3 + y^3 = a^3$ at $P(x_1, y_1)$ meet the curve again at $Q(h, k)$. Put $A = \frac{h}{x_1}$ and $B = \frac{k}{y_1}$, $A \neq B$.

Then answer the following questions.

5. x_1^3 is equal to

(A) $\frac{a^3(1-B)}{A-B}$
(B) $a^3(1-A)$
(C) $\frac{a^3(1-B)}{A+B}$
(D) $\frac{a^3(1+B)}{A-B}$

6. Which of the following is true?

(A) $(A+B)^2 - AB(A+B) + AB + 1 = 0$
(B) $(A+B)^2 + AB(A+B) + AB + 1 = 0$
(C) $(A+B)^2 - AB(A+B) - AB - 1 = 0$
(D) $(A+B)^2 - AB(A+B) + AB - 1 = 0$

7. The value of $\frac{h}{x_1} + \frac{k}{y_1}$ must be

(A) 1
(B) –1
(C) 0
(D) 3

Comprehension 4

A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant $= k > 0$). Suppose that $r(t)$ is the radius of liquid cone at time t.

8. The time after which the cone is empty is

(A) $H/2k$
(B) H/k
(C) $H/3k$
(D) $2H/k$

9. The radius of water cone at $t = 1$ is

(A) $R[1 - k/H]$
(B) $R[1 - H/k]$
(C) $R[1 + H/\mathrm{k}]$
(D) $R[1 + k/H]$

10. The value of $\sum_{i=1}^{10} r(i)$ is equal to

(A) $10R\left[2 - \frac{k}{H}\right]$
(B) $5R\left[2 + \frac{11k}{H}\right]$
(C) $5R\left[2 - \frac{11k}{H}\right]$
(D) $4R\left[2 - \frac{11k}{H}\right]$

Comprehension 5

One of the roots of the equation $f(x) = 0$ is an even prime integer where $f(x) = x^2 - ax + b$ ('a' is an odd positive integer). Also, known that $\sum_{i=1}^{5} f(i) = 20$ and two curves are defined as

$C_1 : y^2 = f(x)$ and $C_2 : y^2 = -f(x)$

11. Number of distinct normal to the curve C_1 which passes through the centre of the curve C_2 is/are

(A) 1
(B) 2
(C) 3
(D) 4

12. Equation of normal to the curve $y = f(x)$ which are parallel to the tangent from origin to the curve C_2 are $\pm\frac{x}{\sqrt{t}} = ty - v$, (where '$t$' and '$v$' are real constants) then 't' equals.

(A) 1
(B) 2
(C) 4
(D) 9

13. The smaller area bounded by $y = f(x)$ and curve C_2 is $\frac{3\pi - \lambda}{6\lambda}$ units then 'λ' equals.

(A) $\frac{4}{3}$
(B) 1
(C) 2
(D) 4

Comprehension 6

A window of fixed perimeter (including the base of the arch) is in the form of a rectangle surmounted by a semicircle. The semicircular portion is fitted with coloured glass, while the rectangular portion is fitted with clear glass. The clear glass transmits three times as much light per square meter as the coloured glass.
Let y be the length (i.e., base of the window) and x be the breadth of the rectangular portion and p be the perimeter

14. The ratio of the sides $y : x$ of the rectangle so that the window transmits the maximum light is

(A) $3:2$
(B) $6:(6+\pi)$
(C) $(6+\pi):6$
(D) $1:2$

15. If μ is the amount of light per square meter for the coloured glass and L is the total light transmitted, then $\left(\frac{dL}{dy}\right)_{y=1}$ is equal to

(A) $\frac{\mu}{2}\left[3p - 12 - \frac{5\pi}{2}\right]$
(B) $\mu\left[2p - 6 - \frac{3\pi}{2}\right]$
(C) $\mu\left[p - 2 - \frac{\pi}{2}\right]$
(D) $2\mu\left[3p - \frac{\pi}{2}\right]$

16. Critical points of L are

(A) $y = \frac{p}{6 + \frac{3\pi}{2}}$
(B) $y = \frac{3p}{12 + \frac{3\pi}{2}}$
(C) $y = \frac{3p}{12 + \frac{5\pi}{2}}$
(D) $y = \frac{2p}{11 + \frac{5\pi}{2}}$

Comprehension 7

By the Rolle's theorem between any two real zeros of a polynomial $p(x)$ lies a zero of $P'(x)$.

17. The number of real roots of $x^3 - 3x + 1 = 0$ lying in the interval (0, 1) is
(A) 0 (B) 1
(C) 2 (D) 3

18. Let $f(x)$ be a polynomial such that α, β with $\alpha < \beta$ are two consecutive zeroes of $f(x)$. The number of zeros of $f'(x)$ lying in the interval $[\alpha, \beta]$ is
(A) At most 1 (B) At most 2
(C) At least 1 (D) At least 2

19. If $a, b, c, \in R$ and $3b^2 - 8ac < 0$, then the equation

$f(x) = ax^4 + bx^3 + cx^2 + 2009x + 2010 = 0$ has
(A) All real roots
(B) All imaginary roots
(C) Cannot have all real roots
(D) Two real and two imaginary roots

Comprehension 8

Let $f(x)$ be a function such that its derivative $f'(x)$ is continuous in $[a, b]$ and derivable in (a, b). Consider a function $\phi(x) = f(b) - f(x) - (b - x) f'(x) - (b - x)^2 A$. If Rolle's theorem is applicable to $\phi(x)$ on $[a, b]$, answer following questions

20. If there exist some number c $(a < c < b)$ such that $\phi'(c) = 0$ and $f(b) = f(a) + (b - a) f'(a) + \lambda(b - a)^2 f''(c)$, then λ is
(A) 1 (B) 0
(C) $\frac{1}{2}$ (D) $-\frac{1}{2}$

21. Let $f(x) = x^3 - 3x + 3$, $a = 1$ and $b = 1 + h$. If there exists $c \in (1, 1 + h)$ such that $\phi'(c) = 0$ and $\frac{f(1+h) - f(1)}{h^2} = \lambda c$, then $\lambda =$
(A) $\frac{1}{2}$ (B) 2
(C) 3 (D) does not exists

22. Let $f(x) = \sin x$, $a = \alpha$ and $b = \alpha + h$. If there exists a real number t such that $0 < t < 1$, $\phi'(\alpha + th) = 0$ and
$\frac{\sin(\alpha + h) - \sin\alpha - h\cos\alpha}{h^2} = \lambda \sin(\alpha + th)$, then $\lambda =$
(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$
(C) $\frac{1}{4}$ (D) $\frac{1}{3}$

Comprehension 9

If function f, g are continuous in a closed interval $[a, b]$ and differentiable in the open interval (a, b), then there exists a number $c \in (a, b)$ such that $[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$.

23. If $f(x) = e^x$ and $g(x) = e^{-x}$, $a \le x \le b$, $a > 0$, then $c =$
(A) $\frac{a+b}{2}$ (B) $\sqrt{ab}$
(C) $\frac{2ab}{a+b}$ (D) $\frac{a+b}{3}$

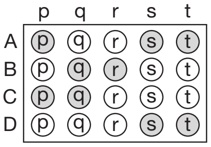
24. If $f(x) = \frac{1}{x^2}$ and $g(x) = \frac{1}{x}$, $a \le x \le b$, $a > 0$, then $c =$
(A) $\frac{a+b}{2}$ (B) $\sqrt{ab}$
(C) $\frac{2ab}{a+b}$ (D) $\frac{a+b}{3}$

25. If $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$, $a \le x \le b$, $a > 0$, then $c =$
(A) $\frac{a+b}{2}$ (B) $\sqrt{ab}$
(C) $\frac{2ab}{a+b}$ (D) $\frac{a+b}{3}$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. Match the following:

COLUMN-I	COLUMN-II
(A) A circular plate is expanded by heat from radius 6 cm 6.06 cm. Approx increase in the area is	(p) 5
(B) If an edge of a cube increase by 2%, then percentage increase in the volume is	(q) 0.72π
(C) If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is thrice the rate of decrease of x, then x is equal to (rate of decrease is non-zero)	(r) 6
(D) The rate of increase in the area of an equilateral triangle of side 30 cm, when each side increase at the rate of 0.1 cm/s is	(s) $\frac{3\sqrt{3}}{2}$

2. Tangent is drawn at any point (x_i, y_i) on the curve $f(x, y) = 0$, which intersects the x-axis at $(x_{i+1}, 0)$. Now again, tangent is drawn at (x_{i+1}, y_{i+1}) which intersects the x-axis at $(x_{i+2}, 0)$ and so on ($i = 1, 2, \dots n$) match the following COLUMN-I with COLUMN-II.

COLUMN-I	COLUMN-II
(A) $xy^2 = 1$	(p) The sequence $x_1, x_2 \dots x_4$ term an G.P.
(B) $y = e^{-3x}$	(q) The sequence $x_1, x_2, x_3 \dots x_4$ is periodic with period 2
(C) $y = -\cot^{-1}x$	(r) The sequence $x_1, x_2, \dots x_4$ terms an increasing A.P.
(D) $y^2 = x$	(s) The sequence $x_1, x_2, x_3 \dots$ is increasing, but not in A.P.

3. Match the following:

COLUMN-I	COLUMN-II
(A) The slope of the tangent to the curve represented parametrically by the equations $x = t^2 - 3t + 1$ and $y = 2t^2 + 3t - 4$ at the point $M(-1, 10)$ is	(p) 8
(B) If the largest possible interval in which $f(x) = x^3 + 6x^2 + px + 7$ is decreasing function is $(-3, -1)$, then p equals	(q) 7
(C) The value of a for which the function $f(\theta) = \frac{7}{3}\sin 3\theta + 2a\sin\theta - 5$ has an extremum at $\theta = \frac{\pi}{3}$ is	(r) 9
(D) Area of the triangle formed by tangent of the curve $x = t^2 + t + 2$, $y = t^2 - t + 2$, $t \geq 0$ which passes through (2, 2) and co-ordinate axes is equal to	(s) 11

(Continued)

4. If $f(x) = \begin{cases} \text{Min}\{2 - x^2, -x^3\} & ; \quad -2 \leq x < 0 \\ x\,|x| & ; \quad 0 \leq x < 1 \\ 3 - x & ; \quad 1 \leq x \leq 2 \end{cases}$ now match the following

COLUMN-I	COLUMN-II
(A) The number of critical points of $f(x)$ in $[-2, 2]$ is	(p) 1
(B) The length of the longest interval in which $f(x)$ satisfies Rolle's theorem is	(q) 2
(C) The number of values of 'c' of Rolle's Mean value theorem in $[-1, 1]$ is	(r) 3
(D) The maximum value of $f(x)$ in $[-2, 2]$ is	(s) 0

5. Match the following:

COLUMN-I	COLUMN-II
(A) If position of the tangent at any point on the curve $x = at^3$, $y = at^4$ between the axes is divided by the abscissa of the point of contact in the ratio $m : n$ externally, then $\lvert n + m \rvert$ is equal to (m and n are coprime)	(p) 1
(B) The area of triangle formed by normal at the point (1, 0) on the curve $x = e\sin y$ with axes is	(q) $\frac{1}{2}$
(C) If the angle between curves $x^2y = 1$ and $y = e^{2(1-x)}$ at the point (1, 1) is θ then $\tan\theta$ is equal to	(r) 7
(D) The length of sub-tangent at any point on the curve $y = be^{x/3}$ is equal to	(s) 3
	(t) 0

6. Match the following:

COLUMN-I	COLUMN-II
(A) $f : [0,4] \to R$ is differentiable and $a, b \in (0,4)$ are some constants then $\frac{(f(4))^2 - (f(0))^2}{f'(a)f(b)} =$	(p) 2

(Continued)

COLUMN-I	COLUMN-II
(B) Let $F(x)=f(x)g(x)h(x)$ for all real x where f, g, h are differentiable functions at some point x_0 then $F'(x_0)=21F(x_0), f'(x_0)=4f(x_0)$, $g'(x_0)=-7g(x_0)$ and $h'(x_0)=kh(x_0)$ then $\frac{k}{6}=$	(q) 4
(C) $f(x)$ is function such that $\lvert f(x_1)-f(x_2)\rvert \le \lvert x_1-x_2\rvert^2$ $\forall$ $x_1, x_2 \in R$ and normal at some point on $y=f(x)$ passes through (1, 2). Equation of normal is $lx+my-n=0$ Then, $lm-mn+nl=$	(r) 8
(D) The normal at $(at_1^2, 2at_1)$ on the curve $y^2=4ax$ meets the curve again at $(at_2^2, 2at_2)$, then t_2^2 takes the least value	(s) 1

7. Match the following:

COLUMN-I	COLUMN-II
(A) The curve $y=2e^{2x}$ intersect the y-axis at an angle $\cot^{-1}\left(\left\lvert\frac{8n-4}{3}\right\rvert\right)$, then the value of n is	(p) 2
(B) The area of triangle formed by normal at the point (1, 0) on the curve $x=e^{\sin y}$ with axes is $\left\lvert\frac{2t+1}{6}\right\rvert$ sq.units, then the value of t is	(q) –1
(C) The length of sub-tangent to the curve $x^2y^2=16$ at the point (–2, 2) is $\lvert K\rvert$, then the value of K is	(r) –2
(D) The slope of the tangent to the curve $y=\sqrt{4-x^2}$ at the point where the ordinate and the abscissa are equal is	(s) 4

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. The number of tangents to the curve $y^2-2x^3-4y+8=0$ which pass through (1, 2) is _______

2. Acute angle between two curves $x^2+y^2=a^2\sqrt{2}$ and $x^2-y^2=a^2$ is π/K, then K is _______

3. If the rate of change in the volume of sphere is equals to the rate of change in its radius then the radius is $\frac{1}{k\sqrt{\pi}}$, then k is _______

4. The tangent at any point on the curve $x^4+y^4=a^4$ cuts intercepts p and q on the axes, then $p^{-4/3}+q^{-4/3}=\frac{1}{\left(a^k\right)^{1/3}}$, then k is _______

5. The number of points lying in $\{(x,y)/\lvert x\rvert\le 10, \lvert y\rvert\le 3\}$, on the curve $y^2=x-\sin x$, at which the tangents to the curve are parallel to x-axis, is\are

6. If the normal to the curve $y=f(x)$ at $x=0$ is $3x-y+3=0$, then the value of $\lim\limits_{x\to 0}\frac{x^2}{\left\{f(x^2)-5f(4x^2)+4f(7x^2)\right\}}$ is $-1/k$, then k is _______

7. If the tangent at the point (a, b) to the curve $x^3+y^3=8$ meets the curve again in (a_1, b_1) and $\frac{a_1}{a}+\frac{b_1}{b}=-k$, then $k=$

8. The segment of the tangent to the curve $x^{2/3}+y^{2/3}=16$, contained between x and y-axis, has length equal to ℓ. Find $\sqrt{\ell}$.

9. Let $f(x)=\begin{cases}5, & x=0\\ x^2+5x+\alpha, & 0<x<2\\ \beta x+\gamma, & 2\le x\le 3\end{cases}$ If $f(x)$ satisfies all the conditions of LMVT in [0, 3] and $P(c, f(c))$ is the point on the curve $f(x)$ in [0, 3] where the tangent is parallel to the chord joining the end points, then find the value of $(\alpha+2\beta+3\gamma+6c)$.

10. The tangent to the curve $y=e^{kx}$ at the point (0, 1) meets the x axis at $(a, 0)$; $a\in[-2,-1]$. If $K\in\left[\frac{\alpha}{8},\frac{\alpha}{4}\right]$, then α is

11. If a variable tangent to the curve $x^2y=c^3$ makes intercepts of length a and b on x and y axis respectively then $\frac{a^2b}{c^3}=\mu$. Then $[\mu]$ is (where [.] is greatest integer function.)

12. The curve $y=ax^3+bx^2+cx+5$, touches the x-axis at $P(-2, 0)$ and cuts the y-axis at a point Q, where its gradient is 3, then find the value of $4b-2a+c$.

13. If the largest positive value of the function $f(x)=\sqrt{8x-x^2}-\sqrt{14x-x^2-48}$ is $m\sqrt{n}$ where $m, n\in N$, then find the least value of $(m+n)$.

14. For a twice differentiable function $f(x), g(x)$ is defined as $g(x)=f'(x)^2+f''(x)f(x)$ on $[a, e]$. If for $a<b<c<d<e$, $f(a)=0, f(b)=2, f(c)=-1, f(d)=2, f(e)=0$, then find the minimum number of zeros of $g(x)$.

15. The minimum value of the expression

$$\left\{(x_1-x_2)^2+\left(12+\sqrt{4-x_1^2}-\sqrt{16x_2}\right)^2\right\}\forall x_1, x_2 \in R$$

is $36-16\sqrt{K}$ then K is ________

16. If p_1 and p_2 are the length of the perpendiculars from origin on the tangent and normal drawn to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=6^{\frac{2}{3}}$, then find the value of $(4p_1^2+p_2^2)$.

17. If the portion of the tangent at any point on the curve $x=at^3$, $y=at^4$ between the axes is divided by the abscissa of the point of contact in the ratio of $m:n$ externally (where m and n are co-prime), then $|n+m|$ is equal to

18. The normal at the point $P\left(2,\frac{1}{2}\right)$ on the curve $xy=1$, meets the curve again at Q. If m is the slope of tangent of the curve at Q, then find $|m|$.

19. If the shortest distance between the line $y = x - 2$ and the parabola $y=x^2+3x+2$ is k, then the value of $2k^2$ is

20. Let f be an odd continuous function defined on set of real number such that $f(p) = (-1)^p\, p$, where p is a prime natural number, then find the minimum number of real roots of $f(x) = 0$.

21. A man 1.6 m high walks at the rate of 30 m per minute away from a lamp which is 4 m above ground. If k is the rate at which the man's shadow is lengthening, then the value of $k/4$ is _______

22. If the function $f(x)=x^3-6x^2+ax+b$ defined on [1, 3] satisfies Rolle's theorem for $c=\frac{2\sqrt{3}+1}{\sqrt{3}}$, then the value of $(a-4)$ is

23. If the curves $ay+x^2=7$ and $x^3=y$ intersect orthogonally at (1, 1), then the value of a is _____________

24. Two aircrafts are flying in two lines of a plane, the lines are inclined at an angle of 120° to each other with equal speed. At a certain moment one aircraft reaches the point of intersection of the lines while the second aircraft is at a distance of 2 km from it. The least possible distance between the aircrafts is

25. Water seeps out of a conical filter at the constant rate of 5 c.c./sec., the height of the filter is 20 cm and radius of the base is 10 cm. The height of the cone of water in the filter is 15 cm. The rate at which the height of the water decreases is $\frac{k}{45\pi}$ cm/sec then k is

26. Tangent lines are drawn at the points P and Q where $f''(x)$ vanishes for the function $f(x) = \cos x$ on $[0, 2\pi]$. The tangent lines at P and Q intersect each other at R so as to form a triangle PQR. If area of triangle PQR is $k\pi^2$, then find the value of $36k$.

27. Let P be the point on the curve $4x^2+\alpha^2y^2=4\alpha^2$, $0<\alpha^2<8$ whose distance from $Q(0, -2)$ is greatest. If R is the reflection of P in the x-axis, then find the least distance of R from the line $3x-4y+7=0$.

28. Let $y=f(x)$ be drawn with $f(0) = 2$ and for each real number 'a', the tangent to $y=f(x)$ at $(a, f(a))$, has x intercept $(a-2)$. If $f(x)$ is of the form $k\,e^{px}$, then $\left(\frac{k}{p}\right)$ has the value equal to

29. Number of points on the curve $y=e^{x^3-3x^2-9x+1}$ where the tangent is parallel to x-axis, is

30. If at each point on the curve $y=ax^3-ax^2+x+1$, the tangent is inclined at an acute angle (measured in anticlockwise direction) with the positive direction of x-axis, then number of integral values of 'a' is

31. If the length of the diagonal of a square is increasing at the rate of 0.2 cm/sec, then the rate of increase of its area is λ cm^2/sec when its side is $\frac{30}{\sqrt{2}}$ cm, then the value of λ is

32. If the curves $x^3-3xy^2+2=0$ and $3x^2y-y^3-2=0$ intersect at an angle θ, then the value of $\sin^2\theta$ is ______

33. If Rolle's theorem is applicable on the function $g(x)=\begin{cases}\dfrac{e^{px}-qx-1}{x}; & 0<x\le 1\\ 2; & x=0\end{cases}$ in interval $x\in[0, 1]$, then find the value of (p^2+q^2).

34. Let $f(x)$ be a derivable function $\forall\ x \in R$, where $f(0) = f(2) = f(4) = 3$ and $f(1) = f(3) = f(5) = -2$, then the minimum number of roots of $x(f(x)f''(x)+(f'(x))^2)+f(x)f'(x)=0$ in [0, 4] is equal to

35. The graph of function $y = f(x)$ has a unique tangent at $(e^a, 0)$ through which the graph passes then $\lim\limits_{x\to e^a}\frac{\log(1+7f(x))-\sin f(x)}{3f(x)}$ is equal to

36. If the curve $x^2=9k(9-y)$ and $x^2=k(y+1)$ intersect orthogonally, then find the value of k.

37. Let $P(a, b)$ be a point on the parabola $y=4x-x^2$ and is the point nearest toe the point $A(-1, 4)$. Find $(a+b)$

38. Let $f(x)=x(x^2+mx+n)+2$, for all $x\in R$ and $m, n\in R$. If Rolle's Theorem holds for $f(x)$ at $x=\frac{4}{3}$ in $x\in[1, 2]$, then $(m+n)$ equals

39. Let $f(x)$ be a non-constant thrice differentiable function defined on $(-\infty, \infty)$ such that $f(3+x)=f(3-x)$ and $f'\left(\frac{3}{2}\right)=0=f'(2)=f'(5)$. Determine the minimum number of zeroes of $g(x)=(f''(x))^2+f'(x)f''(x)$ in the interval [1, 6].

40. If the acute angle between the tangents drawn from (0, 1) to the graph of $f(x)=x^3-3x^2+3x$ is $\tan^{-1}\left(\frac{m}{n}\right)$, where m and n are co-prime natural number. Find the value of $(m+n)$.

41. A differentiable function $y=f(x)$ is such that its graph cuts the curve $y=ax^2+bx+c$ at 10 distinct points. Find the minimum number of distinct roots of $f'''(x)=0$.

42. If d be the minimum distance between curves $C_1 : y^2 = 8x$ and $C_2 : x^2 + y^2 + 12y + 35 = 0$, then find $(d + 1)^2$.

43. If the shortest distance between $2y^2 - 2x + 1 = 0$ and $2x^2 - 2y + 1 = 0$ is d, then find the number of solution of the equation $|\sin\theta| = 2\sqrt{2}\,d$ in the interval $[-\pi, 2\pi]$.

44. Let the function $f : [-4, 4] \to [-1, 1]$ be defined implicitly by the equation $x + 5y - y^5 = 0$. Find the area of triangle formed by tangent and normal to $f(x)$ at $x = 0$ and the line $y = 5$.

45. If the curves $y = \dfrac{(x-a)^2}{4}$ and $y = e^x$ touches each other, then find the sum of square of all possible value of a.

46. Let $f(x)$ be a continuous function on $[1, 3]$, differentiable on $(1, 3)$, $f(1) = 5$, $f(3) = 9$ and $f'(x) \geq 2$ for all x in $(1, 3)$. If $g(x) = \displaystyle\int_0^x f(t)dt \;\; \forall\, x \in [1, 3]$, then find the sum of greatest and least value of $g(x)$ on $[1, 3]$.

ARCHIVE: JEE MAIN

1. **[JEE Mains 2021]**
If the tangent to the curve $y = x^3$ at point $P(t, t^3)$ meets the curve again at Q, then the ordinate of point which divides PQ internally in the ratio 1 : 2 is
(A) $2t^3$ (B) $-t^3$
(C) $-2t^3$ (D) 0

2. **[JEE Mains 2021]**
If the curve $y = ax^2 + bx + c$, $x \in R$, passes through the point $(1, 2)$ and the tangent line to this curve at origin is $y = x$, then the possible value a, b, c are
(A) $a = 1, b = 1, c = 0$ (B) $a = \frac{1}{2}, b = \frac{1}{2}, c = 1$
(C) $a = -1, b = 1, c = 1$ (D) $a = 1, b = 0, c = 1$

3. **[JEE Mains 2021]**
If Rolle's theorem holds for the function $f(x) = x^3 - ax^2 + bx - 4$, $x \in [1, 2]$ with $f'\left(\frac{4}{3}\right) = 0$, then ordered pair (a, b) is equal to
(A) $(5, 8)$ (B) $(-5, 8)$
(C) $(5, -8)$ (D) $(-5, -8)$

4. **[JEE Mains 2021]**
The shortest distance between the line $x - y = 1$ and curve $x^2 = 2y$ and is
(A) 0 (B) $\frac{1}{2\sqrt{2}}$
(C) $\frac{1}{2}$ (D) $\frac{1}{\sqrt{2}}$

5. **[JEE Mains 2021]**
If the curves $x = y^4$ and $xy = k$ cut at right angles, then $(4k)^6$ is equal to ______.

6. **[JEE Mains 2021]**
If the normal to the curve $y(x) = \displaystyle\int_0^x (2t^2 - 15t + 10)\,dt$ at a point (a, b) is parallel to the line $x + 3y = -5$, $a > 1$, then the value of $|a + 6b|$ is equal to ______.

7. **[JEE Mains 2020]**
Let f be a twice differentiable function on $(1, 6)$. If $f(2) = 8$, $f'(2) = 5$. $f'(x) \geq 1$ and $f''(x) \geq 4$, for all $x \in (1, 6)$, then
(A) $f(5) \leq 10$ (B) $f'(5) + f''(5) \leq 20$
(C) $f(5) + f'(5) \geq 28$ (D) $f(5) + f'(5) \leq 26$

8. **[JEE Mains 2020]**
Which of the following points lies on the tangent to the curve $x^4e^y + 2\sqrt{y+1} = 3$ at the point $(1, 0)$?
(A) $(2, 2)$ (B) $(-2, 6)$
(C) $(-2, 4)$ (D) $(2, 6)$

9. **[JEE Mains 2020]**
If the lines $x + y = a$ and $x - y = b$ touch the curve $y = x^2 - 3x + 2$ at the points where the curve intersects the x-axis, then $\frac{a}{b}$ is equal to ______.

10. **[JEE Mains 2020]**
For all twice differentiable functions $f : R \to R$. with $f(0) = f(1) = f'(0) = 0$
(A) $f''(x) = 0$, for some $x \in (0, 1)$
(B) $f''(0) = 0$
(C) $f''(x) \neq 0$, at every point $x \in (0, 1)$
(D) $f''(x) = 0$, at every point $x \in (0, 1)$

11. **[JEE Mains 2020]**
If the tangent to the curve, $y = f(x) = x\log_e x$, $(x > 0)$ at a point $(c, f(c))$ is parallel to the line segment joining the points $(1, 0)$ and (e, e), then c is equal to
(A) $\frac{1}{e-1}$ (B) $e^{\left(\frac{1}{1-e}\right)}$
(C) $e^{\left(\frac{1}{e-1}\right)}$ (D) $\frac{e-1}{e}$

12. **[JEE Mains 2020]**
The position of a moving car at time t is given by $f(t) = at^2 + bt + c$, $t > 0$, where a, b and c are real numbers greater than 1, then the average speed of the car over the time interval $[t_1, t_2]$ is attained at the point
(A) $a(t_2 - t_1) + b$ (B) $(t_2 - t_1)/2$
(C) $2a(t_1 + t_2) + b$ (D) $(t_1 + t_2)/2$

13. **[JEE Mains 2019]**
If θ denotes the acute angle between the curves, $y = 10 - x^2$ and $y = 2 + x^2$ at a point of their intersection, then $|\tan\theta|$ is equal to
(A) $\frac{4}{9}$ (B) $\frac{7}{17}$
(C) $\frac{8}{17}$ (D) $\frac{8}{15}$

14. [JEE Mains 2019]

The shortest distance between the point $\left(\frac{3}{2},0\right)$ and the curve $y=\sqrt{x}$. $(x>0)$, is

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{5}{4}$

(C) $\frac{3}{2}$ (D) $\frac{\sqrt{5}}{2}$

15. [JEE Mains 2019]

The tangent to the curve, $y=xe^{x^2}$ passing through the point $(1, e)$ also passes through the point

(A) $\left(\frac{4}{3},2e\right)$ (B) $(3, 6e)$

(C) $(2, 3e)$ (D) $\left(\frac{5}{3},2e\right)$

16. [JEE Mains 2018]

If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is

(A) $\frac{7}{2}$ (B) 4

(C) $\frac{9}{2}$ (D) 6

17. [JEE Mains 2017]

The normal to the curve $y(x-2)(x-3) = x + 6$ at the point where the curve intersects the y-axis passes through the point.

(A) $\left(\frac{1}{2},\frac{1}{3}\right)$ (B) $\left(-\frac{1}{2},-\frac{1}{2}\right)$

(C) $\left(\frac{1}{2},\frac{1}{2}\right)$ (D) $\left(\frac{1}{2},-\frac{1}{3}\right)$

18. [JEE Mains 2016]

Consider $f(x)=\tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right), x\in\left(0,\frac{\pi}{2}\right)$. A normal to $y=f(x)$ at $x=\frac{\pi}{6}$ also passes through the point

(A) $\left(\frac{\pi}{4},0\right)$ (B) $(0, 0)$

(C) $\left(0,\frac{2\pi}{3}\right)$ (D) $\left(\frac{\pi}{6},0\right)$

19. [JEE Mains 2015]

The normal to the curve, $x^2 + 2xy - 3y^2 = 0$, at $(1, 1)$:

(A) Meets the curve again in the third quadrant

(B) Meets the curve again in the fourth quadrant

(C) Does not meet the curve again

(D) Meets the curve again in the second quadrant

20. [JEE Mains 2014]

If f and g are differentiable functions in $(0, 1]$ satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in [0, 1]$

(A) $2f'(c) = g'(c)$ (B) $2f'(c) = 3g'(c)$

(C) $f'(c) = g'(c)$ (D) $f'(c) = 2g'(c)$

21. [JEE Mains 2013]

The real number k for which the equation $2x^3 + 3x + k$ has two distinct roots in $[0, 1]$

(A) Lies between 1 and 2 (B) Lies between 2 and 3

(C) Lies between –1 and 0 (D) Does not exist

22. [JEE Mains 2013]

The intercepts on x-axis made by tangents to the curve, $y=\int_0^x |t|\,dt$, $x \in R$, which are parallel to the line $y = 2x$, are equal to

(A) ±1 (B) ±2

(C) ±3 (D) ±4

ARCHIVE: JEE ADVANCED

1. [JEE ADV 2020]

For a polynomial $g(x)$ with real coefficients, let m_g denote the number of distinct real roots of $g(x)$. Suppose S is the set of polynomials with real coefficients defined by

$S = [(x^2-1)^2 (a_0 + a_1x + a_2x^2 + a_3x^3) : a_0, a_1, a_2, a_3 \in \mathbb{R}]$.

For a polynomial f, let f' and f'' denotes its first and second order derivatives, respectively. Then the minimum possible value of $(m_{f'} + m_{f''})$, where $f \in S$, is ______

2. [JEE ADV 2018]

For every twice differentiable function $f : R \to [-2, 2]$ with $(f(0)^2) + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE?

(A) There exist $r, s \in R$, where $r < s$, such that f is one-one on the open interval (r, s)

(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| < 1$

(C) $\lim\limits_{x\to\infty} f(x) = 1$

(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

3. [JEE ADV 2018]

Let $f : (0, \pi) \to R$ be a twice differentiable function such that

$\lim\limits_{t\to x}\frac{f(x)\sin t - f(t)\sin x}{t-x} = \sin^2$ for all $x \in (0, \pi)$

If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement (s) is (are) True ?

(A) $f\left(\frac{\pi}{4}\right)=\frac{\pi}{4\sqrt{2}}$

(B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right)+f\left(\frac{\pi}{2}\right)=0$

4. [JEE ADV 2014]

The slope of the tangent to the curve $(y - x^5)^2 = (1 + x^2)^2$ at the point $(1, 3)$ is

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. B	**2.** B	**3.** C	**4.** D	**5.** C	**6.** C	**7.** A	**8.** A	**9.** C	**10.** B
11. D	**12.** C	**13.** D	**14.** C	**15.** A	**16.** A	**17.** B	**18.** C	**19.** C	**20.** C
21. A	**22.** D	**23.** C	**24.** D	**25.** B	**26.** A	**27.** C	**28.** B	**29.** D	**30.** B
31. B	**32.** A	**33.** B	**34.** D	**35.** D	**36.** C	**37.** A	**38.** A	**39.** A	**40.** B
41. C	**42.** D	**43.** B	**44.** A	**45.** A	**46.** A	**47.** D	**48.** A	**49.** D	**50.** B
51. D	**52.** A	**53.** D	**54.** C	**55.** A	**56.** B	**57.** A	**58.** D	**59.** A	**60.** A
61. D	**62.** D	**63.** B	**64.** C	**65.** A	**66.** A	**67.** C	**68.** B	**69.** B	**70.** B
71. B	**72.** B	**73.** B	**74.** D	**75.** D	**76.** B				

Multiple Correct Choice Type Questions

1. A, D	**2.** A, B	**3.** A, D	**4.** A, B, C, D	**5.** B, D
6. A, B	**7.** A, B, D	**8.** B, C	**9.** A, B	**10.** A, B
11. A, C, D	**12.** C, D	**13.** A, B	**14.** B, C	**15.** A, B, C
16. A, B, C	**17.** A, B, C, D	**18.** A, C	**19.** A, B, C, D	**20.** B, C
21. B, C	**22.** A, B, D	**23.** A, B, C, D	**24.** A, B, C	**25.** A, B, C
26. A, C	**27.** B, D			

Linked Comprehension Type Questions

1. B	**2.** D	**3.** A	**4.** D	**5.** A	**6.** C	**7.** B	**8.** B	**9.** A	**10.** C
11. A	**12.** B	**13.** D	**14.** B	**15.** A	**16.** C	**17.** B	**18.** C	**19.** C	**20.** C
21. C	**22.** B	**23.** A	**24.** C	**25.** B					

Matrix Match/Column Match Type Questions

1. A → (q)	B → (r)	C → (p)	D → (s)
2. A → (p)	B → (r)	C → (s)	D → (q)
3. A → (s)	B → (r)	C → (q)	D → (p)
4. A → (q)	B → (q)	C → (p)	D → (q)
5. A → (r)	B → (q)	C → (t)	D → (s)
6. A → (r)	B → (q)	C → (p, q, r, s)	D → (r)
7. A → (p, q)	B → (r)	C → (p, r)	D → (q)

Integer/Numerical Answer Type Questions

1. 2	**2.** 4	**3.** 2	**4.** 4	**5.** 2	**6.** 3	**7.** 1	**8.** 8	**9.** 34	**10.** 4
11. 6	**12.** 1	**13.** 8	**14.** 6	**15.** 2	**16.** 36	**17.** 7	**18.** 64	**19.** 9	**20.** 3
21. 5	**22.** 7	**23.** 6	**24.** 1	**25.** 4	**26.** 9	**27.** 3	**28.** 4	**29.** 2	**30.** 3
31. 6	**32.** 1	**33.** 4	**34.** 7	**35.** 2	**36.** 4	**37.** 4	**38.** 3	**39.** 12	**40.** 31
41. 7	**42.** 8	**43.** 3	**44.** 65	**45.** 4	**46.** 22				

ARCHIVE: JEE MAIN

1. C	**2.** A	**3.** A	**4.** B	**5.** 4	**6.** 406	**7.** C	**8.** B	**9.** 0.50	**10.** A
11. C	**12.** D	**13.** 4	**14.** 4	**15.** 1	**16.** 3	**17.** 3	**18.** 3	**19.** 2	**20.** 4
21. 4	**22.** 1								

ARCHIVE: JEE ADVANCED

1. 5	**2.** A, B, D	**3.** B, C, D	**4.** 8

CHAPTER

7

Monotonicity, Concavity and Maxima-Minima of Functions

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Monotonicity of a Function
(b) Monotonicity of Composite Function
(c) Inequality Using Monotonicity
(d) Concavity and Convexity of Curve
(e) Point of Inflection
(f) Inequality Using Concavity [Jensen Inequality]
(g) Stationary Points and Critical Points
(h) Local or Relative Maxima/Minima
(i) Use of Derivative for Identifying Extrema
(j) Global or Absolute Maxima/Minima
(k) Application of Derivatives in Geometrical/Physical Problems

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

MONOTONICITY OF A FUNCTION

The monotonicity of a function gives an idea about the behaviour of the function, it tells about increasing or decreasing pattern of function.

Monotonicity about a point $x = a$ where δ is a very small positive quantity.

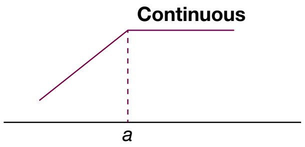
If $f(a - \delta) < f(a) < f(a + \delta)$ then $f(x)$ is strictly monotonically increasing.

If $f(a - \delta) \le f(a) \le f(a + \delta)$ then $f(x)$ is monotonically increasing (non-decreasing).

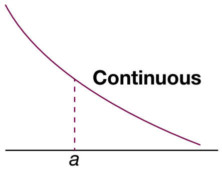
If $f(a - \delta) > f(a) > f(a + \delta)$ then $f(x)$ is strictly monotonically Decreasing.

If $f(a - \delta) \ge f(a) \ge f(a + \delta)$ then $f(x)$ is monotonically decreasing (non-increasing).

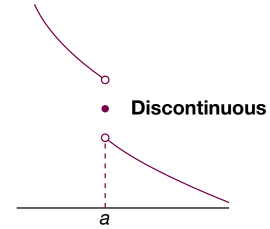
Consider following cases where monotonicity and continuity are explained around point $x = a$.



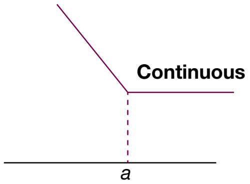
Monotonically increasing (non-decreasing)



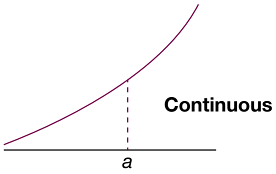
Strictly monotonically decreasing



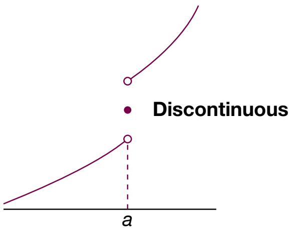
Strictly monotonically decreasing



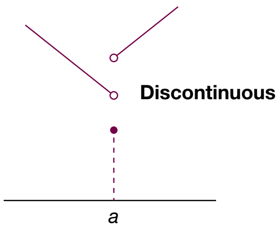
Monotonically decreasing (non-increasing)



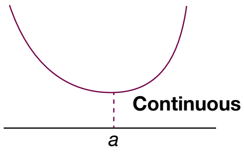
Strictly monotonically increasing



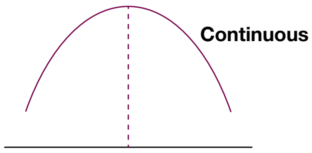
Strictly monotonically increasing



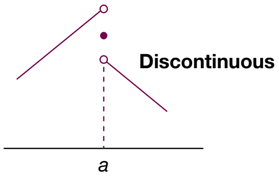
Neither increasing nor decreasing



Neither increasing nor decreasing



Neither increasing nor decreasing



Neither increasing nor decreasing

Note

Continuity and monotonicity are two independent attributes of a function.

ILLUSTRATION 1

Discuss monotonicity of following functions at given point.

(i) $f(x) = x^3$ at $x = 0$ **(ii)** $f(x) = x^2 + x|x|$ at $x = 0$

(iii) $f(x) = x^5 - x^3$ at $x = 0$ **(iv)** $f(x) = \sin^2 x$ at $x = \pi$

(v) $f(x) = \begin{cases} x^2 + x & ; \quad x < 0 \\ \sin x + 1 & ; \quad x \geq 0 \end{cases}$ at $x = 0$

(vi) $f(x) = \begin{cases} 2 - x^2 & ; \quad x < 1 \\ -x & ; \quad x \geq 1 \end{cases}$ at $x = 1$

(vii) $f(x) = \begin{cases} 1 & ; \quad x \leq 0 \\ x^2 + 1 & ; \quad x \geq 0 \end{cases}$ at $x = 0$

SOLUTION

(i) Given $f(x) = x^3$ at $x = 0$

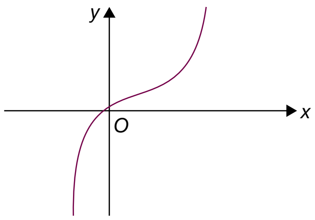
For $h > 0$, $x = 0 - h$, $f(0 - h) = -h^3 < 0$

$x = 0$, $f(0) = 0$

$x = 0 + h$, $f(0 + h) = h^3 > 0$

We have $f(0 - h) < f(0) < f(0 + h)$

Hence, $f(x)$ is strictly increasing function at $x = 0$



From graph also we can say that $f(x)$ is strictly increasing at $x = 0$

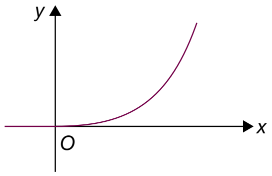
(ii) $f(x) = x^2 + x|x|$ at $x = 0$

$$f(x) = \begin{cases} x^2 + x(+x) & ; \quad x > 0 \\ 0 & ; \quad x = 0 \\ x^2 + x(-x) & ; \quad x < 0 \end{cases} = \begin{cases} 2x^2 & ; \quad x > 0 \\ 0 & ; \quad x = 0 \\ 0 & ; \quad x < 0 \end{cases}$$

$f(0 + h) = 2h^2 > 0$

$f(0) = 0$, $f(0 - h) = 0$

$\because$ $f(0 - h) = f(0) < f(0 + h)$



Therefore, at $x = 0$, $f(x)$ is non-decreasing.

(iii) $f(x) = x^5 - x^3$ at $x = 0$

For $h > 0$, $x = 0 - h$, $f(0 - h) = -h^5 - (-h^3)$

$f(0 - h) = h^3 - h^5 = h^3(1 - h^2) > 0$

$x = 0$, $f(0) = 0 - 0 = 0$

$x = 0 + h$, $f(0 + h) = h^5 - h^3 = h^3(h^2 - 1) < 0$

$\because$ $f(0 - h) > f(0) > f(0 + h)$

Hence, $f(x)$ is strictly decreasing function at $x = 0$.

(iv) $f(x) = \sin^2 x$ at $x = \pi$

$f(\pi - h) = \sin^2(\pi - h) = \sin^2 h > 0$

$f(\pi) = \sin^2 \pi = 0$

$f(\pi + h) = \sin^2(\pi + h) = (-\sin h)^2 = \sin^2 h > 0$

$\because$ $f(\pi - h) > f(\pi) < f(\pi + h)$

So, at $x = \pi$ non-monotonic (neither increasing nor increasing).

(v) $f(x) = \begin{cases} x^2 + x & ; \quad x < 0 \\ \sin x + 1 & ; \quad x \geq 0 \end{cases}$ at $x = 0$

$f(0 - h) = (-h)^2 - h = h^2 - h = h(h - 1) < 0$

$f(0) = 0 + 1 = 1$

$f(0 + h) = \sin h + 1$

$\because$ $f(0 - h) < f(0) < f(0 + h)$

So, at $x = 0$, $f(x)$ is strictly increasing function.

(vi) $f(x) = \begin{cases} 2 - x^2 & ; \quad x < 1 \\ -x & ; \quad x \geq 1 \end{cases}$ at $x = 1$

For $h > 0$,

$f(1 - h) = 2 - (1 - h)^2 = 2 - (1 + h^2 - 2h)$

$= 1 + 2h - h^2 = 1 + h(2 - h)$

$f(1) = -1$

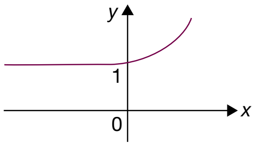
$f(1 + h) = -1 - h$

$\because$ $f(1 - h) > f(1) > f(1 + h)$

Therefore, at $x = 1$, $f(x)$ is strictly increasing function.

(vii) $f(x) = \begin{cases} 1 & ; \quad x < 0 \\ x^2 + 1 & ; \quad x \geq 0 \end{cases}$ check monotonicity at $x = 0$

From the graph of $f(x)$, clearly we can say that



At $x = 0$, $f(x)$ is non-decreasing function

For $x \in [0, \infty)$, $f(x)$ is monotonically increasing function.

MONOTONICITY IN AN INTERVAL

Using Graph

If we plot graph of $f(x)$, from the figure, we can conclude about their increasing and decreasing behaviour.

Mathematical Definition Approach

Let consider function $y = f(x)$ $x \in [a, b]$.

(i) If $x_1 < x_2$ then $f(x_1) < f(x_2)$ for all x_1 and x_2 in interval $[a, b]$ then function is strictly monotonically increasing.

(ii) If $x_1 < x_2$ then $f(x_1) \le f(x_2)$ for all x_1 and x_2 in interval $[a, b]$ then function is monotonically increasing (non- decreasing).

(iii) If $x_1 < x_2$ then $f(x_1) > f(x_2)$ for all x_1 and x_2 in interval $[a, b]$ then function is strictly monotonically decreasing.

(iv) If $x_1 < x_2$ then $f(x_1) \ge f(x_2)$ for all x_1 and x_2 in interval $[a, b]$ then function is monotonically decreasing (non- increasing).

Using Application of Derivatives

Let $f(x)$ is continuous then we should use application of derivative approach to discuss monotonicity of function

(i) $\dfrac{dy}{dx} = f'(x) \ge 0$ then $f(x)$ is monotonically increasing

$\dfrac{dy}{dx} = f'(x) \ge 0$ then $f(x)$ is strictly monotonically increasing provided $f'(x)$ is not zero for an interval. ($f'(x)$ can be zero at discrete points)

(ii) $\dfrac{dy}{dx} = f'(x) \le 0$ then $f(x)$ is monotonically decreasing

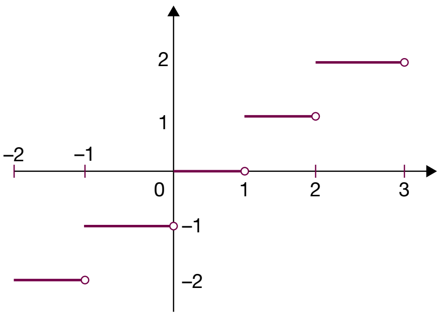
$\dfrac{dy}{dx} = f'(x) \le 0$ then $f(x)$ is strictly monotonically decreasing, provided $f'(x)$ is not zero for an interval ($f'(x)$ can be zero at discrete points)

When $f'(x) = 0$, it means $f(x)$ has become constant for that interval.

Note

If a function is increasing in one interval and decreasing in other interval then function is non-monotonic.

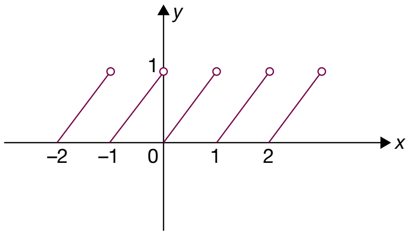
Consider function $f(x) = [x]$ where [.] is GIF



For $0 \le x < 1$ we have $[x] = 0$ for an interval.

Clearly we can conclude that $f(x)$ is non-decreasing function but not strictly increasing function.

For function $f(x) = \{x\}$ where {.} is fractional part function.



As we can see that as $x \in [0, 1)$ then $f(x)$ is increasing but $f(1^-) > f(1)$.

So, $f(x)$ is non-monotonic (neither increasing nor decreasing).

ILLUSTRATION 2

Using derivative analyse monotonicity of following functions.

(i) $f(x) = xe^{x-x^2}$ **(ii)** $f(x) = x^3 - 3x + 2$

(iii) $f(x) = \ln x + \tan^{-1}x$ **(iv)** $f(x) = x \ln x$

SOLUTION

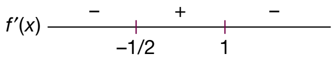
(i) $f(x) = xe^{x-x^2}$

$$f'(x) = e^{x-x^2} + x\left(e^{x-x^2}(1-2x)\right)$$

$$= e^{x-x^2}[1 + x - 2x^2] = e^{x-x^2}(-2x^2 + x + 1)$$

$$f'(x) = -e^{x-x^2}(2x^2 - x - 1)$$

$$= -e^{x-x^2}(2x + 1)(x - 1)$$



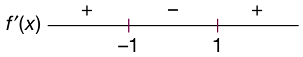
$f(x)$ is strictly increasing for $x \in \left[-\dfrac{1}{2}, 1\right]$

$f(x)$ is strictly decreasing for $x \in \left(-\infty, -\dfrac{1}{2}\right] \cup [1, \infty)$

So, $f(x)$ is non-monotonic in domain as it has both increasing and decreasing behaviour.

(ii) $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$



$x \in (-\infty, -1] \cup [1, \infty)$ strictly increasing function

$x \in [-1, 1]$ strictly monotonically decreasing function

So, $f(x)$ is non-monotonic.

(iii) $f(x) = \ln x + \tan^{-1}x$, domain $x > 0$

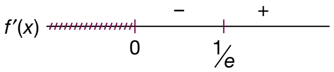
$$f'(x) = \frac{1}{x} + \frac{1}{1+x^2} > 0$$

So, $f(x)$ is monotonically increasing function.

(iv) $f(x) = x \ln x$

$f'(x) = (\ln x) + x\left(\dfrac{1}{x}\right) = \ln x + 1$

$f'(x) = 0 \Rightarrow \ln x = -1 \Rightarrow x = e^{-1} = \dfrac{1}{e}$



$f(x)$ is monotonically increasing in $x \in \left[\dfrac{1}{e}, \infty\right)$ and decreasing $x \in \left(0, \dfrac{1}{e}\right]$

So, we can conclude $f(x)$ as non-monotonic.

ILLUSTRATION 3

Discuss monotonicity of following function

(i) $f(x) = x - \sin x$
(ii) $f(x) = \sin x + \cos x$ where $x \in [0, 2\pi]$
(iii) $f(x) = 3x^4 + 4x^3 + 2x^2 - 7$
(iv) $f(x) = 2x^2 - \ln |x|$

SOLUTION

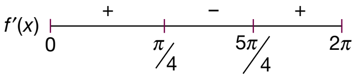
(i) $f(x) = x - \sin x$
$f'(x) = 1 - \cos x \geq 0$
$f'(x) = 0 \Rightarrow \cos x = 1$
At point $x = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi \ldots$, we have $f'(x) = 0$ at discrete points
Hence, $f(x)$ is strictly increasing function in R.

(ii) $f(x) = \sin x + \cos x$ where $x \in [0, 2\pi]$

$f'(x) = \cos x - \sin x = 0$

$\Rightarrow \cos x = \sin x \Rightarrow \tan x = 1 \Rightarrow x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

$f'(x) = \cos x - \sin x$



$f(x)$ is strictly increasing function for

$x \in \left[0, \dfrac{\pi}{4}\right] \cup \left[\dfrac{5\pi}{4}, 2\pi\right]$

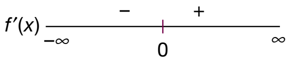
$f(x)$ is strictly decreasing function for $x \in \left[\dfrac{\pi}{4}, \dfrac{5\pi}{4}\right]$

So, $f(x)$ is non-monotonic in $x \in [0, 2\pi]$

(iii) $f(x) = 3x^4 + 4x^3 + 2x^2 - 7$

$f'(x) = 12x^3 + 12x^2 + 4x = 4x(3x^2 + 3x + 1)$

$3x^2 + 3x + 1 > 0$ for all x
As $D = 3^2 - 4 \times 3 = -3 < 0$



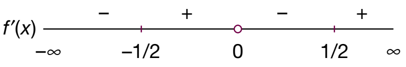
$f(x)$ is decreasing in $x \in (-\infty, 0]$ and increasing in $x \in [0, \infty)$

$f(x)$ is non-monotonic in domain

(iv) $f(x) = 2x^2 - \ln|x|$ domain $x \neq 0$

$$f(x) = \begin{cases} 2x^2 - \ln(+x) & ; \quad x > 0 \\ 2x^2 - \ln(-x) & ; \quad x < 0 \end{cases}$$

$$f'(x) = 4x - \frac{1}{x} = \frac{(2x-1)(2x+1)}{x}$$



$f(x)$ is decreasing in $x \in \left(-\infty, -\dfrac{1}{2}\right] \cup \left(0, \dfrac{1}{2}\right]$

$f(x)$ is increasing in $x \in \left[-\dfrac{1}{2}, 0\right] \cup \left[\dfrac{1}{2}, \infty\right)$

As function is increasing in one interval and decreasing in some other interval so $f(x)$ is non-monotonic function.

ILLUSTRATION 4

Find the intervals of monotonicity of the function $f(x) = 2\sin x + \cos 2x$; $(0 \leq x \leq 2\pi)$.

SOLUTION

$f(x) = 2\sin x + \cos 2x$; $(0 \leq x \leq 2\pi)$

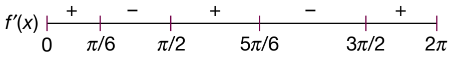
Differentiating w.r.t. x, we get $f'(x) = 2\cos x - 2\sin 2x$

$= 2\cos x - 4\sin x . \cos x$

$f'(x) = 2\cos x\,(1 - 2\sin x)$

Put $\cos x = 0$, we get $x = \dfrac{\pi}{2}$ and $x = \dfrac{3\pi}{2}$

Put $(1 - 2\sin x) = 0 \Rightarrow \sin x = \dfrac{1}{2}$, we get $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$



Hence, $f(x)$ is monotonically increasing function in

$x \in \left[0, \dfrac{\pi}{6}\right] \cup \left[\dfrac{\pi}{2}, \dfrac{5\pi}{6}\right] \cup \left[\dfrac{3\pi}{2}, 2\pi\right]$

$f(x)$ is monotonically decreases function in

$x \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right] \cup \left[\dfrac{5\pi}{6}, \dfrac{3\pi}{2}\right]$

Leibnitz Rule

$f(x) = \displaystyle\int_{h(x)}^{k(x)} g(t)\,dt$, on differentiation,

we get $\dfrac{d(f(x))}{dx} = g(k(x))\left(\dfrac{d(k(x))}{dx}\right) - g(h(x))\left(\dfrac{d(h(x))}{dx}\right)$

ILLUSTRATION 5

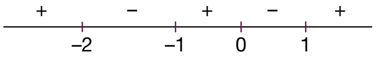
Find the intervals of increase of $f(x)$ defined by

$$f(x)=\int_{-1}^{x}(t^2+2t)(t^2-1)dt.$$

SOLUTION

$$f(x)=\int_{-1}^{x}(t^2+2t)(t^2-1)dt$$

We have, $f'(x) = x(x+2)(x+1)(x-1)$



Sign scheme of $f'(x)$

Clearly $f(x)$ increases in $(-\infty, -2] \cup [-1, 0] \cup [1, \infty)$.

ILLUSTRATION 6

Consider a function $f: R \to R$ defined by

$$f(x)=2x^3+bx^2+4x+\int_{2}^{x}(2\sin^2 t+3\cos^2 t)\,dt \quad \forall x\in R.$$

If $f(x)$ is strictly increasing function for every real x, then find the positive integral values of b.

SOLUTION

$$f(x)=2x^3+bx^2+4x+\int_{2}^{x}(2\sin^2 t+3\cos^2 t)\,dt$$

$$f(x)=2x^3+bx^2+4x+\int_{2}^{x}(2+\cos^2 t)\,dt$$

$$f'(x)=6x^2+2bx+4+2+\cos^2 x\ge 0 \text{ for } \forall\, x\in R$$

$$6x^2+2bx+6+0\ge 0 \Rightarrow 3x^2+bx+3\ge 0$$

Discriminant $D \le 0 \Rightarrow b^2-36\le 0;\ b\in[-6, 6]$
Hence, positive integral $b \in \{1, 2, 3, 4, 5, 6\}$

ILLUSTRATION 7

Let $f(x)=\int_{0}^{x}(t^2+2t+2)\,dt$, where x is set of real numbers satisfying the inequation $\log_2\left(1+\sqrt{6x-x^2-8}\right)\ge 0$. If $f(x)\in[\alpha, \beta]$, then find $|\alpha-\beta|$.

SOLUTION

$$\log_2\left(1+\sqrt{6x-x^2-8}\right)\ge 0 \Rightarrow 1+\sqrt{6x-x^2-8}\ge 2^0$$

$$\Rightarrow \quad \sqrt{6x-x^2-8}\ge 0 \Rightarrow x^2-6x+8\le 0$$

$$\Rightarrow \quad (x-4)(x-2)\le 0 \quad \text{We get } x\in[2, 4]$$

$$f(x)=\int_{0}^{x}(t^2+2t+2)\,dt \text{ for } x\in[2, 4]$$

$$f'(x)=x^2+2x+2=(x+1)^2+1>0 \text{ for } \forall\, x\in R$$

$f(x)$ is increasing function.

$$f(2)=\frac{t^3}{3}+t^2+2t\Big|_0^2=\frac{8}{3}+4+4=\frac{32}{3}$$

$$f(4)=\frac{t^3}{3}+t^2+2t\Big|_0^4=\frac{64}{3}+16+8=\frac{136}{3}$$

Range of function $f(x)\in\left[\frac{32}{3},\frac{136}{3}\right]=[\alpha, \beta]$

Required term $=|\alpha-\beta| = \left|\frac{32-136}{3}\right| = \frac{104}{3}$

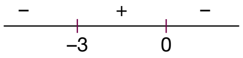
ILLUSTRATION 8

Let $f(x)=(a+2)x^3-3ax^2+9ax-1$ if $f(x)$ is always increasing for $\forall\, x\in R$ find a.

SOLUTION

$$f'(x)=3(a+2)x^2-6ax+9a$$
$$=3[(a+2)x^2-2ax+3a]\ge 0 \text{ for real } x$$
$$A=a+2>0$$

Discriminant of quadratic expression ≤ 0



$$D=B^2-4AC$$
$$=(-2a)^2-4(a+2)(3a)$$
$$=4a[a-3a-6]=-8a(a+3)\le 0$$
$$a\in(-\infty, -3]\cup[0, \infty) \text{ but } a>-2$$

Hence, $a\in[0, \infty)$

ILLUSTRATION 9

If $f(x)=\sin x-a\sin 2x-\frac{1}{3}\sin 3x+2ax$ is monotonically increasing $\forall\, x\in R$ find a.

SOLUTION

$$f(x)=\sin x-a\sin 2x-\frac{1}{3}\sin 3x+2ax$$
$$f'(x)=\cos x-2a\cos 2x-\cos 3x+2a$$
$$=\cos x-\cos 3x+2a(1-\cos 2x)$$
$$=\cos x-(4\cos^3 x-3\cos x)+2a(1-1+2\sin^2 x)$$
$$=4\cos x(1-\cos^2 x)+4a\sin^2 x$$
$$f'(x)=4\sin^2 x(\cos x+a)\ge 0 \text{ for } \forall\, x\in R$$
$$-1+a\ge 0\Rightarrow a\ge 1$$

ILLUSTRATION 10

Show that the function $f(x)=\frac{x}{\sqrt{x+1}}-\ln(1+x)$ is an increasing function for $x>-1$.

SOLUTION

$$f(x)=x/\sqrt{(x+1)}\ -\ln(1+x);\ x\ne -1$$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{1.\sqrt{x+1} - x\dfrac{1}{2\sqrt{x+1}}}{(x+1)} - \frac{1}{1+x}; x \neq -1$$

$$f'(x) = \frac{2(x+1)-x}{2\sqrt{x+1}(x+1)} - \frac{1}{1+x} = \frac{(x+1)+1-2\left(\sqrt{x+1}\right)}{2\sqrt{x+1}(x+1)}$$

$$f'(x) = \frac{\left(\sqrt{x+1}-1\right)^2}{2\sqrt{x+1}(x+1)}$$

$f'(x) > 0$ as given $x > -1$

Hence, $f(x)$ is an increasing function.

ILLUSTRATION 11

If $f(x) = \left(\dfrac{\sqrt{a+4}}{1-a} - 1\right)x^5 - 3x - \ln 5$ is always decreasing $\forall\ x \in R$ find a.

SOLUTION

$$f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^5 - 3x - \ln 5$$

We want $f'(x) = \left[\dfrac{\sqrt{a+4}}{1-a} - 1\right]5x^4 - 3 \le 0\ \forall\ x \in R$

then $\dfrac{\sqrt{a+4}}{1-a} - 1 \le 0 \Rightarrow \dfrac{\sqrt{a+4}}{1-a} \le 1$

where $a + 4 \ge 0 \Rightarrow a \ge -4$

CASE 1: $(a > 1)$

$\dfrac{\sqrt{a+4}}{1-a} \le 1$ As LHS is negative it is always true so $a > 1$ …(1)

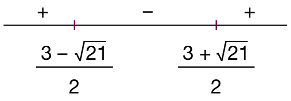
CASE 2: $(-4 \le a < 1)$

$$\frac{\sqrt{a+4}}{1-a} \le 1 \Rightarrow \sqrt{a+4} \le 1-a$$

$$\Rightarrow \quad a + 4 \le 1 + a^2 - 2a \Rightarrow a^2 - 3a - 3 \ge 0$$

$$\Rightarrow \quad \left(a - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{21}}{2}\right)^2 \ge 0$$

$$\Rightarrow \quad \left[a - \frac{3}{2} + \frac{\sqrt{21}}{2}\right]\left[a - \frac{3}{2} - \frac{\sqrt{21}}{2}\right] \ge 0$$



$$a \in \left[-4, \frac{3-\sqrt{21}}{2}\right] \qquad \text{…(2)}$$

Therefore, set of values of a CASE 1 ∪ CASE 2

$$a \in \left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$$

ILLUSTRATION 12

If $f(x) = x^2 + ax^2 + bx + 5\sin^2 x$ is an increasing function for all real number, find conditions in a and b.

SOLUTION

$$f(x) = x^2 + ax^2 + bx + 5\sin^2 x$$

$$f'(x) = 3x^2 + 2ax + b + 5(2\sin x \cos x)$$

$$= 3x^2 + 2ax + b + 5\sin 2x \ge 0$$

[As $-5 \le 5\sin 2x \le 5,\ \forall\ x \in R$]

If $3x^2 + 2ax + b - 5 \ge 0$ for all real x, then $f'(x) \ge 0$

$A > 0$ and discriminant $D \le 0$

$\Rightarrow \quad B^2 - 4AC = 4a^2 - 4(3)(b-5) \le 0$

$\Rightarrow \quad 4[a^2 - 3b + 15] \le 0 \Rightarrow a^2 - 3b + 15 \le 0$ is required condition

ILLUSTRATION 13

Let $f(x) = \sin^4 x + \cos^4 x + bx + c$, If $f(x)$ is monotonic for $\forall\ x \in R$, find b and c.

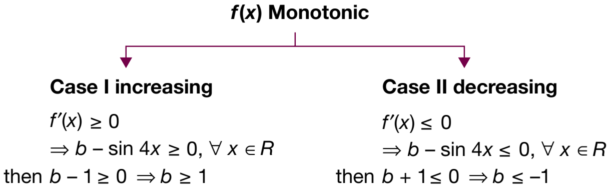
SOLUTION

$$f(x) = \sin^4 x + \cos^4 x + bx + c$$

$$f'(x) = 4\sin^3 x \cos x + 4\cos^3 x(-\sin x) + b$$

$$= 4\sin x \cos x(\sin^2 x - \cos^2 x) + b$$

$$= 2\sin 2x(-\cos 2x) + b = b - \sin 4x$$



$f'(x) \ge 0$

$\Rightarrow b - \sin 4x \ge 0,\ \forall\ x \in R$

then $b - 1 \ge 0 \Rightarrow b \ge 1$

$f'(x) \le 0$

$\Rightarrow b - \sin 4x \le 0,\ \forall\ x \in R$

then $b + 1 \le 0 \Rightarrow b \le -1$

Hence, set of values of $b \in (-\infty, -1] \cup [1, \infty)$ and $c \in R$

ILLUSTRATION 14

Let $f(x) = e^{2x} - ae^x + 1$.

Prove that for any real value of 'a' $f(x)$ can't be decreasing function for every $x \in R$.

SOLUTION

$$f(x) = e^{2x} - ae^x + 1$$

$$f'(x) = 2e^{2x} - ae^x = e^x[2e^x - a] \le 0$$

$$\Rightarrow \quad e^x \le \frac{a}{2} \text{ for } \forall\ x \in R$$

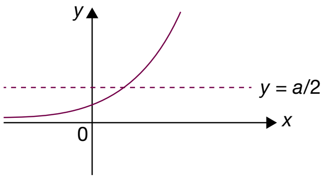
CASE 1: $a \le 0$,

$$e^x \le \frac{a}{2}$$

It is not possible as e^x is a positive number and RHS is negative or zero.

CASE 2: $a > 0$,

$$e^x \le \frac{a}{2}$$



$y = e^x$ cannot be lesser than
$y = a/2$ for all real x
It is not possible.
We can conclude that there exists no real a, for which $f(x)$ can't be decreasing function for every $x \in R$.

ILLUSTRATION 15

Let $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$, for $0 \le x \le \pi$. Find interval of x such that $f(x)$ is decreasing.

SOLUTION

$$f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$$

$$f'(x) = 12\cos^3 x\,(-\sin x) + 30\cos^2 x\,(-\sin x) + 6(2\cos x)(-\sin x) - 0$$

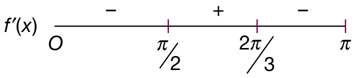
$$= -6\cos x \sin x\,[2\cos^2 x + 5\cos x + 2]$$

$$= -6\cos x \sin x\,[2\cos x + 1]\,[\cos x + 2]$$

Given $x \in [0, \pi]$

$f'(x) = 0$, we have $\cos x = 0$ or $-1/2$

$$x = \frac{\pi}{2} \text{ and } x = \frac{2\pi}{3}$$



Hence, $f(x)$ is decreasing in $x \in \left(0, \frac{\pi}{2}\right] \cup \left[\frac{2\pi}{3}, \pi\right]$

ILLUSTRATION 16

$f(x) = (x-1)\,|(x-2)(x-3)|$ Discuss monotonicity of $f(x)$.

SOLUTION

$$|(x-2)(x-3)| = \begin{cases} -(x-2)(x-3) & ; \quad x \in [2,3] \\ (x-2)(x-3) & ; \quad x \in (-\infty, 2) \cup (3, \infty) \end{cases}$$

$$f(x) = \begin{cases} -(x-1)(x-2)(x-3) & ; \quad x \in [2,3] \\ (x-1)(x-2)(x-3) & ; \quad x \in (-\infty, 2) \cup (3, \infty) \end{cases}$$

$$f(x) = \begin{cases} -x^3 + 6x^2 - 11x + 6 & ; \quad x \in [2,3] \\ x^3 - 6x^2 + 11x - 6 & ; \quad x \in (-\infty, 2) \cup (3, \infty) \end{cases}$$

$$f'(x) = \begin{cases} -3x^2 + 12x - 11; & x \in (2,3) \\ 3x^2 - 12x + 11 & ; \quad x \in (-\infty, 2) \cup (3, \infty) \end{cases}$$

$$3x^2 - 12x + 11 = 0 \Rightarrow x = 2 \pm \frac{1}{\sqrt{3}}$$

CASE 1: $x \in [2, 3]$

$$f'(x) = -(3x^2 - 12x + 11)$$

$$= -3\left(x - \left(2 + \frac{1}{\sqrt{3}}\right)\right)\left(x - \left(2 - \frac{1}{\sqrt{3}}\right)\right)$$

$f'(x) \ge 0$, for $x \in \left[2, 2 + \frac{1}{\sqrt{3}}\right]$, so $f(x)$ is increasing

$f'(x) \le 0$, for $x \in \left[2 + \frac{1}{\sqrt{3}}, 3\right]$, so $f(x)$ is decreasing

CASE 2: $x \in (-\infty, 2] \cup [3, \infty)$

$$f'(x) = 3x^2 - 12x + 11 = 3\left[x - \left(2 + \frac{1}{\sqrt{3}}\right)\right]\left[x - \left(2 - \frac{1}{\sqrt{3}}\right)\right]$$

$f'(x) \ge 0$, for $x \in \left(-\infty, 2 - \frac{1}{\sqrt{3}}\right] \cup [3, \infty)$, so $f(x)$ is increasing

$f'(x) \le 0$, for $x \in \left[2 - \frac{1}{\sqrt{3}}, 2\right]$, so $f(x)$ is decreasing

Hence, set of values of x, for increasing $f(x)$,

$$x \in \left(-\infty, 2 - \frac{1}{\sqrt{3}}\right] \cup \left[2, 2 + \frac{1}{\sqrt{3}}\right] \cup [3, \infty) \text{ and}$$

For decreasing $f(x)$, $x \in \left[2 - \frac{1}{\sqrt{3}}, 2\right] \cup \left[2 + \frac{1}{\sqrt{3}}, 3\right]$

ILLUSTRATION 17

If $g(x) = 2f\left(\frac{x}{2}\right) + f(1-x)$ where $0 \le x \le 1$ and $f''(x) < 0$.

Discuss monotonicity of $g(x)$.

SOLUTION

$$g(x) = 2f\left(\frac{x}{2}\right) + f(1-x)$$

On differentiation, we obtain

$$g'(x) = 2f'\left(\frac{x}{2}\right)\left(\frac{1}{2}\right) + f'(1-x)\,(0-1)$$

$$g'(x) = f'\left(\frac{x}{2}\right) - f'(1-x)$$

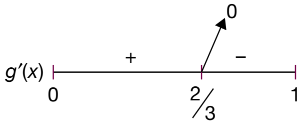
$$g'(x) = 0 \Rightarrow f'\left(\frac{x}{2}\right) = f'(1-x)$$

Given $f''(x) < 0$, it means $f'(x)$ is strictly decreasing.

$\therefore$ $f'(x)$ is one-one function

$$\frac{x}{2} = 1 - x \Rightarrow x = 2 - 3x \Rightarrow x = \frac{2}{3}$$

$$g'(0) = f'(0) - f'(1) > 0;\ g'(1) = f'\left(\frac{1}{2}\right) - f'(0) < 0$$



$g(x)$ is increasing for $x \in \left[0, \frac{2}{3}\right]$ and decreasing for $x \in \left[\frac{2}{3}, 1\right]$

ILLUSTRATION 18

A function $g(x) = \frac{1}{4}f(2x^2-1)+\frac{1}{2}f(1-x^2)$ and $f'(x)$ is increasing function then find $g(x)$ is increasing in the interval.

SOLUTION

Function $g(x) = \frac{1}{4}f(2x^2-1)+\frac{1}{2}f(1-x^2)$

$$g'(x) = \frac{1}{4}\ f'(2x^2-1)\ (4x) + \frac{1}{2}\ f'(1-x^2)\ (-2x)$$

$$g'(x) = x\{f'(2x^2-1) - f'(1-x^2)\} \geq 0$$

$$(\because g(x) \text{ is increasing}) \qquad \ldots(1)$$

CASE 1: Let $x \geq 0$ then $f'(2x^2-1) \geq f'(1-x^2)$

As $f'(x)$ is increasing function, we have $2x^2 - 1 \geq 1 - x^2$

$$\Rightarrow \quad 3x^2 - 2 \geq 0 \Rightarrow 3\left(x-\sqrt{\frac{2}{3}}\right)\left(x+\sqrt{\frac{2}{3}}\right) \geq 0$$

$$x \in \left(-\infty, -\sqrt{\frac{2}{3}}\right] \cup \left[\sqrt{\frac{2}{3}}, \infty\right) \text{ but } x \geq 0$$

We have $x \in \left[\sqrt{\frac{2}{3}}, \infty\right)$

CASE 2: Let $x < 0$ then from (1) $f'(2x^2-1) \leq f'(1-x^2)$

As $f'(x)$ is increasing function, we have $2x^2 - 1 \leq 1 - x^2$

$$\Rightarrow \quad 3x^2 - 2 \leq 0 \Rightarrow 3\left(x-\sqrt{\frac{2}{3}}\right)\left(x+\sqrt{\frac{2}{3}}\right) \leq 0$$

$$\Rightarrow \quad x \in \left[-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right] \text{ but } x < 0 \quad \Rightarrow \quad x \in \left[-\sqrt{\frac{2}{3}}, 0\right)$$

From CASE 1 and CASE 2, $x \in \left[-\sqrt{\frac{2}{3}}, 0\right) \cup \left[\sqrt{\frac{2}{3}}, \infty\right)$

ILLUSTRATION 19

Prove that $f(x) = \frac{2}{3}x^9 - x^6 + 2x^3 - 3x^2 + 6x - 1$ is always increasing $\forall\ x \in R$.

SOLUTION

$$f(x) = \frac{2}{3}x^9 - x^6 + 2x^3 - 3x^2 + 6x - 1$$

$$f'(x) = \frac{2}{3}\ 9x^8 - 6x^5 + 6x^2 - 6x + 6$$

$$= 6x^8 - 6x^5 + 6x^2 - 6x + 6$$

$$= 6[x^8 - x^5 + x^2 - x + 1] \geq 0\ \forall\ x \in R$$

CASE 1: $x < 0$

$f'(x) = 6\ [(x^8 + x^2 + 1) - (x^5 + x)] > 0$ as $(x^8 + x^2 + 1) > 0$, $-(x^5 + x) > 0$ for $x < 0$

$f'(x) > 0$, so $f(x)$ is an increasing function.

CASE 2: $0 \leq x \leq 1$

$f'(x) = 6\ [x^8 + x^2 - x^5 + 1 - x] = 6\ [x^8 + x^2\ (1 - x^3) + (1 - x)] > 0$

As $1 - x^3 \geq 0$ and $1 - x \geq 0$

$f'(x) > 0$, $f(x)$ is an increasing function.

CASE 3: $x > 1$

$f'(x) = 6\ [x^5\ (x^3 - 1) + x(x - 1) + 1\ ] > 0$ as $x^5\ (x^3 - 1) \geq 0$ and $x(x - 1) \geq 0$

$f'(x) > 0$, $f(x)$ is an increasing function.

Clearly from all above cases $f'(x) > 0$, for all real x, hence $f(x)$ is an increasing function.

MONOTONICITY OF COMPOSITE FUNCTION

(i) Let $f(x)$ and $g(x)$ are increasing function

If $x_1 < x_2$ then $f(x_1) \leq f(x_2)$; $g(f(x_1)) \leq g(f(x_2))$ then $g(f(x))$ is increasing

If $f(x)$ and $g(x)$ are differentiable functions then $f'(x) \geq 0$ and $g'(x) \geq 0$ for all x.

$\therefore\ (f(g(x)))' = f'(g(x))\ g'(x) \geq 0 \qquad$ (as $f'(g(x)) \geq 0$)

$\therefore\ f(g(x))$ is increasing function

(ii) Let $f(x)$ and $g(x)$ are decreasing function

If $x_1 < x_2$ then $f(x_1) \geq f(x_2)$; $g(f(x_1)) \leq g(f(x_2))$ then $g(f(x))$ is increasing

If $f(x)$ and $g(x)$ are differentiable functions then $f'(x) \leq 0$ and $g'(x) \leq 0$ for all x.

$\therefore\ (f(g(x)))' = f'(g(x))\ g'(x) \geq 0 \qquad$ (as $f'(g(x)) \leq 0$)

$\therefore\ f(g(x))$ is increasing function.

(iii) Let $f(x)$ is increasing and $g(x)$ is decreasing function

If $x_1 < x_2$ then $f(x_1) \leq f(x_2)$; $g(f(x_1)) \geq g(f(x_2))$ then $g(f(x))$ is decreasing

If $f(x)$ and $g(x)$ are differentiable functions then $f'(x) \geq 0$ and $g'(x) \leq 0$ for all x.

$\therefore\ (f(g(x)))' = f'(g(x))\, g'(x) \leq 0$ (as $f'(g(x)) \leq 0$)

$\therefore\ f(g(x))$ is decreasing function.

(iv) Let $f(x)$ is decreasing and $g(x)$ is increasing function

If $x_1 < x_2$ then $f(x_1) \geq f(x_2)$; $g(f(x_1)) \geq g(f(x_2))$ then $g(f(x))$ is decreasing

If $f(x)$ and $g(x)$ are differentiable functions then $f'(x) \leq 0$ and $g'(x) \geq 0$ for all x.

$\therefore\ (f(g(x)))' = f'(g(x))\, g'(x) \leq 0$ (as $f'(g(x)) \leq 0$)

$\therefore\ f(g(x))$ is decreasing function.

Table 7.1 Functions and their compositions

f	*g*	*fog*(*x*)	*gof*(*x*)	*fof*(*x*)	*gog*(*x*)
↑	↑	↑	↑	↑	↑
↑	↓	↓	↓	↑	↑
↓	↑	↓	↓	↑	↑
↓	↓	↑	↑	↑	↑

Where ↑ sign for increasing function and ↓ sign for decreasing function.

Note

In general $F(x) = f_1(f_2(f_3(\ldots f_n(x))$

(i) If all $f_i(x)$ are increasing then $F(x)$ is increasing

(ii) If all $f_i(x)$ are decreasing, then $F(x)$ is decreasing or increasing as n is odd or even respectively.

ILLUSTRATION 20

Let consider two functions defined for positive real numbers, $f(x) = 1/x$ and $g(x) = e^{-x}$. Discuss monotonicity of $fog(x)$ and $gof(x)$.

SOLUTION

For $x > 0$, $f(x) = \dfrac{1}{x}$ is decreasing

$g(x) = e^{-x}$ is decreasing

$fog(x) = f(e^{-x}) = \dfrac{1}{e^{-x}} = e^{x}$ is an increasing function for $x > 0$

$$gof(x) = g\left(\frac{1}{x}\right) = e^{-\frac{1}{x}} \Rightarrow (gof)'(x) = \frac{e^{-1/x}}{x^2} > 0$$

$\therefore$ $gof(x)$ is an increasing function

ILLUSTRATION 21

Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$.

SOLUTION

$$f(x) = 1 - x - x^3;\ f'(x) = -1 - 3x^2 < 0$$

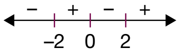
So $f(x)$ is strictly decreasing.

Given inequality $1 - f(x) - f^3(x) > f(1 - 5x)$

$\Rightarrow\quad f(f(x)) > f(1 - 5x) \Rightarrow f(x) < 1 - 5x$

$\Rightarrow\quad 1 - x - x^3 < 1 - 5x$

$\Rightarrow\quad x^3 - 4x > 0 \Rightarrow x(x-2)(x+2) > 0$



Hence, $x \in (-2, 0) \cup (2, \infty)$

ILLUSTRATION 22

Let $f(x)$ and $g(x)$ be two continuous functions defined from $R \to R$, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$, $\forall\, x_1 > x_2$, then find the solution set of $f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4))$.

SOLUTION

$\because\quad f(x_1) > f(x_2);\ \forall\, x_1 > x_2$ then $f(x)$ is strictly increasing

$\because\quad g(x_1) < g(x_2)\ \forall\, x_1 > x_2$ then $f(x)$ is strictly decreasing

$\therefore\quad f(g(x))$ is decreasing function

$f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4)) \Rightarrow \alpha^2 - 2\alpha < 3\alpha - 4$

$\Rightarrow\quad \alpha^2 - 5\alpha + 4 < 0 \Rightarrow (\alpha - 1)(\alpha - 4) < 0 \Rightarrow \alpha \in (1, 4)$

ILLUSTRATION 23

Given that $g(x) = 2f\left(\dfrac{x^2}{2}\right) + f(6 - x^2)\ \forall\, x \in R$ and $f''(x) > 0$ $\forall\, x \in R$, the set of all points of x where $g(x)$ is increasing are $(a, b) \cup (c, \infty)$ then find the value of $[a + b + c]$ is {where [ ] denotes the greatest integer function}

SOLUTION

$$g(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)\ \forall\, x \in R.$$

$$g'(x) = 2f'\left(\frac{x^2}{2}\right)\frac{2x}{2} + f'(6 - x^2)(-2x)$$

$$g'(x) = 2x\, f'\left(\frac{x^2}{2}\right) - 2x\, f'(6 - x^2)$$

$\because\quad f''(x) > 0 \Rightarrow f'(x)$ is strictly increasing $\forall\, x \in R$, hence $f'(x)$ is one-one also

$$g'(x) = 2x\left(f'\left(\frac{x^2}{2}\right) - f'(6 - x^2)\right) > 0$$

CASE 1: If $x > 0$,

$$\left(f'\left(\frac{x^2}{2}\right)-f'(6-x^2)\right)>0 \Rightarrow f'\left(\frac{x^2}{2}\right)>f'(6-x^2)$$

$$\Rightarrow \quad \frac{x^2}{2}>6-x^2 \Rightarrow x^2>4 \Rightarrow x \in (-\infty,-2)\cup(2,\infty)\ (\because\ x>0)$$

$$\Rightarrow \quad x \in (2,\infty) \qquad \ldots(1)$$

CASE 2: If $x \le 0$

$$\left(f'\left(\frac{x^2}{2}\right)-f'(6-x^2)\right)<0 \Rightarrow f'\left(\frac{x^2}{2}\right)<f'(6-x^2)$$

$$\Rightarrow \quad \frac{x^2}{2}<6-x^2 \Rightarrow x^2<4 \Rightarrow x \in (-2,2) \quad (\because\ x\le 0)$$

$$\Rightarrow \quad x \in (-2,0] \qquad \ldots(2)$$

From (1) and (2), we get $x \in (-2, 0] \cup (2, \infty)$

ILLUSTRATION 24

If $f(x)$ is continuous and increasing function such that domain of $g(x)=\sqrt{f(x)-x}$ be R and $h(x)=\frac{1}{1-x}$, then find the domain of $\phi(x)=\sqrt{f(f(f(x)))-h(h(h(x)))}$ is

SOLUTION

$$h(x)=\frac{1}{1-x};\ h(h(x))=\frac{1}{1-\frac{1}{1-x}}=\frac{x-1}{x}$$

$$h(h(h(x)))=\frac{\frac{1}{1-x}-1}{\frac{1}{1-x}}=\frac{1-1+x}{1}=x \text{ but } x\ne 0, 1$$

$g(x)=\sqrt{f(x)-x}$, domain of $g(x)$ is R

$f(x)\ge x$ for $\forall\ x\in R$

Given $f(x)$ is continuous and increasing function

$$f(f(f(x)))\ge f(f(x))\ge f(x)\ge x \qquad \ldots(1)$$

$$\phi(x)=\sqrt{f(f(f(x)))-h(h(h(x)))}$$

$$\phi(x)=\sqrt{f(f(f(x)))-x};\ x\ne 0, 1$$

$f(f(f(x)))-x\ge 0$ is true from (1)

Hence, domain of $\phi(x)\ \forall\ x\in R-\{0, 1\}$

Test Your Concept-I

(Solutions on page H.181)

1. $f(x)=x^2-5x+2$ discuss monotonicity at following points

(i) at $x=1$ (ii) at $x=\frac{5}{2}$ (iii) at $x=3$

2. $f(x)=\begin{cases}\sin x & ;\ x<0\\ \sin^2 x & ;\ x\ge 0\end{cases}$. Check monotonicity at $x=0$

3. $f(x)=x-2\sin x\cdot x\in[0,2\pi]$. Discuss monotonicity

4. The interval on which the following function are monotonic increasing

(i) $f(x)=\log_{1/2}(x^2-2x-3)$

(ii) $f(x)=\sin^4 x+\cos^4 x$, where $x\in\left[0,\frac{\pi}{2}\right]$

5. Let $f(x)=\int_3^{x^2} e^{-t^2}(t-4)(t^2-t+12)dt$. If $f(x)$ is decreasing then find x

6. Function $f(x)=\frac{\log(\pi+x)}{\log(e+x)}$ is decreasing find suitable x

7. Find the number of integer values of x where $f(x)=\frac{x}{\log_e x}$ is decreasing

8. If $f(x)=\frac{x}{\sin x}$ and $g(x)=\frac{x}{\tan x}$, where $0<x\le 1$, show that $f(x)$ is an increasing function and $g(x)$ is decreasing function

9. Find the intervals in which the following function are (a) Increasing (b) Decreasing

(i) $f(x)=x^3+3x^2-105x+25$ (ii) $f(x)=2x^3-24x+7$

10. If $f(x)=2e^x-ae^{-x}+(2a+1)x-3$ monotonically increases for every $x\in R$ then find the range of values of a.

11. Find the set of all values of the parameter 'a' for which the function $f(x)=\sin 2x-8(a+1)\sin x+(4a^2+8a-14)x$ increases for all $x\in R$ and has no critical points for all $x\in R$

12. Find the intervals in which the function $f(x)=\sin(\log_e x)+\cos(\log_e x)$ is decreasing.

13. If $f(x) = 1 + x + \int_1^x (\ln^2 t + 2\ln t)dt$, then x where $f(x)$ increases

14. Find the set of all values of a for which the function $f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^5 - 3x + \log 5$ decreases for all real values of x

15. Find interval of x where following function are decreasing

(i) $f(x) = xe^{-x(x+1)}$ (ii) $f(x) = (x+2)e^{-x}$

(iii) $f(x) = \cos\left(\frac{\pi}{x}\right)$ (iv) $f(x) = |x\log_e x|$

16. Let $f(x) = 2x^3 + ax^2 + bx - 3\cos^2 x$ is an increasing function for all $a, b, x \in R$ then show that $a^2 - 6b + 18 < 0$

Answer key

1. (i) strictly decreasing function (ii) non-monotonic (iii) strictly increasing function

2. strictly increasing function

3. decreasing in $x \in \left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$ and increasing in $x \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$

4. (i) $(-\infty, -1)$ (ii) $\frac{\pi}{4} \le x \le \frac{\pi}{2}$ **5.** $(0, 2)$ **6.** $(0, \infty)$ **7.** 1

9. (i) increasing in $(-\infty, -7] \cup [5, \infty)$ and decreasing in $[-7, 5]$

(ii) increasing in $(-\infty, -2] \cup [2, \infty)$ and decreasing in $[-2, 2]$

10. $a \ge 0$ **11.** $a < -(2 + \sqrt{3})$ or $a > \sqrt{5})$ **12.** $[e^{2n\pi + \pi/4}, e^{2n\pi + 5\pi/4}], n \in Z$

13. $(0, \infty)$ **14.** $\left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$

15. (i) $(-\infty, -1) \cup \left(\frac{1}{2}, \infty\right)$ (ii) $(-1, \infty)$ (iii) $x \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right), \forall n \in N$ (iv) $\left(\frac{1}{e}, 1\right)$

INEQUALITY USING MONOTONICITY

ILLUSTRATION 25

Prove that $\sin x < x < \tan x$ for $x \in \left(0, \frac{\pi}{2}\right)$

SOLUTION

Let consider a function $f(x) = \sin x - x$, for $x \in \left(0, \frac{\pi}{2}\right)$

$$f'(x) = \cos x - 1 < 0$$

$f(x)$ is strictly monotonically decreasing.

$$x > 0 \Rightarrow f(x) < f(0)$$

$$\Rightarrow \quad \sin x - x < 0 - 0 \Rightarrow \sin x < x$$

Let consider function $g(x) = \tan x - x$ for $x \in \left(0, \frac{\pi}{2}\right)$

$$g'(x) = \sec^2 x - 1 = \tan^2 x > 0$$

$g(x)$ is monotonically increasing function.

$$x > 0 \Rightarrow g(x) > g(0) \Rightarrow \tan x - x > 0 - 0 \Rightarrow \tan x > x$$

$$\therefore \quad \sin x < x < \tan x \text{ for } x \in \left(0, \frac{\pi}{2}\right)$$

From graph, we can clearly observe that $\sin x < x < \tan x$ for $x \in \left(0, \frac{\pi}{2}\right)$

Hence proved.

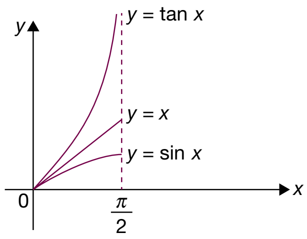


ILLUSTRATION 26

Prove that $x < -\ln(1 - x)\ \forall\ x \in (0, 1)$.

SOLUTION

Let $f(x) = x + \ln(1 - x)$ where $x \in (0, 1)$

$$f'(x) = 1 + \frac{1}{(1-x)}(0-1) = 1 - \frac{1}{1-x} = \frac{1-x-1}{1-x} = \frac{-x}{1-x} < 0$$

$f(x)$ is monotonically decreasing.

$$x > 0 \Rightarrow f(x) < f(0)$$

$$\Rightarrow \quad x + \ln(1 - x) < 0 + \ln(1 - 0) \quad \Rightarrow \quad x + \ln(1 - x) < 0$$

$$\Rightarrow \quad x < -\ln(1 - x) \text{ for } x \in (0, 1)$$

Hence proved.

ILLUSTRATION 27

Prove that $\frac{\pi}{2} < \frac{\sin x}{x} < 1$ where $x \in \left(0, \frac{\pi}{2}\right)$

SOLUTION

Let $f(x) = \frac{\sin x}{x}$ where $x \in \left(0, \frac{\pi}{2}\right)$

$$\Rightarrow \quad f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{\cos x[x - \tan x]}{x^2}$$

(as $\tan x > x$ for $x \in \left(0, \frac{\pi}{2}\right)$)

$\Rightarrow \quad f'(x) < 0 \quad \therefore f(x)$ is decreasing function.

$$0 < x < \frac{\pi}{2} \Rightarrow f(0^+) > f(x) > f\left(\frac{\pi}{2}^-\right)$$

$$\Rightarrow \quad \lim_{x\to 0^+} \frac{\sin x}{x} > \frac{\sin x}{x} > \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}} \Rightarrow 1 > \frac{\sin x}{x} > \frac{2}{\pi}$$

Hence proved.

ILLUSTRATION 28

Prove that $e^x + e^{-x} \geq 2 + x^2$ for $\forall\ x \in R$.

SOLUTION

$$f(x) = e^x + e^{-x} - 2 - x^2$$

$$f'(x) = e^x - e^{-x} - 0 - 2x = e^x - e^{-x} - 2x$$

$$f''(x) = e^x + e^{-x} - 2 = e^x + \frac{1}{e^x} - 2 = \left(\sqrt{e^x} - \frac{1}{\sqrt{e^x}}\right)^2 \geq 0$$

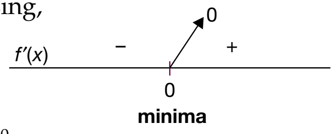
$f''(x) \geq 0; f'(x)$ is strictly increasing

$$f'(x) = e^x - e^{-x} - 2x; f'(0) = 0$$

For $x \in (-\infty, 0)$, $f(x)$ is decreasing

For $x \in (0, \infty)$, $f(x)$ is increasing,

At $x = 0$, $f(x)$ has minima



$$f(x) \geq f(0)$$

$$\Rightarrow \quad e^x + e^{-x} - 2 - x^2 \geq e^0 + e^{-0} - 2 - 0 = 0$$

$$\Rightarrow \quad e^x + e^{-x} - 2 - x^2 \geq 0 \Rightarrow e^x + e^{-x} \geq 2 + x^2 \text{ for real } x$$

Hence proved.

ILLUSTRATION 29

Establish the following inequalities:

(i) $x^2 - 1 > 2x \ln x > 4(x - 1) - 2 \ln x$ for $x > 1$

(ii) $\tan^2 x + 6 \ln \sec x + 2 \cos x + 4 > 6 \sec x$ for $x \in \left(\frac{3\pi}{2}, 2\pi\right)$

(iii) $x - \frac{x^3}{3} < \tan^{-1} x < x - \frac{x^3}{6}$ for $0 < x \leq 1$

(iv) $x^2 > (1 + x)\,[\ln(1 + x)]^2$ for $x > 0$

SOLUTION

(i) Let $f(x) = x^2 - 2x \ln x - 1$ for $x \in [1, \infty)$

$$f'(x) = 2x - 2 - 2 \ln x$$

$$f''(x) = 2 - \frac{2}{x} = \frac{2(x-1)}{x} \geq 0$$

$f'(x)$ is increasing for $\forall\ x \geq 1$

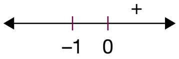
When $x > 1 \Rightarrow f(x) > f(1) \Rightarrow x^2 - 2x \ln x - 1 > 0$

$\Rightarrow \quad x^2 - 1 > 2x \ln x$ …(1)

Now we need to prove $2x \ln x > 4(x - 1) - 2 \ln x$, $x \geq 1$

Let $g(x) = 2x \ln x - 4x + 4 + 2 \ln x$

$$g'(x) = 2 + 2 \ln x - 4 + \frac{2}{x}$$



$$g''(x) = \frac{2}{x} - \frac{2}{x^2} = \frac{2}{x^2}(x - 1) > 0$$

$g'(x)$ is increasing, for $x \geq 1$

When $x > 1 \Rightarrow g'(x) > g'(1) = 0$

$g(x)$ is increasing

When $x > 1 \Rightarrow g(x) > g(1)$

$2x \ln x - 4x + 4 + 2 \ln x > 0$

$\Rightarrow \quad 2x \ln x > 4x - 4 - 2 \ln x$ …(2)

From (1) and (2)

We get, $x^2 - 1 > 2x \ln x > 4(x - 1) - 2 \ln x$ for $x > 1$

Hence proved

(ii) $f(x) = \tan^2 x + 6 \ln \sec x + 2 \cos x + 4 - 6 \sec x;$

$$f'(x) = 2\tan x.\sec^2 x + 6 \tan x - 2\sin x - 6\sec x \tan x$$

$$= \frac{2\sin x}{\cos^3 x} + \frac{6\sin x}{\cos x} - 2 \sin x - \frac{6\sin x}{\cos^2 x}$$

$$= \frac{2\sin x}{\cos^3 x}[1 + 3\cos^2 x - \cos^3 x - 3\cos x]$$

$$= \frac{2\sin x}{\cos^3 x}(1 - \cos x)^3 \leq 0 \text{ for } x \in \left(\frac{3\pi}{2}, 2\pi\right]$$

$f(x)$ is decreasing

For $x < 2\pi \quad \Rightarrow \quad f(x) > f(2\pi) = 0$

$\tan^2 x + 6 \ln \sec x + 2 \cos x + 4 - 6 \sec x > 0$

Hence proved

(iii) Let $f(x) = \tan^{-1} x - x + \frac{x^3}{3}$; $x \in [0, 1]$

$$f'(x) = \frac{1}{1+x^2} - 1 + x^2 = \frac{1 - 1 - x^2 + x^2 + x^4}{1+x^2} = \frac{x^4}{1+x^2} > 0$$

(as $0 < x \leq 1$)

$f(x)$ is increasing

$$x > 0 \Rightarrow f(x) > f(0)$$

$$\Rightarrow \tan^{-1}x - x + \frac{x^3}{3} > 0 \Rightarrow \tan^{-1}x > x - \frac{x^3}{3} \qquad \ldots(1)$$

Let $g(x) = \tan^{-1}x - x + \dfrac{x^3}{6}$

$$g'(x) = \frac{1}{1+x^2} - 1 + \frac{x^2}{2} = \frac{2-2-2x^2+x^2+x^4}{1+x^2}$$

$$g'(x) = \frac{x^2(x^2-1)}{1+x^2} \le 0 \text{ as } 0 < x \le 1$$

$g(x)$ is decreasing function

$$x > 0 \Rightarrow g(x) < g(0)$$

$$\Rightarrow \tan^{-1}x - x + \frac{x^3}{6} < 0 \Rightarrow \tan^{-1}x < x - \frac{x^3}{6} \qquad \ldots(2)$$

From (1) and (2), we obtain

$$x - \frac{x^3}{3} < \tan^{-1}x < x - \frac{x^3}{6} \quad \text{for } 0 < x \le 1$$

Hence proved

(iv) $f(x) = x^2 - (1+x)\,[\ln(1+x)]^2$ where $x \in (0, \infty)$

$$f'(x) = 2x - 2\ln(1+x) - [\ln(1+x)]^2$$

$$f''(x) = 2 - \frac{2}{1+x} - \frac{2\ln(1+x)}{1+x}$$

$$= \frac{2[1+x-1-\ln(1+x)]}{1+x} = \frac{2[x-\ln(1+x)]}{(1+x)} > 0$$

(as $x > \ln(1+x)$ for all $x > 0$)

$f''(x) > 0$ for all $x > 0$

$f'(x)$ is increasing,

$$x > 0 \Rightarrow f'(x) > f'(0) = 0$$

$\Rightarrow$ $f'(x) > 0$, It concludes $f(x)$ is increasing

$x > 0$ we have $f(x) > f(0)$

$\Rightarrow$ $x^2 - (1+x)\,[\ln(1+x)]^2 > 0$

Hence proved

ILLUSTRATION 30

Which one is greater $\sin x \tan x$ or x^2 where $x \in \left(0, \dfrac{\pi}{2}\right)$.

Also find $\lim\limits_{x\to 0}\left[\dfrac{\sin x \tan x}{x^2}\right]$ and $\lim\limits_{x\to 0}\left[\dfrac{x^2}{\sin x \tan x}\right]$ where [.] is GIF.

SOLUTION

$$f(x) = \sin x \tan x - x^2$$

$$f'(x) = \cos x \tan x + \sin x \sec^2 x - 2x$$

$$f'(x) = \sin x + \sin x \sec^2 x - 2x$$

$$f''(x) = \cos x + \cos x \sec^2 x + 2\sin x.\ \sec^2 x \tan x - 2$$

$$= \cos x + \sec x + 2\sec^2 x \sin x \tan x - 2$$

$$= \left(\sqrt{\cos x} - \frac{1}{\sqrt{\cos x}}\right)^2 + 2\sec^2 x \sin x \tan x$$

As $f''(x) > 0$ then $f'(x)$ is increasing.

$$x > 0 \Rightarrow f'(x) > f'(0)$$

$\Rightarrow$ $f'(x) > 0$ (as $f'(0) = 0$)

then $f(x)$ is increasing function

$$x > 0 \Rightarrow f(x) > f(0)$$

$\Rightarrow$ $\sin x \tan x - x^2 > 0$ (as $f(0) = 0$)

$\Rightarrow$ $\sin x \tan x > x^2$ so we have $\dfrac{x^2}{\sin x \tan x} < 1$

$$\lim_{x\to 0}\left[\frac{\sin x \tan x}{x^2}\right] = \lim_{x\to 0}(1) = 1$$

$$\lim_{x\to 0}\left[\frac{x^2}{\sin x \tan x}\right] = \lim_{x\to 0}(0) = 0$$

ILLUSTRATION 31

Prove that $\tan^{-1}e - \dfrac{e}{\sqrt{1+e^2}} > \tan^{-1}\dfrac{1}{e} - \dfrac{1}{\sqrt{1+e^2}}$

SOLUTION

Let $f(x) = \tan^{-1}x + \dfrac{1}{\sqrt{1+x^2}}$ where $x \in \left[\dfrac{1}{e}, e\right]$

$$f'(x) = \frac{1}{1+x^2} - \frac{2x}{2(1+x^2)^{3/2}}$$

$$= \frac{1}{1+x^2} - \frac{x}{(1+x^2)^{3/2}} = \frac{\sqrt{1+x^2} - x}{(1+x^2)^{3/2}} > 0$$

As $f'(x) > 0$ then $f(x)$ is increasing function.

$$e > \frac{1}{e} \Rightarrow f(e) > f\left(\frac{1}{e}\right)$$

$$\Rightarrow \tan^{-1}e + \frac{1}{\sqrt{1+e^2}} > \tan^{-1}\left(\frac{1}{e}\right) + \frac{1}{\sqrt{1+e^2}}$$

Rearranging given terms

$$\tan^{-1}e - \frac{e}{\sqrt{1+e^2}} > \tan^{-1}\frac{1}{e} - \frac{1}{\sqrt{1+e^2}}$$

Hence proved

ILLUSTRATION 32

Using function $f(x) = x^{1/x}$

Compare

(i) $1.01^{\frac{1}{1.01}}$ or $1.02^{\frac{1}{1.02}}$ **(ii)** $100^{\frac{1}{100}}$ or $101^{\frac{1}{101}}$

(iii) $3^{\frac{1}{3}}$ or $4^{\frac{1}{4}}$ **(iv)** $e^{\frac{1}{e}}$ or $\pi^{\frac{1}{\pi}}$ **(v)** e^{π} or e^{ε}

SOLUTION

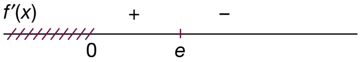
Given $f(x) = x^{1/x}$ where $x > 0$

$$\Rightarrow \quad \ln f(x) = \ln x^{1/x} = \frac{1}{x}\ln x$$

$$\Rightarrow \quad \frac{1}{f(x)} f'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$$f'(x) = 0 \Rightarrow \ln x = 1 \Rightarrow x = e$$

$$\frac{f'(x)}{f(x)} = \frac{1 - \ln x}{x^2}$$



$f(x)$ is increasing function when $x \in (0, e)$ and increasing $x \in (e, \infty)$.

$$f(0^+) = \lim_{x \to 0^+} x^{1/x} = 0^\infty = 0$$

$$f(e) = e^{1/e} \text{ is maximum value of } f(x)$$

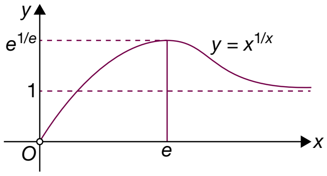
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} x^{1/x} \qquad (\infty^\circ \text{ indeterminant form})$$

$$= \lim_{x \to \infty} e^{\ln x^{1/x}} = \lim_{x \to \infty} e^{\frac{\ln x}{x}} \qquad (\infty / \infty \text{ form})$$

Using L-Hospital Rule

$$= \lim_{x \to \infty} e^{1/x} = e^\circ = 1$$

$$f(x) = x^{1/x} \; x \in (0, \infty)$$



(i) $1.01 < 1.02 \Rightarrow f(1.01) < f(1.02)$

$$\Rightarrow \; 1.01^{\frac{1}{1.01}} < 1.02^{\frac{1}{1.02}}$$

(ii) $100 < 101 \Rightarrow f(100) > f(101) \Rightarrow 100^{\frac{1}{100}} > 101^{\frac{1}{101}}$

(iii) $3 < 4 \Rightarrow f(3) > f(4) \Rightarrow 3^{1/3} > 4^{1/4}$

(iv) $e < \pi \Rightarrow f(e) > f(\pi) \Rightarrow e^{1/e} > \pi^{1/\pi}$

(v) $e^{1/e} > \pi^{1/\pi} \Rightarrow \left(e^{\frac{1}{e}}\right)^{\pi e} > \left(\pi^{\frac{1}{\pi}}\right)^{\pi e} \Rightarrow e^\pi > \pi^e$

ILLUSTRATION 33

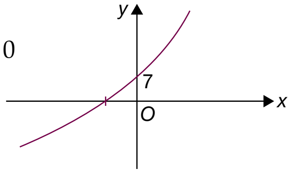
Show that equation $x^5 + 5x + 7 = 0$ has only one real root.

SOLUTION

$$f(x) = x^5 + 5x + 7$$

$$f(0) = 7 > 0$$

$$f(-2) = -32 - 10 + 7 = -35 < 0$$



From Intermediate mean value theorem,

$f(x) = 0$ has atleast one real root.

$f'(x) = 5x^4 + 5 > 0$, $f(x)$ is increasing function.

As $f(x)$ is strictly increasing function, so graph of $f(x)$ cuts x axis only once.

Hence, $f(x) = 0$ has exactly one real roots.

ILLUSTRATION 34

Show that exactly two real values of x satisfy the equation $x^2 = x \sin x + \cos x$.

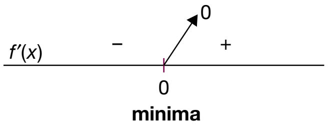
SOLUTION

Let $f(x) = x^2 - x \sin x - \cos x$

As $x \to \infty$, $f(x) \to +\infty$

As $x \to -\infty$, $f(x) \to +\infty$

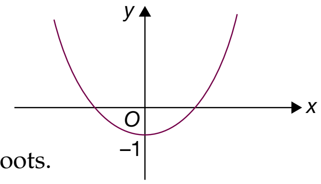
$$f'(x) = x(2 - \cos x)$$



$f(x)$ is decreasing for $x \in (-\infty, 0)$ and increasing for $x \in (0, \infty)$

At $x = 0$ is point of minima.

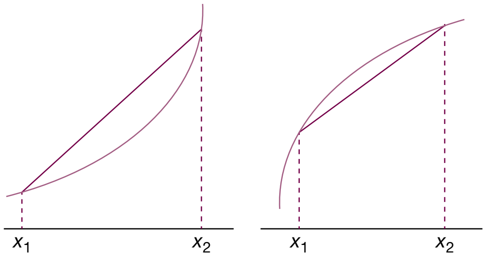
$$f(0) = -1$$



$f(x) = 0$ has exactly two real roots.

CONCAVITY AND CONVEXITY OF CURVE

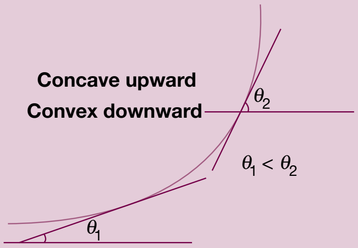
If chord of a curve is above the curve for all points lying in between x_1 and x_2 then it is concave upward (convex downward) and if chord of curve is below the curve for all points lying in between x_1 and x_2 then it is concave downward (convex upward).



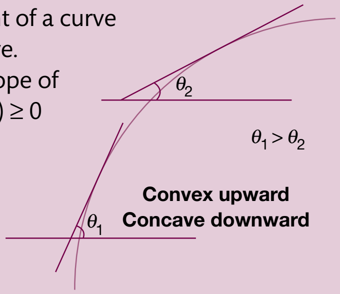
Concave upward
Convex downward

Convex upward
Concave downward

For concave curves, tangent of a curve always lies below the curve. As we can also see that slope of tangent increases so $f''(x) \geq 0$ for concave upward.

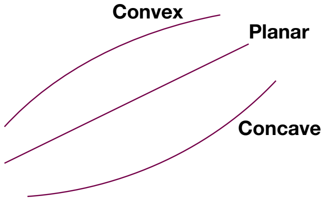


For convex curves, tangent of a curve always lies above the curve.
As we can also see that slope of tangent decreases so $f''(x) \geq 0$

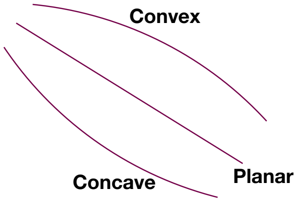

θ_2
$\theta_1 > \theta_2$
Convex upward
Concave downward
θ_1

A function which is monotonically increasing or decreasing can have three pattern of concavity.

(i) Increasing function

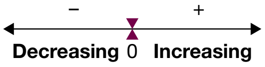

Convex
Planar
Concave

(ii) Decreasing function


Convex
Concave
Planar

Let consider function $y = x^2$,

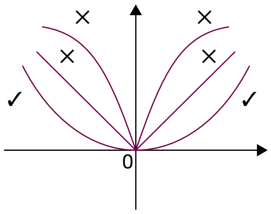
$$\frac{dy}{dx} = 2x$$


–
+
Decreasing
0
Increasing

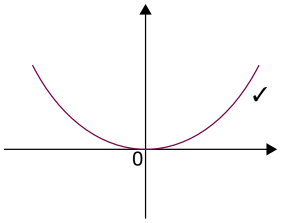
$y = x^2$ is increasing for $x > 0$ and it is decreasing for $x < 0$

as $\dfrac{d^2y}{dx^2} = 2 \Rightarrow f''(x) > 0$

So, curve is concave for all x.

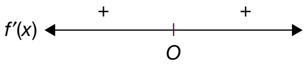


Correct graph of $y = x^2$ should be as shown



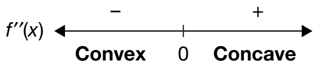
Let consider one more function $f(x) = x^3$

$f'(x) = 3x^2 \geq 0,$

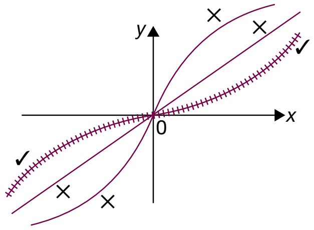


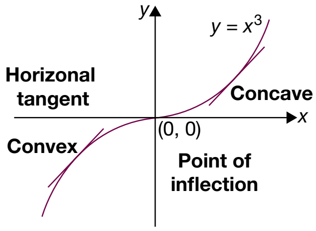
It means $f(x)$ is increasing function for all x.

$f''(x) = 6x,$


f″(x)
–
+
Convex
0
Concave

Clearly $f(x)$ is convex for $x < 0$ and concave for $x > 0$ so we can decide which concavity to follow




y
y = x³
Horizonal tangent
Concave
x
(0, 0)
Convex
Point of inflection

It is the correct representation of $y = x^3$

$f''(x) = 6x = 0$ at point $(0, 0)$

about point (0,0) curve changes it concavity so this point is known as **point of inflection**.

When $x < 0$, tangent lies above the curve and when $x > 0$ tangent lies below the curve and $x = 0$ tangent crosses the curve (horizontal tangent).

POINT OF INFLECTION

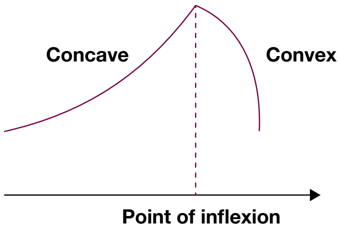
(i) A point where curve changes its concavity from convex to concave or concave to convex

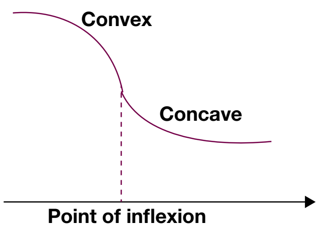
(ii) For continuous and differentiable function, at point of inflection tangent crosses the curve

(iii) For continuous function, let $x = \alpha$ is point of inflexion then

  (a) $y = f''(x)$ is zero or does not exist,

  (b) $y = f''(x)$ must changes it sign across $x = \alpha$


Concave
Convex
Point of inflexion


Convex
Concave
Point of inflexion

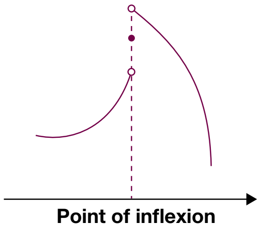

Point of inflexion

ILLUSTRATION 35

Discuss concavity of

(i) Curve $y = \tan^{-1}x$ **(ii)** Curve $y = \sin^{-1}x$

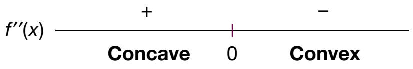
SOLUTION

(i) Curve $y = \tan^{-1}x$

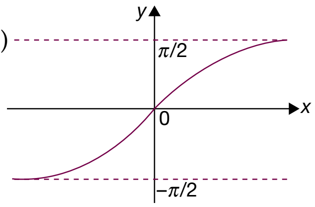
$$f'(x) = \frac{dy}{dx} = \frac{1}{1+x^2} > 0$$

$f(x)$ is strictly increasing function

$$f''(x) = \frac{d^2y}{dx^2} = -\frac{2x}{(1+x^2)^2}$$


f″(x)
+
−
Concave
0
Convex

For $x \in (-\infty, 0)$ $f(x)$ is concave and for $x \in (0, \infty)$ $f(x)$ is convex. At origin (0, 0) curve changes it concavity. Hence, it is point of inflection of curve $y = \tan^{-1} x$

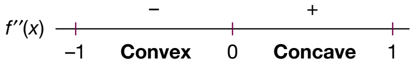


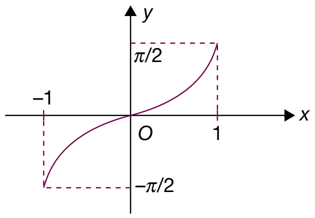
(ii) Curve $y = \sin^{-1} x$

$$f'(x) = \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-1/2} > 0$$

$f(x)$ is increasing

$$f''(x) = \frac{d^2y}{dx^2} = \frac{x}{\left(1-x^2\right)^{3/2}}$$


f″(x)
−
+
−1
Convex
0
Concave
1



$f(x)$ is convex for $x \in (-\infty, 0)$ and $f(x)$ is concave for $x \in (0, \infty)$

Origin (0, 0) curve changes it concavity hence it is point of inflection of curve $y = \sin^{-1} x$

ILLUSTRATION 36

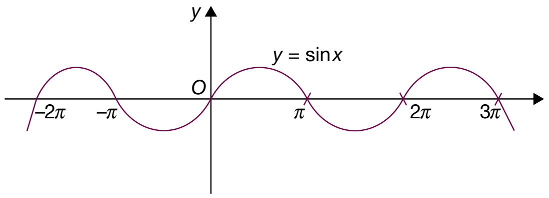
Discuss concavity of following function and point of inflection.

(i) $f(x) = \sin x$ **(ii)** $f(x) = 3x^4 - 4x^3$

(iii) $f(x) = 3x^4 - 8x^3 + 6x^2 + 12$ **(iv)** $f(x) = |x^2 - 16|$

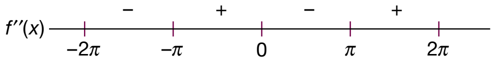
SOLUTION

(i) $f(x) = \sin x$



$f'(x) = \cos x$

$f''(x) = -\sin x = 0$



At $x = n\pi$; $y = 0$ for every integer n,

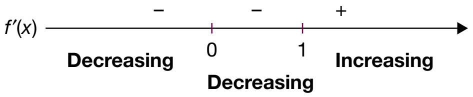
For $x \in ((2n-1)\pi, 2n\pi)$, $f(x)$ is concave

For $x \in (2n\pi, (2n+1)\pi)$, $f(x)$ is convex

Point of inflection are $(n\pi, 0)$

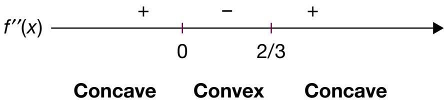
(ii) $f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$


f′(x)
−
−
+
Decreasing
0
1
Increasing
Decreasing

Monotonicity of $f(x)$ can be observed with sign of $f'(x)$

$f''(x) = 36x^2 - 24x = 12x(3x-2)$

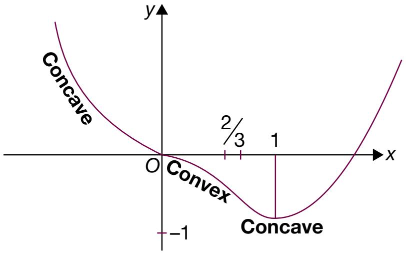

f″(x)
+
−
+
0
2/3
Concave
Convex
Concave

Concavity of $f(x)$ is decided by sign of $f''(x)$

$f(0) = 0 : f(1) = -1$

Point of inflection at point (0, 0) and $\left(\frac{2}{3}, -\frac{16}{27}\right)$

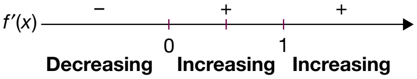
Graph of $f(x)$ is shown



(iii) $f(x) = 3x^4 - 8x^3 + 6x^2 + 12$

$f'(x) = 12x^3 - 24x^2 + 12x$

$f'(x) = 12x(x^2 - 2x + 1) = 12x(x-1)^2$


f′(x)
−
+
+
0
1
Decreasing
Increasing
Increasing

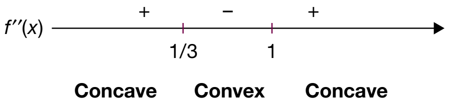
Point origin is point of minima

$$f(0) = 12;\ f(1) = 3 - 8 + 6 + 12 = 13$$

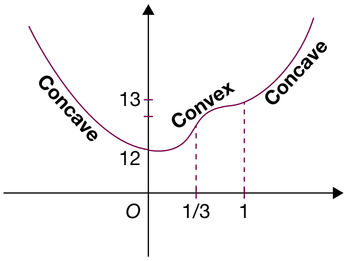
We can decide increasing and decreasing intervals of $f(x)$ from sign change of $f'(x)$ and concavity of $f(x)$ by sign change of $f''(x)$

$$f''(x) = 36x^2 - 48x + 12$$

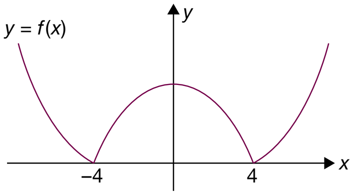
$$= 12(3x^2 - 4x + 1) = 12(3x - 1)(x - 1)$$



Point of inflection at $x = \dfrac{1}{3}$ and $x = 1$



(iv) $f(x) = |x^2 - 16|$ at $x = \pm 4$ not differentiable.



Using the graph transformation, we can plot given function.

From graph $f(x)$ is concave in $x \in (-\infty, -4) \cup (4, \infty)$

Convex in $x \in (-4, 4)$

So we can conclude that point (–4, 0) and (4, 0) are point of inflection, these points are also point of minima.

ILLUSTRATION 37

Curve $y = x^4 + ax^3 + \dfrac{3}{2}x^2 + 1$ be concave for entire real number line find a.

SOLUTION

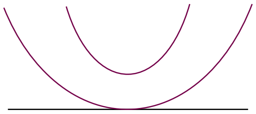
$$y = x^4 + ax^3 + \frac{3}{2}x^2 + 1$$

$$\frac{dy}{dx} = 4x^3 + 3ax^2 + 3x$$

$$\frac{d^2y}{dx^2} = 12x^2 + 6ax + 3 \geq 0\ \forall\ x \in R$$

$$A > 0 \Rightarrow 12 > 0$$

Discriminant of quadratic $D \leq 0$



$$B^2 - 4AC \leq 0$$

$$36a^2 - 4(12)3 \leq 0$$

$$\Rightarrow\quad 36(a^2 - 4) \leq 0$$

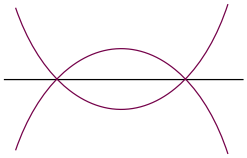
$$\Rightarrow\quad (a - 2)(a + 2) \leq 0$$

We get $-2 \leq a \leq 2$

ILLUSTRATION 38

Find condition on a, b, c such that $y = ax^4 + bx^3 + cx^2 + dx + e$ to have a point of inflection?

SOLUTION



$$y = ax^4 + bx^3 + cx^2 + dx + e$$

$$\frac{dy}{dx} = 4ax^3 + 3bx^2 + 2cx + d$$

$$\frac{d^2y}{dx^2} = 12ax^2 + 6bx + 2c$$

It must change the sign.

It means this quadratic must have real and distinct roots.

$$D > 0 \Rightarrow 36b^2 - 4(12a)\,2c > 0$$

$$\Rightarrow\quad 12[3b^2 - 8ac] > 0 \Rightarrow 3b^2 > 8ac$$

ILLUSTRATION 39

Show that point of inflection of curve $y = x \sin x$ lies on curve $y^2(4 + x^2) = 4x^2$

SOLUTION

Curve $y = x \sin x$, $\dfrac{dy}{dx} = \sin x + x \cos x$

$$\frac{d^2y}{dx^2} = \cos x + \cos x - x \sin x = 0 \Rightarrow 2 \cos x = x \sin x$$

Let point of inflection (x_0, y_0)

$$\frac{d^2y}{dx^2} = 0 \Rightarrow 2 \cos x_0 = x_0 \sin x_0 \Rightarrow x_0 = 2 \cot x_0 \qquad \ldots(1)$$

(x_0, y_0) also lies on curve $y = x \sin x$ so we have

$$y_0 = x_0 \sin x_0 \qquad \ldots(2)$$

Now we want that point of inflection of curve

$$y = x \sin x \text{ lies on curve } y^2(4 + x^2) = 4x^2$$

LHS $= y_0^2(4 + x_0^2)$ using (1) and (2).

$$= x_0^2 \sin^2 x_0\, [4 + 4\cot^2 x_0]$$

$$= x_0^2 \sin^2 x_0 \cdot 4\operatorname{cosec}^2 x_0 = 4x_0^2 = \text{RHS}$$

Hence proved.

ILLUSTRATION 40

If $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)\,x + 2$ passes a negative point of inflection find a.

SOLUTION

$$f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)\,x + 2$$

$$f'(x) = ax^2 + 2(a+2)x + (a-1)$$

For negative point of inflection, we have

$$f''(x) = 2ax + 2(a+2) = 0$$

$$x = -\frac{(a+2)}{a} < 0 \Rightarrow \frac{a+2}{a} > 0$$

$$a \in (-\infty, -2) \cup (0, \infty)$$

ILLUSTRATION 41

Let $f(x) = x^3 - 3px^2 + 3(p^2-1)\,x + 1$, If point of inflection of $f(x)$ lies in $(-2, 4)$ find p.

SOLUTION

$$f(x) = x^3 - 3px^2 + 3(p^2-1)x + 1$$

$$f'(x) = 3x^2 - 6px + 3(p^2-1)$$

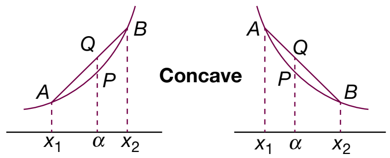
$$f''(x) = 6x - 6p = 0 \Rightarrow x = p$$

Point of inflexion of $f(x)$ lies in $(-2, 4)$, so we have $-2 < p < 4$

INEQUALITY USING CONCAVITY [JENSEN INEQUALITY]

Based on concavity of a curve, we can get inequality.
Lets assume $f(x)$ is concave for some interval, from the given figure clearly we can see that chord AB is above the curve for all the points lying between x_1 and x_2

$$f''(x) = \frac{d^2y}{dx^2} \geq 0$$

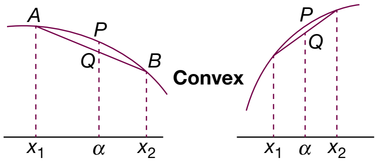


At $x = \alpha$, ordinate of point $P <$ ordinate of point Q.

$$y_P < y_Q$$

Lets assume $f(x)$ is convex for real numbers, from the given figure we can conclude that chord AB is below the curve for all the points lying between x_1 and x_2

$$f''(x) = \frac{d^2y}{dx^2} \leq 0$$



At $x = \alpha$, ordinate of point $P >$ ordinate of point Q.

$$y_P > y_Q$$

ILLUSTRATION 42

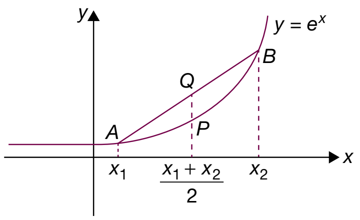
Prove that $\frac{e^{x_1} + e^{x_2}}{2} > e^{\frac{x_1+x_2}{2}}$ where x_1 and x_2 are real distinct numbers $y_P > y_Q$.

SOLUTION

$$f(x) = e^x;\ f'(x) = e^x$$

$$f''(x) = e^x > 0$$

$f(x) = e^x$ is always increasing and concave upwards.



Let two points $A\left(x_1, e^{x_1}\right)$ and $B\left(x_2, e^{x_2}\right)$

Now mid point of AB is $Q\left(\frac{x_1+x_2}{2}, \frac{e^{x_1}+e^{x_2}}{2}\right)$ and point $P\left(\frac{x_1+x_2}{2}, e^{\frac{x_1+x_2}{2}}\right)$ on curve with same abscissa.

$$y_Q > y_P \Rightarrow \frac{e^{x_1}+e^{x_2}}{2} > e^{\frac{x_1+x_2}{2}}$$

Alternative:
For distinct x_1 and x_2,

$$AM > GM$$

$$\frac{e^{x_1}+e^{x_2}}{2} > \sqrt{e^{x_1} \cdot e^{x_2}} \Rightarrow \frac{e^{x_1}+e^{x_2}}{2} > e^{\frac{x_1+x_2}{2}}$$

ILLUSTRATION 43

Prove that $\ln\left(\frac{x_1+x_2+x_3}{3}\right) > \frac{\ln x_1 + \ln x_2 + \ln x_3}{3}$ where $x_1, x_2, x_3 \in R^+$ and all $x_i > 1$

SOLUTION

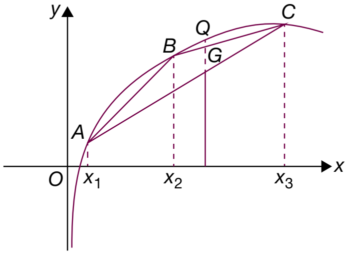
$$f(x) = \ln x;\ f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2} < 0$$

$f(x)$ is always increasing and convex upward.

Let three points on curve $A(x_1, \ln x_1)$, $B(x_2, \ln x_2)$ and $C(x_3, \ln x_3)$

Centroid $G\left(\frac{x_1+x_2+x_3}{3}, \frac{\ln x_1+\ln x_2+\ln x_3}{3}\right)$ and corresponding point $Q\left(\frac{x_1+x_2+x_3}{3}, \ln\left(\frac{x_1+x_2+x_3}{3}\right)\right)$ on curve



$$y_G < y_Q$$

$$\Rightarrow \quad \frac{\ln x_1+\ln x_2+\ln x_3}{3} < \ln\left(\frac{x_1+x_2+x_3}{3}\right)$$

Alternative:

using $AM > GM$

$$\frac{x_1+x_2+x_3}{3} > (x_1x_2x_3)^{\frac{1}{3}}$$

$$\ln\left(\frac{x_1+x_2+x_3}{3}\right) > \ln(x_1x_2x_3)^{\frac{1}{3}} = \frac{1}{3}(\ln x_1 + \ln x_1 + \ln x_3)$$

ILLUSTRATION 44

In an acute angled and scalene triangle Prove that

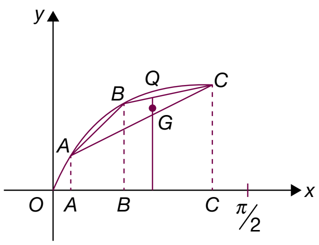
(i) $\sin A + \sin B + \sin C < \frac{3\sqrt{3}}{2}$

(ii) $\tan A + \tan B + \tan C > 3\sqrt{3}$

SOLUTION

(i) Let $f(x) = \sin x$ where $x \in \left(0, \frac{\pi}{2}\right)$

$$f''(x) = -\sin x < 0 \text{ and } A \neq B \neq C$$



As given $O < A, B, C < \frac{\pi}{2}$ for triangle ABC, Point A $(A, \sin A)$, $B(B, \sin B)$ and $C(C, \sin C)$

G is centroid and Q is corresponding point on curve with same abscissa

$$G\left(\frac{A+B+C}{3}, \frac{\sin A+\sin B+\sin C}{3}\right) \text{ and}$$

$$Q\left(\frac{A+B+C}{3}, \sin\left(\frac{A+B+C}{3}\right)\right)$$

$$y_Q > y_G \Rightarrow \sin\left(\frac{A+B+C}{3}\right) > \frac{\sin A+\sin B+\sin C}{3}$$

$$\Rightarrow \frac{\sqrt{3}}{2} > \frac{\sin A+\sin B+\sin C}{3} \qquad (\because A+B+C=\pi)$$

$$\Rightarrow \sin A + \sin B + \sin C < \frac{3\sqrt{3}}{2}$$

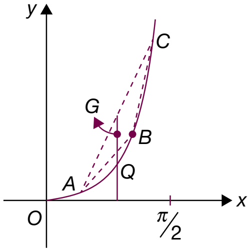
(ii) $f(x) = \tan x$ for $x \in \left(0, \frac{\pi}{2}\right)$

$$f'(x) = \sec^2 x$$

$$f''(x) = 2\sec^2 x \tan x > 0$$

$f(x)$ is increasing and concave upwards

Acute angle ΔABC,
point $A(A, \tan A)$, $B(B, \tan B)$,
$C(C, \tan C)$
G is centroid of ΔABC and
Q is corresponding point on curve for same abscissa



$$G\left(\frac{A+B+C}{3}, \frac{\tan A+\tan B+\tan C}{3}\right)$$

$$Q\left(\frac{A+B+C}{3}, \tan\left(\frac{A+B+C}{3}\right)\right)$$

Clearly $Y_G > Y_Q$

$$\frac{\tan A+\tan B+\tan C}{3} > \tan\left(\frac{A+B+C}{3}\right)$$

$$\frac{\tan A+\tan B+\tan C}{3} > \sqrt{3}$$

$\tan A + \tan B + \tan C > 3\sqrt{3}$. Hence, proved.

ILLUSTRATION 45

If $f(x)$ is twice differentiable monotonic decreasing and $f''(x) < 0$ and $f^{-1}(x)$ exists.

Prove that $f^{-1}\left(\frac{2x_1+x_2}{3}\right) > \frac{2f^{-1}(x_1)+f^{-1}(x_2)}{3}$

SOLUTION

Let $g = f^{-1}(x)$. As we know that if f and g are inverse function of each other then $gof(x) = x$.

On differentiation,

$$g'(f(x))f'(x) = 1 \Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

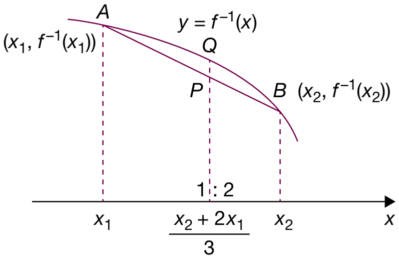
As $f'(x) < 0$ so $g'(f(x)) < 0$, $g'(x) < 0$; $f^{-1}(x)$ is decreasing.

$$g''(f(x))\,f'(x) = -\frac{1}{(f'(x))^2}\,f''(x)$$

$$\Rightarrow \quad g''(f(x)) = \frac{-1}{(f'(x))^3}\,f''(x) < 0$$

(given $f'(x) < 0$ and $f''(x) < 0$)

$f^{-1}(x)$ is decreasing with convex upward.



Lets consider 2 points $A(x_1, f^{-1}(x_1))$ and $B(x_2, f^{-1}(x_2))$ by considering a point, which divides line joining x_1 and x_2 in 1:2 ratio, we have $x = \frac{2x_1 + x_2}{3}$

P divides AB in 1: 2 and point Q is point on curve for same x.

$$p\left(\frac{2x_1 + x_2}{3}, \frac{2f^{-1}(x_1) + f^{-1}(x_2)}{3}\right) \text{ and}$$

$$\text{Point } Q\left(\frac{2x_1 + x_2}{3}, f^{-1}\left(\frac{2x_1 + x_2}{3}\right)\right)$$

Clearly $Y_P < Y_Q$

$$\frac{2f^{-1}(x_1) + f^{-1}(x_2)}{3} < f^{-1}\left(\frac{2x_1 + x_2}{3}\right) \quad \text{Hence proved.}$$

ILLUSTRATION 46

If $f(x)$ is monotonic increasing $\forall\, x \in R$ $f''(x) < 0$ and $f^{-1}(x)$ exists.

Prove that $\dfrac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3} > f^{-1}\left(\dfrac{x_1 + x_2 + x_3}{3}\right)$

SOLUTION

$f(x)$ is monotonic increasing, then $f'(x) > 0$ and $f''(x) < 0$

$$g = f^{-1}(x) \quad \Rightarrow \quad gof(x) = x$$

$$g'(f(x))\,f'(x) = 1 \Rightarrow g'(f(x)) = \frac{1}{f'(x)} > 0$$

$\therefore$ $g(x)$ is increasing function

$$g''(f(x)) = \frac{-1}{(f'(x))^3}\,f''(x) > 0$$

$f^{-1}(x)$ is increasing and concave upward.

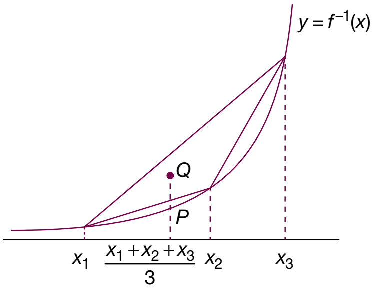
Lets consider 3 points $A(x_1, f^{-1}(x_1))$, $B(x_2, f^{-1}(x_2))$, $C(x_3, f^{-1}(x_3))$.

Q is centroid of ΔABC,

$$Q\left(\frac{x_1 + x_2 + x_3}{3}, \frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3}\right)$$

Point P is point on curve when $x = \frac{x_1 + x_2 + x_3}{3}$,

$$\text{point } P\left(\frac{x_1 + x_2 + x_3}{3}, f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right)\right)$$



Clearly we have $Y_Q > Y_P$

$$\frac{f^{-1}(x_1) + f^{-1}(x_2) + f^{-1}(x_3)}{3} > f^{-1}\left(\frac{x_1 + x_2 + x_3}{3}\right)$$

Hence proved.

Test Your Concept-II

(Solutions on page H.183)

1. If $x_1, x_2 \in \left(0, \frac{\pi}{2}\right)$ then show that $\frac{\tan x_2}{\tan x_1} > \frac{x_1}{x_2}$ where $x_1 < x_2$
2. If $f(x)$ is a differentiable real valued function satisfying $f''(x) - 3f'(x) > 3$ $\forall$ $x \geq 0$ and $f'(0) = -1$, then show that $f(x) + x$, $\forall\, x > 0$ is increasing function
3. Using the result $2(1 - \cos x) < x^2, x \neq 0$.

   Prove that $\sin(\tan x) > x$, for $\forall\, x \in \left(0, \frac{\pi}{4}\right)$.
4. Prove that: $\sin x + 2x \geq \frac{3x(x+1)}{\pi}$, $\forall\, x \in \left[0, \frac{\pi}{4}\right]$

   (Justify the inequality, if any used)
5. If $0 < x < 1$, prove that $y = x \ln x - \frac{x^2}{2} + \frac{1}{2}$ is a function such that $\frac{d^2y}{dx^2} > 0$. Deduce that $x \ln x > \frac{x^2}{2} + \frac{1}{2}$.
6. Show that $xe^x = 2$ has one and only one root between 0 and 1.

7. If k is the least value of k for which the function x^2+kx+1 is an increasing function in the interval $1<x<2$, then the value of $|k|$ is __________

8. Find no of real solutions of $e^x+4x=11$

9. Discuss concavity of following function, also find point of inflection if exists

(i) $f(x)=x^3-6x^2+12x+1$ (ii) $f(x)=x^4+x$

10. Find the equation of the tangent line at an inflection point of $f(x)=x^4-6x^3+12x^2-8x+3$

Answer key

7. 2 **8.** 1

9. (i) convex $x<2$, concave $x>2$, Point of inflection (2, 9) (ii) always concave, no point of inflection

(iii) convex $x<0$, concave $x>0$, origin is point of inflection. **10.** $y=2x$ and $y=3$

STATIONARY POINTS AND CRITICAL POINTS

- Stationary points where $\dfrac{dy}{dx}=f'(x)=0$
- Critical points where $\dfrac{dy}{dx}=f'(x)=0$ or does not exist.
- Stationary points are subset of Critical points.
- Critical points always lie in domain of function.

ILLUSTRATION 47

Find stationary points and critical points of following function.

(i) $f(x)=4x^3-6x^2-24x+9$ **(ii)** $f(x)=(x-1)x^{2/3}$

(iii) $f(x)=|\sin x|$

(iv) $f(x)=\max\{\sin x, \cos x\}$ where $x\in(0,\pi)$

SOLUTION

(i) $f(x)=4x^3-6x^2-24x+9$

$$f'(x)=12x^2-12x-24=12(x^2-x-2)=0$$

$(x-2)(x+1)=0\Rightarrow x=2,-1$ are stationary points and critical points

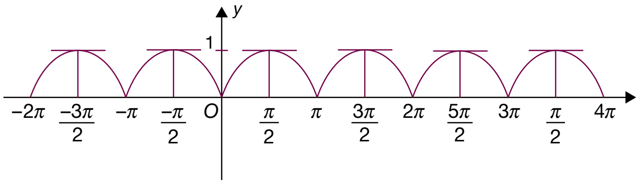
(ii) $f(x)=(x-1)x^{2/3}$ domain $x\in R$

$$f'(x)=x^{2/3}+\frac{2(x-1)}{3x^{1/3}}=\frac{3x+2(x-1)}{3x^{1/3}}=\frac{5x-2}{3x^{1/3}}$$

Stationary point is $x=\dfrac{2}{5}$ only

Critical points are $x=0,\ \dfrac{2}{5}$

(iii) $f(x)=|\sin x|$



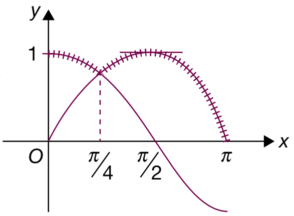
Stationary point where $f'(x)=0$ we get $x=(2n+1)\dfrac{\pi}{2}$

$f(x)$ not differentiable, $f'(x)$ does not exists at $x=n\pi$

$\therefore$ Critical points are $x=\dfrac{n\pi}{2}$

(iv) $f(x)=\max\{\sin x, \cos x\}$ where $x\in(0,\pi)$

$$f(x)=\begin{cases}\cos x & 0<x\le\dfrac{\pi}{4}\\ \sin x & \dfrac{\pi}{4}<x<\pi\end{cases}$$



$f'(x)$ does not exist when $x=\dfrac{\pi}{4}$

For $0<x<\dfrac{\pi}{4}$, $f'(x)=-\sin x\neq0$

For $\dfrac{\pi}{4}<x<\pi$, $f'(x)=\cos x=0$ at $x=\dfrac{\pi}{2}$

Critical points are $x=\dfrac{\pi}{2},\dfrac{\pi}{4}$

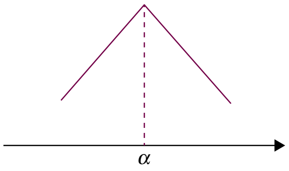
LOCAL OR RELATIVE MAXIMA/MINIMA

Let a function $y=f(x)$ is defined in neighbourhood of point $x=\alpha$ where δ is a very small positive quantity.

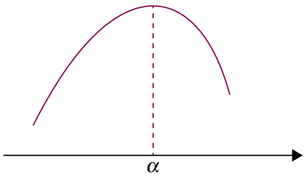
If $f(\alpha-\delta)<f(\alpha)>f(\alpha+\delta)$ then at $x=\alpha$, local maxima (relative maxima).

If $f(\alpha-\delta)>f(\alpha)<f(\alpha+\delta)$ then at $x=\alpha$, local minima (relative minima).

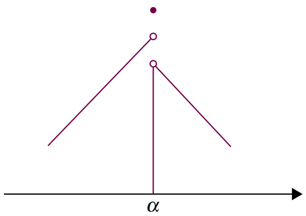
Local maxima or minima are collectively called as an extrema.



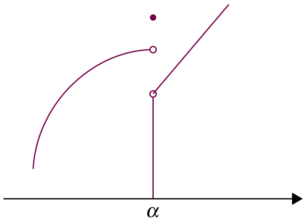
Local maxima



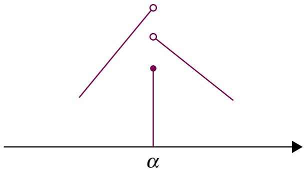
Local maxima



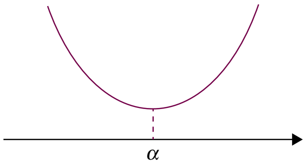
Local maxima



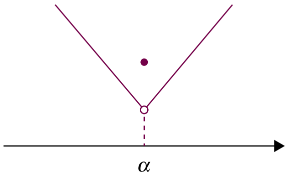
Local maxima



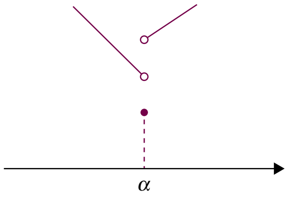
Local minima



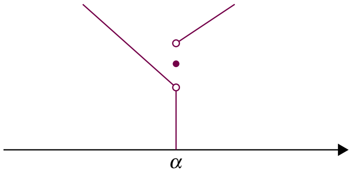
Local minima



Local maxima

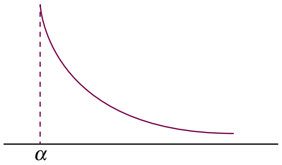


Local minima

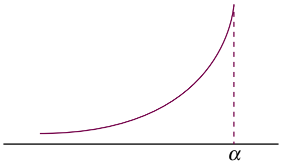


Neither maxima nor minima

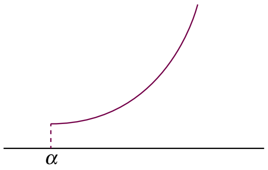
When $x = \alpha$ is a boundary point



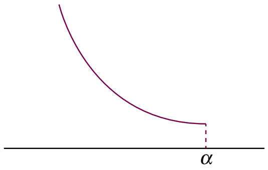
Local maxima



Local maxima



Local minima



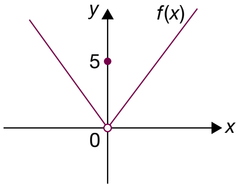
Local minima

ILLUSTRATION 48

$$f(x) = \begin{cases} |x| & ; \quad x \neq 0 \\ 5 & ; \quad x = 0 \end{cases} \text{ and } g(x) = \begin{cases} |x| & ; \quad x \neq 0 \\ -5 & ; \quad x = 0 \end{cases}$$

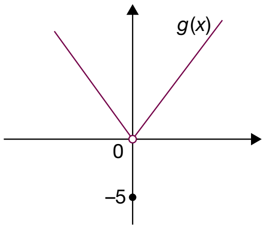
Discuss extrema at $x = 0$

SOLUTION



Given $f(x) = \begin{cases} +x & ; \quad x > 0 \\ 5 & ; \quad x = 0 \\ -x & ; \quad x < 0 \end{cases}$

From graph we can conclude that at $x = 0$ local maxima of $f(x)$ as $f(0 - \delta) < f(0) > f(0 + \delta)$.



$$g(x) = \begin{cases} |x| & ; \quad x \neq 0 \\ -5 & ; \quad x = 0 \end{cases}$$

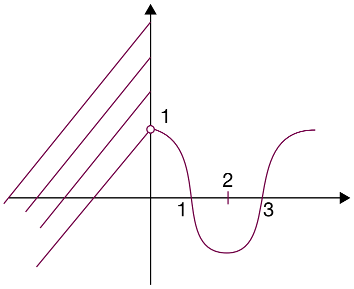
From graph we can conclude that at $x = 0$ local minima of $g(x)$ as $g(0 - \delta) > g(0) < g(0 + \delta)$.

ILLUSTRATION 49

If $f(x) = \begin{cases} \cos\dfrac{\pi}{2}x & ; \quad x > 0 \\ x + a & ; \quad x \leq 0 \end{cases}$ then $f(x)$ has local maxima at point $x = 0$, find a.

SOLUTION

From graph of $f(x)$ we can decide about having local maxima at point $x = 0$



At $x = 0$, for local maxima so $f(0^-) \leq f(0) \geq f(0^+)$.

$f(0) \geq f(0^+) \Rightarrow a \geq 1$ for at $x = 0$ is local minima

ILLUSTRATION 50

Let $f(x) = \begin{cases} -x^3 + \dfrac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} & ; \quad 0 \leq x < 1 \\ 2x - 3 & ; \quad 1 \leq x \leq 3 \end{cases}$ If $f(x)$ has local minima at $x = 1$. Find b.

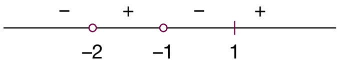
SOLUTION

Here, $x = 1$ is local minima then by using mathematical definition.

$$f(1^-) \geq f(1) \leq f(1^+)$$

$$f(1^-) \geq f(1) \Rightarrow -1 + \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \geq 2 - 3$$

$$\Rightarrow \quad \frac{b^2(b-1)+(b-1)}{(b+1)(b+2)} \geq 0 \Rightarrow \frac{(b^2+1)(b-1)}{(b+1)(b+2)} \geq 0$$



We have $b \in (-2, -1) \cup [1, \infty)$.

ILLUSTRATION 51

$f(x) = \begin{cases} 4x - x^3 + \ln(a^2 - 3a + 3) & ; \quad 0 \leq x < 3 \\ x - 18 & ; \quad x \geq 3 \end{cases}$ $f(x)$ has local minima at $x = 3$, find a?

SOLUTION

For local minima at point $x = 3$, using mathematical definition $f(3^-) \geq f(3) \leq f(3^+)$.

$$f(3^-) \geq f(3) \Rightarrow 12 - 27 + \ln(a^2 - 3a + 3) \geq 3 - 18$$

$$\Rightarrow \quad \ln(a^2 - 3a + 3) \geq 0$$

$$\Rightarrow \quad a^2 - 3a + 3 \geq e^\circ = 1 \Rightarrow a^2 - 3a + 2 \geq 0$$

$$\Rightarrow \quad (a-1)(a-2) \geq 0 \text{ we get } a \in (-\infty, 1] \cup [2, \infty)$$

USE OF DERIVATIVE FOR IDENTIFYING EXTREMA

If function is **continuous** then we can use derivatives for identifying types of extrema.

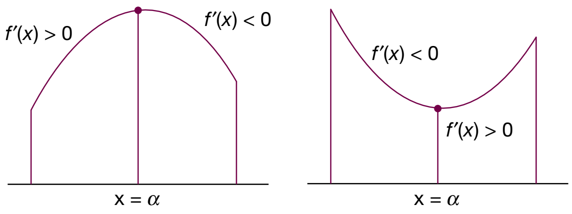
First Derivative Test

(i) Get critical points.

$f'(x) = 0$ or does not exist

Let $x = \alpha, \beta$ are critical points

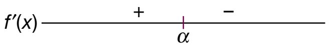
(ii) Observe the sign change of $f'(x)$



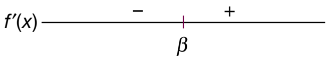
Relative maxima **Relative minima**

An extremum occurs when the derivative changes sign

If $f'(x)$ changing its sign from positive to negative then $x = \alpha$ is a point of local maxima.



If $f'(x)$ changing its sign from negative to positive then $x = \beta$ is a point of local minima.



If $f'(x)$ is not changing its sign across a critical point then it is not an extrema.

Note

For a continuous function having extrema at critical points, $f'(x)$ must change it's sign across it.

Higher Order Derivative Test

(i) For continuous function. First get critical points (let $x = \alpha$), $f'(x)$ is zero or does not exist,

(ii) Now get $f''(x)$

- **(a)** If $f''(\alpha) < 0$ then $x = \alpha$ is a point of local maxima
- **(b)** If $f''(\alpha) > 0$ then $x = \alpha$ is a point of local minima
- **(c)** If $f''(\alpha) = 0$, second derivative is zero at $x = \alpha$, then we should try getting third derivative, fourth derivative etc till we get first non-zero derivative at $x = \alpha$

  Let $f'''(\alpha) = 0, f''''(\alpha) = 0$

  $f^n(\alpha) \neq 0$ (I$^{\text{st}}$ time)

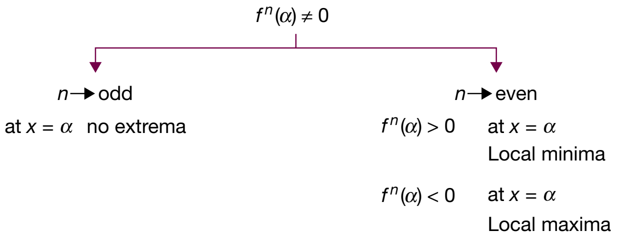


ILLUSTRATION 52

Find out local maxima/minima of following function.

(i) $f(x) = x^2(x-2)^2$ **(ii)** $f(x) = x^5 - 5x^4 + 5x^3 - 1$

(iii) $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$ **(iv)** $f(x) = |x^2 - 2x|$

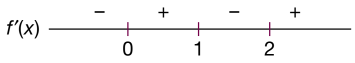
(v) $f(x) = \dfrac{x^2 + x + 1}{x^2 - x + 1}$

SOLUTION

(i) $f(x) = x^2(x-2)^2$

$$f'(x) = 2x(x-2)^2 + x^2(2(x-2))$$

$$= 2x(x-2)(x-2+x) = 4x(x-1)(x-2)$$



(Using First Derivative Test)

Cleary we can see that $f'(x)$ changing its sign from − to + around point $x = 0, 2$ so these are point of local minima.

Around point $x = 1$, $f'(x)$ changing its sign from + to −, so It is a point of local maxima

Alternate:

(Using higher derivative Test)

$f'(x) = 4x(x-1)(x-1) = 0$, critical points are $= 0, 1, 2$

$f'(x) = 4(x^3 - 3x^2 + 2x)$

$f''(x) = 4(3x^2 - 6x + 2)$

$f''(0) = 4(0+2) = 8 > 0$ then at $x = 0$ local minima

$f''(1) = 4(3-6+2) = -4 < 0$ then at $x = 1$ local maxima

$f''(2) = 4(12-12+2) = 8 > 0$ then at $x = 2$ local minima

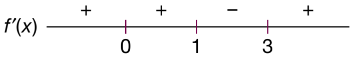
(ii) $f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2[x^2 - 4x + 3]$

$f'(x) = 5x^2(x-1)(x-3) = 0 \Rightarrow x = 0, 1, 3$ are critical points for extrema

Now from first derivative test we need to check sign change of $f'(x)$ around critical points

$f'(x) = 5x^2(x-1)(x-3)$



Cleary we can see that $f'(x)$ changing its sign from − to + around point $x = 3$ so it is a point of local minima. Around point $x = 1$, $f'(x)$ changing its sign from + to −, so, it is a point of local maxima

No sign change around point $x = 0$, it is not an extrema (neither maxima nor minima)

Alternative:

[Using higher derivative test]

$f'(x) = 5x^2[x^2 - 4x + 3] = 5(x^4 - 4x^3 + 3x^2)$

$f''(x) = 5(4x^3 - 12x^2 + 6x)$

$f''(1) = 5(4 - 12 + 6) = -10 < 0$ at $x = 1$ is local maxima

$f''(3) = 5(4 \times 27 - 108 + 18) > 0$ at $x = 3$ is local minima

$f''(0) = 0$

$f''(x) = 5(4x^3 - 12x^2 + 6x)$ we need to differentiate it further till we get non-zero term at $x = 0$

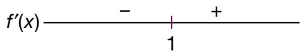
$f'''(x) = 5(12x^2 - 24x + 6)$

$f'''(0) = 30 \neq 0$, as we are getting third derivative non-zero so at $x = 0$ is not an extrema, neither maxima nor minima

(iii) $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

$f'(x) = 4x^3 - 12x^2 + 12x - 4 = 4[x^3 - 3x^2 + 3x - 1]$

$f'(x) = 4(x-1)^3$



Cleary we can see that $f'(x)$ changing its sign from − to + around point $x = 1$ so it is a point of local minima. Using higher derivative test

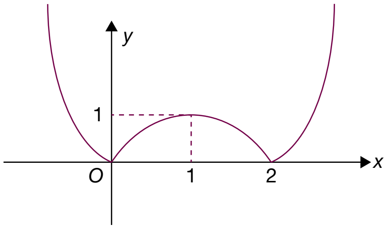
$f''(x) = 12(x-1)^2$, we get $f''(1) = 0$

$f'''(x) = 24(x-1)$, we get $f'''(1) = 0$

$f^{IV}(x) = 24 \neq 0$ first derivative with non-zero value at $x = 1$

$f^{IV}(1) = 24 > 0$, point $x = 1$ is local minima

(iv)



$f(x) = |x^2 - 2x|$

From the graph of $f(x)$, we can clearly state that point $x = 1$ is local minima

and at point $x = 0, 2$ $f(x)$ is having local minima

(v) $f(x) = \dfrac{x^2 + x + 1}{x^2 - x + 1}$

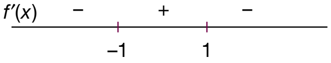
$$f'(x) = \frac{(x^2 - x + 1)(2x+1) - (x^2 + x + 1)(2x-1)}{(x^2 - x + 1)^2}$$

$(x^2 + 1)2x + x^2 + 1 - x(2x+1) -$

$$= \frac{[(x^2+1)2x - (x^2+1) + x(2x-1)]}{(x^2 - x + 1)^2}$$

$$= \frac{2[x^2 + 1 - 2x^2]}{(x^2 - x + 1)^2} = -\frac{2(x^2 - 1)}{(x^2 - x + 1)^2}$$

$$f'(x) = -2\frac{(x-1)(x+1)}{(x^2 - x + 1)^2}$$



Cleary, we can see that $f'(x)$ changing its sign from − to + around point $x = -1$ so it is a point of local minima.

Around point $x = 1$, $f'(x)$ changing its sign from + to −, so It is a point of local maxima.

ILLUSTRATION 53

Curve $y = x^3 + ax^2 + bx + c$ has extrema at $x = -1$ and $x = 3$. Find a, b, c.

SOLUTION

$y = f(x) = x^3 + ax^2 + bx + c$

$\dfrac{dy}{dx} = f'(x) = 3x^2 + 2ax + b$

At $x = -1$, $f'(-1) = 3 - 2a + b = 0$,

At $x = 3$, $f'(3) = 27 + 6a + b = 0$,

By taking difference, we get $-24 - 8a = 0$ or $a = -3$

$3 - 2(-3) + b = 0$ or $b = -9$

Hence, set of values $a = -3$, $b = -9$, $c \in R$.

ILLUSTRATION 54

Let $f(x) = (x^2 - 1)^n\ (x^2 + x - 1)$ If $f(x)$ has local extremum at $x = 1$ then find n where $n \in N$.

SOLUTION

$$f(x) = (x^2 - 1)^n\ (x^2 + x - 1)$$

$$f'(x) = n(x^2 - 1)^{n-1}\ (2x - 1)(x^2 + x - 1) + (x^2 - 1)^n\ (2x + 1)$$

$$= (x^2 - 1)^{n-1}(2nx(x^2 + x - 1) + (x^2 - 1)\ (2x + 1))$$

For $f(x)$ to have local extrema at $x = 1$ $f'(x)$ must change its sign across put $x = 1$, $(x^2 - 1)^{n-1}$ must change its sign, $n - 1$ must be odd. It means n must be even.

ILLUSTRATION 55

Curve $y = \dfrac{ax + b}{(x - 1)(x - 4)}$ has local maxima at point (2, –1). Find a and b.

SOLUTION

Point (2, –1) lies on given curve $y = \dfrac{ax + b}{(x - 1)(x - 4)}$

$$-1 = \frac{2a + b}{(1)(-2)} \Rightarrow 2a + b = 2 \qquad \ldots(1)$$

$$y = \frac{ax + b}{(x - 1)(x - 4)} \text{ or } y(x^2 - 5x + 4) = ax + b$$

On differentiation, $y'(x^2 - 5x + 4) + y(2x - 5) = a$

Point (2, –1) is a local maxima so $x = 2$, $y = -1$, $y' = 0$

$$0 + (-1)\ (4 - 5) = a \text{ or } a = 1$$

From (1), $2a + b = 2$ or $2 + b = 2$, we get $b = 0$

Hence, final values are $a = 1$, $b = 0$

GLOBAL OR ABSOLUTE MAXIMA/MINIMA

Lets consider a function $y = f(x)$, for $x \in [x_1, x_2]$.

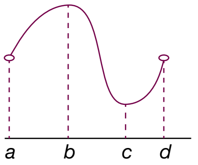
(i) Global maxima or absolute maximum is the greatest value of $f(x)$ in given set values of x, then at $x = \alpha$ is global maxima then $f(x) \le f(\alpha)$ for all given permissible x

(ii) Global minima or absolute minimum is the least value of $f(x)$ in given set of values of x then at $x = \alpha$ is global maxima then $f(x) \ge f(\alpha)$ for all given permissible x

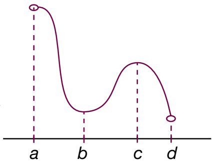
(iii) Global maxima/minima are not necessary to exist, they may or may not co-exist with extrema.

(iv) Global maxima/minima may occur at boundary points also.
Lets consider following function for better understanding:

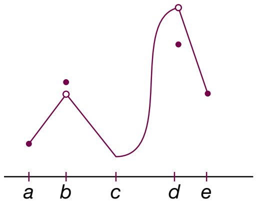
- For function $y = f(x)$
  Local minima at $x = c$
  Local maxima at $x = b$
  Global maxima at $x = b$
  Global minima at $x = c$



- For function $y = g(x)$
  Local minima at $x = b$
  Local maxima at $x = c$
  Global maxima/minima not exist

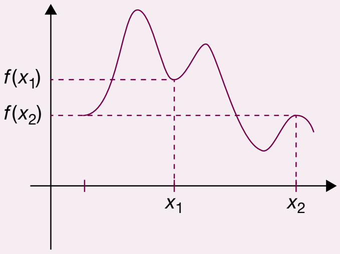


- For function $y = h(x)$
  Local maxima at $x = b$
  Local minima at $x = a, c, d, e$
  Global maxima does not exist
  Global minima at $x = c$



Conceptual Note(s)

(i) Local minima can have higher value than any local maxima
At $x = x_1$ is local minima and $x = x_2$ is local maxima,
Clearly we can see that $f(x_1) > f(x_2)$



(ii) Can global minima have higher values then a local maxima?
Absolutely NO, as we know that global minima is the least value of $f(x)$

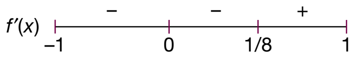
ILLUSTRATION 56

$f(x) = 6x^{4/3} - 3x^{1/3}$ where $x \in [-1, 1]$ find maximum and minimum of $f(x)$.

SOLUTION

$$f'(x) = 6\left(\frac{4}{3}\right)\ x^{1/3} - 3\left(\frac{1}{3}\right)\ x^{-2/3}$$

$$f'(x) = 8x^{1/3} - x^{-2/3} = \frac{8x - 1}{x^{2/3}} = \frac{8x - 1}{\left(x^{1/3}\right)^2}$$



$$f(-1) = 6 + 3 = 9; f(0) = 0; f(1) = 6 - 3 = 3$$

$$f\left(\frac{1}{8}\right) = 6\left(\frac{1}{2}\right)^4 - 3\left(\frac{1}{2}\right) = -\frac{9}{8}$$

$f(x)$ has maximum at point $(-1, 9)$ and minimum at point $\left(\frac{1}{8}, -\frac{9}{8}\right)$

If $x \in (-1, 1]$ then $f(x)$ has no maximum.

ILLUSTRATION 57

$f(x) = 2x^3 - 9x^2 + 12x + 2$ find maximum and minimum of $f(x)$. If

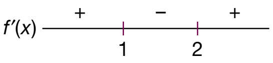
(i) $x \in [0, 2]$ **(ii)** $x\ (-1, 2]$

SOLUTION

$$f(x) = 2x^3 - 9x^2 + 12x + 2$$

$$f'(x) = 6x^2 - 18x + 12$$

$$= 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$$



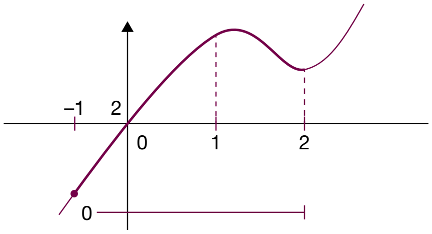
(i) $x \in [0, 2]$

$$f(0) = 2$$

$$f(1) = 2 - 9 + 12 + 2 = 7$$

$$f(2) = 16 - 36 + 24 + 2 = 6$$

When $x \in [0, 2]$, $f(x)$ has maximum at point $(1, 7)$ and minimum at point $(0, 2)$



(ii) $x \in (-1, 2]$

$$f(x) = 2x^3 - 9x^2 + 12x + 2$$

$$f(-1) = -2 - 9 + 12 + 2 = -21$$

$$f(1) = 7, f(2) = 6$$

When $x \in (-1, 2]$, $f(x)$ has maximum at $(1, 7)$ and no minimum

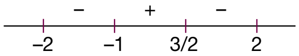
ILLUSTRATION 58

A function $y = f(x)$ is given by the parametric equations $x = t^5 - 5t^3 - 20t + 7$ and $y = 4t^3 - 3t^2 - 18t + 3$, where $-2 < t < 2$. Investigate the maxima and minima.

SOLUTION

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12t^2 - 6t - 18}{5t^4 - 15t^2 - 20}$$

$$\Rightarrow \frac{dy}{dx} = \frac{6(t+1)(2t-3)}{5(t^2+1)(t+2)(t-2)} \text{ as given } -2 < t < 2$$



Minimum at $t = -1$ as $\frac{dy}{dx}$ changes it sign from negative to positive.

Minimum at $x = 31$; $y_{\text{max.}} = 14$

Maximum at $t = \frac{3}{2}$ as $\frac{dy}{dx}$ changes it sign from positive to negative.

Maximum at $x = -\frac{1033}{32}$; $y_{\min} = -\frac{69}{4}$

ILLUSTRATION 59

Find the point(s) of minimum of the function, $f(x) = 4x^3 - x\,|x - 2|$ where $x \in [0, 3]$.

SOLUTION

$$f(x) = \begin{cases} 4x^3 - x(-x+2)) & 0 \le x < 2 \\ 4x^3 - x(x-2) & 2 \le x \le 3 \end{cases}$$

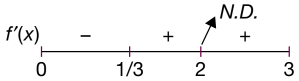
$$f(x) = \begin{cases} 4x^3 + x^2 - 2x & 0 \le x < 2 \\ 4x^3 - x^2 + 2x & 2 \le x \le 3 \end{cases}$$

$$f'(x) = \begin{cases} 12x^2 + 2x - 2 & 0 \le x \le 2 \\ 12x^2 - 2x + 2 & 2 < x \le 3 \end{cases}$$

$$f'(x) = \begin{cases} 2(3x-1)(2x+1) & 0 \le x < 2 \\ 2(6x^2 - x + 1) & 2 < x < 3 \end{cases}$$

$$f'(2^-) = 50; f'(2^+) = 46.$$

$f(x)$ is not differentiable at $x = 2$



Hence, minima at $x = \frac{1}{3}$

ILLUSTRATION 60

The function $f(x) = 2x^3 + \alpha x^2 + \beta x + \gamma$, where $\alpha, \beta, \gamma \in R$ has local minimum at $P(\log_3 t^2, f(\log_3 t^2))$ and local maximum at $Q(\log_3 t, f(\log_3 t))$. If $R\left(\frac{5}{2}, f\left(\frac{5}{2}\right)\right)$ is the point of inflection, then find 't'

SOLUTION

$$f(x) = 2x^3 + \alpha x^2 + \beta x + \gamma$$

$$f'(x) = 6x^2 + 2\alpha x + \beta$$

$$f''(x) = 12x + 2\alpha = 0 \text{ at point of inflection } R\left(\frac{5}{2}, f\left(\frac{5}{2}\right)\right)$$

$$x = -\frac{\alpha}{6} = \frac{5}{2} \Rightarrow \alpha = -15$$

$f'(x) = 6x^2 - 30x + \beta = 0$ for point of extrema

at $x = \log_3 t^2, \log_3 t$

$$\Rightarrow \quad 2\log_3 t + \log_3 t = \frac{30}{6} = 5 \Rightarrow \log_3 t = \frac{5}{3} \Rightarrow t = 3^{5/3}$$

ILLUSTRATION 61

Let $f(x) = \begin{cases} 2-|x^2+5x+6|, & x \neq -2 \\ b^2+1, & x = -2 \end{cases}$. If $f(x)$ has relative maximum at $x = -2$, then find the range of the b.

SOLUTION

At $x = -2$ is local maximum then $f(-2^-) \le f(-2) \ge f(-2^+)$.

$$\Rightarrow \quad 2 - |4 - 10 + 6| \le b^2 + 1$$

$$\Rightarrow \quad b^2 \ge 1 \text{ or } |b| \ge 1$$

ILLUSTRATION 62

Find the values of α for which the points of extremum of the function $f(x) = x^3 - 3\alpha x^2 + 3(\alpha^2 - 1)\,x + 1$ lie in the interval $(-2, 4)$.

SOLUTION

$$f(x) = x^3 - 3\alpha x^2 + 3(\alpha^2 - 1)x + 1$$

$$f'(x) = 3x^2 - 6\alpha x + 3(\alpha^2 - 1) = 3(x^2 - 2\alpha x + \alpha^2 - 1)$$

$$f'(x) = 3(x - \alpha + 1)\,(x - \alpha - 1) = 0$$

At point of extrema, $f'(x) = 0$

We get $x = \alpha - 1, \alpha + 1$ lie in $(-2, 4)$.

$$-2 < \alpha - 1 < 4 \quad \Rightarrow -1 < \alpha < 5$$

$$-2 < \alpha + 1 < 4 \quad \Rightarrow -3 < \alpha < 3$$

Hence, set of values are $-1 < \alpha < 3$

ILLUSTRATION 63

For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\ \cos t\, dt$.

then find extrema of $f(x)$.

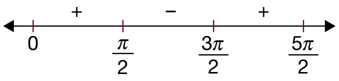
SOLUTION

$$f(x) = \int_0^x \sqrt{t}\ \cos t\, dt$$

We have $f'(x) = \sqrt{x}\cos x, x \in \left(0, \frac{5\pi}{2}\right)$

$$f'(x) = 0 \Rightarrow \sqrt{x}\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Sign of $f'(x)$



$\therefore \quad f(x)$ has local maximum at $\frac{\pi}{2}$ and local minimum at $\frac{3\pi}{2}$.

ILLUSTRATION 64

If curve $y = x^3 + px + q$ has three real and distinct roots then Prove that $4p^3 + 27q^2 < 0$

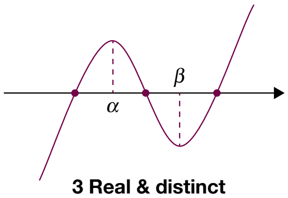
SOLUTION

$$\frac{dy}{dx} = 3x^2 + p = 0$$

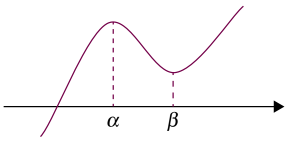
$$\Rightarrow \quad x = \pm\sqrt{-\frac{p}{3}}, \text{ clearly } p < 0$$

$$\Rightarrow \quad \alpha = -\sqrt{-\frac{p}{3}}\,;\ \beta = \sqrt{-\frac{p}{3}}$$

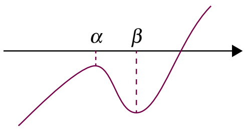
Therefore, $\alpha = -\beta$



3 Real & distinct



One real root



One real root

For having three real and distinct roots of cubic,

$$f(\alpha)\,f(\beta) < 0$$

$$\Rightarrow \quad f(-\beta)f(\beta) < 0 \Rightarrow (-\beta^3 - p\beta + q)\,(\beta^3 + p\beta + q) < 0$$

$$\Rightarrow \quad q^2 - (\beta^3 + p\beta)^2 < 0 \Rightarrow q^2 - \beta^2(\beta^2 + P)^2 < 0$$

$$\Rightarrow \quad q^2 - \left(-\frac{p}{3}\right)\left(-\frac{p}{3} + p\right)^2 < 0$$

$$\Rightarrow \quad q^2 + \frac{p}{3}\cdot\frac{4p^2}{3} < 0 \Rightarrow 27q^2 + 4p^3 < 0$$

Hence proved.

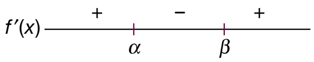
ILLUSTRATION 65

$f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$ If $f(x)$ attains maxima at positive value of x find possible value of a.

SOLUTION

$f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9)$ has two real roots α and β.

$f'(x) = 3(x-\alpha)(x-\beta)$ let $\alpha < \beta$



$x = \alpha$, root which has less value is point of local maxima. If we want local maxima point to be positive then both roots of $f'(x)$ must be positive.

For $f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9) = 0$, both roots one positive.

(i) $D > 0 \Rightarrow 36(a-7)^2 - 36(a^2-9) > 0$

$$\Rightarrow 36(-14a + 49 + 9) > 0 \Rightarrow a < \frac{29}{7}$$

(ii) Sum of the both roots is positive, $\alpha + \beta > 0 \Rightarrow \frac{-B}{A} > 0$ $\Rightarrow -2(a-7) > 0 \Rightarrow a < 7$

(iii) Product of roots is positive, $\alpha\beta > 0 \Rightarrow f'(0) > 0$ $\Rightarrow (a-3\,(a+3) > 0 \Rightarrow a \in (-\infty, -3) \cup (3, \infty)$

final common values of $a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

ILLUSTRATION 66

Let $f(x) = \begin{cases} (x+2)^3 & ; \ -3 < x \le -1 \\ x^{\frac{2}{3}} & ; \ -1 < x \le 2 \end{cases}$ and

$$g(x) = \int_{-3}^{x} f(t)dt \quad ; \ -3 < x < 2$$

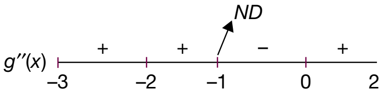
then find the number of extremum points of $g'(x)$.

SOLUTION

$$g(x) = \int_{-3}^{x} f(t)dt, -3 < x < 2$$

$g'(x) = f(x)$, as given function $f(x)$ is continuous.

$$g''(x) = f'(x) = \begin{cases} 3(x+2)^2 & ; \ -3 < x \le -1 \\ \frac{2}{3}x^{-\frac{1}{3}} & ; \ -1 < x < 2 \end{cases}$$



Hence, $x = 0, -1$ are 2 point of extrema of $g'(x)$.

ILLUSTRATION 67

If $P(x)$ be a polynomial satisfying $P(x^2) = x^2P(x)$ and $P(0) = -2$, $P'\left(\frac{3}{2}\right) = 0$ and $P(1) = 0$ then find maximum value of $P(x)$.

SOLUTION

Let $P(x)$ is of degree n, given $P(x^2) = x^2P(x)$.

$P(x^2)$ is of degree $2n$ and $x^2P(x)$ is of degree $n + 2$

$2n = 2 + n$, we get $n = 2$

$P(x) = ax^2 + bx + c$

$P(0) = c = -2$; $P(1) = 0 = a + b + c$ or $a + b = 2$

$P'(x) = 2ax + b$ or $P'\left(\frac{3}{2}\right) = 3a + b = 0$ or $b = -3a$;

On solving we get $a = -1$, $b = 3$, $c = -2$

$P(x) = -x^2 + 3x - 2$

$$P(x)_{\max} = -\frac{D}{4a} = -\frac{(9-4(-1)(-2))}{4(-1)} = \frac{1}{4}$$

ILLUSTRATION 68

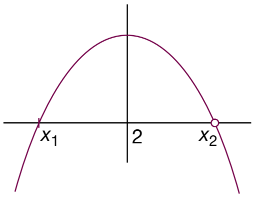
The function $f(x) = 3 + 2(a+1)x + (a^2+1)x^2 - x^3$ has a local minimum at $x = x_1$ and local maximum at $x = x_2$ such that $x_1 < 2 < x_2$, then find a.

SOLUTION

$f(x) = 3 + 2(a+1)x + (a^2+1)x^2 - x^3$

$f'(x) = 2(a+1) + 2(a^2+1)\,x - 3x^2$

As coefficient of x^2 is negative, Clearly $f'(x)$ is opening downward graph.



$f'(2) = 2(a+1) + 4(a^2+1) - 12 > 0$

$\Rightarrow \quad 4a^2 + 2a - 6 > 0$

$\Rightarrow \quad 2(a-1)(2a+3) > 0 \Rightarrow a \in \left(-\infty, -\frac{3}{2}\right) \cup (1, \infty)$

ILLUSTRATION 69

Let $f(x)$ be a function with continuous derivative on (λ, μ) and $2\ln\left(f(x) + \sqrt{(f(x))^2+1}\right).f'(x) - \sqrt{f^2(x)+1} \le 0$ then find minimum value of $(\mu - \lambda)$ if $\lim\limits_{x\to\lambda^+} f(x) = -3$ and $\lim\limits_{x\to\mu^-} f(x) = 3$

SOLUTION

Let $y = f(x)$; $2\ln\left(y + \sqrt{y^2+1}\right)\frac{dy}{dx} \le \sqrt{y^2+1}$

$$\Rightarrow \quad 2\frac{\ln(y+\sqrt{y^2+1})}{\sqrt{y^2+1}}\frac{dy}{dx} \le 1$$

$$\Rightarrow \quad \frac{d}{dx}\left(\ln^2\left(y+\sqrt{y^2+1}\right) - x\right) \le 0$$

$g(x) = \ln^2\left(f(x) + \sqrt{f(x)^2+1}\right) - x$ is decreasing function if $x \in (\lambda, \mu)$

$$g(\lambda^+) = \ln^2\left(-3+\sqrt{10}\right) - \lambda; \ g(\mu^-) = \ln^2\left(3+\sqrt{10}\right) \text{-}\mu$$

$g(x)$ is decreasing.

$$g(\lambda^+) \geq g(\mu^-) \Rightarrow g(\lambda^+) - g(\mu^-) \geq 0$$

$$\ln^2\left(-3+\sqrt{10}\right) - \lambda - \ln^2\left(3+\sqrt{10}\right) + \mu \geq 0$$

$$\mu - \lambda \geq \ln^2\left(3+\sqrt{10}\right) - \ln^2\left(-3+\sqrt{10}\right)$$

$$= (\ln\left(3+\sqrt{10}\right) + \ln\left(-3+\sqrt{10}\right))\,(\ln\left(3+\sqrt{10}\right) - \ln\left(-3+\sqrt{10}\right))$$

$$\mu - \lambda \geq (\ln 1\,(\ln\left(3+\sqrt{10}\right) - \ln\left(-3+\sqrt{10}\right))$$

$$\mu - \lambda \geq 0$$

Hence, minimum value of $(\mu - \lambda) = 0$

ILLUSTRATION 70

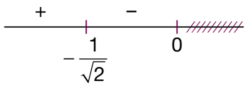
Consider the function $f(x) = \begin{cases} x(x-1)e^{2x} & ; \quad x \leq 0 \\ x(1-x)e^{-2x} & ; \quad x > 0 \end{cases}$. Then find point where $f(x)$ attains its greatest value.

SOLUTION

CASE 1: For $x \leq 0$

$$f'(x) = (x^2 - x)e^{2x}\,2 + (2x-1)e^{2x} = e^{2x}(2x^2 - 2x + 2x - 1)$$

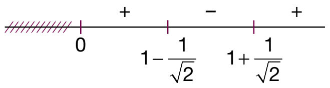
$$f'(x) = e^{2x}(2x^2 - 1) = e^{2x}\left(\sqrt{2}x - 1\right)\left(\sqrt{2}x + 1\right)$$



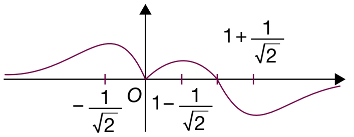
CASE 2: For $x \geq 0$

$$f'(x) = (x - x^2)e^{-2x}(-2) + (1-2x)e^{-2x} = e^{-2x}\,(2(x-1)^2 - 1) = 0$$

$$f'(x) = 2e^{-2x}\left(x - 1 + \frac{1}{\sqrt{2}}\right)\left(x - 1 - \frac{1}{\sqrt{2}}\right)$$



Point on local maxima are $x = -\frac{1}{\sqrt{2}}$ and $1-\frac{1}{\sqrt{2}}$ and at $x = 0$ is point of minima now we can draw graph of $f(x)$.



$$f\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}+1\right)e^{-\sqrt{2}}$$ greatest value of $f(x)$

$$f\left(1-\frac{1}{\sqrt{2}}\right) = \left(1-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)e^{-2+\sqrt{2}}$$

$f(x)$ has greatest value at $x = -\frac{1}{\sqrt{2}}$

ILLUSTRATION 71

Find the number of values of x where the function $f(x) = \cos x + \cos\left(\sqrt{2}x\right)$ attains its maximum.

SOLUTION

$$f(x) = \cos x + \cos(\sqrt{2}\,x)$$

Differentiating w.r.t. x, we get $f'(x) = -\sin x - \sqrt{2}\sin(\sqrt{2}\,x)$

Now, put $f'(x) = 0 \Rightarrow -(\sin x + \sqrt{2}\sin(\sqrt{2}\,x)) = 0$

$\Rightarrow \quad \sin x = -\sqrt{2}\sin(\sqrt{2}\,x)$, It holds only at $x = 0$

Again differentiating w.r.t. x, we get

$$f''(x) = -\cos x - 2\cos\sqrt{2}\,x$$

$$f''(x) = -[\cos x + 2\cos\sqrt{2}\,x]$$

$$f''(x)\big|_{x=0} = -[1 + 2.1] = -3$$

We can conclude that $x = 0$ is point of maxima, $f(0) = 2$

$f(x) = \cos x + \cos(\sqrt{2}\,x)$ is non-periodic function.

Hence, $x = 0$ is the only point of maxima.

Test Your Concepts-III

(Solutions on page H.184)

1. Find no. of critical points.
- (i) $f(x) = \operatorname{sgn}(x)$ (ii) $f(x) = \sin\frac{1}{x}$
- (iii) $f(x) = \frac{|x-2|}{x^2}$ (iv) $f(x) = \frac{1}{x^2 - x}$

2. Let $f(x) = e^{|x^2 - 4x + 3|}$ then show that $f(x)$ has one local maximum point and two local minimum points.

3. Find extrema for following functions
- (i) $f(x) = x^2 \log_e x$ (ii) $f(x) = x\log_e x$
- (iii) $f(x) = \frac{\log_e x}{x}$ (iv) $f(x) = x^{-x}$

4. Let $f(x) = (x - a)^n g(x)$, where $g^n(a) \neq 0$; $n = 1, 2, 3, \ldots$ then $f(x)$ has extrema at $x = a$, find n

5. Find extrema of following function
- (i) $f(x) = \sin x\,(1 + \cos x)$ where $x \in (0, 2\pi)$
- (ii) $f(x) = 10x^6 - 12x^5 + 15x^4 - 20x^3$
- (iii) $f(x) = 2x + 3x^{\frac{2}{3}}$

6. Find the point of local maxima/minima of following functions

(i) $f(x) = 2x^3 - 21x^2 + 36x - 20$

(ii) $f(x) = -(x - 1)^3 (x + 1)^2$

(iii) $f(x) = x \ln x$

7. Find the absolute maxima/minima value of following functions

(i) $f(x) = 4x - \frac{x^2}{2}$; $x \in \left[-2, \frac{9}{2}\right]$

(ii) $f(x) = 3x^4 - 8x^3 + 12x^2 - 48x + 25$; $x \in [0, 3]$

(iii) $f(x) = \sin x + \frac{1}{2}$; $\cos 2x$; $x \in \left[0, \frac{\pi}{2}\right]$

(iv) $f(x) = x + \sqrt{x}$; $x \in (0, 4)$

(v) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$

8. Let $-1 \le p \le 1$. Show that the equation $4x^3 - 3x - p = 0$ has a unique root in the interval $\left[\frac{1}{2}, 1\right]$ and identify it.

9. If $ax + (b/x) \ge c$ for all positive values of x, where a, b and c are positive constants, show that $ab \ge c^2/4$

10. Let $f(x) = \left[\begin{matrix} 3 - x & 0 \le x < 1 \\ x^2 + \ell n b & x \ge 1 \end{matrix}\right.$. Find the set of value of b such that $f(x)$ has a local minima at $x = 1$

11. Let $f(x) = \begin{cases} a - 3x; & -2 \le x < 0 \\ 4x + 3; & 0 \le x < 1 \end{cases}$. If $f(x)$ has smallest value at $x = 0$, then find range of a.

12. If $f(x) = \int_0^1 e^{|t-x|} dt$ where $0 \le x \le 1$, then find maximum value of $f(x)$

13. Find number of critical points of the function $f(x) = (x - 2)^{2/3} (2x + 1)$

14. Show that function $f(x) = x + \sqrt{|x|}$ has two critical points

15. Find maximum value of $\sin(\sin 3x + 4 \sin^3 x)$

16. Let $f(x) = \begin{cases} x^2 + 2x + \frac{a^2 - 1}{a^4 - 5}; & x < 0 \\ x + 2; & x \ge 0 \end{cases}$

Find all possible real values of a such that $f(x)$ possesses the smallest value at $x = 0$.

17. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x \to 0} \left(1 + \frac{p(x)}{x^2}\right) = 2$ then the value of $p(2)$ is

18. If $f(x) = x^3 + ax^2 + bx + c$ has a local maximum at $x = -1$ and a local minimum at $x = 3$. Determine the constants a, b, c.

19. The maximum of $f(x) = 2x^3 - 9x^2 + 12x + 4$ occurs at $x = a$, $g(x) = 4 + 5x^2 + 6x^4$ has number of minimum b, then find $a + b$.

Answer key

1. (i) infinite (ii) infinite (iii) 2 (iv) 1 **3.** (i) $x = \frac{1}{\sqrt{e}}$ (point of minima) (ii) point of minima, $x = \frac{1}{e}$

(iii) $x = e$ (point of maxima) (iv) $x = \frac{1}{e}$ (point of maxima.) **4.** Any positive even number

5. (i) at $x = \frac{\pi}{3}$ is local maxima, at $x = \frac{5\pi}{3}$ is local minima, at $x = \pi$, no extreme

(ii) at $x = 1$ local minima and at $x = 0$ no extreme (iii) at $x = 0$ local minima and at $x = -1$ maxima

6. (i) local maxima at $x = 1$; local minima at $x = 6$, (ii) local maxima at $x = -\frac{1}{5}$; local minima at $x = 1$

(iii) local minima at $x = \frac{1}{e}$; No local maxima

7. (i) maximum = 8; minimum = –10; (ii) maximum = 25; minimum = –39;

(iii) maximum = $\frac{3}{4}$ at $x = \frac{\pi}{6}$; minimum = $\frac{1}{2}$ at $x = 0$ and $\frac{\pi}{2}$)

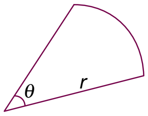
(iv) not defined; (v) Maximum $f(-1) = 18$, minimum $f\left(\frac{1}{8}\right) = -\frac{9}{4}$

10. $b \in (0, e]$ **11.** $[3, \infty)$ **12.** $e - 1$ **13.** 2 **15.** 1 **16.** $a \in \left[-\sqrt{\frac{7}{3}}, -5^{\frac{1}{4}}\right) \cup \left(5^{\frac{1}{4}}, \sqrt{\frac{7}{3}}\right]$

17. 0 **18.** $a = -3$, $b = -9$, $c \in R$ **19.** 2

APPLICATION OF DERIVATIVES IN GEOMETRICAL/PHYSICAL PROBLEMS

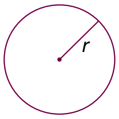
(i) Area of a circular sector $= \frac{1}{2} r^2\theta,$ when θ is in radians.



(ii) Sphere

(a) Volume of a sphere $= \frac{4}{3}\pi r^3$

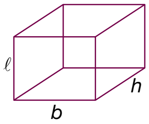
(b) Surface area of a sphere $= 4\,\pi r^2$



(iii) Cuboid

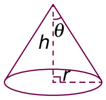
(a) Volume of a cuboid $= lbh.$

(b) Surface area of a cuboid $= 2(lb + bh + hl)$



(iv) Cone

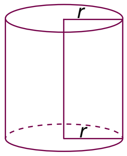
Volume of cone $= \frac{1}{3}\pi r^2 h.$



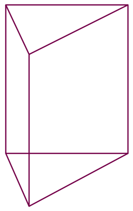
(v) Cylinder

(a) Curved surface area of a cylinder $= 2\pi rh.$

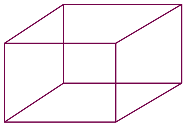
(b) Total surface area of a cylinder $= 2\pi rh + 2\pi r^2$



(vi) Prism



Triangular prism



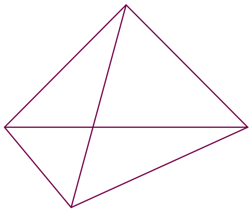
Square base prism

(a) Volume of a prism = area of the base × height

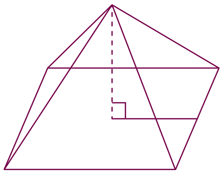
(b) Lateral surface area of prism
= perimeter of the base × height
(Note that lateral surfaces of a prism are all rectangles.)

(c) Total surface area of a prism
= lateral surface area + 2 area of the base

(vii) Pyramid



Triangular pyramid



Square based Pyramid

(a) Volume of a pyramid $= \frac{1}{3}$ area of the base × height.

(b) Curved surface area of a pyramid

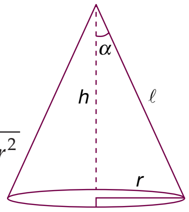
$= \frac{1}{2}$(perimeter of the base) × slant height

(Note that slant surfaces of a pyramid are triangles)

ILLUSTRATION 72

For a given curved surface of a right circular cone when the volume is maximum, prove that the vertical angle is $\sin^{-1}\frac{1}{\sqrt{3}}$.

SOLUTION



Given curved surface $s = \pi r\ell$

Volume of cone $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{\ell^2 - r^2}$

$$V = \frac{1}{3}\pi r^2\sqrt{\frac{s^2}{\pi^2 r^2} - r^2} = \frac{1}{3} r\sqrt{s^2 - \pi^2 r^4}$$

$$\frac{dv}{dr} = \frac{1}{3}\left(\sqrt{s^2 - \pi^2 r^4} + \frac{r(-4\pi^2 r^3)}{2\sqrt{s^2 - \pi^2 r^4}}\right)$$

$$= \frac{(s^2 - \pi^2 r^4 - 2\pi^2 r^4)}{3\sqrt{s^2 - \pi^2 r^4}} = 0$$

We get, $s = 3^{1/2}\pi r^2 = \pi r\ell \Rightarrow \frac{r}{\ell} = \frac{1}{\sqrt{3}} = \sin\alpha$

$\Rightarrow \quad \alpha = \sin^{-1}\frac{1}{\sqrt{3}}$

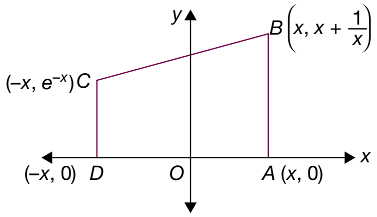
When $\frac{r}{\ell} = \frac{1}{\sqrt{3}}, \frac{d^2v}{dr^2} < 0,$

Hence, it is point of maxima.

ILLUSTRATION 73

If the maximum area of trapezium $ABCD$ formed by points $A(x, 0), B\left(x, x + \frac{1}{x}\right), C(-x, e^{-x}), D(-x, 0)$ is S, where $0 < x \le 1$, then find the value of $[eS]$ where [.] is GIF.

SOLUTION



$$\Delta = \frac{1}{2}\cdot 2x\left(e^{-x} + x + \frac{1}{x}\right)$$

Area of trapezium $\Delta = 1 + x^2 + xe^{-x}$ and

$$\frac{d\Delta}{dx} = 2x + x(-e^{-x}) + e^{-x} = 2x + e^{-x}(1 - x)$$

$$\frac{d\Delta}{dx} \ge 0 \text{ when } x \in (0, 1]$$

Δ is increasing function.

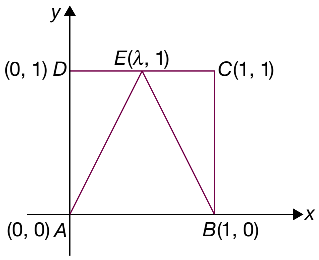
$\therefore \quad S = \Delta_{\max} = 2 + \frac{1}{e}$, when $x = 1$

Now $[eS] = 6$

ILLUSTRATION 74

Let $ABCD$ be a square of side 1 unit. A triangle AEB is formed where E is any point on side CD. If the greatest possible value of perimeter of triangle ABE is $1 + \sqrt{n}$, where $n \in N$, then find the value of n.

SOLUTION



Let Perimeter $(S) = 1 + \sqrt{(\lambda - 0)^2 + 1^2} + \sqrt{(\lambda - 1)^2 + 1^2}$

$$S = 1 + \sqrt{\lambda^2 + 1} + \sqrt{(\lambda - 1)^2 + 1}$$

$$\frac{dS}{d\lambda} = \frac{2\lambda}{2\sqrt{\lambda^2 + 1}} + \frac{2(\lambda - 1)}{2\sqrt{(\lambda - 1)^2 + 1}}; \quad \frac{dS}{d\lambda} = 0$$

$\Rightarrow \quad \lambda\sqrt{(\lambda - 1)^2 + 1} + (\lambda - 1)\sqrt{\lambda^2 + 1} = 0$

$\Rightarrow \quad \lambda^2((\lambda - 1)^2 + 1) = (\lambda - 1)^2 (\lambda^2 + 1)$

$\Rightarrow \quad \lambda^2(\lambda^2 - 2\lambda + 1 + 1) = (\lambda^2 - 2\lambda + 1)(\lambda^2 + 1)$

$\Rightarrow \quad \lambda^4 - 2\lambda^3 + 2\lambda^2 = \lambda^4 + \lambda^2 - 2\lambda^3 - 2\lambda + \lambda^2 + 1 \Rightarrow \lambda = \frac{1}{2}$

For maximum value of perimeter of ΔAEB, $\lambda = \frac{1}{2}$

$\therefore \quad \text{Perimeter} = 2\sqrt{\frac{1}{4} + 1} + 1 = \sqrt{5} + 1 = 1 + \sqrt{n}$

We get $n = 5$

ILLUSTRATION 75

Let (h, k) be a fixed point, where $h > 0$, $k > 0$. A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q. The minimum area of the triangle OPQ, O being the origin is pkh then $p =$

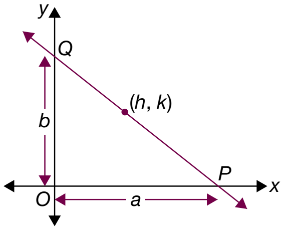
SOLUTION

Let the given line be $\frac{x}{a} + \frac{y}{b} = 1$, so that it makes an intercept of a units on x-axis and b units on y-axis. As it passes through the fixed point (h, k), therefore we must have,

$$\Rightarrow \quad \frac{k}{b} = 1 - \frac{h}{a} \Rightarrow b = \frac{ak}{a - h} \qquad \ldots(1)$$

Now area of $\Delta OPQ = A = \frac{1}{2}ab$

$$\therefore \quad A = \frac{1}{2}a\left(\frac{ak}{a - h}\right) \qquad \text{[using (1)]}$$



or $\quad A = \frac{k}{2}\left(\frac{a^2}{a - h}\right)$

For min value of A, $\frac{dA}{da} = 0$

$$\Rightarrow \quad \frac{k}{2}\left[\frac{2a(a - h) - a^2}{(a - h)^2}\right] = 0$$

$$\Rightarrow \quad \frac{k}{2}\left[\frac{a^2 - 2ah}{(a - h)^2}\right] = 0 \Rightarrow a = 2h$$

Also $\frac{d^2A}{da^2} = \frac{(2a - 2h)(a - h)^2 - 2(a - h)(a^2 - 2ah)}{(a - h)^4}$

$$\therefore \quad \left.\frac{d^2A}{da^2}\right|_{a = 2h} = \frac{2h^3}{h^4} = \frac{2}{h} > 0, \qquad [\because h > 0]$$

$\therefore \quad A$ is min. when $a = 2h$

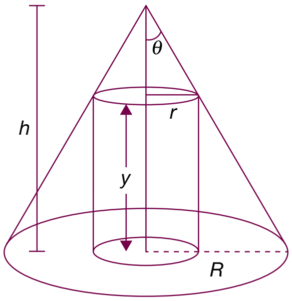
$$\therefore \quad A_{\min} = \frac{k}{2}\left[\frac{4h^2}{h}\right] = 2kh$$

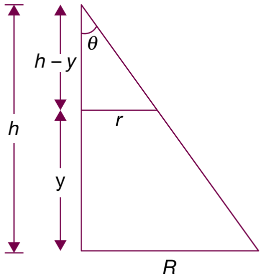
ILLUSTRATION 76

Find altitude of right circular cylinder of maximum volume that can be inscribed in the given right circular cone of height h.

SOLUTION

Let the cone of height h and radius R, a cylinder is inscribed of radius r and height y as shown in figure.





Cone is given h, r, θ

$$\tan\theta = \frac{r}{h - y} = \frac{R}{h} \Rightarrow r = \frac{R}{h}(h - y)$$

Volume of cylinder $V = \pi r^2 y = \pi \dfrac{R^2}{h^2}(h-y)^2 y$

$$\frac{dV}{dy} = \frac{\pi R^2}{h^2}(-2(h-y)y + (h-y)^2)$$

$$\frac{dV}{dy} = \frac{\pi R^2}{h^2}(h-y)(h-3y) = 0$$

We get $y = h$ (not acceptable); $y = \dfrac{h}{3}$

$$\frac{d^2V}{dy^2} = \frac{\pi R^2}{h^2}[(0-1)(h-3y) + (h-y)(0-3)]$$

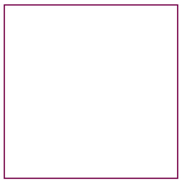
$$= -\frac{\pi R^2}{h^2}[4h - 6y] < 0 \text{ at } y = \frac{h}{3}$$

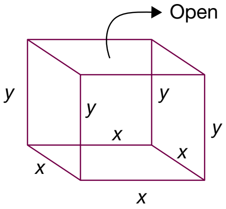
Hence, the volume of cylinder is maximum when height of cylinder is $\dfrac{h}{3}$

ILLUSTRATION 77

40 m^2 area of sheet is converted into an open tank of square base. If volume of tank maximum. Find its dimensions.

SOLUTION





In open tank, let side of square base is x and height is y.

Surface area of open tank $= x^2 + 4xy = 40 =$ area of given sheet

$$\Rightarrow \quad y = \frac{40 - x^2}{4x}$$

Volume of tank $V = x^2 y = x^2\left[\dfrac{40 - x^2}{4x}\right]$

$$V = \frac{40x - x^3}{4} \Rightarrow \frac{dV}{dx} = \frac{1}{4}[40 - 3x^2] = 0 \Rightarrow x = \sqrt{\frac{40}{3}}$$

Tank of volume is maximum as

$$\frac{d^2V}{dx^2} = \frac{1}{4}(0 - 6x) < 0 \text{ at } x = \sqrt{\frac{40}{3}} \text{ and}$$

$$y = \frac{40 - \frac{40}{3}}{4\sqrt{\frac{40}{3}}} = \sqrt{\frac{10}{3}}$$

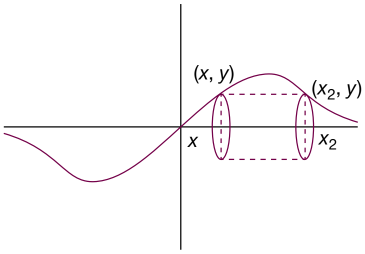
Tank of volume is maximum, for side of square base $\sqrt{\dfrac{40}{3}}$ and height $\sqrt{\dfrac{10}{3}}$

ILLUSTRATION 78

A cylinder is obtained by revolving a rectangle about the x-axis, the base of the rectangle lying on the x-axis and the entire rectangle lying in the region between the curve $y = \dfrac{x}{x^2+1}$ and the x-axis. Find the maximum possible volume of the cylinder.

SOLUTION

Curve $y = \dfrac{x}{x^2+1}$ is an odd function, range $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$



Let consider two points whose y coordinates are same x and x_2

$$\frac{x}{x^2+1} = \frac{x_2}{x_2^2+1} \Rightarrow x x_2 = 1$$

Cylinder is formed by rotating rectangle, with radius $\dfrac{x}{x^2+1}$ and length $x_2 - x = \dfrac{1}{x} - x$.

Volume of cylinder $V = \pi\left(\dfrac{x}{x^2+1}\right)^2\left(\dfrac{1}{x} - x\right) = \pi\dfrac{(x - x^3)}{(x^2+1)^2}$

$$\frac{dV}{dx} = \pi\frac{(1-3x^2)(x^2+1)^2 - 2(x^2+1)2x(x-x^3)}{(x^2+1)^4}$$

$$\frac{dV}{dx} = 0 \Rightarrow (1 - 3x^2)(x^2+1) = 4x^2 - 4x^4$$

$$\Rightarrow \quad 1 - 3x^4 - 2x^2 = 4x^2 - 4x^4$$

$$x^4 - 6x^2 + 1 = 0 \Rightarrow x^2 = 3 - 2\sqrt{2} \Rightarrow x = \sqrt{2} - 1$$

$$\frac{d^2V}{dx^2} < 0 \text{ at } x = \sqrt{2} - 1$$

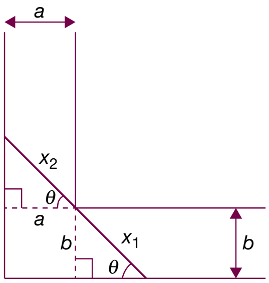
Volume of cylinder $V = \dfrac{\pi(x - x^3)}{(x^2+1)^2}$

$$= \frac{\pi(\sqrt{2}-1)(1-(3-2\sqrt{2}))}{(4-2\sqrt{2})^2} = \frac{\pi}{4}$$

ILLUSTRATION 79

Corridors of width a and b meet at right angle. Find the length of longest pipe that can be passed around the corner horizontally.

SOLUTION



Let length of pipe is ℓ, and x_1 and x_2 are segment of pipe in corridors of length b and a respectively.
Let pipe is inclined at an angle θ,

$$x_2\cos\theta = a \Rightarrow x_2 = a\sec\theta$$

$$x_1\sin\theta = b \Rightarrow x_1 = b\operatorname{cosec}\theta$$

$$\ell = x_1 + x_2 = a\sec\theta + b\operatorname{cosec}\theta$$

$$\frac{d\ell}{d\theta} = a\sec\theta\tan\theta - b\operatorname{cosec}\theta\cot\theta = 0 \text{ for extrema}$$

$$\Rightarrow \quad \frac{a\sin\theta}{\cos^2\theta} = \frac{b\cos\theta}{\sin^2\theta} \Rightarrow \left(\frac{\sin\theta}{\cos\theta}\right)^3 = \frac{b}{a} \Rightarrow \tan\theta = \left(\frac{b}{a}\right)^{1/3}$$

$$\ell = a\sec\theta + b\operatorname{cosec}\theta = a\,(1+\tan^2\theta)^{1/2} + b\,(1+\cot^2\theta)^{1/2}$$

$$\ell_{\max} = a\left(1+\left(\frac{b}{a}\right)^{2/3}\right)^{1/2} + b\left(1+\left(\frac{a}{b}\right)^{2/3}\right)^{1/2}$$

$$= a\left(\frac{b^{2/3}+a^{2/3}}{a^{2/3}}\right)^{1/2} + b\left(\frac{a^{2/3}+b^{2/3}}{b^{2/3}}\right)^{1/2}$$

$$\ell_{\max} = a\left(\frac{b^{2/3}+a^{2/3}}{a^{2/3}}\right)^{1/2} + b\left(\frac{a^{2/3}+b^{2/3}}{b^{2/3}}\right)^{1/2}$$

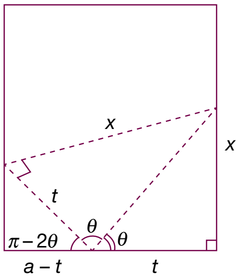
$$\ell_{\max} = (a^{2/3}+b^{2/3})^{3/2}$$

ILLUSTRATION 80

The lower corner of a leaf in a book is folded over so as to just reach the inner edge of the page. Find the fraction of width folder over if the area of the folded part is minimum.

SOLUTION

The lower corner of a leaf of width t in a book is folded over, to reach the inner edge of the page as shown in diagram



$$\tan\theta = \frac{x}{t} \qquad \ldots(1)$$

$$\cos(\pi - 2\theta) = \frac{a-t}{t}$$

$$\Rightarrow \quad \cos 2\theta = \frac{t-a}{t}$$

$$\Rightarrow \quad \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{t-a}{t},$$

Using $\tan\theta = \dfrac{x}{t}$

$$\Rightarrow \quad \frac{t^2-x^2}{t^2+x^2} = \frac{t-a}{t}$$

$$\Rightarrow \quad t^3 - tx^2 = t^3 + x^2t - at^2 - ax^2$$

$$\Rightarrow \quad x^2(2t-a) = at^2 \Rightarrow x^2 = \frac{at^2}{2t-a}$$

Area of folded part $A = \dfrac{1}{2}xt \Rightarrow A^2 = \dfrac{x^2t^2}{4}$

$$A^2 = \frac{a}{4}\left(\frac{t^4}{2t-a}\right)$$

$$\frac{dA^2}{dt} = \frac{a}{4}\left[\frac{(2t-a)4t^3 - t^4(2)}{(2t-a)^2}\right] = \frac{a}{4}\left[\frac{6t^4-4at^3}{(2t-a)^2}\right] = 0$$

$$\Rightarrow \quad t = \frac{2a}{3},\ \frac{t}{a} = \frac{2}{3}$$

$$\frac{d^2A^2}{dt^2} = \frac{a}{4}$$

$$\left[\frac{(2t-a)^2(24t^3-12at^2)-(6t^4-4at^3)4(2t-a)}{(2t-a)^4}\right]_{t=2a/3} > 0$$

It means $t = \dfrac{2a}{3}$ for minimum area.

Now required ratio $= \dfrac{t}{a} = \dfrac{2}{3}$

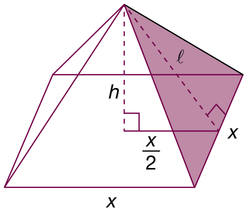
ILLUSTRATION 81

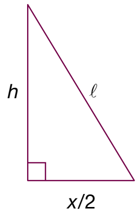
Find pyramid whose lateral surface area is minimum given that it is a square based pyramid with volume $36\sqrt{2}\ \text{cm}^3$.

SOLUTION

Let consider a square base pyramid with base square side x and height h.

Volume of pyramid $V = \dfrac{1}{3}x^2h = 36\sqrt{2}$





$$\Rightarrow \quad x^2h = 108\sqrt{2} \Rightarrow h = \frac{108\sqrt{2}}{x^2}$$

Slant height $\ell = \sqrt{h^2 + \dfrac{x^2}{4}}$

Lateral surface area of pyramid $S = 4\left(\dfrac{1}{2}x\ell\right) = 2x\ell$

$$S = 2x\sqrt{h^2+\frac{x^2}{4}} = 2x\sqrt{\frac{(108)^2 2}{x^4}+\frac{x^2}{4}}$$

$$S^2 = 4x^2\left(\frac{(108)^2 2}{x^4}+\frac{x^2}{4}\right) = \frac{8(108)^2}{x^2} + x^4$$

$$\frac{d(S^2)}{dx} = \frac{-16(108)^2}{x^3} + 4x^3 = 0$$

$\Rightarrow \quad 4x^6 = 16(108)^2 \Rightarrow x = 6$ cm

$x^2h = 108\sqrt{2} \Rightarrow 36\,h = 108\sqrt{2} \Rightarrow h = 3\sqrt{2}$

Pyramid Square Base of side 6 cm, height $3\sqrt{2}$

$$\frac{d^2(S^2)}{dx^2} = \frac{48(108)^2}{x^4} + 12x^2$$

At $x = 6, \dfrac{d^2(S^2)}{dx^2} > 0$ so surface area is minimum when $x = 6$ cm.

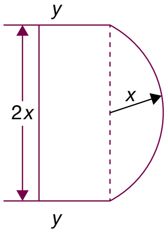
Hence, pyramid sequence base of side 6 cm and height $3\sqrt{2}$

ILLUSTRATION 82

A housing corporation wants to build some flats. The ground plane of a flat consists of a semicircle with a rectangle constructed on its diameter. Given that the perimeter of the flat (excluding the base of arc) is 50 meter, find its dimensions in order that the area covered is maximum. (use π as 22/7).

SOLUTION

Let radius of semi circle be x meter and dimensions of rectangle be y meter and $2x$ metres.



Given perimeter of plate,

$$2x + 2y + \pi x = 50$$

$$\Rightarrow \quad y = 25 - x\left(1 + \frac{\pi}{2}\right)$$

Area of plate $A = \dfrac{\pi x^2}{2} + 2xy$

$$\Rightarrow \quad A = \frac{\pi x^2}{2} + 2x\left[25 - x\left(1 + \frac{\pi}{2}\right)\right]$$

$$\Rightarrow \quad A = 50x - 2x^2 - \frac{\pi x^2}{2}$$

On Differentiation, we get $\dfrac{dA}{dx} = 50 - 4x - \pi x = 50 - (4 + \pi)x$

Again differentiating w.r.t. x, we get $\dfrac{d^2A}{dx^2} = 0 - (4 + \pi) < 0$

Put $\dfrac{dA}{dx} = 0 \Rightarrow 50 - (4 + \pi)\,x = 0 \Rightarrow x = \dfrac{50}{(4 + \pi)}$

$$\Rightarrow \quad x = \frac{50}{4 + \frac{22}{7}} = \frac{50}{\frac{50}{7}} \Rightarrow x = 7 \text{ meters}$$

$$y = 25 - \frac{x}{2}(\pi + 2) = 25 - \frac{7}{2}\left(\frac{22}{7} + 2\right) = 7 \text{ meters.}$$

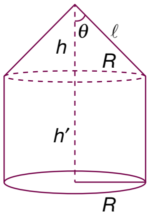
$x = 7$ m and $y = 7$ m for maximum area.

ILLUSTRATION 83

A bell tent consists of a conical portion above a cylindrical portion near the ground. For a given volume and a circular base of a given radius, find the semi-vertical angle of the cone, for which the amount of the canvas used is a minimum.

SOLUTION

Let cylindrical portion has radius R and height h' and cone has radius R and height h as shown in figure.



Given volume $V = \pi R^2h' + \dfrac{1}{3}\pi R^2h$

$$\Rightarrow \quad \frac{V}{\pi R^2} = h' + \frac{h}{3} \qquad \ldots(1)$$

$$\tan\theta = \frac{R}{h} \Rightarrow h = R\cot\theta,$$

From (1), $h' = \dfrac{V}{\pi R^2} - \dfrac{R\cot\theta}{3}$; $\ell\sin\theta = R$

Surface area of bell tent $S = 2\pi Rh' + \pi R\ell$

$$S = 2\pi R\left(\frac{V}{\pi R^2} - \frac{R\cot\theta}{3}\right) + \pi R\cdot\frac{R}{\sin\theta}$$

$$S = \pi R\left[\frac{2V}{\pi R^2} - \frac{2R}{3}\cot\theta + R\,\mathrm{cosec}\,\theta\right]$$

$$\frac{dS}{d\theta} = 0 + \frac{2R}{3}\mathrm{cosec}^2\theta - R\,\mathrm{cosec}\,\theta\cot\theta = 0$$

$$\frac{2}{3}\mathrm{cosec}\,\theta = \cot\theta \Rightarrow \cos\theta = \frac{2}{3} \text{ for minima}$$

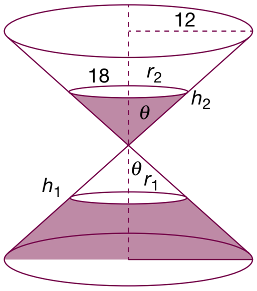
As $\dfrac{d^2S}{d\theta^2} = -\dfrac{2R}{3}2\,\mathrm{cosec}^2\theta\cot\theta + R\,\mathrm{cosec}\,\theta\cot^2\theta + R\,\mathrm{coesc}^3\theta > 0$

For minimum area for which semi vertical angle of cone is $\cos^{-1}\dfrac{2}{3}$

ILLUSTRATION 84

An idealized hourglass is made by joining two identical circular cones, each with base radius 12 cm and height 18 cm, at their vertices. A tiny hole allows water to drop from the upper cone into the lower one. At a certain instant, the depth of water in the upper cone is 6 cm which is decreasing at a rate of 0.2 cm/min, and water in the lower cone is 10 cm deep. Find the rate of increase of depth of water in lower cone.

SOLUTION



$$\tan\theta = \frac{12}{18} = \frac{r_2}{h_2} = \frac{r_1}{h_1}$$

$$\Rightarrow \quad r_2 = \frac{2}{3}h_2 \text{ and } r_1 = \frac{2}{3}h_1$$

Given $h_2 = 6$ cm, $\dfrac{dh_2}{dt} = 0.2$ cm,

$h - h_1 = 10 \Rightarrow h_1 = 8$ cm

volume of water = constant

$$\frac{1}{3}\pi r_2^2h_2 + \left(V - \frac{1}{3}\pi r_1^2h_1\right) = \text{constant}$$

$$\Rightarrow \quad r_2^2h_2 - r_1^2h_1 = k$$

$$\Rightarrow \quad \frac{4}{9}h_2^3 - \frac{4}{9}h_1^3 = k \Rightarrow h_2^3 - h_1^3 = 4k/9$$

$$\Rightarrow \quad 3h_2^2\frac{dh_2}{dt} - 3h_1^2\frac{dh_1}{dt} = 0$$

$$\Rightarrow \quad 36(0.2) = \frac{64dh_1}{dt}$$

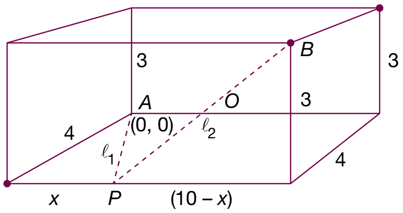
$$\Rightarrow \quad \frac{dh_1}{dt} = \frac{36(0.2)}{64} = \frac{9(0.2)}{16} = \frac{9}{80}\ \text{cm/sec}$$

ILLUSTRATION 85

Consider a rectangular cardboard box of height 3, breadth 4 and length 10 units. There is a lizard in one corner A of the box and an insect in the corner B which is farthest form A. find the length of the shortest path between the lizard and the insect along the surface of the box.

SOLUTION

Lizard first move up to point P from point A and then from point p to point B.



$$D = \ell_1 + \ell_2 = \sqrt{16+x^2} + \sqrt{(10-x)^2+9}$$

$$\frac{dD}{dx} = \frac{1}{2\sqrt{16+x^2}}2x + \frac{2(10-x)(-1)}{2\sqrt{(10-x)^2+9}} = 0$$

$$\frac{x}{\sqrt{16+x^2}} = \frac{10-x}{\sqrt{(10-x)^2+9}}$$

$$\Rightarrow \quad x^2(10-x)^2 + 9x^2 = 16(10-x)^2 + x^2(10-x)^2$$

On solving

$$7x^2 - 320x + 1600 = 0$$

$x = \frac{40}{7}$ accepted but $x = 40$ rejected

$$\text{Shortest path} = \sqrt{16+\frac{1600}{49}} + \sqrt{\frac{900}{49}+9}$$

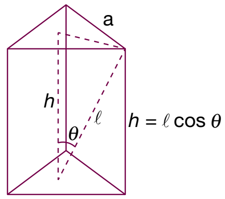
$$= \frac{4\sqrt{149}}{7} + \frac{3}{7}\sqrt{149} = \sqrt{149}\ \text{units.}$$

ILLUSTRATION 86

In a regular triangular prism the distance from the centre of one base to one of the vertices of the other base is ℓ. Find the altitude of the prism for which the volume is greatest.

SOLUTION

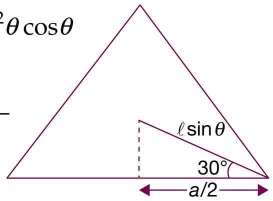
Base of prism 'a' with height l.



Height $h = \ell\cos\theta;\ \frac{a}{2}$

$= \ell\sin\theta \times \cos 30° \Rightarrow a = \sqrt{3}\ \ell\sin\theta$

Volume of prism V = area of base ($h$eight)



$$V = \left(\frac{\sqrt{3}}{4}a^2\right)h = \frac{3\sqrt{3}}{4}\ \ell^2\sin^2\theta\cos\theta$$

$$\frac{dV}{d\theta} = \frac{3\sqrt{3}}{4}\ \ell^2\,[2\sin\theta\cos^2\theta - \sin^3\theta)] = 0$$

$$2\cos^2\theta - \sin^2\theta = 0$$

$$\Rightarrow \quad 3\cos^2\theta - 1 = 0 \Rightarrow \cos\theta = \frac{1}{\sqrt{3}}$$

$$\frac{d^2V}{d\theta^2} = \frac{3\sqrt{3}}{4}\ \ell^2\,[2\cos^3\theta - 4\sin^2\theta\cos\theta - 3\sin^2\theta\cos\theta] < 0$$

when $\cos\theta = \frac{1}{\sqrt{3}}$

Required altitude $h = \ell\cos\theta = \frac{\ell}{\sqrt{3}}$

Test Your Concepts-IV

(Solutions on page H.186)

1. Find dimensions of rectangle of greatest area inscribed in a semicircle of radius r.
2. A wire of 20 cm in length is being broken into two parts. One part is to bent to form a square and other one to a circle what should be the length in which wire is to be broken in order to have sum of their areas as minimum as possible.
3. Manufacturer can sell x item at a price of $\left(5-\frac{x}{100}\right)$₹. each. The cost of x item is $\left(\frac{x}{5}+500\right)$₹. Find number of items, he should sell to earn maximum profit.
4. Find coordinate of point P on the curve $\frac{x^2}{8}+\frac{y^2}{18}=1$ in the first quadrant so that area of triangle formed by tangent at P and coordinate axis is minimum.
5. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the curve $y = 12 - x^2$.
6. The sum of length of the hypotenuse and another side of a right angled triangle is given. The area of triangle will be maximum if the angle between them ______
7. A window is in the shape of a rectangle surmounted by a semi circle .If the perimeter of the window is of fixed length 'l' then find the maximum area of the window.

Answer key

1. $\sqrt{2}\,r, \frac{r}{\sqrt{2}}$ **2.** square of perimeter $\frac{80}{\pi+4}$ and circle with circumference $\frac{20\pi}{\pi+4}$

3. 240 **4.** $P(2, 3)$ **5.** 72 sq units **6.** $\frac{\pi}{3}$ **7.** $\frac{l^2}{2\pi+8}$

SOLVED PROBLEMS

PROBLEM 1

Suppose $f(x)$ is a function satisfying the following conditions.

(i) $f(0) = 2, f(1) = 1$

(ii) f has a minimum value at $x = \frac{5}{2}$ and

(iii) For all x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

where a, b are some constants. Determine the constants a, b and the function $f(x)$.

SOLUTION

$$f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1 - 2R_2$

$$\text{We get, } f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 2ax & 2ax-1 \\ b & b+1 \end{vmatrix}$$

$$\Rightarrow \quad f'(x) = 2ax + b$$

$$f'\left(\frac{5}{2}\right) = 0 \Rightarrow 5a + b = 0 \qquad \ldots(1)$$

Integrating, we get, $f(x) = ax^2 + bx + c$.

where c is an arbitrary constant. Since f has a maximum at $x = \frac{5}{2}$,

Also $f(0) = 2 \Rightarrow c = 2$ and $f(1) = 1 \Rightarrow a + b + c = 1$

$\therefore \quad a + b = -1$

Solving (1) and (2) for a, b we get,

$$a = \frac{1}{4}, b = -\frac{5}{4} \qquad \ldots(2)$$

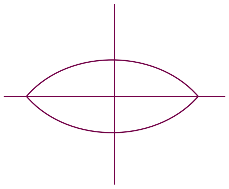
Thus, $f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2$

PROBLEM 2

If $f(x)$ is a decreasing function and attain positive values, then the set of values of 'k', for which the major axis of the ellipse $\frac{x^2}{f(k^2+2k+5)} + \frac{y^2}{f(k+11)} = 1$ is the x-axis, is $(-a, b)$ then find the value of $a + b$

SOLUTION

$$\frac{x^2}{f(k^2+2k+5)} + \frac{y^2}{f(k+11)} = 1$$



Major axis is x axis, it means $a > b$.

$$f(k^2 + 2k + 5) > f(k + 11)$$

As $f(x)$ is positive decreasing function, we have

$$k^2 + 2k + 5 < k + 11$$

$\Rightarrow \quad k^2 + k - 6 < 0 \quad \Rightarrow \quad (k + 3)(k - 2) < 0$

$\Rightarrow \quad k \in (-3, 2) = (-a, b)$

Hence, we have $a + b = 5$

PROBLEM 3

If $x, y \in R$, $x^2 + y^2 + xy = 1$, then the minimum value of $x^3y + xy^3 + 4$ is _______.

SOLUTION

Put $x = r\cos\theta, y = r\sin\theta$

Then, $r^2 \sin\theta\cos\theta = 1 - r^2$

$$r^2 = \frac{2}{2+\sin 2\theta} \Rightarrow \frac{2}{3} \le r^2 \le 2 \qquad (\because \sin 2\theta \in [-1, 1])$$

Let $k = x^3y + xy^3$

$$= r^4 \sin\theta\cos\theta(\sin^2\theta + \cos^2\theta)$$

$$= \frac{r^4}{2}\sin 2\theta = r^4\left(\frac{1-r^2}{r^2}\right)$$

$$k = \left(1 - r^2\right)r^2 \text{ let } r^2 = t, \frac{2}{3} \le t \le 2$$

$$k = (1 - t)t$$

$$\frac{dk}{dt} = 1 - 2t < 0 \text{ for } \frac{2}{3} \le t \le 2$$

k is is a decreasing function, k is minimum when t is max i.e., when $t = 2$, minimum k is -2.

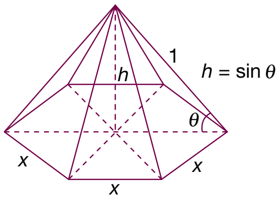
$\therefore \quad x^3y + xy^3 + 4 \ge 2$

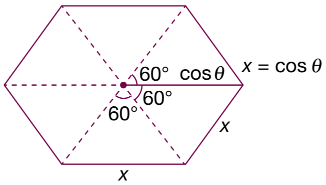
PROBLEM 4

The lateral edge of a regular hexagonal pyramid is 1 cm. If the volume is maximum, then find its height.

SOLUTION

Height of prism $h = \sin\theta$, side of base of prism $x = \cos\theta$





Volume of hexagonal pyramid = $\frac{1}{3}$ (area of base) height

$$V=\frac{1}{3}\left(6\left(\frac{\sqrt{3}}{4}\cdot x^2\right)\right)\sin\theta=\frac{\sqrt{3}}{2}\cos^2\theta\sin\theta$$

$$\frac{dV}{d\theta}=\frac{\sqrt{3}}{2}(-2\cos\theta\sin^2\theta+\cos^3\theta)=0$$

$\Rightarrow$ $\cos\theta(\cos^2\theta-2\sin^2\theta)=0;$

$\Rightarrow$ $\cos^2\theta-2+2\cos^2\theta=0\Rightarrow\cos\theta=\sqrt{\frac{2}{3}}$

$$\frac{d^2V}{d\theta^2}=\frac{\sqrt{3}}{2}(2\sin^3\theta-2\cos\theta 2\sin\theta\cos\theta-3\cos^2\theta\sin\theta)<0$$

For $\cos\theta=\sqrt{\frac{2}{3}}$

Volume is maximum for $\cos\theta=\sqrt{\frac{2}{3}}$

Required height $=\sin\theta=\sqrt{1-\frac{2}{3}}=\frac{1}{\sqrt{3}}$

PROBLEM 5

The lateral edge of a regular rectangular pyramid is 'a' cm long. The lateral edge makes an angle α with the plane of the base. Find the value of α for which the volume of the pyramid is greatest.

SOLUTION

Let the side of base x, and slant height of 'a' which makes an angle α with base height $h = a\sin\alpha$

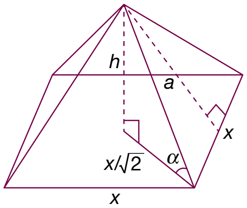
$$\frac{x}{\sqrt{2}}=a\cos\alpha\Rightarrow x=a\sqrt{2}\cos\alpha$$

Volume of pyramid $V=\frac{1}{3}x^2h=\frac{1}{3}(2a^2\cos^2\alpha)a\sin\alpha$

$$V=\frac{2}{3}a^3\cos^2\alpha\sin\alpha$$

$$\frac{dV}{d\alpha}=\frac{2}{3}a^3[-2\cos\alpha\sin^2\alpha+\cos^3\alpha)=0$$

$$\cos\alpha[-2\sin^2\alpha+\cos^2\alpha]=0$$



$\Rightarrow$ $3\sin^2\alpha=1\Rightarrow\sin\alpha=\frac{1}{3}$

$$\frac{d^2V}{d\alpha^2}=\frac{2}{3}a^3[2\sin^3\alpha-4\cos^2\alpha\sin\alpha-3\cos^2\alpha\sin\alpha)<0$$

when $\sin\alpha=\frac{1}{3}$

Hence, for greatest volume, $\sin\alpha=\frac{1}{3}\Rightarrow\alpha=\sin^{-1}\frac{1}{3}$

PROBLEM 6

Let $f(x)=\begin{cases}xe^{ax} & ;\ x\le 0\\ x+ax^2-x^3 & ;\ x>0\end{cases}$; where '$a$' is positive constant. Find the interval in which $f'(x)$ is increasing.

SOLUTION

$$f(x)=\begin{cases}xe^{ax} & ;\ x\le 0\\ x+ax^2-x^3 & ;\ x>0\end{cases};$$

Differentiating above function w.r.t. x, we get

$$f'(x)=\begin{cases}e^{ax}(xa+1) & ;\ x\le 0\\ (1+2ax-3x^2) & ;\ x>0\end{cases}$$

Again differentiating w.r.t. x, we get

$$f''(x)=\begin{cases}ae^{ax}(ax+2) & ;\ x\le 0\\ 2a-6x & ;\ x>0\end{cases}$$

Since, $f'(x)$ is increasing function i.e. $f''(x)\ge 0$

CASE 1: If $x\le 0$

$$f''(x)=ae^{ax}(ax+2)\ge 0;$$

e^{ax} and a both are positive, we get $x\ge-\frac{2}{a}$, we get

$x\in\left[-\frac{2}{a},0\right]$

CASE 2: If $x>0$

$$f''(x)=2a-6x\ge 0\Rightarrow x\le\frac{a}{3}, \text{ we get } x\in\left(0,\frac{a}{3}\right]$$

Hence, required interval is $x\in\left[-\frac{2}{a},\frac{a}{3}\right]$

PROBLEM 7

(i) Find the set of all real values of μ so that the function $f(x)=(\mu+1)x^3+2x^2+3\mu x-7$ is
(a) always increasing and (b) always decreasing

(ii) Find b is $f(x)=\sin x-bx+c$ is always an increasing or a decreasing function.

SOLUTION

(i) $f(x)=(\mu+1)x^3+2x^2+3\mu x-7$

Differentiating w.r.t. x, we get

$f'(x)=3(\mu+1)x^2+4x+3\mu$

(a) For increasing function i.e., $f'(x)\ge 0$

$\Rightarrow 3(\mu+1)x^2+4x+3\mu\ge 0$

Quadratic expression $ax^2 + bx + c \geq 0$, for all real x, then $a > 0$ and $D \leq 0$

Here $a = 3(\mu + 1) > 0 \Rightarrow \mu > -1$

$D \leq 0 \Rightarrow B^2 - 4AC \leq 0$

$\Rightarrow 16 - 4 \times 9\mu (1 + \mu) \leq 0 \Rightarrow 9\mu^2 + 9\mu - 4 \geq 0$

$\Rightarrow (3\mu - 1)(3\mu + 4) \geq 0$

$\mu \in \left(-\infty, -\frac{4}{3}\right] \cup \left[\frac{1}{3}, \infty\right)$ but $\mu > -1$

we get final set, $\mu \in \left[\frac{1}{3}, \infty\right)$

(b) For decreasing function, $f'(x) \leq 0$

$\Rightarrow 3(\mu + 1)x^2 + 4x + 3\mu \leq 0$

quadratic expression $ax^2 + bx + c \leq 0$, for all real x, then $a < 0$ and $D \leq 0$

Here $a = 3(\mu + 1) < 0 \Rightarrow \mu < -1$

$D \leq 0 \Rightarrow B^2 - 4AC \leq 0$

$\Rightarrow 16 - 4 \times 9\mu (1 + \mu) \leq 0 \Rightarrow (3\mu - 1)(3\mu + 4) \geq 0$

$\mu \in \left(-\infty, -\frac{4}{3}\right] \cup \left[\frac{1}{3}, \infty\right)$ but $\mu < -1$

Hence, set of values, $\mu \in \left(-\infty, -\frac{4}{3}\right]$

(ii) $f(x) = \sin x - bx + c$

Differentiating w.r.t. x, we get $f'(x) = \cos x - b$

For increasing function, $f'(x) \geq 0$

$\Rightarrow \cos x - b \geq 0$ (as $-1 \leq \cos x \leq 1$)

$\Rightarrow -1 \geq b \Rightarrow b \leq -1$

For decreasing function, $f'(x) \leq 0$

$\Rightarrow \cos x - b \leq 0$ (as $-1 \leq \cos x \leq 1$)

$\Rightarrow 1 \leq b \Rightarrow b \geq 1$

PROBLEM 8

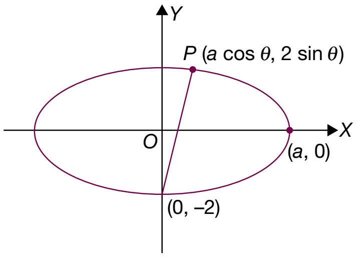
Find the point on the curve $4x^2 + a^2y^2 = 4a^2$, $4 < a^2 < 8$ that is farthest from the point $(0, -2)$.

SOLUTION

The equation of given curve can be expressed as

$$\frac{x^2}{a^2} + \frac{y^2}{4} = 1 \text{ where } 4 < a^2 < 8$$

Clearly it is the equation of an ellipse.



Let us consider a point $P\ (a \cos \theta, 2 \sin\theta)$ on the ellipse. Let the distance of $(a \cos \theta, 2 \sin \theta)$ from $(0, -2)$ is L.

Then, $L^2 = a^2\cos^2\theta + (2 \sin\theta + 2)^2$

For max or min value of L we should have $\frac{d(L^2)}{d\theta} = 0$

$\Rightarrow \cos \theta\,[-2a^2 \sin\theta + 8 \sin\theta + 8] = 0$

$\Rightarrow$ either $\cos\theta = 0$ or $(8 - 2a^2) \sin\theta + 8 = 0$

$\Rightarrow \theta = \frac{\pi}{2}$ or $\sin\theta = \frac{4}{a^2 - 4}$

Since $a^2 < 8 \Rightarrow a^2 - 4 < 4 \Rightarrow \frac{4}{a^2 - 4} > 1 \Rightarrow \sin\theta > 1$

which is not possible.

Also $\frac{d^2(L^2)}{d\theta^2} = \cos\theta\,[-2a^2 \cos\theta + 8 \cos\theta] + (-\sin\theta)\,[-2a^2 \sin\theta + 8 \sin\theta + 8]$

At $\theta = \frac{\pi}{2}$, $\frac{d^2(L^2)}{d\theta^2} = 0 - [16 - 2a^2] = 2(a^2 - 8) < 0$

$\therefore$ L is max. at $\theta = \frac{\pi}{2}$ and the farthest point is $(0, 2)$

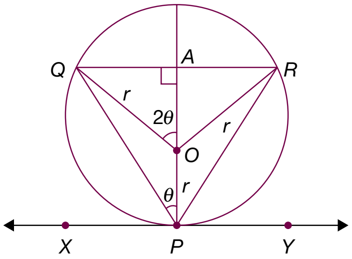
PROBLEM 9

A point P is given on the circumference of a circle of radius r. Chords QR are parallel to the tangent at P. Determine the maximum possible area of the triangle PQR.

SOLUTION

As $QR \parallel XY$, diameter through P is $\perp QR$

Now area of ΔPQR is given by $A = \frac{1}{2}\, QR \cdot AP$



But $QR = 2 \cdot QA = 2r \sin 2\theta$ and $PA = OA + OP = r \cos 2\theta + r$

$\therefore$ $A = \frac{1}{2} \cdot 2r \sin 2\theta \cdot (r + r \cos 2\theta)$

$= r^2 \cdot 2 \sin\theta \cos\theta \cdot 2 \cos^2\theta = 4r^2 \sin\theta \cos^3\theta$

For max,. value of area $\frac{dA}{d\theta} = 0$

$\Rightarrow 4r^2[\cos^4\theta - 3 \sin^2\theta \cos^2\theta] = 0$

$\Rightarrow \cos^2\theta\,(\cos^2\theta - 3 \sin^2\theta) = 0$

$\Rightarrow \tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = 30°$

Also $\dfrac{d^2A}{d\theta^2} = 4r^2[-4\cos^3\theta\sin\theta - 6\sin\theta\cos^3\theta + 6\sin^3\theta\cos\theta]$

$$= 4r^2[-10\sin\theta\cos^3\theta + 6\sin^3\theta\cos\theta]$$

$$\frac{d^2A}{d\theta^2}\Big|_{\theta=30^\circ} = 4r^2\left[-10.\frac{1}{2}.\frac{3\sqrt{3}}{8} + 6.\frac{1}{8}.\frac{\sqrt{3}}{2}\right]$$

$$= 4r^2\left(\frac{-12\sqrt{3}}{8}\right) < 0$$

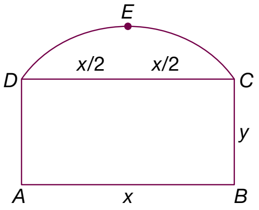
$\therefore$ A is max. at $\theta = 30^\circ$ so $A_{\max} = 4r^2\sin 30^\circ\cos^3 30^\circ$

$$= 4r^2 \times \frac{1}{2} \times \frac{3\sqrt{3}}{8} = \frac{3\sqrt{3}}{4}r^2$$

PROBLEM 10

A window of perimeter P (including the base of the arch) is in the form of a rectangle surmounted by a semi-circle. The semicircular portion is fitted with colored glass while the rectangular part is fitted with clear glass transmits three times as much light per square meter as the colored glass does. What is the ratio for the sides of the rectangle so that the window transmits the maximum light.

SOLUTION



Let $ABCEDA$ be the window as shown in the figure and let

$$AB = x\text{ m};\ BC = y\text{ m}$$

Then its perimeter including the base DC of arch $= \left(2x + 2y + \dfrac{\pi x}{2}\right)$ m

$$\therefore \quad P = \left(2 + \frac{\pi}{2}\right)x + 2y \qquad \ldots(1)$$

Now area of rectangle $ABCD = xy$ and

area of arch $DCED = \dfrac{\pi}{2}\left(\dfrac{x}{2}\right)^2$

Let λ be the light transmitted by colored glass per sq. m, then 3λ will be the light transmitted by clear glass per sq. m

Hence, the area of light transmitted $A = 3\lambda(xy) + \lambda\left[\dfrac{\pi}{2}\left(\dfrac{x}{2}\right)^2\right]$

$$\Rightarrow \quad A = \lambda\left[3xy + \frac{\pi x^2}{8}\right] \qquad \ldots(2)$$

Substituting the value of y from (1) in (2), we get

$$A = \lambda\left[3x\frac{1}{2}\left[P - \left(\frac{4+\pi}{2}\right)x\right] + \frac{\pi x^2}{8}\right]$$

$$= \lambda\left[\frac{3Px}{2} - \frac{3(4+\pi)}{2}x + \frac{\pi x}{4}\right]$$

$$\frac{dA}{dx} = \lambda\left[\frac{3P}{2} - \frac{3(4+\pi)}{4}x + \frac{\pi x}{4}\right]$$

For A to be maximum $\dfrac{dA}{dx} = 0$

$$\Rightarrow \quad x = \frac{\left(\dfrac{3P}{2}\right)}{\left[\dfrac{-\pi}{4} + \left(\dfrac{12+3\pi}{2}\right)\right]} = \frac{3P}{2} \times \frac{4}{5\pi+24}$$

$$\Rightarrow \quad x = \frac{6P}{5\pi+24}$$

Also $\dfrac{d^2A}{dx^2} = \lambda\left[\dfrac{-3(4+\pi)}{2} + \dfrac{\pi}{4}\right] < 0$

$\therefore$ A is max when $x = \dfrac{6P}{5\pi+24}$

$$\Rightarrow \quad 5\pi x + 24x = 6\left[\left(\frac{4+\pi}{2}\right)x + 2y\right]$$

[Using value of P from (1)]

$$\Rightarrow \quad (5\pi + 24 - 12 - 3\pi)x = 12y \Rightarrow (2\pi + 12)x = 12y$$

$$\Rightarrow \quad \frac{y}{x} = \frac{\pi+6}{6}$$

$\therefore$ The required ratio of breadth to length of the rectangle $= \left(\dfrac{6+\pi}{6}\right)$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. If $f'(x)=|x|-\{x\}$, where $\{x\}$ denotes the fractional part of x, then $f(x)$ is decreasing in

(A) $\left(-\frac{1}{2},0\right)$ (B) $\left(\frac{1}{2},2\right)$

(C) $\left(-\frac{1}{2},2\right)$ (D) $\left(\frac{1}{2},\infty\right)$

2. **S-1:** $f: R \to R$, $f(x) = 12x^5 - 15x^4 + 20x^3 - 30x^2 + 60x + 1$ is monotonic and subjective on R.

S-2: Every continuous function defined on R, if strictly monotonic has its range R.

(A) S-1 and S-2 both are true

(B) S-1 and S-2 both are false

(C) S-1 is true and S-2 is false

(D) S-1 is false and S-2 is true

3. The number of critical points on the graph of the function $f(x) = \begin{cases} x^{3/5} & ; \text{ if } x \le 2 \\ -(x-3)^3 & ; \text{ if } x > 2 \end{cases}$ is

(A) 1 (B) 2

(C) 3 (D) 4

4. If $g(x) = \begin{cases} 2-3x^2 & ; -\infty < x \le -1 \\ mx & ; -1 < x < 0 \\ x^2 & ; 0 \le x < \infty \end{cases}$ then the range of m such that $g(x)$ is strictly monotonically increasing in $x \in (-\infty, \infty)$ is

(A) $[0, 1)$ (B) $(0, 1)$

(C) $[0, 1]$ (D) $(0, 1]$

5. If the function $f(x) = \sin(\ln x) - 2\cos(\ln x)$ is increasing in the interval (e^λ, e^μ) then $\sqrt{5}\cos(\mu - \lambda)$ is equal to

(A) $-\sqrt{5}$ (B) $\frac{-\sqrt{5}}{2}$

(C) 0 (D) $\sqrt{5}$

6. Let the function $g: R \to \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(t) = \frac{\pi}{2} - 2\cot^{-1}(3^{-t})$. Then g is

(A) even and is strictly increasing in $(-\infty, \infty)$

(B) odd and is strictly decreasing in $(-\infty, \infty)$

(C) even and is strictly decreasing in $(-\infty, \infty)$

(D) odd and is strictly increasing in $(-\infty, \infty)$

7. The interval in which function $f(x) = \sin^{-1}(x^2 - 3x + 3)$ is increasing is

(A) $[1, 2]$ (B) $\left[\frac{3}{2},\infty\right)$

(C) $\left[\frac{3}{2},2\right]$ (D) None of these

8. If $f(x) = 2e^x - c\ln x$ monotonically increases for every $x \in (0, \infty)$, then the true set of values of c is

(A) $\left(-\infty,-\frac{1}{e}\right)$ (B) $\left(-\infty,-\frac{2}{e}\right)$

(C) $[0, \infty)$ (D) $(-\infty, 0]$

9. If $f''(x) > 0 \ \forall \ x \in R$, $f'(4) = 0$ and $g(x) = f(\cot^2 x - 2\cot x + 5)$, $0 < x < \frac{\pi}{2}$, then

(A) $g(x)$ is increasing in $\left(0,\frac{\pi}{2}\right)$

(B) $g(x)$ is decreasing in $\left(0,\frac{\pi}{4}\right)$

(C) $g(x)$ is increasing in $\left(0,\frac{\pi}{4}\right)$

(D) $g(x)$ is decreasing in $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$

10. If $f(x) = x^5 - 2x^4 + 3x^3 + 100$, then

(A) $f\left(\frac{1}{199}\right) > f\left(\frac{1}{200}\right)$

(B) $f(200) < f(199)$

(C) $f(x+1) < f(x-1)$

(D) $f(2x) < f(2x-3)$

11. If $\alpha, \beta \in \left(\frac{\pi}{2},\pi\right)$ and $\alpha < \beta$, then which one of the following is true

(A) $e^{\cos\alpha-\cos\beta} < \frac{\alpha}{\beta}$ (B) $e^{\cos\beta-\cos\alpha} < \frac{\beta}{\alpha}$

(C) $e^{\cos\alpha-\cos\beta} < \frac{\beta}{\alpha}$ (D) $e^{\cos\beta-\cos\alpha} < \frac{\alpha}{\beta}$

12. Let $f(x) = (a^2 - 3a + 2)\left(\cos^2\frac{x}{4} - \sin^2\frac{x}{4}\right) + (a-1)x + \sin 1$. The largest integral value of 'a' for which $f'(x) \ne 0 \ \forall \ x \in R$

(A) 2 (B) 1

(C) 3 (D) 4

13. Let $f(x)$, $f'(x)$ and $f''(x)$ are all positive $\forall \ x \in [0, 7]$. If $f^{-1}(x)$ exists, then $3f^{-1}(4) - f^{-1}(2) - 2f^{-1}(5)$ is

(A) always positive (B) always negative

(C) non-negative (D) non-positive

14. A straight line cuts a rectangular figure of lengths 8 and breadth 6 units respectively in two parts such that their areas are equal. If s is sum of perimeters of both parts then range of s is

(A) $[34, 38]$ (B) $[28, 38]$

(C) $[40, 44]$ (D) $[40, 48]$

15. The maximum value of $\dfrac{(x^3+3x)-|x^3-x^2+2x-2|}{x^2+x+2}$ is equal to

(A) 2 (B) –1
(C) 0 (D) 1

16. The equation $\sin x + x\cos x = 0$ has at least one root in

(A) $\left(-\dfrac{\pi}{2}, 0\right)$ (B) $(0, \pi)$
(C) $\left(\pi, \dfrac{3\pi}{2}\right)$ (D) $\left(0, \dfrac{\pi}{2}\right)$

17. Which one of the following statements does not hold good for the function $f(x) = \cos^{-1}(2x^2 - 1)$?

(A) f is not differentiable at $x = 0$
(B) f is monotonic
(C) f is even
(D) f has an extremum

18. $f(x) = \displaystyle\int\left(2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}}\right)dx$ then f is

(A) increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$
(B) increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$
(C) increasing in $(-\infty, \infty)$
(D) decreasing in $(-\infty, \infty)$

19. The true set of real values of x for which the function $f(x) = x\ln x - x + 1$ is positive is

(A) $(1, \infty)$ (B) $\left(\dfrac{1}{e}, \infty\right)$
(C) $[e, \infty)$ (D) $(0, 1) \cup (1, \infty)$

20. The maximum value of $\sec^{-1}\left(\dfrac{7-5(x^2+3)}{2(x^2+2)}\right)$ is

(A) $\dfrac{5\pi}{6}$ (B) $\dfrac{5\pi}{12}$
(C) $\dfrac{7\pi}{12}$ (D) $\dfrac{2\pi}{3}$

21. The largest area of a trapezium inscribed in a semi-circle of radius R, if the lower base is on the diameter, is

(A) $\dfrac{3\sqrt{3}}{4}R^2$ (B) $\dfrac{\sqrt{3}}{2}R^2$
(C) $\dfrac{3\sqrt{3}}{8}R^2$ (D) R^2

22. If the function $f(x) = \cos|-x| - 2ax + b$ is strictly increasing $\forall\ x \in R$, then

(A) $a \le \dfrac{-1}{2}$ (B) $a \ge \dfrac{-3}{2}$
(C) $\dfrac{-1}{2} < a < \dfrac{3}{2}$ (D) $a \in \phi$

23. Let $f(x) = x^3 - 3x^2 + 2x$. If the equation $f(x) = k$ has exactly one positive and one negative solution then the value of k equals

(A) $-\dfrac{2\sqrt{3}}{9}$ (B) $-\dfrac{2}{9}$
(C) $\dfrac{2}{3\sqrt{3}}$ (D) $\dfrac{1}{3\sqrt{3}}$

24. The set of all x for which the function $h(x) = -\log_2(-2x-3+x^2)$ is defined and monotonically increasing is

(A) $(1, 3)$ (B) $(-\infty, -1)$
(C) $(-1, 1)$ (D) $(3, \infty)$

25. If $f''(x) > 0,\ \forall\ x \in R, f'(3) = 0$ and $g(x) = f(\tan^2 x - 2\tan x + 4)$ $\forall\ 0 < x < \dfrac{\pi}{2}$ then $g(x)$ is increasing in

(A) $\left(0, \dfrac{\pi}{4}\right)$ (B) $\left(\dfrac{\pi}{6}, \dfrac{\pi}{4}\right)$
(C) $\left(0, \dfrac{\pi}{3}\right)$ (D) $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

26. **S-1:** The function $f(x) = x\ln x$ is increasing in $(1/e, \infty)$
S-2: If $f(x)$ and $g(x)$ both are increasing in (a, b) then $f(x)\cdot g(x)$ must be increasing in (a, b).

(A) S-1 and S-2 both are true
(B) S-1 and S-2 both are false
(C) S-1 is true and S-2 is false
(D) S-1 is false and S-2 is true

27. Let $f(x) = \dfrac{e^x}{x^2}, x \in R - \{0\}$. If $f(x) = k$ has two distinct real roots, then the range of k, is

(A) $\left(0, \dfrac{e^2}{4}\right)$ (B) $\left(\dfrac{e^2}{4}, \infty\right)$
(C) $\left\{\dfrac{e^2}{4}\right\}$ (D) None of these

28. The value of a for which $f(x) = 2e^{2x} - 4ae^x - (4a - 5)x$ is always increasing

(A) $a \in [-5, 1]$ (B) $a \le 1$
(C) $a \in (-\infty, -5] \cup [1, \infty)$ (D) $a < 2$

29. $f(x) = \displaystyle\int_{\cos x}^{\sin x}(1 - t + 2t^3)\,dt\,;\ x \in [0, 2\pi]$ has

(A) a maximum at $\dfrac{\pi}{4}$ and a minimum at $\dfrac{3\pi}{4}$
(B) a maximum at $\dfrac{3\pi}{4}$ and a minimum at $\dfrac{7\pi}{4}$
(C) a maximum at $\dfrac{5\pi}{4}$ and a minimum at $\dfrac{7\pi}{4}$
(D) Neither a maxima nor minima

30. Let $f(x) = \begin{cases} x(x+3) & ;\ -1 \le x < 0 \\ -\sin x & ;\ 0 \le x < \dfrac{\pi}{2} \\ -(1+\cos x); & \dfrac{\pi}{2} \le x \le \pi \end{cases}$. Then

(A) the global maximum value of $f(x)$ is 2
(B) $f(x)$ has maxima but no minima
(C) $f(x)$ has minima but not maxima
(D) $f(x)$ is not differentiable at 2 points

31. For the cubic, $f(x) = x^3 - 3x^2 + 6x + 2006$, the statement which does not hold good, is

(A) $f(x)$ is monotonic increasing $\forall\ x \in R$
(B) $f: R \to R$ is injective as well as subjective

(C) slope of the tangent at the point of inflection is 3

(D) $f(x)$ is monotonic with exactly 3 real zeroes

32. If $y = f(x)$ is twice differentiable function such that $f(a) = f(b) = 0$, and $f(x) > 0\ \forall\ x \in (a, b)$, then

(A) $f''(c) < 0$ for some $c \in (a, b)$

(B) $f''(c) > 0\ \ \forall\ c \in (a, b)$

(C) $f(c) = 0$ for some $c \in (a, b)$

(D) None of these

33. Let f, g and h are differentiable function such that $g(x) = f(x) - x$ and $h(x) = f(x) - x^3$ are both strictly increasing functions, then the function $F(x) = f(x) - \dfrac{\sqrt{3}x^2}{2}$ is

(A) strictly increasing $\forall\ x \in R$

(B) strictly decreasing $\forall\ x \in R$

(C) strictly decreasing on $\left(-\infty, \dfrac{1}{\sqrt{3}}\right)$ and strictly increasing on $\left(\dfrac{1}{\sqrt{3}}, \infty\right)$

(D) strictly increasing on $\left(-\infty, \dfrac{1}{\sqrt{3}}\right)$ and strictly decreasing on $\left(\dfrac{1}{\sqrt{3}}, \infty\right)$

34. Let $f(x) = \ln(2 + x) - \dfrac{2x+2}{x+3}$

S-1: The equation $f(x) = 0$ has a unique solution in the domain of $f(x)$.

S-2: Every continuous function $f(x)$ defined in $[a, b]$ and strictly monotonic in (a, b) possess a unique solution of equation $f(x) = 0$ in (a, b)

(A) S-1 and S-2 both are true

(B) S-1 and S-2 both are false

(C) S-1 is true and S-2 is false

(D) S-1 is false and S-2 is true

35. The function $f(x) = \displaystyle\int_0^{x^2} \frac{t^2 - 1}{e^t + 1}\, dt$ has

(A) $x = 0$ as its point of inflection

(B) one local maxima and one local minima

(C) one local maximum and two local minimum

(D) two local maximum and one local minimum

36. Which of the following statement is true for the function

$$f(x) = \begin{cases} \sqrt{x} & ; \quad x \geq 1 \\ x^3 & ; \quad 0 \leq x \leq 1 \\ \dfrac{x^3}{3} - 4x & ; \quad x < 0 \end{cases}$$

(A) It is monotonic increasing $\forall\ x \in R$

(B) $f'(x)$ fails to exist for 3 distinct real values of x

(C) $f'(x)$ changes its sign twice as x varies from $(-\infty, \infty)$

(D) function attains its extreme values at x_1 and x_2, such that $x_1, x_2 > 0$

37. For all $a, b \in R$ the function $f(x) = 3x^4 - 4x^3 + 6x^2 + ax + b$ has

(A) atleast one extremum (B) exactly one extremum

(C) exactly two extremum (D) atmost one extremum

38. Let $f(x) = \begin{cases} \dfrac{(x-1)(6x-1)}{2x-1} & \text{if} \quad x \neq \dfrac{1}{2} \\ 0 & \text{if} \quad x = \dfrac{1}{2} \end{cases}$ then at $x = \dfrac{1}{2}$

(A) f has a local maxima

(B) f has a local minima

(C) f has no inflection point

(D) f has a removable discontinuity

39. Let $f(x) = x - \dfrac{1}{x}$ then which one of the following statement is true

(A) function is invertible if defined from $R - \{0\} \to R$

(B) $f(x_1) > f(x_2),\ \forall\ x_1 > x_2$ and $x_1, x_2 \neq 0$

(C) Graph of the function has exactly one asymptote

(D) Function is one-one in every continuous interval $[a, b]$ defined on one side of origin

40. If the function $f(x) = \dfrac{t + 3x - x^2}{x - 4}$, where '$t$' is a parameter has a minimum and a maximum then the range of values of 't' is

(A) $(0, 4)$ (B) $(0, \infty)$

(C) $(-\infty, 4)$ (D) $(4, \infty)$

41. P and Q are two points on a circle of centre C and radius α, the angle PCQ being 2θ then the radius of the circle inscribed in the triangle CPQ is maximum when

(A) $\sin\theta = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (B) $\sin\theta = \dfrac{\sqrt{5}-1}{2}$

(C) $\sin\theta = \dfrac{\sqrt{5}+1}{2}$ (D) $\sin\theta = \dfrac{\sqrt{5}-1}{4}$

42. $f(x)$ is twice differentiable function such that $f(x) + f''(x) = -x|\sin x|\ f'(x)$ where $x \geq 0$. Given that $f(0) = -3$, $f'(0) = 4$ the maximum value of $f(x)$ is

(A) 4 (B) 3

(C) 5 (D) Does not exists

43. Read the following mathematical statement carefully :

I. A differentiable function 'f' with maximum at $x = c \Rightarrow f''(c) < 0$

II. Anti-derivative of a periodic function is also a periodic function.

III. If f has a period T then for any $a \in R$.

$$\int_0^T f(x)dx = \int_0^T f(x+a)dx$$

IV. If $f(x)$ has a maximum at $x = c$, then 'f' is increasing in $(c - h, c)$ and decreasing in $(c, c + h)$ as $h \to 0$ for $h > 0$.

Now indicate the correct alternative.

(A) Exactly one statement is correct

(B) Exactly two statement are correct

(C) Exactly three statement are correct

(D) All the four statement are correct

44. If the point of minima of the function, $f(x) = 1 + a^2x - x^3$ satisfy the inequality $\dfrac{x^2 + x + 2}{x^2 + 5x + 6} < 0$, then '$a$' must lie in the interval:

(A) $(-3\sqrt{3}, 3\sqrt{3})$
(B) $(-2\sqrt{3}, -3\sqrt{3})$
(C) $(2\sqrt{3},\ 3\sqrt{3})$
(D) $(-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$

45. Two sides of a triangle are to have lengths 'a' cm and 'b' cm. If the triangle is to have the maximum area, then the length of the median from the vertex containing the sides 'a' and 'b' is

(A) $\frac{1}{2}\sqrt{a^2+b^2}$
(B) $\frac{2a+b}{3}$
(C) $\sqrt{\frac{a^2+b^2}{2}}$
(D) $\frac{a+2b}{3}$

46. A rectangle has one side on the positive y-axis and one side on the positive x-axis. The upper right-hand vertex of the rectangle lies on the curve $y = \frac{\ell nx}{x^2}$. The maximum area of the rectangle is

(A) e^{-1}
(B) $e^{-1/2}$
(C) 1
(D) $e^{1/2}$

47. Let $f(x) = ax^2 - b|x|$, where a and b are constants. Then at $x = 0$, $f(x)$ has

(A) a maxima whenever $a > 0, b > 0$
(B) a maxima whenever $a > 0, b < 0$
(C) minima whenever $a > 0, b > 0$
(D) neither a maxima nor minima whenever $a > 0, b < 0$

48. **S-1:** Suppose $f(x) = 2^x + 1$ and $g(x) = 4^{-x} + 2^{-x}$. The equation $f(x) = g(x)$ has exactly one root.
S-2: If $f(x)$ and $g(x)$ are two differentiable functions defined for all $x \in R$ and if $f(x)$ is strictly increasing and $g(x)$ is strictly decreasing for every $x \in R$ then the equation $f(x) = g(x)$ must have exactly one root.

(A) S-1 and S-2 both are true
(B) S-1 and S-2 both are false
(C) S-1 is true and S-2 is false
(D) S-1 is false and S-2 is true

49. Let $f(x) = \begin{vmatrix} [x] & \{x\} & 1 \\ \{x\} & 1 & [x] \\ 1 & [x] & \{x\} \end{vmatrix}$, where $[x]$ = greatest integer function and $\{x\}$ = fractional part of x. Then which of the following is incorrect

(A) $f(x) \geq 0$ for $x \in (-\infty, -1]$
(B) $f(x)$ has a real root for $x \in (0, 1)$
(C) for $x \in [0, 1)$, range of $f(x)$ is $(-2, -1]$
(D) for $x \in [1, 2)$, $f(x)$ has exactly one local minima

50. Carefully read the following five statements

(a) The function $f(x) = \sec x$ attains a maximum on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(b) If a function is differentiable at $x = c$ then it is continuous at $x = c$
(c) The equation $x^5 - 10x + \sin^5 x = 0$ has at least one non zero solution.
(d) If f is a polynomial such that $f'(3) = 0$ and $f''(3) \neq 0$ then f has a critical point at $x = 3$ which is either a local minimum or a local maximum.
(e) If f is a polynomial such that $f'(2) = 0$ and $f''(2) = 0$ then f has a critical point at $x = 2$ which is neither a local minimum nor a local maximum.

Out of these 5 statements

(A) Exactly 1 is true and 4 are false
(B) Exactly 2 are true and 3 are false
(C) Exactly 3 are true and 2 are false
(D) Exactly 4 are true and 1 is false

51. The minimum value of the function $f(x) = \int_0^x \frac{d\theta}{\cos\theta} + \int_x^{\pi/2} \frac{d\theta}{\sin\theta}$ where $x \in \left(0, \frac{\pi}{2}\right)$, is

(A) $2\ln(\sqrt{2}+1)$
(B) $\ln(2\sqrt{2}+2)$
(C) $\ln(2+\sqrt{3})$
(D) $\ln(3+\sqrt{2})$

52. The range of values of a for which the function $f(x) = \begin{cases} -x^3 + \cos^{-1} a; & 0 \leq x < 1 \\ x & ; \ 1 \leq x \leq 3 \end{cases}$ has the smallest value at $x = 1$, is

(A) $[\cos 2, 0]$
(B) $[-1, \cos 2]$
(C) $[0, 1]$
(D) $[-1, 1]$

53. Let $f(x) = \cos^{-1} x^2 + \cos^{-1} x$, then which one of the following is true?

(A) $f(x)$ is strictly decreasing
(B) $f(x)$ is decreasing in $[0, 1]$ and increasing in $[-1, 0]$
(C) $f(x)$ has only one local maxima
(D) $f(x)$ has only local minima

54. Suppose x_1 and x_2 are the point of maximum and the point of minimum respectively of the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ respectively, then for the equality $x_1^2 = x_2$ to be true the value of 'a' must be

(A) 0
(B) 2
(C) 1
(D) $\frac{1}{4}$

55. The local minimum value of the function $f(x) = \frac{\tan\left(x + \frac{\pi}{6}\right)}{\tan x}$ is

(A) 0
(B) $\frac{1}{2}$
(C) 1
(D) 3

56. Let f and g be two differentiable function defined from $R \to R^+$. If $f(x)$ has a local maximum at $x = c$ and $g(x)$ has a local minimum at $x = c$, then $h(x) = \frac{f(x)}{g(x)}$

(A) has a local maximum at $x = c$
(B) has a local minimum at $x = c$
(C) is monotonic at $x = c$
(D) has a point of inflection at $x = c$

57. Use the following properties of a function $f(x)$ to determine the location of any local extreme.

x	$x < -3$	-3	$-3 < x < 1$	1	$1 < x < 4$	4	$x > 4$
$f'(x)$	$-$	0	$+$	0	$-$	DNE	$-$

NOTE: $f'(4)$ is undefined.

(A) Local maximum at $x = -3$ and local minimum at $x = 1$
(B) No local extreme
(C) No local maximum and local minimum at $x = 4$
(D) Local maximum at $x = 1$ and local minimum at $x = -3$

58. Let $f(x) = \int_1^x \left(t\ln(t) - \frac{\ln t}{t}\right)dt \;\; (x > 1)$ then

(A) $f(x)$ has one point of maxima and no point of minima
(B) $f'(x)$ has two distinct roots
(C) $f(x)$ has one point of minima and no point of maxima
(D) $f(x)$ is monotonic

59. Which of the following six statements are true about the cubic polynomial $P(x) = 2x^3 + x^2 + 3x - 2$?

(i) It has exactly one positive real root
(ii) It has either one or three negative roots
(iii) It has a root between 0 and 1
(iv) It must have exactly two real roots
(v) It has a negative root between –2 and –1
(vi) It has no complex roots

(A) only (i), (iii) and (vi)
(B) only (ii), (iii) and (iv)
(C) only (i) and (iii)
(D) only (iii), (iv) and (v)

60. Write the correct order sequence in respect of the statement given below. **F** stands for **false** and **T** stands for **true**

S-1: Suppose that g is continuous with $g(1) = 5$ and $g(5) = 10$. Then the equation $g(c) = 7$ must have a solution such that $c \in (1, 5)$

S-2: If $f'(x) = g'(x)$ then $f(x) = g(x)$

S-3: Suppose that f is differentiable and $f(2) = f(6)$. Then the must be at least one point $c \in (2, 6)$ with $f'(c) = 0$

(A) TTT　　(B) TFF
(C) FTF　　(D) TFT

61. The range of the function $f(\theta) = \frac{\sin\theta}{\theta} + \frac{\theta}{\tan\theta}$, $\theta \in \left(0, \frac{\pi}{2}\right)$ is equal to

(A) $(0, \infty)$　　(B) $\left(\frac{1}{\pi}, 2\right)$
(C) $(2, \infty)$　　(D) $\left(\frac{2}{\pi}, 2\right)$

62. If a function f is continuous for all x and if f has a relative maximum at $(-1, 4)$ and a relative minimum at $(3, -2)$, then which of the following statements must be true?

(A) The graph of f has a point of inflection somewhere between $x = -1$ and $x = 3$
(B) $f'(-1) = 0$
(C) The graph of f has a horizontal tangent at $x = 3$
(D) The graph of f intersect both coordinate axes

63. Given that $f(x)$ is continuously differentiable on $a \le x \le b$ where $a < b$, $f(a) < 0$ and $f(b) > 0$, which of the following are always true?

(i) $f(x)$ is bounded on $a \le x \le b$
(ii) The equation $f(x) = 0$ has at least one solution in $a < x < b$.
(iii) The maximum and minimum values of $f(x)$ on $a \le x \le b$ occur at points where $f'(c) = 0$.
(iv) There is at least one point c with $a < c < b$ where $f'(c) > 0$
(v) There is at least one point d with $a < d < b$ where $f'(d) < 0$

(A) Only (ii) and (iv) are true
(B) All but (iii) are true
(C) All but (v) are true
(D) Only (i), (ii) and (iv) are true

64. Give the correct order of initials **T** of **F** for following statements. Use **T** if statement is true and **F** if it is false.

S-1: If $f : R \to R$ and $c \in R$ is such that f is increasing in $(c - \delta, c)$ and f is decreasing in $(c, c + \delta)$ then f has a local maximum at c. Where δ is a sufficiently small positive quantity.

S-2: Let $f : (a, b) \to R$, $c \in (a, b)$. Then f can not have both a local maximum and a point of inflection at $x = c$.

S-3: The function $f(x) = x^2\,|x|$ has point of inflection at $x = 0$.

S-4: Let $f : [c - 1, c + 1] \to [a, b]$ be bijective map such that f is differentiable at c then f^{-1} is also differentiable at $f(c)$.

(A) FFFF　　(B) TFTT
(C) TTTT　　(D) TTFT

65. If $f(x) = \begin{cases} 1 + x^2 - 3x, & x < 0 \\ \cos x + 2x, & x \ge 0 \end{cases}$, then the global maximum and local minimum values of $f(x)$ for $x \in [-2, 2]$ are respectively

(A) $4 + \cos 2, 1$　　(B) 11, 1
(C) 11, not exist　　(D) 11, 13

66. If $f(x) = \begin{cases} 3 + |x - k|, & \text{for } x \le k \\ a^2 - 2 + \frac{\sin(x - k)}{x - k}, & \text{for } x > k \end{cases}$ has minimum at $x = k$, then

(A) $a \in R$　　(B) $|a| < 2$
(C) $|a| > 2$　　(D) $1 < |a| < 2$

67. $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + x + 2, g(x) = \frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x$ then $f(g(x))$ has local minima at x

(A) $\{1, 2\}$　　(B) $\{0, 2\}$
(C) $\{2, 3\}$　　(D) $\{1, 3\}$

68. The maximum value of the integral $\int_{a-1}^{a+1} \frac{1}{1 + x^4}dx$ is attained

(A) exactly at two values of a;
(B) only at one value of a which is positive
(C) only at one value of a which is negative
(D) only at $a = 0$

69. If $R \to R$ and $g : R \to R$ are two functions such that $f(x) + f''(x) = -xg(x)f'(x)$ and $g(x) > 0, \forall x \in R$, then the functions $f^2(x) + (f'(x))^2$ has

(A) a maxima at $x=0$
(B) a minima at $x=0$
(C) a point of inflexion at $x=0$
(D) None of these

70. Let $f(x)$ be a positive differentiable function on $[0,a]$ such that $f(0)=1$ and $f(a)=3^{1/4}$ If $f'(x)\geq(f(x))^3+(f(x))^{-1}$, then, maximum value of a is

(A) $\dfrac{\pi}{12}$
(B) $\dfrac{\pi}{24}$
(C) $\dfrac{\pi}{36}$
(D) $\dfrac{\pi}{48}$

71. Let $f:(0,\infty)\to R$ be a (strictly) decreasing function. If $f\left(2a^2+a+1\right)<f\left(3a^2-4a+1\right)$, then the range of $a\in R$ is

(A) $\left(-\infty,\dfrac{1}{3}\right)\cup(1,\infty)$
(B) (0, 5)
(C) $\left(0,\dfrac{1}{3}\right)\cup(1,5)$
(D) [0, 5]

72. Let $f(x)=\begin{cases}(x+1)^3 & , \quad -2<x\leq-1\\ x^{2/3}-1 & , \quad -1<x\leq 1\\ -(x-1)^2 & , \quad 1<x<2\end{cases}$

The total number of maxima and minima of $f(x)$ is

(A) 4
(B) 3
(C) 2
(D) 1

73. The greatest area of the rectangular plot which can be laid out within a triangle of base 36 ft. and altitude 12 ft equals (Assume that one side of the rectangle lies on the base of the triangle)

(A) 90
(B) 108
(C) 72
(D) 126

74. Let $f(x)=\begin{cases}3x+|a^2-4|, & x<1\\ -x^2+2x+7, & x\geq 1\end{cases}$. Then set of values of a for which $f(x)$ has maximum value at $x=1$ is

(A) $(3,\infty)$
(B) $[-3,3]$
(C) $(-\infty,3)$
(D) None of these

75. Let $f(x)$ be a cubic polynomial on R which increases in the interval $(-\infty,0)\cup(1,\infty)$ and decreases in the interval $(0,1)$. If $f'(2)=6$ and $f(2)=2$ then the value of

$\tan^{-1}(f(1))+\tan^{-1}\left(f\left(\dfrac{3}{2}\right)\right)+\tan^{-1}(f(0))$ equals

(A) $\tan^{-1}2$
(B) $\cot^{-1}2$
(C) $-\tan^{-1}2$
(D) $-\cot^{-1}2$

76. The set of values of p for which $f(x)=p^2x-\int 2^{4-x^2}\,dx$ is increasing for all $x\in R$, is

(A) $[-4,4]$
(B) $(-\infty,-16]\cup(16,\infty)$
(C) $(-\infty,-4]\cup(4,\infty)$
(D) $[-16,16]$

77. The set of all real values of b for which the function $f(x)=(b^2-3b+2)(\cos^2x-\sin^2x)+(b-1)x+\sin 2$ does not possesses stationary points on R is equal to

(A) $[1,\infty)$
(B) $(0,1)\cup(1,4)$
(C) $\left(\dfrac{3}{2},\dfrac{5}{2}\right)$
(D) $\left(\dfrac{-7}{2},\dfrac{-3}{2}\right)$

78. The complete set of values of k for which the function

$f(x)=\begin{cases}x & ;\ -\infty<x<2\\ k & ;\ x=2\\ x^2-2x+4 & ;\ 2<x<\infty\end{cases}$ is strictly increasing at $x=2$, is

(A) [2, 4)
(B) (2, 4]
(C) (2, 4)
(D) [2, 4]

79. If the function $f(x)=2\tan x+(2a+1)\ln|\sec x|+(a-2)x$ is increasing on R then a lies in the interval,

(A) (0, 1)
(B) (2, 4)
(C) (–2, 0)
(D) None

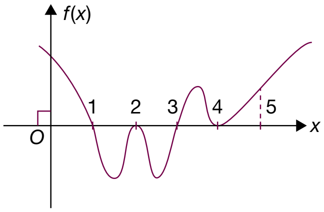
80. Let $f(x)=\left(\dfrac{a^2-4}{a^2+2}\right)x^3-3x+\sin 3$, then the true set of values of for which $f(x)$ is strictly decreasing on R, is

(A) $(-\infty,-2)$
(B) $[2,\infty)$
(C) $[-2,2]$
(D) $(5,\infty)$

81. Let $g:[1,6]\to[0,\infty]$ be a real valued differentiable function satisfying $g'(x)=\dfrac{2}{x+g(x)}$ and $g(1)=0$, then the maximum value of g cannot exceed

(A) ln 2
(B) ln 6
(C) 6 ln 2
(D) 2 ln 6

82. Let $f(x)$ be a differentiable function such that $f(0)=1$, $f(1)=2$, $f(2)=1$ and $f(4)=4$. The graph of $y=f'(x)$ is given below. If $x\in[0,5]$, then the total number of local maximum points and local minimum points are



(A) 1
(B) 2
(C) 3
(D) 4

83. If the equation $x^4+8x^3+18x^2+8x+a=0$ has four distinct real roots, then the range of a is

(A) (0, 9)
(B) (– 9, 0)
(C) (– 8, 1)
(D) (– 1, 8)

84. If function $f(x)=(4-p)x^3+(p-2)x^2+(p^2-25)x+2$ has a minimum at some $x\in(-\infty,0)$ and a maximum at some $x\in(0,\infty)$, then the true set of value of p is

(A) (4, 5)
(B) $(-\infty,-5)\cup(4,5)$
(C) (–5, 5)
(D) $(5,\infty)$

85. Let $f:R\to R$ defined by $f(x)=\dfrac{x}{2}+\dfrac{\sin(2x)}{4}-(3-c)\sin(x)-3cx$ is strictly increasing on R, then the largest integral value of 'c' is equal to

(A) 0
(B) 1
(C) –1
(D) –2

86. Let $f(x)=\begin{cases}-x^3+\log_2 b, & 0\leq x<1\\ 3x, & 1\leq x\leq 3\end{cases}$. The set of real values of b for which $f(x)$ has smallest value at $x=1$, is

(A) (0, 8]
(B) $[8,\infty)$
(C) $[16,\infty]$
(D) (0, 16]

87. The minimum value of the function

$f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$ for all permissible real x, is

(A) –10 (B) –6

(C) –7 (D) –8

88. Let $f : R \to R$ be defined as $f(x) = \begin{cases} e^{1+x} - e^x; & -\infty < x \le 0 \\ e^{1-x} - 1\ ; & 0 < x < 1 \\ x + \ln x\ ; & 1 \le x < \infty \end{cases}$.

Then $f(x)$ has

(A) local maxima at $x = 0$ but $x = 1$ is neither local maxima nor local minima.

(B) local minima at $x = 1$ and $x = 0$ is point of local maxima.

(C) local maxima at $x = 0$ and $x = 1$.

(D) local minima at $x = 0$ and $x = 1$.

89. Consider the function

$f(x) = \begin{cases} x(x-1)e^{2x} & ;\ \text{if } x \le 0 \\ x(1-x)e^{-2x} & ;\ \text{if } x > 0 \end{cases}$. Then $f(x)$ attains its greatest value at

(A) $1 - \frac{1}{\sqrt{2}}$ (B) $1 + \frac{1}{\sqrt{2}}$

(C) $\frac{-1}{\sqrt{2}}$ (D) $\frac{1}{\sqrt{2}}$

90. A differentiable function $f(x)$ has a relative minimum at $x = 0$, then the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$ for

(A) all a and all b (B) all $b > 0$

(C) all b, if $a = 0$ (D) all $a > 0$

91. Let $f(x) = \begin{cases} \frac{\cos(\pi x)}{2} & ;\ \forall\, 0 \le x < 1 \\ 3 + 5x & ;\ \forall\, x \ge 1 \end{cases}$

(A) $f(x)$ has local minimum at $x = 1$

(B) $f(x)$ has local maximum at $x = 1$

(C) $f(x)$ does not have any local maximum or local minimum at $x = 1$

(D) $f(x)$ has a global minimum at $x = 1$

92. The minimum value of $x^2 - x + 1 + \sin x$ is given by

(A) $\frac{1}{4}$ (B) $\frac{3}{4}$

(C) $-\frac{1}{4}$ (D) $-\frac{7}{4}$

93. If the function $f(x) = \frac{ax + b}{x^2 - 1}$ has a local maximum value 1 at $x = 3$, then the value of $(a - b)$ is equal to

(A) 10 (B) 13

(C) 16 (D) 4

94. If $\sin x + x \ge |k|\, x^2$, $\forall\ x \in \left[0, \frac{\pi}{2}\right]$, then the greatest value of k is

(A) $\frac{-2(2+\pi)}{\pi^2}$

(B) $\frac{2(2+\pi)}{\pi^2}$

(C) can't be determined finitely

(D) zero

95. If m is the minimum value of $f(x, y) = x^2 - 4x + y^2 + 6y$ where x and y are subjected to the restrictions $0 \le x \le 1$ and $0 \le y \le 1$, then the value of $|m|$ is

(A) 1 (B) 2

(C) 3 (D) 4

96. Let : $R \to R$ be defined by $f(x) = \begin{cases} b - 2x, & x \le -1 \\ 2x + 3, & x > -1 \end{cases}$. If f has a local minimum at $x = -1$, then a possible value of b is equal to

(A) 0 (B) $-\frac{1}{2}$

(C) –1 (D) 1

97. The global maximum value of $f(x) = \cot x - \sqrt{2}\operatorname{cosec} x$ in interval $(0, \pi)$ is equal to

(A) 1 (B) –1

(C) 0 (D) non-existent

98. Let $f(x)$ be defined as $f(x) = \begin{cases} \tan^{-1}\alpha - 5x^2, & 0 < x < 1 \\ -6x, & x \ge 1 \end{cases}$

$f(x)$ can have a maximum at $x = 1$ if value of α can be

(A) 0 (B) –1

(C) $-\tan 1$ (D) –2

99. The number of critical point of $f(x) = \frac{|x-1|}{x^2}$ is

(A) 1 (B) 2

(C) 3 (D) 0

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. Let $f(x) = 2x^2 - \ln|x|, x \ne 0$, then $f(x)$ is

(A) monotonically increasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

(B) monotonically decreasing in $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

(C) monotonically increasing in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

(D) monotonically decreasing in $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

2. Let $g'(x) > 0$ and $f'(x) < 0$, $\forall\ x \in R$, then

(A) $f(x+1) > f(x-1)$ (B) $f(g(x-1)) > f(g(x+1))$

(C) $g(f(x+1)) > g(f(x-1))$ (D) $g(g(x+1)) > g(g(x-1))$

3. If $f(x) = x^3 - x^2 + 100x + 2002$, then

(A) $f(1000) > f(1001)$ (B) $f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$

(C) $f(x-1) > f(x-2)$ (D) $f(x-3) > f(2x)$

4. Consider the function $f(x) = \sin^5 x + \cos^5 x - 1$, $x \in \left[0, \frac{\pi}{2}\right]$ which of the following is/are correct?

(A) f is monotonic decreasing in $\left(0, \frac{\pi}{4}\right)$

(B) f is monotonic increasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(C) $\exists$ some $c \in \left(0, \frac{\pi}{2}\right)$ for which $f'(c) = 0$

(D) The equation $f(x) = 0$ has two roots in $\left[0, \frac{\pi}{2}\right]$

5. A function $f(x)$ satisfies

$f(x) = \sin x + \int_0^x f'(t)\left(2\sin t - \sin^2 t\right)dt$ then

(A) $f\left(\frac{\pi}{6}\right) = 1$

(B) $g(x) = \int_0^x f(t)dt$ is increasing on $(0, \pi)$

(C) $f(0) = 0$

(D) $f(x)$ is increasing on $(0, \pi)$

6. The function $f(x) = \left(x^2 - 4\right)^n \left(x^2 - x + 1\right)$, $n \in N$ has a local minimum at $x = 2$ then the value of n can be

(A) 2010 (B) 2011

(C) 2009 (D) 2008

7. If $g(x+y) = g(x) + g(y) + 3xy(x+y)\ \forall x, y \in R$ and $g'(0) = -4$ then

(A) Number of local maximum of $g(x)$ is one

(B) Number of local minimum of $g(x)$ is one

(C) Number of points of inflection of $g(x)$ is one

(D) $g(x)$ is a decreasing function on $(-3, 3)$

8. If $f(x) = -1 + |x - 2|$, $0 \le x \le 4$ and $g(x) = 2 - |x|$, $-1 \le x \le 3$ then which of the following is (are) correct?

(A) $(fog)(x)$ is continuous at $x = 0$.

(B) $(fog)(x)$ is non-derivable at $x = 0$

(C) $(fog)(x)$ has relative minimum at $x = 0$.

(D) Range of $(fog)(x)$ equals $[-1, 1]$

9. If $f : \left[-\frac{3\pi}{4}, \pi\right] \to \left[-\sqrt{2}, \frac{9}{2}\right]$ is defined by

$$f(x) = \begin{cases} \sin x + \cos x & \text{for } -\frac{3\pi}{4} \le x \le \frac{\pi}{4} \\ \frac{4}{\pi}x + \frac{1}{2} & \text{for } \frac{\pi}{4} < x \le \pi \end{cases} \text{ then}$$

(A) f is monotonically increasing

(B) f is a one-one function

(C) f is an-onto function

(D) f is continuous on $\left[-\frac{3\pi}{4}, \pi\right]$

10. Let $f(x) = \begin{cases} 2x - 4 & , \ x \le 2 \\ -x^2 + \frac{k^3(k-1)^2}{k^2 - k - 2} + 4 & , \ x > 2 \end{cases}$. $f(x)$ attains local maximum at $x = 2$ if k lies in

(A) $(0, 1)$ (B) $(3, \infty)$

(C) $(-\infty, -1)$ (D) $(1, 2)$

11. $\frac{\pi^e}{x - e} + \frac{e^\pi}{x - \pi} + \frac{\pi^\pi + e^e}{x - \pi - e} = 0$ has

(A) one real root in (e, π) and other in $(\pi - e, e)$

(B) one real root in (e, π) and other in $(\pi, \pi + e)$

(C) two real roots in $(\pi - e, \pi + e)$

(D) no real roots

12. Let $f(x)$ be an increasing function defined on $(0, \infty)$.
If $f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$ then the possible integral value of 'a' is/are

(A) 1 (B) 2

(C) 3 (D) 4

13. If x_1 and x_2 are positive numbers between 0 and 1, then which of the following is/are true?

(A) $\sin\left(\frac{x_1 + x_2}{2}\right) \le \frac{\sin x_1 + \sin x_2}{2}$

(B) $\tan\left(\frac{x_1 + x_2}{2}\right) \le \frac{\tan x_1 + \tan x_2}{2}$

(C) $\log\left(\frac{x_1 + x_2}{2}\right) \le \frac{\log x_1 + \log x_2}{2}$

(D) $\left(\frac{x_1 + x_2}{2}\right)^2 \le \frac{x_1^2 + x_2^2}{2}$

14. For any real number $\lambda \ge 1$, let $f(\lambda)$ denote the real solution of the equation $x(1 + \ln x) = \lambda$ then

(A) $f(\lambda)$ is strictly increasing function in its domain

(B) $\lim_{\lambda \to \infty} f(\lambda) = \infty$

(C) if $h(x) = x(1 + \ln x)$, $h : [1, \infty) \to [1, \infty)$ then $f(\lambda)$ is inverse function of $h(x)$

(D) $\lim_{\lambda \to \infty} \frac{f(\lambda)\ln\lambda}{\lambda} = 1$

15. Let $f : R \to R$ be twice differentiable function for which $f'(x) > 0$ and $f''(x) > 0\ \ \forall\ x \in R$, if a and b are real numbers, then

(A) $f\left(\frac{a+b}{2}\right) < \frac{f(a) + f(b)}{2}$

(B) $f\left(\frac{a+b}{2}\right) > \frac{f(a) + f(b)}{2}$

(C) $f^{-1}\left(\frac{a+b}{2}\right) < \frac{f^{-1}(a) + f^{-1}(b)}{2}$

(D) $f^{-1}\left(\frac{a+b}{2}\right) > \frac{f^{-1}(a) + f^{-1}(b)}{2}$

16. Let f be real-valued function on R defined as $f(x) = x^4(1 - x)^2$, then which of the following statement(s) is (are) correct?

(A) $f'(c) = 0$ for some $c \in (0, 1)$

(B) $f''(x)$ vanishes exactly twice in R

(C) $f(x)$ is an even function

(D) Monotonic increasing in $\left(0, \frac{2}{3}\right) \cup (1, \infty)$

17. Let f be a continuous function on R and satisfies $f(x) = e^x + \int_0^1 e^x \, f(t)dt$, then which of the following is(are) correct?

(A) $f(0) < 0$

(B) $f(x)$ is decreasing function on R

(C) $f(x)$ is an increasing function on R

(D) $\int_0^1 f(x) \, dx > 0$

18. If composite function $f_1 (f_2 (f_3 (….(f_n(x))))$, n times, is an increasing function and if r of $f_i s$ are decreasing function while rest are increasing, then maximum value of $r(n - r)$ is

(A) $\frac{n^2 - 1}{4}$, when n is of $(4k + 2)$ type, $k \in N$

(B) $\frac{n^2}{4}$, when n is of $(4k + 3)$ type, $k \in N$

(C) $\frac{n^2 - 1}{4}$, when n is of $(4k + 1)$ type, $k \in N$

(D) $\frac{n^2}{4}$, when n is of $4k$ type, $k \in N$

19. Let $f(x) = \frac{x^3 - 4}{(x-1)^3}$ then which of the following is (are) correct.

(A) $\lim_{x \to -\infty} f(x) = -1$

(B) $x = -2$ is the point of local minimum

(C) The equation $f(x) = c$ will have three real and distinct roots for $c \in \left(\frac{4}{9}, 1\right) \cup (1, 4)$

(D) The line $y = 1$ intersects the graph of $f(x)$ at exactly one point

20. Consider a real valued continuous function f such that $f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + t f(t)) dt$. If M and m are maximum and minimum values of the function f, then

(A) $\frac{M}{m} = 3$

(B) $M - m = 2\pi + 1$

(C) $M + m = 4(\pi + 1)$

(D) $Mm = 2(\pi^2 + 1)$

21. Let $f(x) = \min. (\phi(t), -3 \le t \le x) \ \forall x \in [-3, \infty]$ where $\phi(x) = \left| \, |x - 1| - |x + 1| \, \right|$, then

(A) $f(x)$ is non-differentiable at $x = -1, 0$

(B) $f(x)$ is non-differentiable at $x = -1, 1$

(C) $f(100) = 0$

(D) $\int_{-3}^{10} f(x) \, dx = 5$

22. Which of the following statement(s) is/are correct for $f(x) = x^{1/3} (x - 1)$

(A) has two points of inflection

(B) is strictly increasing for $x > \frac{1}{4}$

(C) is concave down in $\left(-\frac{1}{2}, 0\right)$

(D) minimum value of $f(x)$ is $-\frac{3}{4^{4/3}}$

23. The greatest value of the function $f(x) = x^4 - 6bx^2 + b$ on the interval $x \in [-2, 1]$ depends on the parameter b. Which of the following statement(s) is/are correct

(A) greatest value is b^2 if $b = 2015$

(B) greatest value is $b^2 - 24b + 16$ if $b = 2016$

(C) greatest value is $b^2 - 24b + 16$ if $b = -2016$

(D) greatest value is b^2 if $b = -2016$

24. Let $f(x) = \frac{e^x}{1 + x^2}$ and $g(x) = f'(x)$ then

(A) $g(x)$ has two local maxima and two local minima points

(B) $g(x)$ has exactly one local maxima and one local minima point

(C) $x = 1$ is point of local maxima for $g(x)$

(D) there is a point of local maxima for $g(x)$ in the interval $(-1, 0)$

25. Let $f(x) = \begin{cases} \cos^{-1} x & ; \ x < 0 \\ \sin^{-1} x & ; \ x \ge 0 \end{cases}$, $g(x) = \begin{cases} \sin^{-1} x & ; \ x < 0 \\ \cos^{-1} x & ; \ x \ge 0 \end{cases}$ and $h(x) = \min \{f(x). g(x)\}$ then

(A) The number of non-differentiable points of $h(x)$ on $(-1, 1)$ is one

(B) $h(x)$ is increasing on $(-1, 1)$

(C) $h(x)$ has local maxima at $x = \frac{1}{\sqrt{2}}$

(D) $h''(x) > 0 \ \forall \ x \in \left(0, \frac{1}{\sqrt{2}}\right)$

26. Consider $f(x) = \begin{cases} \sin(\cos x) + \cos(\sin x), & \frac{-\pi}{2} \le x \le 0 \\ \sin(a \cos x) - x + 1, 0 < x \le \pi, & 0 < x \le \pi \end{cases}$.

Then

(A) difference of maximum and minimum value of $f(x)$ in $\left[\frac{-\pi}{2}, 0\right]$ is greater than 1

(B) difference of maximum and minimum value of $f(x)$ in $\left[\frac{-\pi}{2}, 0\right]$ is smaller than 1

(C) there is exactly one integer $a \in (0, \pi)$ for which $f(x)$ has local maximum at $x = 0$

(D) there is exactly two integers $a \in (0, \pi)$ for which $f(x)$ has local maximum at $x = 0$

27. Let $f(x)$ be a quadratic expression with positive integral coefficients such that for every $\alpha, \beta \in R, \beta > \alpha$, $\int_\alpha^\beta f(x) \, dx > 0$. Let $g(t) = f''(t) f(t)$, and $g(0) = 12$, then

(A) 16 such quadratics are possible
(B) $f(x) = 0$ has either no real root or distinct roots
(C) minimum value of minima of $f(t)$ is $\frac{1}{3}$
(D) maximum value of minima of $f(t)$ is 11

28. Consider $f(x) = e^x \sec x - \sqrt{2}\cos x + x$, where $x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$
(A) minimum value of $f(x)$ is $\left(\sqrt{2}e^{-\frac{\pi}{4}} - 1 - \frac{\pi}{4}\right)$
(B) minimum value of $f(x)$ is $\left(\sqrt{2}e^{-\frac{\pi}{3}} - \frac{1}{\sqrt{2}} - \frac{\pi}{3}\right)$
(C) $f'\left(\frac{\pi}{3}\right) \geq f'(x)\ \forall\ x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$
(D) $f'\left(\frac{\pi}{3}\right) \leq f'(x)\ \forall\ x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

29. The maximum value of the function defined by $f(x) = \min(e^x, 2 + e^2 - x, 8)$ is α then integral value of x satisfying the inequality $\frac{x(x - [\alpha])}{x^2 - [\alpha]x + 12} < 0$, is (where $[k]$ denotes greatest integer function less than or equal to k)
(A) 1
(B) 3
(C) 5
(D) 6

30. Let $f(x) = 1 + x\ln(x + \sqrt{x^2+1}) - \sqrt{1+x^2}$ and $h(x) = f(x) - f^2(x) + f^3(x)$. Then
(A) $h(x)$ is increasing in $(0, \infty)$
(B) $h(x)$ is decreasing in $(-\infty, 0)$
(C) $h(x)$ is increasing on R
(D) $h(x)$ in decreasing on R

31. $F(x)$ is a polynomial satisfying the relation $18f(x) = (f'(x) - 6)f''(x) + 3f'''(x)$. Then which of the following is/are true.
(A) $f(x)$ is increasing in $x \in [-2, -1]$
(B) minimum value of $f(x)$ in $x \in [-1, 1]$ equal to -1.
(C) absolute maxima of the $f(x)$ in $x \in [-3, 3]$ is equal to 19
(D) $\int_{-2}^{2} f(x)dx = 4$

32. Consider, $f(x) = \sin^{-1}\left(1 - 2\sqrt{x}\right) + \sec^{-1}\left(\frac{1}{2\sqrt{\sqrt{x} - x}}\right) + \tan^{-1}\left(\frac{\sqrt{2} - 1 - \sqrt{x}}{1 + \sqrt{2x} - \sqrt{x}}\right)$. Identify correct statement(x) about $f(x)$.
(A) $f(x)$ is a decreasing function
(B) minimum value of $f(x)$ is $\frac{-\pi}{8}$
(C) $f'\left(\frac{1^-}{4}\right) = \frac{-24}{5}$
(D) $f'\left(\frac{1^+}{4}\right) = \frac{-4}{5}$

33. Let $f(x) = 4\sin^3 x + 9\cos^2 x + 6\sin x - 3$ for $0 \leq x \leq \pi$. Maximum value of $f(x)$ occurs at x equals to
(A) 0
(B) $\frac{\pi}{6}$
(C) $\frac{5\pi}{6}$
(D) $\frac{\pi}{2}$

34. Suppose a water bubble loses its volume at a rate proportional to its instantaneous volume. The radius of the bubble decreases from unity to e^{-1} cm in one minute. If instantaneous radius of bubble is 'r' then find correct statements?
(A) Function $t^2 r$ is increasing in the interval (0, 2)
(B) $\int_0^\infty r\,dt$ is equal to unity
(C) Let radius is plotted on y-axis and time t on x-axis, If shortest distance 'p' of this curve from origin is attained at $t = t_0$ then $p = \sqrt{t_0(t_0 + 1)}$
(D) $[r] = 0$ for $t > 0$.

35. Let $f(x) = \frac{(x-1)^2 . e^x}{(1+x^2)^2}$, then which of the following statement(s) is(are) correct?
(A) $f(x)$ is strictly increasing in $(-\infty, -1)$
(B) $f(x)$ is strictly decreasing in $(1, \infty)$
(C) $f(x)$ has two points of local extremum
(D) $f(x)$ has a point of local minimum at some $x \in (-1, 0)$

36. Let $f(x) = \begin{cases} x^3 + x^2 - 10x; & -1 \leq x < 0 \\ \sin x; & 0 \leq x < \frac{\pi}{2} \\ 1 + \cos x; & \frac{\pi}{2} \leq x \leq \pi \end{cases}$ then $f(x)$ has
(A) local maximum at $x = \frac{\pi}{2}$
(B) local minimum at $x = \frac{\pi}{2}$
(C) absolute maximum at $x = 0$
(D) absolute maximum at $x = -1$

37. If x_1, x_2, x_3 are roots of the equation $x^3 - 3x + 1 = 0$ with $x_1 < x_2 < x_3$, then which of the following is/are correct?
(A) All roots are real and distinct
(B) $x_1 \in (-2, -1)$, $x_2 \in \left(0, \frac{1}{2}\right)$, $x_3 \in (1, 2)$
(C) Distance between the local maximum and local minimum of the curve $y = x^3 - 3x + 1$ is 2
(D) $[x_1] + [x_2] + [x_3] = -1$

[**NOTE:** $[k]$ denotes greatest integer less than or equal to k.]

38. Let $f(x) = x^3 + ax^2 + bx + c$ where $a, b, c \in R$, then which of the following statement(s) is (are) correct?
(A) If the equation $f(x) = 0$ has exactly one real root then $f(x)$ must be strictly increasing on R.
(B) If $f(x)$ has a negative point of local minimum then both roots of equation $f'(x) = 0$ must be negative and distinct.
(C) If $f(x_1) \cdot f(x_2) < 0$, $x_1 < x_2$, then the equation $f(x) = 0$ has atleast one real root in (x_1, x_2).
(D) $f(x)$ possesses exactly one point of inflexion.

39. Let $f(x) = \int_0^x e^{-t^2}\,(t-5)\,(t^2-7t+12)dt$ for all $x \in (0, \infty)$, then

(A) f has a local maximum at $x = 4$ and a local minimum at $x = 3$.
(B) f is decreasing on $(3, 4) \cup (5, \infty)$ and increasing on $(0, 3) \cup (4, 5)$.
(C) There exists atleast two $c_1, c_2 \in (0, \infty)$ such that $f''(c_1) = 0$ and $f''(c_2) = 0$.
(D) There exists some $c \in (0, \infty)$ such that $f'''(c) = 0$.

40. For the equation $\dfrac{e^{-x}}{1+x} = p$, which of the following statement(s) is (are) correct?

(A) If $p \in (0, \infty)$ then equation has 2 real and distinct roots.
(B) If $p \in (-\infty, -e^2)$ then equation has 2 real and distinct roots.
(C) If $p \in (0, \infty)$ then equation has 1 real root.
(D) If $p \in (-e, 0)$ then equation has no real root.

41. If $f(x) = \min.\ (1, \cos x, 1 - \sin x),\ -\pi \le x \le \pi$, then

(A) $f(x)$ is not differentiable at $x = 0$
(B) $f(x)$ has local maximum at $x = 0$
(C) $f(x)$ is differentiable at $x = \dfrac{\pi}{4}$
(D) $f(x)$ is continuous and bounded in $x \in [-\pi, \pi]$

42. Consider a function $f(x) = xe^x + \dfrac{1}{xe^x}$ where $x \in R - \{0\}$, then which of the following statement(s) is/are incorrect?

(A) $f(x)$ attains its local minima at $x = -1$
(B) $f(x)$ attains its maxima at $x = \beta$, where $\beta \in (0, 1)$
(C) $f(x)$ attains its local minima at $x = \beta$, where $\beta \in (0, 1)$
(D) $f(x)$ attains its local minima at $x = \beta$, where $\beta \in (1, \infty)$

43. Consider a polynomial $f(x) = x^3 + 2x^2 + x + 1$. Let P and Q denote the point of local maxima and local minima respectively and O is origin, then which of the following statement(s) is/are correct?

(A) Local maximum value of $f(x)$ unity
(B) Local minimum value of $f(x)$ is $\dfrac{23}{27}$
(C) Tangent of the acute angle between OP and OQ is $\dfrac{7}{16}$
(D) $f(x) = 0$ has a root in the interval $[-2, -1]$

44. If $(f(x) - 1)(x^2 + x + 1)^2 - (f(x) + 1)(x^4 + x^2 + 1) = 0$ is true for all $x \in R - \{0\}$, then which of the following statement(s) is (are) correct?

(A) $|f(x)| = 2$ for all $x \in R - \{0\}$
(B) $f(x)$ has local maximum at $x = -1$.
(C) $f(x)$ has local minimum at $x = 1$.
(D) The value of definite integral $\int_{-\pi}^{\pi} (\cos x) f(x)dx = 0$

45. Suppose that $f(x)$ is defined and positive for all real numbers x. If $f(x)$ increases for $x < 0$ and decreases for all $x > 0$, and if $g(x) = \dfrac{1}{f\left(\dfrac{1}{x}\right)}$, then

(A) g increases for all real x
(B) g increases for $x < 0$
(C) g decreases for all real x
(D) g decreases for $x > 0$

46. Let $f(x) = \int_x^{2x} (t\ln t - t)dt$ for all $x > 0$. Then which of the following is/are correct?

(A) $f(x)$ is strictly increasing in $(0, e)$ and strictly decreasing in (e, ∞)
(B) $f(x)$ is strictly decreasing in $\left(0, \dfrac{e}{2^{\frac{4}{3}}}\right)$ and strictly increasing in $\left(\dfrac{e}{2^{\frac{4}{3}}}, \infty\right)$.
(C) $f(x)$ has a local minima but no local maxima.
(D) $\lim\limits_{x\to\infty} \dfrac{f'(x)}{x^2}$ is equal to zero.

47. Let $f(x) = \begin{cases} \sin^{-1}(\sin x), & x > 0 \\ \dfrac{\pi}{2}, & x = 0, \\ \cos^{-1}(\cos x), & x < 0 \end{cases}$ then

(A) $f(x)$ has local maxima at $x = 0$.
(B) $f(x)$ is continuous for all $x \in R$.
(C) $f(x)$ has maximum value π.
(D) $f(x)$ has minimum value 0.

48. Let $f(x) = \left(\dfrac{3x^2 - 12x}{x^2 - 16}\right)$, then which of the following statement(s) is (are) true?

(A) $\lim\limits_{x\to 4} f(x)$ does not exist.
(B) $f(x)$ is monotonic.
(C) The equation $f(x) = k$ has no solution for exactly two values of k.
(D) $f(x)$ is discontinuous at exactly one point.

49. A function is continuous and differentiable on R_0 and satisfies the condition $x f'(x) + f(x) = 1$ throughout its domain, with $f(1) = 2$. Then which of the following hold good?

(A) $f(x)$ is bounded
(B) $f(x)$ is non-monotonic
(C) $f(x)$ has one horizontal and one vertical asymptote
(D) Range of the function is $R - \{1\}$

50. Consider a function $f(x) = \int_{-4\pi^2}^{x^2} \dfrac{\sin\sqrt{\theta}}{1+\cos^2\sqrt{\theta}}\,d\theta$ then which of the following is (are) correct?

(A) $f(x)$ has a local minima at $x = 2\pi$
(B) $f(x)$ has a local maxima at $x = \pi$
(C) $f(x)$ has a local minima at $x = 0$
(D) $f(x)$ has a local maxima at $x = -\pi$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

If $f: R \to R$ is a function defined by

$f(x) = 2\cos^2 2x + \frac{3}{4}\sin 4x + ax$, where $a \in R$.

1. Complete set of values of 'a' for which $f(x)$ is monotonically increasing for all $x \in R$, is

(A) $(-\infty, -5]$ (B) $[5, \infty)$

(C) $(-5, \infty)$ (D) $(-\infty, 5]$

2. Complete set of values of 'a' for which $f(x)$ does not have any critical point is

(A) $(-\infty, -5]$ (B) $[5, \infty)$

(C) $(-5, 5)$ (D) $(-\infty, -5) \cup (5, \infty)$

3. Value of 'a' for which tangent to the curve $y = f(x)$ at $x = 0$ is parallel to first quadrant angle bisector, is

(A) 0 (B) 1

(C) –2 (D) 3

Comprehension 2

Consider $f(x) = [x] + \sqrt{\{x\}}$ where [.] and {.} represents greatest integer and fractional part function respectively.

4. $f(x)$ is

(A) monotonically decreasing

(B) discontinuous at all integers

(C) monotonically increasing

(D) concave upward

5. Identify the correct relation

(A) $f^{-1}(1.2) + f^{-1}(1.8) < 2f^{-1}(1.5)$

(B) $f(-1.3) < f^{-1}(-1.3)$

(C) $f^{-1}(1.2) + f^{-1}(1.8) > 2f^{-1}(1.5)$

(D) $f(1.2) + f(1.8) > 2f(1.5)$

6. If the equation of tangent to the curve $y = f^{-1}(x)$ at $x = 6.5$ is $x + ay + b = 0$, then $a + [b]$ is equal to

(where [.] denotes greatest integer function)

(A) –2 (B) 0

(C) –4 (D) –1

Comprehension 3

Let $f: R \to R$, be real valued differentiable function on R (the set real numbers) and satisfies $\int_3^x f(t)\,dt = \frac{x^2}{2} + \int_x^3 t^2 f(t)\,dt - \frac{9}{2}$.

7. Which one of the following is **incorrect?**

(A) $f(x)$ is neither injective nor surjective

(B) Range of $f(x)$ is $[-1, 1]$

(C) $f(x)$ has local maximum at $x = 1$

(D) $\int_{-1}^{1} f(x)dt = 0$

8. If the equation $f(|x|) = k$ has four distinct real roots, then the range of k is

(A) $[0, 1]$ (B) $(0, 1)$

(C) $\left(0, \frac{1}{2}\right)$ (D) $\left[0, \frac{1}{2}\right]$

9. The distance between horizontal tangents at local maximum and local minimum of the curve $y = f(x)$ is

(A) 1 (B) 2

(C) 3 (D) 4

Comprehension 4

Let $f(x) = a(x - 2)(x - b)$, where $a, b \in R$ and $a \neq 0$.

Also $f(f(x)) = a^3\left(x^2 - (2+b)x + 2b - \frac{2}{a}\right)\left(x^2 - (2+b)x + 2b - \frac{b}{a}\right)$, has exactly one real zero as 5.

10. Which of the following **CORRECT**?

(A) The minimum value of $f(x)$ is 5 and attained at $x = -2$

(B) The maximum value of $f(x)$ is 2 and attained at $x = 5$

(C) The minimum value of $f(x)$ is 10 and attained at $x = 0$

(D) The maximum value of $f(x)$ is 24 and attained at $x = 6$

11. The slope of the straight line passing through $O(0, 0)$ and tangent to $y = f(x)$, can be

(A) 2/9 (B) 2

(C) $\frac{4}{9}$ (D) $\frac{2}{9}$

Comprehension 5

Let $P(x)$ be a polynomial of degree 4 having a relative maximum at $x = 2$ and $\lim_{x \to 0}\left(3 - \frac{P(x)}{x}\right) = 27$. Also $P(1) = -9$ and $P''(x)$ has a local minimum at $x = 2$.

12. The absolute minimum value of function $y = P'(x)$ on the set $A = \{x \mid x^2 + 12 \leq 7x\}$ equals

(A) –1 (B) 0

(C) –9 (D) 3

13. The y-intercept of tangent to the curve $y = P''(x)$ at point M whose abscissa is 3/2, equals

(A) –14 (B) 4

(C) 7 (D) 17

Comprehension 6

Let $P(x)$ be a polynomial of degree 4, vanishes at $x = 0$. Given $P(-1) = 55$ and $P(x)$ has relative maximum/relative minimum at $x = 1, 2, 3$.

14. Area of triangle formed by extremum points of $P(x)$, is

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$

(C) $\frac{1}{8}$ (D) 1

15. Which one of the following statement is correct?

(A) $P(x)$ has two relative maximum points and one relative minimum point

(B) Range of $P(x)$ contains 9 negative integers

(C) Sum of real roots of $P(x) = 0$ is 8

(D) $P(x)$ has exactly one inflection point

Comprehension 7

Consider the real valued function $f : R \to R$ defined as $f(x) = x^2e^{-x}$.

16. Which one of the following statement is true?

(A) $f(x)$ has a local maximum at $x = 0$ and a local minimum at $x = 2$

(B) $f(x)$ has a local minimum at $x = 0$ and a local maximum at $x = 2$

(C) $\lim\limits_{x\to\infty} f(x) = 1$

(D) $f(x)$ is an even function

17. Let $g(x) = \displaystyle\int_0^{2e^x} \frac{f'(t)}{1+t^2}dt$, then

(A) $g(x)$ increases on $(-\infty, 0)$ and decreases on $(0, \infty)$

(B) $g(x)$ has a local minimum at $x = 0$

(C) $g(x)$ decreases on $(-\infty, 0)$ and increases on $(0, \infty)$

(D) $g(x)$ has neither maximum nor minimum at $x = 0$

18. Number of solutions of the equation $4x^2\, e^{-x} - 1 = 0$, is

(A) 0 (B) 1

(C) 2 (D) 3

Comprehension 8

Let $h(x) = f(x) - a(f(x))^2 + a(f(x))^3$ for every real number x.

19. $h(x)$ increases if $f(x)$ increases for all real values of x if

(A) $a \in [0,3]$ (B) $a \in [-2,2]$

(C) $[3,\infty)$ (D) None of these

20. $h(x)$ increases as $f(x)$ decreases for all real values of x if

(A) $a \in [0,3]$ (B) $a \in [-2,2]$

(C) $[3,\infty)$ (D) None of these

21. $h(x)$ is non-monotonic function if

(A) $a \in [0,3]$ (B) $a \in [-2,2]$

(C) $[3,\infty)$ (D) $a \in (-\infty,0] \cup [3,\infty)$

Comprehension 9

If function satisfies: $(x - y)\, f(x + y) - (x + y)\, f(x - y) = 2(x^2y - y^3)$ given $f(1) = 2$

22. Interval in which $f(x)$ is increasing is

(A) $(0, 1)$ (B) $(-1, 0)$

(C) $\left(-\frac{1}{2}\infty\right)$ (D) $\left(-\infty,-\frac{1}{2}\right)$

23. Number of critical points which are also points of local maxima and minima of $y = |f(x)|$ is

(A) 3 (B) 2

(C) 1 (D) Does not exists

24. Equation of tangent to the curve at $\left(-\frac{1}{2},\frac{1}{4}\right)$ to $y = |f(x)|$ is

(A) $x + y = 2$ (B) $y = +\frac{1}{4}$

(C) $x = -\frac{1}{2}$ (D) $2x + y = 4$

Comprehension 10

Consider the cubic $f(x) = 8x^3 + 4\alpha x^2 + 2\beta x + \alpha$ where $\alpha, \beta \in R$

25. For $\alpha = 1$, if $y = f(x)$ is strictly increasing $\forall\, x \in R$ then maximum range of values of 'β' is

(A) $\left(-\infty,\frac{1}{3}\right]$ (B) $\left(\frac{1}{3},\infty\right)$

(C) $\left[\frac{1}{3},\infty\right)$ (D) $(-\infty, \infty)$

26. For $\beta = 1$, if $y = f(x)$ is non-monotonic then the sum of all integral values of $\alpha \in [1,100]$ is

(A) 4950 (B) 5047

(C) 5050 (D) 5049

27. If the sum of the base 2 logarithms of the roots of the cubic $f(x) = 0$ is 5 then the value of 'α' is

(A) –256 (B) –128

(C) –64 (D) –8

Comprehension 11

For a polynomial function $y = f(x)$

Points of extrema are obtained at points where

$f'(x) = 0$

$f''(x_1) > 0 \Rightarrow x_1$ is a point of minima

$f''(x_1) < 0 \Rightarrow x_1$ is a point of maxima

Let $f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)\, x + 2$

28. The values of parameter 'a' if $f(x)$ has a negative point of local minimum are

(A) ϕ (B) $\left(-\infty,\frac{58}{14}\right)$

(C) $(-3, 3)$ (D) None of these

29. The values of parameter 'a' if $f(x)$ has a positive point of local maxima are

(A) ϕ (B) $(-\infty, -3) \cup \left(\frac{58}{14},\infty\right)$

(C) $\left(-\infty,\frac{58}{14}\right)$ (D) None of these

30. The values of parameter 'a' if $f(x)$ has points of extrema which are opposite in sign are

(A) ϕ (B) $(-3, 3)$

(C) $\left(-\infty,\frac{58}{14}\right)$ (D) None of these

Comprehension 12

Let $a \in R$ and $f(x) = 2x^3 - 3(a - 3)x^2 + 6ax + (a + 2)$

31. The set of values of 'a' for which f has no point of extrema is

(A) $(1, 9)$ (B) $[1, 9]$

(C) $(-\infty, 0)$ (D) ϕ

32. The set of values of 'a' for which f has exactly one point of local maxima and one point of local minima is
(A) $(-\infty, 1) \cup (9, \infty)$ (B) $(-\infty, 1] \cup [9, \infty)$
(C) $[1, 9]$ (D) $(1, 9)$

33. The set of values of 'a' for which f has a local maximum at some negative real number and a local minimum at some positive real number is
(A) $(1, 9)$ (B) $[1, 9]$
(C) $(-\infty, 1)$ (D) $(-\infty, 0)$

Comprehension 13

If f be a twice differentiable function such that $f''(x) > 0\ \forall\ x \in R$. Let $h(x)$ is defined by $h(x) = f(\sin^2 x) + f(\cos^2 x)$ where $|x| < \frac{\pi}{2}$

34. The number of critical points of $h(x)$ are
(A) 1 (B) 2
(C) 3 (D) more than 3

35. $f'(\sin^2 x) < f'(\cos^2 x)$ for $x \in$
(A) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ (B) $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(C) $\left(-\frac{\pi}{4}, 0\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (D) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

36. $h(x)$ is increasing for $x \in$
(A) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ (B) $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(C) $\left(-\frac{\pi}{4}, 0\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (D) $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(0, \frac{\pi}{4}\right)$

Comprehension 14

If $f''(x) < 0$ (> 0) on an interval (a, b) then the curve $y = f(x)$ on this interval is convex (concave) i.e. it is situated below (above) any of its tangent lines. If $f''(x_0) = 0$ or does not exist but $f'(x_0)$ does exist and the second derivative changes sign when passing through the point x_0 then the point $(x_0, f(x_0))$ is the point of inflexion of the curve $y = f(x)$

37. If $y = x^4 + x^3 - 18x^2 + 24x - 12$, then
(A) $(-2, -24)$ is a point of inflexion
(B) $\left(\frac{3}{2}, -8\frac{1}{16}\right)$ is a point of inflexion
(C) $\left(-2, \frac{3}{2}\right)$ is a point of inflexion
(D) y is convex on $\left(\frac{3}{2}, \infty\right)$

38. If $y = x \sin(\log x)$, then
(A) y has only two points of inflexion
(B) y has only 4 points of inflexion
(C) $\frac{\pi}{4}$ is only point of inflexion
(D) y has infinite no. of points of inflexion

39. $y = x^4 + ax^3 + \frac{3}{2}x^2 + 1$ is concave along the entire number scale then
(A) $|a| \ge 1$ (B) $|a| \le 1$
(C) $|a| \le 2$ (D) $|a| > 2$

Comprehension 15

Let f and g be continuous functions of x satisfying $f(x+y) = f(x) + f(y) - 8xy$ for all $x, y \in R$ and $f'(0) = 8$ and $g(x+y) = g(x) + g(y) + 3xy(x+y)$ for all $x, y \in R$, $g'(0) = -4$

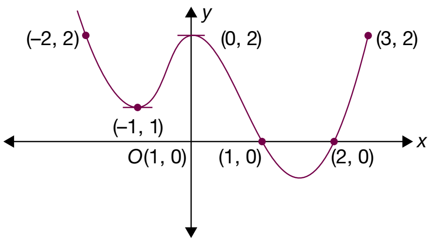
40. $f(x)$ has local maximum at (k, l) then $(k + l)$ is equal to
(A) 5 (B) 8
(C) 15 (D) –12

41. If α, β, γ are the number of local minimum, local maximum and point of inflection of $g(x)$ respectively then $(\alpha + \beta + \gamma)$ is equal to
(A) 0 (B) 1
(C) 2 (D) 3

42. The area bounded by the two curves $f(x)$ and $g(x)$ is
(A) $\frac{28}{3}$ (B) $\frac{25}{3}$
(C) $\frac{18}{5}$ (D) $\frac{25}{7}$

Comprehension 16

In the given figure, graph of $P(x) = x^4 + ax^3 + bx^2 + cx + d$ is given.



43. The minimum number of real roots of equation $(P'(x))^2 + P(x)\,P''(x) = 0$ is equal to
(A) 3 (B) 4
(C) 5 (D) 6

44. If the equation $P(x) + k = 0$ has four distinct real roots $\alpha, \beta, \gamma, \delta$ then $[\alpha] + [\beta] + [\gamma] + [\delta]$ is equal to
[**NOTE:** $[t]$ denotes the greatest integer less than or equal to t]
(A) –1 (B) –2
(C) 0 (D) 1

Comprehension 17

Let $f(x) = e^{(p+1)x} - e^x$ for real number $p > 0$. Answer the following :

45. The value of $x = s_p$ for which $f(x)$ is minimum, is
(A) $-\frac{\ln(p+1)}{p}$ (B) $-\ln(p+1)$
(C) $-\ln p$ (D) $\ln\left(\frac{p+1}{p}\right)$

46. Let $g(t) = \int_t^{t+1} f(x)e^{t-x}dx$. The value of $t = t_p$, for which $g(t)$ is minimum is
(A) $-\ln\left(\frac{e^p - 1}{p}\right)$ (B) $-\frac{1}{p}\ln\left(\frac{e^p - 1}{p}\right)$
(C) $-\frac{1}{p}\ln\left(\frac{(p+1)(e^p - 1)}{p}\right)$ (D) $-\ln((p+1)(e^p - 1))$

47. Use the fact that $1+\frac{p}{2} \le \frac{e^p-1}{p} \le 1+\frac{p}{2}+p^2\ (0<p\le 1)$, the value of $\lim\limits_{p\to 0^+}(s_p-t_p)$ is

(A) 0
(B) $\frac{1}{2}$
(C) 1
(D) non existent

Comprehension 18

$$\text{Let } f(x)=\begin{cases} 2\ln(-x-1); & -e-1\le x\le -2\\ \frac{1}{2}(4-x^2)\ ; & -2<x<0\\ 0\ ; & x=0\\ \frac{1}{2}(x^2-4)\ ; & 0<x<2\\ 2\ln(x-1)\ ; & 2\le x\le e+1\end{cases}$$

Also,

$$g(x)=\begin{cases}\min\{f(t):-e-1\le t\le x\}, & -e-1\le x<0\\ \min\{f(t):0\le t\le x\}, & 0\le x\le e+1\end{cases}$$

48. Which one of the following statement does not hold good?
(A) Range of $g(x)$ is [0, 2]
(B) $g(x)$ is non-monotonic in [–2, 2]
(C) $g(x)$ is a continuous function
(D) $g(x)$ is an odd function

49. If $x = \alpha$ is the point of non-differentiability of $g(x)$ in $(-e-1, e+1]$, then the values of α is
(A) –2, 1
(B) –2, –1
(C) –2, 2
(D) –2, 0

50. If the equation $g(x) = k$ has exactly two distinct solutions in $[-e-1, e+1]$ then the sum of all possible integral values of k is
(A) 0
(B) 1
(C) 2
(D) 3

Comprehension 19

Let $f(x)$ be a cubic polynomial which has local maximum at $x = -1$ and $f'(x)$ has a local minimum at $x = 1$. If $f(-1) = 10$ and $f(0) = 5$.

51. The distance between two horizontal tangents is
(A) 12
(B) 22
(C) 32
(D) 10

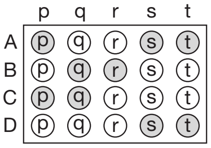
52. Sum of all non-negative integers which lies between the roots of $f(x) = 0$
(A) 9
(B) 10
(C) 12
(D) 13

53. If x_1, x_2, x_3 are the real roots of $f(x) = 0$ then $[x_1]\,[x_2]^2+[x_3]^3$ is equal to
(where [] denotes the greatest integer function)
(A) 20
(B) 21
(C) 22
(D) 25

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. Match the following:

COLUMN-I	COLUMN-II
(A) Integral values of 'b' for which $f(x) = \int_0^x (t^4+(b+1)t^3+(b+2)t^2+(b+1)t+1)\,dt$ Always increases	(p) –1
(B) If $f(x) = x - 2\sin x,\ 0\le x\le 2\pi$ is increasing in $[a\pi, b\pi]$, then $a + b$ is	(q) 2
(C) If function $f(x) = x^3 + ax^2 + bx + c$, where $a, b \in N$ is monotonic increasing, then $\frac{a^2}{b}$ can be	(r) 1
(D) Let $f(x)=\frac{a\sin x+2\cos x}{\sin x+\cos x}$ is monotonically increasing, then integral value of 'a' can be	(s) 0
	(t) 3

(Continued)

2. The function $f(x) = \sqrt{(ax^3 + bx^2 + cx + d)}$ has its non-zero local minimum and maximum values at $x = -2$ and $x = 2$, respectively. If a is a root of $x^2 - x - 6 = 0$.

COLUMN-I	COLUMN-II
(A) The value of a	(p) 0
(B) The value of b	(q) 24
(C) The value of c	(r) 34
(D) the value of d	(s) –2
	(t) 36

3. For the function $f(x) = ax^2 - b\lvert x\rvert$

COLUMN-I	COLUMN-II
(A) $f(x)$ has a local maximum at $x = 0$	(p) $a > 0, b > 0$
(B) $f(x)$ has a local minimum at $x = 0$	(q) $a > 0, b < 0$
(C) $f(x)$ has a local extremum at $x = \frac{b}{2a}$	(r) $a < 0, b < 0$
(D) $f(x)$ is not differentiable at $x = 0$	(s) $a < 0, b > 0$
	(t) cannot determine

4. Match the following:

COLUMN-I	COLUMN-II
(A) The maximum value of the function $f(x) = \dfrac{64\sin^3 x \cdot \cos x}{1 + \tan^2 x}$ is equal to	(p) 10
(B) Let $f : (0, 2\pi) \rightarrow R$ be defined as $f(x) = \sin x \cdot e^{\sin 2x} + \int_0^x e^{\sin 2t}(\sin t - \cos t)dt$. Then the number of points of local minima of $f(x)$ is equal to	(q) 8
(C) If the curves $C_1 : y = \frac{\ln x}{x}$ and $C_2 : y = \lambda x^2$ (where is λ constant) touches each other, then the value of λ is less than	(r) 6
(D) The value of $\lim\limits_{x\to 0} \frac{1}{x}\int_{-x}^{x}(t\sin(10t) + 10t + 5)dt$ is equal to	(s) 3
	(t) 2

5. Match the following:

COLUMN-I	COLUMN-II
(A) The set of values of x satisfying the inequality $1 + \log_2(\sin x) + \log_2(\sin 3x) \geq 0$ in $(0, \pi)$ is/are	(p) $\left(\frac{\pi}{6}, \frac{\pi}{4}\right)$
(B) The set of values of x in $(0, \pi)$ in which $f(x) = \sqrt{3}\sin x - \cos x$ is decreasing is/are	(q) $\left(\frac{\pi}{4}, \frac{\pi}{3}\right)$
(C) Any point of extrema of the function $g(x) = 6\sin x - 4\sin^3 x$ lies in interval	(r) $\left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$
(D) Let $\int_x^{3x} f(t)dt = \int_5^x f(t)dt \;\forall\; x > 0$ and $f(x)$ is a differentiable function such that $f(3) = 3$, then according to LMVT there exists some $c \in (27, 81)$ for which $\lvert 243 f'(c)\rvert$ lies in.	(s) $\left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$
	(t) $\left(\frac{2\pi}{3}, \pi\right)$

6. Match the following:

COLUMN-I	COLUMN-II
(A) Let $f(x) = \begin{cases} x^{3/5} & , \text{ if } x \leq 1 \\ -(x-2)^3 & , \text{ if } x > 1 \end{cases}$ then the number of critical points on the graph of the function is	(p) 5
(B) Number of real solution of the equation, $\log_2{}^2 x + (x - 1)\log_2 x = 6 - 2x$, is	(q) 4
(C) The number of values of c such that the straight line $3x + 4y = c$ touches the curve $\frac{x^4}{2} = x + y$ is	(r) 3
(D) If $f(x) = \int_x^{x^2}(t - 1)\,dt$, $1 \leq x \leq 2$, then global maximum value of $f(x)$ is	(s) 2
	(t) 1

7. Match the following:

COLUMN-I	COLUMN-II
(A) Let $f(x)=x^4-8x^3+22x^2-24x+21$, then	(p) $f(x)$ increases, if $x\in(2,3)$
(B) Let $f(x)=2\ln(x-2)-x^2+4x+1$, then	(q) $f(x)$ decreases, if $x\in(0,1)\cup(2,\infty)$
(C) Let $f(x)=\frac{\lvert x-1\rvert}{x^2}$, then	(r) $f(x)$ increases, if $x\in(1,2)\cup(3,\infty)$
(D) The function $f(x)=-\left(x^3-\frac{15}{2}x^2+18x+2\right)$	(s) $f(x)$ decreases, if $x\in(1,2)\cup(3,\infty)$
	(t) $f(x)$ decreases, if $x\in(-\infty,1)\cup(2,3)$

8. Match the following:

COLUMN-I	COLUMN-II
(A) Interval in which $g(x) = 2\sin x + \cos 2x$ is increasing	(p) $\left(-\frac{\pi}{8},\frac{\pi}{8}\right)$
(B) Interval in which at least one of the point of local maximum of $\cos^2 x + \sin x$ lies	(q) $\left(\frac{\pi}{12},\frac{\pi}{8}\right)$
(C) Interval in which $\tan^{-1}(\sin x + \cos x)$ is increasing	(r) $\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$
(D) Interval in which $x^{99} + \sin x - 1$ is strictly increasing	(s) $\left(0,\frac{\pi}{3}\right)$
	(t) $\left(0,\frac{\pi}{2}\right)$

9. Let f: $R^+ \to R$ be a strictly increasing function such that $f(x)>-\frac{1}{x}\ \forall x>0$ and $f(x)f\left(f(x)+\frac{1}{x}\right)=1\ \forall x>0$. Now match the following

COLUMN-I	COLUMN-II
(A) $f(1)$ =	(p) $\frac{1+\sqrt{5}}{2}$
(B) maximum value of $f(x)$ in [1, 2]	(q) $\frac{1-\sqrt{5}}{2}$
(C) minimum value of $f(x)$ in [1, 2]	(r) $\frac{1-\sqrt{5}}{4}$
(D) $\int_e^{e^2} f(x)dx$	(s) $\frac{1+\sqrt{5}}{4}$

(Continued)

10. Match the following:

COLUMN-I	COLUMN-II
(A) $f(x) = \begin{cases} 4x-x^3+\log_e(a^2-3a+3), & \text{if } 0\le x\le 3 \\ x-18, & \text{if } x\ge 3 \end{cases}$ Then, the complete set of values of 'a' such that $f(x)$ has a local maxima at x = 3 lie in the interval	(p) $(-\infty, -2) \cup (2, \infty)$
(B) The equation $x + \cos x = a$ has exactly a positive root, then, complete set of values of a lie in the interval	(q) [1, 2]
(C) If $f(x) = \begin{cases} 3+\lvert x-k\rvert, & x\le k \\ a^2-2+\frac{\sin(x-k)}{x-k}, & x>k \end{cases}$ has minimum at $x = k$, then, the complete set of values of 'a' lie in the interval	(r) (–20, –16)
(D) If $f(x) = x^3 - 9x^2 + 24x + a$ has three real and distinct roots, then, possible values of a lie in the interval	(s) $(1, \infty)$

11. Match the following:

COLUMN-I	COLUMN-II
(A) Number of values of x lying in $\left(0,\frac{\pi}{2}\right)$ at which $f(x) = \ell n\ (\sin x)$ is not monotonic is	(p) 0
(B) If the greatest interval of decrease of the function $f(x) = x^3 - 3x + 2$ is $[a, b]$, then $a + b$ =	(q) 2
(C) Let $f(x)=\frac{x^2+2}{[x]}$, $1\le x\le 3$ where [.] greatest integer function then least value of $f(x)$ is	(r) –3
(D) Set of all possible values of a such that $f(x) = e^{2x} - (a + 1)e^x + 2x$ is monotonically increasing for all $x \in R$ is $(-\infty, a]$, then a =	(s) 3
	(t) –2

12. Match the following:

COLUMN-I	COLUMN-II
(A) The function $f(x) = (x-1)^2 + (x-2)^2 + (x-3)^2$ has a minimum for $x =$	(p) 0
(B) The least value of the function $f(x) = 2.3^{3x} - 3^{2x}.4 + 2.3^x$ in $[-1, 1]$ is	(q) 2
(C) Let $f(x) = \frac{4}{3}x^3 - 4x, 0 \le x \le 2$, then the global minimum value of the function is	(r) $\frac{8}{27}$
(D) Let $f(x) = 6 - 12x + 9x^2 - 2x^3$, $x \in [1, 4]$, then absolute maximum value of $f(x)$ is	(s) $\frac{-8}{3}$

(*Continued*)

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. If α and β are the number of local maximum and local minimum of the function $f(x) = 2x^3 - 3(\lambda+2)x^2 + 2\lambda x + 5$ for $\lambda \in R$ then $\alpha + \beta =$ ______

2. If $f(x) = \begin{cases} 2x + k & \text{for } x \le 0 \\ 5\cos x & \text{for } x > 0 \end{cases}$ has a local maximum at $x = 0$ then the least value of k is___

3. Let $f(x) = \dfrac{x^2 - 3x + 2}{x^2 + 2x + 1}$ for $x \ne -1$ and $f(-1) = 0$. If m is the number points at which $f(x)$ has a local minimum then $m =$ _____

4. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. If the range of $m(b)$ is $(a, b]$, then $b - a =$ _____

5. If $f(x) = kx^3 - 9x^2 + 9x + 3$ is strictly increasing on R, then the number of integral values of k, $k \in [-5, 100]$ is $100 - n$ where $n =$

6. If $f(x)$ is differentiable and strictly increasing function, $f''(x) > 0$, then the value of $\lim\limits_{x \to 0} \dfrac{f(x) - f(x^2)}{f(x) - f(0)}$ is equal to

7. Range of values of 'b' for which the function $f(x) = \int_0^x (bt^2 + b + \sin t)\,dt \;\forall x \in R$ is either increasing or decreasing is $(-\infty, a_1] \cup [a_2, \infty)$ then $a_1^4 + a_2^4 + 3a_2$ is equal to

8. If the function $k(x) = \log_e \left(\dfrac{x^2 - x + 1}{x^2 + x + 1}\right)$ is strictly decreasing in $x \in \left(\dfrac{-t}{7}, \dfrac{t}{7}\right)$, then find the greatest integral value of t.

9. A polynomial function $P(x)$ of degree 5 with leading coefficient one, increases in the interval $(-\infty, 1)$ and $(3, \infty)$ and decreases in the interval $(1, 3)$. Given that $P'(2) = 0$. Find the value $P'(6)$.

10. Let $f(x) = 2x^3 + 3(1 - 3a)x^2 + 6(a^2 - a)x + b$ where $a, b \in R$. find the smallest integral values of 'a' for which $f(x)$ has positive point of local maximum

11. The greatest value of $f(x) = \left(3 - \sqrt{4 - x^2}\right)^2 + \left(1 + \sqrt{4 - x^2}\right)^3$ is A, then $\dfrac{A}{7} =$

12. Vertices of a variable acute angle triangle ABC lies on a fixed circle. If a, b, c and A, B, C are length of sides and angles of triangle ABC and x_1, x_2, x_3 are distances of orthocentre from A, B, C respectively then find maximum value of $\sqrt{3}\left(\dfrac{dx_1}{da} + \dfrac{dx_2}{db} + \dfrac{dx_3}{dc}\right) + 11$.

13. If M is absolute minimum value of $f(x) = \cos\pi x + 10x + 3x^2 + x^3$, $-2 \le x \le 3$, find the value of $\dfrac{-M}{5} =$ ______

14. If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at P and q, respectively such that $p^2 = q$, then a equal to ________

15. Let $f(x) = \begin{cases} |x^2 - 3x| + a, & 0 \le x < \frac{3}{2} \\ -2x + 3, & x \ge \frac{3}{2} \end{cases}$. If $f(x)$ has a local maxima at $x = \dfrac{3}{2}$, and the greatest value of 'a' is k, then $|4k|$ is ________

16. $f(x) = \dfrac{1}{1 + |x|} + \dfrac{1}{1 + |x - 1|}$ Let x_1, x_2 are points where $f(x)$ attains local minimum and global maximum respectively. Let $k = f(x_1) + f(x_2)$ then $6k - 9$

17. $f(x) = \begin{cases} \left(\sqrt{2} + \sin\dfrac{1}{x}\right)e^{\frac{-1}{|x|}} & ; \text{ if } x \ne 0 \\ 0 & ; \text{ if } x = 0 \end{cases}$

    Number of points where $f(x)$ has local extrema when $x \ne 0$ be $n_1 \cdot n_2$ be the value of global minimum of $f(x)$ then $n_1 + n_2 =$

18. Let $f(x) = \sqrt{x - 1} + \sqrt{2 - x}$ and $g(x) = x^2 - bx + c$ are two given functions such that $f(x)$ and $g(x)$ attain their maximum and minimum values respectively for same value of x then the value of b is

19. Let $f(x)=\left(\sin^{-1}\left(\frac{x}{2}\right)\right)^3+\left(\cos^{-1}\left(\frac{x}{2}\right)\right)^3$. If maximum value of $f(x)$ is at $x = a$ and minimum value of $f(x)$ is at $x = b$, then $a^2 + b^2$ is

20. If $f:[1,10]\to[1,10]$ is a non-decreasing function and $g:[1,10]\to[1,10]$ is non-increasing function and $h(x)=fog(x)$ with $h(1)=1$, then $h(7)=$

21. A L cm long wire is bent to form a triangle with one of it's angle is 60°. The sides of the triangle for which area is largest are a, b, c then $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=$

22. Let $f(x)=\begin{cases}(x+2)^3, & -3<x\le -1\\ x^{2/3}, & -1<x<2\end{cases}$ and $g(x)=\int_{-3}^{x} f(t)dt; -3<x<2$.

Find the number of extremum points of $g'(x)$.

23. Find the sum of all possible integral values of α in [1, 100] for which the function $f(x) = 2x^3 - 3(2 + \alpha)x^2 + 12\alpha x$ has exactly one local maximum and one local minimum.

24. Let $f(x) = \begin{cases}(x+1)^3, & -2<x\le -1\\ x^{\frac{2}{3}}-1, & -1<x\le 1\\ -2(x-1)^2, & 1<x<2\end{cases}$ then find the total number of points at which f attains either a local maximum or a local minimum.

25. Let $f(x)=30-2x-x^3$, then find the number of positive integral values of x which satisfies $f(f(f(x)))>f(f(-x))$

26. Let $P(x)$ be a polynomial of degree 4 having a relative maximum at $x = 2$ and $\lim\limits_{x\to 0}\left(3-\frac{P(x)}{x}\right)=27$. If $P(1) = -9$ and $P''(x)$ has a local minimum at $x = 2$ then find absolute minimum value of $y = P'(x)$ on the set $A = \{x \mid x^2 + 12 \le 7x \mid\}$.

27. Let $f(x) = \begin{cases}(x+2)^3, & -3<x\le -1\\ x^{\frac{2}{3}}, & -1<x\le 2\end{cases}$ and

$g(x) = \int_{-3}^{x} f(t)dt$, $-3 < x < 2$. Find the number of extremum points of $g'(x)$.

28. A polynomial $y = f(x)$ of degree 4 increases in the interval $(-\infty, 1) \cup (2, 3)$ and decreases in the interval $(1, 2) \cup (3, \infty)$ and satisfy $f(0) = 1$ and $f'(0) = 6$. Find the value of $f(2)$.

29. If $f(g(x)) = 4 - x + 5x^{\frac{1}{3}} - x^{\frac{2}{3}}$ and $g(x) = 2 - x^{\frac{1}{3}}$ then find the local maximum value of $f(x)$.

30. Let $F(x) = \begin{cases}-2x+\log_{\frac{1}{2}}(k^2-6k+8), & -2\le x<-1\\ x^3+3x^2+4x+1, & -1\le x\le 3\end{cases}$ find the sum of all possible positive integer(s) in the range of k such that $F(x)$ has the smallest value at $x = -1$.

31. Let $f(x) = \frac{\pi}{2} + \left|\operatorname{sgn}\left(\tan^{-1}\left(\frac{x}{1+x^2}\right)\right)\right|\tan^{-1}x$, where sg$n(y)$ denotes signum function of y, and $g(x)$ is the inverse of $f(x)$. If the complete set of values of k for which the equation $2g(x) + k(\pi - x) = 0$ has three distinct solutions is (a, ∞) then find 'a'.

32. The number of integral values of 'a' for which the function $f(x) = \begin{bmatrix}x^3-x^2+10, & x\le 1\\ -2x+\log_2(a^2-2), & x>1\end{bmatrix}$ has the greatest value at $x = 1$, is

33. Let $f(x) = 2x^3 - 3(2 + p)x^2 + 12px + \ln(16 - p^2)$. If $f(x)$ has exactly one local maxima and one local minima, then the number of integral values of p is

34. Let $f(x) = \begin{cases}x(x-1)(x-2); & 0\le x<n\\ \sin(\pi x); & n\le x\le 2n\end{cases}$ If $n \in N$ then find the least value of n for which $f(x)$ has more points of minima than maxima in $[0, 2n]$.

35. Consider a polynomial $P(x)$ of the least degree that has a maximum equal to 6 at $x = 1$, and a minimum equal to 2 at $x = 3$. Compute the value of $P(2) + P'(0)$.

36. Let $f(x)$ be a cubic polynomial with leading coefficient unity and all its roots real and positive with $f(0) = -64$. Find the maximum value of $|f(-1)|$.

37. Let $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + x + 2$ and $g'(x) = (x^2 - 9)(x^2 - 4x + 3)(x^2 - 3x + 2)(x^2 - 2x - 3)$. If n_1, n_2 and n_3 denote number of points of local minima, number of points of local maxima and number of points of inflection of the function $f(g(x))$ then find the value of $(n_1 + n_2 + n_3)$.

38. Let $f : R \to R^+$ be a differentiable function satisfying $f'(x) = 2f(x)\ \forall\ x \in R$. Also $f(0) = 1$ and $g(x) = f(x) \cdot \cos^2 x$ where $x \in [-\pi, \pi]$. If n_1 represent number of points of local maxima of $g(x)$ and n_2 is the number of points of local minima of $g(x)$ and n_3 is the number of points where $g(x)$ attains its global minimum value, then find the value of $(n_1 + n_2 + n_3)$

39. Find the smallest positive integral value of α for which the function $f(x) = \alpha\cos 2x - 4\cos^3 x$ has exactly one critical point in $x \in (0, \pi)$.

40. Determine the total number of local maxima and local minima of the function

$$f(x) = \left(\frac{2-x}{\pi}\right)\cos\pi(x+3) + \frac{1}{\pi^2}\sin\pi(x+3),\ \text{where } 0<x<4.$$

41. Find the number of point(s) of maxima and minima (whichever exist) of the function f defined by

$$f(x) = xe^{-x} - (1-x)^{\frac{2}{3}} + 2x - x^2 + \int_0^x 2^t \log_{e^{-t}} t\ dt$$

42. Let $f(x)$ be a polynomial; of degree 4 such that $f(1) = 7$ and attains its local minimum value of $f(x)$ is equal to $\frac{p}{q}$ (where p, q are coprime) then find the value of $(p + q)$.

43. If the function $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)\,x - 1$ has a positive point of maxima then find the largest natural value of 'a'.

44. Let $y = p(x)$ be a polynomial function of degree ≤ 5 with leading coefficient unity and satisfying $p(1-x) = p(1+x)\ \forall\ x \in R$. If $y = p(x)$ has global minimum value 4 at $x = 2$ and local maximum value is 5, then find the value of $p(6)$.

45. If all the real values of m for which the function

$$f(x) = \frac{x^3}{3} - (m-3)\frac{x^2}{2} + mx - 2013 \text{ is strictly increasing in}$$

$x \in [0, \infty)$ is $[0, k]$, then find the value of k.

46. If $f(x) = 1 - 3\sqrt{e}x - ex^3$ then find least integral solution satisfying the inequality

$$f\left(\frac{e^{1/x}}{x-2}\right) < f\left(\frac{\left(e^{\frac{1}{10}}\right)}{8}\operatorname{sgn}(1+\operatorname{sgn}(|e^x-1|))\right) \text{ where } \operatorname{sgn}(y)$$

denotes the signum function of y

47. Consider $f(x) = \left\{\frac{x - \sin x}{5}\right\}$. If the number of points in $(0, 20\pi)$ where $f(x)$ is non-derivable are the different values of c of LMVT for the twice differentiable function $g(x)$ i.e., $g'(c) = \frac{g(b)-g(a)}{b-a}$ for some $c \in (a, b)$ then the minimum number of points where $g''(x)$ vanishes is n. Find the value of $\left[\frac{n}{2}\right]$ where $\{y\}$ and $[y]$ denotes fractional part and greatest integer function of y respectively.

48. Let $a_n (n \ge 1)$ be the value of x for which $\int_x^{2x} e^{-t^n}\,dt$ $(x > 0)$ is maximum. If $L = \lim_{n\to\infty} \ln(a_n)$ then find the value of e^{-L}.

49. If the exhaustive range of values of k for which the function $f(x) = kx + (k-4)\ln x - 36\ln(1-x)$ is strictly increasing is $[\alpha, \infty)$, then find α.

50. Let $f(x) = \cos^3 x - a\cos^2 x$, $0 < x < \pi$. Find number of integral values of a so that $f(x)$ has exactly one minimum and exactly one maximum.

51. Find the local maximum value of the function

$$f(x) = (x^2 - 2x)\ln x - \frac{3}{2}x^2 + 4x + \frac{5}{2}.$$

52. If $\Delta(x) = \begin{vmatrix} 0 & 2x-2 & 2x+8 \\ x-1 & 4 & x^2+7 \\ 0 & 0 & x+4 \end{vmatrix}$ and $f(x) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}c_{ij}$ where a_{ij} is the element of i^{th} row and j^{th} column in $\Delta(x)$ and c_{ij} is the cofactor $a_{ij}\ \forall\ I$ and j then find the greatest value of $f(x)$ when $x \in [-3, 18]$.

53. Consider a function $f : R \to R$ defined by

$$f(x) = 2x^3 + bx^2 + 4x + \int_2^x (2\sin^2 t + 3\cos^2 t)\,dt \quad \forall\ x \in R. \text{ If } f(x)$$

is strictly increasing function for every real x, then find the number of integral values of b.

54. Let $f_N(x)$ be a polynomial function which satisfies $f'_N(x) = \prod_{n=1}^{N}(x-n)^n$. If n_1 is sum of all values of x for which $f_{10}(x)$ attains maxima and n_2 denotes the sum of values x for which $f_{10}(x)$ attains minima while n_3 stands for sum of value of x at which inflexion point occurs then find the value of $\frac{6(n_1+n_2)}{n_3}$.

ARCHIVE: JEE MAIN

1. **[JEE Mains 2021]**

The function $f(x) = \frac{4x^3 - 3x^2}{6} - 2\sin x + (2x-1)\cos x$:

(A) increases in $\left[\frac{1}{2}, \infty\right)$ (B) decreases in $\left[\frac{1}{2}, \infty\right)$

(C) decreases in $\left(-\infty, \frac{1}{2}\right]$ (D) increases in $\left(-\infty, \frac{1}{2}\right]$

2. **[JEE Mains 2021]**

Let $f : R \to R$ be defined as

$$f(x) = \begin{cases} -55x, & \text{if } x < -5 \\ 2x^3 - 3x^2 - 120x, & \text{if } -5 \le x \le 4\,. \\ 2x^3 - 3x^2 - 36x - 336, & \text{if } x > 4 \end{cases}$$

Let $A = \{x \in R : f \text{ is increasing}\}$. Then A is equal to

(A) $(-\infty, -5) \cup (-4, \infty)$ (B) $(-5, -4) \cup (4, \infty)$

(C) $(-\infty, -5) \cup (4, \infty)$ (D) $(-5, \infty)$

3. **[JEE Mains 2021]**

Let $f(x)$ be a polynomial of degree 6 in x, in which the coefficient of x^6 is unity and it has extrema at $x = -1$ and $x = 1$. If $\lim_{x\to 0}\frac{f(x)}{x^3} = 1$, then $5f(2)$ is equal to ______.

4. **[JEE Mains 2021]**

The number of distinct real roots of the equation $3x^4 + 4x^3 - 12x^2 + 4 = 0$ is ________.

5. **[JEE Mains 2021]**

Let M and m respectively be the maximum and minimum values of the function $f(x) = \tan^{-1}(\sin x + \cos x)$ in $\left[0, \frac{\pi}{2}\right]$, Then the value of $\tan(M - m)$ is equal to:

(A) $2 + \sqrt{3}$ (B) $2 - \sqrt{3}$

(C) $3 + 2\sqrt{2}$ (D) $3 - 2\sqrt{2}$

6. **[JEE Mains 2021]**
A box open from top is made from a rectangular sheet of dimension $a \times b$ by cutting squares each of side x from each of the four corners and folding up the flaps. If the volume of the box is maximum, then x is equal to:

(A) $\frac{a+b-\sqrt{a^2+b^2-ab}}{12}$

(B) $\frac{a+b-\sqrt{a^2+b^2+ab}}{6}$

(C) $\frac{a+b-\sqrt{a^2+b^2-ab}}{6}$

(D) $\frac{a+b+\sqrt{a^2+b^2-ab}}{6}$

7. **[JEE Mains 2021]**
The number of real roots of the equation $e^{4x}+2e^{3x}-e^x-6=0$ is:

(A) 2
(B) 4
(C) 1
(D) 0

8. **[JEE Mains 2021]**
If 'R' is the least value of 'a' such that the function $f(x)=x^2+ax+1$ is increasing on [1, 2] and 'S' is the greatest value of 'a' such that the function $f(x)=x^2+ax+1$ is decreasing on [1, 2], then the value of $|R-S|$ is ________.

9. **[JEE Mains 2021]**
Let $f(x)$ be a cubic polynomial with $f(1) = -10$, $f(-1) = 6$, and has a local minima at $x = 1$, and $f'(x)$ has a local minima at $x = -1$. Then $f(3)$ is equal to ________.

10. **[JEE Mains 2021]**
The function $f(x)=x^3-6x^2+ax+b$ is such that $f(2)=f(4)=0$. Consider two statements.

(S1) there exists $x_1, x_2 \in (2, 4)$, $x_1 < x_2$, such that $f'(x_1) = -1$ and $f'(x_2) = 0$.

(S2) there exists $x_3, x_4 \in (2, 4)$, $x_3 < x_4$, such that f is decreasing in $(2, x_4)$, increasing in $(x_4, 4)$ and $2f'(x_3) = \sqrt{3}\, f(x_4)$.

Then

(A) both (S1) and (S2) are true
(B) (S1) is false and (S2) is true
(C) both (S1) and (S2) are false
(D) (S1) is true and (S2) is false

11. **[JEE Mains 2020]**
Let $f(x)$ be a polynomial of degree 5 such that $x = \pm 1$ are its critical points. If $\lim_{x\to 0}\left(2+\frac{f(x)}{x^3}\right)=4$, then which one of the following is not true?

(A) $x = 1$ is a point of maxima and $x = -1$ is a point of minimum of f.
(B) $f(1) - 4f(-1) = 4$.
(C) f is an odd function.
(D) $x = 1$ is a point of maxima and $x = -1$ is a point of maxima of f.

12. **[JEE Mains 2021]**
Let $f(x) = x\cos^{-1}(-\sin|x|)$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then which of the following is true?

(A) f' is decreasing in $\left(-\frac{\pi}{2}, 0\right)$ and increasing in $\left(0, \frac{\pi}{2}\right)$
(B) $f'(0) = -\frac{\pi}{2}$
(C) f' is increasing in $\left(-\frac{\pi}{2}, 0\right)$ and decreasing in $\left(0, \frac{\pi}{2}\right)$
(D) f is not differentiable at $x = 0$

13. **[JEE Mains 2020]**
Let $f(x)$ be a polynomial of degree 3 such that $f(-1) = 10$, $f(1) = -6$, $f(x)$ has a critical point at $x = -1$ and $f'(x)$ has a critical point at $x = 1$. Then $f(x)$ has a local minima at $x =$ ________.

14. **[JEE Mains 2020]**
If $p(x)$ be a polynomial of degree three that has a local maximum value 8 at $x = 1$ and a local minimum value 4 at $x = 2$; then $p(0)$ is equal to:

(A) 12
(B) 24
(C) 6
(D) 12

15. **[JEE Mains 2020]**
Let $f : (-1, \infty) \to R$ be defined by $f(0) = 1$ and $f(x)=\frac{1}{x}\log_e(1+x)$, $x \neq 0$. Then the function f:

(A) decreases in $(-1, \infty)$
(B) decreases in $(-1, 0)$ and increases in $(0, \infty)$
(C) increases in $(-1, \infty)$
(D) increases in $(-1, 0)$ and decreases in $(0, \infty)$

16. **[JEE Mains 2020]**
Suppose $f(x)$ is a polynomial of degree four, having critical points at -1, 0, 1. If $T = \{x \in R \mid f(x) = f(0)\}$, then the sum of squares of all the elements of T is:

(A) 6
(B) 8
(C) 4
(D) 2

17. **[JEE Mains 2020]**
The function, $f(x) = (3x - 7)x^{2/3}$, $x \in R$, is increasing for all x lying in:

(A) $(-\infty, 0) \cup \left(\frac{3}{7}, \infty\right)$
(B) $(-\infty, 0) \cup \left(\frac{14}{15}, \infty\right)$
(C) $\left(-\infty, \frac{14}{15}\right)$
(D) $\left(-\infty, -\frac{14}{15}\right) \cup (0, \infty)$

18. **[JEE Mains 2019]**
Let $f(x) = e^x - x$ and $g(x) = x^2 - x$, $\forall\, x \in R$. Then the set of all $x \in R$, where the function $h(x) = (fog)(x)$ is increasing, is

(A) $[0, \infty)$
(B) $\left[-1, \frac{-1}{2}\right] \cup \left[\frac{1}{2}, \infty\right)$
(C) $\left[\frac{-1}{2}, 0\right] \cup [1, \infty]$
(D) $\left[0, \frac{1}{2}\right] \cup [1, \infty)$

19. **[JEE Mains 2019]**
If m is the minimum value of k for which the function $f(x) = x\sqrt{kx - x^2}$ is increasing in the interval [0, 3] and M is the maximum value of f in [0, 3] when $k = m$, then the ordered pair (m, M) is equal to:

(A) $(5, 3\sqrt{6})$ (B) $(4, 3\sqrt{3})$
(C) $(4, 3\sqrt{2})$ (D) $(3, 3\sqrt{3})$

20. [JEE Mains 2019]
Let $f(x) = 5 - |x-2|$ and $g(x) = |x+1|$, $x \in R$. If $f(x)$ attains maximum value at α and $g(x)$ attains minimum value at β, then $\lim\limits_{x \to -\alpha\beta} \dfrac{(x-1)(x^2-5x+6)}{x^2-6x+8}$ is equal to:

(A) $\frac{1}{2}$ (B) $-\frac{1}{2}$
(C) $\frac{3}{2}$ (D) $-\frac{3}{2}$

21. [JEE Mains 2018]
Let $f(x) = x^2 + \dfrac{1}{x^2}$ and $g(x) = x - \dfrac{1}{x}$, $x \in R - \{-1, 0, 1\}$. If $h(x) = \dfrac{f(x)}{g(x)}$. Then the local minimum value of $h(x)$ is:

(A) -3 (B) $-2\sqrt{2}$
(C) $2\sqrt{2}$ (D) 3

22. [JEE Mains 2017]
Twenty metres of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum are (in sq. m) of the flower-bed is

(A) 30 (B) 12.5
(C) 10 (D) 25

23. [JEE Mains 2016]
A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then:

(A) $2x = r$ (B) $2x = (\pi + 4)r$
(C) $(4 - \pi)x = \pi r$ (D) $x = 2r$

24. [JEE Mains 2015]
Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim\limits_{x \to 0}\left[1 + \dfrac{f(x)}{x^2}\right] = 3$, then $f(2)$ is equal to:

(A) 0 (B) 4
(C) -8 (D) -4

25. [JEE Mains 2014]
If $x = -1$ and $x = 2$ are extreme points of
$f(x) = \alpha \log|x| + \beta x^2 + x$ then:

(A) $\alpha = -6, \beta = \frac{1}{2}$ (B) $\alpha = -6, \beta = -\frac{1}{2}$
(C) $\alpha = 2, \beta = -\frac{1}{2}$ (D) $\alpha = 2, \beta = \frac{1}{2}$

ARCHIVE: JEE ADVANCED

Multiple Correct Choice Type Problems

1. [JEE ADV 2021]
Let $f : R \to R$ be defined by $f(x) = \dfrac{x^2 - 3x - 6}{x^2 + 2x + 4}$. Then which of the following statements is (are) **TRUE**?

(A) f is decreasing in the interval $(-2, -1)$
(B) f is increasing in the interval $(1, 2)$
(C) f is onto
(D) Range of f is $\left[-\frac{3}{2}, 2\right]$

2. [JEE ADV 2020]
Consider all rectangles lying in the region $\left\{(x, y) \in R \times R : 0 \le x \le \dfrac{\pi}{2} \text{ and } 0 \le y \le 2\sin(2x)\right\}$ and having one side on the x-axis. The area of the rectangle which has the maximum perimeter among all such rectangles, is

(A) $\frac{3\pi}{2}$ (B) π
(C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi\sqrt{3}}{2}$

3. [JEE ADV 2020]
Let b be a non-zero real number. Suppose $f : R \to R$ is a differentiable function such that $f(0) = 1$ If the derivative f' of f satisfies the equation $f'(x) = \dfrac{f(x)}{b^2 + x^2}$ for all $x \in R$, then which of the following statements is/are TRUE?

(A) If $b > 0$, then f is an increasing function
(B) If $b < 0$, then f is an decreasing function
(C) $f(x)\,f(-x) = 1$ for all $x \in R$
(D) $f(x) - f(-x) = 0$ for all $x \in R$

4. [JEE ADV 2019]
Let $f : R \to R$ be given by

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1 & ; \quad x < 0; \\ x^2 - x + 1 & ; \quad x \le x < 1 \\ \dfrac{2}{3}x^3 - 4x^2 + 7x - \dfrac{8}{3} & ; \quad 1 \le x < 3 \\ (x-2)\ln(x-2) - x + \dfrac{10}{3} & ; \quad x \ge 3 \end{cases}$$

Then which of the following option is/are correct?

(A) f is increasing on $(-\infty, 0)$
(B) f' has a local maximum at $x = 1$
(C) f is onto
(D) f' is NOT differentiable at $x = 1$

5. [JEE ADV 2019]
Let $f(x) = \dfrac{\sin \pi x}{x^2}$, $x > 0$
Let $x_1 < x_2 < x_3 < \ldots. < x_n < \ldots$ be all the points of local maximum of f.
And $y_1 < y_2 < y_3 < \ldots. < y_n < \ldots$ be all the points of local minimum of f.
Then which of the following option is/are correct?

(A) $x_1 < y_1$
(B) $x_{n+1} - x_n > 2$ for every n
(C) $x_n \in \left(2n, 2n + \frac{1}{2}\right)$ for every n
(D) $|x_n - y_n| > 1$ for every n

Linked Comprehension Type Questions

Comprehension 1

Answer the following by appropriately matching the list based on the information given in the paragraph

Let $f(x) = \sin(\pi \cos x)$ and $g(x) = \cos(2\pi \sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order.

$$X = \{x : f(x) = 0\},\ Y = \{x : f'(x) = 0\}$$

$$Z = \{x : g(x) = 0\},\ W = \{x : g'(x) = 0\}.$$

LIST-I contains the sets X, Y, Z and W. LIST-II contains some information regarding these sets.

LIST-I	LIST-II
(I) X	(p) $\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II) Y	(q) an arithmetic progression
(III) Z	(r) NOT an arithmetic progression
(IV) W	(s) $\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
	(t) $\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
	(u) $\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

6. **[JEE ADV 2019]**
Which of the following is the only **CORRECT** combination?
(A) (I), (p), (r) (B) (II), (q), (t)
(C) (I), (q), (u) (D) (II), (r), (s)

7. **[JEE ADV 2019]**
Which of the following is the only **CORRECT** combination?
(A) (III), (r), (u) (B) (IV), (p), (r), (s)
(C) (III),(p), (q), (u) (D) (IV), (q), (t)

Comprehension 2

Answer Q.8, Q.9, and Q.10 by appropriately matching the information given in the three columns of the following table

Let $f(x) = x + \log_e x - x\log_e x,\ x \in (0, \infty)$

- COLUMN-I contains information about zeros of $f(x), f'(x)$ and $f''(x)$.
- COLUMN-II contains information about the limiting behaviour of $f(x), f'(x)$ and $f''(x)$ at infinity.
- COLUMN-III contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

COLUMN-I	COLUMN-II	COLUMN-III
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim\limits_{x\to\infty} f(x) = 0$	(p) f is increasing in (0, 1)
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim\limits_{x\to\infty} f(x) = -\infty$	(q) f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim\limits_{x\to\infty} f'(x) = -\infty$	(r) f' is increasing in (0, 1)
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim\limits_{x\to\infty} f''(x) = 0$	(s) f' is decreasing in (e, e^2)

8. **[JEE ADV 2017]**
Which of the following options is the only CORRECT combination?
(A) (I)(i)(p) (B) (II)(ii)(q)
(C) (III)(iii)(r) (D) (IV)(iv)(s)

9. **[JEE ADV 2017]**
Which of the following options is the only CORRECT combination?
(A) (I)(ii)(r) (B) (II)(iii)(s)
(C) (III)(iv)(p) (D) (IV)(i)(s)

10. **[JEE ADV 2017]**
Which of the following options is the only **INCORRECT** combination?
(A) (I)(iii)(p) (B) (II)(iv)(q)
(C) (III)(i)(r) (D) (II)(iii)(p)

11. **[JEE ADV 2017]**
If $f : R \to R$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in R$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}, f(1) = 1,$ then
(A) $f'(1) \le 0$ (B) $0 < f'(1) \le \frac{1}{2}$
(C) $\frac{1}{2} < f'(1) \le 1$ (D) $f'(1) > 1$

12. **[JEE ADV 2017]**
If $f : R \to R$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in R$, and $f(0) = 1$, then
(A) $f(x)$ is increasing in $(0, \infty)$
(B) $f(x)$ is decreasing in $(0, \infty)$
(C) $f(x) > e^{2x}$ in $(0, \infty)$
(D) $f'(x) < e^{2x}$ in $(0, \infty)$

13. **[JEE ADV 2017]**

$$\text{If } f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}, \text{ then}$$

(A) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$
(B) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$
(C) $f(x)$ attains its maximum at $x = 0$
(D) $f(x)$ attains its minimum at $x = 0$

14. [JEE ADV 2016]

The least value of $\alpha \in R$ for which $4\alpha x^2+\frac{1}{x}\ge 1$, for all $x > 0$, is

(A) $\frac{1}{64}$
(B) $\frac{1}{32}$
(C) $\frac{1}{27}$
(D) $\frac{1}{25}$

15. [JEE ADV 2016]

Let $f : R \to (0, \infty)$ and $g : R \to R$ be twice differentiable functions such that f'' and g'' are continuous functions on R. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim\limits_{x\to 2}\frac{f(x)g(x)}{f'(x)g'(x)}=1$, then

(A) f has a local minimum at $x = 2$
(B) f has a local maximum at $x = 2$
(C) $f''(2) > f(2)$
(D) $f(x) - f''(x) = 0$ for at least one $x \in R$

16. [JEE ADV 2015]

A cylindrical container is to be made from certain solid material with the following constraints : It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container. If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is

17. [JEE ADV 2015]

Consider the hyperbola $H : x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$, Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of triangle ΔPMN, then the correct expression(s) is(are)

(A) $\frac{dl}{dx_1}=1-\frac{1}{3x_1^2}$ for $x_1 > 1$
(B) $\frac{dm}{dx_1}=\frac{x_1}{3\left(\sqrt{x_1^2-1}\right)}$ for $x_1 > 1$
(C) $\frac{dl}{dx_1}=1+\frac{1}{3x_1^2}$ for $x_1 > 1$
(D) $\frac{dm}{dy_1}=\frac{1}{3}$ for $y_1 > 0$

18. [JEE ADV 2015]

Let $f, g : [-1, 2] \to R$ be continuous function which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points –1, 0 and 2 be as given in the following table:

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	–1

In each of the intervals $(-1, 0)$ and $(0, 2)$ the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are)

(A) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$
(B) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$
(C) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$
(D) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

19. [JEE ADV 2014]

Let $a \in \mathbb{R}$ and let $f : R \to R$ be given by $f(x) = x^5 - 5x + a$. Then

(A) (x) has three real roots if $a > 4$
(B) (x) has only one real root if $a > 4$
(C) (x) has three real roots if $a < -4$
(D) (x) has three real roots if $-4 < a < 4$

20. [JEE ADV 2014]

Let $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to R$ be given by $f(x) = (\log(\sec x + \tan x))^3$ then

(A) $f(x)$ is an odd function
(B) $f(x)$ is a one-one function
(C) $f(x)$ is an onto function
(D) $f(x)$ is an even function

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A	**2.** C	**3.** C	**4.** D	**5.** A	**6.** B	**7.** C	**8.** D	**9.** B	**10.** A
11. B	**12.** C	**13.** A	**14.** D	**15.** D	**16.** B	**17.** B	**18.** C	**19.** D	**20.** D
21. A	**22.** A	**23.** A	**24.** B	**25.** D	**26.** C	**27.** C	**28.** B	**29.** B	**30.** D
31. D	**32.** A	**33.** A	**34.** C	**35.** C	**36.** C	**37.** B	**38.** C	**39.** D	**40.** C
41. B	**42.** C	**43.** A	**44.** D	**45.** A	**46.** A	**47.** A	**48.** C	**49.** B	**50.** C
51. A	**52.** B	**53.** C	**54.** B	**55.** D	**56.** A	**57.** D	**58.** D	**59.** C	**60.** D
61. D	**62.** D	**63.** D	**64.** A	**65.** B	**66.** C	**67.** D	**68.** D	**69.** A	**70.** B
71. C	**72.** B	**73.** B	**74.** B	**75.** D	**76.** C	**77.** C	**78.** D	**79.** A	**80.** C
81. D	**82.** B	**83.** C	**84.** D	**85.** C	**86.** C	**87.** A	**88.** A	**89.** C	**90.** C
91. A	**92.** C	**93.** C	**94.** B	**95.** C	**96.** C	**97.** B	**98.** D	**99.** B	

Multiple Correct Choice Type Questions

1. A, D	**2.** B, D	**3.** B, C	**4.** A, B, C, D	**5.** A, B, C
6. A, D	**7.** A, B, C	**8.** A, B, C, D	**9.** A, B	**10.** A, C, D
11. B, C	**12.** B, C, D	**13.** B, D	**14.** A, B, C, D	**15.** A, D
16. A, D	**17.** A, B	**18.** C, D	**19.** B, C	**20.** A, C
21. A, C, D	**22.** A, B, C, D	**23.** A, C	**24.** B, D	**25.** A, C, D
26. A, D	**27.** A, C	**28.** A, C	**29.** A, C, D	**30.** A, B
31. A, B, C, D	**32.** A, C, D	**33.** B, C	**34.** A, B, C, D	**35.** A, C
36. A, D	**37.** A, B, D	**38.** B, C, D	**39.** A, C, D	**40.** B, C, D
41. A, B, C, D	**42.** A, B, D	**43.** A, B, C, D	**44.** A, B, C, D	**45.** B, D
46. B, C, D	**47.** A, C	**48.** B, C	**49.** C, D	**50.** A, B, C, D

Linked Comprehension Type Questions

1. B	**2.** D	**3.** C	**4.** C	**5.** C	**6.** A	**7.** B	**8.** C	**9.** A	**10.** B
11. C	**12.** B	**13.** D	**14.** D	**15.** B	**16.** B	**17.** A	**18.** D	**19.** A	**20.** D
21. D	**22.** C	**23.** A	**24.** B	**25.** B	**26.** D	**27.** A	**28.** A	**29.** B	**30.** B
31. B	**32.** A	**33.** D	**34.** C	**35.** A	**36.** C	**37.** B	**38.** D	**39.** C	**40.** A
41. D	**42.** A	**43.** B	**44.** A	**45.** A	**46.** C	**47.** B	**48.** D	**49.** C	**50.** D
51. C	**52.** B	**53.** D							

Matrix Match/Column Match Type Questions

1. A → (p, q, r, s)	B → (q)	C → (q, r, t)	D → (q, t)
2. A → (s)	B → (p)	C → (q)	D → (r, t)
3. A → (p, s)	B → (q, r)	C → (p, r)	D → (p, q, r, s)
4. A → (q)	B → (t)	C → (p, q, r, s, t)	D → (p)
5. A → (p)	B → (s, t)	C → (r, t)	D → (r, s, t)
6. A → (r)	B → (s)	C → (t)	D → (q)
7. A → (r, t)	B → (p, r)	C → (p, r)	D → (p)
8. A → (p, q)	B → (r, s, t)	C → (p, q, r)	D → (p, q, r, s, t)
9. A → (q)	B → (r)	C → (q)	D → (q)

10. A → (q)	B → (s)	C → (p)	D → (r)
11. A → (p)	B → (p)	C → (s)	D → (s)
12. A → (q)	B → (p)	C → (s)	D → (s)

Integer/Numerical Answer Type Questions

1. 2	**2.** 5	**3.** 2	**4.** 1	**5.** 2	**6.** 1	**7.** 5	**8.** 7	**9.** 1200	**10.** 2
11. 4	**12.** 2	**13.** 3	**14.** 2	**15.** 9	**16.** 8	**17.** 0	**18.** 3	**19.** 6	**20.** 1
21. 3	**22.** 2	**23.** 5048	**24.** 3	**25.** 2	**26.** 0	**27.** 2	**28.** 3	**29.** 7	**30.** 12
31. 1	**32.** 20	**33.** 6	**34.** 3	**35.** 13	**36.** 125	**37.** 5	**38.** 8	**39.** 3	**40.** 2
41. 1	**42.** 65	**43.** 0	**44.** 580	**45.** 9	**46.** 3	**47.** 5	**48.** 2	**49.** 4	**50.** 2
51. 5	**52.** 0	**53.** 13	**54.** 5						

ARCHIVE: JEE MAIN

1. A	**2.** B	**3.** 144	**4.** 4	**5.** D	**6.** C	**7.** C	**8.** 2	**9.** 22	**10.** A
11. B	**12.** A	**13.** 3	**14.** D	**15.** A	**16.** C	**17.** B	**18.** D	**19.** B	**20.** A
21. C	**22.** D	**23.** D	**24.** A	**25.** C					

ARCHIVE: JEE ADVANCED

Multiple Correct Choice Type Questions

1. A, B	**2.** C	**3.** A, C	**4.** B, C, D	**5.** B, C, D

Linked Comprehension Type Questions

6. B	**7.** B	**8.** B	**9.** B	**10.** C	**11.** D	**12.** A, C	**13.** B, C	**14.** C	**15.** A, D
16. 4	**17.** A, B, D	**18.** B, C	**19.** B, D	**20.** A, B, C					

CHAPTER

8

Indefinite Integration

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Standard Integration

(b) Integration Using Substitution

(c) Integration of $\frac{1}{Q}, \frac{1}{\sqrt{Q}}$ Form

(d) Integration by Parts

(e) Integral of $\frac{L}{Q}$ or $\frac{L}{\sqrt{Q}}$ Form

(f) Integration Using Partial Fraction

(g) Integration of Irrational Function

(h) Integration of Trigonometric Functions

(i) Reduction Integration and Euler's Substitution

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

Integration is a reverse process of differentiation. The *integral or primitive* of a function $f(x)$ with respect to x is that function $\phi(x)$ whose derivative with respect to x is the given function $f(x)$. It is expressed symbolically as $\int f(x)\,dx = \phi(x) + C$

Thus, $\frac{d}{dx}[\phi(x)+C] = f(x)$

The process of finding the integral of a function is called *integration* and the given function is called *integrand*. Now, it is obvious that the operation of integration is the inverse operation of differentiation. Hence, integral of a function is also named as *anti-derivative* of that function.

Further, we observe that:

$$\left.\begin{array}{ll} \frac{d}{dx}(x^3) & = 3x^2 \\ \frac{d}{dx}(x^3+2) & = 3x^2 \\ \frac{d}{dx}(x^3+k) & = 3x^2 \end{array}\right\} \Rightarrow \int 3x^2 = x^3 + C$$

So, we always add a constant C to the integral of a function, which is called *constant of integration*. It is generally denoted by C. Due to the presence of this constant such an integral is called an *indefinite integral*.

$$\left.\begin{array}{l} \text{Integration of } f(x) \text{ with respect to } x \\ \text{Primitive of } f(x) \text{ with respect to } x \\ \text{Integral of } f(x) \text{ about } x \\ \text{Anti-derivative of } f(x) \text{ with respect to } x \end{array}\right\} = \int f(x)dx$$

BASIC THEOREMS ON INTEGRATION

If $f(x)$, $g(x)$ are two functions of a variable x and k is a scalar, then:

(i) $\int k\, f(x)\, dx = k \int f(x)\, dx$

(ii) $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$

(iii) $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

(iv) $\int \left(\frac{d}{dx} f(x)\right) dx = f(x) + k$

Table 8.1 Formulae for differentiation and integration

Differentiation Formula	Integration Formula
(i) $\frac{d}{dx}k = 0$	**(i)** $\int k \cdot dx = kx + C \ (k \in R)$
(ii) $\frac{d}{dx}x^n = nx^{n-1}$	**(ii)** $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \ (n \neq -1)$
(iii) $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$	**(iii)** $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C \ (n \neq -1)$
(iv) $\frac{d}{dx}(\log_\alpha(ax+b)) = \frac{a}{(ax+b)\log_e \alpha}$ For base e $\frac{d}{dx}(\log_e(ax+b)) = \frac{a}{ax+b}$	**(iv)** $\int \frac{1}{ax+b}\,dx = \frac{\log_e(ax+b)}{a} + C$
(v) $\frac{d}{dx}a^{kx+\lambda} = k \,.\, a^{kx+\lambda}\log a$	**(v)** $\int a^{kx+\lambda}\,dx = \frac{a^{kx+\lambda}}{k\log_e a} + C = \frac{a^{kx+\lambda}\log_a e}{k} + C$
(vi) $\frac{d}{dx}e^{ax+b} = a.\, e^{ax+b}$	**(vi)** $\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$
(vii) $\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b),$	**(vii)** $\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} + C$
(viii) $\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$	**(viii)** $\int \cos(ax+b)\,x\,dx = \frac{\sin(ax+b)}{a} + C$
(ix) $\frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)$	**(ix)** $\int \tan(ax+b)\,dx = \frac{\ln\lvert\sec(ax+b)\rvert}{a} + C$
(x) $\frac{d}{dx}\cot(ax+b) = -a\,\text{cosec}^2(ax+b)$	**(x)** $\int \cot(ax+b)\,dx = \frac{\ln\lvert\sin(ax+b)\rvert}{a} + C$
(xi) $\frac{d}{dx}\sec(ax+b) = a\sec(ax+b)\tan(ax+b)$	**(xi)** $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{\sec(ax+b)}{a} + C$
(xii) $\frac{d}{dx}\text{cosec}(ax+b) = -a\,\text{cosec}(ax+b)\cot(ax+b)$	**(xii)** $\int \text{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{\text{cosec}(ax+b)}{a} + C$
	(xiii) $\int \sec^2(ax+b)\,dx = \frac{\tan(ax+b)}{a} + C$
	(xiv) $\int \text{cosec}^2(ax+b)\,dx = -\frac{\cot(ax+b)}{a} + C$
(Students are advised to learn these standard integrations)	**(xv)** $\int \sec(ax+b)\,dx = \frac{1}{a}\ln\lvert\sec(ax+b)+\tan(ax+b)\rvert + C$ $= -\frac{1}{a}\ln\lvert\sec(ax+b)-\tan(ax+b)\rvert + C$ $= \frac{1}{a}\ln\left\lvert\tan\left(\frac{\pi}{4}+\frac{ax+b}{2}\right)\right\rvert + C$
	(xvi) $\int \text{cosec}(ax+b)dx = -\frac{1}{a}\ln\lvert\text{cosec}(ax+b)+\cot(ax+b)\rvert + C$ $= \frac{1}{a}\ln\lvert\text{cosec}(ax+b)-\cot(ax+b)\rvert + C = \frac{1}{a}\ln\left\lvert\tan\left(\frac{ax+b}{2}\right)\right\rvert + C$

ILLUSTRATION 1

Evaluate: $\int (2^x - 4)^2\, dx$

SOLUTION

$$I = \int (2^x - 4)^2\, dx$$

$$I = \int 2^{2x} + 16 - 8.2^x\ dx = \int 4^x + 16 - 8.2^x\ dx$$

$$I = \frac{4^x}{\ln 4} + 16x - \frac{8.2^x}{\ln 2} + C$$

ILLUSTRATION 2

Evaluate: $\int \frac{(1+x)^3}{x}\, dx$

SOLUTION

$$\int \frac{(1+x)^3}{x}\, dx = \int \frac{1+x^3+3x(1+x)}{x}\, dx$$

$$= \int \frac{1}{x} + x^2 + 3(1+x)\ dx$$

$$= \ln x + \frac{x^3}{3} + 3\left[x + \frac{x^2}{2}\right] + C$$

ILLUSTRATION 3

Evaluate: $\int \frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}}\, dx$

SOLUTION

$$I = \int \frac{\sqrt[3]{x^2} - \sqrt[4]{x}}{\sqrt{x}}\, dx = \int \frac{\left(x^2\right)^{1/3}}{x^{1/2}} - \frac{x^{1/4}}{x^{1/2}}\, dx$$

$$I = \int x^{1/6} - x^{-1/4}\ dx$$

$$I = \frac{x^{7/6}}{\frac{7}{6}} - \frac{x^{3/4}}{\frac{3}{4}} + C = \frac{6}{7}x^{7/6} - \frac{4}{3}x^{3/4} + C$$

ILLUSTRATION 4

Evaluate: $\int \frac{x^4+x^2+1}{x^2-x+1}\, dx$

SOLUTION

$$\int \frac{x^4+x^2+1}{x^2-x+1}\, dx = \int \frac{x^4+2x^2+1-x^2}{x^2-x+1}\, dx$$

$$= \int \frac{(x^2+1)^2 - x^2}{x^2-x+1}\, dx$$

$$= \int \frac{(x^2+1+x)(x^2+1-x)}{(x^2-x+1)}\, dx$$

$$= \int \left(x^2+1+x\right)\, dx = \frac{x^3}{3} + x + \frac{x^2}{2} + C$$

ILLUSTRATION 5

Evaluate: $\int e^{x\ln a} + e^{a\ln x} + e^{a\ln a}\ dx$

SOLUTION

$$I = \int e^{x\ln a} + e^{a\ln x} + e^{a\ln a}\ dx$$

$$I = \int e^{\ln a^x} + e^{\ln x^a} + e^{\ln a^a}\ dx$$

$$I = \int a^x + x^a + a^a\ dx = \frac{a^x}{\ln a} + \frac{x^{a+1}}{a+1} + a^a x + C$$

ILLUSTRATION 6

Evaluate: $\int \frac{(a^x - b^x)^2}{a^x b^x}\, dx$

SOLUTION

$$I = \int \frac{(a^x - b^x)}{a^x b^x}\, dx$$

$$I = \int \frac{a^{2x} + b^{2x} - 2a^x b^x}{a^x b^x}\, dx$$

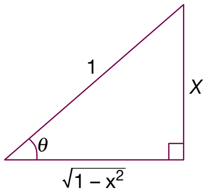
$$I = \int \left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x - 2\ dx = \frac{\left(\frac{a}{b}\right)^x}{\ln\left|\frac{a}{b}\right|} + \frac{\left(\frac{b}{a}\right)^x}{\ln\left(\frac{b}{a}\right)} - 2x + C$$

ILLUSTRATION 7

Evaluate: $\int \sin\left(\tan^{-1}\frac{x}{\sqrt{1-x^2}}\right) dx$

SOLUTION

$$I = \int \sin\left(\tan^{-1}\frac{x}{\sqrt{1-x^2}}\right) dx$$



Let $\theta = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$

$$I = \int \sin\theta\, dx = \int x\, dx = \frac{x^2}{2} + C$$

INTEGRATION OF ELEMENTARY TRIGONOMETRIC EXPRESSION

It covers the integration of basic trigonometric functions where allied angled formula, substitution are used.

(i) $\sin(A+B) = \sin A\cos B + \cos A\sin B$
(ii) $\sin(A-B) = \sin A\cos B - \cos A\sin B$
(iii) $\cos(A+B) = \cos A\cos B - \sin A\sin B$
(iv) $\cos(A-B) = \cos A\cos B + \sin A\sin B$

(v) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

(vi) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

(vii) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

(viii) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

(ix) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

(x) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

(xi) $\sin C+\sin D=2\sin\left(\dfrac{C+D}{2}\right)\cdot\cos\left(\dfrac{C-D}{2}\right)$

(xii) $\sin C-\sin D=2\cos\left(\dfrac{C+D}{2}\right)\cdot\sin\left(\dfrac{C-D}{2}\right)$

(xiii) $\cos C+\cos D=2\cos\left(\dfrac{C+D}{2}\right)\cdot\cos\left(\dfrac{C-D}{2}\right)$

(xiv) $\cos C-\cos D=2\sin\left(\dfrac{C+D}{2}\right)\cdot\sin\left(\dfrac{D-C}{2}\right)$

(xv) $\sin 2\theta=2\sin\theta\cos\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}$

(xvi) $\cos 2\theta=\cos^2\theta-\sin^2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(xvii) $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

(xviii) $\sin 3\theta=3\sin\theta-4\sin^3\theta$

(xix) $\cos 3\theta=4\cos^3\theta-3\cos\theta$

(xx) $\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

(xxi) $\sqrt{1+\sin 2A}=|\sin A+\cos A|$

(xxii) $\sqrt{1-\sin 2A}=|\sin A-\cos A|$

ILLUSTRATION 8

Evaluate: $\int \sin 3x.\cos 5x\ dx$

SOLUTION

$$I=\int \sin 3x.\cos 5x\ dx$$

$$I=\frac{1}{2}\int \sin(3x+5x)+\sin(3x-5x)\ dx$$

(using 2 sin A cos B = sin(A + B) + sin (A – B))

$$I=\frac{1}{2}\int \sin 8x-\sin 2x\ dx=\frac{1}{2}\left[-\frac{\cos 8x}{8}+\frac{\cos 2x}{2}\right]+C$$

ILLUSTRATION 9

Evaluate: $\int \dfrac{dx}{1-\cos x}$

SOLUTION

$$I=\int\frac{dx}{1-\cos x}$$

$$I=\int\frac{dx}{1-\left(1-2\sin^2\dfrac{x}{2}\right)}=\int\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\ dx$$

$$I=-\cot\frac{x}{2}+C \qquad \ldots(1)$$

Alternate:

$$I=\int\frac{(1+\cos x)}{(1-\cos x)(1+\cos x)}dx=\int\frac{1+\cos x dx}{\sin^2 x}$$

$$=\int \operatorname{cosec}^2 x+\cot x\operatorname{cosec} x\ dx$$

$$I=-\cot x-\operatorname{cosec} x+C \qquad \ldots(2)$$

On further simplifying $-\left(\dfrac{\cos x+1}{\sin x}\right)+C$

$$=-\left(\frac{2\cos^2\dfrac{x}{2}-1+1}{2\sin^2\dfrac{x}{2}\cos\dfrac{x}{2}}\right)=-\cot\frac{x}{2}+C$$

Both integrals (1) and (2) are the same.

ILLUSTRATION 10

Evaluate: $\int\dfrac{1-\cos x}{1+\cos x}dx$

SOLUTION

$$I=\int\frac{1-\cos x}{1+\cos x}dx$$

$$I=\int\frac{2-(1+\cos x)}{1+\cos x}\ dx=\int\frac{2}{1+\cos x}\ dx-\int dx$$

$$I=\int\frac{2(1-\cos x)}{1-\cos^2 x}\ dx-x+C$$

$$I=2\int \operatorname{cosec}^2 x-\cot x\operatorname{cosec}x\ dx-x+C$$

$$=2(-\cot x-\operatorname{cosec} x)-x+C$$

Alternate:

$$I=\int\frac{1-\cos x}{1+\cos x}\ dx=\int\frac{1-\left(1-2\sin^2\dfrac{x}{2}\right)}{1+2\cos^2\dfrac{x}{2}-1}\ dx$$

$$=\int\tan^2\frac{x}{2}\ dx=\int\sec^2\frac{x}{2}-1\ dx=2\tan\frac{x}{2}-x+C$$

ILLUSTRATION 11

Show that $\int\sqrt{1-\sin 2x}\ dx$ = (sin x + cos x) sgn (cos x – sin x) + C; where sgn(x) is signum function.

SOLUTION

$$I(x)=\int\sqrt{1-\sin 2x}\ dx$$

$$=\int\sqrt{\sin^2 x+\cos^2 x-2\sin x\cos x}\ dx$$

$$=\int\sqrt{(\sin x-\cos x)^2}\ dx$$

$$I(x)=\int|\sin x-\cos x|\ dx$$

CASE 1:

If $\sin x - \cos x > 0$

$$I(x) = \int \sin x - \cos x \; dx = -\cos x - \sin x + C$$

CASE 2:

If $\sin x - \cos x = 0$,

$$I(x) = 0$$

CASE 3:

If $\sin x - \cos x < 0$,

$$I(x) = \int \cos x - \sin x \; dx = \sin x + \cos x + C$$

Summarizing all cases, we get

$$I(x) = (\sin x + \cos x)\operatorname{sgn}(\cos x - \sin x) + C$$

ILLUSTRATION 12

Evaluate: $\displaystyle\int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x\cos^2 x} dx$

SOLUTION

$$I = \int \frac{\sin^8 x - \cos^8 x}{1 - 2\sin^2 x\cos^2 x} dx$$

$$= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{1 - 2\sin^2 x\cos^2 x} dx$$

$$(\sin^2 x + \cos^2 x)(\sin^2 x - \cos^2 x)$$

$$= \int \frac{\left(\left(\sin^2 x + \cos^2 x\right)^2 - 2\sin^2 x\cos^2 x\right)}{(1 - 2\sin^2 x\cos^2 x)} dx$$

$$= \int (1)(\sin^2 x - \cos^2 x)\, dx = -\int \cos 2x \; dx = -\frac{\sin 2x}{2} + C$$

ILLUSTRATION 13

Evaluate: $\displaystyle\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x\cos^2 x} dx$

SOLUTION

$$I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x\cos^2 x} dx$$

$$I = \int \frac{(\sin^2 x)^3 + (\cos^2 x)^3 dx}{\sin^2 x\cos^2 x}$$

Using $A^3 + B^3 = (A + B)^3 - 3AB(A + B)$

$$(\sin^2 x)^3 + (\cos^2 x)^3$$

$$= (\sin^2 x + \cos^2 x)^3 - 3\sin^2 x \cos^2 x\,(\sin^2 x + \cos^2 x)$$

$$= 1 - 3\sin^2 x \cos^2 x$$

$$I = \int \frac{1 - 3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x} dx = \int \operatorname{cosec}^2 x \sec^2 x - 3 \; dx$$

$$= \int (1 + \cot^2 x)(1 + \tan^2 x) - 3 \; dx$$

$$= \int (1 + \cot^2 x + \tan^2 x + 1) - 3 \; dx$$

$$= \int \operatorname{cosec}^2 x + \sec^2 x - 3 dx \; = -\cot x + \tan x - 3x + C$$

ILLUSTRATION 14

Evaluate: $\displaystyle\int \sin^3 x + \cos^3 x \; dx$

SOLUTION

$$I = \int \sin^3 x + \cos^3 x \; dx$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \Rightarrow \cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4}$$

$$\sin 3\theta = 3\sin\,\theta - 4\sin^3\theta \Rightarrow \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

$$I = \int \sin^3 x + \cos^3 x \;\; dx$$

$$= \int \frac{3\sin x - \sin 3x}{4} + \frac{\cos 3x + 3\cos x}{4} dx$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3} + \frac{\sin 3x}{3} + 3\sin x\right] + C$$

ILLUSTRATION 15

Evaluate: $\displaystyle\int \sin^3 x\cos^3 x \;\; dx$

SOLUTION

$$I = \int \sin^3 x\cos^3 x \;\; dx$$

$$= \frac{1}{8}\int (2\sin x\cos x)^3 \; dx = \frac{1}{8}\int \sin^3 2x \;\; dx$$

Using $\sin 3\theta = 3\,\sin\theta - 4\,\sin^3\theta \Rightarrow \sin^3\theta = \dfrac{3\sin\theta - \sin 3\theta}{4}$

$$I = \frac{1}{8}\int \frac{3\sin 2x - \sin 6x}{4} \; dx = \frac{1}{32}\int 3\sin 2x - \sin 6x \;\; dx$$

$$= \frac{1}{32}\left[-\frac{3\cos 2x}{2} + \frac{\cos 6x}{6}\right] + C$$

ILLUSTRATION 16

Evaluate: $\displaystyle\int \frac{dx}{\sin(x - \alpha)\sin(x - \beta)}$

SOLUTION

$$I = \int \frac{dx}{\sin(x - \alpha)\sin(x - \beta)}$$

$$= \frac{1}{\sin(\alpha - \beta)} \int \frac{\sin[(x - \beta) - (x - \alpha)]dx}{\sin(x - \alpha)\sin(x - \beta)}$$

$$= \operatorname{cosec}(\alpha - \beta)\int \frac{\sin(x - \beta)\cos(x - \alpha) - \cos(x - \beta)\sin(x - \alpha)}{\sin(x - \alpha)\sin(x - \beta)} dx$$

$$= \operatorname{cosec}(\alpha-\beta)\int \cot(x-\alpha)-\cot(x-\beta)\,dx$$

$$= \operatorname{cosec}(\alpha-\beta)[\ln|\sin(x-\alpha)| - \ln|\sin(x-\beta)|] + C$$

$$= \operatorname{cosec}(\alpha-\beta)\ln\left|\frac{\sin(x-\alpha)}{\sin(x-\beta)}\right| + C$$

ILLUSTRATION 17

Find the value of

$$\int\left(\sin^3 x+\sin^3\left(x+\frac{2\pi}{3}\right)+\sin^3\left(x+\frac{4\pi}{3}\right)\right)dx$$

SOLUTION

$$I=\int \sin^3 x+\sin^3\left(x+\frac{2\pi}{3}\right)+\sin^3\left(x+\frac{4\pi}{3}\right)dx$$

We know $\sin 3x = 3\sin x - 4\sin^3 x$

$$I=\int\frac{3\sin x-\sin 3x}{4}dx+\frac{3\sin\left(x+\frac{2\pi}{3}\right)}{4}$$

$$\frac{-\sin(3x+2\pi)}{4}dx+\frac{3\sin\left(x+\frac{4\pi}{3}\right)-\sin(3x+4\pi)}{4}dx$$

$$I=-\frac{3}{4}\int\sin 3x\,dx+\frac{3}{4}\int\sin x+\sin\left(x+\frac{2\pi}{3}\right)$$

$$-\sin\left(x+\frac{\pi}{3}\right)dx$$

$$I=\frac{\cos 3x}{4}+\frac{3}{4}\int\sin x+2\cos\left(x+\frac{\pi}{2}\right)\sin\left(\frac{\pi}{6}\right)dx$$

$$I=\frac{\cos 3x}{4}+C$$

ILLUSTRATION 18

Evaluate: $\displaystyle\int\frac{\cot x+\operatorname{cosec}x-1}{\cot x-\operatorname{cosec}x+1}dx$

SOLUTION

$$I=\int\frac{\cot x+\operatorname{cosec}x-1}{\cot x-\operatorname{cosec}x+1}dx=\int\frac{\cos x-\sin x+1}{\cos x+\sin x-1}dx$$

$$I=\int\frac{2\cos^2\left(\frac{x}{2}\right)-2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)-2\sin^2\left(\frac{x}{2}\right)}dx$$

$$=\int\frac{\cos\left(\frac{x}{2}\right)\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)}{\sin\left(\frac{x}{2}\right)\left(\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right)}dx$$

$$I=\int\cot\left(\frac{x}{2}\right)dx=2\ln\left|\sin\left(\frac{x}{2}\right)\right|+C=\ln\left|\sin^2\left(\frac{x}{2}\right)\right|+C$$

Test Your Concepts-I

(Solutions on page H.229)

Evaluate the following integrals:

1. $\displaystyle\int(x^2-1)(x+1)\,dx$

2. $\displaystyle\int\frac{x^3-1}{x-1}dx$

3. $\displaystyle\int\cos 6x\cos 3x\,dx$

4. $\displaystyle\int 20\sin 2x\sin 6x\,dx$

5. $\displaystyle\int\sec^2 5x-\operatorname{cosec}^2 7x+\operatorname{cosec}3x\cot 3x\,dx$

6. $\displaystyle\int\frac{1+\cos^2 x}{1+\cos 2x}dx$

7. $\displaystyle\int\frac{dx}{\sin(x-\alpha)\cos(x-\beta)}$

8. $\displaystyle\int\frac{2-3\sin x}{\cos^2 x}dx$

9. $\displaystyle\int(1-\tan 3x)^2\,dx$

10. Show that $\displaystyle\int\left(\sin\alpha\cdot\sin(x-\alpha)+\sin^2\left(\frac{x}{2}-\alpha\right)\right)dx$ is independent of α.

11. $\displaystyle\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx$

12. $\displaystyle\int\frac{1+\cos^2 x}{1+\cos 2x}dx$

13. $\displaystyle\int 4\cos\frac{x}{2}\cdot\cos x\cdot\sin\frac{21}{2}x\,dx$

14. $\displaystyle\int\frac{x^6-1}{x^2+1}dx$

15. $\displaystyle\int\frac{\sin 2x+\sin 5x-\sin 3x}{\cos x+1-2\sin^2 2x}dx$

16. $\displaystyle\int\frac{1+\sin x}{1-\sin x}dx$

17. Evaluate: $\displaystyle\int\frac{dx}{(x+\sqrt{x^2-1})^2}$

18. Evaluate: $\displaystyle\int\{1+2\tan x(\tan x+\sec x)\}^{1/2}\,dx$

Answer key

1. $\frac{x^4}{4} - \frac{x^2}{2} + \frac{x^3}{3} - x + C$ **2.** $\frac{x^3}{3} + \frac{x^2}{2} + x + C$ **3.** $\frac{1}{2}\left[\frac{\sin 3x}{3} + \frac{\sin 9x}{9}\right] + C$

4. $10\left[\frac{\sin 4x}{4} - \frac{\sin 8x}{8}\right] + C$ **5.** $\frac{1}{5}\tan 5x + \frac{1}{7}\cot 7x - \frac{1}{3}\operatorname{cosec} 3x + C$ **6.** $\frac{1}{2}[\tan x + x] + C$

7. $\sec(\alpha - \beta)\ln|\sin(x - \alpha) . \sec(x - \beta)| + C$ **8.** $2\tan x - 3\sec x + C$

9. $\frac{1}{3}[\tan 3x + \ln(\cos^2 3x)] + C$ **10.** $\frac{x}{2} - \frac{\sin x}{2} + C$ **11.** $2(\sin x + x\cos\alpha) + C$ **12.** $\frac{1}{2}(\tan x + x) + C$

13. $-\frac{\cos 12x}{12} - \frac{\cos 10x}{10} - \frac{\cos 11x}{11} - \frac{\cos 9x}{9} + C$ **14.** $\frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1}x + C$ **15.** $-2\cos x + C$

16. $2\tan x + 2\sec x - x + C$ **17.** $\frac{2}{3}x^3 - x - \frac{2}{3}(x^2 - 1)^{3/2} + C$ **18.** $\ln \sec x . (\sec x + \tan x) + C$

INTEGRATION USING SUBSTITUTION

If $g(x)$ is a continuously differentiable function, then to evaluate integrals of the form,

$I = \int f(g(x))\, g'(x)dx$, we substitute $g(x) = t$ then $g'(x)dx = dt$.

The substitution reduces the integral

$$I = \int f(g(x))\, g'(x)dx = \int f(t)\, dt.$$

After evaluating this integral, we substitute back the value of $t = g(x)$

I. Let $\int f(x)\, dx = F(x) + C$.

Then $\int f(ax+b)\, dx = \frac{F(ax+b)}{a} + C$.

Proof: We have $I = \int f(ax+b)\, dx$.

Let $ax + b = t$, then $a\, dx = dt$.

$$\therefore I = \int f(ax+b)\, dx = \int f(t)\, \frac{dt}{a}$$

$$I = \frac{1}{a}\int f(t)\, dt = \frac{1}{a}F(t) + C$$

$$\int f(ax+b)\, dx = \frac{1}{a}F(ax+b) + C$$

II. $\int \frac{f'(x)}{2\sqrt{f(x)}}dx = \sqrt{f(x)} + C$

Proof: Let $f(x) = t$, then $dt = f'(x)dx$.

$$\therefore \int \frac{f'(x)}{2\sqrt{f(x)}}dx = \int \frac{dt}{2\sqrt{t}} = \sqrt{t} + C = \sqrt{f(x)} + C$$

III. $\int \frac{f'(x)}{f(x)}dx = \log_e |f(x)| + C$

Proof: Let $f(x) = t$, then $dt = f'(x)dx$.

$$\therefore \int \frac{f'(x)}{f(x)}dx = \int \frac{dt}{t} = \log_e|t| + C = \log_e|f(x)| + C$$

IV. $\int f'(x)(f(x))^n\, dx = \frac{(f(x))^{n+1}}{n+1} + C$

Proof: Let $f(x) = t$, then $dt = f'(x)\, dx$.

$$\therefore \int f'(x)(f(x))^n\, dx = \int t^n\, dt = \frac{t^{n+1}}{n+1} + C = \frac{(f(x))^{n+1}}{n+1} + C$$

SOME IMPORTANT SUBSTITUTION

Table 8.2 Integral and their substitution

Integral	**Substitution**
(i) $a^2 + x^2$;	$x = a\tan\theta$ or $x = a\cot\theta$
(ii) $a^2 - x^2$;	$x = a\sin\theta$ or $x = a\cos\theta$
(iii) $x^2 - a^2$;	$x = a\sec\theta$ or $x = a\operatorname{cosec}\theta$
(iv) $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$;	$x = a\cos^2\theta$
(v) $\sqrt{\frac{x-a}{b-x}}$ or $\sqrt{(x-a)(b-x)}$;	$x = a\cos^2\theta + b\sin^2\theta$
(vi) $\sqrt{\frac{x}{x-a}}$ or $\sqrt{\frac{x-a}{x}}$;	$x = a\sec^2\theta$

ILLUSTRATION 19

Evaluate: $\int x^3 \sin x^4 \, dx$

SOLUTION

$$I = \int x^3 \sin x^4 \, dx$$

Put $x^4 = t$ on differentiation $4x^3dx = dt \Rightarrow x^3dx = \frac{dt}{4}$

$$I = \int \sin t \cdot \frac{dt}{4} = \frac{1}{4}\int \sin t \, dt = \frac{1}{4}(-\cos t) + C$$

$$I = -\frac{1}{4}\cos x^4 + C$$

ILLUSTRATION 20

Evaluate: $\int \frac{5x^4 + 5^x \ln 5}{x^5 + 5^x} dx$

SOLUTION

$$I = \int \frac{5x^4 + 5^x \ln 5}{x^5 + 5^x} dx$$

Put $x^5 + 5^x = t \Rightarrow (5x^4 + 5^x \ln 5)dx = dt$

$$I = \int \frac{dt}{t} = \ln t + C \quad \Rightarrow \quad I = \ln |\, x^5 + 5^x \,| + C$$

ILLUSTRATION 21

Evaluate: $\int (x^3 - 1)^{1/3} x^5 dx$

SOLUTION

$$I = \int (x^3 - 1)^{1/3} x^3 \cdot x^2 \, dx$$

Put $x^3 - 1 = t \Rightarrow 3x^2dx = dt$

$$I = \int t^{1/3}(1+t)dt \;\frac{dt}{3} = \frac{1}{3}\int t^{1/3} + t^{4/3} \, dt$$

$$I = \frac{1}{3}\left[\frac{t^{4/3}}{\frac{4}{3}} + \frac{t^{7/3}}{\frac{7}{3}}\right] + C = \frac{1}{4}t^{4/3} + \frac{1}{7}t^{7/3} + C$$

$$I = \frac{1}{4}(x^3 - 1)^{4/3} + \frac{1}{7}(x^3 - 1)^{7/3} + C$$

ILLUSTRATION 22

Evaluate: $\int \left(\sin(\ln x) + \sec^2(\ln x) - 5^{\ln x}\right) \frac{dx}{x}$

SOLUTION

$$I = \int \left(\sin(\ln x) + \sec^2(\ln x) - 5^{\ln x}\right) \frac{dx}{x}$$

Put $\ln x = t \Rightarrow \frac{dx}{x} = dt$

$$I = \int \left(\sin(t) + \sec^2(t) - 5^t\right) dt$$

$$I = -\cos t + \tan t - \frac{5^t}{\ln 5} + C$$

$$I = -\cos(\ln x) + \tan(\ln x) - \frac{5^{\ln x}}{\ln 5} + C$$

ILLUSTRATION 23

Evaluate the following integrals:

(i) $\int \sin^3 x \; d(\sin x)$ **(ii)** $\int \cos^2 x \; d(\sin x)$

(iii) $\int \sin e^x + \frac{1}{5e^x - 1} d(e^x)$ **(iv)** $\int \frac{d(f(x))}{\sin^2(100 + f(x))}$

SOLUTION

(i) $I = \int \sin^3 x \; d(\sin x)$

Put $\sin x = t$, $d(\sin x) = dt$

$$I = \int t^3 \, dt = \int \frac{t^4}{4} + C = \frac{\sin^4 x}{4} + C$$

(ii) $I = \int \cos^2 x \; d(\sin x)$

$$I = \int (1 - \sin^2 x) d(\sin x)$$

Put $\sin x = t$, $d(\sin x) = dt$

$$I = \int 1 - t^2 \, .dt = t - \frac{t^3}{3} + C$$

$$I = \sin x - \frac{\sin^3 x}{3} + C$$

(iii) $I = \int \sin e^x + \frac{1}{5e^x - 1} d(e^x)$

Put $e^x = t$, $d(e^x) = dt$

$$I = \int \sin t + \frac{1}{5t - 1} dt$$

$$I = -\cos t + \frac{1}{5}\ln(5t - 1) + C$$

$$I = -\cos(e^x) + \frac{1}{5}\ln|5e^x - 1| + C$$

(iv) $I = \int \frac{d(f(x))}{\sin^2(100 + f(x))}$

Put $f(x) = t$; $d(f(x)) = dt$

$$I = \int \frac{dt}{\sin^2(100 + t)} = \int \operatorname{cosec}^2(100 + t) \, dt$$

$$= -\cot(100 + t) + C = -\cot(100 + f(x)) + c$$

ILLUSTRATION 24

Evaluate: $\displaystyle\int \frac{x^2+x}{(e^x+x+1)^2}\,dx$

SOLUTION

$$I = \int \frac{x^2+x}{(e^x+x+1)^2}\,dx$$

$$= \int \frac{(x^2+x)e^{-2x}}{(1+xe^{-x}+e^{-x})^2}\,dx$$

$$= \int \frac{(x+1)e^{-x}\cdot xe^{-x}dx}{(1+xe^{-x}+e^{-x})^2}$$

Put $1 + xe^{-x} + e^{-x} = t \Rightarrow (e^{-x} - xe^{-x} - e^{-x})dx = dt$

$$= \int \frac{(t-1)(-dt)}{t^2} = \int \frac{1-t}{t^2}\,dt$$

$$= \int \frac{1}{t^2} - \frac{1}{t}\,dt = -\frac{1}{t} - \ln t + c$$

$$= -\frac{1}{(1+xe^{-x}+e^{-x})} - \ln(1 + xe^{-x} + e^{-x}) + C$$

$$= x - \ln(1 + x + e^x) - \frac{e^x}{e^x+x+1} + c$$

ILLUSTRATION 25

Evaluate: $\displaystyle\int \frac{x^3dx}{\sqrt{1+x^2}}$

SOLUTION

$$I = \int \frac{x^3\,dx}{\sqrt{1+x^2}}$$

Put $\sqrt{1+x^2} = t \Rightarrow 1 + x^2 = t^2 \Rightarrow x\,dx = t\,dt$

$$I = \int \frac{x^2.x\,dx}{\sqrt{1+x^2}} = \int \frac{(t^2-1)t\,dt}{t}$$

$$I = \frac{t^3}{3} - t + C = t\left(\frac{t^2}{3} - 1\right) + C$$

$$I = \sqrt{1+x^2}\left(\frac{1+x^2}{3} - 1\right) + C = \sqrt{1+x^2}\left(\frac{x^2-2}{3}\right) + C$$

ILLUSTRATION 26

Evaluate: $\displaystyle\int \left(\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right) \ln x\,dx$

SOLUTION

$$I = \int \left(\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right) \ln x\,dx$$

Put $\left(\dfrac{x}{e}\right)^x = t \Rightarrow x \ln \left(\dfrac{x}{e}\right) = x(\ln x - \ln e) = \ln t$

$$\Rightarrow \left[1(\ln x - 1) + x\left(\frac{1}{x}\right)\right] dx = \frac{1}{t}\,dt \Rightarrow \ln x\,dx = \frac{dt}{t}$$

$$I = \int\left(t + \frac{1}{t}\right)\frac{dt}{t} = \int\left(1 + \frac{1}{t^2}\right)dt = t - \frac{1}{t} + C$$

$$I = \left(\frac{x}{e}\right)^x - \left(\frac{e}{x}\right)^x + C$$

ILLUSTRATION 27

Evaluate: $\displaystyle\int \frac{(x^3-1)dx}{(x^4+1)(x+1)}$

SOLUTION

$$I = \int \frac{\left(x^3-1\right)dx}{\left(x^4+1\right)\left(x+1\right)} = \int \frac{x^4+x^3-x^4-1}{(x^4+1)(x+1)}\,dx$$

$$I = \int \frac{x^3}{1+x^4} - \frac{1}{x+1}\,dx = \frac{1}{4}\ \ln(1 + x^4) - \ln(x + 1) + C$$

ILLUSTRATION 28

Evaluate: $\displaystyle\int \frac{\sin^3 x\,dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}$

SOLUTION

$$I = \int \frac{\sin^3 x\,dx}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}$$

Let $\tan^{-1}(\sec x + \cos x) = t$

$$\Rightarrow \frac{\sec x \tan x - \sin x}{(1+(\sec x + \cos x)^2)}\,dx$$

$$= \frac{\sin^3 x}{(\cos^4 x + 3\cos^2 x + 1)}\,dx = dt$$

$$I = \int \frac{1}{t}\,dt = \log t + C$$

$$I = \log \tan^{-1} (\sec x + \cos x) + C$$

ILLUSTRATION 29

Evaluate: $\displaystyle\int \sqrt{\frac{(2012)^{2x}}{1-(2012)^{2x}}}\ (2012)^{\sin^{-1}(2012)^x}\,dx$

SOLUTION

$$I = \int \frac{(2012)^x}{\sqrt{1-(2012)^{2x}}}\ (2012)^{\sin^{-1}(2012)^x}\,dx$$

Let $(2012)^{\sin^{-1}(2012)^x} = t;$

On differentiation

$$(2012)^{\sin^{-1}(2012)^x}\ \frac{(2012)^x(\log_e 2012)^2}{\sqrt{1-(2012)^{2x}}}\,dx = dt$$

$$I = \int \frac{dt}{(\log_e 2012)^2} = (\log_{2012}e)^2\ (2012)^{\sin^{-1}(2012)^x} + C$$

ILLUSTRATION 30

Evaluate: $\displaystyle\int \frac{(x-\sin x)^{3/2}}{\sqrt{x}}\left\{\frac{6x^2\sin^2\left(\frac{x}{2}\right)}{x-\sin x}+3x\right\}dx$

SOLUTION

$$I=\int \frac{(x-\sin x)^{3/2}}{\sqrt{x}}\left\{\frac{6x^2\sin^2\left(\frac{x}{2}\right)}{x-\sin x}+3x\right\}dx$$

$$I=\int\left(\frac{x-\sin x}{x}\right)^{1/2}\left\{6x^2\sin^2\frac{x}{2}+3x^2-3x\sin x\right\}dx$$

$$I=3\int(x(x-\sin x))^{1/2}\left\{2x\sin^2\frac{x}{2}+x-\sin x\right\}$$

Let $x(x-\sin x)=t$, $(x-\sin x+x(1-\cos x))dx$

$=(x-\sin x+x(2\sin^2 x/2))dx=dt$

$$I=3\int t^{1/2}dt=2t^{3/2}+c=2\left(x\left(x-\sin x\right)\right)^{3/2}+C$$

ILLUSTRATION 31

If $\displaystyle\int\frac{x+1}{(x^2+4x+5)^2}dx = f(x)+g(x)+C$ where $f(0)=-\frac{3}{10}$, $g(-1)=-\frac{\pi}{8}$ and C is any arbitrary constant, then find the value of $f(1)+g(\sqrt{3}-2)$.

SOLUTION

$$I=\int\frac{x+1}{(x^2+4x+5)^2}dx$$

$$I=\int\frac{x+2}{\left(x^2+4x+5\right)^2}-\frac{1}{\left(x^2+4x+5\right)^2}dx$$

$$I=-\frac{1}{2\left(x^2+4x+5\right)}-\int\frac{dx}{\left(x^2+4x+5\right)^2}$$

$$I=-\frac{1}{2\left(x^2+4x+5\right)}-I_1 \qquad \ldots(1)$$

$$I_1=\int\frac{dx}{\left(\left(x+2\right)^2+1\right)^2}$$

Put $x+2=\tan\theta \Rightarrow dx=\sec^2\theta\, d\theta$

$$I_1=\int\frac{\sec^2\theta\, d\theta}{\sec^4\theta}=\int\cos^2\theta d\theta$$

$$I_1=\int\frac{1+\cos 2\theta}{2}d\theta=\frac{\theta}{2}+\frac{\sin 2\theta}{4}+C$$

$$I_1=\frac{1}{2}\tan^{-1}(x+2)+\frac{\tan\theta}{2\left(1+\tan^2\theta\right)}+C$$

$$I_1=\frac{1}{2}\tan^{-1}(x+2)+\frac{x+2}{2(1+(x+2)^2)}+C$$

From (1)

$$I=-\frac{1}{2(x^2+4x+5)}-\frac{1}{2}\tan^{-1}(x+2)-\frac{x+2}{2(1+(x+2)^2)}+C$$

$$=-\frac{x+3}{2(x^2+4x+5)}-\frac{1}{2}\tan^{-1}(x+2)+c=f(x)+g(x)+C$$

$$f(x)=-\frac{x+3}{2(x^2+4x+5)},\ g(x)=-\frac{1}{2}\tan^{-1}(x+2)$$

$$f(0)=\frac{-3}{10}\ \text{and}\ g(-1)=\frac{-\pi}{8};$$

We get $f(1)+g\left(\sqrt{3}-2\right)=-\frac{1}{5}-\frac{\pi}{6}$

ILLUSTRATION 32

If $\displaystyle\int\frac{\pi dx}{2(\cos^{-1}x)^2\sqrt{1-x^2}}=f(x)+C$, where $f(x)$ is non-constant function of x, then find the value of $\displaystyle\lim_{x\to\frac{1}{\sqrt{2}}}[f(x)]$ is (where [.] denotes greatest integer function).

SOLUTION

$$I=\int\frac{\pi dx}{2(\cos^{-1}x)^2\sqrt{1-x^2}}$$

Let $\cos^{-1}x=t \Rightarrow \dfrac{-dx}{\sqrt{1-x^2}}=dt$

$$I=\frac{\pi}{2}\int-\frac{dt}{t^2}=\frac{\pi}{2t}+C=\frac{\pi}{2\cos^{-1}x}+C=f(x)+C$$

$$\lim_{x\to\frac{1}{\sqrt{2}}}[f(x)]=\lim_{x\to\frac{1}{\sqrt{2}}}\left[\frac{\pi}{2\cos^{-1}x}\right]$$

RHL = 2, LHL = 1

$\displaystyle\lim_{x\to\frac{1}{\sqrt{2}}}[f(x)]$ does not exist

Test Your Concepts-II

(Solutions on page H.230)

Evaluate the following integrals:

1. $\int \frac{\ln^2 x}{x}\,dx$

2. $\int e^{-x^3} x^2\,dx$

3. $\int \frac{x^3 dx}{\sqrt{1-x^8}}$

4. $\int \frac{\cos x}{2\sin x - 1}\,dx$

5. $\int e^x \text{cosec}(e^x)\cot(e^x)\,dx$

6. $\int \sin x\, d(\sin x) + \int \tan^3 x\, d(\tan x)$

7. $\int (4x+2)\sqrt{x^2+x+1}\,dx$

8. $\int \frac{1}{x-\sqrt{x}}\,dx$

9. $\int \frac{\sin^2 x \cos x}{(1+\sin^2 x)}\,dx$

10. $\int \frac{\ln(x+1)-\ln x}{x(x+1)}\,dx$

11. $\int \frac{d(\ln x)}{\cos^2(1+\ln x)}$

12. $\int \sqrt{e^x - 1}\,dx$

13. $\int \sqrt{\frac{\cos x - \cos^3 x}{1-\cos^3 x}}\,dx$

14. The value of $\int \frac{ax^2 - b}{x\sqrt{c^2x^2-(ax^2+b)^2}}\,dx$ is

Answer key

1. $\frac{\ln^3 x}{3} + C$

2. $-\frac{1}{3}e^{-x^3} + C$

3. $\frac{1}{4}$ arc sin $x^4 + C$

4. $\frac{1}{2}\ln|2\sin x - 1| + c$

5. $-$ cosec $(e^x) + C$

6. $\frac{\sin^2 x}{2} + C$

7. $\frac{4}{3}\left(x^2+x+1\right)^{3/2} + C$

8. $2\ln\left|\sqrt{x}-1\right| + C$

9. $\sin x$ – arc tan $\sin x + C$

10. $\frac{1}{2}\ln^2\left(\frac{x}{x+1}\right) + C$

11. $\tan(1 + \ln x) + C$

12. $2\sqrt{e^x - 1}$ – 2 arc tan $\sqrt{e^x - 1} + C$

13. $\frac{2}{3}\cos^{-1}(\cos^{3/2} x) + C$

14. $\sin^{-1}\left(\frac{ax+\frac{b}{x}}{c}\right) + k$

INTEGRATION OF $\frac{1}{Q}, \frac{1}{\sqrt{Q}}$ FORM

(I) $\left(\int \frac{dx}{Q}\text{ Type}\right) : \int \frac{dx}{ax^2+bx+c}$ form where Q is quadratic expression

(i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$ (Put $x = a\tan\theta$)

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$ (Put $x = a\sec\theta$)

(iii) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$ (Put $x = a\sec\theta$)

(II) $\left(\int \frac{dx}{\sqrt{Q}}\text{ Type}\right) : \int \frac{dx}{\sqrt{ax^2+bx+c}}$ form where Q is quadratic expression

(i) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x+\sqrt{x^2+a^2}\right| + C$ (Put $x = a\tan\theta$)

(ii) $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x+\sqrt{x^2-a^2}\right| + C$ (Put $x = a\sec\theta$)

(iii) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$ (Put $x = a\sin\theta$)

(III) $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$ or $-\frac{1}{a}\text{cosec}^{-1}\frac{x}{a} + C$

(Put $x = a\sec\theta$)

Note

Always make the coefficient of x^2 as 1

For proof of the above integrals, suitable substitutions are mentioned, let consider a few of them:

(I) (i) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

Proof: $I = \int \frac{dx}{x^2+a^2}$

Put, $x = a\tan\theta \Rightarrow \tan\theta = \frac{x}{a} \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right)$

$x = a\tan\theta; dx = a\sec^2\theta\, d\theta;$

$$I = \int \frac{a\sec^2\theta\, d\theta}{a^2\tan^2\theta + a^2} = \int\frac{a\sec^2\theta\, d\theta}{a^2(\tan^2\theta+1)} = \frac{1}{a}\int d\theta$$

$$I = \frac{1}{a}\theta + C = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$$

(ii) $\int\frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$

Proof:

$$I=\int\frac{dx}{x^2-a^2} = \int\frac{dx}{(x+a)(x-a)} = \frac{1}{2a}\int\frac{(x+a)-(x-a)dx}{(x+a)(x-a)}$$

$$I = \frac{1}{2a}\left(\int\frac{1}{x-a}-\frac{1}{x+a}dx\right)$$

$$I=\frac{1}{2a}[\ln|x-a| - \ln|x+a|]+C$$

$$I = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

Alternate

$I=\int\frac{dx}{x^2-a^2}$ put $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\, d\theta$

$$I = \int\frac{a\sec\theta\tan\theta\, d\theta}{a^2\sec^2\theta-a^2} = \int\frac{\sec\theta\tan\theta\, d\theta}{a\tan^2\theta} = \frac{1}{a}\int \text{cosec}\,\theta\, d\theta$$

$$= \frac{1}{a}\ln|\,\text{cosec}\,\theta - \cot\theta|+C$$

$$=\frac{1}{a}\ln\left|\frac{1-\cos\theta}{\sin\theta}\right|+C = \frac{1}{a}\ln\left|\frac{1-\frac{a}{x}}{\sqrt{1-\left(\frac{a}{x}\right)^2}}\right|+C$$

$$=\frac{1}{a}\ln\left|\sqrt{\frac{x-a}{x+a}}\right|+C = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

II (i) $\int\frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)+C$

$$=\ln\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+C$$

Proof: $I=\int\frac{dx}{\sqrt{a^2+x^2}}$

Put $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$

$$I=\int\frac{a\sec^2\theta\, d\theta}{\sqrt{a^2\tan^2\theta+a^2}} = \int\frac{a\sec^2\theta\, d\theta}{a\sec\theta} = \int\sec\theta\, d\theta$$

$$I = \ln|\sec\theta+\tan\theta|+C$$

$$I=\ln\left|\sqrt{1+\frac{x^2}{a^2}}+\frac{x}{a}\right|+C = \ln\left|\frac{\sqrt{x^2+a^2}+x}{a}\right|+C$$

(ii) $\int\frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right)+C$

Proof: $I=\int\frac{dx}{\sqrt{x^2-a^2}}$

Put, $x=a\sec\theta\Rightarrow\sec\theta=\frac{x}{a}$

On diff. $dx = a\sec\theta\tan\theta\, d\theta$

$$I=\int\frac{a\sec\theta\tan\theta\, d\theta}{\sqrt{a^2\sec^2\theta-a^2}} = \int\frac{a\sec\theta\tan\theta\, d\theta}{a\tan\theta} = \int\sec\theta\, d\theta$$

$$=\ln|\sec\theta+\tan\theta|+C = \ln\left|\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right|+C$$

$$=\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right|+C$$

(iii) $\int\frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C$

Proof: Put, $x=a\sin\theta$, $dx = a\cos\theta\, d\theta$

$$I=\int\frac{dx}{\sqrt{a^2-x^2}}$$

$$I=\int\frac{a\cos\theta\, d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int\frac{a\cos\theta\, d\theta}{\sqrt{a\cos\theta}} = \int d\theta = \theta+C$$

$$\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\left(\frac{x}{a}\right)+C$$

ILLUSTRATION 33

Evaluate the following integrals:

(i) $\int\frac{dx}{x^2-2x+9}$ **(ii)** $\int\frac{dx}{x^2-5x-7}$

(iii) $\int\frac{dx}{2x^2-3x+11}$ **(iv)** $\int\frac{dx}{3x^2-24x+48}$

SOLUTION

(i) $I=\int\frac{dx}{x^2-2x+9}$

$$I=\int\frac{dx}{(x-1)^2+8} = \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{x-1}{2\sqrt{2}}\right)+c$$

(ii) $I = \int \frac{dx}{x^2-5x-7} = \int \frac{dx}{\left(x-\frac{5}{2}\right)^2-\left(\frac{\sqrt{53}}{2}\right)^2}$

$$I = \frac{1}{2\left(\frac{\sqrt{53}}{2}\right)} \ln\left(\frac{x-\frac{5}{2}-\frac{\sqrt{53}}{2}}{x-\frac{5}{2}+\frac{\sqrt{53}}{2}}\right)+C$$

$$I = \frac{1}{\sqrt{53}} \ln\left|\frac{2x-5-\sqrt{53}}{2x-5+\sqrt{53}}\right|+C$$

(iii) $I = \int \frac{dx}{2x^2-3x+11}$

$$I = \frac{1}{2}\int \frac{dx}{x^2-\frac{3}{2}x+\frac{11}{2}} = \frac{1}{2}\int \frac{dx}{\left(x-\frac{3}{4}\right)^2+\frac{11}{2}-\frac{9}{16}}$$

$$I = \frac{1}{2}\int \frac{dx}{\left(x-\frac{3}{4}\right)^2+\left(\frac{\sqrt{79}}{4}\right)^2}$$

$$I = \frac{1}{2}\left(\frac{1}{\frac{\sqrt{79}}{4}}\tan^{-1}\left(\frac{x-\frac{3}{4}}{\frac{\sqrt{79}}{4}}\right)\right)+C$$

$$I = \frac{2}{\sqrt{79}}\tan^{-1}\left(\frac{4x-3}{\sqrt{79}}\right)+C$$

(iv) $I = \int \frac{dx}{3x^2-24x+48}$

$$I = \frac{1}{3}\int \frac{dx}{x^2-8x+16}$$

$$I = \frac{1}{3}\int \frac{dx}{(x-4)^2} = -\frac{1}{3(x-4)}+C$$

ILLUSTRATION 34

Evaluate the following integrals:

(i) $\int \frac{dx}{\sqrt{x^2-4x+11}}$ **(ii)** $\int \frac{dx}{\sqrt{3x^2+2x-11}}$

SOLUTION

(i) $I = \int \frac{dx}{\sqrt{x^2-4x+11}}$

$$I = \int \frac{dx}{\sqrt{(x-2)^2+7}} = \ln\left|x-2+\sqrt{x^2-4x+11}\right|+C$$

(ii) $I = \int \frac{dx}{\sqrt{3x^2+2x-11}}$

$$I = \int \frac{dx}{\sqrt{3}\sqrt{x^2+\frac{2x}{3}-\frac{11}{3}}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\left(x+\frac{1}{3}\right)^2-\left(\frac{\sqrt{34}}{3}\right)^2}}$$

$$= \frac{1}{\sqrt{3}}\ln\left|x+\frac{1}{3}+\sqrt{x^2+\frac{2x}{3}-\frac{11}{3}}\right|+C$$

ILLUSTRATION 35

Evaluate: $\int \frac{x\,dx}{x^4+x^2+1}$

SOLUTION

$$I = \int \frac{x\,dx}{x^4+x^2+1}$$

Put $x^2 = t$, $2x\,dx = dt$

$$I = \int \frac{\frac{dt}{2}}{t^2+t+1} = \frac{1}{2}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)+C = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x^2+1}{\sqrt{3}}\right)+C$$

ILLUSTRATION 36

Evaluate: $\int \frac{e^x dx}{\sqrt{4e^x-e^{2x}}}$

SOLUTION

Put $e^x = t$, $e^x dx = dt$

$$I = \int \frac{dt}{\sqrt{4t-t^2}} = \int \frac{dt}{\sqrt{4-(t-2)^2}}$$

$$I = \sin^{-1}\left(\frac{t-2}{2}\right)+C = \sin^{-1}\left(\frac{e^x-2}{2}\right)+C$$

ILLUSTRATION 37

Evaluate: $\int \frac{x\,dx}{\sqrt{x^4-2x^2+7}}$

SOLUTION

Put $x^2 = t$, $2x\,dx = dt$

$$I = \int \frac{\frac{dt}{2}}{\sqrt{t^2-2t+7}} = \frac{1}{2}\int \frac{dt}{\sqrt{(t-1)^2+6}}$$

$$I=\frac{1}{2}\ln\left|t-1+\sqrt{t^2-2t+7}\right|+C$$

$$I=\frac{1}{2}\ln\left|x^2-1+\sqrt{x^4-2x^2+7}\right|+C$$

ILLUSTRATION 38

Evaluate $\displaystyle\int\frac{\cos x\,dx}{\sin^2 x-4\sin x+3}$

SOLUTION

$$I=\int\frac{\cos x\,dx}{\sin^2 x-4\sin x+3}$$

Put $\sin x=t\Rightarrow\cos x\,dx=dt$

$$I=\int\frac{dt}{t^2-4t+3}=\int\frac{dt}{(t-2)^2-1}$$

$$=\frac{1}{2}\ln\left|\frac{t-2-1}{t-2+1}\right|+C=\frac{1}{2}\ln\left|\frac{\sin x-3}{\sin x-1}\right|+C$$

ILLUSTRATION 39

Evaluate: $\displaystyle\int\frac{x^2(1-\ln x)}{\ln^4 x-x^4}\,dx$

SOLUTION

$$I=\int\frac{x^2(1-\ln x)}{\ln^4 x-x^4}=\int\frac{x^2(1-\ln x)dx}{(\ln^2 x+x^2)(\ln x+x)(\ln x-x)}$$

$$I=\int\frac{(1-\ln x)dx}{x^2\left(\left(\frac{\ln x}{x}\right)^2+1\right)\left(\frac{\ln x}{x}-1\right)\left(\frac{\ln x}{x}+1\right)}$$

Put $\displaystyle\frac{\ln x}{x}=t\Rightarrow\frac{x\left(\frac{1}{x}\right)-\ln x}{x^2}dx=dt$

$$I=\int\frac{dt}{(t^2+1)(t^2-1)}=\int\frac{1}{2}\left(\frac{1}{t^2-1}-\frac{1}{t^2+1}\right)dt$$

$$I=\frac{1}{4}\ln\left|\frac{t-1}{t+1}\right|-\frac{1}{2}\tan^{-1}t+C$$

Replacing value of t, we get

$$I=\frac{1}{4}\ln\left|\frac{\ln x-x}{\ln x+x}\right|-\frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right)+C$$

ILLUSTRATION 40

Evaluate: $\displaystyle\int\left(\frac{x^2+\cos^2 x}{x^2+1}\right)\operatorname{cosec}^2 x\,dx$

SOLUTION

$$I=\int\left(\frac{x^2+\cos^2 x}{x^2+1}\right)\operatorname{cosec}^2 x\,dx$$

$$I=\int\left(\frac{x^2+1-\sin^2 x}{x^2+1}\right)\operatorname{cosec}^2 x\,dx$$

$$=\int\left(1-\frac{\sin^2 x}{x^2+1}\right)\operatorname{cosec}^2 x\,dx$$

$$=\int\operatorname{cosec}^2 x-\frac{1}{x^2+1}\,dx=-\cot x-\tan^{-1}x+C$$

ILLUSTRATION 41

Evaluate $\displaystyle\int\frac{x\,dx}{\sqrt{1+x^2}\left(\sqrt{x^2+1}+1\right)}$

SOLUTION

$$I=\int\frac{x\,dx}{\sqrt{1+x^2}\left(\sqrt{x^2+1}+1\right)}$$

Put $\sqrt{x^2+1}=t\Rightarrow x^2+1=t^2\Rightarrow x\,dx=t\,dt$

$$I=\int\frac{t\,dt}{t(t+1)}=\int\frac{dt}{t+1}$$

$$I=\ln(t+1)+C=\ln\left(\sqrt{x^2+1}+1\right)+C$$

ILLUSTRATION 42

Evaluate: $\displaystyle\int\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}\quad(\beta>\alpha)$

SOLUTION

$$I=\int\frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}\quad(\beta>\alpha)$$

Put $x=\alpha\cos^2\theta+\beta\sin^2\theta$

$$dx=2\alpha\cos\theta\,(-\sin\theta)\,d\theta+\beta(2\sin\theta)\cos\theta\,d\theta$$

$$dx=(\beta-\alpha)\sin 2\theta\,d\theta$$

$$I=\int\frac{(\beta-\alpha)\sin 2\theta d\theta}{\sqrt{(\alpha\cos^2\theta+\beta\sin^2\theta-\alpha)(\beta-\alpha\cos^2\theta-\beta\sin^2\theta)}}$$

$$I=(\beta-\alpha)\int\frac{\sin 2\theta d\theta}{\sqrt{(\beta-\alpha)\sin^2\theta(\beta-\alpha)\cos^2\theta}}$$

$$I=(\beta-\alpha)\int\frac{2\sin\theta\cos\theta}{(\beta-\alpha)\sin\theta\cos\theta}\,d\theta$$

$$I=2\int d\theta=2\theta+C\qquad\ldots(1)$$

As $x=\alpha\cos^2\theta+\beta\sin^2\theta$; $x=\alpha-\alpha\sin^2\theta+\beta\sin^2\theta$

$$\sin^2\theta=\frac{x-\alpha}{\beta-\alpha};\ \sin\theta=\sqrt{\frac{x-\alpha}{\beta-\alpha}}$$

From (1), we get, $I=2\sin^{-1}\sqrt{\dfrac{x-\alpha}{\beta-\alpha}}+C$ (A)

As $x = \alpha\cos^2\theta + \beta\sin^2\theta$

$\Rightarrow \quad x = \alpha\left(\frac{1+\cos 2\theta}{2}\right) + \beta\left(\frac{1-\cos 2\theta}{2}\right)$

$\Rightarrow \quad 2x - \alpha - \beta = (\alpha - \beta)\cos 2\theta$

$\Rightarrow \quad \cos 2\theta = \frac{\alpha+\beta-2x}{\beta-\alpha} \Rightarrow 2\theta = \cos^{-1}\left(\frac{\alpha+\beta-2x}{\beta-\alpha}\right)$

From (1), we get, $I = \cos^{-1}\left(\frac{\alpha+\beta-2x}{\beta-\alpha}\right) + C$ (B)

Both (A) and (B) are correct integrals.

Alternate:

$I = \int \frac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ $(\beta > \alpha)$ as It is $\int \frac{dx}{\sqrt{Q}}$ form

$$I = \int \frac{dx}{\sqrt{-\alpha\beta + x(\alpha+\beta) - x^2}}$$

$$= \int \frac{dx}{\sqrt{\left(\left(\frac{\beta+\alpha}{2}\right)^2 - \alpha\beta\right) - \left(x - \left(\frac{\alpha+\beta}{2}\right)\right)^2}}$$

$$I = \int \frac{dx}{\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2 - \left(x - \left(\frac{\alpha+\beta}{2}\right)\right)^2}}$$

$$I = \sin^{-1}\left(\frac{x - \left(\frac{\alpha+\beta}{2}\right)}{\frac{\beta-\alpha}{2}}\right) + C = \sin^{-1}\left(\frac{2x-\alpha-\beta}{\beta-\alpha}\right) + C$$

$$= -\sin^{-1}\left(\frac{\alpha+\beta-2x}{\beta-\alpha}\right) + C$$

$$I = -\left(\frac{\pi}{2} - \cos^{-1}\left(\frac{\alpha+\beta-2x}{\beta-\alpha}\right)\right) + C$$

(using $\sin^{-1}x + \cos^{-1}x = \pi/2$)

$$I = \cos^{-1}\left(\frac{\alpha+\beta-2x}{\beta-\alpha}\right) + K$$

Test Your Concepts-III

(Solutions on page H.231)

Evaluate the following integrals:

1. $\int \frac{dx}{\sqrt{(x+1)(x-2)}}$
2. $\int \frac{dx}{\sqrt{x^2-3x+2}}$
3. $\int \frac{dx}{\sqrt{-4x^2-12x-8}}$
4. $\int \frac{(1+x)^2}{x(1+x^2)}dx$
5. $\int \frac{x}{x^4-1}dx$
6. $\int \frac{dx}{20-6x-x^2}$
7. $\int \frac{xdx}{x^4-3x^2+2}$
8. $\int \frac{(2x^2-5)dx}{x^4-5x^2+6}$
9. $\int \frac{e^x \cdot dx}{\sqrt{e^{2x}-3e^x+1}}$

Answer key

1. $\ln\left(x - \frac{1}{2} + \sqrt{x^2-x-2}\right) + C$

2. $\ln\left(x - \frac{3}{2} + \sqrt{x^2-3x+2}\right) + C$

3. $\frac{1}{2}\text{arc}\sin(2x+3) + C$

4. $\ln|x| + 2\,\text{arc}\tan x + C$

5. $\frac{1}{4}\ln\left|\frac{x^2-1}{x^2+1}\right| + C$

6. $-\frac{1}{2\sqrt{29}}\ln\left|\frac{x+3-\sqrt{29}}{x+3+\sqrt{29}}\right| + C$

7. $\ln\sqrt{\frac{x^2-2}{x^2-1}} + C$

8. $\frac{1}{2\sqrt{3}}\ln\left(\frac{x-\sqrt{3}}{x+\sqrt{3}}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right) + C$

9. $\log\left|2e^x - 3 + \sqrt{e^{2x}-3e^x+1}\right| + c$

INTEGRATION BY PARTS

$$\int \underset{\text{I}}{f(x)}\cdot \underset{\text{II}}{g(x)}\,dx = f(x)\left(\int g(x)dx\right) - \int\left(\frac{df(x)}{dx}\right)\left(\int g(x)dx\right)dx$$

Differentiate both sides

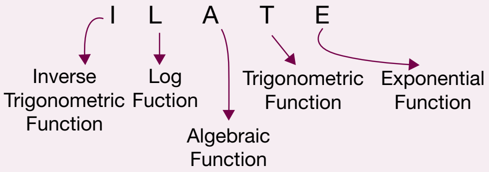
$$f(x)\,g(x) = \left(\frac{df(x)}{dx}\right)\int g(x)\,dx + f(x)\,g(x) - \left(\frac{df(x)}{dx}\right)\int g(x)\,dx$$

$= f(x)\,g(x)$ which is same as LHS

Conceptual Note(s)

$$\int I\cdot II\,dx = I\int II\,dx - \int\left(\frac{dI}{dx}\right)\left(\int IIdx\right)dx$$

In most of the cases



is generally used to decide the first and second function for using integration by parts. However, it is not always necessary to use this trick.

ILLUSTRATION 43

Evaluate: $\int x.\sin x\ dx$

SOLUTION

$$I = \int \underset{\text{I}}{x}.\underset{\text{II}}{\sin x}\ dx$$

x is linear function (Algebra) and sin x is trigonometric function, so we can consider x as first and sin x as second function.

$$I = x\int \sin x\ dx - \int 1\left(\int \sin x\ dx\right)dx$$

$$I = x(-\cos x) - \int(-\cos x)\ dx$$

$$I = -x\cos x + \int \cos x\ dx = -x\cos x + \sin x + C$$

ILLUSTRATION 44

Evaluate: $\int x.e^x\ dx$

SOLUTION

$$I = \int \underset{\text{I}}{x}.\underset{\text{II}}{e^x}\ dx$$

x is linear function (Algebra) and e^x is exponential function, so we can consider x as first and e^x as second function.

$$I = x\int e^x\ dx - \int 1\left(\int e^x dx\right)dx$$

$$= x(e^x) - \int e^x\ dx = xe^x - e^x + C = e^x(x-1) + C$$

ILLUSTRATION 45

Evaluate: $\int x^3.e^x\ dx$

SOLUTION

$$I = \int \underset{\text{I}}{x^3}.\underset{\text{II}}{e^x}\ dx = x^3e^x - \int 3x^2e^x\ dx$$

$$I = x^3e^x - 3\int \underset{\text{I}}{x^2}\,\underset{\text{II}}{e^x}\ dx$$

$$I = x^3e^x - 3\left[x^2e^x - \int 2xe^x dx\right]$$

$$I = x^3e^x - 3x^2e^x + 6\int \underset{\text{I}}{x}\,\underset{\text{II}}{e^x}\ dx$$

$$I = x^3e^x - 3x^2e^x + 6\left(xe^x - \int 1.e^x dx\right)$$

$$I = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

$$I = e^x(x^3 - 3x^3 + 6x - 6x + 6) + C$$

ILLUSTRATION 46

Evaluate: $\int \cos\sqrt{x}\ dx$

SOLUTION

Put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\ dt$

$$I = \int \cos t\cdot 2t\ dt = 2\int \underset{\text{I}}{t}\,\underset{\text{II}}{\cos t}\ dt$$

$$I = 2\left[t\sin t - \int 1.\sin t\ dt\right]$$

$$I = 2[\text{t}\sin t - (-\cos t)] + C$$

$$I = 2t\sin t + 2\cos t + C$$

$$I = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$$

ILLUSTRATION 47

Evaluate: $\int e^{ax}\sin bx\ dx$

SOLUTION

$$I = \int \underset{\text{II}}{e^{ax}}\,\underset{\text{I}}{\sin bx}\ dx$$

$$I = \left(\frac{e^{ax}}{a}\right)\sin bx - \int\left(\frac{e^{ax}}{a}\right)b\cos bx\ dx$$

$$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\int \underset{\text{II}}{e^{ax}}.\underset{\text{I}}{\cos bx}\,dx$$

$$= \frac{e^{ax}\sin bx}{a} - \frac{b}{a}\left[\frac{e^{ax}}{a}\cos bx - \int \frac{e^{ax}}{a}(-b\sin bx)\right] + C$$

$$I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a^2}e^{ax}\cos bx - \frac{b^2}{a^2}\int e^{ax}\sin bx\,dx + C$$

$$I + \frac{b^2}{a^2}I = \frac{e^{ax}\sin bx}{a} - \frac{b}{a^2}e^{ax}\cos bx + C$$

$$\Rightarrow \quad I\,\frac{(a^2+b^2)}{a^2} = \frac{e^{ax}[a\sin bx - b\cos bx]}{a^2} + C$$

$$\Rightarrow \quad I = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$$

ILLUSTRATION 48

Evaluate: $\int e^x(x^2-4x+7)\,dx$

SOLUTION

$$I = \int \underset{\text{II}}{e^x}(x^2 - \underset{\text{I}}{4}x + 7)\,dx$$

$$I = e^x(x^2-4x+7) - \int \underset{\text{II}}{e^x}(2x\underset{\text{I}}{-}4)\,dx$$

$$I = e^x(x^2-4x+7) - [e^x(2x-4) - \int e^x(2)\,dx]$$

$$I = e^x(x^2-4x+7-2x+4) + 2\int e^x\,dx$$

$$I = e^x(x^2-6x+11) + 2e^x + C = e^x(x^2-6x+13) + C$$

ILLUSTRATION 49

Evaluate: $\int x^2\ln(1+x)\,dx$

SOLUTION

$$I = \int x^2\ln(1+x)\,dx = \frac{x^3}{3}\ln(1+x) - \int \frac{x^3}{3}\frac{dx}{1+x}$$

$$I = \frac{x^3}{3}\ln(1+x) - \frac{1}{3}\int \frac{x^3+1-1}{1+x}\,dx$$

$$I = \frac{x^3}{3}\ln(1+x) - \frac{1}{3}\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right) + \frac{1}{3}\ln(1+x) + C$$

$$= \frac{(x^3+1)}{3}\ln(1+x) - \frac{x^3}{9} + \frac{x^2}{6} - \frac{x}{3} + C$$

ILLUSTRATION 50

Evaluate: $\int (\text{arc}\sin x)^2\,dx$

SOLUTION

$$I = \int (\sin^{-1}x)^2\,dx = \int 1\cdot(\sin^{-1}x)^2\,dx$$

$$I = x(\sin^{-1}x)^2 - \int x.2\sin^{-1}x.\frac{1}{\sqrt{1-x^2}}\,dx$$

Put $\sin^{-1}x = t \Rightarrow \dfrac{dx}{\sqrt{1-x^2}} = dt;\ x = \sin t$

$$I = x(\sin^{-1}x)^2 - 2\int \sin t\cdot t\,dt$$

$$I = x(\sin^{-1}x)^2 - 2((-\cos t)t - \int(-\cos t)dt)$$

$$I = x(\sin^{-1}x)^2 + 2\sin^{-1}x\cos(\sin^{-1}x) - 2\sin t + C$$

$$I = x(\sin^{-1}x)^2 + 2\sin^{-1}x\sqrt{1-x^2} - 2x + C$$

ILLUSTRATION 51

Evaluate: $\int (x^2+3x+5)\cos 2x\,dx$

SOLUTION

$$I = \int (x^2+3x+5)\cos 2x\,dx$$

$$I = (x^2+3x+5)\left(\frac{\sin 2x}{2}\right) - \int (2x+3)\left(\frac{\sin 2x}{2}\right)dx$$

$$I = \frac{(x^2+3x+5)}{2}\sin 2x - \frac{1}{2}\left[(2x+3)\left(\frac{-\cos 2x}{2}\right) - \int 2\left(\frac{-\cos 2x}{2}\right)dx\right]$$

$$I = \frac{(x^2+3x+5)\sin 2x}{2} + \frac{(2x+3)\cos 2x}{4} - \frac{1}{2}\left(\frac{\sin 2x}{2}\right) + C$$

$$I = \left(\frac{2x+3}{4}\right)\cos 2x + \left(\frac{x^2}{2} + \frac{3x}{2} + \frac{9}{4}\right)\sin 2x + C$$

ILLUSTRATION 52

Evaluate: $\displaystyle\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx$

SOLUTION

$$I = \int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx$$

$$I = \int \frac{\sin^{-1}\sqrt{x} - \left(\frac{\pi}{2} - \sin^{-1}\sqrt{x}\right)}{\frac{\pi}{2}}\cdot dx$$

$$= \frac{2}{\pi}\int\left\{2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right\}dx$$

$$I = \frac{4}{\pi}\int \sin^{-1}\sqrt{x}\,dx - x + C \qquad \ldots(1)$$

Now, $I_1 = \int \sin^{-1}\sqrt{x}\,dx = \int \theta\sin 2\theta\,d\theta$, where $x = \sin^2\theta$

$$I_1 = -\frac{\theta \cos 2\theta}{2} + \int \frac{1}{2} \cos 2\theta \, d\theta$$

$$I_1 = -\frac{\theta}{2} \cos 2\theta + \frac{1}{4} \sin 2\theta + C$$

$$I_1 = -\frac{1}{2} \theta (1 - 2 \sin^2\theta) + \frac{1}{2} \sin\theta \sqrt{1 - \sin^2 \theta} + C$$

$$I_1 = -\frac{1}{2} \sin^{-1} \sqrt{x} \, (1 - 2x) + \frac{1}{2} \sqrt{x} \sqrt{1 - x} \qquad \ldots(2)$$

From (1) and (2), we get

$$I = \frac{4}{\pi} \left\{ -\frac{1}{2}(1 - 2x)\sin^{-1} \sqrt{x} + \frac{1}{2}\sqrt{x - x^2} \right\} - x + C$$

$$I = \frac{2}{\pi} \{ \sqrt{x - x^2} - (1 - 2x) \sin^{-1} \sqrt{x} \} - x + C$$

ILLUSTRATION 53

Evaluate: $\displaystyle\int \frac{x^2 \cos x}{(1 + \sin x)^2} dx$

SOLUTION

$$I = \int x^2 \cdot \frac{\cos x}{(1 + \sin x)^2} dx$$

$$I = x^2 \int \frac{\cos x \, dx}{(1 + \sin x)^2} - \int 2x \int \frac{\cos x \, dx}{(1 + \sin x)^2} dx$$

$$I = x^2 \left(\frac{-1}{1 + \sin x} \right) + \int \frac{2x}{1 + \sin x} dx$$

$$I = \frac{-x^2}{1 + \sin x} + \int \frac{2x(1 - \sin x)}{1 - \sin^2 x} dx$$

$$= \frac{-x^2}{1 + \sin x} + \int 2x \, (\sec^2 x - \tan x \sec x) \, dx$$

$$I = \frac{-x^2}{(1 + \sin x)} + 2x(\tan x - \sec x) - \int 2(\tan x - \sec x) \, dx$$

$$I = \frac{-x^2}{(1 + \sin x)} + 2x(\tan x - \sec x) - 2 \ln \sec x + 2 \ln (\sec x + \tan x) + C$$

ILLUSTRATION 54

Evaluate: $\displaystyle\int \frac{x e^x dx}{\sqrt{1 + e^x}}$

SOLUTION

Put $\sqrt{1 + e^x} = t$; $x = \ln(t^2 - 1)$

$\Rightarrow \quad 1 + e^x = t^2 \Rightarrow e^x \, dx = 2t \, dt$

$$I = \int \frac{\ln(t^2 - 1) 2t \, dt}{t} = 2 \int \ln(t^2 - 1) dt$$

$$I = 2t \ln(t^2 - 1) - \int \frac{2t}{t^2 - 1} \cdot 2t \, dt$$

$$I = 2t \ln(t^2 - 1) - 4 \int \frac{t^2 - 1 + 1}{t^2 - 1} dt$$

$$= 2t \ln(t^2 - 1) - 4 \left(\int dt + \int \frac{dt}{t^2 - 1} \right)$$

$$= 2t \ln(t^2 - 1) - 4t - 2 \ln \left| \frac{t - 1}{t + 1} \right| + C \text{ where } t = \sqrt{1 + e^x}$$

$$= 2x \sqrt{1 + e^x} - 4 \sqrt{1 + e^x} - 2 \ln \left| \frac{\sqrt{e^x + 1} - 1}{\sqrt{e^x + 1} + 1} \right| + C$$

ILLUSTRATION 55

Evaluate: $\displaystyle\int \frac{x^2 dx}{(x \sin x + \cos x)^2}$

SOLUTION

$$I = \int \frac{x^2 dx}{(x \sin x + \cos x)^2}$$

ILATE is not used here

$$I = \int \underset{\text{I}}{\frac{x}{\cos x}} \cdot \underset{\text{II}}{\frac{(x \cos x)}{(x \sin x + \cos x)^2}} dx$$

$$I = \frac{x}{\cos x} \left\{ -\frac{1}{x \sin x + \cos x} \right\} - \int \frac{\cos x + x \sin x}{\cos^2 x} \, \frac{-1}{x \sin x + \cos x} dx$$

$$I = \frac{-x}{\cos x (x \sin x + \cos x)} + \int \sec^2 x \, dx$$

$$I = \frac{-x}{\cos x (x \sin x + \cos x)} + \tan x + C$$

$$I = \frac{-x}{\cos x (x \sin x + \cos x)} + \frac{\sin x}{\cos x} + C$$

$$I = \frac{\sin x - x \cos x}{x \sin x + \cos x} + C$$

ILLUSTRATION 56

Evaluate: $\displaystyle\int \tan^{-1} x \cdot \ln(1 + x^2) \, dx$

SOLUTION

$$I = \int 1 \cdot (\tan^{-1} x \cdot \ln(1 + x^2)) \, dx$$

$$I = x \, (\tan^{-1} x \ln (1 + x^2)) - \int x \left(\frac{1}{1 + x^2} \ln(1 + x^2) + \frac{\tan^{-1} x}{(1 + x^2)} 2x \right) dx$$

$$= x\tan^{-1}x\ln(1+x^2) - \int \frac{x}{1+x^2}\ln(1+x^2)dx - 2\int \frac{(x^2+1-1)}{x^2+1}\tan^{-1}x\,dx$$

$$= x\tan^{-1}x\ln(1+x^2) - \frac{\ln^2(1+x^2)}{4} - 2\int \tan^{-1}x\,dx + 2\int \frac{\tan^{-1}x}{x^2+1}dx$$

$$= x\tan^{-1}x\ln(1+x^2) - \frac{\ln^2(1+x^2)}{4} - 2x\tan^{-1}x + 2\int \frac{x}{1+x^2}dx + 2\frac{(\tan^{-1}x)^2}{2} + C$$

$$= x\tan^{-1}x\ln(1+x^2) - \frac{\ln^2(1+x^2)}{4} - 2x\tan^{-1}x + \ln(1+x^2) + (\tan^{-1}x)^2 + C$$

INTEGRATION BY CANCELLATION

When we integrate the addition of two functions, then in process of integration of one part of function cancels second integral. This is the reason we call this method '*Integration by cancellation*'.

(i) $\int [f(x) + xf'(x)]\,dx = xf(x) + C$

Proof: LHS $= \int f(x)dx + \int \underset{\text{I}}{x}\,\underset{\text{II}}{f'(x)}dx$

$= \int f(x)dx + (xf(x) - \int f(x)dx + C)$

$= xf(x) + C$

(ii) $\int e^x(f(x) + f'(x))\,dx = e^x f(x) + C$

Proof: LHS $= \int e^x f(x)\,dx + \int \underset{\text{I}}{e^x}\,\underset{\text{II}}{f'(x)}dx$

$= \int e^x f(x)\,dx + f(x)e^x - \int f(x)e^x dx\,dx + C$

$= e^x f(x) + C$

ILLUSTRATION 57

Evaluate: $\int xe^{\tan x}(2 + x\sec^2 x)\,dx$

SOLUTION

Let $I = \int (xe^{\tan x} + xe^{\tan x}(1 + x\sec^2 x))$

Using $\int (f(x) + xf'(x))\,dx = xf(x) + C$

$I = x^2 e^{\tan x} + C$

ILLUSTRATION 58

Evaluate: $\int \frac{x^2+2x-2}{(x+1)^2}dx$

SOLUTION

$$I = \int \frac{x^2+2x-2}{(x+1)^2}dx$$

$$I = \int \left(\frac{x-2}{x+1} + x\frac{3}{(x+1)^2}\right)dx$$

Using $\int (f(x) + xf'(x))\,dx = xf(x) + C$

$$I = x\left(\frac{x-2}{x+1}\right) + C$$

ILLUSTRATION 59

Evaluate: $\int e^x(x^2+5x+5)\,dx$

SOLUTION

$$I = \int e^x(x^2+5x+5)\,dx$$

$$I = \int e^x\left[\underset{f}{(x^2+3x+2)} + \underset{f'}{2x+3}\right]dx = e^x(x^2+3x+2) + C$$

ILLUSTRATION 60

Evaluate: $\int e^x(x^5 - 20x^3)\,dx$

SOLUTION

$$I = \int e^x(x^5 - 20x^3)\,dx$$

$$I = \int e^x\left(\underset{f}{x^5} + \underset{f'}{5x^4} - \underset{g}{5x^4} - \underset{g'}{20x^3}\right)dx$$

$$I = \int e^x(x^5+5x^4)\,dx - \int e^x(5x^4+20x^3)\,dx$$

$$I = e^x(x^5) - e^x(5x^4) + C = e^x(x^5 - 5x^4) + C$$

ILLUSTRATION 61

Evaluate: $\int e^x\left(\ln x + \frac{1}{x^2}\right)dx$

SOLUTION

$$I = \int e^x\left(\ln x + \frac{1}{x^2}\right)dx$$

$$I=\int e^x\left(\ln x+\frac{1}{x}-\frac{1}{x}+\frac{1}{x^2}\right)dx$$

$$I=\int e^x\left(\underbrace{\ln x}_{f}+\underbrace{\frac{1}{x}}_{f'}\right)dx+\int e^x\left(\underbrace{-\frac{1}{x}}_{g}+\underbrace{\frac{1}{x^2}}_{g'}\right)dx$$

$$I=e^x(\ln x)+e^x\left(-\frac{1}{x}\right)+C=e^x\left(\ln x-\frac{1}{x}\right)+C$$

ILLUSTRATION 62

Evaluate: $\int\frac{xe^x}{(x+1)^2}dx$

SOLUTION

$$I=\int\frac{xe^x}{(x+1)^2}dx=\int e^x\left(\frac{x+1-1}{(x+1)^2}\right)dx$$

$$I=\int e^x\left[\underbrace{\frac{1}{x+1}}_{f}-\underbrace{\frac{1}{(x+1)^2}}_{f'}\right]dx=e^x\left(\frac{1}{x+1}\right)+C$$

ILLUSTRATION 63

Evaluate: $\int\frac{e^x(1-x)^2}{(1+x^2)^2}dx$

SOLUTION

$$I=\int\frac{e^x(1-x)^2}{(1+x^2)^2}dx=\int e^x\frac{(1+x^2-2x)}{(1+x^2)^2}dx$$

$$=\int e^x\left[\underbrace{\frac{1}{1+x^2}}_{f}-\underbrace{\frac{2x}{(1+x^2)^2}}_{f'}\right]dx=\frac{e^x}{1+x^2}+C$$

ILLUSTRATION 64

Evaluate: $\int e^{2x}\left[\frac{1+\sin 2x}{1+\cos 2x}\right]dx$

SOLUTION

$$I=\int e^{2x}\left[\frac{1+\sin 2x}{1+\cos 2x}\right]dx$$

Put $2x=t$, $2dx=dt$

$$I=\int e^t\left[\frac{1+\sin t}{1+\cos t}\right]\frac{dt}{2}=\frac{1}{2}\int e^t\left[\frac{1+2\sin\frac{t}{2}\cos\frac{t}{2}}{1+2\cos^2\frac{t}{2}-1}\right]dt$$

$$I=\frac{1}{4}\int e^t\left[\underbrace{\sec^2\frac{t}{2}}_{f'}+\underbrace{2\tan\frac{t}{2}}_{f}\right]dt$$

$$I=\frac{1}{4}\left[e^t.2\tan\frac{t}{2}\right]+C=\frac{1}{2}e^t\tan\frac{t}{2}+C$$

$$I=\frac{1}{2}e^{2x}\tan x+C$$

ILLUSTRATION 65

Evaluate: $\int\frac{e^{\cos x}(x\sin^3 x+\cos x)}{\sin^2 x}dx$

SOLUTION

$$I=\int\frac{e^{\cos x}(x\sin^3 x+\cos x)}{\sin^2 x}dx$$

$$I=\int e^{\cos x}\frac{(x\sin^3 x+\cos x)}{\sin^3 x}\sin x\,dx$$

$$=\int e^{\cos x}(x+\cot x\,\mathrm{cosec}^2 x).\sin x\,dx$$

Put $\cos x=t\Rightarrow-\sin x\,dx=dt$

$$I=-\int e^t\left(\cos^{-1}t+\frac{t}{(1-t^2)^{3/2}}\right)dt$$

$$I=-\int e^t\left(\cos^{-1}t-\frac{1}{\sqrt{1-t^2}}+\frac{1}{\sqrt{1-t^2}}+\frac{t}{(1-t^2)^{3/2}}\right)dt$$

$$I=-\left[e^t\cos^{-1}t+e^t\frac{1}{\sqrt{1-t^2}}\right]+C$$

[using $\int e^x(f(x)+f'(x))\,dx=e^x$]

$$I=-e^{\cos x}\left(x+\frac{1}{\sin x}\right)+C=C-e^{\cos x}(x+\mathrm{cosec}\,x)$$

Test Your Concepts-IV

(Solutions on page H.232)

Evaluate following integrals:

1. $\int\ln x\ dx$

2. $\int\sin^{-1}x\,dx$

3. $\int x\ln^2 x\,dx$

4. $\int\cos(\ln x)\,dx$

5. $\int e^{\sqrt{x}}\ dx$

6. $\int\tan^{-1}x\,dx$

7. $\int x \tan^{-1} x \, dx$

8. $\int x^3 \sin x \, dx$

9. $\int x^n \ln x \, dx \; (n \neq -1)$

10. $\int \frac{x^2 dx}{(1+x^2)^2}$

11. $\int \frac{\text{arc} \sin \sqrt{x}}{\sqrt{1-x}} dx$

12. $\int \frac{(x+\sin x)dx}{1+\cos x}$

13. $\int e^{3x} (\sin 2x - \cos 2x) \, dx$

14. $\int \frac{\ln^2 x}{\sqrt{x^5}} dx$

15. $\int e^x (x^2 + \cos x + 2x - \sin x) \, dx$

16. $\int e^x (\sec x + \sec^2 x + (\sec x + 1) \tan x) dx \, dx$

17. $\int e^{\tan^2 x} \cdot \sin 4x \, dx$

18. $\int e^{\sin x} \frac{x \cos^3 x - \sin x}{\cos^2 x} dx$

19. If $\int \frac{x \tan^{-1} x}{\sqrt{1+x^2}} dx = \sqrt{1+x^2} \, (\tan^{-1} x) + K \log\left(x + \sqrt{x^2+1}\right)$ $+ C$, then find K.

20. $\int e^x \left(\frac{x^3 - x + 2}{(x^2+1)^2} \right) dx$

Answer key

1. $x \ln x - x + C$

2. $x \sin^{-1} x + \sqrt{1-x^2} + c$

3. $\frac{x^2}{2} \ln^2 x - \frac{x^2}{2} \ln x + \frac{x^2}{4} + C$

4. $\frac{x}{2} (\cos \ln x + \sin \ln x) + C$

5. $2e^{\sqrt{x}} (\sqrt{x} - 1) + C$

6. $x \tan^{-1} x - \frac{1}{2} \ln(1 + x^2) + C$

7. $\frac{x^2}{2} \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + C$

8. $C - x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x.$

9. $\frac{x^{n+1}}{n+1} \left(\ln x - \frac{1}{n+1} \right) + C$

10. $C - \frac{x}{2(1+x^2)} + \frac{1}{2} \text{arc} \tan x.$

11. $2(\sqrt{x} - \sqrt{1-x} \; \text{arc} \sin \sqrt{x}) + C$

12. $x \tan \frac{x}{2} + C$

13. $\frac{e^{3x}}{13} (\sin 2x - 5 \cos 2x) + C$

14. $C - \frac{8}{27\sqrt{x^3}} \left(\frac{9}{4} \ln^2 x + 3 \ln x + 2 \right)$

15. $e^x (x^2 + \cos x) + C$

16. $e^x (\sec x + \tan x) + C$

17. $-2 \cos^4 x \cdot e^{\tan^2 x} + C$

18. $e^{\sin x} (x - \sec x) + C$

19. -1

20. $e^x \left(\frac{x+1}{x^2+1} \right) + C$

INTEGRAL OF $\frac{L}{Q}$ OR $\frac{L}{\sqrt{Q}}$ FORM

$L = px + q$ (linear expression) and $Q = ax^2 + bx + c$ (quadratic expression)

We are looking for integrals of type $\int \frac{px+q}{ax^2+bx+c} dx$ or $\int \frac{px+q}{\sqrt{ax^2+bx+c}} dx$

Put, $L = \alpha \left(\frac{dQ}{dx} \right) + \beta$ or $px + q = \alpha \frac{d}{dx} (ax^2 + bx + c) + \beta$

Now get, $\alpha + \beta$ by equating:

(i) $\int \frac{L}{Q} dx = \int \frac{\alpha Q' + \beta}{Q} dx = \alpha \ln |Q| + \beta \int \frac{dx}{Q}$

(ii) $\int \frac{L}{\sqrt{Q}} dx = \int \frac{\alpha Q' + \beta}{\sqrt{Q}} dx = 2\alpha \sqrt{Q} + \beta \int \frac{dx}{\sqrt{Q}}$

where Integrals of $\int \frac{dx}{Q}$ and $\int \frac{dx}{\sqrt{Q}}$ we have studied earlier

ILLUSTRATION 66

Evaluate $\int \frac{x\,dx}{\sqrt{x^2+3x+2}}$

SOLUTION

$$\frac{L}{\sqrt{Q}} = \frac{x}{\sqrt{x^2+3x+2}}$$

where $L = x$, $Q = x^2 + 3x + 2$

$$L = \alpha Q' + \beta;\ x = \alpha(2x+3) + \beta$$

$$x = 2\alpha x + 3\alpha + \beta;\ \text{we have } 2\alpha = 1;\ \beta + 3\alpha = 0$$

We get $\alpha = \frac{1}{2};\ \beta = -\frac{3}{2}$

$$I = \int \frac{\alpha Q' + \beta dx}{\sqrt{Q}} = 2\alpha\sqrt{Q} + \beta\int \frac{dx}{\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$I = \sqrt{x^2+3x+2} - \frac{3}{2}\ln\left(x + \frac{3}{2} + \sqrt{x^2+3x+2}\right) + C$$

ILLUSTRATION 67

Evaluate: $\int \frac{5x-2\,dx}{3x^2+2x+1}$

SOLUTION

$$\frac{L}{Q} = \frac{5x-2}{3x^2+2x+1}$$

Where $L = 5x - 2$, $Q = 3x^2 + 2x + 1$

$$L = \alpha Q' + \beta;$$

$$5x - 2 = \alpha(6x+2) + \beta = 6\alpha x + 2\alpha + \beta;$$

$$6\alpha = 5 \text{ and } 2\alpha + \beta = -2$$

$$\alpha = \frac{5}{6};\ \frac{5}{3} + \beta = -2 \Rightarrow \beta = -2 - \frac{5}{3} = -\frac{11}{3}$$

$$I = \int \frac{5x-2}{3x^2+2x+1}dx = \int \frac{\alpha Q' + \beta}{Q}dx$$

$$I = \frac{5}{6}\int \frac{Q'}{Q}dx - \frac{11}{3}\int \frac{dx}{3x^2+2x+1}$$

$$I = \frac{5}{6}\ln|3x^2+2x+1| - \frac{11}{9}\int \frac{dx}{x^2 + \frac{2}{3}x + \frac{1}{3}}$$

$$= \frac{5}{6}\ln|3x^2+2x+1| - \frac{11}{9}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2}$$

$$I = \frac{5}{6}\ln|3x^2+2x+1| - \frac{11}{9}\cdot\frac{3}{\left(\sqrt{2}\right)}\tan^{-1}\left(\frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}}\right) + C$$

$$I = \frac{5}{6}\ln|3x^2+2x+1| - \frac{11}{3\sqrt{2}}\tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + C$$

ILLUSTRATION 68

Evaluate: $\int \frac{(2\cos x - 3)\sin x}{\cos^2 x - 2\cos x + 3}dx$

SOLUTION

$$I = \int \frac{(2\cos x - 3)\sin x}{\cos^2 x - 2\cos x + 3}dx$$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$I = \int \frac{(2t-3)(-dt)}{t^2-2t+3} = \int \frac{3-2t\,dt}{t^2-2t+3}$$

$$\frac{L}{Q} = \frac{3-2t}{t^2-2t+3}$$

$$L = \alpha Q' + \beta$$

$$3 - 2t = \alpha(2t-2) + \beta \quad\Rightarrow\quad 2\alpha = -2,\ \beta - 2\alpha = 3$$

We get $\alpha = -1$, $\beta = 1$

$$I = \int \frac{\alpha Q' + \beta}{Q}dt = \alpha\ln|Q| + \beta\int \frac{dt}{(t-1)^2+2}$$

$$I = -\ln|t^2-2t+3| + \int \frac{dt}{(t-1)^2+2}$$

$$I = -\ln|t^2-2t+3| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-1}{\sqrt{2}}\right) + C$$

Where $t = \cos x$

$$I = -\ln|\cos^2 x - 2\cos x + 3| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\cos x - 1}{\sqrt{2}}\right) + C$$

ILLUSTRATION 69

Evaluate: $\int \frac{(2e^{2x} - e^x)dx}{\sqrt{3e^{2x} - 4e^x + 1}}$

SOLUTION

$$I = \int \frac{(2e^x - 1)e^x\,dx}{\sqrt{3e^{2x} - 4e^x + 1}}$$

Put $e^x = t \Rightarrow e^x\,dx = dt$

$$I = \int \frac{2t-1}{\sqrt{3t^2-4t+1}}dt = \frac{1}{\sqrt{3}}\int \frac{2t-1\,dt}{\sqrt{t^2 - \frac{4t}{3} + \frac{1}{3}}}$$

$$\frac{L}{Q} = \frac{2t-1}{\sqrt{t^2 - \frac{4t}{3} + \frac{1}{3}}}$$

$$L = \alpha Q' + \beta \Rightarrow 2t - 1 = \alpha(2t - 4/3) + \beta$$

We get $\alpha = 1, \beta = \frac{1}{3}$

$$I = \frac{1}{\sqrt{3}}\int \frac{\alpha Q' + \beta}{\sqrt{Q}}\,dt = \frac{1}{\sqrt{3}}\left(\int \frac{Q'}{\sqrt{Q}}\,dt + \frac{1}{3}\int \frac{dt}{\sqrt{Q}}\right)$$

$$= \frac{1}{\sqrt{3}}\left(2\sqrt{Q} + \frac{1}{3}\int \frac{dt}{\sqrt{Q}}\right)$$

$$I = \frac{1}{\sqrt{3}}\left(2\sqrt{t^2 - \frac{4t}{3} + \frac{1}{3}} + \frac{1}{3}\int \frac{dt}{\sqrt{\left(t - \frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2}}\right)$$

$$I = \frac{2}{\sqrt{3}}\sqrt{t^2 - \frac{4t}{3} + \frac{1}{3}} + \frac{1}{3\sqrt{3}}\ln\left(t - \frac{2}{3} + \sqrt{t^2 - \frac{4t}{3} + \frac{1}{3}}\right) + C$$

where $t = e^x$

ILLUSTRATION 70

Let $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 5 & 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ x^2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 2\alpha x + \beta x^2 \\ 5x + \gamma x^2 + 3 \end{bmatrix}$, $\forall\, x \in R$ and $f(x)$ is a differentiable functions satisfying, $f(xy) = f(x) + x^2(y^2 - 1) + x(y - 1)$; $\forall\, x, y \in R$ and $f(1) = 3$. Evaluate $\int \frac{\alpha x^2 + \beta x + \gamma}{f(x)}\,dx$

SOLUTION

Given $\begin{bmatrix} 1 & 0 & 0 \\ 6 & 2 & 0 \\ 5 & 4 & 3 \end{bmatrix}\begin{bmatrix} x \\ x^2 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 2\alpha x + \beta x^2 \\ 5x + \gamma x^2 + 3 \end{bmatrix}$,

$$\Rightarrow \begin{bmatrix} x \\ 6x + 2x^2 \\ 5x + 4x^2 + 3 \end{bmatrix} = \begin{bmatrix} x \\ 2\alpha x + \beta x^2 \\ 5x + \gamma x^2 + 3 \end{bmatrix}$$

$\alpha = 3;\ \beta = 2;\ \gamma = 4.$

Given relation $f(xy) = f(x) + x^2(y^2 - 1) + x(y - 1)\ \forall\, x, y \in R$

Let $x = 1$, $f(y) = f(1) + (y^2 - 1) + y - 1$

$f(y) = y^2 + y + 1 \qquad (\because f(1) = 3)$

$f(x) = x^2 + x + 1$

Required integral $I = \int \frac{\alpha x^2 + \beta x + \gamma}{f(x)}\,dx = \int \frac{3x^2 + 2x + 4}{x^2 + x + 1}\,dx$

$$I = \int \frac{3(x^2 + x + 1) - x + 1}{x^2 + x + 1}\,dx$$

$$I = \int 3\,dx - \frac{1}{2}\int \frac{(2x + 1)}{x^2 + x + 1}\,dx + \frac{3}{2}\int \frac{dx}{x^2 + x + 1}$$

$$I = 3x - \frac{1}{2}\ln(x^2 + x + 1) + \frac{3}{2}\int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$I = 3x - \frac{1}{2}\ln(x^2 + x + 1) + \frac{3}{2}\,\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$I = 3x - \frac{1}{2}\ln(x^2 + x + 1) + \sqrt{3}\,\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + C$$

ILLUSTRATION 71

Evaluate: $\int \frac{x + (\text{arc}\cos 3x)^2}{\sqrt{1 - 9x^2}}\,dx$

SOLUTION

$$I = \int \frac{x + (\cos^{-1} 3x)^2}{\sqrt{1 - 9x^2}}\,dx$$

$$= \int \frac{x}{\sqrt{1 - 9x^2}}\,dx + \int \frac{(\cos^{-1} 3x)^2}{\sqrt{1 - 9x^2}}\,dx$$

In first integral

Put $\sqrt{1 - 9x^2} = t \Rightarrow 1 - 9x^2 = t^2$

$\Rightarrow \quad 0 - 18x\,dx = 2t\,dt \Rightarrow x\,dx = -\frac{t}{9}\,dt$

In second integral

Put $\cos^{-1}3x = \alpha \Rightarrow \frac{-1}{\sqrt{1 - 9x^2}}(3\,dx) = d\alpha$

$$\Rightarrow \quad \frac{dx}{\sqrt{1 - 9x^2}} = -\frac{d\alpha}{3}$$

$$I = \int -\frac{t\,dt}{9t} + \int \alpha^2\left(-\frac{d\alpha}{3}\right)$$

$$I = -\frac{t}{9} - \frac{\alpha^3}{9} = -\frac{1}{9}\left(\sqrt{1 - 9x^2} + (\cos^{-1} 3x)^3\right) + C$$

$$I = C - \frac{1}{9}\left[\sqrt{1 - 9x^2} + (\text{arc}\cos 3x)^3\right]$$

INTEGRATION USING PARTIAL FRACTION

Let $P(x)$ and $Q(x)$ are polynomials in x, then $\frac{P(x)}{Q(x)}$ is a rational function.

(i) If degree $P(x) <$ degree $Q(x)$, then $\frac{P(x)}{Q(x)}$ is called *proper rational function*.

(ii) If degree $P(x) \geq$ degree $Q(x)$, then $\frac{P(x)}{Q(x)}$ is called an *improper rational function*, we should use division process to change it into proper rational function

$\frac{P(x)}{Q(x)} = R(x) + \frac{T(x)}{Q(x)}$, where $R(x)$ is polynomial in x and $\frac{T(x)}{Q(x)}$ is proper rational function.

Any proper rational fraction $\frac{P(x)}{Q(x)}$ can be expressed as the sum of rational functions. Each such function is known as a *partial function* and the process of obtaining them is called resolving of the given function into partial function. Now we firstly resolve the denominator into the simplest factors according to the following:

Let L is a linear expression and Q is a quadratic expression

(i) $$\frac{P(x)}{L_1L_2} = \frac{A}{L_1} + \frac{B}{L_2}$$

(ii) $$\frac{P(x)}{L_1L_2L_3} = \frac{A}{L_1} + \frac{B}{L_2} + \frac{C}{L_3}$$

(iii) $$\frac{P(x)}{L_1^2L_2} = \frac{A}{L_1} + \frac{B}{L_1^2} + \frac{C}{L_2}$$

(iv) $$\frac{P(x)}{L_1L_2^3} = \frac{A}{L_1} + \frac{B}{L_2} + \frac{C}{L_2^2} + \frac{D}{L_2^3}$$

(v) $$\frac{P(x)}{L^kL_1L_2L_3\ldots L_{r-1}L_r} = \frac{A_1}{L} + \frac{A_2}{L^2} + \frac{A_3}{L^3} + \ldots + \frac{A_k}{L^k} + \frac{B_1}{L_1} + \frac{B_2}{L_2} + \frac{B_3}{L_3} + \ldots + \frac{B_r}{L_r}$$

(vi) $$\frac{P(x)}{LQ} = \frac{A}{L} + \frac{Bx+C}{Q}$$

(vii) $$\frac{P(x)}{Q_1Q_2} = \frac{Ax+B}{Q_1} + \frac{Cx+D}{Q_2}$$

(viii) $$\frac{P(x)}{L^2Q} = \frac{A}{L} + \frac{B}{L^2} + \frac{Cx+D}{Q}$$

ILLUSTRATION 72

Evaluate: $\int \frac{x^2+3x+4\,dx}{(x+1)(x+2)(x+3)}$

SOLUTION

$$\frac{x^2+3x+4}{\underset{L_1}{(x+1)}\underset{L_2}{(x+2)}\underset{L_3}{(x+3)}} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$\Rightarrow \quad x^2 + 3x + 4 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

$$\Rightarrow \quad x^2 + 3x + 4 = A(x^2+5x+6) + B(x^2+4x+3) + C(x^2+3x+2)$$

$$\Rightarrow \quad x^2 + 3x + 4 = (A+B+C)x^2 + (5A+4B+3C)x + (6A+3B+2C)$$

$$A + B + C = 1;\ 5A + 4B + 3C = 3;\ 6A + 3B + 2C = 4$$

On solving, we get: $A = 1$; $B = -2$; $C = 2$

Alternate:

$$\frac{x^2+3x+4}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{(x+2)} + \frac{C}{x+3}$$

$$\Rightarrow \quad x^2 + 3x + 4 = A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2)$$

We get an identity:

Put $x = -2$; $\quad 4 - 6 + 4 = 0 + B(-1)(1) + 0 \Rightarrow B = -2$

Put $x = -1$; $\quad 1 - 3 + 4 = A(1)(2) + 0 + 0 \Rightarrow A = 1$

Put $x = -3$; $\quad 9 - 9 + 4 = 0 + 0 + C(-2)(-1) \Rightarrow C = 2$

Now $$\int \frac{x^2+3x+4}{(x+1)(x+2)(x+3)}dx = \int \frac{1}{x+1} - \frac{2}{x+2} + \frac{2}{x+3}dx$$

$$= \ln(x+1) - 2\ln(x+2) + 2\ln(x+3) + K$$

$$= \ln\left(\frac{(x+1)(x+3)^2}{(x+2)^2}\right) + K$$

ILLUSTRATION 73

Evaluate: $\int \frac{dx}{(x-1)(x^2-3x+2)}$

SOLUTION

$$\frac{1}{(x-1)(x^2-3x+2)} = \frac{1}{(x-1)(x-1)(x-2)}$$

$$= \frac{1}{(x-1)^2(x-2)} \qquad \left(\frac{1}{L_1^2L_2}\text{Form}\right)$$

$$\frac{1}{(x-1)^2(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-2)}$$

$$\Rightarrow \quad 1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

Put $x = 1$; $1 = 0 - B + 0 \Rightarrow B = -1$

Put $x = 2$; $1 = 0 + 0 + C \Rightarrow C = 1$

Put $x = 0$; $1 = 2A - 2B + C \Rightarrow 1 = 2A + 2 + 1$ or $A = -1$

$$I = \int -\frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{x-2}dx$$

$$I = -\ln(x-1) + \frac{1}{x-1} + \ln(x-2) + C$$

$$I = \ln\left(\frac{x-2}{x-1}\right) + \frac{1}{x-1} + C$$

ILLUSTRATION 74

Evaluate: $\displaystyle\int \frac{(3x-1)dx}{(x-1)^2(x+1)(x+2)}$

SOLUTION

It is of form $\dfrac{P(x)}{L_1^2L_2L_3}$

$$\frac{(3x-1)}{(x-1)^2(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{x+2}$$

$$\Rightarrow \quad (3x-1) = A(x-1)\,(x+1)\,(x+2) + B(x+1)\,(x+2) + C(x-1)^2\,(x+2) + D(x-1)^2\,(x+1)$$

Put $x = 1;\ 2 = 0 + B(2)\,(3) + 0 + 0 \Rightarrow B = \dfrac{1}{3}$

Put $x = -1;\ -4 = 0 + 0 + C(4)\,(1) + 0 \Rightarrow C = -1$

Put $x = -2;\ -7 = 0 + 0 + 0 + D(9)\,(-1) + 0 \Rightarrow D = \dfrac{7}{9}$

Put $x = 0;\ -1 = -2A + 2B + 2C + D \Rightarrow A = \dfrac{2}{9}$

$$I = \int \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{1}{(x+1)} + \frac{7}{9(x+2)}\,dx$$

$$= \frac{2}{9}\ln|x-1| - \frac{1}{3(x-1)} - \ln|x+1| + \frac{7}{9}\ln|x+2| + C$$

ILLUSTRATION 75

Evaluate: $\displaystyle\int \frac{(2x-1)dx}{(x+1)(x^2+2)}$

SOLUTION

Algebraic expression is of form $\dfrac{P(x)}{LQ}$

$$I = \int \frac{(2x-1)dx}{(x+1)(x^2+2)} = \int -\frac{1}{x+1} + \frac{x+1}{x^2+2}\,dx$$

$$I = -\ln|x+1| + \int \frac{\frac{1}{2}(2x)+1}{x^2+2}\,dx \qquad (L/Q \text{ form})$$

$$I = -\ln|x+1| + \frac{1}{2}\int \frac{2x}{x^2+2}\,dx + \int \frac{dx}{x^2+2}$$

$$I = -\ln|x+1| + \frac{1}{2}\ln(x^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$$

ILLUSTRATION 76

Evaluate: $\displaystyle\int \frac{2x^3-3x^2-8x-26}{2x^2-5x-12}\,dx$

SOLUTION

$\dfrac{2x^3-3x^2-8x-26}{2x^2-5x-12}$ is not partial fraction as numerator has more degree than denominator

$$2x^3-3x^2-8x-26 = (x+1)(2x^2-5x-12)+9x-14$$

$$I = \int \frac{(x+1)(2x^2-5x-12)+9x-14}{2x^2-5x-12}\,dx$$

$$= \int x+1+\frac{9x-14}{2x^2-5x-12}\,dx$$

$$I = \frac{x^2}{2} + x + \frac{1}{2}\int \frac{\frac{9}{2}\left(2x-\frac{5}{2}\right)-14+\frac{45}{4}}{x^2-\frac{5}{2}x-6}\,dx$$

Writing numerator as $L = \alpha Q' + \beta$

$$I = \frac{x^2}{2} + x + \frac{9}{4}\int \frac{2x-\frac{5}{2}}{x^2-\frac{5}{2}x-6}\,dx - \frac{11}{8}\int \frac{dx}{\left(x-\frac{5}{4}\right)^2 - 6 - \frac{25}{16}}$$

$$= \frac{x^2}{2} + x + \frac{9}{4}\ln\left|x^2-\frac{5}{2}x-6\right| - \frac{11}{8}\int \frac{dx}{\left(x-\frac{5}{4}\right)^2 - \left(\frac{11}{4}\right)^2}$$

$$= \frac{x^2}{2} + x + \frac{9}{4}\ln\left|x^2-\frac{5}{2}x-6\right| - \frac{1}{4}\ln\left|\frac{x-4}{x+\frac{3}{2}}\right| + C$$

ILLUSTRATION 77

Evaluate: $\displaystyle\int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}\,dx$

SOLUTION

$$I = \int \frac{x^3+3x+2}{(x^2+1)^2(x+1)}\,dx$$

$$I = \int \frac{x(x^2+1)+2(x+1)}{(x^2+1)^2(x+1)}\,dx$$

$$= \int \frac{x}{(x^2+1)(x+1)}\,dx + \int \frac{2}{(x^2+1)^2}\,dx$$

$$I = I_1 + I_2 \qquad \ldots(1)$$

For I_1 using partial fraction, we get

$$I_1 = \int \frac{x}{(x^2+1)(x+1)}\,dx$$

$$= \int -\frac{1}{2(x+1)} + \frac{x}{2(x^2+1)} + \frac{1}{2(x^2+1)}\,dx$$

$$I_1 = -\frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{1}{2}\int \frac{x}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$I_1 = -\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\tan^{-1}x + C_1$$

$$I_2 = \int \frac{2}{(x^2+1)^2}\,dx$$

Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$I_2 = 2\int \frac{\sec^2\theta d\theta}{(\tan^2\theta + 1)^2} = 2\int \cos^2\theta\, d\theta$$

$$= \int (1+\cos 2\theta)\, d\theta = \theta + \frac{\sin 2\theta}{2} + c_2$$

$$I_2 = \theta + \frac{\tan\theta}{1+\tan^2\theta} + C_2 = \tan^{-1}x + \frac{x}{1+x^2} + C_2$$

From (1)

$$I = -\frac{1}{2}\ln|x+1| + \frac{1}{4}\ln|x^2+1| + \frac{3}{2}\tan^{-1}x + \frac{x}{x^2+1} + C$$

ILLUSTRATION 78

Evaluate: $\displaystyle\int \frac{x^5 + 2x^3 + 4x + 4}{x^4 + 2x^3 + 2x^2}\, dx$

SOLUTION

$$I = \int \frac{x^5 + 2x^3 + 4x + 4}{x^4 + 2x^3 + 2x^2}\, dx$$

$$I = \int \left[x - 2 + \frac{4x^3 + 4x^2 + 4x + 4}{x^2(x^2 + 2x + 2)} \right] dx$$

$$I = \int (x-2)\, dx + \int \frac{2x^2(2x+2) + 2(x^2+2x+2) - 2x^2}{x^2(x^2+2x+2)}\, dx$$

$$I = \int (x-2)\, dx + \int \frac{2(2x+2)}{(x^2+2x+2)}\, dx + \int \frac{2}{x^2}\, dx - \int \frac{2}{1+(1+x)^2}\, dx$$

$$I = \frac{x^2}{2} - 2x + 2\ln(x^2+2x+2) - \frac{2}{x} - 2\tan^{-1}(1+x) + C$$

ILLUSTRATION 79

Evaluate: $\displaystyle\int \frac{(x^2-4)dx}{(x^2+1)(x^2+2)(x^2+3)}$

SOLUTION

Let $x^2 = t$, then given expression is $\dfrac{t-4}{(t+1)(t+2)(t+3)}$

$$\frac{t-4}{(t+1)(t+2)(t+3)} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{C}{t+3}$$

$\Rightarrow \quad (t-4) = A(t+2)\,(t+3) + B(t+1)\,(t+3) + C(t+1)\,(t+2)$

$\Rightarrow \quad (t-4) = A(t^2 + 5t + 6) + B(t^2 + 4t + 3) + C(t^2 + 3t + 2)$

$\Rightarrow \quad t - 4 = t^2\,(A + B + C) + t\,(5A + 4B + 3C) + 6A + 3B + 2C$

We get: $A + B + C = 0$, $5A + 4B + 3C = 1$, $6A + 3B + 2C = -4$

$$A = -\frac{5}{2},\ B = 6,\ C = -\frac{7}{2}$$

$$I = \int \frac{(x^2-4)dx}{(x^2+1)(x^2+2)(x^2+3)}$$

$$= \int \frac{-5}{2(x^2+1)} + \frac{6}{x^2+2} - \frac{7}{2(x^2+3)}\, dx$$

$$I = -\frac{5}{2}\tan^{-1}x + \frac{6}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{7}{2\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$

ILLUSTRATION 80

Evaluate: $\displaystyle\int \frac{e^x - 2\, dx}{(e^x+3)(e^x-4)}$

SOLUTION

Let $e^x = t$, then given expression can be written as

$$\frac{t-2}{(t+3)(t-4)} = \frac{A}{t+3} + \frac{A}{t-4}$$

$\Rightarrow \quad t - 2 = A(t-4) + B(t+3)$

$\Rightarrow \quad t - 2 = t(A + B) - 4A + 3B$

$A + B = 1;\ 3B - 4A = -2$

We get $A = \dfrac{5}{7}$, $B = \dfrac{2}{7}$

$$I = \int \frac{e^x - 2\, dx}{(e^x+3)(e^x-4)} = \int \frac{5}{7(e^x+3)} + \frac{2}{7(e^x-4)}\, dx$$

$$I = \frac{5}{7}\int \frac{e^{-x}}{1+3e^{-x}}\, dx + \frac{2}{7}\int \frac{e^{-x}}{1-4e^{-x}}\, dx$$

Put $1 + 3e^{-x} = t$ | Put $1 - 4\,e^{-x} = z$

$0 - 3e^{-x}\, dx = dt$ | $4e^{-x}\, dx = dz$

$$I = \frac{5}{7}\int \frac{dt}{(-3)t} + \frac{2}{7}\int \frac{dz}{4(z)} = -\frac{5}{21}\ln t + \frac{1}{14}\ln z + C$$

$$I = -\frac{5}{21}\ln(1 + 3e^{-x}) + \frac{1}{14}\ln(1 - 4e^{-x}) + C$$

ILLUSTRATION 81

Evaluate: $\displaystyle\int \frac{2e^{5x} + e^{4x} - 4e^{3x} + 4e^{2x} + 2e^x}{(e^{2x}+4)(e^{2x}-1)^2}\, dx$

SOLUTION

$$\int \frac{2e^{5x} + e^{4x} - 4e^{3x} + 4e^{2x} + 2e^x}{(e^{2x}+4)(e^{2x}-1)^2}\, dx$$

Let $e^x = t \Rightarrow e^x\, dx = dt$

$$I = \int \frac{2t^4 + t^3 - 4t^2 + 4t + 2}{(t^2+4)(t^2-1)^2}\, dt$$

$$I = \int \frac{2(t^4 - 2t^2 + 1) + t(t^2+4)}{(t^2+4)(t^2-1)^2}\, dt$$

$$I = \int \frac{2}{t^2+4}\, dt + \int \frac{t}{(t^2-1)^2}\, dt$$

$$I = \tan^{-1}\frac{t}{2} + \left(-\frac{1}{2}\right)\frac{1}{(t^2-1)} + C$$

$$I = \tan^{-1}\frac{e^x}{2} - \frac{1}{2}\cdot\frac{1}{(e^{2x}-1)} + C$$

Test Your Concepts-V

(Solutions on page H.234)

Evaluate following integrals:

1. $\int \frac{x+3}{x^2-2x-5}\,dx$

2. $\int \frac{x+2}{\sqrt{x^2-1}}\,dx$

3. $\int \frac{x\ln x}{(x^2-1)^{3/2}}\,dx$

4. $\int \frac{(6x-5)dx}{2\sqrt{3x^2-5x+6}}$

5. $\int \frac{e^x(1+e^x)dx}{\sqrt{1-e^{2x}}}$

6. $\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx$

7. $\int \frac{xdx}{(x+1)(2x+1)}$

8. $\int \frac{2x^2+41x-91}{(x+1)(x+3)(x-4)}\,dx$

9. $\int \frac{x^3-6x^2+9x+7}{(x-2)^3(x-5)}\,dx$

10. $\int \frac{(3+x^2)^2 x^3 dx}{(1+x^2)^3}$

11. $\int \frac{x^2+3}{x^6\cdot(x^2+1)}\,dx$

12. $\int \frac{x+1}{x(1+xe^x)^2}\,dx$

13. $\int \frac{dx}{(x-1)\,(1+x^2)}$

14. $\int \frac{3x+1}{(x-1)^3\cdot(x+1)}\,dx$

Answer key

1. $\frac{1}{2}\ln|x^2-2x-5|+\frac{2}{\sqrt{6}}\ln\left|\frac{x-1-\sqrt{6}}{x-1+\sqrt{6}}\right|+C$

2. $\sqrt{x^2-1}+2\ln\left|x+\sqrt{x^2-1}\right|+C$

3. $-\frac{\ln x}{\sqrt{x^2-1}}+\sec^{-1}x+C$

4. $\sqrt{3x^2-5x+6}+C$

5. $\text{arc sin } e^x-\sqrt{1-e^{2x}}+C$

6. $2\sqrt{1+\ln x}+\ln\left|\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right|+C$

7. $\ln\frac{|x+1|}{\sqrt{2x+1}}+C$

8. $\ln\left|\frac{(x-1)^4(x-4)^5}{(x+3)^7}\right|+C$

9. $\frac{3}{2(x-2)^2}+\ln|x-5|+C$

10. $\frac{1}{2}x^2+\frac{3}{2}\ln(1+x^2)+\frac{1}{(1+x^2)^2}+C$

11. $C-\frac{2}{x}+\frac{2}{3}\frac{1}{x^3}-\frac{3}{5}\frac{1}{x^5}-2\tan^{-1}x$

12. $\frac{1}{1+xe^x}+\log\frac{xe^x}{1+xe^x}+C$

13. $\frac{1}{2}\log(x-1)-\frac{1}{4}\log(1+x^2)-\frac{1}{2}\tan^{-1}x+C$

14. $\frac{1}{4}\log\frac{x+1}{x-1}-\frac{1}{2(x-1)}-\frac{1}{(x-1)^2}+C$

INTEGRATION OF IRRATIONAL FUNCTION

Type I: $\sqrt{Q}$ form

(i) $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

(Put $x=\sin\theta$)

(ii) $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln(x+\sqrt{x^2-a^2})+C$

(Put $x=a\sec\theta$)

(iii) $\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})+C$

(Put $x=a\tan\theta$)

Note

Always make coefficient of x^2 as 1

Proof

(i) $I=\int\sqrt{a^2-x^2}\,dx$

Put $x=a\sin\theta$; $dx=a\cos\theta\,d\theta$

$$I=\int\sqrt{a^2-a^2\sin^2\theta}\,(a\cos\theta\,d\theta)=\int a\cos\theta\cdot a\cos\theta\,d\theta$$

$$I=a^2\int\cos^2 d\theta=a^2\int\frac{1+\cos 2\theta}{2}\,d\theta$$

$$I=\frac{a^2}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]+C$$

As $\sin\theta=\frac{x}{a}\Rightarrow\theta=\sin^{-1}\left(\frac{x}{a}\right)$

$$I=\frac{a^2}{2}\left[\sin^{-1}\frac{x}{a}+\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right]+C$$

$$I=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$$

(ii) $I=\int\sqrt{x^2-a^2}\,dx$

Put $x=a\sec\theta \Rightarrow dx=a\sec\theta\tan\theta\,d\theta$

$$I=\int\sqrt{a^2\sec^2\theta-a^2}\,.(a\sec\theta\tan\theta)\,d\theta$$

$$I=\int(a\tan\theta)(a\sec\theta\tan\theta)\,d\theta=a^2\int\sec\theta\tan^2\theta\,d\theta$$

$$I=a^2\int\underset{\text{II}}{(\sec\theta\tan\theta)}\underset{\text{I}}{\tan\theta}\,d\theta$$

$$=a^2\left[\sec\theta.\tan\theta-\int\sec\theta\cdot\sec^2\theta\,d\theta\right]$$

$$I=a^2\left[\sec\theta\tan\theta-\int\sec\theta(1+\tan^2\theta)d\theta\right]$$

$$I=a^2\left[\sec\theta\tan\theta-\int\sec\theta\,d\theta-\int\sec\theta\tan^2\theta\,d\theta\right]$$

$$\Rightarrow 2I=a^2\sec\theta\tan\theta-a^2\ln(\sec\theta+\tan\theta)+C$$

As $\sec\theta=\frac{x}{a}\Rightarrow\theta=\sec^{-1}\left(\frac{x}{a}\right)$

$$\Rightarrow 2I=a^2\left(\frac{x}{a}\right)\sqrt{\frac{x^2}{a^2}-1}-a^2\ln\left(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right)+C$$

$$\Rightarrow I=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)+C$$

(iii) $I=\int\sqrt{x^2+a^2}\,dx$

$$I=\int\sqrt{x^2+a^2}\,dx=\int\underset{\text{II}}{1}\cdot\underset{\text{I}}{\sqrt{x^2+a^2}}\,dx$$

Using integration by parts, we get

$$I=x\sqrt{x^2+a^2}-\int x\left(\frac{1}{2\sqrt{x^2+a^2}}(2x)\right)dx$$

$$I=x\sqrt{x^2+a^2}-\int\frac{(x^2+a^2-a^2)}{\sqrt{x^2+a^2}}dx$$

$$I=x\sqrt{x^2+a^2}-\int\sqrt{x^2+a^2}\,dx+a^2\int\frac{dx}{\sqrt{x^2+a^2}}$$

$$2I=x\sqrt{x^2+a^2}+a^2\ln(x+\sqrt{x^2+a^2})+C$$

$$I=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\ln(x+\sqrt{x^2+a^2})+C$$

ILLUSTRATION 82

Evaluate: $\int\sqrt{x^2+4x-5}\,dx$

SOLUTION

$$I=\int\sqrt{x^2+4x-5}\,dx$$

$$I=\int\sqrt{(x+2)^2-3^2}\,dx$$

$$I=\frac{(x+2)}{2}\sqrt{x^2+4x-5}-\frac{9}{2}\ln\left(x+2+\sqrt{x^2+4x-5}\right)+C$$

ILLUSTRATION 83

Evaluate: $\int\sqrt{15+2x-x^2}\,dx$

SOLUTION

$$I=\int\sqrt{15+2x-x^2}\,dx=\int\sqrt{4^2-(x-1)^2}\,dx$$

$$I=\frac{(x-1)}{2}\sqrt{15+2x-x^2}+\frac{16}{2}\sin^{-1}\left(\frac{x-1}{4}\right)+C$$

ILLUSTRATION 84

Evaluate: $\int\sqrt{2x^2-4x-3}\,dx$

SOLUTION

$$I=\int\sqrt{2x^2-4x-3}\,dx=\sqrt{2}\int\sqrt{x^2-2x-\frac{3}{2}}\,dx$$

$$I=\sqrt{2}\int\sqrt{(x-1)^2-\left(\sqrt{\frac{5}{2}}\right)^2}\,dx$$

$$I=\sqrt{2}\left[\frac{(x-1)}{2}\sqrt{x^2-2x-\frac{3}{2}}-\frac{5}{4}\ln\left(x-1+\sqrt{(x-1)^2-\frac{5}{2}}\right)+C\right]$$

$$I=\frac{x-1}{\sqrt{2}}\sqrt{x^2-2x-\frac{3}{2}}-\frac{5}{2\sqrt{2}}\ln\left(x-1+\sqrt{x^2-2x-\frac{3}{2}}\right)+C$$

TYPE II: *L* is linear polynomial and *Q* is quadratic expression

(i) $\int\frac{dx}{L_1\sqrt{L_2}}$ Put $\sqrt{L_2}=t\Rightarrow L_2=t^2$

(ii) $\int\frac{dx}{Q\sqrt{L}}$ Put $\sqrt{L}=t\Rightarrow L=t^2$

(iii) $\int\frac{dx}{L\sqrt{Q}}$ Put $L=\frac{1}{t}$

(iv) $\int\frac{dx}{Q_1\sqrt{Q_2}}$ Put $x=\frac{1}{t}$

ILLUSTRATION 85

Evaluate: $\displaystyle\int \frac{dx}{(x-3)\sqrt{x+1}}$

SOLUTION

Given expression is of $\dfrac{1}{L_1\sqrt{L_2}}$ type

Put $\sqrt{x+1} = t \Rightarrow x+1 = t^2 \Rightarrow x = t^2 - 1 \Rightarrow dx = 2t\,dt$

$$I = \int \frac{2t\,dx}{(t^2-4)t} = 2\int \frac{dt}{t^2-4}$$

$$= 2\left(\frac{1}{2(2)}\ln\left|\frac{t-2}{t+2}\right|\right) + C = \frac{1}{2}\ln\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right| + C$$

ILLUSTRATION 86

Evaluate: $\displaystyle\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$

SOLUTION

Given expression is of $\dfrac{1}{L\sqrt{Q}}$ type

Put $x+1 = \dfrac{1}{t} \Rightarrow dx = -\dfrac{dt}{t^2}$

$$I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2+\frac{1}{t}}} = -\int \frac{dt}{\sqrt{(1-t)^2+t}}$$

$$I = -\int \frac{dt}{\sqrt{t^2-t+1}} = -\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$I = -\ln\left(t-\frac{1}{2}+\sqrt{t^2-t+1}\right) + C$$

where $t = \dfrac{1}{x+1}$

ILLUSTRATION 87

Evaluate $\displaystyle\int \frac{dx}{(2x-3)\sqrt{4x-x^2}}$

SOLUTION

$$I = \int \frac{dx}{(2x-3)\sqrt{4x-x^2}} \qquad \left(\frac{1}{L\sqrt{Q}}\text{form}\right)$$

Put $L = \dfrac{1}{t}$; $2x-3 = \dfrac{1}{t} \Rightarrow 2\,dx = -\dfrac{1}{t^2}\,dt$

$$I = \int \frac{\frac{-1}{2t^2}dt}{\frac{1}{t}\sqrt{\frac{4}{2}\left(\frac{1}{t}+3\right)-\frac{1}{4}\left(\frac{1}{t}+3\right)^2}}$$

$$= -\int \frac{dt}{2t\sqrt{\frac{2}{t}+6-\frac{1}{4}\left(\frac{1}{t^2}+9+\frac{6}{t}\right)}}$$

$$I = -\int \frac{dt}{2t\sqrt{\frac{8t+24t^2-1-9t^2-6t}{4t^2}}} = -\int \frac{dt}{\sqrt{15t^2+2t-1}}$$

$$= -\frac{1}{\sqrt{15}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{15}\right)^2-\left(\frac{4}{15}\right)^2}}$$

$$= -\frac{1}{\sqrt{15}}\ln\left|t+\frac{1}{15}+\sqrt{t^2+\frac{2t}{15}-\frac{1}{15}}\right| + C$$

$$= -\frac{1}{\sqrt{15}}\ln\left|\frac{1}{2x-3}+\frac{1}{15}+\sqrt{\frac{1}{(2x-3)^2}+\frac{2}{15(2x-3)}-\frac{1}{15}}\right| + C$$

$$= -\frac{1}{\sqrt{15}}\ln\left|\frac{2x+12+\sqrt{240x-60x^2}}{15(2x-3)}\right| + C$$

ILLUSTRATION 88

Evaluate: $\displaystyle\int \frac{\sqrt{2-x-x^2}}{x^2}dx$

SOLUTION

$$I = \int \frac{\sqrt{2-x-x^2}}{x^2}dx = \int \sqrt{2-x-x^2}\left(\frac{1}{x^2}\right)dx$$

$$I = \sqrt{2-x-x^2}\left(-\frac{1}{x}\right) - \int \frac{(-1-2x)}{2\sqrt{2-x-x^2}}\left(-\frac{1}{x}\right)dx$$

$$I = -\frac{\sqrt{2-x-x^2}}{x} - \int \frac{dx}{\sqrt{2-x-x^2}} - \int \frac{dx}{2x\sqrt{2-x-x^2}}$$

$$I = -\frac{\sqrt{2-x-x^2}}{x} - \int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}} - I_1$$

$$I = -\frac{\sqrt{2-x-x^2}}{x} - \sin^{-1}\left(\frac{x+\frac{1}{2}}{\frac{3}{2}}\right) - I_1 \qquad \ldots(1)$$

$$I_1 = \int \frac{dx}{2x\sqrt{2-x-x^2}} \qquad \text{Put } x = \frac{1}{t} \Rightarrow dx = -\frac{dt}{t^2}$$

$$I_1 = \int \frac{-dt}{t^2\left(\frac{2}{t}\sqrt{2-\frac{1}{t}-\frac{1}{t^2}}\right)} = -\int \frac{dt}{2\sqrt{2t^2-t-1}}$$

$$I_1 = -\int \frac{dt}{2\sqrt{2}\sqrt{t^2-\frac{t}{2}-\frac{1}{2}}}$$

$$I_1 = -\frac{1}{2\sqrt{2}} \int \frac{dt}{\sqrt{\left(t-\frac{1}{4}\right)^2 - \left(\frac{3}{4}\right)^2}}$$

$$= -\frac{1}{2\sqrt{2}} \ln\left(t - \frac{1}{4} + \sqrt{t^2 - \frac{t}{2} - \frac{1}{2}}\right) + C$$

From (1) we have:

$$I = -\frac{\sqrt{2-x-x^2}}{x} - \sin^{-1}\left(\frac{2x+1}{3}\right) +$$

$$\frac{1}{2\sqrt{2}} \ln\left(\frac{1}{x} - \frac{1}{4} + \sqrt{\frac{1}{x^2} - \frac{1}{2x} - \frac{1}{2}}\right)$$

$$I = -\frac{\sqrt{2-x-x^2}}{x} - \sin^{-1}\left(\frac{2x+1}{3}\right) +$$

$$\frac{1}{2\sqrt{2}} \ln\left(\frac{4-x+2\sqrt{2}\sqrt{2-x-x^2}}{4x}\right) + C$$

Type $\int \frac{x^2 \pm 1}{x^4 \pm kx^2 + 1} dx$

(i) Take x^2 common from numerator and denominator,

$$\int \frac{x^2\left(1 \pm \frac{1}{x^2}\right)}{x^2\left(x^2 \pm k + \frac{1}{x^2}\right)} dx = \int \frac{1 \pm \frac{1}{x^2}}{\left(x \mp \frac{1}{x}\right)^2 \pm k \pm 2} dx.$$

(ii) Put $x \mp \frac{1}{x} = t$, $\left(1 \pm \frac{1}{x^2}\right) dx = dt$

ILLUSTRATION 89

Evaluate: $\int \frac{1-x^2}{1-x^2+x^4} dx$

SOLUTION

$$I = \int \frac{1-x^2}{1-x^2+x^4} dx = \int \frac{\left(\frac{1}{x^2} - 1\right) x^2\, dx}{x^2\left[\frac{1}{x^2} - 1 + x^2\right]}$$

$$I = \int \frac{\frac{1}{x^2} - 1\, dx}{\frac{1}{x^2} - 1 + x^2} = -\int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x}\right)^2 - 3}$$

Put $x + \frac{1}{x} = \alpha$, $\left(1 - \frac{1}{x^2}\right) dx = d\alpha$

$$I = -\int \frac{d\alpha}{\alpha^2 - 3} = -\frac{1}{2\sqrt{3}} \ln\left|\frac{\alpha - \sqrt{3}}{\alpha + \sqrt{3}}\right| + C$$

$$I = \frac{1}{2\sqrt{3}} \ln\left|\frac{x^2 + \sqrt{3}x + 1}{x^2 - \sqrt{3}x + 1}\right| + C$$

ILLUSTRATION 90

Evaluate: $\int (x^2 + a^2)(x^4 + a^2x^2 + a^4)^{-1} dx$

SOLUTION

$$I = \int \frac{(x^2 + a^2)}{x^4 + a^2x^2 + a^4} dx = \int \frac{\frac{a^2}{x^2} + 1}{\frac{a^4}{x^2} + x^2 - 2a^2 + 3a^2} dx$$

$$I = \int \frac{\frac{a^2}{x^2} + 1}{\left(\frac{a^2}{x} - x\right)^2 + 3a^2} dx$$

Let $\frac{a^2}{x} - x = t \Rightarrow \left(\frac{a^2}{x^2} + 1\right) dx = -dt$

$$I = -\int \frac{dt}{t^2 + (\sqrt{3}a)^2} = -\frac{1}{\sqrt{3}a} \tan^{-1} \frac{t}{\sqrt{3}a} + C$$

$$I = \frac{1}{\sqrt{3}a} \tan^{-1} \frac{x^2 - a^2}{\sqrt{3}ax} + C$$

ILLUSTRATION 91

Evaluate: $\int \frac{(1+x^2)dx}{1 - 2x^2 \cos\alpha + x^4}$

SOLUTION

$$I = \int \frac{(1+x^2)dx}{1 - 2x^2 \cos\alpha + x^4} = \int \frac{\left(\frac{1}{x^2} + 1\right) dx}{\frac{1}{x^2} + x^2 - 2\cos\alpha}$$

$$I = \int \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + 2 - 2\cos\alpha}$$

Put $x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$

$$I = \int \frac{dt}{t^2 + 2\left(1 - 1 + 2\sin^2\frac{\alpha}{2}\right)} = \int \frac{dt}{t^2 + 4\sin^2\frac{\alpha}{2}}$$

$$I = \frac{1}{2\sin\frac{\alpha}{2}} \tan^{-1}\left(\frac{t}{2\sin\frac{\alpha}{2}}\right) + C$$

$$I = \frac{\operatorname{cosec}\frac{\alpha}{2}}{2} \tan^{-1}\left(\frac{x^2 - 1}{2x\sin\frac{\alpha}{2}}\right) + C$$

ILLUSTRATION 92

Evaluate: $\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx$

SOLUTION

$$I=\int(\sqrt{\tan x}+\sqrt{\cot x})\,dx=\int\frac{\tan x+1}{\sqrt{\tan x}}\,dx,$$

Put $\tan x=t^2\Rightarrow(\sec^2 x)dx=2t\,dt$ or $dx=\dfrac{2t}{1+t^4}\,dt.$

$$I=\int\frac{t^2+1}{\sqrt{t^2}}\,\frac{2t}{t^4+1}\,dt=2\int\frac{t^2+1}{t^4+1}\,dt=2\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2}\,dt$$

$$I=2\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+\left(\sqrt{2}\right)^2}\,dt=2\int\frac{du}{u^2+\left(\sqrt{2}\right)^2},$$

where $u=t-\dfrac{1}{t}$; $du=\left(1+\dfrac{1}{t^2}\right)dt$

$$\Rightarrow\quad I=\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+C$$

$$\Rightarrow\quad =\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right)+C$$

ILLUSTRATION 93

Evaluate: $\displaystyle\int\frac{dx}{(x^2+2x+3)\sqrt{x+1}}$

SOLUTION

Given expression is of $\dfrac{1}{Q\sqrt{L}}$ type

$$I=\int\frac{dx}{((x+1)^2+2)\sqrt{x+1}}$$

Put $\sqrt{L}=t$, $\sqrt{x+1}=t\Rightarrow x=t^2-1$

$\Rightarrow\quad x+1=t^2\Rightarrow dx=2t\,dt$

$$I=\int\frac{2t\,dt}{(t^4+2)t}=2\int\frac{dt}{t^4+2}$$

$$I=\frac{1}{\sqrt{2}}\int\frac{2\sqrt{2}\,dt}{t^2\left(t^2+\frac{2}{t^2}\right)}$$

$$=\frac{1}{\sqrt{2}}\int\frac{\left(1+\frac{\sqrt{2}}{t^2}\right)-\left(1-\frac{\sqrt{2}}{t^2}\right)dt}{t^2+\frac{2}{t^2}}$$

$$I=\frac{1}{\sqrt{2}}\left[\int\frac{\left(1+\frac{\sqrt{2}}{t^2}\right)dt}{\left(t-\frac{\sqrt{2}}{t}\right)^2+2\sqrt{2}}-\int\frac{\left(1-\frac{\sqrt{2}}{t^2}\right)}{\left(t+\frac{\sqrt{2}}{t}\right)^2-2\sqrt{2}}\,dt\right]$$

Put $t-\dfrac{\sqrt{2}}{t}=\alpha\Rightarrow\left(1+\dfrac{\sqrt{2}}{t^2}\right)dt=d\alpha$

Put $t+\dfrac{\sqrt{2}}{t}=k\Rightarrow\left(1-\dfrac{\sqrt{2}}{t^2}\right)dt=dk$

$$I=\frac{1}{\sqrt{2}}\left(\int\frac{d\alpha}{\alpha^2+2\sqrt{2}}-\int\frac{dk}{k^2-2\sqrt{2}}\right)$$

$$I=\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2\sqrt{2}}}\tan^{-1}\frac{\alpha}{\sqrt{2\sqrt{2}}}-\frac{1}{2\sqrt{2\sqrt{2}}}\ln\left|\frac{k-\sqrt{2\sqrt{2}}}{k+\sqrt{2\sqrt{2}}}\right|\right]+C$$

$$I=\frac{2}{\left(2\sqrt{2}\right)^{\frac{3}{2}}}\tan^{-1}\left(\frac{t-\frac{\sqrt{2}}{t}}{\sqrt{2\sqrt{2}}}\right)-$$

$$\frac{1}{\left(2\sqrt{2}\right)^{\frac{3}{2}}}\ln\left(\frac{t^2+\sqrt{2}-\sqrt{2\sqrt{2}}t}{t^2+\sqrt{2}+\sqrt{2\sqrt{2}}t}\right)+C$$

where $t=\sqrt{x+1}$

ILLUSTRATION 94

Evaluate: $\int\sqrt{\tan\theta}\,d\theta$

SOLUTION

$$I=\int\sqrt{\tan\theta}\,d\theta$$

Put $\sqrt{\tan\theta}=t\Rightarrow\tan\theta=t^2\Rightarrow\sec^2\theta\,d\theta=2t\,dt$

$\Rightarrow\quad(1+t^4)d\theta=2t\,dt\Rightarrow d\theta=\dfrac{2t\,dt}{1+t^4}$

$$I=\int t\left(\frac{2t\,dt}{1+t^4}\right)=\int\frac{(t^2+1)+(t^2-1)}{1+t^4}\,dt$$

$$I=\int\frac{t^2+1}{t^4+1}\,dt+\frac{t^2-1}{t^4+1}\,dt=\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt+\int\frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\,dt$$

$$I=\int\frac{1+\frac{1}{t^2}\,dt}{\left(t-\frac{1}{t}\right)^2+2}+\int\frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)^2-2}$$

Put $t-\dfrac{1}{t}=\alpha\Rightarrow\left(1+\dfrac{1}{t^2}\right)dt=d\alpha$

Put $t+\frac{1}{t}=\beta \Rightarrow \left(1-\frac{1}{t^2}\right)dt=d\beta$

$$I=\int\frac{d\alpha}{\alpha^2+2}+\int\frac{d\beta}{\beta^2-2}$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\frac{\alpha}{\sqrt{2}}+\frac{1}{2\sqrt{2}}\ln\left|\frac{\beta-\sqrt{2}}{\beta+\sqrt{2}}\right|+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+\frac{1}{2\sqrt{2}}\ln\left(\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right)+C$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right)+\frac{1}{2\sqrt{2}}\ln\left(\frac{t^2+1-\sqrt{2}t}{t^2+1+\sqrt{2}t}\right)+C$$

where $t=\sqrt{\tan\theta}$

ILLUSTRATION 95

Evaluate: $\int\sqrt[3]{\tan x}\,dx$

SOLUTION

$$I=\int\sqrt[3]{\tan x}\,dx$$

Put $\tan^{1/3}x=t\Rightarrow \tan x=t^3\Rightarrow \sec^2x\,dx=3t^2\,dt$

$\Rightarrow \quad (1+t^6)dx=3t^2\,dt\Rightarrow dx=\frac{3t^2}{1+t^6}\,dt$

$$I=\int t\left(\frac{3t^2}{1+t^6}\right)dt=3\int\frac{t^3\,dt}{t^2+2}$$

$$I=3\int\frac{(t^2)t\,dt}{(t^2)^3+1} \qquad \text{Put } t^2=z\Rightarrow 2t\,dt=dz$$

$$I=\frac{3}{2}\int\frac{zdz}{(z^3+1)}$$

$$\frac{z}{(z^3+1)}=\frac{z}{(z+1)(z^2-z+1)}=\frac{A}{z+1}+\frac{Bz+C}{z^2-z+1}$$

$\Rightarrow \quad z=A(z^2-z+1)+(Bz+C)\,(z+1)$

$\Rightarrow \quad z=z^2(A+B)+z(B+C-A)+A+C$

$A+B=0;\ B+C-A=1;\ A+C=0$

Solving $A=-\frac{1}{3}$; $B=C=\frac{1}{3}$

$$I=\frac{3}{2}\left[-\frac{1}{3}\int\frac{dz}{z+1}+\frac{1}{3}\int\frac{z+1\,dz}{z^2-z+1}\right]$$

$$I=-\frac{1}{2}\ln(z+1)+\frac{1}{2}\int\frac{\frac{1}{2}(2z-1)+\frac{3}{2}}{z^2-z+1}dz$$

$$I=-\frac{1}{2}\ln(z+1)+\frac{1}{4}\ln(z^2-z+1)+\frac{3}{4}\int\frac{dz}{z^2-z+1}$$

$$I=-\frac{1}{2}\ln(z+1)+\frac{1}{4}\ln(z^2-z+1)+\frac{3}{4}\int\frac{dz}{\left(z-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$I=-\frac{1}{2}\ln(z+1)+\frac{1}{4}\ln(z^2-z+1)+\frac{3}{4}\cdot\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\tan^{-1}\left(\frac{z-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+C$$

where $z=t^2=(\tan^{1/3}x)^2=\tan^{2/3}x$

Type $\int\frac{dx}{x(x^n+1)}$ or $\int\frac{dx}{x^p(x^m+1)^{k/m}}$

In this form, a suitable term is taken as common and the remaining term is substituted, in order to make it integrable using the formulas we have learnt earlier.

ILLUSTRATION 96

Evaluate: $\int\frac{dx}{x(x^n+1)}$

SOLUTION

$$I=\int\frac{dx}{x(x^n+1)}=\int\frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)}$$

Put $1+\frac{1}{x^n}=t\Rightarrow 0-\frac{n}{x^{n+1}}dx=dt$

$$I=\int\frac{-\frac{dt}{n}}{t}=-\frac{1}{n}\int\frac{dt}{t}$$

$$I=-\frac{1}{n}\ln t+C=-\frac{1}{n}\ln\left(1+\frac{1}{x^n}\right)+C$$

ILLUSTRATION 97

Evaluate: $\int\frac{dx}{x^2(x^7+1)^{6/7}}$

SOLUTION

$$I=\int\frac{dx}{x^2(x^7+1)^{6/7}}=\int\frac{dx}{x^2(x^7)^{6/7}\left(1+\frac{1}{x^7}\right)^{6/7}}$$

$$I=\int\frac{dx}{x^8\left(1+\frac{1}{x^7}\right)^{6/7}}$$

Put $1+\frac{1}{x^7}=t\Rightarrow 0-7x^{-8}\,dx=dt\Rightarrow\frac{dx}{x^8}=-\frac{dt}{7}$

$$I=\int \frac{-dt}{7(t)^{6/7}}=\frac{-1}{7}\int t^{-6/7}\,dt$$

$$I=-\frac{1}{7}\left(\frac{t^{1/7}}{\frac{1}{7}}\right)+C=-\left(1+\frac{1}{x^7}\right)^{1/7}+C$$

ILLUSTRATION 98

Evaluate: $\displaystyle\int \frac{dx}{x^3\left(x^5+1\right)^{3/5}}$

SOLUTION

$$I=\int\frac{dx}{x^3\left(x^5+1\right)^{3/5}}=\int\frac{dx}{x^6\left(1+\frac{1}{x^5}\right)^{3/5}}$$

Put $1+\frac{1}{x^5}=t\Rightarrow 0-\frac{5}{x^6}\,dx=dt$

$$I=\int\frac{-\frac{dt}{5}}{t^{3/5}}=-\frac{1}{5}\int t^{-3/5}\,dt$$

$$I=-\frac{1}{5}\left(\frac{t^{2/5}}{\frac{2}{5}}\right)+C=-\frac{1}{2}\left(1+\frac{1}{x^5}\right)^{2/5}+C$$

ILLUSTRATION 99

Evaluate: $\displaystyle\int\frac{x^2-1}{x\sqrt{x^4+4x^3-6x^2+4x+1}}\,dx$

SOLUTION

$$I=\int\frac{(x^2-1)dx}{x^2\sqrt{x^2+4x-6+\frac{4}{x}+\frac{1}{x^2}}}$$

$$I=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\sqrt{\left(x+\frac{1}{x}\right)^2+4\left(x+\frac{1}{x}\right)-8}}$$

Put $x+\frac{1}{x}=t\Rightarrow\left(1-\frac{1}{x^2}\right)dx=dt$

$$I=\int\frac{dt}{\sqrt{t^2+4t-8}}=\int\frac{dt}{\sqrt{(t+2)^2-12}}$$

$$I=\ln\left(t+2+\sqrt{(t+2)^2-12}\right)+C$$

$$I=\ln\left(x+\frac{1}{x}+2+\sqrt{\left(x+\frac{1}{x}+2\right)^2-12}\right)+C$$

ILLUSTRATION 100

Evaluate: $\displaystyle\int\frac{x^2-x^3}{(x+1)(x^3+x^2+x)^{3/2}}\,dx$

SOLUTION

$$I=\int\frac{x^2(1-x)}{(1+x)\cdot x^3\left(1+\frac{1}{x}+x\right)^{3/2}}\,dx$$

$$I=\int\frac{(1-x)}{x(1+x)\left(x+\frac{1}{x}+1\right)^{3/2}}\,dx$$

$$f(x)=-\int\frac{(x-1)\cdot(x+1)}{x(x+1)^2\left(x+\frac{1}{x}+1\right)^{3/2}}\,dx$$

$$f(x)=-\int\frac{(x^2-1)\cdot dx}{x^2\left(x+\frac{1}{x}+2\right)\left(x+\frac{1}{x}+1\right)^{3/2}}$$

Let $x+\frac{1}{x}+1=t^2\Rightarrow\left(1-\frac{1}{x^2}\right)dx=2t\,dt$

$\Rightarrow\left(\frac{x^2-1}{x^2}\right)dx=2t\,dt$ …(1)

$$f(x)=-\int\frac{2t\cdot dt}{(t^2+1)(t^3)}=-\int\frac{2\cdot dt}{(t^2+1)(t^2)}$$

$$f(x)=-2\int\frac{dt}{t^2}+2\int\frac{dt}{t^2+1}=2\tan^{-1}t+\frac{2}{t}+C$$

$$f(x)=2\tan^{-1}\left(\sqrt{x+\frac{1}{x}+1}\right)+\frac{2}{\sqrt{\left(x+\frac{1}{x}+1\right)}}+C$$

ILLUSTRATION 101

Evaluate: $\displaystyle\int\frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}\,dx$

SOLUTION

$$I=\int\frac{\sqrt{x^2+1}\left[\log(x^2+1)-2\log x\right]}{x^4}\,dx$$

$$I=\int\frac{\sqrt{x^2+1}\log\left(\frac{x^2+1}{x^2}\right)}{x^4}\,dx$$

$$=\int\frac{\sqrt{1+\frac{1}{x^2}}\log\left(1+\frac{1}{x^2}\right)}{x^3}\,dx$$

Let $1+\frac{1}{x^2}=t^2\Rightarrow-\frac{1}{x^3}\,dx=t\,dt$

$$I = -\int t^2 \log t^2 \; dt = -2\int t^2 \log t \; dt$$

$$I = -2\left[\frac{t^3}{3}\log t - \int \frac{t^3}{3}.\frac{1}{t}dt\right] = -2\left[\frac{t^3}{3}\log t - \frac{t^3}{9}\right] + C$$

$$I = -\frac{2}{3}t^3\left(\log t - \frac{1}{3}\right) + c = -\frac{1}{3}t^3\left(\log t^2 - \frac{2}{3}\right) + C$$

$$I = -\frac{1}{3}\left(1+\frac{1}{x^2}\right)^{3/2}\left[\log\left(1+\frac{1}{x^2}\right) - \frac{2}{3}\right] + C$$

ILLUSTRATION 102

If m is a non-zero real number and $\int \frac{x^{5m-1}+2x^{4m-1}}{(x^{2m}+x^m+1)^3}dx =$ $f(x) + C$, where C is constant of integration, then show that $f(x) = \frac{x^{4m}}{2m(x^{2m}+x^m+1)^2}$.

SOLUTION

$$I = \int \frac{x^{5m-1}+2x^{4m-1}}{(x^{2m}+x^m+1)^3}dx = \int \frac{x^{5m-1}+2x^{4m-1}}{x^{6m}\left(1+\frac{1}{x^m}+\frac{1}{x^{2m}}\right)^3}dx$$

$$= \int \frac{x^{-m-1}+2x^{-2m-1}}{\left(1+\frac{1}{x^m}+\frac{1}{x^{2m}}\right)^3}dx$$

Put $1 + \frac{1}{x^m} + \frac{1}{x^{2m}} = t \Rightarrow (-mx^{-m-1} - 2mx^{-2m-1})dx = dt$

$$I = -\frac{1}{m}\int \frac{dt}{t^3} = \frac{1}{2mt^2} + C = \frac{x^{4m}}{2m(x^{2m}+x^m+1)^2} + C$$

ILLUSTRATION 103

Evaluate: $\int \frac{dx}{(x^3+3x^2+3x+1)\sqrt{x^2+2x-3}}$

SOLUTION

$$I = \int \frac{dx}{(x+1)^3\sqrt{(x+1)^2-4}}$$

Put $x + 1 = 2\sec\theta$, $dx = 2\sec\theta\tan\theta\, d\theta$

$$I = \int \frac{2\sec\theta\tan\theta\, d\theta}{8\sec^3\theta(2\tan\theta)} = \frac{1}{8}\int \cos^2\theta\, d\theta$$

$$I = \frac{1}{16}\int (1+\cos 2\theta)\; d\theta$$

$$I = \frac{1}{16}\left(\theta + \frac{1}{2}\sin 2\theta\right) = \frac{\theta}{16} + \frac{\sin\theta\cos\theta}{16} + C$$

$$I = \frac{1}{16}\cos^{-1}\left(\frac{2}{x+1}\right) + \frac{1}{16}\sqrt{1-\frac{4}{(x+1)^2}}\,.\,\frac{2}{x+1} + C$$

$$I = \frac{1}{16}\cos^{-1}\left(\frac{2}{x+1}\right) + \frac{\sqrt{x^2+2x-3}}{8(x+1)^2} + C$$

Test Your Concepts-VI

(Solutions on page H.235)

1. Evaluate: $\int \frac{x^7 dx}{(1-x^2)^5}$

2. Evaluate: $\int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx$

3. Evaluate: $\int \frac{1}{x^{11}\sqrt{1+x^4}}dx.$

4. Evaluate: $\int \frac{(x^3-1)dx}{x^4\sqrt{6x^6-4x^3+2}}$

5. Evaluate: $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx$

6. Evaluate: $\int \frac{dx}{(x-1)\sqrt{x^2+4}}$

7. Evaluate: $\int \frac{(x^2+4x+7)}{\sqrt{x^2+x+1}}dx$

8. Evaluate: $\int \frac{x^2+1}{x^4+2x^2+1}dx$

9. Evaluate: $\int \sqrt{2x^2+3x+4}\; dx$

10. Evaluate: $\int \frac{dx}{1+x^4}$

Answer key

1. $\frac{x^8}{8(1-x^2)^4} + C$

2. $-\frac{x+1}{x^5+x+1} + C$ or $\frac{x^5}{x^5+x+1} + C$

3. $-\frac{1}{2}\left[\frac{t^5}{5} - \frac{2t^3}{3} + t\right] + C$ where $t^2 = 1 + \frac{1}{x^4}$.

4. $\frac{\sqrt{6x^6-4x^3+2}}{6x^3} + C$

5. $\sqrt{\left(x+\frac{1}{x}\right)^2 - 1} + C$

6. $-\frac{1}{\sqrt{5}}\log\left|\left(\frac{1}{x-1}+\frac{1}{5}\right) + \sqrt{\frac{1}{(x+1)^2} + \frac{2}{5(x-1)} + \frac{1}{5}}\right| + C$

7. $\left(\dfrac{x+\frac{1}{2}}{2}\right)\sqrt{x^2+x+1}+3\sqrt{x^2+x+1}+\dfrac{9}{2}\ln\left(x+\dfrac{1}{2}+\sqrt{x^2+x+1}\right)+C$

8. $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x^2-1}{2x}\right)+C$ **9.** $\dfrac{4x+3}{8\sqrt{2}}\sqrt{2x^2+3x+4}+\dfrac{23}{32\sqrt{2}}\log\left[\dfrac{4x+3}{4}+\dfrac{\sqrt{2x^2+3x+4}}{\sqrt{2}}\right]+C$

10. $I=\dfrac{1}{2\sqrt{2}}\tan^{-1}\dfrac{1}{\sqrt{2}}\left(x-\dfrac{1}{x}\right)-\dfrac{1}{4\sqrt{2}}\log\left|\dfrac{x^2-\sqrt{2}+1}{x^2+\sqrt{2}x+1}\right|+C$

INTEGRATION OF TRIGONOMETRIC FUNCTIONS

Type I: $\dfrac{1}{a+b\sin x},\ \dfrac{1}{a+b\cos x},\ \dfrac{1}{a\sin x+b\cos x+C}$

(i) Change in $\tan\dfrac{x}{2}$

$$\sin x=\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}};\ \cos x=\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}};\ \tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

(ii) Put $\tan\dfrac{x}{2}=t$

ILLUSTRATION 104

Evaluate: $\displaystyle\int\frac{dx}{4+5\cos x}$

SOLUTION

$$I=\int\frac{dx}{4+5\cos x}=\int\frac{dx}{4+5\left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$I=\int\frac{\sec^2\frac{x}{2}\,dx}{4+4\tan^2\frac{x}{2}+5-5\tan^2\frac{x}{2}}=\int\frac{\sec^2\frac{x}{2}\,dx}{9-\tan^2\frac{x}{2}}$$

Put $\tan\dfrac{x}{2}=t;\ \dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx=dt$

$$I=\int\frac{2\,dt}{9-t^2}=-2\int\frac{dt}{t^2-9}=-\frac{2}{2(3)}\ln\left|\frac{t-3}{t+3}\right|+C$$

$$I=\frac{1}{3}\ln\left|\frac{\tan\frac{x}{2}+3}{\tan\frac{x}{2}-3}\right|+C$$

ILLUSTRATION 105

Evaluate: $\displaystyle\int\frac{dx}{5-4\sin x}$

SOLUTION

$$I=\int\frac{dx}{5-4\sin x}=\int\frac{dx}{5-4\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$I=\int\frac{\sec^2\frac{x}{2}\,dx}{5+5\tan^2\frac{x}{2}-8\tan\frac{x}{2}}$$

Put $\tan\dfrac{x}{2}=t;\ \dfrac{1}{2}\sec^2\dfrac{x}{2}\,dx=dt$

$$I=\int\frac{2\,dt}{5+5t^2-8t}=\frac{2}{5}\int\frac{dt}{t^2-\frac{8}{5}t+1}$$

$$I=\frac{2}{5}\int\frac{dt}{\left(t-\frac{4}{5}\right)^2+\left(\frac{3}{5}\right)^2}=\frac{2}{5}\cdot\frac{1}{\frac{3}{5}}\tan^{-1}\left(\frac{t-\frac{4}{5}}{\frac{3}{5}}\right)+C$$

$$I=\frac{2}{3}\tan^{-1}\left(\frac{5\tan\frac{x}{2}-4}{3}\right)+C$$

ILLUSTRATION 106

Evaluate: $\displaystyle\int\frac{dx}{3\sin x-4\cos x}$

SOLUTION

$$I=\int\frac{dx}{3\sin x-4\cos x}$$

$$I=\int\frac{dx}{3\left(\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)-4\left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$I=\int\frac{\sec^2\frac{x}{2}dx}{6\tan\frac{x}{2}-4+4\tan^2\frac{x}{2}}$$

Put $\tan\frac{x}{2}=t\Rightarrow\frac{1}{2}\sec^2\frac{x}{2}dx=dt$

$$I=\int\frac{2\,dt}{6t-4+4t^2}=\frac{1}{2}\int\frac{dt}{t^2+\frac{3}{2}t-1}$$

$$I=\frac{1}{2}\int\frac{dt}{\left(t+\frac{3}{4}\right)^2-\left(\frac{5}{4}\right)^2}=\frac{1}{2.2\left(\frac{5}{4}\right)}\ln\left|\frac{t+\frac{3}{4}-\frac{5}{4}}{t+\frac{3}{4}+\frac{5}{4}}\right|+C$$

$$I=\frac{1}{5}\ln\left|\frac{\tan\frac{x}{2}-\frac{1}{2}}{\tan\frac{x}{2}+2}\right|+C$$

Alternate:

$$I=\int\frac{dx}{3\sin x-4\cos x}=\frac{1}{5}\int\frac{dx}{\frac{3}{5}\sin x-\frac{4}{5}\cos x}$$

$$I=\frac{1}{5}\int\frac{dx}{\sin(x-\alpha)}=\frac{1}{5}\int\text{cosec}(x-\alpha)\,dx$$

Let $\sin\alpha=\frac{4}{5}$; $\cos\alpha=\frac{3}{5}$

$$I=\frac{1}{5}\ln|\text{ cosec }(x-\alpha)-\cot(x-\alpha)\,|+C$$

$$I=\frac{1}{5}\ln\left|\tan\left(\frac{x}{2}-\frac{\alpha}{2}\right)\right|+C\quad\text{where }\alpha=\sin^{-1}\left(\frac{4}{5}\right)$$

ILLUSTRATION 107

Evaluate: $\int\sqrt{\frac{\text{cosec}\,x-\cot x}{\text{cosec}\,x+\cot x}}\,\frac{\sec x}{\sqrt{1+2\sec x}}\,dx$

SOLUTION

$$I=\int\sqrt{\frac{\text{cosec}\,x-\cot x}{\text{cosec}\,x+\cot x}}\cdot\frac{\sec x}{\sqrt{1+2\sec x}}\,dx$$

$$I=\int\sqrt{\frac{1-\cos x}{1+\cos x}}\cdot\frac{\sec x}{\sqrt{1+2\sec x}}\,dx$$

$$I=\int\sqrt{\frac{1-1+2\sin^2\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}}\cdot\frac{\sec x}{\sqrt{1+2\sec x}}\,dx$$

$$I=\int\frac{\tan\frac{x}{2}dx}{\sqrt{\cos x}\sqrt{\cos x+2}}=\int\frac{\tan\frac{x}{2}dx}{\sqrt{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\sqrt{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}+2}}$$

$$I=\int\frac{\tan\frac{x}{2}\left(\sec^2\frac{x}{2}\right)dx}{\sqrt{\left(1-\tan^2\frac{x}{2}\right)\left(3+\tan^2\frac{x}{2}\right)}}$$

Put $\tan^2\frac{x}{2}=t\Rightarrow 2\tan\frac{x}{2}\sec^2\frac{x}{2}\cdot\frac{dx}{2}=dt$

$$I=\int\frac{dt}{\sqrt{(1-t)(3+t)}}=\int\frac{dt}{\sqrt{3-2t-t^2}}$$

$$I=\int\frac{dt}{\sqrt{4-(t+1)^2}}=\sin^{-1}\left(\frac{t+1}{2}\right)+C$$

$$I=\sin^{-1}\left(\frac{\tan^2\frac{x}{2}+1}{2}\right)+C=\sin^{-1}\left(\frac{1}{2}\sec^2\frac{x}{2}\right)+C$$

Type II: $\int\frac{\sin x}{a\sin x+b\cos x}dx$, $\int\frac{\cos x\,dx}{a\sin x+b\cos x}$

$\int\frac{a\sin x+b\cos x+c}{A\sin x+B\cos x+K}dx$

Numerator $=\alpha$(Denominator) $+\beta\frac{d}{dx}$(Denominator)

$N=\alpha(D)+\beta\frac{d}{dx}(D)$ get α and β on comparison

$$\int\frac{\alpha D+\beta\frac{d}{dx}(D)}{D}dx=\alpha x+\beta\ln|D|+C$$

ILLUSTRATION 108

Evaluate: $\int\frac{3\sin x+2\cos x}{3\cos x+2\sin x}dx$

SOLUTION

$3\sin x+2\cos x=\alpha(3\cos x+2\sin x)+\beta(-3\sin x+2\cos x)$

$3\sin x+2\cos x=(3\alpha+2\beta)\cos x+(2\alpha-3\beta)\sin x$

$3\alpha+2\beta=2$ and $2\alpha-3\beta=3$

we get $\alpha=\frac{12}{13}$, $\beta=-\frac{5}{13}$

$$I=\int\frac{\alpha D+\beta D'}{D}dx=\alpha x+\beta\ln(D)+C$$

$$I=\frac{12}{13}x-\frac{5}{13}\ln|3\cos x+2\sin x|+C$$

ILLUSTRATION 109

If $\int\frac{3e^x+5e^{-x}}{4e^x-5e^{-x}}dx=Ax+B\ln|4e^{2x}-5|+C$ then (C is constant of integration) then compute A and B

SOLUTION

Let $3e^x + 5e^{-x} = a(4e^x - 5e^{-x}) + b(4e^x + 5e^{-x})$

$3 = 4a + 4b$ and $5 = -5a + 5b$

$a = -1/8,\ b = 7/8$

$$\int \frac{3e^x+5e^{-x}}{4e^x-5e^{-x}}\,dx = \int -\frac{1}{8}\frac{(4e^x-5e^{-x})}{4e^x-5e^{-x}} + \frac{7}{8}\frac{(4e^x+5e^{-x})}{4e^x-5e^{-x}}\,dx$$

$$= -\frac{x}{8}+\frac{7}{8}\int\frac{(4e^x+5e^{-x})}{4e^x-5e^{-x}}\,dx = -\frac{x}{8}+\frac{7}{8}\ln(4e^x-5e^{-x})+C$$

$$= -\frac{x}{8} + \frac{7}{8}\ln(4e^{2x}-5) - \frac{7x}{8} + C$$

$$= -x + \frac{7}{8}\ln(4e^{2x}-5) + c = Ax + B\ln|4e^{2x}-5| + C$$

$A = -1,\ B = \frac{7}{8}$

Type III: $\displaystyle\int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$

(i) Divide Numerator and Denominator by $\cos^2 x$

(ii) Put $\tan x = t$

ILLUSTRATION 110

Evaluate: $\displaystyle\int \frac{dx}{\sin^2 x + \sin 2x}$

SOLUTION

$$I = \int \frac{dx}{\left(\frac{\sin^2 x + 2\sin x\cos x}{\cos^2 x}\right)\cos^2 x}$$

$$I = \int \frac{\sec^2 x\,dx}{\tan^2 x + 2\tan x}$$

Put tan $= t \Rightarrow \sec^2 x\,dx = dt$

$$I = \int \frac{dt}{t^2+2t} = \int \frac{dt}{(t+1)^2-1} = \frac{1}{2}\ln\left|\frac{t+1-1}{t+1+1}\right| + C$$

$$I = \frac{1}{2}\ln\left|\frac{\tan x}{\tan x + 2}\right| + C$$

ILLUSTRATION 111

Evaluate: $\displaystyle\int \frac{dx}{4\sin^2 x + 4\sin x\cos x + 5\cos^2 x}$

SOLUTION

$$I = \int \frac{dx}{4\sin^2 x + 4\sin x\cos x + 5\cos^2 x}$$

$$I = \int \frac{\sec^2 x\,dx}{4\tan^2 x + 4\tan x + 5}$$

Put $\tan x = t;\ \sec^2 x\,dx = dt$

$$I = \int \frac{dt}{4t^2+4t+5} = \frac{1}{4}\int \frac{dt}{\left(t+\frac{1}{2}\right)^2+1}$$

$$I = \frac{1}{4}\tan^{-1}\left(\tan x + \frac{1}{2}\right) + C$$

ILLUSTRATION 112

Evaluate: $\displaystyle\int \frac{(\cos 2x - 3)dx}{\cos^4 x\sqrt{4-\cot^2 x}}$

SOLUTION

$$I = \int \frac{(\cos 2x - 3)dx}{\cos^4 x\sqrt{4-\cot^2 x}}$$

$$= \int \frac{\left(\frac{1-\tan^2 x}{1+\tan^2 x} - 3\right)\tan x}{\cos^4 x\sqrt{4\tan^2 x - 1}}\,dx$$

$$I = \int \frac{(-2-4\tan^2 x)\tan x\sec^2 x\,dx}{\sqrt{4\tan^2 x - 1}}$$

Put $\sqrt{4\tan^2 x - 1} = t \Rightarrow 4\tan^2 x - 1 = t^2$

$\Rightarrow \quad 8\tan x\sec^2 x\,dx = 2t\,dt$

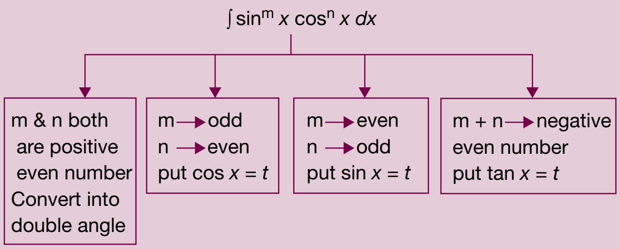
$$I = -\int \frac{\left(1+\frac{1+t^2}{2}\right)\frac{t}{2}dt}{t} = -\frac{1}{2}\int\left(\frac{t^2+3}{2}\right)dt$$

$$I = -\frac{1}{4}\left(\frac{t^3}{3}+3t\right) + C = -\frac{t}{4}\left(\frac{t^2}{3}+3\right) + C$$

$$I = -\frac{\sqrt{4\tan^2 x - 1}}{4}\left(\frac{4\tan^2 x - 1}{3}+3\right) + C$$

$$I = C - \frac{\sqrt{4\tan^2 x - 1}(\tan^2 x + 2)}{3}$$

Type IV:


∫ sin^m x cos^n x dx
m & n both are positive even number Convert into double angle
m → odd n → even put cos x = t
m → even n → odd put sin x = t
m + n → negative even number put tan x = t

$$\cos^2\theta = \frac{1+\cos 2\theta}{2};\quad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

ILLUSTRATION 113

Evaluate: $\int \cos^4 x \sin^3 x \, dx$

SOLUTION

$$I = \int \cos^4 x \sin^3 x \, dx$$

$$I = \int \cos^4 x \sin^2 x . \sin x \, dx$$

Put $\cos x = t; -\sin x \, dx = dt$

$$I = \int \cos^4 x(1-\cos^2 x)\sin x \, dx$$

$$I = \int t^4(1-t^2)\,(-dt) = \int t^6 - t^4 \, dt$$

$$I = \frac{t^7}{7} - \frac{t^5}{5} + C = \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$$

ILLUSTRATION 114

Evaluate: $\int \sin^{1/3} x \cos^{-7/3} x \, dx$

SOLUTION

$$I = \int \sin^{1/3} x \cos^{-7/3} x \, dx$$

Sum of powers $= \frac{1}{3} - \frac{7}{3} = -2$

$$I = \int \left(\frac{\sin x}{\cos x}\right)^{1/3} \frac{1}{\cos^2 x} \, dx = \int \tan^{1/3} x . \sec^2 x \, dx$$

Put, $\tan x = t$, $\sec^2 x \, dx = dt$

$$I = \int \mathrm{t}^{1/3} \, dt = \frac{3}{4} t^{4/3} + C = \frac{3}{4} \tan^{4/3} x + C$$

ILLUSTRATION 115

Evaluate: $\int \sin^2 x \cos^2 x \, dx$

SOLUTION

$$I = \int \sin^2 x \cos^2 x \, dx = \frac{1}{4} \int (2\sin x \cos x)^2 \, dx$$

$$I = \frac{1}{4} \int \sin^2 2x \, dx = \frac{1}{4} \int \left(\frac{1-\cos 4x}{2}\right) dx$$

$$I = \frac{1}{8}\left(x - \frac{\sin 4x}{4}\right) + C$$

ILLUSTRATION 116

Evaluate: $\int \frac{dx}{\sin^5 x \cos^5 x}$

SOLUTION

$$I = \int \frac{dx}{\sin^5 x \cos^5 x}$$

$$I = \int \frac{dx}{\tan^5 x \cos^{10} x} = \int \frac{(1+\tan^2 x)^4 \sec^2 x}{\tan^5 x} \, dx$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$I = \int \frac{(1+t^2)^4 \, dt}{t^5} = \int \frac{t^8 + 4t^6 + 6t^4 + 4t^2 + 1}{t^5} \, dt$$

$$I = \int t^3 + 4t + \frac{6}{t} + \frac{4}{t^3} + \frac{1}{t^5} \, dt$$

$$I = \frac{t^4}{4} + 2t^2 + 6 \ln t - \frac{2}{t^2} - \frac{1}{4t^4} + C$$

$$I = \frac{\tan^4 x}{4} + 2 \tan^2 x + 6 \ln |\tan x| - \frac{2}{\tan^2 x} - \frac{1}{4\tan^4 x} + C$$

$$I = \frac{1}{4}(\tan^4 x - \cot^4 x) + 2(\tan^2 x - \cot^2 x) + 6 \ln |\tan x| + C$$

ILLUSTRATION 117

Evaluate: $\int \cos^4 \theta \, d\theta$

SOLUTION

$$I = \int \cos^4 \theta \, d\theta = \int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{1}{4} \int (1 + 2\cos 2\theta + \cos^2 2\theta) \, d\theta$$

$$= \frac{1}{4}\left[\theta + \sin 2\theta + \int \frac{1+\cos 4\theta}{2} d\theta\right]$$

$$= \frac{1}{4}\left[\theta + \sin 2\theta + \frac{\theta}{2} + \frac{\sin 4\theta}{8}\right] + C$$

ILLUSTRATION 118

Evaluate: $\int \cos^8 \theta \, d\theta$

SOLUTION

Here, $\cos x$ has bigger even power; so, we can also use complex number to simplify it.

Let $z = e^{i\theta} = \cos\theta + i\sin\theta;\ \overline{z} = \frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

$$2\cos\theta = z + \frac{1}{z} = z + \overline{z}$$

$$2^8\cos^8\theta = \left(z + \frac{1}{z}\right)^8 = {}^8C_0 z^8 + {}^8C_1 z^6 + {}^8C_2 z^4 + {}^8C_3 z^2 + {}^8C_4 + {}^8C_5 \frac{1}{z^2} + {}^8C_6 \frac{1}{z^4} + {}^8C_7 \frac{1}{z^6} + {}^8C_8 \frac{1}{z^8}$$

$$2^8\cos^8\theta = \left(z^8 + \frac{1}{z^8}\right) + 8\left(z^6 + \frac{1}{z^6}\right) + 28\left(z^4 + \frac{1}{z^4}\right) + 56\left(z^2 + \frac{1}{z^2}\right) + 70$$

$$= (z^8 + z^{-8}) + 8(z^6 + z^{-6}) + 28(z^4 + z^{-4}) + 56(z^2 + z^{-2}) + 70$$

$$= (z^8 + \bar{z}^8) + 8(z^6 + \bar{z}^6) + 28(z^4 + \bar{z}^4) + 56(z^2 + \bar{z}^2) + 70$$

$$2^8\cos^8\theta = 2\cos 8\theta + 16\cos 6\theta + 56\cos 4\theta + 112\cos 2\theta + 70$$

$$\int \cos^8\theta \, d\theta =$$

$$\frac{1}{2^8}\left[\int 2\cos 8\theta + 16\cos 6\theta + 56\cos 4\theta + 112\cos 2\theta + 70 \, d\theta\right]$$

$$= \frac{1}{256}\left[\frac{\sin 8\theta}{4} + \frac{16\sin 6\theta}{6} + \frac{56\sin 4\theta}{4} + \frac{112\sin 2\theta}{2} + 70\theta\right] + C$$

Type V:

$$\int f(\sin x \pm \cos x, \sin x \cos x)\, dx$$

Put $\sin x \pm \cos x = t$

On square $1 \pm 2\sin x \cos x = t^2$

ILLUSTRATION 119

Evaluate: $\displaystyle\int \frac{\sqrt{\cot x} - \sqrt{\tan x}}{\sqrt{2}(\cos x + \sin x)}\, dx$

SOLUTION

$$I = \int \frac{\sqrt{\frac{\cos x}{\sin x}} - \sqrt{\frac{\sin x}{\cos x}}}{\sqrt{2}(\cos x + \sin x)}\, dx$$

$$= \int \frac{(\cos x - \sin x)\, dx}{\sqrt{2\sin x \cos x}(\cos x + \sin x)}$$

Put $\cos x + \sin x = t \Rightarrow (-\sin x + \cos x)dx = dt$

$$\cos^2 x + \sin^2 x + 2\sin x \cos x = t^2 \Rightarrow 2\sin x \cos x = t^2 - 1$$

$$I = \int \frac{dt}{t\sqrt{t^2 - 1}}$$

$$I = \sec^{-1} t + C = \sec^{-1}(\cos x + \sin x) + C$$

ILLUSTRATION 120

Evaluate: $\displaystyle\int \frac{\cos x - \sin x}{\sqrt{8 - \sin 2x}}\, dx$

SOLUTION

$$I = \int \frac{\cos x - \sin x}{\sqrt{9 - (1 + \sin 2x)}}\, dx$$

$$I = \int \frac{\cos x - \sin x}{\sqrt{9 - (\sin x + \cos x)^2}}\, dx$$

Put $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)dx = dt$

$$I = \int \frac{dt}{\sqrt{3^2 - t^2}} = \sin^{-1}\frac{t}{3} + C$$

$$I = \sin^{-1}\left(\frac{\sin x + \cos x}{3}\right) + C$$

ILLUSTRATION 121

Evaluate: $\displaystyle\int \frac{\sec x \, dx}{\sqrt{\sin(2x + \theta) + \sin\theta}}$

SOLUTION

$$I = \int \frac{\sec x \, dx}{\sqrt{\sin(2x + \theta) + \sin\theta}}$$

$$= \int \frac{\sec x \, dx}{\sqrt{\sin 2x \cos\theta + \cos 2x \sin\theta + \sin\theta}}$$

$$I = \int \frac{\sec x \, dx}{\sqrt{2\sin x \cos x \cos\theta + (2\cos^2 x - 1 + 1)\sin\theta}}$$

$$I = \int \frac{\sec^2 x \, dx}{\sqrt{2\tan x \cos\theta + 2\sin\theta}} = \int \frac{1}{2}\frac{\sec\theta \, dt}{\sqrt{t}}$$

Put $2\tan x \cos\theta + 2\sin\theta = t \Rightarrow 2\sec^2 x \cos\theta \, d\theta = dt$

$$I = \sec\theta\sqrt{t} + C$$

$$I = \sec\theta\sqrt{2\tan x \cos\theta + 2\sin\theta} + C$$

$$I = \sqrt{2(\tan x + \tan\theta)\sec\theta} + C$$

ILLUSTRATION 122

Evaluate: $\displaystyle\int \frac{(\cos 2x - 3)}{\cos^4 x\sqrt{4 - \cot^2 x}}\, dx$

SOLUTION

$$I = \int \frac{\cos 2x - 3}{\cos^4 x\sqrt{4 - \cot^2 x}}\, dx$$

$$I = \int \frac{\left(\frac{1 - \tan^2 x}{1 + \tan^2 x} - 3\right)}{\sqrt{4 - \frac{1}{\tan^2 x}}}(1 + \tan^2 x)\sec^2 x \, dx$$

$$I = \int \frac{(-2 - 4\tan^2 x)}{\sqrt{4\tan^2 x - 1}}\tan x \sec^2 x \, dx$$

Put $4\tan^2 x - 1 = t^2$; $8\tan x \sec^2 x \, dx = 2t \, dt$

$$I = \int \frac{(-2 - 1 - t^2)}{t}\frac{t\,dt}{4} = \frac{1}{4}\left(-3t - \frac{t^3}{3}\right) + C$$

$$I = -\frac{t}{12}(9 + t^2) + C = -\frac{\sqrt{4\tan^2 x - 1}}{12}(4\tan^2 x + 8) + C$$

$$I = -\frac{\sqrt{4\tan^2 x - 1}}{3}(\tan^2 x + 2) + C$$

ILLUSTRATION 123

Evaluate: $\displaystyle\int \frac{x}{(7x - 10 - x^2)^{3/2}}\, dx$

SOLUTION

$$I = \int \frac{x \, dx}{((x - 2)(5 - x))^{3/2}}$$

Put $x = 2\sin^2\theta + 5\cos^2\theta \Rightarrow dx = -6\sin\theta\cos\theta\, d\theta$

$$x\sec^2\theta = 2\tan^2\theta + 5 \Rightarrow x + x\tan^2\theta = 2\tan^2\theta + 5$$

$$\Rightarrow \quad \tan\theta = \sqrt{\frac{5-x}{x-2}}$$

$$I = \int \frac{(2\sin^2\theta + 5\cos^2)(-6\sin\theta\cos\theta\, d\theta)}{(3\cos^2\theta)(3\sin^2\theta)^{3/2}}$$

$$I = \frac{-6}{27}\int \frac{(2\sin^2\theta + 5\cos^2\theta)}{(\sin\theta\cos\theta)^3}\sin\theta\cos\theta\, d\theta$$

$$I = -\frac{2}{9}\int (2\sec^2\theta + 5\operatorname{cosec}^2\theta)\, d\theta$$

$$I = -\frac{2}{9}(2\tan\theta - 5\cot\theta) + C = \frac{10\cot\theta - 4\tan\theta}{9} + C$$

$$I = \frac{1}{9}\left(10\sqrt{\frac{x-2}{5-x}} - 4\sqrt{\frac{5-x}{x-2}}\right) + C$$

Test Your Concepts-VII

(Solutions on page H.237)

1. Evaluate: $\int \frac{dx}{1+\cot x}$

2. Evaluate: $\int \frac{2\cos x - 3\sin x}{6\cos x + 4\sin x}dx$

3. Evaluate: $\int \frac{2-\sin x}{2+\cos x}dx$

4. Evaluate: $\int \frac{5\cos x + 6}{2\cos x + \sin x + 3}dx$

5. Evaluate: $\int \frac{dx}{\sin x(2+\cos x)}$

6. Evaluate: $\int \frac{\cos^2 x dx}{\sin x \cdot \cos 3x}$

7. Evaluate: $\int \frac{dx}{1+\tan x}$

8. Evaluate: $\int \frac{dx}{\tan x \cdot \cos 2x}$

9. Evaluate: $\int \frac{dx}{\sin^2 x + \tan^2 x}$

10. Evaluate: $\int \frac{\sin 2x dx}{\cos^4 x + \sin^4 x}$

11. Evaluate: $\int \frac{\tan x\, dx}{1+\tan x + \tan^2 x}$

12. Evaluate: $\int \frac{3e^x + 4e^{-x}}{2e^x - 3e^{-x}}dx$

13. Evaluate: $\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x}dx$

14. Evaluate: $\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

15. Evaluate: $\int \frac{3 + 2\cos x + \sin x}{6 + 3\cos x + 4\sin x}dx$

Answer key

1. $\frac{1}{2}[x - \ln(\sin x + \cos x)] + C$

2. $\frac{1}{2}\ln|3\cos x + 2\sin x| + \lambda$

3. $\ln(2+\cos x) + \frac{4}{\sqrt{3}}\text{arc tan}\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) + C$

4. $2x + \log(2\cos x + \sin x + 3) + C$

5. $\frac{1}{6}\log|1 - \cos x| - \frac{1}{2}\log|1 + \cos x| + \frac{1}{3}\log|2 + \cos x| + C$

6. $\ln\frac{|C\sin x|}{\sqrt{1-4\sin^2 x}}$

7. $\frac{1}{2}(x + \ln|\sin x + \cos x|) + C$

8. $\ln\frac{|C\sin x|}{\sqrt{\cos 2x}}$

9. $C - \frac{1}{2}\left[\cot x + \frac{1}{\sqrt{2}}\text{arc tan}\frac{\tan x}{\sqrt{2}}\right]$

10. $\text{arc tan}(\tan^2 x) + C$

11. $\tan^{-1}(\tan x) - \frac{2}{\sqrt{3}}\text{arc tan}\left(\frac{1+2\tan x}{\sqrt{3}}\right) + C$

12. $\frac{x}{12} + \frac{17}{12}\ln|2e^x - 3e^{-x}| + C$

13. $-\left[\sin x + \frac{1}{2}\sin 2x\right] + C$

14. $-2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} + C$

15. $\frac{1}{5}\log(6 + 3\cos x + 4\sin x) + \frac{2}{5}x + \frac{5}{5\sqrt{11}}\tan^{-1}\left(\frac{3\tan\frac{x}{2} + 4}{\sqrt{11}}\right) + C$

REDUCTION INTEGRATION

ILLUSTRATION 124

Evaluate: $I_n = \int \sin^n x\,dx$

SOLUTION

$$I_n = \int \sin^n x\,dx = \int \underset{\text{II}}{\sin x} \cdot \underset{\text{I}}{\sin^{n-1}} x\,dx$$

$$I_n = (-\cos x)\sin^{n-1}x - \int (-\cos x)(n-1)\sin^{n-2} x \cdot \cos x\,dx$$

$$I_n = -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2} x \cos^2 x\,dx$$

$$I_n = -\cos x \sin^{n-1}x + (n-1)\int \sin^{x-2} x(1-\sin^2 x)\,dx$$

$$I_n = -\cos x \sin^{n-1}x + (n-1)\left[\int \sin^{n-2} x\,dx - \int \sin^n x\,dx\right]$$

$$I_n = -\cos x \sin^{n-1}x + (n-1)\,I_{n-2} - (n-1)\,I_n$$

$$I_n(1+n-1) = -\cos x \sin^{n-1}x + (n-1)\,I_{n-2}$$

$$I_n = -\frac{\cos x \cdot \sin^{n-1} x}{n} + \frac{(n-1)}{n} I_{n-2},$$ is true for n

Let $n = 100$

$$I_{100} = -\frac{\cos x.\sin^{99} x}{100} + \frac{99}{100} I_{98}$$

Let $n = 2000$

$$I_{2000} = \frac{-\cos x.\sin^{1999} x}{2000} + \frac{1999}{2000} I_{1998}$$

ILLUSTRATION 125

Evaluate: $\int \tan^n x\,dx$

SOLUTION

$$I_n = \int \tan^n x\,dx$$

$$I_n = \int \tan^{n-2} x \cdot \tan^2 x\,dx$$

$$I_n = \int \tan^{n-2} x(\sec^2 x - 1)\,dx$$

$$I_n = \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$I_n = \int \tan^{n-2} x \sec^2 x\,dx - I_{n-2}$$

Put $\tan x = t$; $\sec^2 x\,dx = dt$

$$I_n = \int t^{n-2}\,dt - I_{n-2} = \frac{t^{n-1}}{n-1} - I_{n-2}$$

$$I_n + I_{n-2} = \frac{\tan^{n-1} x}{n-1}$$

ILLUSTRATION 126

Evaluate: $I_n = \int \sec^n x\,dx$

SOLUTION

$$I_n = \int \underset{\text{I}}{\sec^{n-2}} x \cdot \underset{\text{II}}{\sec^2} x\,dx$$

$$I_n = \sec^{n-2}x(\tan x) - \int (n-2)\sec^{n-3} x(\sec x \tan x)\tan x\,dx$$

$$I_n = \sec^{n-2}x.\tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\,dx$$

$$I_n = \sec^{n-2}x \tan x - (n-2)\left[\int \sec^n x dx - \int \sec^{n-2} x\,dx\right]$$

$$I_n = \sec^{n-2}x \tan x - (n-2)\,[I_n - I_{n-2}]$$

$$I_n + (n-2)\,I_n = \sec^{n-2}x \tan x + (n-2)\,I_{n-2}$$

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{(n-2)}{n-1} I_{n-2}$$

ILLUSTRATION 127

Evaluate $I_n = \int x^n e^x\,dx$

SOLUTION

$$I_n = \int \underset{\text{I}}{x^n}\,\underset{\text{II}}{e^x}\,dx$$

$$I_n = x^n e^x - \int n x^{n-1} e^x\,dx$$

$$I_n = x^n e^x - n\,I_{n-1}$$

ILLUSTRATION 128

Evaluate: $I_{n,\,m} = \int \frac{(\sin x)^n}{(\cos x)^m}\,dx$

SOLUTION

$$I_{n,\,m} = \int \frac{(\sin x)^n}{(\cos x)^m}\,dx = \int \sin^{n-1} x \frac{\sin x}{(\cos x)^m}\,dx$$

Let $u = \sin^{n-1} x$; $v = \int \frac{\sin x}{\cos^m x}\,dx = \frac{1}{(m-1)\cos^{m-1} x}$

$$I_{n,\,m} = \int u\,dv = uv - \int v\,du$$

$$I_{n,\,m} = \frac{\sin^{n-1} x}{(m-1)\cos^{m-1} x} - \frac{n-1}{m-1}\int \frac{\sin^{n-2} x \cos x\,dx}{\cos^{m-1} x}$$

$$I_{n,\,m} = \frac{\sin^{n-1} x}{(m-1)\cos^{m-1} x} - \frac{n-1}{m-1}\int \frac{\sin^{n-2} x\,dx}{\cos^{m-2} x}$$

$$I_{n,\,m} = \frac{\sin^{n-1} x\,dx}{(m-1)\cos^{m-1} x} - \frac{n-1}{m-1} I_{n-2,\,m-2}\ (m \neq 1)$$

Which is the required reduction formula.

ILLUSTRATION 129

Evaluate: $I_n = \int (a^2 - x^2)^n \, dx$

SOLUTION

$$I_n = \int 1.(a^2 - x^2)^n \, dx$$

$$I_n = x(a^2 - x^2)^n - \int x(-2\,nx\,(a^2 - x^2)^{n-1})dx;$$

$$\therefore \quad I_n = x(a^2 - x^2)^n + 2n \int x^2\,(a^2 - x^2)^{n-1}\, dx$$

$$I_n = x(a^2 - x^2)^n + 2n \int (x^2 - a^2 + a^2)\,(a^2 - x^2)^{n-1}\, dx$$

$$I_n = x(a^2 - x^2)^n - 2n \int (a^2 - x^2)^n\, dx + 2n\,a^2 \int (a^2 - x^2)^{n-1}\, dx$$

$$I_n = x\,(a^2 - x^2)^n - 2nI_n + 2na^2 I_{n-1}.$$

$$\Rightarrow \quad (1 + 2n)\, I_n = x\,(a^2 - x^2)^n + 2na^2 I_{n-1}.$$

$$\Rightarrow \quad I_n = \frac{x(a^2 - x^2)^n}{2n+1} + \frac{2na^2}{2n+1}\, I_{n-1}.$$

ILLUSTRATION 130

If $\int \sin 101x.\sin^{99} x\, dx = \dfrac{\sin 100x.\sin^{\lambda} x}{\mu} + C$ find $\lambda\mu$

SOLUTION

$$I = \int \sin 101x.\sin^{99} x\, dx$$

$$I = \int \sin(100+1)x.\sin^{99} x\, dx$$

$$I = \int (\sin 100x\cos x + \cos 100x \sin x)\sin^{99} x\, dx$$

$$I = \int \underset{\text{I}}{\sin 100x}\,(\underset{\text{II}}{\sin^{99} x\cos x})\, dx + \int \cos 100x.\sin^{100} x\, dx$$

$$I = \sin 100x\left(\frac{\sin^{100} x}{100}\right) - \int 100\cos 100x.\left(\frac{\sin^{100} x}{100}\right)dx + \int \cos 100x.\sin^{100} x\, dx$$

$$I = \frac{\sin 100x.\sin^{100} x}{100} + C = \frac{\sin 100x.\sin^{\lambda} x}{\mu} + C$$

We get $\lambda = \mu = 100$;
Now required term $= \lambda\mu = 10000$

ILLUSTRATION 131

If $I_n = \int \cot^n x\, dx$, , then Show that $I_0 + I_1 + 2(I_2 + I_3 + \ldots + I_8)$ $+ I_9 + I_{10} = -\left(u + \dfrac{u^2}{2} + \ldots + \dfrac{u^9}{9}\right)$; where $u = \cot x$.

SOLUTION

$$I_n = \int \cot^n x\, dx$$

$$I_n = \int \cot^{n-2} x(\operatorname{cosec}^2 x - 1)\, dx$$

$$I_n = \int \cot^{n-2} x\operatorname{cosec}^2 x\, dx - \int \cot^{n-2} x\, dx;$$

$$I_n + I_{n-2} = \int \cot^{n-2} x\operatorname{cosec}^2 x\, dx$$

Put, $u = \cot x$, $du = -\operatorname{cosec}^2 x\, dx$

$$I_n + I_{n-2} = -\int u^{n-2} du = -\frac{u^{n-1}}{n-1}$$

$$I_2 + I_0 = -u$$

$$I_3 + I_1 = -\frac{u^2}{2}$$

……………

$$I_9 + I_7 = -\frac{u^8}{8}$$

$$I_{10} + I_8 = -\frac{u^9}{9}$$

By adding them,

$$I_0 + I_1 + 2(I_2 + \cdots I_8) + I_9 + I_{10} = -\left(u + \frac{u^2}{2} + \cdots + \frac{u^9}{9}\right)$$

Where u is $\cot x$

ILLUSTRATION 132

Let $I_{(m,\,n)} = \int (\cos 2x)^m (1 - \tan^2 x)^n dx$, such that

$$4I_{(0,\,10)} + \frac{I_{(1,10)}}{5} = A\ \sin 2x\,(1 - \tan^2 x)^{10} + BI_{(0,\,9)} + C$$

(where C is integration constant), find A and B

SOLUTION

$$I_{(1,\,10)} = \int \cos 2x \cdot (1 - \tan^2 x)^{10}\ dx$$

$$= \frac{\sin 2x}{2}(1 - \tan^2 x)^{10} - \int \frac{\sin 2x}{2} 10(1 - \tan^2 x)^9(-2\tan x\sec^2 x)\, dx$$

$$= \frac{\sin 2x}{2}\ (1 - \tan^2 x)^{10} + 20\int (1 - (1 - \tan^2 x))(1 - \tan^2 x)^9\, dx$$

$$= \frac{\sin 2x(1 - \tan^2 x)^{10}}{2} + 20\int (1 - \tan^2 x)^9\ dx - 20\int (1 - \tan^2 x)^{10}\ dx$$

$$I_{(1,\,10)} = \frac{\sin 2x(1-\tan^2 x)^{10}}{2} + 20\, I_{(0,\,9)} - 20\, I_{(0,\,10)} + C$$

$$4I_{(0,\,10)} + \frac{I_{(1,10)}}{5} = \frac{\sin 2x(1-\tan^2 x)^{10}}{10} + 4\, I_{(0,\,9)} + C$$

We get $A = \dfrac{1}{10}$; $B = 4$,

EULER'S SUBSTITUTION

Integrals of the form $\int f(x, \sqrt{ax^2+bx+c})\, dx$ are calculated with the aid of the following substitutions.

(i) $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$ if $a > 0$

(ii) $\sqrt{ax^2+bx+c} = tx \pm \sqrt{c}$ if $c > 0$

(iii) $\sqrt{ax^2+bx+c} = (x-\alpha)\, t$

if $ax^2+bx+c = a(x-\alpha)(x-\beta)$,
i.e., if $(x-\alpha)$ is a real factor of the ax^2+bx+c

ILLUSTRATION 133

Evaluate: $\displaystyle\int \frac{dx}{(x-1)\sqrt{-x^2+3x-2}}$

SOLUTION

Let $-x^2+3x-2 = t^2(x-2)^2$

$$\Rightarrow \quad t = \sqrt{\frac{1-x}{x-2}} \Rightarrow x = \frac{2t^2+1}{t^2+1}$$

$$\Rightarrow \quad dx = \frac{2t\,dt}{(t^2+1)^2}$$

$$\frac{1}{(x-1)\sqrt{-x^2+3x-2}} = \frac{1}{t(x-1)(2-x)}$$

$$= \frac{1}{t\left(\dfrac{2t^2+1}{t^2+1}-1\right)\left(2-\dfrac{2t^2+1}{t^2+1}\right)} = \frac{(t^2+1)^2}{t^3}$$

$$I = \int \frac{dx}{(x-1)\sqrt{-x^2+3x-2}} = \int \frac{2}{t^2}\,dt = -\frac{2}{t} + C$$

$$I = -2\sqrt{\frac{x-2}{1-x}} + C$$

ILLUSTRATION 134

Evaluate: $\displaystyle\int \frac{dx}{x\sqrt{x^2+2x-1}}$

SOLUTION

Put $t = x + \sqrt{x^2+2x-1}$

$$\Rightarrow \quad x = \frac{t^2+1}{2(t+1)} \Rightarrow dx = \frac{1}{2}\,\frac{t^2+2t-1}{(1+t)^2}\,dt$$

$$\frac{1}{x\sqrt{x^2+2x-1}} = \frac{1}{x(t-x)} = \frac{1}{\dfrac{t^2+1}{2(t+1)}\left(t-\dfrac{(t^2+1)}{2(t+1)}\right)}$$

$$= \frac{4(t+1)^2}{(t^2+1)(t^2+2t-1)}$$

$$I = \int \frac{dx}{x\sqrt{x^2+2x-1}} = 2\int \frac{dt}{(1+t^2)}$$

$$I = 2\tan^{-1}t + C = 2\tan^{-1}\left(x+\sqrt{x^2+2x-1}\right) + C$$

ILLUSTRATION 135

Evaluate: $\displaystyle\int \frac{dx}{x+\sqrt{x^2-x+1}}$

SOLUTION

Here, $c = 1 > 0$, we can apply the second Euler substitution.

$$\sqrt{x^2-x+1} = tx - 1 \Rightarrow (2t-1)x = (t^2-1)x^2$$

$$\Rightarrow \quad x = \frac{2t-1}{t^2-1};\ dx = -2\frac{t^2-t+1}{(t^2-1)^2}\,dt;$$

$$x + \sqrt{x^2-x+1} = x + tx - 1 = \frac{t}{t-1}$$

Putting the value of x and dx in I.

$$\int \frac{dx}{x+\sqrt{x^2-x+1}} = \int \frac{-2t^2+2t-2}{t(t-1)(t+1)^2}\,dt,$$

By partial fraction

$$\frac{-2t^2+2t-2}{t(t-1)(t+1)^2} = \frac{2}{t} + \frac{-1}{2(t+1)} + \frac{-3}{(t+1)^2} + \frac{-3}{2(t+1)}$$

Hence,

$$I = 2\int\frac{dt}{t} - \frac{1}{2}\int\frac{dt}{t-1} - 3\int\frac{dt}{(t+1)^2} - \frac{3}{2}\int\frac{dt}{t+1}$$

$$= 2\ln|t| - \frac{1}{2}\ln|t-1| + \frac{3}{t+1} - \frac{3}{2}\ln|t+1| + C$$

where $t = \dfrac{\sqrt{x^2-x+1}+1}{x}$

ILLUSTRATION 136

Evaluate: $\displaystyle\int \frac{dx}{1+x\sqrt{1-x^2}}$

SOLUTION

$$I = \int \frac{dx}{1+x\sqrt{1-x^2}} = \int \frac{1-x\sqrt{1-x^2}}{1-x^2(1-x^2)}\,dx$$

$$= \int \frac{1-x\sqrt{1-x^2}}{x^4-x^2+1}\,dx$$

$$I = \int \frac{dx}{x^4 - x^2 + 1} - \int \frac{x\sqrt{1-x^2}}{x^4 - x^2 + 1}\, dx$$

$$I = \int \frac{\frac{1}{x^2}dx}{x^2 - 1 + \frac{1}{x^2}} + \int \frac{t^2 dt}{(1-t^2)^2 + t^2}$$

Put $\sqrt{1-x^2} = t$; $1 - x^2 = t^2$; $-x\,dx = t\,dt$

$$I = \frac{1}{2}\int \frac{1 + \frac{1}{x^2} - \left(1 - \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2} - 1}\, dx + \int \frac{t^2 dt}{t^4 - t^2 + 1}$$

$$I = \frac{1}{2}\int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 1} dx - \frac{1}{2}\int \frac{\left(1 - \frac{1}{x^2}\right)dx}{\left(x + \frac{1}{x}\right)^2 - 3} + \int \frac{dt}{t^2 - 1 + \frac{1}{t^2}}$$

$$I = \frac{1}{2}\tan^{-1}\left(x - \frac{1}{x}\right) - \frac{1}{4\sqrt{3}}\ln\left(\frac{x + \frac{1}{x} - \sqrt{3}}{x + \frac{1}{x} + \sqrt{3}}\right) +$$

$$\frac{1}{2}\int \frac{\left(1 + \frac{1}{t^2}\right) + \left(1 - \frac{1}{t^2}\right)dt}{t^2 + \frac{1}{t^2} - 1}$$

$$I = \frac{1}{2}\tan^{-1}\left(x - \frac{1}{x}\right) - \frac{1}{4\sqrt{3}}\ln\left(\frac{x + \frac{1}{x} - \sqrt{3}}{x + \frac{1}{x} + \sqrt{3}}\right) +$$

$$\frac{1}{2}\int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 1} dt + \frac{1}{2}\int \frac{\left(1 - \frac{1}{t^2}\right)dt}{\left(t + \frac{1}{t}\right)^2 - 3}$$

$$I = \frac{1}{2}\tan^{-1}\left(x - \frac{1}{x}\right) - \frac{1}{4\sqrt{3}}\ln\left(\frac{x + \frac{1}{x} - \sqrt{3}}{x + \frac{1}{x} + \sqrt{3}}\right) +$$

$$\frac{1}{2}\tan^{-1}\left(t - \frac{1}{t}\right) + \frac{1}{4\sqrt{3}}\ln\left(\frac{t + \frac{1}{t} - \sqrt{3}}{t + \frac{1}{t} + \sqrt{3}}\right) + C$$

Where $t = \sqrt{1-x^2}$

$$= \frac{1}{2}\tan^{-1}\left(x - \frac{1}{x}\right) - \frac{1}{4\sqrt{3}}\ln\left(\frac{x^2 + 1 - \sqrt{3}x}{x^2 + 1 + \sqrt{3}x}\right) -$$

$$\frac{1}{2}\tan^{-1}\left(\frac{x^2}{\sqrt{1-x^2}}\right) + \frac{1}{4\sqrt{3}}\left(\frac{2 - x^2 - \sqrt{3(1-x^2)}}{2 - x^2 + \sqrt{3(1-x^2)}}\right) + C$$

Till now we have discussed many approaches to integrate functions, still there are integrals which we cannot integrate, no matter how hard we try using the tricks and concepts we have learnt in JEE syllabus, e.g., $\int e^{t^2}\, dx$ $\int \frac{e^x}{x} dx$, $\int \sin(x^2)\, dx$, $\int \cos(e^x)\, dx$, $\int \sqrt{x^3 + 1}\, dx$, $\int \frac{1}{\ln x} dx$, $\int \frac{\sin x}{x} dx$

SOLVED PROBLEMS

PROBLEM 1

If $u = \int e^{ax} \cos bx \, dx$, $v = \int e^{ax} \sin bx \, dx$, Prove that

$\tan^{-1}\dfrac{v}{u} + \tan^{-1}\dfrac{b}{a} = bx$, and that $(a^2+b^2)(u^2+v^2) = e^{2ax}$

SOLUTION

$$u = \int e^{ax} \cos bx \, dx$$

We put, $a = r\cos\alpha$, $b = r\sin\alpha$

$$\Rightarrow \quad r^2 = a^2 + b^2 \text{ and } \tan\alpha = \frac{b}{a}$$

$$u = \int e^{ax} \cos bx \, dx = \frac{e^{ax}\cos bx}{a} + \frac{b}{a}\int e^{ax} \sin bx \; dx$$

$$u = \frac{e^{ax}\cos bx}{a} + \frac{be^{ax}\sin bx}{a^2} - \frac{b^2}{a^2}\int e^{ax}\cos bx \, dx$$

$$u = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2} - \frac{b^2 u}{a^2}$$

$$u = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$u = \frac{e^{ax}}{r^2}(r\cos\alpha\cos bx + r\sin\alpha\sin bx)$$

$$u = \frac{e^{ax}}{r}\cos(bx-\alpha) \qquad \ldots(1)$$

Similarly, $v = \dfrac{e^{ax}}{r}\sin(bx-\alpha)$ …(2)

Squaring and adding we get,

$$u^2 + v^2 = \frac{e^{2ax}}{r^2} = \frac{e^{2ax}}{a^2+b^2}$$

$$\Rightarrow \quad (a^2+b^2)(u^2+v^2) = e^{2ax}$$

Divide (2) and (1) we have,

$$\frac{v}{u} = \tan(bx-\alpha) \Rightarrow \tan^{-1}\frac{v}{u} + \alpha = bx$$

$$\Rightarrow \quad \tan^{-1}\frac{v}{u} + \tan^{-1}\frac{b}{a} = bx$$

Hence proved.

PROBLEM 2

Evaluate: $\int \ln(\sqrt{1-x}+\sqrt{1+x})\,dx$

SOLUTION

$$I = \int 1.\ln(\sqrt{1-x}+\sqrt{1+x}) \; dx$$

$$I = x\ln(\sqrt{1-x}+\sqrt{1+x}) - \int x\frac{1}{\sqrt{1-x}+\sqrt{1+x}}\cdot\left(-\frac{1}{2\sqrt{1-x}}+\frac{1}{2\sqrt{1+x}}\right)dx$$

$$I = x\ln(\sqrt{1-x}+\sqrt{1+x}) - \int \frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\cdot\frac{x}{2\sqrt{1-x^2}}\,dx \qquad \ldots(1)$$

Let $I_1 = -\int \dfrac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\cdot\dfrac{x}{2\sqrt{1-x^2}}\,dx$

Put, $x = \cos 2\theta \Rightarrow dx = -2\sin 2\theta\, d\theta$ and solving we get

$$I_1 = -\int \frac{\sqrt{1-\cos 2\theta}-\sqrt{1+\cos 2\theta}}{\sqrt{1-\cos 2\theta}+\sqrt{1+\cos 2\theta}}\cdot\frac{\cos 2\theta}{2\sqrt{1-\cos^2 2\theta}}(-2\sin 2\theta)d\theta$$

$$I_1 = \int \frac{\sin\theta-\cos\theta}{\sin\theta+\cos\theta}(\cos^2\theta-\sin^2\theta)d\theta$$

$$I_1 = -\int(\sin\theta-\cos\theta)^2\,d\theta$$

$$I_1 = \int \sin 2\theta - 1\;d\theta = -\frac{\cos 2\theta}{2} - \theta + c$$

$$I_1 = -\frac{x}{2} - \frac{\cos^{-1}x}{2} + C = -\frac{x}{2} - \frac{\pi}{4} + \frac{\sin^{-1}x}{2} + C$$

So, from (1), we get:

$$I = x\ln(\sqrt{1-x}+\sqrt{1+x}) - \frac{x}{2} + \frac{\sin^{-1}x}{2} + K$$

PROBLEM 3

Evaluate: $\displaystyle\int \frac{1+x^{-2/3}}{1+x}\,dx$

SOLUTION

$$I = \int \frac{1+x^{-2/3}}{1+x}\,dx$$

$$I = \int \frac{1}{1+x}\,dx + \int \frac{dx}{x^{2/3}(1+x)}$$

$$I = \ln(1+x) + \int \frac{dx}{x^{2/3}(1+x)} \qquad \ldots(1)$$

Let $I_1 = \displaystyle\int \frac{dx}{x^{2/3}(1+x)}$; Put $x = t^3 \Rightarrow dx = 3t^2\,dt$

$$I_1 = \int \frac{3dt}{1+t^3}$$

Using partial fraction we get $I_1 = \displaystyle\int\left(\frac{1}{1+t} - \frac{t-2}{1-t+t^2}\right)dt$

$$I_1 = \ln(1+t) - \frac{1}{2}\int \frac{2t-1}{1-t+t^2}\,dt + \frac{3}{2}\int \frac{1}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,dt$$

$$I_1 = \ln(1+t) - \frac{1}{2}\ln(1-t+t^2) + \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right) + C$$

where $t = x^{1/3}$

From (1),

We get $I = \ln(1+x^{1/3}) -$

$$\frac{1}{2}\ln(1-x^{1/3}+x^{2/3}) + \sqrt{3}\,\tan^{-1}\left(\frac{2x^{1/3}-1}{\sqrt{3}}\right) + C$$

PROBLEM 4

Evaluate: $\displaystyle\int \frac{2-x^2}{(1+x)\sqrt{1-x^2}}\,dx$

SOLUTION

$$I = \int \frac{2-x^2}{(1+x)\sqrt{1-x^2}}\,dx = \int \frac{1-x^2+1}{(1+x)\sqrt{1-x^2}}\,dx$$

$$I = \int \sqrt{\frac{1-x}{1+x}}\,dx + \int \frac{1}{(1+x)\sqrt{1-x^2}}\,dx$$

$$\Rightarrow \quad I = I_1 + I_2 \qquad \ldots(1)$$

Where, $I_1 = \displaystyle\int \frac{1-x}{\sqrt{1-x^2}}\,dx$

$$= \int \frac{1}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + \sqrt{1-x^2} + C_1$$

For $I_2 = \displaystyle\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

Let $1 + x = \dfrac{1}{t} \Rightarrow dx = -\dfrac{1}{t^2}\,dt$

$$I_2 = \int \frac{-1}{\sqrt{2t-1}}\,dt = -\sqrt{2t-1} + C_2 = -\sqrt{\frac{1-x}{1+x}} + C_2$$

From (1), we have:

$$I = \sin^{-1}x + \sqrt{1-x^2} - \sqrt{\frac{1-x}{1+x}} + C$$

PROBLEM 5

Integrate: $\displaystyle\int \frac{\sin\theta-\cos\theta}{(\sin\theta+\cos\theta)\sqrt{\sin\theta\cos\theta+\sin^2\theta\cos^2\theta}}\,d\theta$

SOLUTION

$$I = \int \frac{\sin\theta-\cos\theta}{(\sin\theta+\cos\theta)\sqrt{\sin\theta\cos\theta+\sin^2\theta\cos^2\theta}}\,d\theta$$

$$I = \int \frac{\sin^2\theta-\cos^2\theta}{(\sin\theta+\cos\theta)^2\sqrt{\sin\theta\cos\theta+\sin^2\theta\cos^2\theta}}\,d\theta$$

$$I = -\int \frac{2\cos 2\theta}{(1+\sin 2\theta)\sqrt{\sin 2\theta(2+\sin 2\theta)}}\,d\theta$$

Put, $\sin 2\theta = t$; $2\cos 2\theta\,d\theta = dt$

$$I = -\int \frac{dt}{(1+t)\sqrt{t^2+2t}} = -\int \frac{dt}{(1+t)\sqrt{(t+1)^2-1}}$$

$$I = \operatorname{cosec}^{-1}(1+t) + C = \operatorname{cosec}^{-1}(1+\sin 2\theta) + C$$

$$I = \sin^{-1}\left(\frac{1}{1+\sin 2\theta}\right) + C$$

PROBLEM 6

Evaluate: $\displaystyle\int \sqrt{x+\sqrt{x^2+2}}\,dx$

SOLUTION

$$I = \int \sqrt{x+\sqrt{x^2+2}}\,dx$$

$$\text{Put } x + \sqrt{x^2+2} = t^2 \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{x+\sqrt{x^2+2}}{\sqrt{x^2+2}}\,dx = 2t\,dt \Rightarrow \frac{1}{\sqrt{x^2+2}}\,dx = \frac{2}{t}\,dt$$

$$\text{But } x + \sqrt{x^2+2} = \frac{x^2-(x^2+2)}{x-\sqrt{x^2+2}} = t^2$$

$$\Rightarrow \quad -\frac{2}{t^2} = x - \sqrt{x^2+2} \qquad \ldots(2)$$

Subtracting (1) and (2), we get: $\sqrt{x^2+2} = \dfrac{1}{2}\left(t^2 + \dfrac{2}{t^2}\right)$

So, $dx = \dfrac{2}{t}\sqrt{x^2+2}\;dt \Rightarrow dx = \dfrac{1}{t}\left(t^2+\dfrac{2}{t^2}\right)dt$;

$$I = \int \sqrt{x+\sqrt{x^2+2}}\,dx = \int t.\frac{1}{t}\left(t^2+\frac{2}{t^2}\right)dt$$

$$I = \int \left(t^2+\frac{2}{t^2}\right)dt = \frac{1}{3}t^3 - \frac{2}{t} + C$$

$$I = \frac{1}{3}\left(x+\sqrt{x^2+2}\right)^{3/2} - 2\left(x+\sqrt{x^2+2}\right)^{-1/2} + C$$

PROBLEM 7

Evaluate: $\displaystyle\int \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2}\frac{dx}{x}$

SOLUTION

$$I = \int \left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)^{1/2}\frac{dx}{x}$$

Put $x = t^2 \Rightarrow dx = 2t\,dt$

$$I = \int \sqrt{\frac{1-t}{1+t}}.\frac{2t}{t^2}\,dt = 2\int \frac{1-t}{t\sqrt{1-t^2}}\,dt$$

$$= 2\int \frac{1}{t\sqrt{1-t^2}}\,dt - 2\int \frac{dt}{\sqrt{1-t^2}}$$

$$I = I_1 - 2\sin^{-1}t + c = I_1 - 2\sin^{-1}\sqrt{x} + C_2 \qquad \ldots(1)$$

Where $I_1 = 2\int \frac{1}{t\sqrt{1-t^2}}\,dt$

Put $t = \frac{1}{z} \Rightarrow dt = -\frac{1}{z^2}\,dz$

$$I_1 = -2\int \frac{1}{\sqrt{z^2-1}}\,dz. = -2\ln[z + \sqrt{z^2-1}\,]$$

where $z = 1/\sqrt{x}$

$$I_1 = -2\ln\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right) + C_1$$

$$I = -2\ln\left(\frac{1+\sqrt{1-x}}{\sqrt{x}}\right) - 2\sin^{-1}\sqrt{x} + C$$

PROBLEM 8

Find the indefinite integral $\int \cos 2\theta \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)d\theta.$

SOLUTION

$$I = \int \cos 2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)dx$$

We integrate it by parts taking $\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)$ as first function

$$I = \frac{\sin 2\theta}{2}\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) -$$

$$\frac{1}{2}\int \frac{d}{d\theta}\left(\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right).\sin 2\theta\, d\theta \qquad \ldots(1)$$

But, $\frac{d}{d\theta}\left[\ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right)\right]$

$$= \frac{d}{d\theta}[\ln(\cos\theta+\sin\theta) - \ln(\cos\theta-\sin\theta)]$$

$$= \frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta} - \frac{(-\sin\theta-\cos\theta)}{\cos\theta-\sin\theta}$$

$$= \frac{2}{(\cos\theta+\sin\theta)(\cos\theta-\sin\theta)} = \frac{2}{\cos 2\theta}$$

Therefore, putting this value in (1), we get:

$$I = \frac{1}{2}\sin^2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) - \frac{1}{2}\int \sin 2\theta \frac{2}{\cos 2\theta}\,d\theta$$

$$I = \frac{1}{2}\sin 2\theta \ln\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) + \frac{1}{2}\ln(\cos 2\theta) + C$$

PROBLEM 9

Evaluate: $\int \frac{\cos 2x.\sin 4x}{\cos^4 x(1+\cos^2 2x)}\,dx$

SOLUTION

After simplification

$$I = 4\int \frac{2\cos^2 2x \sin 2x\, dx}{(1+\cos 2x)^2(1+\cos^2 2x)}$$

Let $\cos 2x = t \Rightarrow 2\sin 2x\, dx = -dt$

$$I = -4\int \frac{t^2}{(1+t)^2(1+t^2)}\,dt$$

Using partial fraction we get:

$$I = -4\int\left[\frac{1}{2(1+t)^2} - \frac{1}{2(1+t)} + \frac{t}{2(1+t^2)}\right]dt$$

$$I = -2\left[-\frac{1}{1+t} - \ln(1+t) + \frac{1}{2}\ln(1+t^2)\right] + C$$

$$I = \frac{2}{2\cos^2 x} + 2\ln(1+\cos 2x) - \ln(1+\cos^2 2x) + C$$

$$I = \sec^2 x + 2\ln(1+\cos 2x) - \ln(1+\cos^2 2x) + C$$

PROBLEM 10

Evaluate: $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

SOLUTION

$$I = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+9}}\right)dx$$

Let $2x+2 = 3\tan\theta \Rightarrow dx = \frac{3}{2}\sec^2\theta\, d\theta$

$$I = \frac{3}{2}\int \sin^{-1}\left(\frac{3\tan\theta}{3\sec\theta}\right)\sec^2\theta\, d\theta = \frac{3}{2}\int \theta\sec^2\theta\, d\theta$$

$$I = \frac{3}{2}\left[\theta\tan\theta - \int \tan\theta\, d\theta\right] = \frac{3}{2}[\theta\tan\theta - \ln\sec\theta] + C$$

$$I = \frac{3}{2}\left[\tan^{-1}\left(\frac{2x+2}{3}\right)\frac{2x+2}{3} - \ln\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\,\right] + C$$

$$I = (x+1)\tan^{-1}\frac{2x+2}{3} - \frac{3}{2}\ln\sqrt{4x^2+8x+13} + C$$

PROBLEM 11

For any natural m, evaluate

$$\int (x^{3m}+x^{2m}+x^m)(2x^{2m}+3x^m+6)^{1/m}\,dx,\ x>0.$$

SOLUTION

$$I = \int (x^{3m} + x^{2m} + x^m)\,(2x^{2m} + 3x^m + 6)^{1/m}\ dx$$

$$I = \int \frac{(x^{3m} + x^{2m} + x^m)}{x}\left(2x^{3m} + 3x^{2m} + 6x^m\right)^{1/m} dx$$

Let $2x^{3m} + 3x^{2m} + 6x^m = t \Rightarrow \dfrac{6m(x^{3m} + x^{2m} + x^m)}{x}\ dx = dt$

$$I = \frac{1}{6m}\int t^{1/m}\ dt = \frac{1}{6m}\ \frac{t^{1/m+1}}{\frac{1}{m}+1} + C$$

$$I = \frac{1}{6(m+1)}\left(2x^{3m} + 3x^{2m} + 6x^m\right)^{\frac{m+1}{m}} + C$$

PROBLEM 12

Evaluate: $\displaystyle\int\left(\frac{1}{\sqrt[3]{x} + \sqrt[4]{x}} + \frac{\log(1+\sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\right) dx$

SOLUTION

Let $I = I_1 + I_2$; where $I_1 = \displaystyle\int\left(\frac{1}{\sqrt[3]{x} + \sqrt[6]{x}}\right) dx,\ I_2 = \frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\ dx$

$$I_1 = \int \frac{1}{\sqrt[3]{x} + \sqrt[4]{x}}\ dx$$

Now, let $x = t^{12} \Rightarrow dx = 12t^{11}\ dt$

$$I_1 = 12\int \frac{t^{11}}{t^4 + t^3}\ dt = 12\int \frac{t^8 dt}{t+1}$$

$$= 12\int (t^7 - t^6 + t^5 - t^4 + t^3 - t^2 + t - 1)\ dt + 12\int \frac{dt}{t+1}$$

$$I_1 = 12.\left(\frac{t^8}{8} - \frac{t^7}{7} + \frac{t^6}{6} - \frac{t^5}{5} + \frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} - t\right) + 12\ \ln\ (t+1)$$

where $t = x^{1/12}$

$$I_1 = \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + 2x^{1/2} - \frac{12}{5}x^{5/12} + 3x^{1/3} -$$

$$4x^{1/4} + 6x^{1/6} - 12x^{1/12} + 12\ \ln(x^{1/12} + 1)$$

$$I_2 = \int\left\{\frac{\ln(1+\sqrt[6]{x})}{\sqrt[3]{x} + \sqrt{x}}\right\} dx$$

Put, $x = u^6 \Rightarrow dx = 6u^5\ du$

$$I_2 = \int \frac{\ln(1+u)}{u^2 + u^3}\ 6u^5\ du$$

$$I_2 = 6\int \frac{u^3}{(u+1)}\ \ln(1+u)\ du$$

$$= 6\int\left(\frac{u^3 + 1 - 1}{u+1}\right)\ln(1+u)du$$

$$I_2 = 6\int\left(u^2 - u + 1 - \frac{1}{u+1}\right)\ln\ (1+u)\ du$$

$$= 6\int (u^2 - u + 1)\ln(1+u)du\ - 6\int \frac{\ln(1+u)}{(u+1)}\ du$$

Integrating the first integral by parts, we get:

$$I_2 = 6\left(\frac{u^3}{3} - \frac{u^2}{2} + u\right)\ln(1+u) -$$

$$\int \frac{2u^3 - 3u^2 + 6u}{u+1}\ du - 6.\ \frac{1}{2}[\ln(1+u)]^2$$

$$= (2u^3 - 3u^2 + 6u)\ \ln\ (1+u) -$$

$$\int\left(2u^2 - 5u + 11 - \frac{11}{u+1}\right)du - 3[\ln(1+u]^2$$

$$= (2u^3 - 3u^2 + 6u)\ \ln\ (1+u) -$$

$$\left(\frac{2u^3}{3} - \frac{5}{2}u^2 + 11u - 11\ln(u+1)\right) - 3(\ln(1+u))^2$$

where $u = x^{1/6}$

$$I_2 = (2x^{1/2} - 3x^{1/3} + 6x^{1/6})\ \ln\ (1 + x^{1/6}) -$$

$$\left(\frac{2}{3}x^{1/2} - \frac{5}{2}x^{1/3} + 11x^{1/6} - 11\ln(1+x^{1/6})\right) -$$

$$3[\ln(1 + x^{1/6})]^2 + C$$

Therefore, $I = I_1 + I_2$

$$I = \frac{3}{2}x^{2/3} - \frac{12}{7}x^{7/12} + \frac{4}{3}x^{1/2} - \frac{12}{5}x^{5/12} +$$

$$\frac{1}{2}\ x^{1/3} - 4x^{1/4} - 5x^{1/6} - 12x^{1/12} +$$

$$(2x^{1/2} - 3x^{1/3} + 6x^{1/6} + 11)\ \ln\ (1 + x^{1/6}) +$$

$$12\ \ln(1 + x^{12}) - 3[\ln(1 + x^{1/6})]^2 + C$$

PROBLEM 13

Evaluate: $\displaystyle\int \frac{(\cos 2x)^{1/2}}{\sin x}\ dx$

SOLUTION

$$I = \frac{\sqrt{\cos 2x}}{\sin x}\ dx = \int \frac{\sqrt{\cos^2 x - \sin^2 x}}{\sin x}\ dx$$

$$= \int \sqrt{\cot^2 x - 1}\ dx$$

Putting, $\cot x = \sec\theta \Rightarrow -\text{cosec}^2 x\ dx = \sec\theta\tan\theta\, d\theta$.

We get, $I = \displaystyle\int \sqrt{\sec^2\theta - 1}\ .\ \frac{\sec\theta.\tan\theta}{-(1+\sec^2\theta)}\ d\theta.$

$$I = -\int \frac{\sec\theta.\tan^2\theta}{1+\sec^2\theta}\ d\theta = -\int \frac{\sin^2\theta}{\cos\theta + \cos^3\theta}\ d\theta$$

$$I = -\int \frac{1-\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\,d\theta$$

$$= -\int \frac{(1+\cos^2\theta)-2\cos^2\theta}{\cos\theta(1+\cos^2\theta)}\,d\theta$$

$$I = -\int \sec\theta\;d\theta + 2\int \frac{\cos\theta}{1+\cos^2\theta}\;d\theta$$

$$= -\log\;|\sec\theta+\tan\theta|\;+2\int \frac{\cos\theta}{2-\sin^2\theta}\,d\theta$$

$$I = -\log\;|\;\sec\theta+\tan\theta|\;+\;\int \frac{dt}{2-t^2},$$

where $\sin\theta = t$.

$$I = -\log\;|\sec\theta+\tan\theta|\;+2.\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right|+C$$

where $\cot x = \sec\theta$

$$I = -\log\;|\cot x + \sqrt{\cot^2 x - 1}\,|\;+$$

$$\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2}+\sqrt{1-\tan^2 x}}{\sqrt{2}-\sqrt{1-\tan^2 x}}\right|+C$$

PROBLEM 14

Evaluate: $\int \sqrt{\dfrac{1-\sqrt[3]{x}}{1+\sqrt[3]{x}}}\,dx$

SOLUTION

Let $x = t^3 \Rightarrow dx = 3t^2dt$

$$I = \int 3t^2\;\sqrt{\frac{1-t}{1+t}}\;dt = \int \frac{3t^2(1-t)}{\sqrt{1-t^2}}\,dt$$

$$= 3\int \sin^2\theta(1-\sin\theta)\,d\theta$$

Again, let $t = \sin\theta \Rightarrow dt = \cos\theta\,d\theta$

Put this value and solving we get:

$$I = 3\int \frac{1-\cos 2\theta}{2}\,d\theta - 3\int \sin\theta(1-\cos^2\theta)\;d\theta$$

$$I = \frac{3\theta}{2} - \frac{3}{4}\sin 2\theta + 3\left(\cos\theta - \frac{\cos^3\theta}{3}\right) + C$$

By solving, we get:

$$I = \frac{3\theta}{2} + \frac{1}{2}\cos\theta\,(2\sin^2\theta + 4 - 3\sin\theta) + C$$

$$I = \frac{3}{2}\sin^{-1}t + \frac{1}{2}\sqrt{1-t^2}\;(2t^2 - 3t + 4) + C$$

where $t = x^{1/3}$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. Let $\int \tan^{-1}\left(\frac{\tan x}{2}\right) dx = \alpha$, then the value of $\int \tan^{-1}\left(\frac{\tan x - 2\cot x}{3}\right) dx$ equals to: $\left(0 < x < \frac{\pi}{2}\right)$

(A) $\alpha + \frac{\pi x}{2} + \frac{x^2}{2} + c$
(B) $\alpha - \frac{\pi x}{2} + \frac{x^2}{2} + c$
(C) $-\alpha - \frac{\pi x}{2} + \frac{x^2}{2} + c$
(D) $-\alpha + \frac{\pi x}{2} - \frac{x^2}{2} + c$

2. $\int \frac{1}{1-x^8}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\} dx; x > 1$

(A) constant
(B) $\frac{\pi}{8}\tan^{-1} x + c$
(C) $-\frac{\pi}{8}\sin^{-1} x + c$
(D) $\frac{\pi}{2} + x + c$

3. $\int\left(\int e^x \left(\ln x + \frac{2}{x} - \frac{1}{x^2}\right) dx\right) dx$

(A) $e^x \ln x + C_1 x + C_2$
(B) $e^x \ln x + \frac{1}{x} + C_1 x + C_2$
(C) $\frac{\ln x}{x} + C_1 x + C_2$
(D) None of these

4. Let $A = \begin{bmatrix} x^2 & 6 & 0 \\ 1 & -5 & 1 \\ 2 & 0 & x \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$. If a function is defined as $f(x) = \text{trace}(AB)$, then $\int \frac{3dx}{f(x)}$ is equal to:

(A) $\frac{1}{4}\ln\left|\frac{2x-1}{2x+5}\right| + c$
(B) $\frac{1}{4}\ln\left|\frac{2x+5}{2x-1}\right| + c$
(C) $\frac{1}{3}\ln\left|\frac{1-2x}{2x+5}\right| + c$
(D) $\frac{1}{3}\ln\left|\frac{1-2x}{2x+3}\right| + c$

5. $\int \frac{(x^3-1)}{(x^4+1)(x+1)} dx$, is:

(A) $\frac{1}{4}\ln(1+x^4) + \frac{1}{3}\ln(1+x^3) + C$
(B) $\frac{1}{4}\ln(1+x^4) - \frac{1}{3}\ln(1+x^3) + C$
(C) $\frac{1}{4}\ln(1+x^4) - \ln(1+x) + C$
(D) $\frac{1}{4}\ln(1+x^4) + \ln(1+x) + C$

6. If $g(x) = \left(4\cos^4 x - 2\cos 2x - \frac{1}{2}\cos 4x - x^7\right)^{\frac{1}{7}}$, then $\int g(g(x))dx$ is:

(A) $2x + C$
(B) $\frac{x^2}{2} + C$
(C) $-2\cos x + C$
(D) $2\sin x + C$

7. If $\int \frac{(x+1)}{x(1+xe^x)^2} dx = \log|1 - f(x)| + f(x) + C$ (where C is constant of integration), then $f(x)$ is equal to:

(A) $\frac{1}{x+e^x}$
(B) $\frac{1}{1+xe^x}$
(C) $\frac{1}{(1+xe^x)^2}$
(D) $\frac{1}{(1+e^x)^2}$

8. $\int_0^1 \frac{2}{(x^3+6x^2+11x+6)} dx$

(A) $5\ln 2 - 3\ln 3$
(B) $3\ln 3 - 5\ln 2$
(C) $5\ln 2 + 3\ln 3$
(D) $5\ln 5 - 2\ln 2$

9. $\int \frac{1}{(\tan x + 1)\sin^2 x} dx = f(x) + c$, where $f\left(\frac{\pi}{2}\right) = 0$, then the value of $f\left(\frac{\pi}{4}\right)$ is:

(A) $1 - \ln 2$
(B) $\ln 2 - 1$
(C) $1 + \ln 2$
(D) $-1 - \ln 2$

10. $\int \frac{(x^3-1)dx}{\sqrt{x^6+2x^3}}$

(A) $\sqrt{x^4+2x} + C$
(B) $\sqrt{\frac{x^3+2}{x^2}} + C$
(C) $\sqrt{\frac{x^6+2x^3}{x^3}} + C$
(D) $\sqrt{\frac{x^3+2}{x}} + C$

11. $\int x^{-7}(1+x^4)^{-1/2} dx$ is equal to:

(A) $\frac{\sqrt{x^4+1}(2x^4-1)}{6x^6} + C$
(B) $\frac{\sqrt{x^4+1}(2x^4+1)}{6x^6} + C$
(C) $\frac{\sqrt{x^4+1}(2x^4+1)}{4x^6} + C$
(D) $\frac{\sqrt{x^4+1}(2x^4-1)}{4x^6} + C$

12. If the integral $\int \frac{1+\cos 8x}{\cot 2x - \tan 2x} dx = A\cos(8x) + k$, where k is an arbitrary constant, then A is equal to:

(A) $\frac{1}{8}$ (B) $\frac{1}{16}$

(C) $\frac{-1}{16}$ (D) $\frac{-1}{8}$

13. If $\int\left(\frac{x^2-x+1}{x^2+1}\right)\cdot e^{\cot^{-1}x}\,dx = A(x)\cdot e^{\cot^{-1}x}+C$, where C is constant of integration, then $A(x)$ is equal to:

(A) $-x$ (B) $\sqrt{1-x}$

(C) x (D) $\sqrt{1+x}$

14. If $f(x) = \frac{1}{x.\ell(x)\ell^2(x).\ell^3(x).......\ell^n(x)}$, , where $l^r(x)$ means log(log(log…(log x))), the log being repeated r times, then $\int f(x)\,dx =$

(A) $\ell^n(x)+C$ (B) $\ell^{n+1}(x)+C$

(C) $\ell^{n-1}(x)+C$ (D) None of these

15. $\int\frac{dx}{x(x+1)(\ln(x+1)-\ln x)^{11}}$ equals

(A) $\frac{1}{10(\ln(x+1)-\ln x)^{10}}+C$

(B) $\frac{(\ln(x+1)-\ln x)^{10}}{10}+C$

(C) $\frac{1}{11(\ln(x+1)-\ln x)^{11}}+C$

(D) $\frac{(\ln(x+1)-\ln x)^{11}}{11}+C$

16. $\int 4\cos\left(x+\frac{\pi}{6}\right)\cos 2x\cos\left(\frac{5x}{6}+x\right)dx$ is:

(A) $-\left(x+\frac{\sin 4x}{4}+\frac{\sin 2x}{2}\right)+C$

(B) $-\left(x+\frac{\sin 4x}{4}-\frac{\sin 2x}{2}\right)+C$

(C) $-\left(x-\frac{\sin 4x}{4}+\frac{\sin 2x}{2}\right)+C$

(D) $-\left(x-\frac{\sin 4x}{4}-\frac{\sin 2x}{2}\right)+C$

17. If $\int e^{1-\cos x}(1+x\sin x)\,dx = f(x).\ e^{1-g(x)}+c$, then find the number of solutions of the equation $|f(x)| = g(x)$.

(A) 1 (B) 2

(C) 3 (D) zero

18. $\int\frac{(\cos x+x\sin x)}{x(\cos x-x)}dx$ is equal to:

(A) $\ln\left|\frac{Cx}{\cos x-x}\right|$

(B) $\ln|x|+\ln|\cos x+x|+2\ln|\cos x|+C$

(C) $\ln\left|\frac{\cos x-x}{Cx}\right|$

(D) $\ln|\cos x|+C$

19. Let f be a polynomial function such that for all real x. $f(x^2+1) = x^4+5x^2+2$, then $\int f(x)\,dx$ is:

(A) $\frac{x^3}{3}+\frac{3x^2}{2}-2x+C$ (B) $\frac{x^3}{3}+\frac{3x^2}{2}+2x+C$

(C) $\frac{x^3}{3}-\frac{3x^2}{2}-2x+C$ (D) $\frac{x^3}{3}-\frac{3x^2}{2}+2x+C$

20. $\int\sec^2\theta\,(\sec\theta+\tan\theta)^2\,d\theta$ equals:

(A) $\frac{(\sec\theta+\tan\theta)}{2}[2+\tan\theta\,(\sec\theta+\tan\theta)]+C$

(B) $\frac{(\sec\theta+\tan\theta)}{3}[2+4\tan\theta\,(\sec\theta+\tan\theta)]+C$

(C) $\frac{(\sec\theta+\tan\theta)}{3}[2+\tan\theta\,(\sec\theta+\tan\theta)]+C$

(D) $\frac{3(\sec\theta+\tan\theta)}{2}[2+\tan\theta\,(\sec\theta+\tan\theta)]+C$

21. If $\int e^x\left(\frac{1-x^6-6x^2}{(1+x^3)^2}\right)dx = e^x.\frac{1-x^p}{1+x^q}+c$, then $p+q$ equals:

(A) 3 (B) 6

(C) 9 (D) 12

22. If $\int\frac{x-\sin x\cos x}{x^2\cos^2 x}dx = f(x)+c$, then $\lim\limits_{x\to 0} f(x)$ equals:

(A) 1 (B) $\sqrt{2}$

(C) $\frac{1}{\sqrt{2}}$ (D) $\sqrt{3}$

23. The value of integral $\int e^x\left(\frac{2\tan x}{1+\tan x}+\cot^2\left(x+\frac{\pi}{4}\right)\right)dx$ is equal to:

(A) $e^x\tan\left(\frac{\pi}{4}-x\right)+C$ (B) $e^x\tan\left(x-\frac{\pi}{4}\right)+C$

(C) $e^x\tan\left(\frac{3\pi}{4}-x\right)+C$ (D) $e^x\tan\left(x-\frac{3\pi}{4}\right)+C$

24. $\int\left(\frac{\ln x\cdot\ln(x\cdot\ln x)\cdot\ln(x.\ln(x\ln x))+1+\ln x}{\ln x\cdot\ln(x\ln x)}\right)dx$ equals:

(A) $x\ln x - x\ln(\ln(x\ln x))+x+C$

(B) $x\ln x - x\ln(\ln(x\ln x))-x+C$

(C) $x\ln x + x\ln(\ln(x\ln x))-x+C$

(D) $x\ln x + x\ln(\ln(x\ln x))+x+C$

25. For any natural number m,

$\int(x^{7m}+x^{2m}+x^m)(2x^{6m}+7x^m+14)^{1/m}dx$ (where $x>0$), equals:

(A) $\frac{(7x^{7m}+2x^{2m}+14x^m)^{\frac{m+1}{m}}}{14(m+1)}+C$

(B) $\frac{(2x^{7m}+14x^{2m}+7x^m)^{\frac{m+1}{m}}}{14(m+1)}+C$

(C) $\dfrac{(2x^{7m}+7x^{2m}+14x^{m})^{\frac{m+1}{m}}}{14(m+1)}+C$

(D) $\dfrac{(7x^{7m}+2x^{2m}+x^{m})^{\frac{m+1}{m}}}{14(m+1)}+C$

26. $\displaystyle\int \frac{e^{x}(x-1)(x-\ln x)}{x^{2}}\,dx$ is equal to:

(A) $e^{x}\left(\dfrac{x-\ln x}{x}\right)+c$ (B) $e^{x}\left(\dfrac{x-\ln x+1}{x}\right)+c$

(C) $e^{x}\left(\dfrac{x-\ln x}{x}\right)+c$ (D) $e^{x}\left(\dfrac{x-\ln x-1}{x}\right)+c$

27. The value of $\displaystyle\int \frac{\cot x}{\sqrt{5+9\cot^{2}x}}\,dx$ is equal to:

(A) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{2\sin x}{3}\right)+C$ (B) $\dfrac{1}{2}\sin^{-1}\left(\dfrac{3\sin x}{2}\right)+C$

(C) $\dfrac{1}{3}\sin^{-1}\left(\dfrac{3\sin x}{2}\right)+C$ (D) $\dfrac{1}{3}\sin^{-1}\left(\dfrac{2\sin x}{3}\right)+C$

28. Primitive of $\dfrac{3x^{4}-1}{(x^{4}+x+1)^{2}}$ w.r.t. x is:

(A) $\dfrac{x}{x^{4}+x+1}+c$ (B) $-\dfrac{x}{x^{4}+x+1}+c$

(C) $\dfrac{x+1}{x^{4}+x+1}+c$ (D) $-\dfrac{x+1}{x^{4}+x+1}+c$

29. Let $f:\left[0,.\dfrac{\pi}{2}\right]\to R$ be continuous and satisfy $f'(x)=\dfrac{1}{1+\cos x}$ for all $x\in\left(0,\dfrac{\pi}{2}\right)$.

If $f(0)=3$, then $f\left(\dfrac{\pi}{2}\right)$ has the value equal to:

(A) $\dfrac{13}{4}$ (B) 2

(C) 4 (D) None of these

30. Let $A=2^{-\frac{\log_{10}\left(\frac{100}{x}-1\right)}{\log_{10}2}}$, then $\displaystyle\int \ln 10\cdot\log_{10}\left(\frac{1}{A}\right)dx$ is equal to:

(A) $(x-100)\ln(100-x)-x\ln x+C$

(B) $(x-100)\ln(100-x)+x\ln x+C$

(C) $(100-x)\ln(100-x)-x\ln x+C$

(D) $(100-x)\ln(100-x)+x\ln x+C$

31. The integral $\displaystyle\int \frac{x}{2-x^{2}+\sqrt{2-x^{2}}}\,dx$ equals:

(A) $-x\ln\left|1-\sqrt{2-x^{2}}\right|+C$ (B) $\ln\left|1+\sqrt{2+x^{2}}\right|+C$

(C) $-\ln\left|1+\sqrt{2-x^{2}}\right|+C$ (D) $x\ln\left|1-\sqrt{2+x^{2}}\right|+C$

32. The value of integral $\displaystyle\int e^{\sin^{2}x}(\cos x+\cos^{3}x)\ \sin x\,dx$ is equal to:

(A) $\dfrac{1}{2}\,e^{\sin^{2}x}(3-\sin^{2}x)+C$

(B) $\dfrac{1}{2}\,e^{\sin^{2}x}\left(1-\dfrac{1}{2}\cos^{2}x\right)+C$

(C) $e^{\sin^{2}x}(3\cos^{2}x+2\sin^{2}x)+C$

(D) $e^{\sin^{2}x}(2\cos^{2}x+3\sin^{2}x)+C$

33. Primitive of $f(x)=x.2^{\ln(x^{2}+1)}$ w.r.t. x is:

(A) $\dfrac{2^{\ln(x^{2}+1)}}{2(x^{2}+1)}+C$ (B) $\dfrac{(x^{2}+1)2^{\ln(x^{2}+1)}}{\ln 2+1}+C$

(C) $\dfrac{(x^{2}+1)^{\ln 2+1}}{2(\ln 2+1)}+C$ (D) $\dfrac{(x^{2}+1)^{\ln 2}}{2(\ln 2+1)}+C$

34. $\displaystyle\int \frac{\sin 2x+\sin 4x-\sin 6x}{1+\cos 2x+\cos 4x+\cos 6x}\,dx$ is equal to (where C is indefinite integration constant):

(A) $\dfrac{1}{3}\ln|\sec 3x|+\dfrac{1}{2}\ln|\sec 2x|+\ln|\sec x|+C$

(B) $\dfrac{1}{3}\ln|\sec 3x|-\dfrac{1}{2}\ln|\sec 2x|-\ln|\sec x|+C$

(C) $C-\dfrac{1}{3}\ln|\sec 3x|+\dfrac{1}{2}\ln|\sec 2x|+\ln|\sec x|$

(D) $C-\dfrac{1}{3}\ln|\sec 3x|-\dfrac{1}{2}\ln|\sec 2x|+\ln|\sec x|$

35. $\displaystyle\int \frac{\sin x}{\cos^{2}x\cdot\sqrt{\cos 2x}}\,dx$ equals:

(A) $C-\sqrt{1-\tan^{2}x}$ (B) $C-\sqrt{1+\sec^{2}x}$

(C) $C-\sqrt{1+\cos^{2}x}$ (D) $C-\sqrt{1+\sin^{2}x}$

36. $\displaystyle\int x^{x}\left(\frac{1}{x}+\ln^{2}x+\ln x\right)dx$ is equal to (where C is constant of integration):

(A) $x^{x}\left(\ln^{2}x-\dfrac{1}{x}\right)+C$ (B) $x^{x}(\ln x-x)$

(C) $\dfrac{x^{x}\ln^{2}x}{2}+C$ (D) $x^{x}\ln x+C$

37. If $\displaystyle\int e^{x}\left(\frac{x+2}{x+4}\right)^{2}dx=\frac{f(x)e^{x}}{x+4}+C$ then the sum $f(x)+f^{2}(x)+f^{3}(x)\ldots\infty$ at $x=\dfrac{1}{2}$, is:

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{4}$

(C) 1 (D) $\dfrac{3}{2}$

38. $\displaystyle\int \frac{(2x+1)}{(x^{2}+4x+1)^{3/2}}\,dx$ equals

(A) $\dfrac{x^{3}}{(x^{2}+4x+1)^{1/2}}+C$

(B) $\dfrac{x}{(x^{2}+4x+1)^{1/2}}+C$

(C) $\dfrac{x^{2}}{(x^{2}+4x+1)^{1/2}}+C$

(D) $\dfrac{1}{(x^{2}+4x+1)^{1/2}}+C$

39. The evaluation of $\int \frac{px^{p+2q-1} - qx^{q-1}}{x^{2p+2q} + 2x^{p+q} + 1} dx$ is:

(A) $-\frac{x^p}{x^{p+q}+1} + C$ (B) $\frac{x^q}{x^{p+q}+1} + C$

(C) $-\frac{x^q}{x^{p+q}+1} + C$ (D) $\frac{x^p}{x^{p+q}+1} + C$

40. If $f(x) = \int 2e^x \cos^2 x\,(-\tan^2 x + \tan x + 1)dx$ and $f(x)$ passes through $(\pi, 0)$, then $(f(0) + f'(0))$ equals:

(A) $2e$ (B) $3e$

(C) 2 (D) 3

41. $\int \left(\frac{g'(x)}{g(x)} - \frac{f'(x)}{f(x)}\right)(\ln(g(x)) - \ln(f(x)))dx$ equals to

(A) $\ln\left(\frac{g(x)}{f(x)}\right) + C$ (B) $\frac{1}{2}\left(\ln\left(\frac{g(x)}{f(x)}\right)\right)^2 + C$

(C) $\frac{g(x)}{f(x)} \ln\left(\frac{g(x)}{f(x)}\right) + C$ (D) $\ln\left(\frac{f(x)}{g(x)}\right) + C$

42. $\int e^{\sin x}\left(\frac{x \operatorname{cosec} x - \sec^3 x}{\operatorname{cosec} x . \sec x}\right) dx$ equals

(A) $e^{\sin x} - e^{\sin x} \sec x + C$ (B) $xe^{\sin x} + e^{\sin x} \sec x + C$

(C) $xe^{\sin x} - e^{\sin x} \sec x + C$ (D) $xe^{\sin x} - \sec x + C$

43. $\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}}$ equals

(A) $2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6 \ln|\sqrt[6]{x} - 1| + C$

(B) $2\sqrt{x} + 3\sqrt[3]{x} - 6\sqrt[6]{x} + 6 \ln|\sqrt[6]{x} - 1| + C$

(C) $3\sqrt{x} + 2\sqrt[3]{x} + 6\sqrt[6]{x} + 6 \ln|\sqrt[6]{x} - 1| + C$

(D) $3\sqrt{x} + 2\sqrt[3]{x} - 6\sqrt[6]{x} - \ln|\sqrt[6]{x} - 1| + C$

44. If $\int x^2 e^{-2x} dx = e^{-2x}(ax^2 + bx + c) + d$, , then:

(A) $a = -\frac{1}{2}, b = -\frac{1}{2}, c = -\frac{1}{4}$

(B) $a = -\frac{1}{2}, b = -\frac{1}{2}, c = \frac{1}{4}$

(C) $a = -\frac{1}{2}, b = -1, c = -\frac{1}{2}$

(D) $a = 1, b = 1, c = -\frac{1}{2}$

45. $\int \sec^2\theta(\sec\theta + \tan\theta)^2 d\theta$

(A) $\frac{(\sec\theta + \tan\theta)}{2}[2 + \tan\theta(\sec\theta + \tan\theta)] + C$

(B) $\frac{(\sec\theta + \tan\theta)}{3}[2 + 4\tan\theta(\sec\theta + \tan\theta)] + C$

(C) $\frac{(\sec\theta + \tan\theta)}{3}[1 + \sec\theta(\sec\theta + \tan\theta)] + C$

(D) $\frac{3(\sec\theta + \tan\theta)}{2}[2 + \tan\theta(\sec\theta + \tan\theta)] + C$

46. $\int e^x \frac{(1 + n.x^{n-1} - x^{2n})}{(1 - x^n)\sqrt{1 - x^{2n}}} dx$ is equal to:

(A) $e^x \sqrt{\frac{1 - x^n}{1 + x^n}} + C$ (B) $e^x \sqrt{\frac{1 + x^n}{1 - x^n}} + C$

(C) $-e^x \sqrt{\frac{1 - x^n}{1 + x^n}} + C$ (D) $-e^x \sqrt{\frac{1 + x^n}{1 - x^n}} + C$

47. The value of the integral $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x} dx$ is:

(A) $\sin x - 6 \tan^{-1}(\sin x) + c$

(B) $\sin x - 2 (\sin x)^{-1} + c$

(C) $\sin x - 2 (\sin x)^{-1} - 6 \tan^{-1}(\sin x) + c$

(D) $\sin x - 2 (\sin x)^{-1} + 5 \tan^{-1}(\sin x) + c$

48. If $\int \frac{dx}{x\sqrt{1 - x^{2003}}} = k. \log\left|\frac{\sqrt{1 - x^{2013}} - 1}{\sqrt{1 - x^{2013}} + 1}\right| + C$, then the value of $\left(\frac{1}{k}\right)$ is equal to (where C is constant of integration):

(A) 2010 (B) 2011

(C) 2012 (D) 2013

49. If $\int \frac{x^3 - 8}{x^3\sqrt{x^4 - x^3 + 2}} dx = \frac{k.\sqrt{x^4 - x^3 + 2}}{x^2} + C$, then the value of k is (where C is constant of integration):

(A) 3 (B) 2

(C) 7 (D) 4

50. $\int \frac{\cos^6 x.dx}{\sin^4 x(\cos^7 x + \sin^7 x)^{\frac{4}{7}}}$ is equal to:

(A) $\frac{-1}{3}\left(1 + \cot^7 x\right)^{\frac{3}{7}} + C$ (B) $\frac{-1}{3}\left(1 + \cot^7 x\right)^{\frac{7}{3}} + C$

(C) $\frac{-1}{21}\left(1 + \cot^7 x\right)^{\frac{7}{3}}$ (D) $\frac{1}{21}\left(1 + \cot^7 x\right)^{\frac{7}{3}} + C$

51. $\int \frac{\sec x}{\sqrt{-6\operatorname{cosec}^2 x + 9\cot^2 x + 5}} dx$, is equal to (where C is constant of integration):

(A) $\sin^{-1}\left(\frac{\sec x}{2}\right) + C$ (B) $\cos^{-1}(\tan x) + C$

(C) $\frac{1}{2}\sec^{-1}\left(\frac{\sec x}{2}\right) + C$ (D) $\frac{1}{2}\sin^{-1}\left(\frac{\sec x}{2}\right) + C$

52. $f(x) = \int\left(\sin\sqrt{x} + \cos\sqrt{x}\right)dx$ such that $f(0) = 0$, then $f(\pi^2) + f\left(\frac{\pi^2}{4}\right)$ is equal to:

(A) $2\pi + 1$ (B) $3\pi - 4$

(C) $\pi + 2$ (D) 0

53. If $\int \frac{2\sec x.\tan x}{(\sec x - \tan x)^{10}} dx = (\sec x + \tan x)^{11}(m(\sec x - \tan x)^2 + n)$ $+ C$, then $\frac{1}{m} + \frac{1}{n}$ is equal to:

(A) 1 (B) –1

(C) –2 (D) 2

54. $\int e^{x^2} \cdot e^x (2x^2 + x + 1)\, dx$ is equal to:

(A) $xe^{x^2+x} + C$

(B) $(x^2 + 1)\, e^{x^2+x} + C$

(C) $x^2 \cdot e^{x^2+x} + C$

(D) $xe^{x^2} + x^2 e^x + C$

55. If $\int (x^{24} + x^{16} + x^8)(2x^{16} + 3x^8 + 6)^{1/8}\, dx =$ $\frac{1}{\alpha}\left(2x^{24} + 3x^{16} + 6x^8\right)^{\frac{\beta}{\gamma}} + C$ (where C is constant of integration and β, γ are coprime numbers), then the value of $(\alpha + \beta + \gamma)$ is:

(A) 51

(B) 61

(C) 71

(D) 81

56. If $\int \frac{x^2 \tan^{-1}(x^3)}{1+x^6}\, dx = \frac{1}{k}(\tan^{-1}(x^{\alpha}))^{\beta} + c$, (where c is constant of integration), then the value of triplet (α, β, k) is:

(A) (3, 6, 2)

(B) (2, 3, 6)

(C) (2, 6, 3)

(D) (3, 2, 6)

57. The value of the integral $\int \frac{x(2-x)}{e^x(1+x^2e^{-x})^2}\, dx$ is:

(A) $\frac{e^{-x}}{e^{-x}+x^2} + C$

(B) $-\frac{e^{x}}{e^{-x}+x^2} + C$

(C) $-\frac{e^{-x}}{e^{-x}+x^2} + C$

(D) $-\frac{e^{x}}{e^{x}+x^2} + C$

58. $\int \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cos^2 x\sqrt{\tan^3 x + \tan^2 x + \tan x}}\, dx$

(A) $-2\tan^{-1}\sqrt{\tan x + 1 + \frac{1}{\tan x}} + k$

(B) $2\tan^{-1}\sqrt{\tan x + 1 + \frac{1}{\tan x}} + k$

(C) $-3\tan^{-1}\sqrt{\tan x + 1 + \frac{1}{\tan x}} + k$

(D) $3\tan^{-1}\sqrt{\tan x + \frac{1}{\tan x}} + k$

59. $\int \frac{\sec(\tan^2 x - x) \cdot (8\sin x - 3\cos x - \cos 3x)}{4\cos^3 x}\, dx$ is equal to:

(A) $\ln|\sec(\tan^2 x - x)| + c$

(B) $\ln|\sin(\tan^2 x - x)| + c$

(C) $\ln|\sec(\tan^2 x - x) + \tan(\tan^2 x - x)| + c$

(D) $\ln|\sec(\tan^2 x - x) - \tan(\tan^2 x - x)| + c$

60. $\int e^{x^3+\sin x+x}(x + 1 + x\cos x + 3x^3)\, dx$ is equal to:

(A) $x^2 e^{x^3+\sin x+x} + C$

(B) $x^3 e^{x^3+\sin x+x} + C$

(C) $(x + x^2)\, e^{x^3+\sin x+x} + C$

(D) $xe^{x^3+\sin x+x} + C$

61. $\int \frac{(\sin 2x)^{1/3} d((\tan x)^{1/3})}{\sin^{2/3} x + \cos^{2/3} x}$ is equal to:

(A) $\ln|1 + \cot^{1/3} x| + c$

(B) $2^{1/3}.\ln|1 + \tan^{2/3} x| + c$

(C) $\frac{1}{2^{2/3}}\ln|1 + \tan^{1/3} x| + c$

(D) $\frac{1}{2^{2/3}}\ln|1 + \tan^{2/3} x| + c$

62. $\int \frac{(1-\tan^2 x).\mathrm{cosec} 2x\, dx}{\sec^2 x}$ is equal to:

(A) $\frac{\ln|\sin 2x|}{2} + C$

(B) $\frac{\ln|\cos 2x|}{2} + C$

(C) $\frac{\ln(\tan x)}{2} + C$

(D) $\frac{\ln|\cos x + \sin x|}{2} + C$

63. $\int (2 - x\tan x)\sqrt{\cos x}\, dx$ is equal to:

(A) $x\sqrt{\sin x} + C$

(B) $-x\sqrt{\sin x} + C$

(C) $2x\sqrt{\cos x} + C$

(D) $-x\sqrt{\cos x} + C$

64. $\int e^{x\sin x+\cos x}\left(\frac{x^4\cos^3 x - x\sin x + \cos x}{x^2\cos^2 x}\right)dx$

(A) $e^{x\sin x+\cos x}\left(x - \frac{1}{\cos x}\right) + C$

(B) $e^{x\sin x+\cos x}\left(x - \frac{1}{x\cos x}\right) + C$

(C) $e^{x\sin x+\cos x}\left(1 - \frac{1}{x\cos x}\right) + C$

(D) $e^{x\sin x+\cos x}\left(1 - \frac{x}{\cos x}\right) + C$

65. If $\int \frac{x(x-1)dx}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}} = \frac{1}{2}\log\left|\frac{\sqrt{x+\frac{1}{x}+1}-1}{\sqrt{x+\frac{1}{x}+1}+1}\right| - A + C,$ then the value of A is equal to:

(A) $\cos^{-1}\sqrt{1+\frac{1}{x}}$

(B) $\tan^{-1}\sqrt{x+\frac{1}{x}+1}$

(C) $\cot^{-1}\sqrt{x+\frac{1}{x}}$

(D) $\sin^{-1}\sqrt{x+\frac{1}{x}+1}$

66. $\int \frac{\cot^2 x\cos x(1+\cos^2 x)}{1+\sin^2 x}\, dx$ is equal to:

(A) $\cos x - \frac{2}{\cos x} - 6\tan^{-1}(\cos x) + C$

(B) $\sin x - \frac{2}{\sin x} - 6\tan^{-1}(\sin x) + C$

(C) $\cos x + \frac{2}{\sin x} - 6\cos^{-1}(\tan x) + C$

(D) $\sin x + \frac{2}{\sin x} - 6\tan^{-1}(\cos x) + C$

67. $\int x^{-7}(1+x^4)^{-1/2}\, dx$ is equal to:

(A) $\frac{\sqrt{x^4+1}(2x^4-1)}{6x^6} + C$

(B) $\frac{\sqrt{x^4+1}(2x^4+1)}{6x^6} + C$

(C) $\frac{\sqrt{x^4+1}(2x^4+1)}{4x^6} + C$

(D) $\frac{\sqrt{x^4+1}(2x^4-1)}{4x^6} + C$

68. If $\int \frac{\cos 9x + \cos 6x}{-1 + 2\cos 5x} dx = A \sin 4x + B \sin x + C$, where C is constant of integration, then $(A + B)$ is equal to:

(A) $\frac{1}{2}$

(B) $\frac{3}{4}$

(C) $\frac{5}{4}$

(D) None of these

69. $\int \frac{1 + x + \sqrt{x + x^2}}{\sqrt{x} + \sqrt{1 + x}} dx$ is equal to:

(A) $\frac{1}{2}\sqrt{1 + x} + c$

(B) $\frac{2}{3}(1 + x)^{3/2} + c$

(C) $\sqrt{1 + x} + c$

(D) $2(1 + x)^{3/2} + c$

70. If $\int \frac{\sec^2 x dx}{(\tan^{101} x + \tan x)} = g(x) + c$, where $g\left(\frac{\pi}{4}\right) = -\frac{\ln 2}{100}$, then $\lim_{x \to \frac{\pi}{2}} g(x)$ is equal to:

(A) 0

(B) 1

(C) –1

(D) 2

71. $\int \left(\log_e(\log_e x) + (\log_e x)^{-1}\right) dx$ is equal to:

(A) $x \log_e(\log_e x) + c$

(B) $x \log_e(\log_e x) + \frac{x}{\log_e x} + c$

(C) $x \log_e(\log_e x) - \frac{x}{\log_e x} + c$

(D) None of these

72. $\int \frac{\sin 5x}{\cos 7x \cos 2x} dx =$

(A) $\log |\sec 7x \sec 2x| + c$

(B) $\log \left|\frac{\cos^2 2x}{\cos^7 7x}\right| + c$

(C) $\log |\cos^2 2x \cos^7 7x| + c$

(D) None

73. If $\int \frac{\sec^2 x - 2010}{\sin^{2010} x} dx = \frac{P(x)}{\sin^{2010} x} + C$, then value of $P\left(\frac{\pi}{3}\right)$ is:

(A) 0

(B) $\frac{1}{\sqrt{3}}$

(C) $\sqrt{3}$

(D) $-\sqrt{3}$

74. Let $y = y(x)$, $y(1) = 1$ and $y(e) = e^2$. Consider $J = \int \frac{x + y}{xy} dy$, $I = \int \frac{x + y}{x^2} dx$, $J - I = g(x)$ and $g(1) = 1$, then the value of $g(e)$ is:

(A) $3e + 1$

(B) $e + 1$

(C) $e^2 - e + 2$

(D) $e^2 + e - 2$

75. $\int \frac{\sqrt{x^2 + 8x + 15}}{x + 4} dx$ is equal to:

(A) $\sqrt{x^2 + 8x + 15} + \tan^{-1}\sqrt{x^2 + 8x + 15} + C$

(B) $\sqrt{x^2 + 8x + 15} - \tan^{-1}\sqrt{x^2 + 8x + 15} + C$

(C) $\sqrt{x + 4} + \tan^{-1}\sqrt{x^2 + 8x + 15} + C$

(D) $\sqrt{x + 4} - \tan^{-1}\sqrt{x^2 + 8x + 15} + C$

76. $\int \frac{3 + \sqrt{x}}{(x + 2\sqrt{x} + 3)^2} dx$ equals:

(A) $\frac{2x}{(x + 2\sqrt{x} + 3)^2} + C$

(B) $\frac{2x}{(x + 2\sqrt{x} + 3)} + C$

(C) $\frac{x}{(x + 2\sqrt{x} + 3)^2} + C$

(D) $\frac{x}{(x + 2\sqrt{x} + 3)} + C$

77. $\int \frac{\operatorname{cosec} x\, dx}{(2\operatorname{cosec} x + \cot x)}$ is equal to:

(A) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\tan \frac{x}{2}\right) + c$

(B) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\tan \frac{x}{2}\right) + c$

(C) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan \frac{x}{2}}{\sqrt{3}}\right) + c$

(D) $\frac{1}{\sqrt{3}} \tan^{-1}\left(\tan \frac{\frac{x}{2}}{\sqrt{3}}\right) + c$

78. $\int e^{\tan^{-1} x}(1 + x + x^2) d(\cot^{-1} x) =$

(A) $e^{\tan^{-1} x} + c$

(B) $-e^{\tan^{-1} x} + c$

(C) $-xe^{\tan^{-1} x} + c$

(D) $xe^{\tan^{-1} x} + c$

79. Let $f(x) = \int \frac{\cos x - \sin x}{e^{-x} + \cos x} dx$ and $f(0) = \ln 2$, then $f(x)$ is:

(A) $\ln |(e^x + \cos x)|$

(B) $x + \ln |(e^{-x} + \cos x)|$

(C) $\ln |(1 + \sin x)|$

(D) $\ln |(e^{-x} + \cos x)|$

80. $\int e^{\tan x}\left\{\left(3x^2 + \sec^2 x\right)e^{x^3} + \left(2x + \sec^2 x\right)e^{x^2}\right\} dx$ is equal to:

(A) $e^{\tan x + x^3} + e^{\sec x + x^2} + c$

(B) $e^{\tan x}\left\{e^{\sec x} + e^{x^3} + e^{x^2}\right\} + c$

(C) $e^{\tan x + x^3} + e^{\tan x + x^2} + c$

(D) $e^{\tan x}\left(e^{x^3} + e^{x^2} + 1\right) + c$

81. $\int \left(\sqrt{1 + \sin x} + \sqrt{1 - \sin x}\right) dx$ is equal to $\left\{x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)\right\}$:

(A) $-4 \cos \frac{x}{2} + C$

(B) $4 \sin \frac{x}{2} + C$

(C) $2\sqrt{1 - \sin x + C}$

(D) $\sin \frac{x}{2} - \cos \frac{x}{2} + C$

82. Suppose $\begin{vmatrix} f'(x) & f(x) \\ f''(x) & f'(x) \end{vmatrix} = 0$, where $f(x)$ is continuously differentiable function with $f'(x) \neq 0$ and satisfies $f(0) = 1$ and $f'(0) = 2$, then $f(x)$ is:

(A) $x^2 + 2x + 1$

(B) $2e^x - 1$

(C) e^{2x}

(D) $4e^{x/2} - 3$

83. $\int \frac{\operatorname{cosec}^2 x - 2005}{\cos^{2005} x}\,dx$ is equal to:

(A) $\frac{\cot x}{(\cos x)^{2005}} + c$ (B) $\frac{\tan x}{(\cos x)^{2005}} + c$

(C) $\frac{-(\tan x)}{(\cos x)^{2005}} + c$ (D) None of these

84. Let $f(x) = \int \frac{x^2 dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $f(0) = 0$, then the value of $f(1)$ be:

(A) $\log(1+\sqrt{2})$ (B) $\log(1+\sqrt{2}) - \frac{\pi}{4}$

(C) $\log(1+\sqrt{2}) + \frac{\pi}{4}$ (D) None of these

85. Let $f(x) = \frac{2\sin^2 x - 1}{\cos x} + \frac{\cos x(2\sin x + 1)}{1 + \sin x}$, then $\int e^x (f(x) + f'(x))\,dx$ (where c is the constant of integration):

(A) $e^x \tan x + c$ (B) $e^x \cot x + c$

(C) $e^x \operatorname{cosec}^2 x + c$ (D) $e^x \sec^2 x + c$

86. If $\int \frac{dx}{x\sqrt{1-x^3}} := a \ln \cdot \left(\frac{\sqrt{1-x^3}+b}{\sqrt{1-x^3}+1}\right) + k$, then

(A) $b = 1, a = 1$ (B) $b = -1, a = \frac{1}{3}$

(C) $b = 1, a = -\frac{2}{3}$ (D) None of these

87. If $I = \int \frac{x^2 + 20}{(x\sin x + 5\cos x)^2}\,dx$, then I equals:

(A) $-\frac{x}{\cos x(x\sin x + 5\cos x)} + \tan x + c$

(B) $\frac{x}{\sin x(x\sin x + 5\cos x)} + \cot x + c$

(C) $(x\sin x - 5\cos x)^{-1}\sin x + 7x + c$

(D) none of these

88. $\int \frac{\sqrt{1-x^2} - x}{\sqrt{1-x^2}\left(1 + x\sqrt{1-x^2}\right)}\,dx$ is:

(A) $2\tan^{-1}\left(x + \sqrt{1-x^2}\right) + c$ (B) $\tan^{-1}\left(x + \sqrt{1-x^2}\right) + c$

(C) $2\tan^{-1}\left(x - \sqrt{1-x^2}\right) + c$ (D) $2\cot^{-1}\left(x + \sqrt{1-x^2}\right) + c$

89. Let $\int \frac{dx}{x^{2008} + x} = \frac{1}{p}\ln\left(\frac{x^q}{1+x^r}\right) + C$ where $p, q, r \in N$ and need not be distinct, then the value of $(p + q + r)$ equals:

(A) 6024 (B) 6022

(C) 6021 (D) 6020

90. $\int \left(\sqrt{\sin x} + \sqrt{\cos x}\right)^{-4} dx =$

(A) $\frac{2}{\left(\sqrt{\tan x}+1\right)^2}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)} - \frac{1}{2}\right] + c$

(B) $\frac{2}{\left(\sqrt{\tan x}+1\right)}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)} + \frac{1}{2}\right] + c$

(C) $\frac{2}{\left(\sqrt{\tan x}+1\right)^2}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)} + \frac{1}{2}\right] + c$

(D) $\frac{2}{\left(\sqrt{\tan x}+1\right)}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)} - \frac{1}{2}\right] + c$

91. $\int \left(\frac{x-1}{x+1}\right)\frac{dx}{\sqrt{x^3 + x^2 + x}} = 2\tan^{-1}\sqrt{f(x)} + C$ then find $f(x)$.

(A) $x + \frac{1}{x} + 1$ (B) $x + \frac{1}{x} + 2$

(C) $x - \frac{1}{x} + 1$ (D) $x - \frac{1}{x} - 2$

92. Let $x^2 \ne n\pi - 1, n \in N$. Then, the value of $\int x\sqrt{\frac{2\sin(x^2+1) - \sin 2(x^2+1)}{2\sin(x^2+1) + \sin 2(x^2+1)}}\,dx$ is equal to:

(A) $\log\left|\frac{1}{2}\sec(x^2+1)\right| + C$

(B) $\log\left|\sec\left(\frac{x^2+1}{2}\right)\right| + C$

(C) $\frac{1}{2}\log|\sec(x^2+1)| + C$

(D) None

93. $\int \frac{e^{\tan^{-1} x}}{(1+x^2)}\left[\left(\sec^{-1}\sqrt{1+x^2}\right)^2 + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]dx \; (x > 0)$

(A) $e^{\tan^{-1} x}.\tan^{-1} x + C$

(B) $\frac{e^{\tan^{-1} x}\cdot(\tan^{-1} x)^2}{2} + C$

(C) $e^{\tan^{-1} x}.\left(\sec^{-1}\left(\sqrt{1+x^2}\right)\right)^2 + C$

(D) $e^{\tan^{-1} x}.\left(\operatorname{cosec}^{-1}\left(\sqrt{1+x^2}\right)\right)^2 + C$

94. If y is implicit differentiable function of x such that $y(x+y)^2 = x$, then $\int \frac{dx}{x + 3y} =$

(A) $\ln\left((x+y)^2 + 1\right) + c$ (B) $\frac{1}{2}\ln\left((x-y)^2 + 1\right) + c$

(C) $\ln\left((x-y)^2 + 1\right) + c$ (D) $\frac{1}{2}\ln\left((x+y)^2 + 1\right) + c$

95. $\int \frac{(1+\sqrt{\tan x})(1+\tan^2 x)}{2\tan x}\,dx$ equal to:

(A) $\log \tan^{1/2} x + \sqrt{\tan x} + c$

(B) $\log \tan^2 x + \frac{1}{2\sqrt{\tan x}} + c$

(C) $\log|\tan x| + 2\sqrt{\tan x} + c$

(D) $\log|\tan| + \sqrt{\tan x} + c$

96. $\displaystyle\int \frac{dx}{(x-3)^{4/5}(x+1)^{6/5}} =$

(A) $((x-3)\,(x+1))^{1/5}+c$ (B) $\dfrac{5}{4}\left(\dfrac{x-3}{x+1}\right)^{1/5}+c$

(C) $\left(\dfrac{x+1}{x-3}\right)^{1/5}+c$ (D) $(x-3)^{6/5}\,(x+1)^{4/5}+c$

97. Let $S(x) = \displaystyle\int \frac{dx}{e^{x}+8e^{-x}+4e^{-3x}}$, $R(x) = \displaystyle\int \frac{dx}{e^{3x}+8e^{x}+4e^{-x}}$ and $M(x) = S(x) - 2R(x)$.

If $M(x) = \dfrac{1}{2}\tan^{-1}(f(x)) + c$ where c is an arbitrary constant, then $f(\log_e 2) =$

(A) $\dfrac{3}{2}$ (B) $\dfrac{1}{2}$

(C) $\dfrac{5}{2}$ (D) $\dfrac{7}{2}$

98. If $\displaystyle\int (2-3\sin^2 x)\sqrt{\sec x}\,dx = 2f(x)\sqrt{g(x)} + c$ and $f(x)$ is non constant function, then:

(A) $f^2(x)+g^2(x)=1$ (B) $f^2(x)-g^2(x)=1$

(C) $f(x)g(x)=1$ (D) $f(x)=g(x)$

99. If $\displaystyle\int \frac{x\,dx}{\sqrt[2012]{\left(1+x^2\right)^{1012}\left(2+x^2\right)^{3012}}} = \frac{\alpha}{\beta}\left(1-f(x)\right)^{\frac{\beta}{2\alpha}} + k$, then which is true?

(A) $\alpha=503;\ \beta=500,\ f\left(\sqrt{2}\right) = \dfrac{1}{\beta-\alpha}$

(B) $\alpha=503;\ \beta=250,\ f\left(\sqrt{2}\right) = \dfrac{1}{\alpha-\beta}$

(C) $\alpha=503;\ \beta=500,\ f(1) = \dfrac{1}{\alpha-\beta}$

(D) $\alpha=503;\ \beta=225,\ f\left(\sqrt{3}\right) = \dfrac{1}{\alpha-\beta}$

100. $\displaystyle\int\left(\frac{\sin\theta}{\cos 3\theta}+\frac{\sin 3\theta}{\cos 9\theta}+\frac{\sin 9\theta}{\cos 27\theta}\right)d\theta =$

(A) $\dfrac{1}{2}\log\left|\dfrac{\sec 27\theta}{\sec\theta}\right|$ (B) $\dfrac{1}{2}\log\left|\dfrac{\sec\theta}{\sec 27\theta}\right|$

(C) $\dfrac{1}{2}\log\left|\dfrac{\sqrt[27]{\sec 27\theta}}{\sec\theta}\right|$ (D) none of these

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. A primitive of sin 6x is:

(A) $\dfrac{1}{3}(\sin^6 x-\sin^3 x)+c$

(B) $-\dfrac{1}{3}\cos^2 3x+c$

(C) $\dfrac{1}{3}\sin^2 3x+c$

(D) $\dfrac{1}{3}\sin\left(3x+\dfrac{\pi}{7}\right)\sin\left(3x-\dfrac{\pi}{7}\right)+c$

2. If $f(x) = \lim\limits_{n\to\infty} e^{x\tan(1/n)\ln(1/n)}$ and $\displaystyle\int \frac{f(x)}{\sqrt[3]{(\sin^{11}x\cos x)}}\,dx = g(x)+c$, then:

(A) $g\left(\dfrac{\pi}{4}\right)=\dfrac{3}{2}$

(B) $g(x)$ is continuous for all x

(C) $g\left(\dfrac{\pi}{4}\right)=-\dfrac{15}{8}$

(D) $g(x)$ is non differentiable at infinitely many points

3. If the primitive of sin(ln x) is $f(x)$ {sin $g(x)$ – cos $h(x)$} + c (c being the constant of integration), then:

(A) $\lim\limits_{x\to 2} f(x)=1$ (B) $\lim\limits_{x\to 1}\dfrac{g(x)}{h(x)}=1$

(C) $g(e^3)=3$ (D) $h(e^5)=5$

4. Suppose $J = \displaystyle\int \frac{\sin^2 x+\sin x}{1+\sin x+\cos x}\,dx$ and $K = \displaystyle\int \frac{\cos^2 x+\cos x}{1+\sin x+\cos x}\,dx$. If C is an arbitrary constant of integration, then which of the following is/are correct?

(A) $J = \dfrac{1}{2}(x-\sin x+\cos x)+C$

(B) $J = K-(\sin x+\cos x)+C$

(C) $J = x-K+C$

(D) $K = \dfrac{1}{2}(x-\sin x+\cos x)+C$

5. $\displaystyle\int(\tan x+2\tan 2x+4\tan 4x+8\cot 8x)\,dx$ is equal to (where C is constant of integration):

(A) ln(sin x) + C

(B) –ln(cosec x) + C

(C) ln(cos x) + C

(D) ln(sin 2x) – ln(2 cos x) + C

6. $\displaystyle\int\left(1+\tan x.\tan\frac{x}{2}\right)dx$ is equal to:

(A) $\displaystyle\int\left(\tan x.\cot\frac{x}{2}-1\right)dx$ (B) $\displaystyle\int \sec x\,dx$

(C) $-\ln|\sec x-\tan x|+C$ (D) $-\ln\left|\dfrac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right|+C$

7. If $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx = \alpha x^{\beta} + \gamma \tan^{-1}(x^{\delta}) + C$, where C is constant of integration, then which of the following is(are) correct?

(A) $\alpha = \frac{2}{3}$

(B) $\alpha\beta = 1$

(C) $(\alpha\beta + \gamma) = 7$

(D) Sum of all coefficients of even power of x in the expansion of $\left(\alpha\beta + \frac{x}{\delta}\right)^{\gamma}$ is equal to $\frac{(\gamma+1)^{\gamma}+(\gamma-1)^{\gamma}}{2}$

8. Let $f(x) = \int \frac{e^{\sin x}}{2\sin^3 x}(\sin x - 2)\cos x\, dx$ such that $f\left(\frac{\pi}{2}\right) = \frac{e}{2}$, then:

(A) $\lim\limits_{x\to 0}\frac{2\sin^2 x(f(x))-1}{\tan\left(\frac{x}{2}\right)} = 2$

(B) If $g'(x) = f(x)$, then $g: R - \{n\pi\} \to R$ is an injective function

(C) $\lim\limits_{x\to 0} x\, f(x)$ is not defined

(D) $\lim\limits_{x\to 0} x\, f(x) = 1$

9. If $\int \frac{3\cot 3x - \cot x}{\tan x - 3\tan 3x}dx = pf(x) + qg(x) + c$, where $'c'$ is a constant of integration, then:

(A) $p = 1; q = \frac{1}{\sqrt{3}}; f(x) = x; g(x) = \ln\left|\frac{\sqrt{3}-\tan x}{\sqrt{3}+\tan x}\right|$

(B) $p = 1; q = -\frac{1}{\sqrt{3}}; f(x) = x; g(x) = \ln\left|\frac{\sqrt{3}-\tan x}{\sqrt{3}+\tan x}\right|$

(C) $p = 1; q = -\frac{2}{\sqrt{3}}; f(x) = x; g(x) = \ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|$

(D) $p = 1; q = -\frac{1}{\sqrt{3}}; f(x) = x; g(x) = \ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|$

10. If $\int \frac{x^2+3}{x^4+x^3-x^2-3x+9}dx = \frac{m}{\sqrt{n}}\tan^{-1}\left(\frac{m(x-\alpha)(x-\beta)}{\sqrt{n}x}\right)+C$, where $\alpha > \beta$ and $\alpha, \beta \in R$ and m and n are prime numbers, then (where C is constant of integration):

(A) $n - m$ is a prime number

(B) $n + m$ is a prime number

(C) $m + \beta = 0$

(D) $|\alpha + \beta| = m^{-1}$

11. Let $f(x) = \int (2x + \sec x(1 + x\tan x))\,dx$ such that $f(0) = 0$. Then:

(A) $\lim\limits_{x\to 0}\frac{f(x)}{x} = 1$

(B) $\lim\limits_{x\to \frac{\pi}{2}}\frac{(f(x)-x^2)(\pi-2x)}{x} = 2$

(C) $\lim\limits_{x\to 0}\frac{f(x)}{x}$ does not exist

(D) $\lim\limits_{x\to 0}\frac{f(x)\cos x - x}{x}$ does not exist

12. $\int \frac{x\cos x + 1}{\sqrt{2x^3 e^{\sin x} + x^2}}dx$ is equal to: $(x > 0)$

(A) $\log\left|\frac{\sqrt{2xe^{\sin x}+1}-1}{\sqrt{2xe^{\sin x}+1}+1}\right| + C$

(B) $2\ln(\sqrt{2xe^{\sin x}+1}-1) - \sin x - \ln x + C$

(C) $\ln 2x + \ln\sin x + \cos x + C$

(D) $\ln x\cos x + 2\ln\left(\sqrt{2xe^{\sin x}+1}-1\right) + C$

13. $\int \sqrt{x - \sqrt{x^2-4}}\,dx$ is equal to:

(A) $\left[\frac{1}{3\sqrt{2}}\sqrt{x-\sqrt{x^2-4}} + \frac{8}{\sqrt{2}}(\sqrt{x+2}-\sqrt{x-2})\right] + C$

(B) $\sqrt{2}\left[\frac{1}{3}\left(\sqrt{x+2}-\sqrt{x-2}\right)^3 + \sqrt{x^2-4}(\sqrt{x+2}-\sqrt{x-2})\right] + C$

(C) $\frac{2}{3\sqrt{2}}(x+2)^{3/2} - \frac{2}{3\sqrt{2}}(x-2)^{3/2} + C$

(D) $\frac{\sqrt{2}}{3}(\sqrt{x+2}-\sqrt{x-2})(2x+\sqrt{x^2-4}) + C$

(where C is constant of integration)

14. $\int \frac{\cos^{-1}\frac{x}{a}}{x^2}dx$ is equal to:

(A) $-\frac{\pi}{2x} - \int \frac{\sin^{-1}\frac{x}{a}}{x^2}dx$

(B) $\frac{1}{a}\ln\left(\frac{a+\sqrt{a^2+x^2}}{x}\right) - \frac{1}{x}\cos^{-1}\frac{x}{a} + C$

(C) $\cos^{-1}\frac{x}{a}\ln x - x(\ln x - 1) + C$

(D) $\sqrt{a^2-x^2}\ln x + x(\ln x - 1) + C$

(where C is constant oaf integration)

15. If $\int \frac{x\cot x + 1}{\sqrt{\operatorname{cosec}^2 x - x^2}}dx = f(x) + C$, where $\lim\limits_{x\to 0^+} f(x) = 0$ and $\operatorname{cosec} x > 0$ (where C is constant of integration), then which of the following is/are incorrect?

(A) $f(1) = 1$

(B) $\lim\limits_{x\to 0}\frac{f(x)}{|x|\tan x}$ does not exist

(C) $f(1) = \frac{\pi}{2} - 1$

(D) $f(x)$ is periodic with period 2π

16. $\displaystyle\int \frac{x^2}{(x^2-1)\sin 2x + 2x\cos 2x}dx =$

(A) $\displaystyle\frac{1}{2}\left|\frac{x\cos x - \sin x}{x\sin x + \cos x}\right| + C$ (B) $\displaystyle -2\ \ln\left|\frac{x\cos x - \sin x}{x\sin x + \cos x}\right| + C$

(C) $\displaystyle\frac{1}{2}\ln\left|\frac{x\sin x + \cos x}{x\cos x - \sin x}\right| + C$ (D) $\displaystyle\frac{-1}{2}\ln\left|\frac{x\cos x - \sin x}{x\sin x + \cos x}\right| + C$

17. $\displaystyle I_n = \int \frac{dx}{x(1+x^n)}$ is equal to:

(A) $\displaystyle I_5 = \ln x - \frac{1}{5}\ln(1+x^5) + C$

(B) $\displaystyle nI_n = \ln\left(\frac{x^n}{1+x^n}\right) + C$

(C) $\displaystyle 101_{10} = \ln\left(\frac{x^{10}}{x(1+x^n)}\right) + C$

(D) $\displaystyle 1001_{100} = \ln\left(\frac{x^{100}}{(1+x^{100})}\right) + C$

18. If $\displaystyle\int \frac{dt}{(1+\sqrt{t})^8} = \frac{-1}{3(1+\sqrt{t})^{p_1}} + \frac{2}{7(1+\sqrt{t})^{p_2}} + C$, where C is constant of integration, then:

(A) $p_1 = 5$ (B) $p_1 = 6$

(C) $p_2 = 7$ (D) $p_2 = 8$

19. If $\displaystyle\int \frac{3e^x + 5e^{-x}}{4e^x - 5e^{-x}}dx = Ax + B\ln|4e^{2x} - 5| + C$, then:

(A) $\displaystyle A = -\frac{1}{8}, B = \frac{7}{8}$ (B) $\displaystyle A = -1, B = \frac{7}{8}$

(C) $A = -1$, C can be -100 (D) $\displaystyle B = \frac{7}{8}$, C can be same as A

20. Let $f(x)$ be a function satisfying $f'(x) = 2f(x)$ and $f(0) = 3$. If $\displaystyle\int \frac{f(x)}{12 + 4f(x)}dx = \frac{1}{k}\ln(1 + ae^{bx}) + C$, then:

(A) $a + b = 3$ (B) $k + b = 10$

(C) $ak = 4b$ (D) $a + b + k = 0$

21. $\displaystyle I = \int e^{x\sin x + \cos x}\left(\frac{x^4\cos^3 x - x\sin x + \cos x}{x^2\cos^2 x}\right)dx$, then I equals:

(A) $\displaystyle e^{x\sin x} + \cos x\left(x - \frac{\sec x}{x}\right) + C$

(B) $\displaystyle e^{x\sin x} + \cos x\left(x\sin x - \frac{\cos x}{x}\right) + C$

(C) $\displaystyle e^{x\sin x} + \cos x\left(\frac{x}{\tan x} - \frac{\sec x}{x}\right) + C$

(D) $\displaystyle xe^{x\sin x + \cos x} - \int e^{x\sin x + \cos x}\left(1 - \frac{\cos x - x\sin x}{x^2\cos^2 x}\right)dx$

22. If $\displaystyle\int \frac{\cos 8x - \cos 7x}{1 + 2\cos 5x}\ dx = K\ \sin 3x + M\sin 2x + C$, then:

(A) $\displaystyle K = -\frac{1}{3}$ (B) $\displaystyle K = \frac{1}{3}$

(C) $\displaystyle M = -\frac{1}{2}$ (D) $\displaystyle M = \frac{1}{2}$

23. If $\displaystyle\int xe^{-5x^2}\sin 4x^2 dx = Ke^{-5x^2}(A\sin 4x^2 + B\cos 4x^2) + C$. Then:

(A) $\displaystyle K = -\frac{1}{82}$ (B) $\displaystyle K = \frac{1}{82}$

(C) $A = 5$ (D) $B = 4$

24. Let $f(x)$ be a function such that $f''(x) = f(x)$ and $f'(0) = f(0) = 0$. The value of $\displaystyle\int \frac{f(x)}{4 - f(x)}dx$ is equal to:

(A) $\log_e(4 - f(x)) + c$ (B) $-\log_e(4 - f(x)) + c$

(C) $\displaystyle\log_e\left(\frac{1 + f(x)}{4 - f(x)}\right) + c$ (D) $\displaystyle -\log_e\left(\frac{1 + f(x)}{4 - f(x)}\right) + c$

25. If $\displaystyle\int \frac{dx}{x^{2010}(1 + x^{2010})^{\frac{1}{2010}}} = -\frac{1}{a}(1 + x^{-b})^{\frac{c-2}{b}} + k$, then:

(A) $\displaystyle\frac{a + b - c}{1004} = 2$ (B) a, b, c are in A.P.

(C) a, b, c are in H.P. (D) $a, -b, c$ are in A.P.

26. $\displaystyle\int x^5\cos^2\log\sqrt{x}dx =$

(A) $\displaystyle\frac{x^6}{12} + \frac{x^6}{74}$ (6 cos log x + sin log x) + c

(B) $\displaystyle\frac{x^6}{12} - \frac{x^6}{37}$ (6 cos log x – sin log x) + c

(C) $\displaystyle\frac{(x^3 - 1)(x^3 + 1)}{12} + \frac{x^6}{37}$ (6 cos log x + sin log x) + c

(D) $\displaystyle\frac{(x^3 - 1)(x^3 + 1)}{12} + \frac{x^6}{74}$ (6 cos log x + sin log x) + c

27. If $\displaystyle\int \sqrt{x + \sqrt{x^2 + 2}}\ dx = A\left\{x + \sqrt{x^2 + 2}\right\}^{3/2} + \frac{B}{\sqrt{x + \sqrt{x^2 + 2}}} + c,$ then:

(A) $\displaystyle A = \frac{1}{3}$ (B) $B = -2$

(C) $\displaystyle A = \frac{2}{3}$ (D) $B = -1$

28. If $\displaystyle\int \frac{dx}{\cos^3 x - \sin^3 x} = A\tan^{-1}(\sin x + \cos x) + B\ln\left|\frac{\cos x + \sin x - \sqrt{2}}{\cos x + \sin x + \sqrt{2}}\right|$ $+ C$, then:

(A) $\displaystyle A = \frac{2}{3}$ (B) $\displaystyle B = -\frac{2}{3\sqrt{2}}$

(C) $\displaystyle A = -\frac{2}{3}$ (D) $\displaystyle B = -\frac{1}{3\sqrt{2}}$

29. $\displaystyle\int \frac{1}{(\sin x + 2\cos x)(\cos x + 2\sin x)}dx =$

(A) $\displaystyle\frac{1}{3}\ln\left|\frac{2\tan x + 1}{\tan x + 2}\right| + c$ (B) $\displaystyle\frac{1}{3}\ln\left|\frac{\tan x + 2}{\tan x + 1}\right| + c$

(C) $\displaystyle\frac{1}{3}\ln\left|\frac{\sin x + \cos x}{2\sin x + \cos x}\right| + c$ (D) $\displaystyle\frac{1}{3}\ln\left|\frac{2\sin x + \cos x}{\sin x + 2\cos x}\right| + c$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Let $\int \frac{x^2(x\sec^2 x+\tan x)}{(x\tan x+1)^2}\,dx = 2\ln|f(x)| - \frac{x^2}{g(x)}+k$ (where k is constant of integration):

1. $f(x)$ is:
(A) a bounded function (B) an even function
(C) an odd function (D) a periodic function

2. $\lim_{x\to 0}\frac{g(x)-1}{f(x)-1}$ is equal to:
(A) 0 (B) 1
(C) 2 (D) does not exist

Comprehension 2

Let $f(x)$, $g(x)$ $h(x)$ are three differentiable functions satisfying $\int (f(x)+g(x))\,dx = \frac{x^3}{3}+C_1$, $\int (f(x)-g(x))\,dx = x^2-\frac{x^3}{3}+C_2$ and $\int \frac{f(x)}{h(x)}dx = -\frac{1}{x}+C_3$ (where C_1, C_2, C_3 are integration constants):

3. Number of points of non-differentiability of function $\phi(x) = \min\{f(x), f(x)+g(x), h(x)\}$ is equal to:
(A) 0 (B) 1
(C) 2 (D) 3

4. If number of distinct terms in the expansion of $\left((1+f(x))+\left(\frac{f(x)+g(x)}{h(x)}\right)\right)^{\sum n}$, $(n \in N)$ is 31, then the value of n is equal to:
(A) 17 (B) 5
(C) 6 (D) 15

Comprehension 3

Let $\int (2x^6+15x^4+2x^2+3)\cos 2x\;dx = f(x)\sin(2x)+g(x)\cos(2x)+K$; where $f(x)$ and $g(x)$ are polynomial functions of x and K is the constant of integration.

5. $\int \frac{g(x)}{f(x)}dx$ equals:
(A) $\frac{1}{2}\ln(x^6+x^2+1)+C$
(B) $\frac{1}{2}\ln(x^6+x^4+1)+C$
(C) $\frac{1}{2}\tan^{-1}(x^6+x^2+1)+C$
(D) $\frac{1}{2}\tan^{-1}(x^6+x^4+1)+C$

6. $f(x)$ and $g(x)$ are respectively:
(A) odd and even (B) even and odd
(C) both are odd (D) both are even

Comprehension 4

Let f, g, h be 3 functions such that $f'(x) > 0$ and $g(x) \ge 0\ \forall\ x \in R$, where $\int f(x).g(x)dx = \frac{x^4}{4}+C$ and $\int \frac{f(x)}{g(x)}dx = \int \frac{g(x)}{h(x)}dx = \ln|x|+C, x \ne 0$.

On the basis of above information, answer the following questions:

7. $\int f(x).g(x).h(x)dx$ is equal to:
(A) $\frac{x^5}{5}+C$ (B) $\frac{x^6}{6}+C$
(C) $\frac{x^7}{7}+C$ (D) $\frac{x^8}{8}+C$

8. $f(x)$, $g(x)$, $h(x)$ are in:
(A) A.P. (B) G.P.
(C) H.P. (D) None of these

Comprehension 5

Let $2I_1+I_2 = \int \frac{e^x}{e^{2x}+e^{-2x}}dx$ and $I_1+2I_2 = \int \frac{e^{-x}}{e^{2x}+e^{-2x}}dx$.

9. I_1 is equal to:
(A) $\frac{1}{6\sqrt{2}}\tan^{-1}\left(\frac{e^x-e^{-x}}{\sqrt{2}}\right)+\frac{1}{4\sqrt{2}}\ln\left|\frac{e^x+e^{-x}-\sqrt{2}}{e^x+e^{-x}+\sqrt{2}}\right|+C$
(B) $\frac{1}{6\sqrt{2}}\tan^{-1}\left(\frac{e^x-e^{-x}}{\sqrt{2}}\right)-\frac{1}{4\sqrt{2}}\ln\left|\frac{e^x+e^{-x}-\sqrt{2}}{e^x+e^{-x}+\sqrt{2}}\right|+C$
(C) $\frac{1}{6\sqrt{2}}\tan^{-1}\left(\frac{e^x+e^{-x}}{\sqrt{2}}\right)+\frac{1}{4\sqrt{2}}\ln\left|\frac{e^x-e^{-x}-\sqrt{2}}{e^x-e^{-x}+\sqrt{2}}\right|+C$
(D) $\frac{1}{6\sqrt{2}}\tan^{-1}\left(\frac{e^x+e^{-x}}{\sqrt{2}}\right)-\frac{1}{4\sqrt{2}}\ln\left|\frac{e^x-e^{-x}-\sqrt{2}}{e^x-e^{-x}+\sqrt{2}}\right|+C$

10. I_2 is equal to:
(A) $\frac{1}{6\sqrt{2}}\tan^{-1}\left(\frac{e^x-e^{-x}}{\sqrt{2}}\right)+\frac{1}{4\sqrt{2}}\ln\left|\frac{e^x+e^{-x}-\sqrt{2}}{e^x+e^{-x}+\sqrt{2}}\right|+C$
(B) $\frac{1}{6\sqrt{2}}\tan^{-1}\left(\frac{e^x-e^{-x}}{\sqrt{2}}\right)-\frac{1}{4\sqrt{2}}\ln\left|\frac{e^x+e^{-x}-\sqrt{2}}{e^x+e^{-x}+\sqrt{2}}\right|+C$
(C) $\frac{1}{6\sqrt{2}}\tan^{-1}\left(\frac{e^x+e^{-x}}{\sqrt{2}}\right)+\frac{1}{4\sqrt{2}}\ln\left|\frac{e^x-e^{-x}-\sqrt{2}}{e^x-e^{-x}+\sqrt{2}}\right|+C$
(D) $\frac{1}{6\sqrt{2}}\tan^{-1}\left(\frac{e^x+e^{-x}}{\sqrt{2}}\right)-\frac{1}{4\sqrt{2}}\ln\left|\frac{e^x-e^{-x}-\sqrt{2}}{e^x-e^{-x}+\sqrt{2}}\right|+C$

Comprehension 6

Repeated application of integration by parts gives us, the reduction formula if the integrand is depended of n, $n \in N$. On the basis of above information, answer the following question:

11. If $I_{m-2,\,n+2} = \int \sin^{m-2} x \cos^{n+2} x \, dx$

and $I_{m,\,n} = -\dfrac{\sin^{m-1} x \cos^{n+1} x}{(n+1)} + f(m, n) I_{m-2,\; n+2}$, then $f(2, 3)$ is equal to:

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) $\frac{1}{4}$ (D) $\frac{1}{5}$

12. $I_{m.p} = \int x^m \left(a + bx^n\right)^p dx$ and

$I_{m,p} = \dfrac{x^{m+1}\left(a+bx^n\right)^p}{(m+1)} - f(m,n,p,b) I_{m+n,p-1}$, then $f(1,2,3,4) =$

(A) 8 (B) 10

(C) 12 (D) 6

13. If $I_{m,n} = \int x^m (\ln x)^n \, dx$ where m and n are positive integers, then $I_{m,n} - \dfrac{x^{m+1}}{m+1} (\ln x)^n =$

(A) $\dfrac{n}{m+1} I_{m,\,n-1}$ (B) $\dfrac{m}{n+1} I_{m,\,n-1}$

(C) $-\dfrac{n}{m+1} I_{m,\,n-1}$ (D) none of these

14. If $I_{n-2,\,n} = \int \cos^{n-2} x \sin nx \, dx$, where $n > 2$ be a positive integer, then $I_{n-2,\,n} + \dfrac{\cos^{n-2} x \cos nx}{2(n-1)} =$

(A) $-\dfrac{n-2}{2(n-1)} I_{n-3,\,n-1}$ (B) $\dfrac{n-2}{2(n-1)} I_{n-3,\,n-1}$

(C) $\dfrac{n-2}{n-1} I_{n-3,\,n-1}$ (D) none of these

Comprehension 7

Let $I_r(x) = \int \dfrac{x^r \, dx}{x^4 + 4}$ $(x > 0)$

15. If $I_1(\sqrt{2}) = \dfrac{9\pi}{16}$, then $\lim\limits_{x \to 0} \dfrac{\cos(I_1(x))}{x^2}$ is equal to:

(A) $-\frac{1}{4}$ (B) $-\frac{1}{8}$

(C) $\frac{1}{8}$ (D) 1

16. $I_2(x) + 2I_0(x)$ is equal to:

(A) $\tan^{-1}(x+1) - \frac{1}{4}\tan^{-1}\left(\frac{x^2}{2}\right) + C$

(B) $\tan^{-1}(x-1) - \frac{1}{4}\tan^{-1}\left(\frac{x^2}{2}\right) + C$

(C) $\tan^{-1}(x+1) + \frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right) + C$

(D) $\tan^{-1}(x-1) + \frac{1}{2}\tan^{-1}\left(\frac{x^2}{2}\right) + C$

Comprehension 8

For $x \in \left(0, \frac{\pi}{2}\right)$, let $f_n(x) = \int n \sin 2x (\sin^{2n-2} x - \cos^{2n-2} x) \, dx - \dfrac{1}{2^{n-1}}$, $n \in N$ and $f_n\left(\frac{\pi}{4}\right) = \dfrac{1}{2^{n-1}}$.

17. If $g(x) = \lim\limits_{n \to \infty} \sum\limits_{n=1}^{n} f_n(x)$ then minimum value of $g(x)$, is:

(A) 1 (B) 2

(C) 3 (D) 4

18. If $n = \lim\limits_{t \to \infty} [\cot^{-1} t]$, then $\int \dfrac{f_3(x)}{\sin^2(x).\cos^2(x)} dx$ equals:

(Where [·] is GIF)

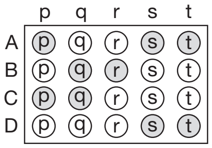
(A) $\tan x - \cot x + 3x + C$ (B) $\tan x + \cot x - 3x + C$

(C) $\tan x + \cot x + 3x + C$ (D) $\tan x - \cot x - 3x + C$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. Match the following:

COLUMN-I	COLUMN-II
(A) If $\int \frac{dx}{x^{1/3}+2} = g(x) + C$, where $g(0) = 12 \ln 2$, then $g(-1)$ is equal to:	(p) 1
(B) If $\int (\sec x)^{9/5} (\operatorname{cosec} x)^{1/5}\, dx = k \tan^m x + C$, then km is equal to:	(q) 3
(C) Let $\int \frac{dx}{\cot^2(x)-1} = \frac{1}{\ell} \ln \lvert \sec(2x) + \tan(2x) \rvert - \frac{x}{m} + c$, then $\mathrm{l} + m$ is equal to:	(r) 6
(D) Let $\int \frac{\left(1+\frac{1}{x}\right)dx}{\sqrt{1+xe^x}} = \ln\left\lvert \frac{g(x)-1}{g(x)+1} \right\rvert + C$, then $[g^2(1)]$ is equal to: (where [.] denotes greatest integer function)	(s) Rational number
	(t) Irrational number

2. If $x \in (0, 1)$ then match the entries of COLUMN-I with COLUMN-II considering 'c' as an arbitrary constant of integration.

COLUMN-I	COLUMN-II
(A) $\int \tan\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}}\right)dx$	(p) $\frac{4}{3}x^{3/4} + c$
(B) $\int \cot\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-\sqrt[4]{x}}{\sqrt{1+\sqrt{x}}+\sqrt[4]{x}}}\right)\frac{dx}{\sqrt{x}}$	(q) $\frac{4}{5}x^{5/4} + c$
(C) $\int \left(\frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}\right)\frac{dx}{2\sqrt{x}}$	(r) $\frac{2}{3}x^{3/4} + c$
(D) $\int \frac{\sqrt{x}}{2}\tan\left(2\tan^{-1}\left(\frac{\sqrt{\sqrt{1+\sqrt{x}}+1}-\sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1}+\sqrt{\sqrt{1+\sqrt{x}}-1}}\right)\right)dx$	(s) $\frac{2}{5}x^{5/4} + c$

3. Match the following:

COLUMN-I	COLUMN-II
(A) If $\int e^x\left(\log_e(3x+2) + \frac{9}{(3x+2)^2}\right)dx = e^x\left(\log_e(3x+2) + \frac{m}{nx+r}\right) + C$ Then $m + n + r$ (where m, n, r are coprime numbers), where $m < 0$, $n > 0$, $r > 0$.	(p) 5
(B) If $\int \frac{\cos 9x + \cos 6x}{2\cos 5x - 1}\, dx = A \sin 4x + B \sin x + C$ (where C is constant of integration), then $A + B$	(q) 1
(C) If $\int \sqrt[3]{\frac{\sin^{2016} x}{\cos^{2022} x}}\, dx = \frac{1}{p}(\tan x)^q + C$, then $p/q =$	(r) $\frac{5}{4}$
(D) If $\int \frac{(x^2+1)^{3/2} \cdot (x^2-1)}{x^{7/2}}\, dx = \frac{1}{m}\left(x + \frac{1}{x}\right)^n + C$, then $m + n =$	(s) 2

4. Match the following:

COLUMN-I	COLUMN-II
(A) $\int \frac{f'(x)}{f(x).\log(f(x))}\,dx = K\log(\log f(x)) + C$ (where $f(x)$ is differentiable function and $f(x) > 0$) where K	(p) 5
(B) If $\int \frac{3x^3-2x^2}{\sqrt{(x^2-3x+1)^5}}\,dx = \frac{nx^3}{m\sqrt{(x^2-3x+1)^3}}+C$, then $m-n$ where $m>0,\ n<0$	(q) 1
(C) If $f(x)=\int \frac{1+2^x\ln(e.2^{1-x})}{(x+2^x)^2}\,dx$ and $f(0)=2$, then $f(1)$ is	(r) 3
(D) If $\int \frac{x^2}{(x^2-1)\sin 2x+2x\cos 2x}\,dx = \frac{1}{\lambda}\ln\left\lvert\frac{x\sin x+\cos x}{x\cos x-\sin x}\right\rvert+C$, then $\lambda =$	(s) 2

5. Match the following:

COLUMN-I	COLUMN-II
(A) $\int \frac{x^2(x^6+x^5-1)}{(2x^6+3x^5+2)^2}\,dx$	(p) $\frac{x}{(x^2+4x+1)^{1/2}}+c$
(B) $\int \frac{(x^5+x^4+x^2)}{\sqrt{4x^7+5x^6+10x^4}}\,dx$	(q) $\frac{1}{2}(1+x^{-2}+x^{-5})^{-2}+c$
(C) $\int \frac{(2x^{12}+5x^9)}{(x^5+x^3+1)^3}\,dx$	(r) $-\frac{1}{6}(2x^3+3x^2+2x^{-3})^{-1}+c$
(D) $\int \frac{(2x+1)}{(x^2+4x+1)^{3/2}}\,dx$	(s) $x\left(\frac{x^3}{25}+\frac{x^2}{20}+\frac{1}{10}\right)^{1/2}+c$

6. Let $I=\int \frac{e^x}{e^{4x}+1}\,dx$ and $J=\int \frac{e^{-x}}{e^{-4x}+1}\,dx$ then for any arbitrary constant c.

Match the following:

COLUMN-I	COLUMN-II
(A) I	(p) $\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)+c$
(B) $J+I$	(q) $\frac{1}{2\sqrt{2}}\ln\left(\frac{e^{2x}-\sqrt{2}e^x+1}{e^{2x}+\sqrt{2}e^x+1}\right)+c$
(C) $J-I$	(r) $\frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)-\frac{1}{2}\ln\left(\frac{e^{2x}-\sqrt{2}e^x+1}{e^{2x}+\sqrt{2}e^x+1}\right)\right)+c$
(D) J	(s) $\frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)+\frac{1}{2}\ln\left(\frac{e^{2x}-\sqrt{2}e^x+1}{e^{2x}+\sqrt{2}e^x+1}\right)\right)+c$

7. COLUMN-I below gives values of integrals involving parameters A and B while COLUMN-II gives values of these parameters for which the results given are correct. Match the integrals in COLUMN-I with the values of parameters A, B in COLUMN-II so that the given result is correct.

COLUMN-I	**COLUMN-II**
(A) $\int \frac{e^{2x} - 2e^{x}}{e^{2x} + 1} dx = A \ln (e^{2x} + 1) + B \tan^{-1}(e^{x}) + c$	(p) $A = -\frac{1}{2}, B = -\frac{1}{4}$
(B) $\int \sqrt{x + \sqrt{x^2 + 2}}\, dx = A \left\{x + \sqrt{x^2 + 2}\right\}^{3/2} + \frac{B}{\sqrt{x + \sqrt{x^2 + 2}}} + c$	(q) $A = \frac{1}{3}, B = -2$
(C) $\int \frac{\cos 8x - \cos 7x}{1 + 2\cos 5x} dx = A \sin 3x + B \sin 2x + c$	(r) $A = \frac{1}{2}, B = -2$
(D) $\int \frac{\ln x}{x^3} dx = A \frac{\ln x}{x^2} + \frac{B}{x^2} + c$	(s) $A = \frac{1}{3}, B = -\frac{1}{2}$

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. Let $f(x) = \int (x\cos x + 1)e^{\sin x} dx$ and $g(x) = \int (x \sin x - 1)e^{\cos x} dx$. Further suppose that $f(0) = g(0) = 0$. Find the number of distinct solution(s) of the equation $f(x).\, g(x) = 0$.

2. $\int \frac{(1 + x^{1/4})^{1/3}}{\sqrt{x}} dx = \frac{a}{b}(1 + x^{1/4})^{7/3} - c(1 + x^{1/4})^{4/3} + k;$ a and b are coprime. Find value of $(a - b + c)$

3. If $\int \frac{(3x^2 + 2x)}{(x^6 + 2x^5 + x^4 + 2x^3 + 2x^2 + 5)} dx = Af\left(\frac{g(x)}{B}\right) + c$, where $g(x)$ is a polynomial function with leading coefficient 1 and c is integration constant, then find the value of B/A.

4. If $f(x) = \lim\limits_{n \to \infty} n(x^{1/n} - 1)$ $(x > 0)$ and $\int x^x (f(x) + 1) dx = g(x)^{h(x)} + C$, where $g(x)$ and $h(x)$ both are linear functions, then $\lim\limits_{x \to 0} (g(x + 4))^{h(x+4)}$ is equal to:

5. If $\int \left(\log_{ex} e\right)\left(\log_{e^2 x} e\right)\left(\log_{e^3 x} e\right)\frac{dx}{x} =$ $A\log\left(\log ex\right) + B\log\left(\log e^2x\right) + C\log\left(\log e^3x\right)$, then the value of $A + C$ is:

6. Let $\int (\sin 3\theta + \sin\theta)\cos\theta \cdot e^{\sin\theta} \cdot d\theta$ $= (A \cdot \sin^3\theta + B \cdot \cos^2\theta +$ $C \cdot \sin\theta + D \cdot \cos\theta + E) \cdot e^{\sin\theta} + K$; where K is constant of integration. Find the value of $\frac{B}{A} + \frac{D}{E}$.

7. If $\int (\sin x)^{\frac{-3}{4}} (\cos x)^{\frac{-21}{4}} dx = \frac{(\tan x)^{a_1}}{b_1} + \frac{(\tan x)^{a_2}}{b_2} + \frac{(\tan x)^{a_3}}{b_3} + C$, where C is integration constant, then $\left[\frac{a_1 + a_2 + a_3 + b_1 + b_2 + b_3}{2}\right]$ equals (where $[\cdot]$ denotes greatest integer function):

8. If $\int \frac{(1 + x\cos x - \sin x)dx}{(x + \sin x + \cos x)^{1/2}.(x - \sin x + \cos x)^{3/2}}$ $= (x + \ell \sin x + m \cos x)^{n}.(x + p \sin x + q \cos x)^{r} + C$ (where C is constant of integration and ℓ, $n > 0$), then $\left(\frac{\ell + m}{n}\right)(p - q)r$ is equal to:

9. For $x \in \left(0, \frac{\pi}{2}\right)$, if

$$\int \sqrt{\sec x + \tan x} + \sqrt{\sec x - \tan x}\, dx = \sqrt{2}\sin^{-1}\left(1 - g(x)\right) + c,$$

then find $\left[g\left(\frac{\pi}{6}\right)\right]$. (where [.] denotes G.I.F)

10. For $x \in \left(0, \frac{\pi}{2}\right)$, if

$$\int \frac{1 + \sin 2x}{\sqrt{\tan x} + \sqrt{\cot x}} dx = \frac{1}{2\sqrt{2}}\left[f(x)\sqrt{\sin 2x} + \sin^{-1}\left(f(x)\right)\right] + c$$

then find $\left[f\left(\frac{3\pi}{4}\right)\right]$. (where [.] denotes G.I.F)

11. For $x \in \left(0, \frac{\pi}{2}\right)$, If

$$\int \frac{\tan x\sqrt{\sec x}\left(1 + \cos^2 x\right)}{\sqrt{\cos x + \sin^2 x}} dx = 2\sqrt{f(x) + 1} + c,$$ then find

$\left[f\left(\frac{\pi}{4}\right)\right]$. (where [.] denotes G.I.F)

12. If $\int \frac{\ln(ex^{x+1}) + (\ln(x^{\sqrt{x}}))^2}{1 + (x\ln x)(\ln(e^2x^x))} dx = f(x) + C$, where $f(1) = 0$, then $e^{(e^{f(2)} - 1)}$ is equal to:

13. If $\int (x^6+7x^5+6x^4+5x^3+4x^2+3x+1)e^x \; dx$ is equal to $\sum_{k=1}^{\alpha} \beta x^k.e^x$ + C (where C is constant of integration), then $(\alpha+\beta)$ is:

14. If $\int \frac{\sqrt{e^{7x}}}{\sqrt[4]{e^{2x}+e^{-2x}}}dx = \frac{1}{k}\left(e^{\alpha x}+1\right)^{\frac{\beta}{\gamma}}$ + C (where C is constant of integration and k, α, β, γ are positive integers and β, γ are relatively prime), then $\left(\frac{k}{\beta}+\frac{\gamma}{\alpha}\right)$ is:

15. Suppose $\int \frac{1-7\cos^2 x}{\sin^7 x.\cos^2 x}dx = \frac{g(x)}{\sin^7 x}$ + C, where C is arbitrary constant of integration. Then find the value of $g'(0)+g''\left(\frac{\pi}{4}\right)$.

16. Let $\frac{d}{dx}(f(x)) = \frac{(\sin^{12}x-\cos^{12}x)}{(\sin^2 x-\cos^2 x)(1-3\sin^2 x\cos^2 x)}$ be such that $f(0)=0$, then $f\left(\frac{\pi}{2}\right)$ equals $\frac{k\pi}{16}$. Find the value of $k(k \in N)$.

17. For $y < 0$, if y is a differentiable function of x such that $y(x+y)=x$ and $\int \frac{dx}{x+2y} = -\ln(k-y)+c$ where $k \in N$ then $k=$

18. If $\int \frac{(\sin 2x+2\tan x)dx}{\cos^6 x+6\cos^2 x+4} = \frac{1}{a}\log\left(1+\frac{b}{\cos^4 x}+\frac{d}{\cos^6 x}\right)$ + C then $(a-b-d)$ is equal to:

19. Let $\int \frac{(x^6-4)dx}{(x^6+2)^{1/4}\cdot x^4} = \frac{\ell(x^6+2)^m}{x^n}+C$, then $\frac{n}{\ell m}$ is equal to:

20. If $\int \frac{dx}{2x\sqrt{1-x}\sqrt{2-x+\sqrt{1-x}}} =$

$$-\frac{1}{2\sqrt{3}}\log\left|u+\frac{1}{2}+\sqrt{u^2+u+\frac{1}{3}}\right|+\frac{K}{16}\log\left|v-\frac{1}{2}+\sqrt{v^2-v+1}\right|+C;$$

where $u=\frac{1}{\sqrt{1-x}-1}$ $v=\frac{1}{\sqrt{1-x}+1}$, then K is equal to:

21. If $\int \frac{\cos x}{3+3\sin(x)-\cos^2(x)}dx = \ln(1-f(x))+C$, then number of integral value of $f(x)$ can take is:

22. Let $f(x) = \int\left(x-{}^{10}C_1x^2+{}^{10}C_2x^3-{}^{10}C_3x^4+\ldots+{}^{10}C_{10}x^{11}\right)dx$ and $f(0)=\frac{263}{132}$, then $|f(1)|$ is:

23. If $F(x) = \int \frac{(1+x)((1-x+x^2)(1+x+x^2)+x^2)}{1+2x+3x^2+4x^3+3x^4+2x^5+x^6}dx$, then find the value of $[F(99)] - F(3)]$. (where [.] denotes greatest integer function)

24. $\int (x^{12}+x^8+x^4)(2x^8+3x^4+6)^{1/4}\,dx = a(2x^{12}+3x^8+6x^4)^{5/4}+C$ (where C is integration constant), then the value of $(60a+1)$ is:

25. If the primitive of the function $f(x)=\frac{\sin^4 x}{\cos^2 x}$ with respect to x is $\tan(x)+g(x)+C$, where $g(0)=0$ and C is indefinite integration constant, then find the value of $8\left(\frac{\pi}{8}+g\left(\frac{\pi}{12}\right)\right)$.

26. If $\int \frac{\{f'(x)g(x)-g'(x)f(x)\}dx}{\{f(x)+g(x)\}\sqrt{g(x)(f(x)-g(x))}}$ is equal to $\sqrt{m}\,\tan^{-1}\left(\sqrt{\frac{f(x)-g(x)}{ng(x)}}\right)$ + c, where m, $n \in R$ and 'c' constant of integration $(g(x)>0)$, then m^2+n^2 is equal to:

27. Let $f(x)=\int \frac{x^2+4}{x^4+x^3-7x^2-4x+16}dx =$ $\frac{A}{\sqrt{3}}\tan^{-1}\left(\frac{Ax^2+Bx+2C}{\sqrt{3}x}\right)+\lambda$ such that $f(2)=\frac{\pi}{3\sqrt{3}}$ (where $A, B, C \in R$), then find value of $|A+B+C|$

28. If $\int \frac{(2x+3)dx}{x(x+1)(x+2)(x+3)+1}dx = -\frac{1}{f(x)}+C$ where $f(x)$ is of the form of ax^2+bx+c, then find $(a+b+c)$.

29. Let $F(x)=\int e^{\sin^{-1}x}\left(1-\frac{x}{\sqrt{1-x^2}}\right)dx$ and $F(0)=1$.

If $F\left(\frac{1}{2}\right)=\frac{k\sqrt{3}e^{\frac{\pi}{6}}}{\pi}$, then find $4k/\pi$.

30. Let $I_{10} = \int (1-x^5)^{10}dx$ and $I_{10}=\frac{x(1-x^5)^{10}}{m}+\frac{n}{m}I_9$. then value of $\frac{n}{10}+\frac{m}{17}$ is:

31. $\int \frac{2e^{5x}+e^{4x}-4e^{3x}+4e^{2x}+2e^x}{(e^{2x}+4)(e^{2x}-1)^2}dx = \tan^{-1}\left(\frac{e^x}{2}\right)+\frac{k}{(e^{2x}-1)}+C,$ (where C is constant of integration). Then $\left|\frac{1}{k}\right|$ is equal to:

32. $\int \frac{x^2-3}{x^3-2x^2-x+2}dx = \frac{A}{9}\ln\left|C.\frac{(x-2)(x-1)^3}{(x+1)}\right|$. Where C is constant of integration), then A is equal to:

33. If $f_1(x)$ and $f_2(x)$ are two antiderivatives of $f(x) = ax + b + \sin(x)$ such that $\int_2^4 (f_1(x)-f_2(x))\,dx = 8$, then find the area of the triangle formed by joining the points $(3, f_1(3), (3, f_2(3))$ and the origin.

34. Let $\int \frac{(x^6-4)dx}{(x^6+2)^{1/4}x^4} = \left(\frac{\ell(x^6+2)^m}{x^n}+C,\right)(\ell, m, n \in R)$, then $\frac{n}{\ell m}$ is equal to:

35. Let $\int \frac{dx}{x^{2008}+x} = \frac{1}{p}\log\left(\frac{x^q}{1+x^r}\right)+c$, where p, q, $r \in N$. and need not be distinct then the sum of the digits in $p+q+r$ is:

36. If $\int \frac{(1-\cos x)^{2/7}}{(1+\cos x)^{9/7}}dx = a\left(\tan\frac{x}{2}\right)^b + c$, then the value of product ab equals:

37. If $\int \frac{dx}{\left(x^2+1\right)^2} = a\tan^{-1}x + b\frac{x}{\left(x^2+1\right)} + c$, then $a+b=$ ________.

38. Let $f(x)$ be a function satisfying $f(0) = 2$, $f'(0) = 3$, $f''(x) = f(x)$, $f(4) = \frac{k\left(e^8-1\right)}{e^4}$, then $[k]$ is ____. (where $[x]$ is greatest integer function)

39. $\int \frac{x^2dx}{(x\sin x+\cos x)^2} = \frac{\sin^k x - x\cos^p x}{(x\sin x+\cos x)} + c$ where $k+p$.

40. If $\int \frac{dx}{\left(\sec x+\operatorname{cosec} x+\tan x+\cot x\right)^2} =$

$\frac{x}{a} + \frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{b} + \frac{\cos 2x}{c} + d$ then $|a+b+c|$ is

41. If $f(x) = \lim\limits_{n\to\infty}\left[\sin^n\left(\frac{2\pi n}{3n+1}\right) + \frac{x^n - x^{-n}}{x^n + x^{-n}}(x>1)\right]$ and

$\int \frac{xf(x).\log\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$ is $l\left[\sqrt{1+x^2}\log(x+\sqrt{1+x^2}) - x\right]$

+ C. Then l is:

42. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ and $f(0) = 2$. and $\int \frac{f(x)}{3+4f(x)}dx$ is $K\log(3+8e^x) + C$, then $\frac{1}{K} =$

43. Let $\int \frac{dx}{(x+1)\sqrt{x^2+x+1}} = f(x) + C$ where $f(0) = -\ln 2$ and $f\left(-\frac{1}{2}\right) = \ell n\left(\sqrt{k} - \frac{3}{2}\right)$, then k is equal to:

44. Let $\int \frac{(4+3x)dx}{(x^3-2x-4)} = \ln\left|\frac{x-k}{\sqrt{x^2+kx+\ell}}\right| + C$, then $k^2+\ell^2$ is equal to:

45. $\int \frac{\cos^4 x dx}{\sin^3 x(\sin^5 x+\cos^5 x)^{3/5}} = -\frac{1}{2}(1+\cot^A x)^B + c$, then find AB.

46. Let $I = \int \frac{x+x^{2/3}+x^{1/6}}{x(1+x^{1/3})}dx = \frac{a}{b}x^{2/3} + c\tan^{-1}(x^{1/6}) + \lambda$ there a and b are relatively prime, then the value of $(100a + 10b + c - 320)$ is:

47. If $\int \frac{\operatorname{cosec}^2 x - 2009}{\cos^{2009} x}dx = \frac{-A(x)}{(B(x))^{2009}} + C$, then number of solution of the equation $\frac{A(x)}{B(x)} = \{x\}$ in $[0, 2\pi]$ is: (where $\{.\}$ represents fractional part function)

48. Let $f(x) = \int (x\cos x+1)e^{\sin x}dx$ and $g(x) = \int (x\sin x-1)e^{\cos x}dx$. Further suppose that $f(0) = g(0) = 0$. Find the number of distinct solution (*s*) of the equation $f(x).g(x) = 0$.

49. Let $\int \frac{f'(x)g(x) - g'(x)f(x)}{(f(x)+g(x))\sqrt{f(x)g(x)-g^2(x)}}dx =$

$\sqrt{m}\tan^{-1}\left(\sqrt{\frac{f(x)-g(x)}{ng(x)}}\right) + C$, where $m, n \in N$ and 'C' is constant of integration ($g(x) > 0$) then value of (m^2+n^2) is:

ARCHIVE: JEE MAIN

1. **[JEE Mains 2021]**

If $\int \frac{\cos x - \sin x}{\sqrt{8-\sin 2x}}dx = a\sin^{-1}\left(\frac{\sin x+\cos x}{b}\right) + c$, where c is a constant of integration, then the ordered pair (a, b) is equal to:

(A) (1, –3) (B) (3, 1)
(C) (–1, 3) (D) (1, 3)

2. **[JEE Mains 2021]**

The value of the integral

$$\int \frac{\sin\theta.\sin 2\theta(\sin^6\theta+\sin^4\theta+\sin^2\theta)\sqrt{2\sin^4\theta+3\sin^2\theta+6}}{1-\cos 2\theta}d\theta$$

is: (where c is a constant of integration)

(A) $\frac{1}{18}[11 - 18\sin^2\theta + 9\sin^4\theta - 2\sin^6\theta]^{3/2} + c$

(B) $\frac{1}{18}[9 - 2\cos^6\theta - 3\cos^4\theta - 6\cos^2\theta]^{3/2} + c$

(C) $\frac{1}{18}[9 - 2\sin^6\theta - 3\sin^4\theta - 6\sin^2\theta]^{3/2} + c$

(D) $\frac{1}{18}[11 - 18\cos^2\theta + 9\cos^4\theta - 2\cos^6\theta]^{3/2} + c$

3. **[JEE Mains 2021]**

The integral $\int \frac{e^{3\log_e 2x} + 5e^{2\log_e 2x}}{e^{4\log_e x} + 5e^{3\log_e x} - 7e^{2\log_e x}}dx$, $x > 0$, is equal to:

(where c is a constant of integration)

(A) $\log_e|x^2+5x-7| + c$

(B) $4\log_e|x^2+5x-7| + c$

(C) $\log_e\sqrt{x^2+5x-7} + c$

(D) $\frac{1}{4}\log_e|x^2+5x-7| + c$

4. **[JEE Mains 2021]**

For real numbers α, β, γ and δ, if

$$\int \frac{(x^2-1)+\tan^{-1}\left(\frac{x^2+1}{x}\right)}{(x^4+3x^2+1)\tan^{-1}\left(\frac{x^2+1}{x}\right)}dx = \alpha \log_e\left(\tan^{-1}\left(\frac{x^2+1}{x}\right)\right) +$$

$\beta \tan^{-1}\left(\frac{\gamma(x^2-1)}{x}\right) + \delta \tan^{-1}\left(\frac{x^2+1}{x}\right) + C$ where C is an arbitrary constant, then the value of $10(\alpha+\beta\gamma+\delta)$ is equal to ______.

5. **[JEE Mains 2021]**

The integral $\int \frac{(2x-1)\cos\sqrt{(2x-1)^2+5)}}{\sqrt{4x^2-4x+6}}dx$ is equal to:

(where c is a constant of integration)

(A) $\frac{1}{2}\sin\sqrt{(2x-1^2)+5}+c$ (B) $\frac{1}{2}\cos\sqrt{(2x+1^2)+5}\ +c$

(C) $\frac{1}{2}\cos\sqrt{(2x-1^2)+5}+c$ (D) $\frac{1}{2}\sin\sqrt{(2x+1^2)+5}\ +c$

6. **[JEE Mains 2021]**

If $f(x) = \int \frac{5x^8+7x^6}{(x^2+1+2x^7)^2}dx$, $(x \geq 0)$, $f(0) = 0$ and $f(1) = \frac{1}{K}$, then the value of K is

7. **[JEE Mains 2021]**

If $\int \frac{dx}{(x^2+x+1)^2} = a\ \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + b\left(\frac{2x+1}{x^2+x+1}\right) + C$, $x > 0$, where C is the constant of integration, then the value of $9\left(\sqrt{3}a+b\right)$ is equal to _______.

8. **[JEE Mains 2020]**

If $\int \frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)} = \lambda\tan\theta + 2\log_e|f(\theta)| + C$ where C is a constant of integration, then the ordered pair $(\lambda, f(\theta))$ is equal to:

(A) $(-1, 1-\tan\theta)$ (B) $(1, 1+\tan\theta)$
(C) $(1, 1-\tan\theta)$ (D) $(-1, 1+\tan\theta)$

9. **[JEE Mains 2020]**

If $\int \sin^{-1}\left(\sqrt{\frac{x}{1+x}}\right)dx = A(x)\tan^{-1}\left(\sqrt{x}\right) + B(x) + C$, where C is a constant of integration, then the ordered pair $(A(x).B(x))$ can be:

(A) $\left(x-1, \sqrt{x}\right)$ (B) $\left(x+1, \sqrt{x}\right)$

(C) $\left(x+1, -\sqrt{x}\right)$ (D) $\left(x-1, -\sqrt{x}\right)$

10. **[JEE Mains 2020]**

Let $f(x) = \int \frac{\sqrt{x}}{(1+x)^2}dx\ (x \geq 0)$. Then $f(3) - f(1)$ is equal to:

(A) $-\frac{\pi}{6}+\frac{1}{2}+\frac{\sqrt{3}}{4}$ (B) $\frac{\pi}{6}+\frac{1}{2}-\frac{\sqrt{3}}{4}$

(C) $-\frac{\pi}{12}+\frac{1}{2}+\frac{\sqrt{3}}{4}$ (D) $\frac{\pi}{12}+\frac{1}{2}-\frac{\sqrt{3}}{4}$

11. **[JEE Mains 2020]**

The integral $\int\left(\frac{x}{x\sin x+\cos x}\right)^2 dx$ is equal to:

(where C is a constant of integration)

(A) $\sec x + \frac{x\tan x}{x\sin x+\cos x} + C$

(B) $\sec x - \frac{x\tan x}{x\sin x+\cos x} + C$

(C) $\tan x + \frac{x\sec x}{x\sin x+\cos x} + C$

(D) $\tan x - \frac{x\sec x}{x\sin x+\cos x} + C$

12. **[JEE Mains 2020]**

If $\int \frac{\cos\theta}{5+7\sin\theta-2\cos^2\theta}d\theta = A\log_e|B(\theta)| + C$, where C is a constant of integration, then $\frac{B(\theta)}{A}$ can be:

(A) $\frac{2\sin\theta+1}{5(\sin\theta+3)}$ (B) $\frac{2\sin\theta+1}{\sin\theta+3}$

(C) $\frac{5(\sin\theta+3)}{2\sin\theta+1}$ (D) $\frac{5(2\sin\theta+1)}{\sin\theta+3}$

13. **[JEE Mains 2020]**

If $\int(e^{2x}+2e^x-e^{-x}-1)e^{(e^x+e^{-x})}dx = g(x)\ e^{(e^x+e^{-x})} + c$, where c is a constant of integration, then $g(0)$ is equal to:

(A) 2 (B) e^2
(C) e (D) 1

14. **[JEE Mains 2019]**

The integral $\int \sec^{2/3}x\ \text{cosec}^{4/3}x\ dx$ is equal to:

(here, C is a constant of integration)

(A) $-3\tan^{-1/3}x + C$ (B) $-\frac{3}{4}\tan^{-4/3}x + C$

(C) $3\tan^{-1/3}x + C$ (D) $-3\cot^{-1/3}x + C$

15. **[JEE Mains 2019]**

If $\int e^{\sec x}\left(\sec x\tan x f(x) + (\sec x\tan x+\sec^2 x)\right)dx = e^{\sec x}f(x) + C$, then a possible choice of $f(x)$ is:

(A) $\sec x - \tan x - \frac{1}{2}$ (B) $\sec x + x\tan x - \frac{1}{2}$

(C) $\sec x + \tan x + \frac{1}{2}$ (D) $x\sec x + \tan x + \frac{1}{2}$

16. **[JEE Mains 2019]**

If $\int \frac{dx}{(x^2-2x+10)^2} = A\left(\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{f(x)}{x^2-2x+10}\right) + C$, where C is a constant of integration, then:

(A) $A = \frac{1}{54}$ and $f(x) = 9(x-1)^2$

(B) $A = \frac{1}{54}$ and $f(x) = 3(x-1)$

(C) $A = \frac{1}{81}$ and $f(x) = 3(x-1)$

(D) $A = \frac{1}{27}$ and $f(x) = 9(x-1)$

17. [JEE Mains 2019]

If $\int x^5 e^{-x^2}dx = g(x)\,e^{-x^2} + c$, where c is a constant of integration, then $g(-1)$ is equal to:

(A) -1 (B) $-\frac{5}{2}$

(C) 1 (D) $-\frac{1}{2}$

18. [JEE Mains 2019]

The integral $\int \frac{2x^3-1}{x^4+x}dx$ is equal to:

(here C is a constant of integration)

(A) $\log_e \frac{|x^3+1|}{x^2} + C$ (B) $\frac{1}{2}\log_e \frac{|x^3+1|}{x^2} + C$

(C) $\log_e \left|\frac{x^3+1}{x}\right| + C$ (D) $\frac{1}{2}\log_e \frac{(x^3+1)^2}{|x^3|} + C$

19. [JEE Mains 2019]

Let $\alpha \in \left(0, \frac{\pi}{2}\right)$ be fixed. If the integral $\int \frac{\tan x + \tan\alpha}{\tan x - \tan\alpha}dx =$ $A(x)\cos 2\alpha + B(x)\sin 2\alpha + C$, where C is a constant of integration, then the functions $A(x)$ and $B(x)$ are respectively:

(A) $x - \alpha$ and $\log_e|\cos(x-\alpha)|$

(B) $x + \alpha$ and $\log_e|\sin(x-\alpha)|$

(C) $x + \alpha$ and $\log_e|\cos(x+\alpha)|$

(D) $x - \alpha$ and $\log_e|\sin(x-\alpha)|$

20. [JEE Mains 2018]

The integral $\int \frac{\sin^2 x\cos^2 x}{(\sin^5 x + \cos^3 x\sin^2 x + \sin^3 x\cos^2 x + \cos^5 x)^2}dx$ is equal to:

(where C is a constant of integration)

(A) $\frac{-1}{3(1+\tan^3 x)} + C$ (B) $\frac{1}{1+\cot^3 x} + C$

(C) $\frac{-1}{1+\cot^3 x} + C$ (D) $\frac{1}{3(1+\tan^3 x)} + C$

21. [JEE Mains 2017]

Let $I_n = \int \tan^n x\,dx, (n > 1)$. $I_4 + I_6 = a\tan^5 x + bx^5 + C$, where C is constant of integration, then the ordered pair (a, b) is equal to:

(A) $\left(-\frac{1}{5}, 0\right)$ (B) $\left(-\frac{1}{5}, 1\right)$

(C) $\left(\frac{1}{5}, 0\right)$ (D) $\left(\frac{1}{5}, 1\right)$

22. [JEE Mains 2015]

The integral $\int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}}$ equals:

(A) $-(x^4+1)^{\frac{1}{4}} + c$ (B) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$

(C) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}} + c$ (D) $(x^4+1)^{\frac{1}{4}} + c$

23. [JEE Mains 2014]

The integral $\int \left(1 + x - \frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to:

(A) $(x-1)e^{x+\frac{1}{x}} + c$ (B) $xe^{x+\frac{1}{x}} + c$

(C) $(x+1)e^{x+\frac{1}{x}} + c$ (D) $-xe^{x+\frac{1}{x}} + c$

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. [IIT-JEE 2012]

The integral $\int \frac{2x}{(\sec x + \tan x)^{9/2}}dx$ equals:

(for some arbitrary constant K)

(A) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(B) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(C) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(D) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

2. [IIT-JEE 2008]

Let, $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1}dx$, $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1}dx$, then for an arbitrary constant C, the value of $J - I$ equals:

(A) $\frac{1}{2}\log\left(\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1}\right) + C$

(B) $\frac{1}{2}\log\left(\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1}\right) + C$

(C) $\frac{1}{2}\log\left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right) + C$

(D) $\frac{1}{2}\log\left(\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1}\right) + C$

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. B	**2.** A	**3.** A	**4.** A	**5.** C	**6.** B	**7.** B	**8.** A	**9.** B	**10.** D
11. A	**12.** C	**13.** C	**14.** B	**15.** A	**16.** A	**17.** B	**18.** A	**19.** A	**20.** C
21. B	**22.** A	**23.** B	**24.** C	**25.** C	**26.** D	**27.** A	**28.** B	**29.** C	**30.** A
31. C	**32.** A	**33.** C	**34.** B	**35.** A	**36.** D	**37.** C	**38.** B	**39.** C	**40.** C
41. B	**42.** C	**43.** A	**44.** A	**45.** C	**46.** B	**47.** C	**48.** D	**49.** B	**50.** A
51. A	**52.** B	**53.** D	**54.** A	**55.** C	**56.** D	**57.** D	**58.** **A**	**59.** C	**60.** D
61. D	**62.** A	**63.** C	**64.** B	**65.** B	**66.** B	**67.** A	**68.** C	**69.** B	**70.** A
71. A	**72.** D	**73.** C	**74.** B	**75.** B	**76.** D	**77.** C	**78.** C	**79.** B	**80.** C
81. B	**82.** C	**83.** D	**84.** B	**85.** A	**86.** B	**87.** A	**88.** A	**89.** C	**90.** A
91. A	**92.** B	**93.** C	**94.** D	**95.** A	**96.** B	**97.** A	**98.** A	**99.** C	**100.** C

Multiple Correct Choice Type Questions

1. B, C, D	**2.** C, D	**3.** A, B, C, D	**4.** B, C	**5.** A, B, D
6. A, B, C, D	**7.** B, C, D	**8.** A, B, C	**9.** A, D	**10.** A, C, D
11. A, B	**12.** A, B	**13.** B, C, D	**14.** A, B	**15.** B, C, D
16. C, D	**17.** A, B, C, D	**18.** B, C	**19.** B, C, D	**20.** A, B, C
21. A, D	**22.** B, C	**23.** A, C, D	**24.** A, B	**25.** A, B
26. A, D	**27.** A, B	**28.** A, D	**29.** A, D	

Linked Comprehension Type Questions

1. B	**2.** C	**3.** D	**4.** B	**5.** A	**6.** B	**7.** C	**8.** B	**9.** A	**10.** B
11. C	**12.** C	**13.** C	**14.** B	**15.** B	**16.** C	**17.** B	**18.** D		

Matrix Match/Column Match Type Questions

1. A → (s)	B → (p, s)	C → (r, s)	D → (q, s)
2. A → (q)	B → (p)	C → (r)	D → (s)
3. A → (s)	B → (r)	C → (q)	D → (p)
4. A → (q)	B → (p)	C → (r)	D → (s)
5. A → (r)	B → (s)	C → (q)	D → (p)
6. A → (r)	B → (p)	C → (q)	D → (s)
7. A → (r)	B → (q)	C → (s)	D → (p)

Integer/Numerical Answer Type Questions

1. 1	**2.** 8	**3.** 4	**4.** 256	**5.** 1	**6.** 3	**7.** 6	**8.** 4	**9.** 1	**10.** 1
11. 0	**12.** 4	**13.** 7	**14.** 2	**15.** 5	**16.** 7	**17.** 1	**18.** 2	**19.** 6	**20.** 8
21. 0	**22.** 2	**23.** 3	**24.** 3	**25.** 1	**26.** 8	**27.** 1	**28.** 5	**29.** 2	**30.** 8
31. 2	**32.** 3	**33.** 6	**34.** 6	**35.** 9	**36.** 1	**37.** 1	**38.** 2	**39.** 2	**40.** 4
41. 1	**42.** 4	**43.** 3	**44.** 8	**45.** 2	**46.** 6	**47.** 0	**48.** 1	**49.** 8	

ARCHIVE: JEE MAIN

1. D	**2.** D	**3.** B	**4.** 6	**5.** A	**6.** 4	**7.** 15	**8.** D	**9.** C	**10.** D
11. D	**12.** D	**13.** A	**14.** A	**15.** C	**16.** B	**17.** B	**18.** C	**19.** D	**20.** A
21. C	**22.** B	**23.** **B**							

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. C	**2.** C

CHAPTER

9

Definite Integration

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Mathematical and Geometrical Definition
(b) Properties of Definite Integration
(c) Definite Integration Using Substitution
(d) Definite Integration for Periodic Function
(e) Walli's Formula
(f) Integration Involving $F(x)$ and $F^{-1}(x)$
(g) Definite Integration with two variables
(h) Newton-Leibnitz Rule
(i) Definite Integration using limit of sum
(j) Solving limit using Integration
(k) Estimation of Definite Integration

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

MATHEMATICAL DEFINITION

Let $f(x)$ is a function defined for $x \in (a, b)$

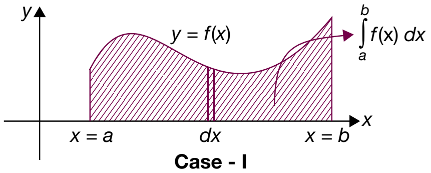
$$\int_a^b f(x)\,dx = \int_{x=a^+}^{x=b^-} f(x)\,dx = g(x)\Big|_{x=a^+}^{x=b^-} = g(b^-) - g(a^+)$$

where Integral of $f(x)$ with respect to x is $g(x)$

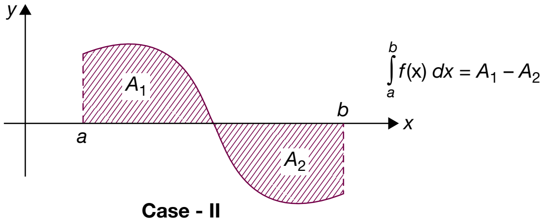
GEOMETRICAL DEFINITION

$\int_a^b f(x)\,dx$ = Algebraic sum of area enclosed by curve $y = f(x)$ from ordinates $x = a$ to $x = b$ with x-axis where algebraic sum means that area which is above the x-axis will be added in this sum with + sign and area which is below the x-axis will be added in this sum with – sign hence, the value of the definite integral may be positive, zero or negative.

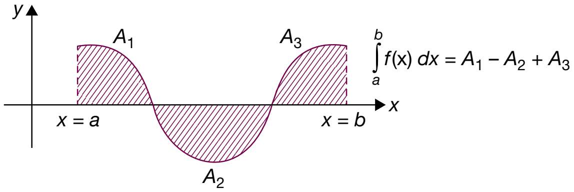
Let A_1, A_2, A_3 are the numeric value of the area shown in the following figures:



Case - I



Case - II



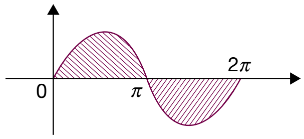
Case - III

Let consider integral

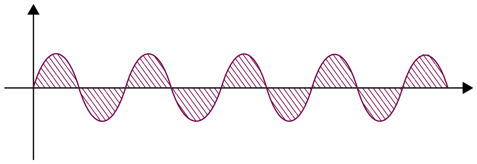
$$\int_0^{2\pi} \sin x\;dx = (-\cos x)_0^{2\pi} = (-1) - (-1) = 0.$$

Mathematically, we are getting the value of integral as zero, now let us understand it graphically.
sin x is above x -axis when $0 < x < \pi$, sin x is below x-axis when $\pi < x < 2\pi$ as shown in diagram.

sin x is forming two equal areas, one loop above x-axis and one loop below x-axis, so the algebraic sum of these areas is zero, hence, the given integral should be zero.



For computing $\int_0^{2000\pi} \sin x\, dx$, we need to observe the graph of sin x, It is a periodic function with 2π as period. sin x has 1000 loops above x-axis and 1000 loops below x-axis where each loop is of π length.



Mathematically, we can directly compute it using indefinite integration of function and followed by limits substitution.

$$\int_0^{2000\pi} \sin x\, dx = (-\cos x)_0^{2\pi} = -1-(-1) = 0$$

We can always use the concepts which we learnt from the Indefinite Integration chapter, for evaluating Definite Integration, let us go through the following illustrations:

ILLUSTRATION 1

Evaluate: $\int_0^5 \sin\left(\frac{\pi}{2}x\right)dx + \int_0^{1/2} \frac{dx}{\sqrt{4-x^2}}$

SOLUTION

$$I = \int_0^5 \sin\left(\frac{\pi}{2}x\right)dx + \int_0^{1/2} \frac{dx}{\sqrt{4-x^2}}$$

$$I = -\left.\frac{\cos\frac{\pi}{2}x}{\frac{\pi}{2}}\right|_0^5 + \left.\sin^{-1}\frac{x}{2}\right|_0^{1/2}$$

$$= -\frac{2}{\pi}\left[\cos\frac{5\pi}{2} - \cos 0\right] + \sin^{-1}\frac{1}{4}$$

$$I = \frac{2}{\pi} + \sin^{-1}\frac{1}{4}$$

ILLUSTRATION 2

Let $f(x) = \lim\limits_{n\to\infty} \frac{\cos x}{1+\left(\tan^{-1} x\right)^n}$, then evaluate $\int_0^\infty f(x)dx$.

SOLUTION

$$f(x) = \lim_{n\to\infty} \frac{\cos x}{1+(\tan^{-1} x)^n} = \begin{cases} \cos x & ; \quad 0 \le x < \tan 1 \\ \frac{\cos(\tan 1)}{2} & ; \quad x = \tan 1 \\ 0 & ; \quad x > \tan 1 \end{cases}$$

$$\int_0^\infty f(x)dx = \int_0^{\tan 1} \cos x\, dx + \int_{\tan 1}^\infty 0\, dx = \sin(\tan 1)$$

ILLUSTRATION 3

Evaluate: $\int_0^{\pi/4} x \cdot \cos x . \cos 3x \; dx$

SOLUTION

$$I = \int_0^{\pi/4} x \cdot \cos x \cdot \cos 3x \; dx = \int_0^{\pi/4} \frac{x}{2}[\cos 4x + \cos 2x] \; dx$$

$$I = \frac{1}{2}\int_0^{\pi/4} \underset{\text{I}}{\underset{\downarrow}{x}} \underset{\text{II}}{\underset{\downarrow}{\cos 4x}} dx + \frac{1}{2}\int_0^{\pi/4} \underset{\text{I}}{\underset{\downarrow}{x}} \underset{\text{II}}{\underset{\downarrow}{\cos 2x}} dx$$

(using integration by parts)

$$I = \frac{1}{2}\left(x\left.\frac{\sin 4x}{4}\right|_0^{\pi/4} - \int_0^{\pi/4} \frac{\sin 4x}{4} dx\right) + \frac{1}{2}\left(x\left.\frac{\sin 2x}{2}\right|_0^{\pi/4} - \int_0^{\pi/4} \frac{\sin 2x}{2} dx\right)$$

$$I = \frac{1}{2}\left[0 - 0 - \frac{1}{16}(-\cos 4x)_0^{\pi/4}\right] + \frac{1}{2}\left[\frac{\pi}{8}\cdot(1) - 0 - \frac{1}{2}\left(-\frac{\cos 2x}{2}\right)_0^{\pi/4}\right]$$

$$I = \frac{1}{2}\left[\frac{1}{16}(-1-1)\right] + \frac{\pi}{16} + \frac{1}{8}[0-1] = \frac{\pi-3}{16}$$

ILLUSTRATION 4

Evaluate: $\int_0^{3\pi/4} [(1+x)\sin x + (1-x)(\cos x)] \; dx$

SOLUTION

$$I = \int_0^{3\pi/4} [(1+x)\sin x + (1-x)\cos x] \; dx$$

$$I = \int_0^{3\pi/4} \sin x + \cos x \, dx + \int_0^{3\pi/4} \underset{\overset{\downarrow}{\text{I}}}{x} (\sin x - \underset{\overset{\downarrow}{\text{II}}}{\cos x}) \, dx$$

(using by integration by parts)

$$I = -\cos x + \sin x \Big|_0^{3\pi/4} + x(-\cos x - \sin x)\Big|_0^{3\pi/4} - \int_0^{3\pi/4} (-\cos x - \sin x) \, dx$$

$$I = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 - 0 + 0 - 0 + (\sin x - \cos x)\Big|_0^{3\pi/4}$$

$$I = (\sqrt{2} + 1) + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 = 2\left(\sqrt{2} + 1\right)$$

ILLUSTRATION 5

Let $f(x)$, $f'(x)$ and $f''(x)$ are continuous on $[0, \ln 2]$, and $f(0) = 0, f'(0) = 3, f'(\ln 2) = 4, f(\ln 2) = 6$ and $\int_0^{\ln 2} e^{-2x}.f(x)\,dx = 3$. Find the value of $\int_0^{\ln 2} e^{-2x} f''(x)\ dx$.

SOLUTION

$$\int_0^{\ln 2} e^{-2x} f''(x)\ dx = e^{-2x} f'(x)\Big|_0^{\ln 2} - \int_0^{\ln 2} (e^{-2x}(-2) f'(x)\ dx$$

(using by parts integration)

$$= \frac{1}{4} f'(\ln 2) - f'(0) + 2 \int_0^{\ln 2} e^{-2x} f'(x)\, dx$$

$$= \frac{1}{4}(4) - 3 + 2 \int_0^{\ln 2} e^{-2x} f'(x)\, dx$$

$$= -2 + 2\left(e^{-2x} f(x)\Big|_0^{\ln 2} - \int_0^{\ln 2} (-2) e^{-2x} f(x) dx \right)$$

$$= -2 + 2\left(\frac{1}{4} 6 - 0 + 2.3 \right) = -2 + 3 + 12 = 13.$$

ILLUSTRATION 6

Solve: $\int_{-1}^{x} \left(8t^2 + \frac{28}{3} t + 4 \right) dt = \frac{1.5x + 1}{\log_{x+1} \sqrt{x+1}}$.

SOLUTION

For log to be defined $x > -1, x \neq 0$

$$\int_{-1}^{x} \left(8t^2 + \frac{28}{3} t + 4 \right) dt = \frac{1.5x + 1}{\log_{x+1} \sqrt{x+1}}$$

$$\Rightarrow \quad \frac{8}{3}(x^3 + 1) + \frac{14}{3}(x^2 - 1) + 4(x + 1) = \frac{\frac{3}{2}x + 1}{\frac{1}{2}}$$

$$\Rightarrow \quad 8x^3 + 8 + 14x^2 - 14 + 12x + 12 = 9x + 6$$

$$\Rightarrow \quad 8x^3 + 14x^2 + 3x = 0$$

$$\Rightarrow \quad x(8x^2 + 14x + 3) = 0$$

$$\Rightarrow \quad x(8x^2 + 2x + 12x + 3) = 0$$

$$\Rightarrow \quad x(2x + 3)\,(4x + 1) = 0$$

We get, $x = 0, -\frac{3}{2}, -\frac{1}{4}$ but we know that $x > -1, x \neq 0$

Final set of value of $x = -\frac{1}{4}$ only.

Note

$$\int_a^b g^2(x)\ dx = 0 \text{ then } g(x) = 0.$$

As integral of a non-negative continuous function is zero then function is zero function.

ILLUSTRATION 7

Let $f(x)$ be a continuous function satisfying $\int_0^1 f(x)(4x^2 - f(x))\,dx = \frac{4}{5}$, then find $f(3)$.

SOLUTION

$$\int_0^1 f(x)(4x^2 - f(x))\,dx = \frac{4}{5}$$

$$\Rightarrow \quad \int_0^1 (f(x))^2 - 2(2x^2) f(x) + (2x^2)^2 - 4x^4\,dx = -\frac{4}{5}$$

$$\Rightarrow \quad \int_0^1 (f(x) - 2x^2)^2\,dx = \frac{-4}{5} + \int_0^1 4x^4\,dx = \frac{-4}{5} + \frac{4}{5}$$

$$\therefore \quad \int_0^1 (f(x) - 2x^2)^2\,dx = 0$$

We have $f(x) - 2x^2 = 0 \Rightarrow f(x) = 2x^2 \Rightarrow f(3) = 18$

ILLUSTRATION 8

If for non-zero x, $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5$, where $a \neq b$, then compute $\int_1^2 f(x)\ dx$.

SOLUTION

Given $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \quad \ldots(1)$

Replace x by $\frac{1}{x}$ in (1), we get:

$$af\left(\frac{1}{x}\right) + bf(x) = x - 5 \quad \ldots(2)$$

Multiply (1) by a and (2) by b and substract we get:

$$(a^2 - b^2)\, f(x) = \frac{a}{x} - bx - 5a + 5b$$

$$\Rightarrow \quad f(x) = \frac{1}{(a^2 - b^2)}\left[\frac{a}{x} - bx - 5a + 5b\right]$$

Now, $\int\limits_1^2 f(x)\,dx = \frac{1}{(a^2 - b^2)}\int_1^2\left[\frac{a}{x} - bx - 5a + 5b\right]$

$$= \frac{1}{(a^2 - b^2)}\left[a\log|x| - \frac{b}{2}x^2 - 5(a-b)x\right]_1^2$$

$$= \frac{1}{(a^2 - b^2)}\left[a\log 2 - 2b - 10(a-b) - a\log 1 + \frac{b}{2} + 5(a-b)\right]$$

$$= \frac{1}{(a^2 - b^2)}\left[a\log 2 - 5a + \frac{7}{2}b\right].$$

ILLUSTRATION 9

Evaluate: $\int\limits_0^\infty e^{-ax}\cos bx\,dx.$

SOLUTION

We know $\int e^{-ax}\cos bx\,dx = \frac{e^{-ax}}{a^2 + b^2}(-a\cos bx + b\sin bx)$

Given $I = \int\limits_0^\infty e^{-ax}\cos bx\,dx = \lim\limits_{\alpha\to\infty}\int\limits_0^\alpha e^{-ax}\cos bx\,dx$

$$= \lim_{\alpha\to\infty}\left[\frac{e^{-ax}}{a^2 + b^2}(-a\cos bx + b\sin bx)\right]_0^\alpha$$

$$= \lim_{\alpha\to\infty}\frac{1}{a^2 + b^2}\left[e^{-a\alpha}(-a\cos b\alpha + b\sin b\alpha) + a\right]$$

$$= \frac{1}{a^2 + b^2}\left[\lim_{\alpha\to\infty}\left(\frac{-a\cos b\alpha + b\sin b\alpha}{e^{a\alpha}}\right) + a\right]$$

$$= \frac{1}{a^2 + b^2}[0 + a] = \frac{a}{a^2 + b^2}$$

ILLUSTRATION 10

Evaluate: $\sum\limits_{r=2}^{16}\int\limits_r^{r+1}\frac{dx}{(2r - x)(2r + 2 - x)}$

SOLUTION

$$\sum_{r=2}^{16}\int_r^{r+1}\frac{dx}{(2r - x)(2r + 2 - x)}$$

$$= \sum_{r=2}^{16}\int_r^{r+1}\frac{1}{2}\left(\frac{1}{x - 2r - 2} - \frac{1}{x - 2r}\right)dx$$

$$= \frac{1}{2}\sum_{r=2}^{16}\left[\ln(x - 2r - 2) - \ln(x - 2r)\Big|_r^{r+1}\right]$$

$$= \frac{1}{2}\sum_{r=2}^{16}\left(\ln\left(\frac{x - 2r - 2}{x - 2r}\right)\Bigg|_r^{r+1}\right)$$

$$= \frac{1}{2}\sum_{r=2}^{16}\left(\ln\left(\frac{r + 1 - 2r - 2}{r + 1 - 2r}\right) - \ln\left(\frac{r - 2r - 2}{r - 2r}\right)\right)$$

$$= \frac{1}{2}\sum_{r=2}^{16}\left(\ln\left(\frac{r+1}{r-1}\right) - \ln\left(\frac{r+2}{r}\right)\right)$$

$$= \frac{1}{2}\sum_{r=2}^{16}\ln\left(\frac{r+1}{r-1}\frac{r}{r+2}\right) = \frac{1}{2}\sum_{r=2}^{16}\ln\left(\frac{r}{r-1}\cdot\frac{r+1}{r+2}\right)$$

$$= \frac{1}{2}\left[\left(\sum_{r=2}^{16}\ln\frac{r}{r-1}\right) + \sum_{r=2}^{16}\ln\left(\frac{r+1}{r+2}\right)\right]$$

$$= \frac{1}{2}\left(\ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{15}{14}\cdot\frac{16}{15}\right) + \ln\left(\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdots\frac{17}{18}\right)\right)$$

$$= \frac{1}{2}\left[\ln\frac{16}{1} + \ln\frac{3}{18}\right] = \frac{1}{2}\ln\frac{8}{3} = \ln 2\sqrt{\frac{2}{3}}$$

ILLUSTRATION 11

If $I_1 = \int\limits_0^1(1 - x^{10})^{20}\,dx$ and $I_2 = \int\limits_0^1(1 - x^{10})^{19}\,dx$ then find $I_1 : I_2$

SOLUTION

$$I_1 = \int_0^1(1 - x^{10})^{20}\,.1\,dx;$$

Using integration by parts

$$I_1 = (1 - x^{10})^{20}x\Big|_{x=0}^{x=1} - \int_0^1 20(1 - x^{10})^{19}(0 - 10x^9)x\,dx$$

$$I_1 = 0 + 200\int_0^1 (1-x^{10})^{19}(1-(1-x^{10}))dx$$

$$I_1 = 0 + 200\int_0^1 (1-x^{10})^{19} - (1-x^{10})^{20}dx$$

$$I_1 = 200\, I_2 - 200\, I_1 \Rightarrow 201\, I_1 = 200\, I_2$$

Hence, $I_1 : I_2 = 200 : 201$

PROPERTIES OF DEFINITE INTEGRATION

Property P-I

Change of Variable: The definite integral $\int_a^b f(x)\ dx$ is a number, it does not depend on x.

In fact, we could use any letter in place of x without changing the value of the integral

$$\int_{x=a}^{x=b} f(x)\,dx = \int_{t=a}^{t=b} f(t)\,dt = \int_{p=a}^{p=b} f(p)\,dp$$

$$= \int_{\alpha=a}^{\alpha=b} f(\alpha)\,d\alpha = \int_{\lambda=a}^{\lambda=b} f(\lambda)d(\lambda)$$

Property P-II

Reversal of limits: $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Property P-III

Breakdown of integration limits:

If a function $f(x)$ changes its definition depending upon different interval of x or function is discontinuous at several points then we can break limits of integration and integrate the function in permissible interval.

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^d f(x)\,dx + \int_d^b f(x)\,dx$$

$$\int_a^b f(x)dx = \int_a^{c_1} f(x)dx + \int_{c_1}^{c_2} f(x)dx + \ldots + \int_{c_n}^{b} f(x)dx$$

ILLUSTRATION 12

Let $f(x) = \begin{cases} 2x+3 \ ; & x<3 \\ 3x^2+1 \ ; & x \geq 3 \end{cases}$, find $\int_2^5 f(x)\,dx$

SOLUTION

$$\int_2^5 f(x)\,dx = \int_2^3 2x+3\,dx + \int_3^5 3x^2+1\,dx$$

$$= x^2+3x\Big|_2^3 + \left(x^3+x\right)_3^5$$

$$= 9+9-(4+6)+(125+5-27-3)$$

$$= 8+100 = 108$$

ILLUSTRATION 13

Evaluate: $\int_0^3 [2x]\,dx$ where [·] is greatest integer function

SOLUTION

$$\int_0^3 [2x]\,dx = \int_0^{1/2} 0\,dx + \int_{1/2}^{1} 1\,dx + \int_1^{3/2} 2\,dx + \int_{3/2}^{2} 3\,dx + \int_2^{5/2} 4\,dx + \int_{5/2}^{3} 5\,dx$$

$$= 0 + x\Big|_{1/2}^{1} + 2x\Big|_1^{3/2} + 3x\Big|_{3/2}^{2} + 4x\Big|_2^{5/2} + 5x\Big|_{5/2}^{3}$$

$$= 1 - \frac{1}{2} + 2\left(\frac{3}{2}-1\right) + 3\left(2-\frac{3}{2}\right) + 4\left(\frac{5}{2}-2\right) + 5\left(3-\frac{5}{2}\right)$$

$$= \frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} = 7{\cdot}5$$

Working Rule

(i) If [.] is greatest integer function and {.} is fractional part function.
For $[f(x)]$ and $\{f(x)\}$, we should break limit of integration for set of values of x where $f(x)$ = Integer.

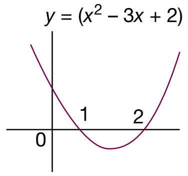
(ii) For sgn($f(x)$), we should break limit of integration for set of values of x where $f(x) = 0$.

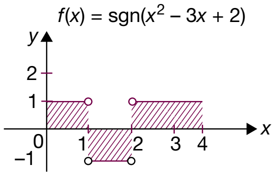
(iii) For $|f(x)|$, we should break limit of integration for set of values of x where $f(x) = 0$ or $f(x)$ changes its sign.

ILLUSTRATION 14

Evaluate $\int_0^4 \text{sgn}(x^2-3x+2)\,dx$; where sgn(x) is a signum function

SOLUTION





$$\int_0^4 \text{sgn}(x^2-3x+2)\,dx = \int_0^1 1\,dx + \int_1^2 -dx + \int_2^4 1\,dx$$

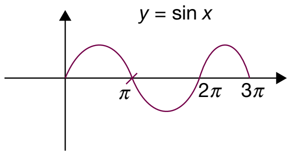
$$= x\Big|_0^1 - x\Big|_1^2 + x\Big|_2^4$$

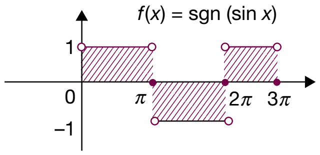
$$= 1 - 0 - (2 - 1) + (4 - 2) = 2$$

ILLUSTRATION 15

Evaluate: $\int_0^{3\pi} \operatorname{sgn}(\sin x)\, dx$ where $\operatorname{sgn}(x)$ is a signum function

SOLUTION





$$\int_0^{3\pi} \operatorname{sgn}(\sin x)\, dx = \int_0^{\pi} 1\, dx + \int_{\pi}^{2\pi} -1\, dx + \int_{2\pi}^{3\pi} 1\, dx = \pi - \pi + \pi = \pi$$

ILLUSTRATION 16

Evaluate the following integrals, where [.] is greatest integer function.

(i) $\int_0^9 \left[\sqrt{t}\right] dt$ **(ii)** $\int_0^2 \left[t^2\right] dt$

(iii) $\int_{-1}^3 \left[x + \frac{1}{2}\right] dx$ **(iv)** $\int_0^{2.5} \left[\frac{x}{4}\right] dx$

SOLUTION

(i) $I = \int_0^9 \left[\sqrt{t}\right] dt$

Given $0 < t < 9 \Rightarrow 0 < \sqrt{t} < 3$

Integral values of $\sqrt{t}$ are 0, 1, 2, 3

We get, $t = 0, 1, 4, 9$ accordingly

$$I = \int_0^1 0\, dt + \int_1^4 1\, dt + \int_4^9 2\, dt$$

$$= 0 + t\Big|_1^4 + 2t\Big|_4^9 = (4 - 1) + 2(3 - 4) = 13$$

(ii) $I = \int_0^2 \left[t^2\right] dt$

Given $0 < t < 2 \Rightarrow 0 < t^2 < 4$

Integral values of t^2 are 0, 1, 2, 3, 4

We get, $t = 0, 1, \sqrt{2}, \sqrt{3}, 2$ accordingly

$$I = \int_0^1 0\, dt + \int_1^{\sqrt{2}} 1\, dt + \int_{\sqrt{2}}^{\sqrt{3}} 2\, dt + \int_{\sqrt{3}}^{2} 3\, dt$$

$$I = 0 + \sqrt{2} - 1 + 2\left(\sqrt{3} - \sqrt{2}\right) + 3\left(2 - \sqrt{3}\right)$$

$$I = 5 - \sqrt{2} - \sqrt{3}$$

(iii) $\int_{-1}^3 \left[x + \frac{1}{2}\right] dx$

Given $-1 < x < 3$; $-0.5 < x + 0.5 < 3.5$

Integral values of $x + 0.5$ are 0, 1, 2, 3;

We get, $x = -0.5, 0.5, 1.5, 2.5$ accordingly.

$$I = \int_{-1}^3 \left[x + \frac{1}{2}\right] dx$$

$$= \int_{-1}^{-0.5} -1\, dx + \int_{-0.5}^{0.5} 0\, dx + \int_{0.5}^{1.5} 1\, dx + \int_{1.5}^{2.5} 2\, dx + \int_{2.5}^{3} 3\, dx$$

$$I = -x\Big|_{-1}^{-0.5} + 0 + x\Big|_{0.5}^{1.5} + 2x\Big|_{1.5}^{2.5} + 3x\Big|_{2.5}^{3}$$

$$I = -(0.5) + 1 + 2(1) + 3(0.5) = 4$$

(iv) $\int_0^{2.5} \left[\frac{x}{4}\right] dx$

Given $0 < x < 25$; $0 < \frac{x}{4} < \frac{25}{4}$

Integral value of $\frac{x}{4}$ are 0, 1, 2, 3, 4, 5, 6;

We get, $x = 0, 4, 8, 12, 16, 20, 24, 25$ accordingly.

$$I = \int_0^{2.5} \left[\frac{x}{4}\right] dx = \int_0^4 0\, dx + \int_4^8 1\, dx + \int_8^{12} 2\, dx + \int_{12}^{16} 3\, dx + \int_{16}^{20} 4\, dx + \int_{20}^{24} 5\, dx + \int_{24}^{25} 6\, dx$$

$$I = 0 + (8 - 4) + 2(4) + 3(4) + 4(4) + 5(4) + 6(1)$$

$$I = 4[1 + 2 + 3 + 4 + 6] + 6 = 66$$

ILLUSTRATION 17

Evaluate: $\int_1^3 \frac{dx}{x^2 + [x]^2 + 1 - 2x[x]}$, where [.] denotes the greatest integer function.

SOLUTION

$$I = \int_1^3 \frac{dx}{x^2 + [x]^2 + 1 - 2x[x]}$$

If $1 < x < 2 \Rightarrow [x] = 1$ and If $2 < x < 3 \Rightarrow [x] = 2$

$$I = \int_1^2 \frac{dx}{x^2 + 1 + 1 - 2x} + \int_2^3 \frac{dx}{x^2 + 4 + 1 - 4x}$$

$$I = \int_1^2 \frac{dx}{(x-1)^2+1} + \int_2^3 \frac{dx}{(x-2)^2+1}$$

$$I = \left[\tan^{-1}(x-1)\right]_1^2 + \left[\tan^{-1}(x-2)\right]_2^3$$

$$I = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

ILLUSTRATION 18

Evaluate $\int\limits_0^{2\pi} [2\sin x]\,dx$, where [.] represents the greatest integral function.

SOLUTION

Let $I = \int\limits_0^{2\pi} [2\sin x]\,dx$

For $0 \le x < \frac{\pi}{6} \Rightarrow 0 \le 2\sin x < 1 \Rightarrow [2\sin x] = 0$

For $x \ne \frac{\pi}{2}$ and $\frac{\pi}{6} < x < \frac{5\pi}{6}$

$\Rightarrow \quad 1 < 2\sin x < 2 \Rightarrow [2\sin x] = 1$

$\frac{5\pi}{6} < x < \pi \Rightarrow 0 < 2\sin x < 1 \Rightarrow [2\sin x] = 0$

$\pi < x < \frac{7\pi}{6} \Rightarrow -1 < 2\sin x < 0 \Rightarrow [2\sin x] = -1$

$\frac{7\pi}{6} < x < \frac{11\pi}{6} \Rightarrow -2 < 2\sin x < -1 \Rightarrow [2\sin x] = -2$

$\frac{11\pi}{6} < x < 2\pi \Rightarrow -1 < 2\sin x < 0 \Rightarrow [2\sin x] = -1$

$$\therefore \quad I = \int_{\pi/6}^{5\pi/6} 1\,dx + \int_{\pi}^{7\pi/6} -1\,dx + \int_{7\pi/6}^{11\pi/6} -2\,dx + \int_{11\pi/6}^{2\pi} -1\,dx$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{6}\right) + \left(-\frac{7\pi}{6} + \pi\right) + 2\left(-\frac{11\pi}{6} + \frac{7\pi}{6}\right) + \left(-2\pi + \frac{11\pi}{6}\right)$$

$$= -\pi$$

ILLUSTRATION 19

Evaluate the following integrals where |.| is modulus function.

(i) $\int\limits_{-2}^{3} |x|\,dx$ **(ii)** $\int\limits_{-3}^{6} |x-2|\,dx$

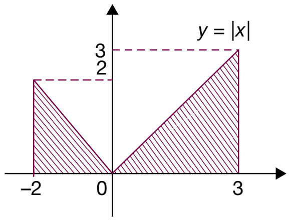
(iii) $\int\limits_0^{\pi/2} |\sin x - \cos x|\,dx$

SOLUTION

(i) $\int\limits_{-2}^{3} |x|\,dx = \int\limits_{-2}^{0} -x\,dx + \int\limits_0^3 x\,dx = -\left.\frac{x^2}{2}\right|_{-2}^{0} + \left.\frac{x^2}{2}\right|_0^3$

$$= \left(0 + \frac{4}{2}\right) + \left(\frac{1}{2}(9-0)\right) = \frac{13}{2}$$

Alternate (using area)



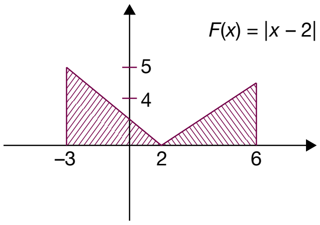
$$\int\limits_{-2}^{3} |x|\,dx = \text{shaded area} = \frac{1}{2}(2)2 + \frac{1}{2}(3)(3) = \frac{13}{2}$$

(ii) $\int\limits_{-3}^{6} |x-2|\,dx = \int\limits_{-3}^{2} -(x-2) + \int\limits_2^6 (x-2)\,dx$

$$= 2x - \left.\frac{x^2}{2}\right|_{-3}^{2} + \left.\frac{x^2}{2} - 2x\right|_2^6$$

$$= 4 - \frac{4}{2} - \left(-6 - \frac{9}{2}\right) + \frac{36}{2} - 12 - (2-4) = \frac{41}{2}$$

Alternate (Using Area)



$$\int\limits_{-3}^{6} |x-2|\,dx = \text{shaded area} = \frac{1}{2}(5)5 + \frac{1}{2}(4)4$$

$$= \frac{25}{2} + 8 = \frac{41}{2}$$

(iii) $\int\limits_0^{\pi/2} |\sin x - \cos x|\,dx$

$|\sin x - \cos x| = 0 \Rightarrow \sin x = \cos x;\ x = \frac{\pi}{4}$

$$I = \int\limits_0^{\pi/2} |\sin x - \cos x|\,dx$$

$$= \int\limits_0^{\pi/4} (\cos x - \sin x)\,dx + \int\limits_{\pi/4}^{\pi/2} \sin x - \cos x\,dx$$

$$I = \sin x + \cos x\Big|_0^{\pi/4} + (-\cos x - \sin x)_{\pi/4}^{\pi/2}$$

$$I = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (0+1) + \left(-0-1-\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)\right)$$

$$I = 2\left(\sqrt{2}-1\right)$$

ILLUSTRATION 20

Evaluate: $\displaystyle\int_{e^{-1}}^{e^2} \left|\frac{\log_e x}{x}\right| dx$

SOLUTION

$$\left|\frac{\log_e x}{x}\right| = 0, \text{ for } x = 1$$

$$I = \int_{e^{-1}}^{1} \left|\frac{\log_e x}{x}\right| dx + \int_{1}^{e^2} \left|\frac{\log_e x}{x}\right| dx$$

$$I = -\int_{e^{-1}}^{1} \frac{\log_e x}{x} dx + \int_{1}^{e^2} \frac{\log_e x}{x} dx$$

$$I = -\frac{1}{2}\left[(\log_e x)^2\right]_{e^{-1}}^{1} + \frac{1}{2}\left[(\log_e x)^2\right]_{1}^{e^2}$$

$$I = -\frac{1}{2}\{0-(-1)^2\} + \frac{1}{2}(2^2-0) = \frac{5}{2}$$

ILLUSTRATION 21

Evaluate: $\displaystyle\int_0^2 |(1-x)\ln x|\, dx$

SOLUTION

$$I = -\int_0^2 (1-x)\ln x \; dx = \int_0^2 (x \underset{\text{II}}{\underset{\downarrow}{-}} 1)\, \underset{\text{II}}{\underset{\downarrow}{\ln}}\, x \; dx$$

$$I = \left(\frac{x^2}{2}-x\right)\ln x\Big|_0^2 - \int_0^2 \left(\frac{x^2}{2}-x\right)\frac{dx}{x}$$

$$I = 0 - \lim_{x\to 0^+} \frac{x(1-2x)}{2}\ln x - \int_0^2 \frac{x}{2} - 1\, dx$$

$$I = \lim_{x\to 0^+} \frac{\ln x}{\frac{2}{(2x^2-x)}} + \int_0^2 1 - \frac{x}{2}\, dx$$

$$I = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{2(4x-1)}{(2x^2-x)^2}} + x - \frac{x^2}{4}\Bigg|_0^2$$

$$I = -\lim_{x\to 0^+} \frac{x(2x-1)^2}{2(4x-1)} + (2-1) = 1$$

ILLUSTRATION 22

Let $\displaystyle\int_0^{2\pi} \sqrt{1-2\sin 2x + 3\cos^2 x}\, dx = \sqrt{p} - \sqrt{q}$ where $p, q, \in N$, find the value of $(p+q)$.

SOLUTION

$$I = \int_0^{\pi/2} \sqrt{1-2\sin 2x + 3\cos^2 x}\, dx$$

$$= \int_0^{\pi/2} \sqrt{\sin^2 x - 4\sin\cos x + 4\cos^2 x}\, dx$$

$$I = \int_0^{\pi/2} |\sin x - 2\cos x| dx$$

when $|\sin x - 2\cos x| = 0$

we get $\sin x = 2\cos x$ or $\tan x = 2 \Rightarrow x = \tan^{-1}(2)$

$$x = \sin^{-1}\left(\frac{2}{\sqrt{5}}\right) \Rightarrow x = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

$$I = \int_0^{\tan^{-1}2} (2\cos x - \sin x)\, dx + \int_{\tan^{-1}2}^{\pi/2} \sin x - 2\cos x\, dx$$

$$I = (2\sin x + \cos x)\Big|_0^{\tan^{-1}2} + (-\cos x - 2\sin x)_{\tan^{-1}2}^{\pi/2}$$

$$I = 2\left(\frac{2}{\sqrt{5}}\right) + \frac{1}{\sqrt{5}} - (0+1) + (0-2) - \left(-\frac{1}{\sqrt{5}} - 2.\frac{2}{\sqrt{5}}\right)$$

$$I = \frac{4}{\sqrt{5}} + \frac{1}{\sqrt{5}} - 3 + \frac{5}{\sqrt{5}} = \frac{10}{\sqrt{5}} - 3 = \sqrt{20} - \sqrt{9}$$

$$I = \sqrt{p} - \sqrt{q} \qquad \therefore\ p + q = 29$$

ILLUSTRATION 23

Evaluate:

$$\int_0^{\pi} \sqrt{(\cos x + \cos 2x + \cos 3x)^2 + (\sin x + \sin 2x + \sin 3x)^2}\ dx$$

SOLUTION

$$(\cos x + \cos 2x + \cos 3x)^2 + (\sin x + \sin 2x + \sin 3x)^2$$

$$= \cos^2 x + \cos^2 2x + \cos^2 3x + \sin^2 x + \sin^2 2x + \sin^2 3x + 2(\cos x\cos 2x + \cos 2x\cos 3x + \cos 3x\cos x) + 2(\sin x\sin 2x + \sin 2x\sin 3x + \sin 3x\sin x)$$

$$= 3 + 2((\cos x\cos 2x + \sin x\sin 2x) + (\cos 3x\cos 2x + \sin 3x\sin 2x) + (\cos x\cos 3x + \sin x\sin 3x))$$

$$= 3 + 2(\cos x + \cos x + \cos 2x)$$

$$= 3 + 4\cos x + 2\cos 2x$$

$$I = \int_2^{\pi} \sqrt{(\cos x + \cos 2x + \cos 3x)^2 + (\sin x + \sin 2x + \sin 3x)^2}\, dx$$

$$I = \int_2^{\pi} \sqrt{3 + 4\cos x + 2\cos 2x}\ .\ dx$$

$$= \int_0^{\pi} \sqrt{4\left(\cos^2 x + \cos x + \frac{1}{4}\right)}\ .\ dx$$

$$I = \int_0^{\pi} 2\left|\cos x + \frac{1}{2}\right|\ .\ dx$$

$$= 2\int_0^{2\pi/3} \left(\cos x + \frac{1}{2}\right) dx - 2\int_{2\pi/3}^{\pi} \left(\cos x + \frac{1}{2}\right) dx$$

$$I = 2(\sin x)_0^{2\pi/3} + (x)_0^{2\pi/3} - 2(\sin x)_{2\pi/3}^{\pi} - (x)_{2\pi/3}^{\pi}$$

$$I = 2\left[\frac{\sqrt{3}}{2} - 0\right] + \frac{2\pi}{3} - 0 - 2\left(0 - \frac{\sqrt{3}}{2}\right) - \left(\pi - \frac{2\pi}{3}\right)$$

$$I = \sqrt{3} + \frac{2\pi}{3} + \sqrt{3} - \frac{\pi}{3} = 2\sqrt{3} + \frac{\pi}{3}$$

ILLUSTRATION 24

Evaluate: $\int_3^5 \left(\sqrt{x + 2\sqrt{2x-4}} + \sqrt{x - 2\sqrt{2x-4}}\right) dx$

SOLUTION

$$I = \frac{1}{\sqrt{2}} \int_3^5 \left(\sqrt{2x - 4 + 4 + 4\sqrt{2x-4}} + \sqrt{2x - 4 + 4 - 4\sqrt{2x-4}}\right) dx$$

$$= \frac{1}{\sqrt{2}} \int_3^5 \left|\left(\sqrt{2x-4} + 2\right)\right| + \left|\left(\sqrt{2x-4} - 2\right)\right| dx$$

$$= \frac{1}{\sqrt{2}} \left(\sqrt{2}\int_3^5 \sqrt{x-2}\,dx + 2(5-3) + \int_3^4 2 - \sqrt{2x-4}\,dx + \int_4^5 \sqrt{2x-4} - 2\, dx\right)$$

$$\frac{1}{\sqrt{2}}\left(\sqrt{2}\left.\frac{(x-2)^{3/2}}{\frac{3}{2}}\right|_3^5 + 4 - \sqrt{2}\left.\frac{(x-2)^{3/2}}{\frac{3}{2}}\right|_3^4 + \sqrt{2}\left.\frac{(x-2)^{3/2}}{\frac{3}{2}}\right|_4^5\right.$$

$$= \frac{1}{\sqrt{2}}\left[\frac{2\sqrt{2}}{3}\left(3\sqrt{3} - 1\right) + 4 - \frac{2\sqrt{2}}{3}\left[2\sqrt{2} - 1\right] + \frac{2\sqrt{2}}{3}\left[3\sqrt{3} - 2\sqrt{2}\right]\right)$$

$$= 2\sqrt{3} - \frac{2}{3} + 2\sqrt{2} - \frac{4\sqrt{2}}{3} + \frac{2}{3} + \frac{2}{3}\left(3\sqrt{3} - 2\sqrt{2}\right)$$

$$= 4\sqrt{3} - \frac{2\sqrt{2}}{3}$$

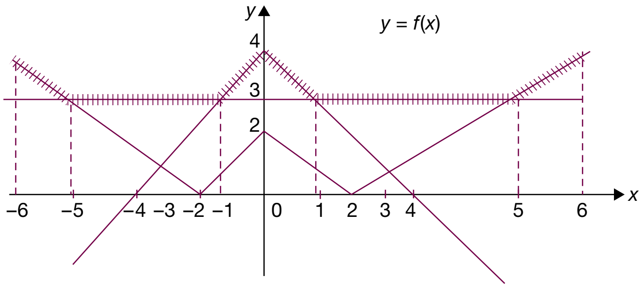
ILLUSTRATION 25

Evaluate: $\int_{-6}^{6} \max(|2 - |x||, 4 - |x|, 3)\,dx$

SOLUTION

We have drawn diagram of

$f(x)$ max $(|2 - |x||, 4 - |x|, 3)$ for $x \in (-6, 6)$



$$\int_{-6}^{6} \max(|2 - |x||, 4 - |x|, 3)\,dx = 2\int_0^6 f(x)\,dx$$

= 2(Area of trapezium from $x = 0$ to $x = 1$ + Area of rectangle from $x = 1$ to $x = 5$ + Area of trapezium from $x = 5$ to $x = 6$)

$$= 2\left[\frac{1}{2}(4+3)\times 1 + 4\times 3 + \frac{1}{2}(1)(3+4)\right]$$

$$= 2\left[\frac{7}{2} + 12 + \frac{7}{2}\right] = 2(19) = 38$$

ILLUSTRATION 26

Evaluate $\int_{-3/2}^{2} f(x)\,dx$, where $f(x)$ is given by

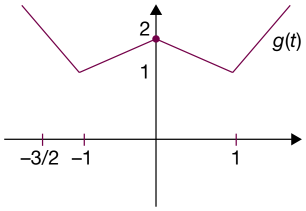
$f(x) = \max.(|t-1| - |t| + |t+1|); -\frac{3}{2} \le t \le x.$

SOLUTION

$g(t) = |t-1| - |t| + |t+1|$

$$g(t)=\begin{cases}(t-1)-t+(t+1) & \text{for } t\ge 1\\ (1-t)-t+(t+1) & \text{for } 0<t<1\\ (1-t)+t+(t+1) & \text{for } -1\le t\le 0\\ (1-t)+t-(t+1) & \text{for } t<-1\end{cases}$$

$$g(t)=\begin{cases}t & \text{for } t\ge 1\\ 2-t & \text{for } 0<t<1\\ 2+t & \text{for } -1\le t\le 0\\ -t & \text{for } t<-1\end{cases}$$



$$f(x)=\max(|t-1|-|t|+|t+1|);-\frac{3}{2}\le t\le x$$

$$f(x)=\begin{cases}g\left(-\frac{3}{2}\right) & \text{for } -\frac{3}{2}\le x\le -\frac{1}{2}\\ x+2 & \text{for } -\frac{1}{2}<x<0\\ g(0) & \text{for } 0\le x\le 2\end{cases}$$

We have $g(0)=2,\ g\left(-\frac{3}{2}\right)=\frac{3}{2}$

$$\int_{-3/2}^{2} f(x)\,dx=\int_{-3/2}^{-1/2}\frac{3}{2}dx+\int_{-1/2}^{0}x+2dx+\int_{0}^{2}2dx$$

$$=\frac{3}{2}+\left(\frac{x^2}{2}+2x\right)_{-1/2}^{0}+2x\,\Big|_0^2$$

$$=\frac{3}{2}+0-\left(\frac{1}{8}-1\right)+4-0=\frac{51}{8}$$

ILLUSTRATION 27

Let $f(x)=\begin{cases}1-x & ; & 0\le x\le 1\\ 0 & ; & 1<x\le 2\\ (2-x)^2 & ; & 2<x\le 3\end{cases}$. Define $F(x)=\int_0^x f(t)\,dt$

SOLUTION

$$F(x)=\int_0^x f(t)\,dt$$

$$F(x)=\begin{cases}\int_0^x(1-t)dt & ; & 0\le x\le 1\\ \int_0^1(1-t)dt+\int_1^x 0dt & ; & 1<x\le 2\\ \int_0^1(1-t)dt+\int_1^2 0dt+\int_2^x(2-t)^2dt & ; & 2<x\le 3\end{cases}$$

$$F(x)=\begin{cases}x-\frac{x^2}{2} & ; & 0\le x\le 1\\ \frac{1}{2} & ; & 1<x\le 2\\ \frac{(x-2)^3}{3}+\frac{1}{2} & ; & 2<x\le 3\end{cases}$$

ILLUSTRATION 28

Prove that $\int_0^x[x]dx=[x]x-\frac{[x]([x]+1)}{2}$ where [ ] denotes greatest integer functions.

SOLUTION

Let $x=[x]+\{x\}=I+f$; where $0<f<I$

$$\int_0^{I+f}[x]dx$$

$$=0+\int_1^2 1dx+\int_2^3 2dx+\int_3^4 3dx+\ldots+\int_{I-1}^{I}(I-1)dx+\int_I^{I+f}Idx$$

$$=1(2-1)+2(3-2)+3(4-3)+\ldots+$$
$$(I-1)\,(I-(I-1))+I(I+f-I)$$

$$=1+2+3+\ldots+(I-1)+If$$

$$=\frac{(I-1)(I-1+1)}{2}+If=\frac{I(I-1)}{2}+If$$

$$=\frac{[x]([x]-1)}{2}+[x]\,(x-[x])=[x]x-\frac{[x]([x]+1)}{2}$$

Hence proved.

ILLUSTRATION 29

Find number of $x\in(5,10)$ satisfying equation $\int_0^x[t]dt=\int_0^{[x]}t\,dt$ where [.] is greatest integer function.

SOLUTION

As we know $x=[x]+\{x\}$

$$\int_0^x[t]dt=\int_0^{[x]}[t]\,dt+\int_{[x]}^{x}[t]\,dt$$

$$= \int_0^1 0 \cdot dt + \int_1^2 1\ dt + \int_2^3 2\ dt + \ldots +$$

$$\int_{[x]-1}^{[x]} ([x]-1)\,dt + [x]\int_{[x]}^{x} dt$$

$$= (1 + 2 + \ldots + ([x] - 1)) + [x]\,(x - [x])$$

$$= \frac{[x]([x]-1)}{2} + [x](x-[x]) = \frac{[x]}{2}(2x-[x]-1)$$

Given equation $\int\limits_0^x [t]dt = \int\limits_0^{[x]} t\,dt$

$$\Rightarrow \quad \frac{[x]}{2}(2x-[x]-1) = \frac{[x]^2}{2} \Rightarrow [x]\left(x-[x]-\frac{1}{2}\right) = 0$$

As $5 < x < 10 \Rightarrow [x] \neq 0$

Now $x = [x] + \frac{1}{2} = k + \frac{1}{2}$ for $5 \leq k \leq 9$

The number of solutions of the given equation are 5.

Test Your Concepts-I

(Solutions on page H.268)

1. Evaluate $\int\limits_0^{\pi/4} \sin x d(x-[x])$ is equal to (where [.] denotes the greatest integer function).

2. If $F(x) = \begin{cases} e^{\cos x}\sin x & ; \ |x| \leq 2 \\ 2 & ; \ \text{otherwise} \end{cases}$, then $\int\limits_{-2}^{3} F(x)\,dx =$

3. If $k = \int\limits_0^1 \frac{e^t}{1+t}dt$, then $\int\limits_0^1 e^t \ln(1+t)dt$ is equal to ____.

4. Evaluate: $\int_0^{\pi} \cos^{-1}(\sin x)\,dx =$

5. Evaluate: $\int\limits_1^3 \left[\tan^{-1}\left(\frac{x}{x^2+1}\right) + \tan^{-1}\left(\frac{x^2+1}{x}\right)\right]dx$

6. Evaluate $\int\limits_0^{1.5} [x^2]\ dx$, where [.] denotes the greatest function

7. Evaluate: $\int\limits_{-4}^{3} |x^2-4|\,dx$

8. The value of $\int\limits_{-2}^{2} |1-x^2|\,dx$ is ______

9. Evaluate: $\int\limits_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}\,dx$

10. Evaluate: $\int\limits_{0.3}^{4.5} [x]\,dx$

11. Evaluate: $\int\limits_0^{n^2} \left[\sqrt{x}\right]dx$, where $n \in$ N, [.] is the greatest integer function

12. Evaluate $\int\limits_{\pi/4}^{\pi/2} e^x[\cot x + \log(\sin x)]dx$,

13. Evaluate $\int\limits_0^{\infty}\left[\frac{2}{e^x}\right]dx$ is, [where [ ] represents the greatest integer function]

14. Evaluate $\int_1^5 \left(\sqrt{x+2\sqrt{x-1}} + \sqrt{x-2\sqrt{(x-1)}}\right)dx$

15. Given $f(x) = \begin{cases} x+1 & ; \ \text{for } x < 0 \\ \cos \pi x & ; \ \text{for } x \geq 0 \end{cases}$ then Evaluate $\int\limits_{-1}^{1} f(x)\,dx$.

16. Let $f: R \to R$ be defined as $f(x) = \begin{cases} |x-[x]| & ; \ [x] \text{ is odd} \\ |x-[x+1]| & ; \ [x] \text{ is even} \end{cases}$ where [x] denotes the greatest integer less than or equal to x, then find $\int\limits_{-2}^{4} f(x)\,dx$.

17. Let $I_n = \int\limits_0^1 (x\ln x)^n\,dx$, if $I_4 = k\int\limits_0^1 x^4(\ln x)^3\,dx$, then find k.

18. Let $f(x) = \frac{1}{x+1}$ and $f(g(x)) = \frac{x-1}{x}$, then find $\int\limits_2^1 g(f(x))\,dx$.

19. Evaluate: $\int\limits_{3\pi/4}^{4\pi/3} \frac{\cos x \cdot \sin x}{|\cos x|}\,dx$

Answer key

1. $1-\frac{1}{\sqrt{2}}$ **2.** 2 **3.** $e\ln 2-k$ **4.** $\frac{\pi^2}{4}$ **5.** π **6.** $2-\sqrt{2}$ **7.** $\frac{71}{3}$ **8.** 4 **9.** 2

10. 8 **11.** $\frac{n(n-1)(4n+1)}{6}$ **12.** $e^{\pi/4}\log\sqrt{2}$ **13.** $\log_e 2$ **14.** $\frac{34}{3}$ **15.** $\frac{1}{2}$ **16.** 3

17. $-\frac{4}{5}$ **18.** $1+\ln 2$ **19.** $\frac{\sqrt{2}-1}{2}$

DEFINITE INTEGRATION USING SUBSTITUTION

$$I=\int_a^b f(g(x)).g'(x)\,dx \text{ where } x\in(a,b)$$

Put $g(x)=t \Rightarrow g'(x)dx=dt$

Lower limit: when $x=a$, $t=g(a)$

Upper limit: when $x=b$, $t=g(b)$

$$I=\int_a^b f(g(x)).g'(x)\,dx=\int_{g(a)}^{g(b)} f(t)\,dt$$

Note

(i) Always change limit of integration according to new parameter.

(ii) $g(x)$ must be continuous in $x\in(a,b)$

ILLUSTRATION 30

Evaluate $\int_0^{17}\left[\frac{x}{4}\right]dx$ where $[\cdot]$ is greatest integer function.

SOLUTION

$$I=\int_0^{17}\left[\frac{x}{4}\right]dx$$

We can also take help of substitution in case of GIF function

Put $\frac{x}{4}=t$; $x=4t$; $dx=4dt$

When $x=0$; $t=0$ (lower limit)

When $x=17$; $t=\frac{17}{4}$ (upper limit)

$$I=4\int_0^{17/4}[t]\,dt$$

$$=4\left[\int_0^1 0\,dt+\int_1^2 1\,dt+\int_2^3 2\,dt+\int_3^4 3\,dt+\int_4^{17/4} 4\,dt\right]$$

$$=4\left[0+1+2(1)+3(1)+4\left(\frac{1}{4}\right)\right]=4(7)=28$$

ILLUSTRATION 31

Evaluate $\int_0^2[2x+0.5]\,dx$ where $[.]$ is greatest integer function.

SOLUTION

$$I=\int_0^2[2x+0.5]\,dx$$

Put $2x+0.5=t \Rightarrow 2dx=dt \Rightarrow dx=\frac{dt}{2}$

When $x=0$, $t=2x+0.5=0.5$ (lower limit)

When $x=2$, $t=2x+0.5=4.5$ (upper limit)

$$I=\int_{0.5}^{4.5}[t]\frac{dt}{2}$$

$$I=\frac{1}{2}\left(\int_{0.5}^1 0dt+\int_1^2 1dt+\int_2^3 2dt+\int_3^4 3dt+\int_4^{4.5} 4dt\right)$$

$$I=\frac{1}{2}\left[0+1+2+3+4\left(\frac{1}{2}\right)\right]=4$$

ILLUSTRATION 32

Evaluate: $\int_0^{\pi/2}\sqrt{\sin\phi}\,\cos^5\phi\,d\phi$

SOLUTION

$$I=\int_0^{\pi/2}\sqrt{\sin\phi}\,\cos^5\phi\,d\phi$$

$$I = \int_0^{\pi/2} \sqrt{\sin\phi}\,(1-\sin^2\phi)^2 \cos\phi\, d\phi$$

Put $\sin\phi = t \Rightarrow \cos\phi\, d\phi = dt$

When $\phi = 0$; $t = \sin 0 = 0$ (lower limit.)

When $\phi = \frac{\pi}{2}$; $t = \sin\frac{\pi}{2} = 1$ (upper limit)

$$I = \int_0^1 \sqrt{t}(1-t^2)^2\,dt = \int_0^1 \sqrt{t}(t^4-2t^2+1)$$

$$I = \int_0^1 t^{4.5} - 2t^{2.5} + t^{0.5}\,dt = \left.\frac{t^{5.5}}{5.5} - \frac{2t^{3.5}}{3.5} + \frac{t^{1.5}}{1.5}\right|_0^1$$

$$I = \frac{1}{5.5} - \frac{2}{3.5} + \frac{1}{1.5} - 0 = \frac{2}{11} - \frac{4}{7} + \frac{2}{3} = \frac{64}{231}$$

ILLUSTRATION 33

If $\int_0^{10} f(x)dx = 5$, then compute $\sum_{K=1}^{10}\int_0^1 f(k-1+x)dx$

SOLUTION

$$\sum_{k=1}^{10}\int_0^1 f(k-1+x)dx = \int_0^1 f(x)dx + \underset{\text{Put } x+1=t}{\int_0^1 f(1+x)dx} +$$

$$\underset{\text{Put } x+2=t}{\int_0^1 f(2+x)dx} + \ldots + \underset{\text{Put } x+9=t}{\int_0^1 f(9+x)dx}$$

$$= \int_0^1 f(x)dx + \int_1^2 f(t)dt + \int_2^3 f(t)dt + \ldots + \int_9^{10} f(t)dt$$

$$= \int_0^{10} f(x)\,dx = 5$$

ILLUSTRATION 34

Let $I_1 = \int_0^1 \frac{e^x dx}{x+1}$ and $I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)}$, then compute $\frac{I_1}{I_2}$

SOLUTION

$$I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)}$$

Let $1 - x^3 = t \Rightarrow x^3 = 1 - t \Rightarrow x^2\,dx = -\frac{dt}{3}$

$$I_2 = \int_1^0 \frac{-dt}{3e^{1-t}(1+t)} = \frac{1}{3e}\int_0^1 \frac{e^t}{1+t}dt = \frac{I_1}{3e} \Rightarrow \frac{I_1}{I_2} = 3e$$

ILLUSTRATION 35

If $f(x) = \int_0^x \frac{e^t}{t}dt$; $x > 0$ and $\int_1^x \frac{e^t}{t+a}dt = e^{-a}[f(x+a) - f(k)]$, then compute k.

SOLUTION

$$f(x) = \int_0^x \frac{e^t}{t}dt \Rightarrow f'(x) = \frac{e^x}{x}$$

$$\int_1^x \frac{e^t}{t+a}dt = \int_{1+a}^{a+x} \frac{e^{z-a}}{z}dz$$

Put $t + a = z \Rightarrow dt = dz$

$$= e^{-a}\left(\int_{1+a}^{a+x} \frac{e^z}{z}dz\right) = e^{-a}\left(\int_{1+a}^{0} \frac{e^z}{z}dz + \int_0^{a+x} \frac{e^z}{z}dz\right)$$

$$= e^{-a}\left(\int_0^{a+x} \frac{e^z}{z}dz - \int_0^{1+a} \frac{e^z}{z}dz\right) \text{ using } f(x) = \int_0^x \frac{e^t}{t}dt$$

$$= e^{-a}(f(x+a) - f(1+a))$$

We get $k = 1 + a$

ILLUSTRATION 36

Let $\frac{d}{dx}F(x) = \frac{e^{\sin x}}{x}$, $x > 0$. If $\int_1^4 \frac{2e^{\sin x^2}}{x}dx = F(k) - F(1)$, then find k.

SOLUTION

$$I = \int_1^4 \frac{2e^{\sin x^2}}{x}dx = \int_1^4 \frac{2xe^{\sin x^2}}{x^2}dx$$

Put $x^2 = t$, $2x\,dx = dt$

$$I = \int_1^{16} \frac{e^{\sin t}}{t}dt = \int_1^{16} F'(t)dt = F(16) - F(1)$$

We get $k = 16$

ILLUSTRATION 37

For $x > 0$, let $f(x) = \int_1^x \frac{\ln t}{1+t}dt$. Find the function $f(x) + f\left(\frac{1}{x}\right)$ and show that $f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}$.

SOLUTION

$$f(x) = \int_1^x \frac{\ln t}{1+t}dt \qquad \ldots(1)$$

Replace x by $\frac{1}{x}$

$$f\left(\frac{1}{x}\right)=\int_1^{1/x}\frac{\ln t}{1+t}$$

Let $t=\frac{1}{p}\Rightarrow dt=-\frac{1}{p^2}dp$

$$\Rightarrow\quad f\left(\frac{1}{x}\right)=\int_1^x\frac{\log\left(\frac{1}{p}\right)}{\left(1+\frac{1}{p}\right)}\left(-\frac{1}{p^2}\right)dp$$

$$\Rightarrow\quad f\left(\frac{1}{x}\right)=\int_1^x\frac{\ln p}{(1+p)p}dp=\int_1^x\frac{\ln t}{(1+t)t}dt \qquad \ldots(2)$$

Now adding (1) and (2), we get

$$f(x)+f\left(\frac{1}{x}\right)=\int_1^x\frac{\ln t}{1+t}\left(\frac{1+t}{t}\right)dt=\int_1^x\frac{\ln t}{t}dt$$

$$\Rightarrow\quad f(x)+f\left(\frac{1}{x}\right)=\left(\frac{(\ln t)^2}{2}\right)_1^x=\frac{(\ln x)^2}{2}$$

$$\therefore\quad f(e)+f\left(\frac{1}{e}\right)=\frac{(\ln e)^2}{2}=\frac{1}{2}$$

ILLUSTRATION 38

Evaluate: $\displaystyle\int_{1/2}^{2}\frac{1}{x}\sin\left(x-\frac{1}{x}\right)dx$

SOLUTION

Put $x=\frac{1}{\theta}\Rightarrow dx=-\frac{1}{\theta^2}d\theta.$

$\therefore$ After substitution,

$$I=\int_2^{1/2}\theta\sin\left(\frac{1}{\theta}-\theta\right)\left(-\frac{d\theta}{\theta^2}\right)$$

$$I=-\int_{1/2}^{2}\frac{1}{\theta}\sin\left(\theta-\frac{1}{\theta}\right)d\theta=-I$$

$\Rightarrow\quad 2I=0$ or $I=0$

ILLUSTRATION 39

Evaluate: $\displaystyle\int_a^\infty\frac{dx}{x^4(a^2+x^2)^{1/2}}$

SOLUTION

Let $x=a\tan\theta$, so that $dx=a\sec^2\theta\,d\theta$

$$\therefore\quad I=\int_{\pi/4}^{\pi/2}\frac{a\sec^2\theta d\theta}{a^4\tan^4\theta(a^2+a^2\tan^2\theta)^{1/2}}=\int_{\pi/4}^{\pi/2}\frac{1}{a^4}\frac{\sec\theta}{\tan^4\theta}d\theta$$

$$I=\frac{1}{a^4}\int_{\pi/4}^{\pi/2}\cot^3\theta\,\text{cosec}\,\theta\,d\theta$$

$$I=\frac{1}{a^4}\int_{\pi/4}^{\pi/2}(\text{cos}\,ec^2\theta-1)\cot\theta\,\text{cosec}\,\theta\,d\theta$$

Put cosec $\theta=y\Rightarrow-\text{cosec}\,\theta\cot\theta\,d\theta=dy,$

$$\therefore\quad I=-\frac{1}{a^4}\int_{\sqrt{2}}^{1}(y^2-1)dy$$

$$I=-\frac{1}{a^4}\left[\frac{y^3}{3}-y\right]_{\sqrt{2}}^{1}=\frac{2-\sqrt{2}}{3a^4}$$

ILLUSTRATION 40

Evaluate: $\displaystyle\int_0^{\pi/2}\frac{dx}{1+\cos\theta.\cos x}$; $\theta\in(0,\pi)$

SOLUTION

$$I=\int_0^{\pi/2}\frac{dx}{1+\cos\theta\cos x}\ \text{converting into tan}\ \frac{x}{2}$$

$$I=\int_0^{\pi/2}\frac{dx}{1+\cos\theta\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$I=\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}dx}{(1+\cos\theta)+(1-\cos\theta)\tan^2\frac{x}{2}}$$

$$I=\frac{1}{(1-\cos\theta)}\int_0^{\pi/2}\frac{\sec^2\frac{x}{2}dx}{\left(\cot\frac{\theta}{2}\right)^2+\tan^2\frac{x}{2}}$$

Let $\tan\frac{x}{2}=t$ then $\sec^2\frac{x}{2}dx=2\,dt$

$$I=\frac{1}{(1-\cos\theta)}\int_0^1\frac{2dt}{t^2+\left(\cot\frac{\theta}{2}\right)^2}$$

$$I=\frac{2}{1-\cos\theta}\cdot\frac{1}{\cot\frac{\theta}{2}}\tan^{-1}\frac{t}{\cot\frac{\theta}{2}}\Bigg|_0^1$$

$$I=\frac{2}{2\sin^2\frac{\theta}{2}\cot\frac{\theta}{2}}\left[\tan^{-1}\tan\frac{\theta}{2}-0\right]=\frac{\theta}{\sin\theta}$$

ILLUSTRATION 41

If $m = \displaystyle\int_{-2}^{0} \frac{|\sin x|}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}\,dx$ and $n = \displaystyle\int_{0}^{2} \frac{|\sin x|}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}\,dx$, where [·] represent greatest integer function then find $m + n$.

SOLUTION

As $-2 < x < 0 \Rightarrow -\frac{2}{\pi} < \frac{x}{\pi} < 0 \Rightarrow \left[\frac{x}{\pi}\right] = -1$

$$m = \int_{-2}^{0} \frac{|\sin x|}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}\,dx = \int_{-2}^{0} \frac{|\sin x|}{-1+\frac{1}{2}}\,dx = -2\int_{-2}^{0} |\sin x|\,dx$$

Let $x = -t \Rightarrow dx = -dt$

$$= -2\int_{2}^{0} |\sin(-t)|\,(-dt) = -2\int_{0}^{2} |\sin t|\,dt$$

As $0 < x < 2 \Rightarrow 0 < \frac{x}{\pi} < \frac{2}{\pi} < 1 \Rightarrow \left[\frac{x}{\pi}\right] = 0$

$$n = \int_{0}^{2} \frac{|\sin t|}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}\,dx = 2\int_{0}^{2} |\sin x|\,dx$$

We get $m + n = 0$

ILLUSTRATION 42

If $\displaystyle\int_{0}^{\infty}\left(\frac{\sin x}{x}\right)^3 dx = A$ and $\displaystyle\int_{0}^{\infty}\left(\frac{x-\sin x}{x^3}\right)dx = \frac{aA}{b}$, where a and b are relative prime, then find the value of $(a + b)$.

SOLUTION

$$\int_{0}^{\infty} \frac{\sin^3 x}{x^3}\,dx = \int_{0}^{\infty} \frac{3\sin x - \sin 3x}{4x^3}\,dx$$

$\because$ $\sin 3x = 3\sin x - 4\sin^3 x \Rightarrow \sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$

$$= \int_{0}^{\infty} \frac{-(3x - 3\sin x) + 3x - \sin 3x}{4x^3}\,dx$$

$$= -\frac{3}{4}\left(\int_{0}^{\infty} \frac{x-\sin x}{x^3}\,dx\right) + \int_{0}^{\infty} \frac{3x-\sin 3x}{4x^3}\,dx$$

Put $3x = t$ in second integral

$$A = -\frac{3}{4}\left(\frac{aA}{b}\right) + \int_{0}^{\infty} \frac{t-\sin t}{4\left(\frac{t}{3}\right)^3}\left(\frac{dt}{3}\right);$$

$$A = -\frac{3a}{4b}A + \frac{9}{4}\int_{0}^{\infty} \frac{t-\sin t}{t^3}\,dt$$

$$A = -\frac{3aA}{4b} + \frac{9}{4}\cdot\frac{aA}{b} = \frac{3a}{2b}A$$

$3a = 2b \Rightarrow \frac{a}{b} = \frac{2}{3}$, we get $a + b = 5$

ILLUSTRATION 43

Let $f(x)$ be a continuous function on R.

If $\displaystyle\int_{0}^{1}(f(x) - f(2x))\,dx = 5$ and $\displaystyle\int_{0}^{2}(f(x) - f(4x))\,dx = 10$, then find the value of $\displaystyle\int_{0}^{1}(f(x) - f(8x))\,dx$.

SOLUTION

$$\int_{0}^{2}(f(x) - f(4x))\,dx = 10 \qquad \text{Put } x = 2t,\ dx = 2dt$$

$$\int_{0}^{1}(f(2t) - f(8t))\,dt = 5 \qquad \ldots(1)$$

Given $\displaystyle\int_{0}^{1} f(x) - f(2x)\,dx = 5$ ...(2)

Adding (1) + (2), we get

$$\int_{0}^{1}(f(x) - f(8x))\,dx = 10$$

ILLUSTRATION 44

Evaluate: $\displaystyle\lim_{n\to\infty}\int_{0}^{(n!+1)\pi/2}\left(\frac{\cos x + x\sin x}{x^2 + \cos^2 x}\right)dx$ where $n > 1$

SOLUTION

$$I = \lim_{n\to\infty}\int_{0}^{(n!+1)\frac{\pi}{2}}\left(\frac{\cos x + x\sin x}{x^2 + \cos^2 x}\right)dx$$

$$I = \lim_{n\to\infty}\int_{0}^{(n!+1)\frac{\pi}{2}}\frac{\cos x + x\sin x}{x^2\left(1+\left(\frac{\cos x}{x}\right)^2\right)}\,dx$$

Let $\dfrac{\cos x}{x} = t \Rightarrow \dfrac{x(-\sin x) - \cos x}{x^2}\,dx = dt$

$$I = -\int_{\infty}^{0}\frac{dt}{1+t^2} = \tan^{-1} t\Big|_{0}^{\infty} = \frac{\pi}{2}$$

ILLUSTRATION 45

If $f(x)$ is polynomial function such that $f(x) + f'(x) + f''(x) + f'''(x) = x^3$ and $g(x) = \int \frac{f(x)}{x^3} dx$ and $g(1) = 1$, then find $g(e)$.

SOLUTION

Let $f(x) = x^3 + ax^2 + bx + c$

$$f(x) + f'(x) + f''(x) + f'''(x) = x^3$$

$$\Rightarrow \quad (x^3 + ax^2 + bx + c) + (3x^2 + 2ax + b) + (6x + 2a) + 6 = x^3$$

$$\Rightarrow \quad x^3 + x^2(a+3) + (b + 2a + 6)\, x + c + b + 2a + 6 = x^3$$

We get $a = -3$, $b = c = 0$

$$f(x) = x^3 - 3x^2$$

$$g(x) = \int \frac{f(x)}{x^3}\, dx = \int \frac{x^3 - 3x^2}{x^3}\, dx$$

$$g(x) = \int 1 - \frac{3}{x}\, dx = x - 3\ln x + C$$

$$g(1) = 1 - 3\ln 1 + C = 1 \Rightarrow C = 0$$

$$g(e) = e - 3\ln e = e - 3$$

ILLUSTRATION 46

Evaluate: $\lim\limits_{n\to\infty} \frac{1}{n^2} \sum\limits_{k=0}^{n-1} \left[k \int\limits_{k}^{k+1} \sqrt{(x-k)(k+1-x)}dx \right]$

SOLUTION

$$f(x) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \left[k \int_{k}^{k+1} \sqrt{(x-k)(1-(x-k))} \cdot dx \right]$$

$$P = \int_{k}^{k+1} \sqrt{(x-k)(1-(x-k))}\; dx$$

Put $x - k = t$, $dx = dt$

$$P = \int_0^1 \sqrt{t(1-t)}\; dt = \int_0^1 \sqrt{\frac{1}{4} - \left(t - \frac{1}{2}\right)^2} dt\; dt$$

$$P = \left(\frac{1}{2}\left(t - \frac{1}{2}\right)\sqrt{t(1-t)} + \frac{1}{8}\sin^{-1}\left(\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)\right)_0^1$$

$$P = \frac{1}{8}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{8}$$

$$\therefore \quad f(x) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=0}^{n-1} \left[k \int_{k}^{k+1} \sqrt{(x-k)(1-(x-k))} \cdot dx \right]$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \frac{\pi}{8} \sum_{k=0}^{n-1} k = \frac{\pi}{8} \lim_{n\to\infty} \left[\frac{n(n-1)}{2n^2}\right] = \frac{\pi}{16}$$

ILLUSTRATION 47

Evaluate: $\int\limits_{\sqrt{\frac{3a^2+b^2}{2}}}^{\sqrt{\frac{a^2+b^2}{2}}} \frac{x \cdot dx}{\sqrt{(x^2-a^2)(b^2-x^2)}}$ where $b > a$

SOLUTION

$$f(x) = \int_{\sqrt{\frac{3a^2+b^2}{2}}}^{\sqrt{\frac{a^2+b^2}{2}}} \frac{x \cdot dx}{\sqrt{(x^2-a^2)(b^2-x^2)}}$$

Put $x^2 = a^2\sin^2\theta + b^2\cos^2\theta$

$$2x\, dx = (a^2 - b^2)\sin 2\theta\, d\theta$$

$$f(x) = \int_{\pi/3}^{\pi/4} \frac{1}{2}\left(\frac{(a^2-b^2)\sin 2\theta . d\theta}{\sqrt{(b^2-a^2)(\cos^2\theta)(b^2-a^2)\sin^2\theta}}\right.$$

$$= \int_{\pi/3}^{\pi/4} \frac{(a^2-b^2)(2)\sin\theta\cos\theta . d\theta}{2(b^2-a^2)\cos\theta\sin\theta}$$

$$= \int_{\pi/3}^{\pi/4} -d\theta = \left(-\frac{\pi}{4} + \frac{\pi}{3}\right) = \frac{\pi}{12}$$

ILLUSTRATION 48

Evaluate: $\int\limits_0^1 \frac{x^2 \cdot \ln x}{\sqrt{1-x^2}}\, dx$

SOLUTION

$$I = \int_0^1 \frac{x^2 \ln x}{\sqrt{1-x^2}} \cdot dx \qquad \text{Put } x = \sin\theta,\ dx = \cos\theta \cdot d\theta$$

$$I = \int_0^{\pi/2} \frac{\sin^2\theta \cdot \ln(\sin\theta)}{\cos\theta} \cos\theta \cdot d\theta = \int_0^{\pi/2} \sin^2\theta \ln\sin\theta\, d\theta$$

$$= \int_0^{\pi/2} \ln\sin\theta\, d\theta - \int_0^{\pi/2} \cos^2\theta \ln(\sin\theta)\, d$$

$$= \int_0^{\pi/2} \ln\sin\theta\, d\theta - \int_0^{\pi/2} \ln(\sin\theta)\left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= \int_0^{\pi/2} \frac{\ln(\sin\theta)}{2} \cdot d\theta - \int_0^{\pi/2} \frac{\ln(\sin\theta)}{2} \cdot \cos 2\theta \cdot d\theta$$

$$I = -\frac{\pi}{4}\ln 2 - \frac{1}{2}\left[\ln\sin\theta \cdot \frac{\sin 2\theta}{2} - \int \frac{\cos\theta}{\sin\theta} \cdot \frac{\sin 2\theta}{2} \cdot d\theta\right]_0^{\pi/2}$$

$$I = -\frac{\pi}{4}\ln 2 - 0 + \frac{1}{2}\int_0^{\pi/2} \cos^2\theta \cdot d\theta$$

$$I = -\frac{\pi}{4}\ln 2 + \frac{\pi}{8} = \frac{\pi}{8}(1-\ln 4)$$

ILLUSTRATION 49

Evaluate: $\int_0^\infty \frac{dx}{(x+\sqrt{1+x^2})^n}$, If $n>1$

SOLUTION

$$I = \int_0^\infty \frac{dx}{(x+\sqrt{1+x^2})^n}$$

Let $x+\sqrt{1+x^2} = t$

⇒ $1+x^2 = (t-x)^2 \Rightarrow 1 = t^2 - 2xt$

⇒ $x = \frac{t^2-1}{2t} = \frac{1}{2}\left(t-\frac{1}{t}\right) \Rightarrow dx = \frac{1}{2}\left(1+\frac{1}{t^2}\right)dt$

$$I = \int_1^\infty \frac{1}{t^n}\cdot\frac{1}{2}\left(1+\frac{1}{t^2}\right)dt = \frac{1}{2}\int_1^\infty \left(t^{-n}+t^{-n-2}\right)dt$$

$$I = \frac{1}{2}\left[\frac{t^{-n+1}}{-n+1}+\frac{t^{-n-1}}{-n-1}\right]_1^\infty = \frac{1}{2}\left[\frac{t^{1-n}}{1-n}-\frac{t^{-(n+1)}}{n+1}\right]_1^\infty$$

$$I = \frac{1}{2}\left[0-\left\{\frac{1}{1-n}-\frac{1}{n+1}\right\}\right] = \frac{n}{n^2-1}$$

ILLUSTRATION 50

Evaluate: $\int_0^{\pi/6}\frac{\sqrt{3\cos 2x-1}}{\cos x}dx$

SOLUTION

$$I = \int_0^{\pi/6}\frac{\sqrt{3\cos 2x-1}}{\cos x}dx = \int_0^{\pi/6}\frac{\sqrt{3(1-2\sin^2 x)-1}}{(1-\sin^2 x)}\cos x\,dx$$

$$I = \int_0^{\pi/6}\frac{\sqrt{2-6\sin^2 x}}{(1-\sin^2 x)}\cos x\,dx$$

Put $\sin x = \frac{1}{t}$ then $\cos x\,dx = -\frac{1}{t^2}\,dt$

$$I = \int_\infty^2 \frac{\sqrt{2-\frac{6}{t^2}}}{\left(1-\frac{1}{t^2}\right)}\left(-\frac{1}{t^2}dt\right) = \int_2^\infty \frac{\sqrt{2}\sqrt{t^2-3}}{t^2(t^2-1)}t\,dt$$

Put $t^2-3=z^2$; $2t\,dt = 2z\,dz$

$$I = \int_1^\infty \frac{\sqrt{2}z^2}{(z^2+3)(z^2+2)}dz$$

$$= 6\sqrt{2}\int_1^\infty \frac{\frac{(z^2+2)}{2}-\frac{(z^2+3)}{3}}{(z^2+3)(z^2+2)}dz$$

$$I = 6\sqrt{2}\left[\int_1^\infty\frac{1}{2(z^2+3)}dz - \frac{1}{3}\int_1^\infty\frac{dz}{z^2+2}\right]$$

$$I = 6\sqrt{2}\left[\frac{1}{2\sqrt{3}}\tan^{-1}\frac{z}{\sqrt{3}}\bigg|_1^\infty - \frac{1}{3\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}\bigg|_1^\infty\right]$$

$$I = 6\sqrt{2}\left[\frac{1}{2\sqrt{3}}\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{\sqrt{3}}\right)-\frac{1}{3\sqrt{2}}\left(\frac{\pi}{2}-\tan^{-1}\frac{1}{\sqrt{2}}\right)\right]$$

$$I = 6\sqrt{2}\left[\frac{1}{2\sqrt{3}}\cdot\frac{\pi}{3}-\frac{1}{3\sqrt{2}}\cot^{-1}\frac{1}{\sqrt{2}}\right]$$

$$I = \sqrt{6}\,\frac{\pi}{3} - 2\cot^{-1}\frac{1}{\sqrt{2}}$$

ILLUSTRATION 51

Let $f(x)$ be a function satisfying $f(x) = f\left(\frac{100}{x}\right)$ ∀ $x > 0$.

If $\int_1^{10}\frac{f(x)}{x}dx = 5$, then find the value of $\int_1^{100}\frac{f(x)}{x}dx$.

SOLUTION

$$I = \int_1^{100}\frac{f(x)}{x}dx = \int_1^{10}\frac{f(x)}{x}dx + \int_{10}^{100}\frac{f(x)}{x}dx$$

In second integral Put $x = \frac{100}{t}$, $dx = -\frac{100}{t^2}dt$

$$I = \int_1^{10}\frac{f(x)}{x}dx + \int_{10}^{1}\frac{f\left(\frac{100}{t}\right)}{\frac{100}{t}}\left(\frac{-100}{t^2}\right)dt$$

Using $f(x) = f\left(\frac{100}{x}\right)$, we get

$$I = \int_1^{10}\frac{f(x)}{x}dx + \int_1^{10}\frac{f(t)}{t}dt = 2\int_1^{10}\frac{f(x)}{x}dx = 2(5) = 10$$

ILLUSTRATION 52

If $\int_{\pi/4}^{\pi/3}\frac{(\sin^3\theta-\cos^3\theta-\cos^2\theta)(\sin\theta+\cos\theta+\cos^2\theta)^{2007}}{(\sin\theta)^{2009}(\cos\theta)^{2009}}d\theta$

$= \frac{(a+\sqrt{b})^n-(1+\sqrt{c})^n}{d}$ where a, b, c and d are all positive integers. Find the value $(a+b+c+d)$.

SOLUTION

$$I=\int_{\pi/4}^{\pi/3}\frac{(\sin^3\theta-\cos^3\theta-\cos^2\theta)(\sin\theta+\cos\theta+\cos^2\theta)^{2007}}{(\sin\theta)^{2009}\cdot(\cos\theta)^{2009}}.d\theta$$

$$I=\int_{\pi/4}^{\pi/3}\frac{(\sin^3\theta-\cos^3\theta-\cos^2\theta)}{\sin^2\theta\cdot\cos^2\theta}\left(\frac{\sin\theta+\cos\theta+\cos^2\theta)}{\sin\theta\cdot\cos\theta}\right)^{2007}d\theta$$

$$I=\int_{\pi/4}^{\pi/3}[\sin\theta\cdot\sec^2\theta-\cos\theta\operatorname{cosec}^2\theta-\operatorname{cosec}^2\theta]$$

$$(\sec\theta+\operatorname{cosec}\theta+\cot\theta)^{2007}d\theta$$

Put $\sec\theta+\operatorname{cosec}\theta+\cot\theta=t$

$$(\sec\theta\tan\theta-\operatorname{cosec}\theta\cot\theta-\operatorname{cosec}^2\theta)\,d\theta=dt$$

$$\Rightarrow\quad(\sec^2\theta\sin\theta-\operatorname{cosec}^2\theta\cos\theta-\operatorname{cosec}^2\theta)d\theta=dt$$

$$\therefore\quad\int t^{2007}\,dt=\left(\frac{t^{2008}}{2008}\right)$$

$$I=\frac{1}{2008}\left[(\sec\theta+\operatorname{cosec}\theta+\cot\theta)^{2008}\right]_{\pi/4}^{\pi/3}$$

$$I=\frac{1}{2008}\left[\left(2+\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)^{2008}-\left(\sqrt{2}+\sqrt{2}+1\right)^{2008}\right]$$

$$I=\frac{1}{2008}\left[(2+\sqrt{3})^{2008}-(1+\sqrt{8})^{2008}\right]$$

$$I=\frac{(a+\sqrt{b})^n-(1+\sqrt{c})^n}{d}$$

On comparing, we get

$\therefore\quad a=2,\ b=3,\ c=8,\ d=2008,\ n=2008$

$\therefore\quad a+b+c+d=2+3+8+2008=2021$

ILLUSTRATION 53

Evaluate: $\displaystyle\int_0^1\frac{1-x}{1+x}\cdot\frac{dx}{\sqrt{x+x^2+x^3}}$

SOLUTION

$$I=\int_0^1\frac{1-x}{1+x}\cdot\frac{dx}{\sqrt{x+x^2+x^3}}$$

Put $x=\tan^2\theta\Rightarrow dx=2\tan\theta\sec^2\theta\,d\theta$

$$I=\int_0^{\pi/4}\frac{(1-\tan^2\theta)}{(1+\tan^2\theta)}\frac{2\tan\theta\cdot\sec^2\theta\cdot d\theta}{\tan\theta\sqrt{1+\tan^2\theta+\tan^4\theta}}$$

$$I=\int_0^{\pi/4}\frac{2(1-\tan^2\theta)}{\sqrt{1+\tan^2\theta+\tan^4\theta}}d\theta$$

Let $\tan\theta=t,\ \sec^2\theta\,d\theta=dt$

$$I=2\int_0^1\frac{(1-t^2)}{\sqrt{1+t^2+t^4}}\ \frac{dt}{(1+t^2)}$$

$$I=-2\int_0^1\frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)\sqrt{\left(t+\frac{1}{t}\right)^2-1}}$$

Put $t+\dfrac{1}{t}=z,\left(1-\dfrac{1}{t^2}\right)dt=dz$

$$I=\int_2^\infty\frac{2dz}{z\sqrt{z^2-1}}=2\left(\sec^{-1}z\right)_2^\infty=2\left(\frac{\pi}{2}-\frac{\pi}{3}\right)=\frac{\pi}{3}$$

ILLUSTRATION 54

Let $\displaystyle\int_1^2\frac{(x^2-1)dx}{x^3\cdot\sqrt{2x^4-2x^2+1}}=\frac{u}{v}$ where u and v are in their lowest form. Find the value of $\dfrac{(1000)u}{v}$.

SOLUTION

$$I=\int_1^2\frac{(x^2-1)dx}{x^3\cdot\sqrt{2x^4-2x^2+1}}$$

$$I=\int_1^2\frac{(x^2-1)\,x\,dx}{x^4\sqrt{2x^4-2x^2+1}}\qquad\text{Put } x^2=t,\ 2x\,dx=dt$$

$$I=\int_1^4\frac{\frac{(t-1)dt}{2}}{t^2\sqrt{2t^2-2t+1}}=\int_1^4\frac{(t-1)\,dt}{2t^3\sqrt{2-2t^{-1}+t^{-2}}}$$

$$I=\int_1^4\frac{(t-1)\,dt}{2t^3\sqrt{2-2t^{-1}+t^{-2}}}=\int_1^4\frac{(t^{-2}-t^{-3})\,dt}{2\sqrt{2-2t^{-1}+t^{-2}}}$$

Put $2-2t^{-1}+t^{-2}=z^2\Rightarrow(2t^{-2}-2t^{-3})dt=2z\,dz$

$$I=\int_1^{5/4}\frac{zdz}{2z}=\frac{z}{2}\bigg|_1^{5/4}=\frac{1}{2}\left(\frac{5}{4}-1\right)=\frac{1}{8}$$

$$\therefore\quad\frac{u}{v}=\frac{1}{8}$$

We get $1000\times\dfrac{u}{v}=1000\times\dfrac{1}{8}=125.$

ILLUSTRATION 55

Let a, b, c be non-zero real numbers such that $\displaystyle\int_0^1(5+\tan^6 x)(ax^2+bx+c)\,dx=\int_0^2(5+\tan^6 x)(ax^2+bx+c)\,dx,$ then show that the quadratic equation $ax^2+bx+c=0$ has atleast one root in (0, 1).

SOLUTION

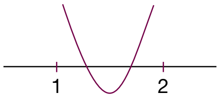
$$\int_0^2 (5+\tan^6 x)(ax^2+bx+c)\,dx$$

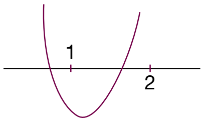
$$=\int_0^1 (5+\tan^6 x)(ax^2+bx+c)\,dx+\int_1^2 (5+\tan^6 x)(ax^2+bx+c)\,dx$$

We have $\int_1^2 (5+\tan^6 x)(ax^2+bx+c)\,dx=0$

$$(5+\tan^6 x)>0 \quad \text{for } x\in(1,2)$$

It means $ax^2+bx+c=0$ has atleast one root $x\in(1,2)$ with possibilities shown as in diagram.





Test Your Concepts-II

(Solutions on page H.269)

1. If $\int_0^1 \frac{e^t dt}{t+1}=a$, then $\int_{b-1}^{b} \frac{e^{-t}dt}{t-b-1}$ is equal to ____.

2. Evaluate: $\int_{\pi}^{5\pi/4} \frac{\sin 2x\,dx}{\sin^4 x+\cos^4 x}$

3. Evaluate: $\int_2^3 \frac{dx}{x\sqrt{4x^2+1}}$

4. Evaluate: $\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}dx$

5. If $I_1=\int_x^1 \frac{1}{1+t^2}dt$ and $I_2=\int_1^{1/x}\frac{1}{1+t^2}dt$ for $x>0$, then relate I_1 and I_2.

6. Evaluate: $\int_1^{e^{37}} \frac{\pi\sin(\pi\log x)}{x}dx$

7. Evaluate: $\int_{e^{-1}}^{e^2}\left|\frac{\log x}{x}\right|dx$

8. Evaluate: $\int_0^{\pi/2}\frac{4x\sin x+x^2\cdot\cos x}{2\sqrt{\sin x}}dx$

9. Evaluate: $\int_1^e \frac{dx}{\sqrt{x^2\ln x+(x\ln x)^2}}$

10. Evaluate: $\int_2^3 \frac{(x^2-2)}{(x-1)^3}dx$

11. Evaluate: $\int_0^1 \frac{dx}{\sqrt{(x+1)^3\cdot(3x+1)}}$

12. Let $I_1=\int_0^{\infty}\frac{x^2\sqrt{x}}{(1+x)^6}dx$, $I_2=\int_0^{\infty}\frac{x\sqrt{x}}{(1+x)^6}dx$, then show that $I_1=I_2$

13. Let $f(x)$ be a function satisfying $xf(x)+\frac{f\left(\frac{1}{x}\right)}{x}=0$, then value of $\int_{\cos\theta}^{\sec\theta} f(x)\,dx$ is ____.

14. If $\int_1^3 \frac{dx}{(x+1)\sqrt{x^2+x}}=\sqrt{a}-\sqrt{b}$ where $a, b\in N$, then find $(a-b)$.

15. Evaluate absolute value of $\dfrac{\int_0^{\pi/2}(x\cos x+1)e^{\sin x}dx}{\int_0^{\pi/2}(x\sin x-1)e^{\cos x}dx}$.

Answer key

1. $(-ae^{-b})$ **2.** $\frac{\pi}{4}$ **3.** $\ln\left(\frac{3}{2}\left(\frac{\sqrt{17}+1}{\sqrt{37}+1}\right)\right)$ **4.** $4-\pi$ **5.** $I_1=I_2$ **6.** 2 **7.** $\frac{5}{2}$

8. $\frac{\pi^2}{4}$ **9.** $\ln(3+2\sqrt{2})$ **10.** $\ln 2+1-\frac{3}{8}$ **11.** $\sqrt{2}-1$ **13.** 0

14. 1 **15.** e

Property P-IV

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a}[f(x)+f(-x)]dx$$

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)dx \;; & f(x) \text{ is even function,} \\ 0 \;; & f(x) \text{ is an odd function,} \end{cases}$$

It is also commonly named as **Jack Rule**.

PROOF:

$$I = \int_{-a}^{a} f(x)\ dx = \int_{-a}^{0} f(x)\ dx + \int_{0}^{a} f(x)\ dx$$

Put $x = -t$; $dx = -dt$ in first integral

$$I = \int_{a}^{0} f(-t)(-dt) + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(-t)\,dt + \int_{0}^{a} f(x)\ dx$$

$$I = \int_{0}^{a} f(-x)\ dx + \int_{0}^{a} f(x)\ dx = \int_{0}^{a}[f(x)+f(-x)]\ dx$$

ILLUSTRATION 56

Evaluate: $\int_{-\pi}^{\pi}(x^3 + x\cos x + \tan^5 x + 4)\,dx$

SOLUTION

Using $\left[\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)+f(-x)dx\right]$

$$I = \int_{6}^{\pi}(x^3 + x\cos x + \tan^5 x + 4) + (-x^3 - x\cos(-x) + \tan^5(-x) + 4)dx$$

$$I = \int_{0}^{\pi} 8\,dx = 8x\Big|_0^{\pi} = 8\pi$$

ILLUSTRATION 57

Evaluate: $\int_{-\pi/4}^{\pi/4}\frac{x^7 - 3x^5 + 3x^3 - x + 1}{\cos^2 x}\,dx$

SOLUTION

Using $\left[\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)+f(-x)dx\right]$

$$I = \int_{0}^{\pi/4}\frac{x^7 - 3x^5 + 3x^3 - x + 1}{\cos^2 x} + \frac{-x^7 + 3x^5 - 3x^3 + x + 1}{\cos^2(-x)}dx$$

$$I = \int_{0}^{\pi/4}\frac{2}{\cos^2 x}\,dx = 2\int_{0}^{\pi/4}\sec^2 x\,dx$$

$$I = 2\tan x\Big|_0^{\pi/4} = 2(1-0) = 2$$

ILLUSTRATION 58

Evaluate: $\int_{-1}^{1}\frac{e^x + e^{-x}}{1+e^x}\,dx$

SOLUTION

Using $\left[\int_{-a}^{a} f(x)dx = \int_{0}^{a} f(x)+f(-x)dx\right]$

$$I = \int_{0}^{1}\frac{e^x + e^{-x}}{1+e^x} + \frac{e^{-x}+e^x}{1+e^{-x}}\ dx$$

$$I = \int_{0}^{1}(e^x + e^{-x})\left[\frac{1}{1+e^x} + \frac{1}{1+\frac{1}{e^x}}\right]dx$$

$$I = \int_{0}^{1}(e^x + e^{-x})\left[\frac{1+e^x}{1+e^x}\right]dx = \int_{0}^{1} e^x + e^{-x}\ dx$$

$$I = e^x - e^{-x}\Big|_0^1 = (e - e^{-1}) - (e^0 - e^{-0}) = e - \frac{1}{e}$$

ILLUSTRATION 59

Evaluate: $\int_{-3}^{-1}(x+1)^4(x+2)^3(x+3)^4\ dx$

SOLUTION

Let $I = \int_{-3}^{-1}(x+1)^4(x+2)^3(x+3)^4\ dx$

Put $x + 2 = t \Rightarrow dx = dt$

$$I = \int_{-1}^{1}(t-1)^4\cdot t^3\cdot(t+1)^4\ dt = \int_{-1}^{1} t^3\cdot(t^2-1)^4\ \ dt = 0$$

ILLUSTRATION 60

Evaluate: $\displaystyle\int_{-2010}^{2010} x^{2010}\cdot\cot^{-1}(2010x)\,dx$

SOLUTION

Let $a = 2010$

$$I=\int_{-a}^{a} x^{a}\cdot\cot^{-1}(ax)\,dx=\int_{-a}^{a}(-x)^{a}\cdot\cot^{-1}(-ax)\,dx$$

$$I=\int_{-a}^{a}(x)^{a}\cdot(\pi-\cot^{-1}ax)\,dx$$

$$2I=\pi\int_{-a}^{a}x^{a}\,dx=2\pi\int_{0}^{a}x^{a}\,dx$$

$$I=\pi\int_{0}^{a}x^{a}\,dx=\pi\cdot\left.\frac{x^{a+1}}{a+1}\right|_{0}^{a}=\pi\cdot\frac{a^{a+1}}{a+1}=\pi\frac{(2010)^{2011}}{2011}.$$

ILLUSTRATION 61

Evaluate: $\displaystyle\int_{-1}^{1}\frac{(2x^{332}+x^{998}+4x^{1668}\cdot\sin x^{691})}{1+x^{666}}\,dx$

SOLUTION

$$I=\int_{-1}^{1}\frac{(2x^{332}+x^{998}+4x^{666}\cdot\sin x^{691})}{1+x^{666}}\,dx$$

$$I=2\int_{0}^{1}\frac{(2x^{332}+x^{998})}{(1+x^{666})}\,dx=2\int_{0}^{1}\frac{x^{332}(2+x^{666})}{(1+x^{666})}\,dx$$

$$I=2\int_{0}^{1}\frac{x^{332}dx}{1+(x^{333})^{2}}+2\int_{0}^{1}x^{332}dx$$

$$I=\frac{2}{333}\tan^{-1}x^{333}\Big|_{0}^{1}+\frac{2}{333}x^{333}\Big|_{0}^{1}$$

$$=\frac{2}{333}\left(\frac{\pi}{4}\right)+\frac{2}{333}=\frac{\pi+4}{666}$$

ILLUSTRATION 62

Evaluate: $\displaystyle\int_{-\pi/3}^{\pi/3}\frac{\pi+4x^{3}}{2-\cos\left(|x|+\frac{\pi}{3}\right)}\,dx$

SOLUTION

$$I=\int_{-\pi/3}^{\pi/3}\frac{\pi}{2-\cos\left(|x|+\frac{\pi}{3}\right)}\,dx+\int_{-\pi/3}^{\pi/3}\frac{4x^{3}}{2-\cos\left(|x|+\frac{\pi}{3}\right)}\,dx$$

The second integral becomes zero as function $f(x)$ is odd.

$$I=2\pi\int_{0}^{\pi/3}\frac{1}{2-\cos\left(x+\frac{\pi}{3}\right)}\,dx$$

Put $x+\dfrac{\pi}{3}=t\Rightarrow dx=dt$

$$I=2\pi\int_{\pi/3}^{2\pi/3}\frac{1}{2-\cos t}\,dt=2\pi\int_{\pi/3}^{2\pi/3}\frac{1}{2-\left(\dfrac{1-\tan^{2}\left(\frac{t}{2}\right)}{1+\tan^{2}\left(\frac{t}{2}\right)}\right)}\,dt$$

$$I=2\pi\int_{\pi/3}^{2\pi/3}\frac{\sec^{2}\frac{t}{2}}{1+3\tan^{2}\left(\frac{t}{2}\right)}\,dt$$

$$=\frac{2\pi}{3}\int_{\pi/3}^{2\pi/3}\frac{\sec^{2}\frac{t}{2}}{\tan^{2}\frac{t}{2}+\left(\frac{1}{\sqrt{3}}\right)^{2}}\,dt$$

$$I=\frac{4\pi\sqrt{3}}{3}\left[\tan^{-1}\left(\sqrt{3}\tan\frac{t}{2}\right)\right]_{\pi/3}^{2\pi/3}$$

$$I=\frac{4\pi}{\sqrt{3}}\left[\tan^{-1}3-\frac{\pi}{4}\right]$$

ILLUSTRATION 63

Evaluate: $\displaystyle\int_{-1/2}^{1/2}(\sin^{-1}(3x-4x^{3})-\cos^{-1}(4x^{3}-3x)\,dx$

SOLUTION

Using $\left[\displaystyle\int_{-a}^{a}f(x)dx=\int_{0}^{a}f(x)+f(-x)dx\right]$

$$I=\int_{0}^{1/2}\sin^{-1}(3x-4x^{3})-\cos^{-1}(4x^{3}-3x)+\sin^{-1}(-3x+4x^{3})-\cos^{-1}(-4x^{3}+3x)dx$$

$$I=\int_{0}^{1/2}-\cos^{-1}(4x^{3}-3x)-(\pi-\cos^{-1}(4x^{3}-3x))\,dx$$

$$I=-\pi\int_{0}^{1/2}dx=-\frac{\pi}{2}$$

ILLUSTRATION 64

For a positive integer n, let $I_n=\displaystyle\int_{-\pi}^{\pi}\left(\frac{\pi}{2}-|x|\right)\cos nx\,dx$. Find the value of $9(I_1+I_2+I_3+I_4)$.

SOLUTION

$$I_n = \int_{-\pi}^{\pi}\left(\frac{\pi}{2} - |x|\right)\cos nx\,dx$$

$$I_n = 2\int_{0}^{\pi}\left(\frac{\pi}{2} - x\right)\cos nx\,dx$$

$$I_n = 2\left[\left(\frac{\pi}{2} - x\right)\frac{\sin nx}{n}\Bigg|_0^{\pi} - \int_0^{\pi}(-1)\frac{\sin nx}{n}dx\right]$$

$$I_n = 2\left[0 - 0 + \frac{1}{n}\frac{(-\cos nx)}{n}\Bigg|_0^{\pi}\right] = -\frac{2}{n^2}((-1)^n - 1)$$

$$I_1 = 4;\ I_2 = -\frac{1}{2}(1-1) = 0$$

$$I_3 = -\frac{2}{9}(-2) = \frac{4}{9};\ I_4 = 0$$

$$9(I_1 + I_2 + I_3 + I_4) = 36 + 4 = 40$$

Useful Concepts from ITF

$\sin^{-1}(-x) = -\sin^{-1}x;\ \cos^{-1}(-x) = \pi - \cos^{-1}x$

$\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x;\ \sec^{-1}(-x) = \pi - \sec^{-1}x$

$\tan^{-1}(-x) = -\tan^{-1}x;\ \cot^{-1}(-x) = \pi - \cot^{-1}x$

ILLUSTRATION 65

Evaluate: $\displaystyle\int_{-1/\sqrt{3}}^{1/\sqrt{3}} x^2\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

SOLUTION

$$I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} x^2\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$$

Using $\left[\displaystyle\int_{-a}^{a} f(x)dx = \int_0^a f(x) + f(-x)dx\right]$

$$I = \int_0^{1/\sqrt{3}} x^2\cos^{-1}\left(\frac{2x}{1+x^2}\right) + (-x)^2\cos^{-1}\left(\frac{-2x}{1+x^2}\right)dx$$

$$I = \int_0^{1/\sqrt{3}} x^2\left[\cos^{-1}\left(\frac{2x}{1+x^2}\right) + \cos^{-1}\left(\frac{-2x}{1+x^2}\right)\right]dx$$

$$I = \int_0^{1/\sqrt{3}} x^2\left[\cos^{-1}\left(\frac{2x}{1+x^2}\right) + \pi - \cos^{-1}\left(\frac{2x}{1+x^2}\right)\right]dx$$

$(\because\ \cos^{-1}(-x) = \pi - \cos^{-1}x)$

$$I = \int_0^{1/\sqrt{3}} x^2\pi\,dx = \frac{\pi x^3}{3}\Bigg|_0^{1/\sqrt{3}} = \frac{\pi}{3}\left(\frac{1}{3\sqrt{3}} - 0\right) = \frac{\pi}{9\sqrt{3}}$$

ILLUSTRATION 66

Evaluate: $\displaystyle\lim_{n\to\infty} n^2\int_{-1/n}^{1/n}(2007\sin x + 2008\cos x)\,|x|\,dx$

SOLUTION

$$f(x) = \lim_{n\to\infty} n^2\int_{-1/n}^{1/n}(2007\sin x + 2008\cos x)\ |x|\ \cdot dx$$

$$= \lim_{n\to\infty} n^2\int_{-1/n}^{1/n}|x|\,2007\sin x\,dx + \int_{-1/n}^{1/n}|x|\,2008\cos x\,dx$$

(as first is integral of odd function in given limits)

$$= 0 + 2\lim_{n\to\infty} n^2\,2008\int_0^{1/n} x\cos x\,.dx$$

$$= 4016\lim_{n\to\infty} n^2\int_0^{1/n} x\cdot\cos x\,dx$$

$$= 4016\lim_{n\to\infty} n^2\left[x\sin x + \cos x\right]_0^{1/n}$$

$$= 4016\lim_{n\to\infty}\left[\frac{\sin\frac{1}{n}}{\frac{1}{n}} + n^2\left(\cos\frac{1}{n} - 1\right)\right]$$

$$= (4016)\lim_{n\to\infty}\left[\frac{\sin\frac{1}{n}}{\frac{1}{n}} - \frac{2}{4}\left(\frac{\sin\frac{1}{2n}}{\frac{1}{2n}}\right)^2\right]$$

$$= 4016\left[1 - \frac{1}{2}\times 1\right] = 4016\left(1 - \frac{1}{2}\right) = 2008$$

Property V

$\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ Replace x by $a + b - x$

It is also known as **King Rule**.

PROOF:

$$\text{RHS} = \int_a^b f(a+b-x)\,dx$$

Put $a + b - x = t \Rightarrow -dx = dt$

When $x = a;\quad t = a + b - a = b\quad$ lower limit

When $x = b$; $t = a + b - b = a$ upper limit

$$\text{RHS} = \int_b^a f(t)\,(-dt) = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx = \text{LHS}$$

Note

When $b = 0$; we have $\int_0^a f(x)dx = \int_0^a f(a-x)dx$.

ILLUSTRATION 67

Evaluate: $\displaystyle\int_3^6 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx$

SOLUTION

$$I = \int_3^6 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}}\,dx \qquad \ldots(1)$$

Replace x by $a + b - x = 9 - x$

$$I = \int_3^6 \frac{\sqrt{9-x}\,dx}{\sqrt{9-x}+\sqrt{9-(9-x)}} = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{9-x}+\sqrt{x}} \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2I = \int_3^6 \frac{\sqrt{x}+\sqrt{9-x}}{\sqrt{x}+\sqrt{9-x}}\,dx = \int_3^6 dx = x\Big|_3^6 = 6 - 3$$

$$\Rightarrow \quad I = \frac{3}{2}$$

ILLUSTRATION 68

Evaluate: $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

SOLUTION

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} \qquad \ldots(1)$$

Replace x by $a + b - x = \dfrac{\pi}{3} + \dfrac{\pi}{6} - x = \dfrac{\pi}{2} - x$

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\dfrac{\pi}{2}-x\right)}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{\sqrt{\tan x}+1}\,dx \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{1+\sqrt{\tan x}}{\sqrt{\tan x}+1}\,dx = x\Big|_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

$$\Rightarrow \quad I = \frac{\pi}{12}$$

ILLUSTRATION 69

Evaluate: $\displaystyle\int_0^1 \sqrt[3]{2x^3-3x^2-x+1}\,dx$

SOLUTION

Let $I = \displaystyle\int_0^1 \sqrt[3]{2x^3-3x^2-x+1}\,dx$

Using $\displaystyle\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$$I = \int_0^1 \sqrt[3]{2(1-x)^3-3(1-x)^2-(1-x)+1}\,dx$$

$$I = \int_0^1 \sqrt[3]{-2x^3+3x^2+x-1}\,dx = -\int_0^1 \sqrt[3]{2x^3-3x^2-x+1}\,dx$$

$$I = -I \Rightarrow I = 0\,.$$

ILLUSTRATION 70

Evaluate: $\displaystyle\int_0^\pi \frac{dx}{1+5^{\tan x}}$

SOLUTION

$$I = \int_0^\pi \frac{dx}{1+5^{\tan x}} \qquad \ldots(1)$$

Replace x by $0 + \pi - x$

$$I = \int_0^\pi \frac{dx}{1+5^{\tan(\pi-x)}} = \int_0^\pi \frac{dx}{1+5^{-\tan x}} = \int_0^\pi \frac{5^{\tan x}}{5^{\tan x}+1}\,dx \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2I = \int_0^\pi \frac{1+5^{\tan x}}{1+5^{\tan x}}\,dx = \int_0^\pi 1\,dx = x\Big|_0^\pi = \pi - 0$$

$$\Rightarrow \quad I = \frac{\pi}{2}$$

ILLUSTRATION 71

Evaluate: $\displaystyle\int_0^{14} \frac{[x^2]\,dx}{[x^2-28x+196]+[x^2]}$ where $[\cdot]$ is greatest integer function

SOLUTION

$$I=\int_0^{14}\frac{[x^2]\,dx}{[x^2-28x+196]+[x^2]}=\int_0^{14}\frac{[x^2]\,dx}{[(14-x)^2]+[x^2]} \quad \ldots(1)$$

Replace x by $14-x$

$$I=\int_0^{14}\frac{[(14-x)^2]\,dx}{[(14-14+x)^2]+[(14-x)^2]}$$

$$I=\int_0^{14}\frac{[(14-x)^2]\,dx}{[x^2]+[(14-x)^2]} \quad \ldots(2)$$

Adding (1) + (2), we get

$$2I=\int_0^{14}\frac{[x^2]+[(14-x)^2]}{[(14-x)^2]+[x^2]}\,dx=\int_0^{14}dx \;=\; x\Big|_0^{14}\;=14-0$$

$$\Rightarrow \quad I=7$$

ILLUSTRATION 72

Let $f(x)$ be a continuous function on [0, 4] satisfying $f(x)\,f(4-x)=1$, then find the value of the definite integral $\displaystyle\int_0^4\frac{1}{1+f(x)}\,dx.$

SOLUTION

$$I=\int_0^4\frac{1}{1+f(x)}\,dx \quad \ldots(1)$$

$$I=\int_0^4\frac{1}{1+f(4-x)}\,dx$$

Using king rule, given $f(x)\,f(4-x)=1$

$$I=\int_0^4\frac{1}{1+\dfrac{1}{f(x)}}\,dx=\int_0^4\frac{f(x)}{f(x)+1}\,dx \quad \ldots(2)$$

Eq. (1) + (2), we get

$$2I=\int_0^4 1\,dx=4;\ \text{we get } I=2$$

ILLUSTRATION 73

Evaluate: $\displaystyle\int_0^a \ln(\cot a+\tan x)dx$, where $a\in\left(0,\dfrac{\pi}{2}\right)$

SOLUTION

$$I=\int_0^a \ln(\cot a+\tan x)\,dx \quad \ldots(1)$$

$$I=\int_0^a \ln(\cot a+\tan(a-x))\,dx$$

$$=\int_0^a \ln\left(\cot a+\frac{\tan a-\tan x}{1+\tan a\tan x}\right)dx$$

$$I=\int_0^a \ln\frac{(\cot a+\tan x+\tan a-\tan x)}{1+\tan a\tan x}\,dx$$

$$I=\int_0^a \ln\left(\frac{\cot^2 a+1}{\cot a+\tan x}\right)dx \quad \ldots(2)$$

By adding (1) + (2), we get

$$2I=\int_0^a \ln\ \text{cosec}^2 a\,dx$$

$$\Rightarrow \quad I=\ln\text{cosec}\,a\,(a)=-a\ln\sin a$$

ILLUSTRATION 74

Evaluate: $\displaystyle\int_0^{\pi/2}\sin x\ \sin 2x\,\sin 3x\,dx$

SOLUTION

$$I=\int_0^{\pi/2}\sin x\ \sin 2x\,\sin 3x\,dx \quad \ldots(1)$$

Using King Rule $x\to a+b-x=\dfrac{\pi}{2}-x$

$$I=\int_0^{\pi/2}\cos x\sin 2x(-\cos 3x)\,dx \quad \ldots(2)$$

Adding (1) + (2), we get

$$2I=\int_0^{\pi/2}\sin 2x(-(\cos x\cos 3x-\sin x\sin 3x))\,dx$$

$$\Rightarrow \quad 2I=-\int_0^{\pi/2}\sin 2x\cos 4x\,dx$$

$$\Rightarrow \quad I = \frac{-1}{4}\int_0^{\pi/2}(2\sin 2x\cos 4x)dx = \frac{1}{4}\int_0^{\pi/2}(\sin 2x - \sin 6x)\,dx$$

$$I = \frac{1}{4}\left[\frac{\cos 6x}{6} - \frac{\cos 2x}{2}\right|_0^{\pi/2}\Bigg)$$

$$I = \frac{1}{4}\left[\frac{1}{6}(-1-1) - \frac{1}{2}(-1-1)\right] = \frac{1}{4}\left[-\frac{1}{3}+1\right] = \frac{1}{6}$$

ILLUSTRATION 75

Let f be a positive function, let $I_1 = \int_{1-k}^{k} x\cdot f[x(1-x)]\,dx$ and $I_2 = \int_{1-k}^{k} f[x(1-x)]\,dx$, where $(2k-1) > 0$, then compute $\dfrac{I_1}{I_2}$.

SOLUTION

$$I_1 = \int_{1-k}^{k} x\cdot f[x(1-x)]\,dx$$

Applying $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$= \int_{1-k}^{k}(1-x)\cdot f[x(1-x)]\,dx$$

$$I_1 = \int_{1-k}^{k} f[x(1-x)]\,dx - \int_{1-k}^{k} x\cdot f[x(1-x)]\,dx = I_2 - I_1$$

$$2I_1 = I_2 \Rightarrow \frac{I_1}{I_2} = \frac{1}{2}$$

ILLUSTRATION 76

Let $u = \int_0^{\pi/2}\cos\left(\frac{2\pi}{3}\sin^2 x\right)dx$ and $v = \int_0^{\pi/2}\cos\left(\frac{\pi}{3}\sin x\right)dx$, then show that $2u = v$

SOLUTION

$$u = \int_0^{\pi/2}\cos\left(\frac{2\pi}{3}\sin^2 x\right)dx \qquad \ldots(1)$$

$$x \to \frac{\pi}{2} - x$$

$$u = \int_0^{\pi/2}\cos\left(\frac{2\pi}{3}\cos^2 x\right)dx \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2u = \int_0^{\pi/2} 2\cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\cos 2x\right)dx$$

$$u = \frac{1}{2}\int_0^{\pi/2}\cos\left(\frac{\pi}{3}\cos 2x\right)dx \Rightarrow 2x = t,\ 2dx = dt$$

$$u = \frac{1}{4}\int_0^{\pi}\cos\left(\frac{\pi}{3}\cos t\right)dt$$

$$= \frac{1}{4}\int_0^{\pi/2}\cos\left(\frac{\pi}{3}\cos t\right) + \cos\left(-\frac{\pi}{3}\cos t\right)dt$$

$$= \frac{1}{2}\int_0^{\pi/2}\cos\left(\frac{\pi}{3}\cos t\right)dt = \frac{1}{2}\int_0^{\pi/2}\cos\left(\frac{\pi}{3}\sin t\right)dt$$

$$u = \frac{v}{2} \Rightarrow 2u = v$$

ILLUSTRATION 77

If $\alpha, \beta\ (\beta > \alpha)$ are the roots of $g(x) = ax^2 + bx + c = 0$ and $f(x)$ is a even function, then evaluate $\displaystyle\int_\alpha^\beta \frac{e^{f\left(\frac{g(x)}{x-\alpha}\right)}}{e^{f\left(\frac{g(x)}{x-\alpha}\right)} + e^{f\left(\frac{g(x)}{x-\beta}\right)}}\,dx.$

SOLUTION

$$g(x) = ax^2 + bx + c = a(x-\alpha)(x-\beta)\ \beta > \alpha$$

$$f(-x) = f(x)$$

$$I = \int_\alpha^\beta \frac{e^{f(a(x-\beta))}}{e^{f(a(x-\beta))} + e^{f(a(x-\alpha))}}\,dx \qquad \ldots(1)$$

Replace x by $\alpha + \beta - x$

$$I = \int_\alpha^\beta \frac{e^{f(a(\alpha-x))}}{e^{f(a(\alpha-x))} + e^{f(a(\beta-x))}}\,dx$$

$f(x)$ is an even function

$$I = \int_\alpha^\beta \frac{e^{f(a(x-\alpha))}}{e^{f(a(x-\alpha))} + e^{f(a(x-\beta))}}\,dx \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2I = \int_\alpha^\beta 1\,dx = x\Big|_\alpha^\beta = \beta - \alpha$$

$$I = \frac{\beta-\alpha}{2} = \frac{\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}}{2}$$

$$I = \frac{\sqrt{\left(-\frac{b}{a}\right)^2 - 4\frac{c}{a}}}{2} = \frac{\sqrt{b^2-4ac}}{2|a|}$$

ILLUSTRATION 78

Evaluate: $f(\theta) = \int\limits_0^\infty \frac{\tan^{-1}x}{x^2+2x\cos\theta+1}dx$ where $\theta \in (0, \pi)$

SOLUTION

$$f(\theta)=\int\limits_0^\infty \frac{\tan^{-1}(x).dx}{x^2+2x\cos\theta+1}$$

Put $x=\tan t$, $dx=\sec^2 t\cdot dt$

$$=\int\limits_0^{\pi/2}\frac{t\sec^2 t\,dt}{1+\tan^2 t+2\tan t\cos\theta}=\int\limits_0^{\pi/2}\frac{t\sec^2 t\,dt}{\sec^2 t+2\tan t\cos\theta}$$

$$f(\theta)=\int\limits_0^{\pi/2}\frac{t\,dt}{1+\sin 2t\cos\theta} \qquad \ldots(1)$$

Using King Rule

$$f(\theta)=\int\limits_0^{\pi/2}\frac{\left(\frac{\pi}{2}-t\right)dt}{1+\sin(\pi-2t)\cos\theta} \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2f(\theta)=\frac{\pi}{2}\int\limits_0^{\pi/2}\frac{dt}{1+\sin 2t\cos\theta}$$

$$f(\theta)=\frac{\pi}{4}\int\limits_0^{\pi/2}\frac{\sec^2 t\,dt}{1+\tan^2 t+2\tan t\cos\theta}$$

Put $\tan t = z$; $\sec^2 t\ dt = dz$

$$f(\theta)=\frac{\pi}{4}\int\limits_0^\infty\frac{dz}{(z+\cos\theta)^2+\sin^2\theta}$$

$$f(\theta)=\left(\frac{\pi}{4}\right)\frac{1}{\sin\theta}\left[\tan^{-1}\left(\frac{z+\cos\theta}{\sin\theta}\right)\right]_0^\infty$$

$$f(\theta)=\frac{\pi}{4\sin\theta}\left[\tan^{-1}(\infty)-\tan^{-1}(\cot\theta)\right]$$

$$f(\theta)=\frac{\pi}{4\sin\theta}\left[\frac{\pi}{2}-\frac{\pi}{2}+\theta\right]=\frac{\pi\theta}{4\sin\theta}$$

ILLUSTRATION 79

Evaluate: $\int\limits_{\sin^{-1}(x^2-4x+5)-2x}^{\cos^{-1}(x^2-4x+5)+x^2}\frac{\sqrt{f(x)}}{\sqrt{f\left(\frac{\pi}{2}-x\right)}+\sqrt{f(x)}}dx$

SOLUTION

$(x^2-4x+5)=(x-2)^2+1$, for $x=2$ only, $\sin^{-1}(x^2-4x+5)$ and $\cos^{-1}(x^2-4x+5)$ are defined.

Upper limit $=\cos^{-1}1+4=4$

Lower limit $=\sin^{-1}1-4=\frac{\pi}{2}-4$

$$I=\int\limits_{\frac{\pi}{2}-4}^{4}\frac{\sqrt{f(x)}dx}{\sqrt{f\left(\frac{\pi}{2}-x\right)}+\sqrt{f(x)}} \qquad \ldots(1)$$

Replace x by $a+b-x=\frac{\pi}{2}-x$

$$I=\int\limits_{\frac{\pi}{2}-4}^{4}\frac{\sqrt{f\left(\frac{\pi}{2}-x\right)}}{\sqrt{f(x)}+\sqrt{f\left(\frac{\pi}{2}-x\right)}}dx \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2I=\int\limits_{\frac{\pi}{2}-4}^{4}dx\ =x\Big|_{\frac{\pi}{2}-4}^{4}=4-\frac{\pi}{2}+4=8-\frac{\pi}{2}$$

$$\Rightarrow \quad I=\frac{16-\pi}{4}$$

ILLUSTRATION 80

Evaluate: $\int\limits_0^1\frac{\sin^{-1}\sqrt{x}}{x^2-x+1}dx$

SOLUTION

$$f(x)=\int\limits_0^1\frac{\sin^{-1}\sqrt{x}}{x^2-x+1}dx=\int\limits_0^1\frac{\sin^{-1}\sqrt{x}\,dx}{x(x-1)+1} \qquad \ldots(1)$$

Replace x by $1-x$, we obtain

$$f(x)=\int\limits_0^1\frac{\sin^{-1}\sqrt{1-x}\cdot dx}{(x^2-x+1)} \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2f(x)=\int\limits_0^1\frac{\sin^{-1}\sqrt{x}}{x^2-x+1}+\int\limits_0^1\frac{\sin^{-1}\sqrt{1-x}\cdot dx}{(x^2-x+1)}$$

$$=\int\limits_0^1\frac{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}\cdot dx}{(x^2-x+1)}=\frac{\pi}{2}\int\limits_0^1\frac{dx}{(x^2-x+1)}$$

$$2f(x)=\left(\frac{\pi}{2}\right)\int\limits_0^1\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$2f(x) = \frac{\pi}{2}\cdot\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1$$

$$2f(x) = \left(\frac{\pi}{\sqrt{3}}\right)\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$$

$$\Rightarrow \quad 2f(x) = \frac{\pi}{\sqrt{3}}\left(\frac{\pi}{3}\right) \Rightarrow f(x) = \frac{\pi^2}{6\sqrt{3}}$$

Property VI

$$\int_0^{2a} f(x)\,dx = \int_0^{a}[f(x)+f(2a-x)]\,dx$$

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)dx & \text{if } f(2a-x)=f(x) \\ 0 & \text{if } f(2a-x)=-f(x) \end{cases}$$

It is known as **Queen Rule**.

PROOF:

$$\text{LHS} = \int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$$

In second integral
Put $x = 2a - t \Rightarrow t = 2a - x \Rightarrow dt = -dx$
when $x = a$; $t = a$; when $x = 2a$; $t = 0$

$$\text{LHS} = \int_0^{a} f(x)\,dx + \int_a^{0} f(2a-t)\,(-dt)$$

$$= \int_0^{a} f(x) + \int_0^{a} f(2a-t)\,dt$$

$$= \int_0^{a} f(x)\,dx + \int_0^{a} f(2a-x)\,dx = \text{RHS}$$

Working Rule

Jack Rule: $\int_{-a}^{a} f(x)\,dx = \int_0^{a}(f(x)+f(-x))\,dx$

King Rule: $\int_a^{b} f(x)\,dx = \int_a^{b} f(a+b-x)\,dx$

Queen Rule: $\int_0^{2a} f(x)\,dx = \int_0^{a}(f(x)+f(2a-x))\,dx$

ILLUSTRATION 81

Evaluate: $\int_0^{2\pi} \sin^3 x\cos^3 x\,dx$

SOLUTION

Using $\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)+f(2a-x)\,dx$

$$I = \int_0^{\pi} \sin^3 x\cos^3 x + \sin^3(2\pi-x)\cos^3(2\pi-x)\,dx$$

$$I = \int_0^{\pi} \sin^3 x\cos^3 x + (-\sin x)^3(\cos x)^3\,dx$$

$$I = \int_0^{\pi} 0\,dx = 0$$

ILLUSTRATION 82

Prove that

$$\int_0^{\pi/2} \ln\sin x\,dx = \int_0^{\pi/2} \ln\cos x\,dx = \int_0^{\pi/2} \ln\sin 2x\,dx = -\frac{\pi}{2}\ln 2$$

SOLUTION

$$I = \int_0^{\pi/2} \ln\sin x\,dx \qquad \ldots(1)$$

Replace x by $\frac{\pi}{2} - x$, we obtain

$$I = \int_0^{\pi/2} \ln\sin\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi/2} \ln\cos x\,dx \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2I = \int_0^{\pi/2} \ln\sin x + \ln\cos x = \int_0^{\pi/2} \ln\sin x\cos x\,dx$$

$$2I = \int_0^{\pi/2} \ln\left(\frac{\sin 2x}{2}\right)dx = \int_0^{\pi/2} \ln\sin 2x\,dx - \int_0^{\pi/2} \ln 2\,dx$$

$$2I = \int_0^{\pi/2} \ln\sin 2x\,dx - \frac{\pi}{2}\ln 2 \qquad \ldots(3)$$

Let $I_1 = \int_0^{\pi/2} \ln\sin 2x\,dx$

Put $2x = t \Rightarrow 2dx = dt$,

When $x=0$, $t=0$ and $x=\dfrac{\pi}{2}$, $t=\pi$

$$I_1=\frac{1}{2}\int_0^{\pi}\ln\sin t\,dt=\frac{1}{2}\left[\int_0^{\pi/2}\ln\sin t+\ln\sin(\pi-t)dt\right]$$

$$I_1=\frac{1}{2}\int_0^{\pi/2}2\ln\sin t\,dt=\int_0^{\pi/2}\ln\sin t\,dt$$

We get $I_1=I$

From (3), we have $2I=I_1-\dfrac{\pi}{2}\ln 2=I-\dfrac{\pi}{2}\ln 2$

$\Rightarrow$ $I=-\dfrac{\pi}{2}\ln 2$

$$\int_0^{\pi/2}\ln\sin x\,dx=\int_0^{\pi/2}\ln\sin 2x\,dx=-\frac{\pi}{2}\ln 2$$

It is very commonly used integral.

ILLUSTRATION 83

Evaluate: $\displaystyle\int_0^{\pi}\frac{x^2\sin x\,dx}{(2x-\pi)(1+\cos^2 x)}$

SOLUTION

Using $\displaystyle\int_0^{2a}f(x)dx=\int_0^{a}f(x)+f(2a-x)dx$

$$I=\int_0^{\pi/2}\frac{x^2\sin x}{(2x-\pi)(1+\cos^2 x)}+\frac{(\pi-x)^2\sin(\pi-x)\,dx}{(2(\pi-x)-\pi)(1+\cos^2(\pi-x))}$$

$$I=\int_0^{\pi/2}\frac{x^2\sin x}{(2x-\pi)(1+\cos^2 x)}+\frac{(\pi-x)^2\sin x\,dx}{(\pi-2x)(1+\cos^2 x)}$$

$$=\int_0^{\pi/2}\frac{(x^2-(\pi-x)^2)\sin x\,dx}{(2x-\pi)(1+\cos^2 x)}$$

$$I=\int_0^{\pi/2}\frac{(2x-\pi)\pi\sin x\,dx}{(2x-\pi)(1+\cos^2 x)}=\pi\int_0^{\pi/2}\frac{\sin x\,dx}{1+\cos^2 x}$$

Put $\cos x=t\Rightarrow -\sin x\,dx=dt$

When $x=0$; $t=\cos 0=1$ (lower limit)

When $x=\dfrac{\pi}{2}$; $t=\cos\dfrac{\pi}{2}=0$ (upper limit)

$$I=\pi\int_1^{0}-\frac{dt}{1+t^2}=\pi\int_0^{1}\frac{dt}{1+t^2}=\pi\left(\tan^{-1}t\right)_0^1$$

$$I=\pi\left(\frac{\pi}{4}-0\right)=\frac{\pi^2}{4}$$

ILLUSTRATION 84

For any integer n, evaluate $\displaystyle\int_0^{\pi}e^{\cos^2 x}\cos^3(2n+1)x\,dx$

SOLUTION

Using $\displaystyle\int_0^{2a}f(x)dx=\int_0^{a}(f(x)+f(2a-x))dx$

where, $f(x)=e^{\cos^2 x}\cdot\cos^3\{(2n+1)x\}$

$\therefore$ $f(2a-x)=f(\pi-x)=(e^{\cos^2 x})(-\cos^3(2n+1)x)=-f(x)$

$f(2a-x)+f(x)=0$

we get $I=0$

ILLUSTRATION 85

Show that $\displaystyle\int_0^{t}e^{tx}\cdot e^{-x^2}\,dx=e^{t^2/4}\int_0^{t}e^{-t^2/4}\,dt$

SOLUTION

Using $\displaystyle\int_0^{2a}f(x)\,dx=2\int_0^{a}f(x)\,dx$ if $f(2a-x)=f(x)$

$$\text{LHS}=2\int_0^{t/2}e^{tx-x^2}\,dx=2\int_0^{t/2}e^{x(t-x)}\,dx$$

Using $\displaystyle\int_a^{b}f(x)\,dx=\int_a^{b}f(a+b-x)\,dx$

$$\text{LHS}=2\int_0^{t/2}e^{(t/2-x)(t-(t/2-x))}\,dx=2\int_0^{t/2}e^{t^2/4}e^{-x^2}\,dx$$

$$=2e^{t^2/4}\int_0^{t/2}e^{-x^2}\,dx\qquad\text{Put }x=\frac{u}{2}\Rightarrow dx=\frac{du}{2}$$

$$=2e^{t^2/4}.\int_0^{u}e^{-u^2/4}\frac{du}{2}=e^{t^2/4}\int_0^{t}e^{-t^2/4}\,dt=\text{RHS}$$

Hence proved.

ILLUSTRATION 86

Evaluate: $\displaystyle\int_{-\pi}^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}\,dx$

SOLUTION

Using $\displaystyle\int_{-a}^{a}f(x)\,dx=\int_0^{a}(f(x)+f(-x))\,dx$

$$I=\int_0^{\pi}\frac{2x(1+\sin x)}{1+\cos^2 x}+\frac{(-2x)(1+\sin(-x))}{1+\cos^2(-x)}\,dx$$

$$I=\int_0^{\pi}\frac{2x(1+\sin x)-2x(1-\sin x)dx}{1+\cos^2 x}=\int_0^{\pi}\frac{4x\sin x\,dx}{1+\cos^2 x}$$

$$\left(\text{using}\int_0^{2a}f(x)dx=\int_0^{a}f(x)+f(2a-x)dx\right)$$

$$I=\int_0^{\pi/2}\frac{4x\sin x}{1+\cos^2 x}+\frac{4(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$$

$$=\int_0^{\pi/2}\frac{4(x\sin x+(\pi-x)\sin x)}{1+\cos^2 x}\,dx$$

$$I=4\int_0^{\pi/2}\frac{\pi\sin x\,dx}{1+\cos^2 x}$$

Put $\cos x=t\Rightarrow-\sin x dx=dt$

When $x=0$, $t=1$ and when $x=\frac{\pi}{2}$, $t=0$

$$I=4\pi\int_0^{1}\frac{dt}{1+t^2}=4\pi\left(\tan^{-1}t\right)_0^1=4\pi\left(\frac{\pi}{4}-0\right)=\pi^2$$

ILLUSTRATION 87

Evaluate: $\displaystyle\int_0^{\infty}\frac{x\log x}{(1+x^2)^2}dx$

SOLUTION

Put $x=\tan\theta$; $dx=\sec^2\theta\,d\theta$

$$I=\int_0^{\pi/2}\frac{\tan\theta\ln\tan\theta}{\sec^4\theta}\sec^2\theta\,d\theta=\int_0^{\pi/2}\frac{\sin 2\theta}{2}\ln\tan\theta\,d\theta$$

$$\text{using}\left(\because\int_0^{2a}f(x)dx=\int_0^{a}f(x)+f(2a-x)dx\right)$$

$$I=\int_0^{\pi/4}\frac{1}{2}(\sin 2\theta\ln\tan\theta+\sin 2\theta\ln\cot\theta)\,d\theta$$

$$I=\frac{1}{2}\int_0^{\pi/4}\sin 2\theta(\ln\tan\theta\cot\theta)\,d\theta=0$$

ILLUSTRATION 88

Evaluate: $\displaystyle\int_0^{\infty}\ln\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}$

SOLUTION

$$I=\int_0^{\infty}\ln\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}$$

Put $x=\tan\theta$, $dx=\sec^2\theta\,d\theta$

When $x=0$; $\theta=0$ and when $x\to\infty$; $\theta\to\frac{\pi}{2}$

$$I=\int_0^{\pi/2}\ln(\tan\theta+\cot\theta)\frac{\sec^2\theta d\theta}{1+\tan^2\theta}$$

$$I=\int_0^{\pi/2}\ln(\tan\theta+\cot\theta)\,d\theta=\int_0^{\pi/2}\ln\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$I=\int_0^{\pi/2}\ln\left(\frac{1}{\cos\theta\sin\theta}\right)d\theta$$

$$I=-\int_0^{\pi/2}\ln\sin\theta\,d\theta-\int_0^{\pi/2}\ln\cos\theta\,d\theta$$

$$\left(\because\int_0^{\pi/2}\ln\sin\theta d\theta=\int_0^{\pi/2}\ln\cos\theta d\theta=-\frac{\pi}{2}\ln 2\right)$$

$$I=-\left(-\frac{\pi}{2}\ln 2\right)-\left(-\frac{\pi}{2}\ln 2\right)=\pi\ln 2$$

ILLUSTRATION 89

Evaluate: $\displaystyle\int_0^{1}\frac{\sin^{-1}x}{x}dx$

SOLUTION

Put $\sin^{-1}x=t\Rightarrow x=\sin t\Rightarrow dx=\cos t\,dt$

When $x=0$; $t=0$ and when $x=1$; $t=\sin^{-1}1=\frac{\pi}{2}$

$$I=\int_0^{\pi/2}\frac{t}{\sin t}\cdot\cos t\,dt=\int_0^{\pi/2}\underset{\text{I}}{t}\cdot\underset{\text{II}}{\cot t}\,dt$$

$$I=t\ln\,\sin t\Big|_0^{\pi/2}-\int_0^{\pi/2}1\cdot\ln\sin t\,dt$$

$$=\frac{\pi}{2}\ln(1)-\lim_{t\to0^+}t\ln\sin t-\left(-\frac{\pi}{2}\ln 2\right)$$

(limit has $0\times\infty$ form)

$$I=-\lim_{t\to0^+}\frac{\ln\sin t}{\frac{1}{t}}+\frac{\pi}{2}\ln 2\qquad\left(\frac{\infty}{\infty}\text{ form}\right)$$

Using L-Hospital Rule, we obtain

$$I = -\lim_{t\to 0^+} \frac{\left(\frac{1}{\sin t}.\cos t\right)}{-\frac{1}{t^2}} + \frac{\pi}{2}\ln 2$$

$$= -\lim_{t\to 0^+}\left(\frac{t}{\tan t}\right)t + \frac{\pi}{2}\ln 2 = (1)\,0 + \frac{\pi}{2}\ln 2$$

$$I = \frac{\pi}{2}\ln 2$$

ILLUSTRATION 90

If $f(x)$ is differentiable function such that $f'(x)$ is symmetrical about line $2x = 3$ and $f(1) = 0$, $f(2) = 8$, then compute $\int_0^2 f(x)[x]\,dx$ is (where [.] represent G.I.F.).

SOLUTION

$f'(x)$ is symmetric about line $x = \frac{3}{2}$

$$f'\left(x+\frac{3}{2}\right)=f'\left(\frac{3}{2}-x\right)\Rightarrow f\left(x+\frac{3}{2}\right)+f\left(\frac{3}{2}-x\right)=C$$

When $x = \frac{1}{2}$, we have $f(2) + f(1) = C$, as $f(1) = 0, f(2) = 8$

We get $C = 8$, $f\left(x+\frac{3}{2}\right)+f\left(\frac{3}{2}-x\right) = 8$ …(1)

Put $x = t + \frac{3}{2}$; $dx = dt$

$$I=\int_0^2 f(x)[x]\,dx = \int_0^1 0\,d\alpha + \int_1^2 f(x)\,dx$$

Put $x = t + \frac{3}{2}$; $dx = dt$

$$I=\int_{-1/2}^{1/2} f\left(t+\frac{3}{2}\right)dt$$

$$I=\int_0^{1/2} f\left(t+\frac{3}{2}\right)+f\left(-t+\frac{3}{2}\right)dt = \int_0^{1/2} 8\ \ dt = 4.$$

ILLUSTRATION 91

Evaluate: $\int_0^{\pi}\frac{x\,dx}{1+\cos\alpha\sin x}$ where $0<\alpha<\pi$

SOLUTION

Let $I=\int_0^{\pi}\frac{x}{1+\cos\alpha\sin x}\,dx$ …(1)

Using $\int_a^b f(x)\,dx = \int_0^b f(a+b-x)\,dx$

$$I=\int_0^{\pi}\frac{(\pi-x)}{1+\cos\alpha\sin x}\,dx \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2I=\pi\int_0^{\pi}\frac{dx}{1+\cos\alpha\sin x}$$

$$\therefore\quad 2I=\pi\int_0^{\pi}\frac{\left(1+\tan^2\frac{x}{2}\right)dx}{\left(1+\tan^2\frac{x}{2}\right)+2\cos\alpha\tan\frac{x}{2}}$$

$$\Rightarrow\quad 2I=\pi\int_0^{\infty}\frac{2dt}{1+t^2+2t\cos\alpha}\text{, where } t=\tan\frac{x}{2}$$

$$\Rightarrow\quad 2I=2\pi\int_0^{\infty}\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$$

$$\Rightarrow\quad I=\frac{\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^{\infty}$$

$$\Rightarrow\quad I=\frac{\pi}{\sin\alpha}\left(\tan^{-1}(\infty)-\tan^{-1}(\cot\alpha)\right)$$

$$\Rightarrow\quad I=\frac{\pi}{\sin\alpha}\left(\frac{\pi}{2}-\left(\frac{\pi}{2}-\alpha\right)\right)=\frac{\alpha\pi}{\sin\alpha}$$

ILLUSTRATION 92

Evaluate: $\int_1^{\frac{1+\sqrt{5}}{2}}\frac{x^2+1}{x^4-x^2+1}\ln\left(1+x-\frac{1}{x}\right)dx$

SOLUTION

$$I=\int_1^{(1+\sqrt{5})/2}\frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}-1\right)}\ln\left(x-\frac{1}{x}+1\right)dx$$

$$I=\int_1^{(1+\sqrt{5})/2}\frac{\left(1+\frac{1}{x^2}\right)\ln\left(x-\frac{1}{x}+1\right)}{\left(\left(x-\frac{1}{x}\right)^2+1\right)}\,dx$$

Put $x-\frac{1}{x}=t$; $\left(1+\frac{1}{x^2}\right)dx=dt$

$$I=\int_0^1\frac{\ln(t+1)}{(t^2+1)}\,dt \qquad \text{Let } t=\tan\theta\Rightarrow dt=\sec^2\theta\,d\theta$$

$$I=\int_0^{\pi/4}\ln(1+\tan\theta)\,d\theta$$

$$I = \int_0^{\pi/8} \ln(1+\tan\theta) + \ln(1+\tan(45^\circ-\theta))d\theta$$

(using Queen Rule)

$$I = \int_0^{\pi/8} \ln(2)d\theta = \frac{\pi}{8}\ln 2$$

ILLUSTRATION 93

For $n > 0$, Evaluate $\displaystyle\int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$

SOLUTION

$$I = \int_0^{2\pi} \frac{x\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx \qquad \ldots(1)$$

Using $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$I = \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2I = 2\pi\int_0^{2\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx \qquad \text{(using Queen Rule)}$$

$$I = 2\pi\int_0^{\pi} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx \qquad \text{(using Queen Rule)}$$

$$I = 4\pi\int_0^{\pi/2} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$$

Using $\displaystyle\int_0^{2a} f(x)\,dx = \int_0^{a}(f(x)+f(2a-x))\,dx$

$$I = 4\pi\int_0^{\pi/4} \frac{\sin^{2n}x}{\sin^{2n}x+\cos^{2n}x}+\frac{\cos^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$$

$$I = 4\pi\int_0^{\pi/4} dx = 4\pi\cdot\frac{\pi}{4} = \pi^2$$

ILLUSTRATION 94

Evaluate: $\displaystyle\int_0^{\pi} \frac{x^2\sin 2x\cdot\sin\left(\frac{\pi}{2}\cdot\cos x\right)}{2x-\pi}dx$

SOLUTION

$$I = \int_0^{\pi} \frac{x^2\sin 2x\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx \qquad \ldots(1)$$

Replace x by $\pi - x$

$$I = \int_0^{\pi} \frac{(\pi-x)^2\cdot(-\sin 2x)\left(-\sin\left(\frac{\pi}{2}\cos x\right)\right)}{-(2x-\pi)}dx$$

$$I = \int_0^{\pi} \frac{-(\pi^2+x^2-2\pi x)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}dx \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2I = \int_0^{\pi} \frac{(2\pi x-\pi^2)\sin 2x\cdot\sin\left(\frac{\pi}{2}\cdot\cos x\right)}{(2x-\pi)}dx$$

$$2I = \int_0^{\pi} \pi\cdot\sin 2x\cdot\sin\left(\frac{\pi}{2}\cos x\right)dx$$

$$I = \int_0^{\pi/2} (\pi)\sin 2x\cdot\sin\left(\frac{\pi}{2}\cdot\cos x\right)dx$$

$$I = 2\pi\int_0^{\pi/2} \sin x\cos x\cdot\sin\left(\frac{\pi}{2}\cos x\right)dx$$

Put $\cos x = t$, $\sin x\,dx = -dt$

$$I = 2\pi\int_0^{1} t\sin\left(\frac{\pi}{2}t\right)dt$$

$$I = 2\pi\left(\left.\frac{t\left(-\cos\frac{\pi t}{2}\right)}{\frac{\pi}{2}}\right|_0^1 + \int_0^1\frac{1\cos\frac{\pi t}{2}dt}{\frac{\pi}{2}}\right)$$

$$I = 2\pi\left[-\frac{2}{\pi}(0-0)+\left(\frac{4}{\pi^2}\right)\left[\sin\frac{\pi}{2}t\right]_0^1\right] = \frac{8}{\pi}$$

Test Your Concepts-III

(Solutions on page H.271)

1. Evaluate: $\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx$

2. Evaluate: $\int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin x}+1}$ is equal to

3. Evaluate: $\int_{0}^{1} x(1-x)^{98}\,dx$

4. Evaluate: $\int_{0}^{1} \tan^{-1}(1-x+x^2)\,dx$

5. Evaluate: $\int_{-1}^{1} e^{-x^4}(1+\ln(x+\sqrt{x^2+1})+5x^3-4x^4)\,dx$

6. Evaluate: $\int_{0}^{\pi/2} \frac{\cos\theta-\sin\theta}{(1+\cos\theta)(1+\sin\theta)}\,d\theta$

7. Evaluate: $\int_{-1/2}^{1/2} \sin\left(\tan^{-1}\left(\frac{x^2}{\sqrt{1-x^4}}\right)\right)dx$

8. Evaluate: $\int_{-a}^{a} \frac{x^2\cos x+e^x}{e^x+1}\,dx$

9. Evaluate: $\int_{-3\pi/4}^{5\pi/4} \frac{\sin x+\cos x}{1+e^{x-\frac{\pi}{4}}}\,dx$

10. Evaluate $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx,\ a>0$

11. Evaluate $\int_{-\pi/4}^{\pi/4} \frac{e^x\cdot\sec^2 x\,dx}{e^{2x}-1}$

12. Show that $\int_{0}^{\pi} x\,f(\sin x)\,dx=\frac{\pi}{2}\int_{0}^{\pi} f(\sin x)\,dx$

13. Evaluate $\int_{0}^{\pi} \theta\sin^2\theta\cos^2\theta\,d\theta.$

14. Let $I_n=\int_{0}^{1}(1-x^5)^n\,dx$, then show that $\frac{I_{10}}{I_{11}}=\frac{56}{55}$.

15. Let $I_1=\int_{0}^{1}(1-x^{50})^{100}\,dx$ and $I_2=\int_{0}^{1}(1-x^{50})^{101}\,dx$, then evaluate $\frac{I_1}{I_2}$.

16. Evaluate: $\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-7x^3-12x^2+x+1}{x^2+2}\,dx$

17. Evaluate: $\int_{0}^{\pi/2} \frac{x\sin x\cos x}{\cos^4 x+\sin^4 x}\,dx$

18. Evaluate: $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

19. Evaluate: $\int_{0}^{\pi} \frac{x\sin(2x)\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}\,dx$

20. Evaluate: $\int_{0}^{\pi/2} \frac{\sin 8x\log\cot x}{\cos 2x}\,dx$

Answer key

1. 0

2. $\frac{\pi}{2}$

3. $\frac{1}{9900}$

4. ln 2

5. $\frac{2}{e}$

6. 0

7. $\frac{1}{12}$

8. $2a\cos a+a+(a^2-2)\sin a$

9. 0

10. $\frac{\pi}{2}$

11. 0

13. $\frac{\pi^2}{16}$

15. $\frac{5051}{5050}$

16. $\frac{\pi}{2\sqrt{2}}-\frac{16\sqrt{2}}{5}$

17. $\frac{\pi^2}{16}$

18. $\frac{\pi}{2}\left[\frac{\pi}{6}+\frac{1}{2}\ln\left|\frac{\sqrt{3}+1}{\sqrt{3}-1}\right|-\frac{2}{\sqrt{3}}\right]$

19. $\frac{8}{\pi^2}$

20. 0

DEFINITE INTEGRATION FOR PERIODIC FUNCTION

Let function $f(x + T) = f(x)$ is a periodic function with period T. $\sin x$, $\cos x$, $\operatorname{cosec} x$, $\sec x$ have period $T = 2\pi$ and $\tan x$, $\cot x$ have period $T = \pi$. $\{x\}$ where $\{.\}$ is fractional part function has period $T = 1$

(i) $f(x)$ has period T. (least) then $\sqrt{f(x)}$, $\dfrac{1}{f(x)}$, $(f(x))^n$ also have T as period

(ii) $f(x)$ has period T. (least) then $f(ax + b)$ has period $\dfrac{T}{|a|}$

Let $f(x)$ is periodic function with period T where m and n are Integers

(a) $\displaystyle\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$ where $n \in N$

PROOF:

$$\int_0^{nT} f(x)\,dx = \int_0^{T} f(x)\,dx + \int_T^{2T} f(x)\,dx + \int_{2T}^{3T} f(x)\,dx + \ldots + \int_{(n-1)T}^{nT} f(x)\,dx$$

$$= \int_0^{T} f(x)\,dx + \int_0^{T} f(x+T)\,dx + \int_0^{T} f(x+2T)\,dx + \ldots + \int_0^{T} f(x+(n-1)T)\,dx$$

$$= \underbrace{\int_0^{T} f(x)dx + \int_0^{T} f(x)dx + \int_0^{T} f(x)dx + \ldots + \int_0^{T} f(x)dx}_{n \text{ times}}$$

$$\{\because f(x) = f(x+T) = f(x+2T) = \ldots = f(x+(n-1)T)\}$$

$$\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$$

(b) $\displaystyle\int_a^{a+nT} f(x)\,dx = \int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$ where $n \in N$

PROOF:

$$I = \int_a^{a+nT} f(x)\,dx$$

$$= \int_a^{0} f(x)\,dx + \int_0^{nT} f(x)\,dx + \underbrace{\int_{nT}^{a+nT} f(x)dx}_{\text{Put } x=y+nT}$$

$$= \int_a^{0} f(x)\,dx + \int_0^{nT} f(x)\,dx + \int_0^{a} f(y)\,dy$$

$$= n\int_0^{T} f(x)\,dx$$

(c) $\displaystyle\int_{nT}^{mT} f(x)\,dx = (m-n)\int_0^{T} f(x)\,dx$ where $m, n \in N$

PROOF:

$$\int_{nT}^{mT} f(x)\,dx = \int_{nT}^{0} f(x)\,dx + \int_0^{mT} f(x)$$

$$= -\int_0^{nT} f(x)\,dx + \int_0^{mT} f(x)\,dx$$

$$= -n\int_0^{T} f(x)\,dx + m\int_0^{T} f(x)\,dx = (m-n)\int_0^{T} f(x)\,dx$$

(d) $\displaystyle\int_{a+nT}^{b+nT} f(x)\,dx = \int_a^{b} f(x)\,dx$ where $n \in N$

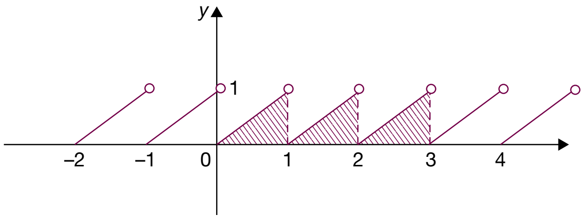
(e) $\displaystyle\int_{nT}^{a+nT} f(x)\,dx = \int_0^{a} f(x)\,dx$ where $n \in N$

Similarly, we can try proving (d) and (e) also.

ILLUSTRATION 95

Evaluate $\displaystyle\int_0^{3} \{x\}\,dx$ where $\{.\}$ is fractional part function.

SOLUTION



$y = \{x\}$ has period $T = 1$

$$I = \int_0^{3} \{x\}\,dx \qquad \left(\text{using } \int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx\right)$$

$$I = 3\int_0^{1} \{x\}\,dx = 3\int_0^{1} x\,dx = 3\left(\frac{x^2}{2}\right)_{x=0}^{x=1} = \frac{3}{2}$$

ILLUSTRATION 96

Evaluate: $\int_{-1}^{2} e^{\{5x\}}\,dx$ where $\{\cdot\}$ is fractional part function

SOLUTION

$f(x) = e^{\{x\}}$ is periodic with period 1,

$g(x) = e^{\{5x\}}$ is periodic with period $T = \frac{1}{5}$

$$I = \int_{-1}^{2} e^{\{5x\}}\,dx = \int_{-5T}^{10T} e^{\{5x\}}\,dx$$

$$I = (10-(-5))\int_{0}^{T=\frac{1}{5}} e^{\{5x\}}\,dx$$

$$\left(\text{using } \int_{nT}^{mT} f(x)\,dx = (m-n)\int_{0}^{T} f(x)\,dx\right)$$

We know that $0 < x < \frac{1}{5}$; $0 < 5x < 1$, then $[5x] = 0$;

$$5x = [5x] + \{5x\} = \{5x\}$$

$$I = 15\int_{0}^{T=\frac{1}{5}} e^{5x}\,dx = 15\left(\frac{e^{5x}}{5}\right)_0^{1/5} = 3(e-e^{0}) = 3(e-1)$$

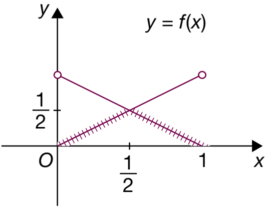
ILLUSTRATION 97

Evaluate: $\int_{-2}^{2} \min(x-[x],\,-x-[-x])\,dx$

SOLUTION

$$f(x) = \min(x-[x],\,-x-[-x])$$

$$f(x) = \min(\{x\},\{-x\})$$



When x is non-integer,
$\{-x\} = 1-\{x\}$

It is a periodic function, with period $T = 1$.

We can observe $f(x)$ when $0 < x < 1$ from diagram

$$\int_{-2}^{2} \min(\{x\},\{-x\})\,dx = 4\int_{0}^{1} \min(\{x\},\{-x\})\,dx$$

$$= 4\left(\frac{1}{2}(1)\frac{1}{2}\right) = 1$$

ILLUSTRATION 98

Evaluate: $\int_{-\pi}^{199\pi} \sqrt{\frac{1+\cos 2x}{2}}\,dx$

SOLUTION

$$I = \int_{-\pi}^{199\pi} \sqrt{\frac{1+\cos 2x}{2}}\,dx = \int_{-\pi}^{199\pi} \sqrt{\frac{1+2\cos^2 x-1}{2}}\,dx$$

$$I = \int_{-\pi}^{199\pi} \sqrt{\cos^2 x}\,dx = \int_{-\pi}^{199\pi} |\cos x|\,dx \qquad \left(\because \sqrt{x^2} = |x|\right)$$

Let $f(x) = |\cos x|$;

$f(\pi + x) = |\cos(\pi + x)| = |-\cos x| = f(x)$, so period $T = \pi$

$$I = \int_{-T}^{199T} |\cos x|\,dx = (199-(-1))\int_{0}^{T=\pi} |\cos x|\,dx$$

$$\left(\text{using } \int_{nT}^{mT} f(x)\,dx = (m-n)\int_{0}^{T} f(x)\,dx\right)$$

$$I = 200\left[\int_{0}^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx\right]$$

$$I = 200\left[\sin x\Big|_0^{\pi/2} + (-\sin x)\Big|_{\pi/2}^{\pi}\right]$$

$$= 200[1-0+(0+1)] = 400$$

ILLUSTRATION 99

Evaluate: $\int_{0}^{2000\pi} \frac{dx}{1+e^{\sin x}}$

SOLUTION

Let $f(x) = \frac{1}{1+e^{\sin x}}$ $\qquad f(2\pi + x) = f(x)$ so period $T = 2\pi$

$$I = \int_{0}^{2000\pi} \frac{dx}{1+e^{\sin x}} = \int_{0}^{1000T} \frac{dx}{1+e^{\sin x}}$$

$$I = 1000\int_{0}^{T=2\pi} \frac{dx}{1+e^{\sin x}}$$

$$I = 1000\left[\int_{0}^{\pi} \frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{\sin(2\pi - x)}}\,dx\right]$$

$$I = 1000\left[\int_{0}^{\pi} \frac{1}{1+e^{\sin x}} + \frac{1}{1+e^{-\sin x}}\,dx\right]$$

$$I = 1000\left[\int_0^{\pi} \frac{1}{1+e^{\sin x}} + \frac{e^{\sin x}}{e^{\sin x}+1}\,dx\right]$$

$$I = 1000\int_0^{\pi} dx \;= 1000\,(x)_0^{\pi} \;= 1000\pi$$

ILLUSTRATION 100

If $\displaystyle\int_0^{50\pi} \frac{x\,|\sin x\,|}{\{x\}+\{-x\}}\,dx = k\pi\ (k \in N)$, where $\{\cdot\}$ is fractional part function of x then find the value of k

SOLUTION

As we know $\{x\} + \{-x\} = 1$ in domain of given function

$$I = \int_0^{50\pi} \frac{x\,|\sin x\,|}{\{x\}+\{-x\}}\,dx = \int_0^{50\pi} x\,|\sin x\,|\,dx$$

(Using Queen Rule)

$$I = \int_0^{25\pi} (x + 50\pi - x)\,|\sin x\,|\,dx = 50\pi\int_0^{25T} |\sin x\,|\,dx$$

$$I = (50\pi)\ 25\int_0^{\pi} \sin x\,dx \;= 1250\pi(-\cos x)_0^{\pi} \;= 2500\pi$$

ILLUSTRATION 101

Evaluate: $\displaystyle\sqrt{\pi\left(\int_0^{2008} x\,|\sin \pi x\,|\,dx\right)}$

SOLUTION

$$I = \int_0^{2008} x\,|\sin \pi x\,|\,dx \qquad \text{(Using Queen Rule)}$$

$$I = \int_0^{1004} x\,|\sin \pi x\,| + (2008 - x)\ |\ \sin(2008\pi - \pi x)\,|\,dx$$

$$I = \int_0^{1004} 2008\,|\sin \pi x\,|\,dx = 2008(1004)\int_0^{1} \sin \pi x\ dx$$

(as period $T = 1$)

$$I = (1004)^2\,2\cdot\frac{(-\cos \pi x)_0^1}{\pi} = \frac{2}{\pi}(1004)^2(2)$$

Now required term = $\displaystyle\sqrt{\pi\left(\frac{4}{\pi}(1004)^2\right)}$ = 2008.

ILLUSTRATION 102

Evaluate: $\displaystyle\int_0^{n\pi+v} |\cos x\,|\,dx$ where $\dfrac{\pi}{2} < v < \pi;\ n \in I$

SOLUTION

$f(x) = |\cos x\,|$ has period $T = \pi$

$$I = \int_0^{n\pi+v} |\cos x\,|\,dx = \int_0^{nT} |\cos x\,|\,dx + \int_{nT}^{nT+v} |\cos x\,|\,dx$$

$$= n\int_0^{T=\pi} |\cos x\,|\,dx + \int_0^{v} |\cos x\,|\,dx$$

$$|\cos x\,| = \begin{cases} +\cos x & ; \quad 0 \le x \le \dfrac{\pi}{2} \\ -\cos x & ; \quad \dfrac{\pi}{2} < x < \pi \end{cases}$$

$$I = n\left(\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} (-\cos x)\,dx\right) + \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{V} (-\cos x)\,dx$$

$$I = n\left(\sin x\big|_0^{\pi/2} + (-\sin x)_{\pi/2}^{\pi}\right) + \sin x\big|_0^{\pi/2} + (-\sin x)_{\pi/2}^{v}$$

$$I = n[1 - 0 + (0 + 1)] + [1 - 0] + (-\sin v + 1)$$

$$I = 2n + 2 - \sin v$$

ILLUSTRATION 103

Let $f(x)$ be an odd periodic function in $\left[-\dfrac{T}{2}, \dfrac{T}{2}\right]$ where T be its period, then show that $g(x) = \displaystyle\int_a^x f(x)\,dx$ will also be periodic function with period T.

SOLUTION

$f(x)$ is odd, periodic

$$f(-x) = -f(x) \text{ and } f(x + T) = f(x)$$

$$f(-x + T) = f(-x) = -f(x);$$

$$I_1 = \int_0^{T} f(x)\,dx$$

using $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx;$

$$I_1 = \int_0^{T} f(T-x)\,dx = \int_0^{T} -f(x)\,dx = -I_1$$

$$2I_1 = 0 \Rightarrow I_1 = \int_0^T f(x)\,dx = 0 \qquad \ldots(1)$$

$$g(x+T) = \int_a^{x+T} f(x)\,dx = \int_a^x f(x)\,dx + \int_x^{x+T} f(x)\,dx$$

$$g(x+T) = g(x) + \int_0^T f(x) \qquad \text{from (1)}$$

$g(x+T) = g(x)$, hence $g(x)$ is periodic.

Hence proved

ILLUSTRATION 104

Let g be a non-negative continuous function such that $g(x) + g\left(x + \frac{5}{2}\right) = 7, \forall\, x \in \left(0, \frac{5}{2}\right)$. Find $4\int_0^5 g(t)\,dt$.

SOLUTION

$$g(x) + g\left(x+\frac{5}{2}\right) = 7 \;\forall\; x \in \left(0, \frac{5}{2}\right) \qquad \ldots(1)$$

Replace x by $\left(x + \frac{5}{2}\right)$

$$g\left(x+\frac{5}{2}\right) + g(x+5) = 7 \qquad \ldots(2)$$

Subtracting (1) – (2), we get

$$g(x) - g(x+5) = 0 \Rightarrow g(x) = g(x+5)$$

$$\int_0^5 g(t)\,dt = \int_0^{5/2} g(t)\,dt + \int_{5/2}^5 g(t)\,dt$$

In second integral Put $t = z + \frac{5}{2}$

$$\int_0^5 g(t)\,dt = \int_0^{5/2} g(t)\,dt + \int_0^{5/2} g\left(z+\frac{5}{2}\right)dz$$

$$\int_0^5 g(t)\,dt = \int_0^{5/2} g(t) + g\left(t+\frac{5}{2}\right)dt = \int_0^{5/2} 7\,dt = \frac{35}{2}$$

$$4\int_0^5 g(t)\,dt = 4.\frac{35}{2} = 70$$

ILLUSTRATION 105

If $f(x)$ be a real valued function,

$$f(x) + f(x+4) = f(x+2) + f(x+6),\; g(x) = \int_x^{x+8} f(t)\,dt,$$

then find $g'(x)$.

SOLUTION

$$f(x) + f(x+4) = f(x+2) + f(x+6) \qquad \ldots(1)$$

Put $x \to x + 2$

$$f(x+2) + f(x+6) = f(x+4) + f(x+8) \qquad \ldots(2)$$

Add equation (1) and (2) $\Rightarrow f(x) = f(x+8)$

$$g(x) = \int_x^{x+8} f(t)\,dt$$

$$\Rightarrow \quad g'(x) = f(x+8) - f(x) \Rightarrow g'(x) = 0$$

ILLUSTRATION 106

If $k \in N$ and $I_k = \int_{-2k\pi}^{2k\pi} |\sin x|[\sin x]\,dx$, (where [.] denotes the greatest integer function) then compute $\sum_{k=1}^{100} I_k$.

SOLUTION

$$I_k = \int_{-2k\pi}^{2k\pi} |\sin x|[\sin x]\,dx$$

$$I_k = \int_0^{2k\pi} |\sin x|([\sin x]+[-\sin x])\,dx$$

$$\because \quad [\sin x]+[-\sin x] = \begin{cases} 0 \;;\; x \in \dfrac{n\pi}{2} \\ -1 \;;\; x \notin \dfrac{n\pi}{2} \end{cases}$$

$$I_k = -\int_0^{2k\pi} |\sin x|\,dx\,,$$

We know that $|\sin x|$ is periodic function π as period

$$I_k = -2k\int_0^{\pi} (\sin x)\,dx = -2k\left(-\cos x\right)_0^{\pi} = 2k(-2) = -4k$$

$$\text{Now } \sum_{k=1}^{100} I_k = -4\sum_{k=1}^{100} k = -4\left(\frac{100-101}{2}\right) = -\,20200.$$

ILLUSTRATION 107

Find the number of values of x satisfying

$\int_0^{2[x+14]} \left\{\frac{x}{2}\right\} dx = \int_0^{\{x\}} [x+14]\,dx$ where [.] and {.} denote the greatest integer and fractional part of x.

SOLUTION

$$\int_0^{2[x+14]} \left\{\frac{x}{2}\right\} dx = \int_0^{\{x\}} [x+14]\,dx$$

$$\Rightarrow \quad \int_0^{2[x+14]} \left\{\frac{x}{2}\right\} dx = 14\int_0^{\{x\}} dx = 14\{x\}$$

For $\left\{\frac{x}{2}\right\}$; Period $T = 2$

$$\Rightarrow \quad [x+14]\int_0^2 \left\{\frac{x}{2}\right\} dx = 14\{x\}$$

$$\left(\because 0<x<2 \Rightarrow 0<\frac{x}{2}<1 \Rightarrow \left\{\frac{x}{2}\right\} = \frac{x}{2}\right)$$

$$\Rightarrow \quad [x+14]\int_0^2 \frac{x}{2}\,dx = 14\{x\} \Rightarrow [x+14]\left.\frac{x^2}{4}\right|_0^2 = 14\{x\}$$

$$\Rightarrow \quad \{x\} = \frac{[x+14]}{14} = \frac{[x]}{14}+1 \qquad \ldots(1)$$

We know that $0 \le \{x\} < 1$

$$0 \le \frac{[x]}{14}+1<1 \Rightarrow -1 \le \frac{[x]}{14} < 0$$

$\Rightarrow \quad -14 \le [x] < 0$

$[x] = -1, -2, -3, \ldots, -14$

From (1)

$$\{x\} = x - [x] = \frac{[x]}{14}+1$$

$\Rightarrow \quad x = [x]\frac{15}{14}+1$ where $[x] = -1, -2, \ldots, -14$

∴ Given equation has 14 solutions.

WALLI'S FORMULA

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \frac{\left[(m-1)(m-3)\ldots 2 \text{ or } 1\right]\left[(n-1)(n-3)\ldots 2 \text{ or } 1\right]}{(m+n)(m+n-2)(m+n-4)\ldots 2 \text{ or } 1}\, p;$$

for m and $n \in W$;

$$\text{where } p = \begin{cases} \frac{\pi}{2} & ; \quad m, n \text{ both even} \\ 1 & ; \quad \text{else} \end{cases}.$$

ILLUSTRATION 108

Evaluate: $\displaystyle\int_0^{\pi/2} \sin^6 x \cos^7 x \, dx$

SOLUTION

$$I = \int_0^{\pi/2} \sin^6 x \cos^7 x \, dx = \int_0^{\pi/2} \sin^6 x (1-\sin^2 x)^3 \cos x \, dx$$

Put $\sin x = t$, $\cos x\, dx = dt$

$$I = \int_0^1 t^6 (1-t^2)^3 \, dt = \int_0^1 t^6(1-t^6-3t^2+3t^4)dt$$

$$I = \int_0^1 t^6 - t^{12} - 3t^8 + 3t^{10} dt = \left(\frac{t^7}{7} - \frac{t^{13}}{13} - \frac{t^9}{3} + \frac{3t^{11}}{11}\right)_{t=0}^{t=1}$$

$$I = \frac{1}{7} - \frac{1}{13} - \frac{1}{3} + \frac{3}{11} = \frac{6}{13.7} - \frac{2}{11.3} = \frac{16}{13.11.7.3}$$

Alternate:

$$I = \int_0^{\pi/2} \sin^6 x \cos^7 x \, dx \qquad \text{(Using Walli's Formula)}$$

$$I = \frac{(5\cdot 3\cdot 1)(6\cdot 4\cdot 2)}{13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1} \times 1 = \frac{16}{13.11.7.3} = \frac{16}{3003}$$

ILLUSTRATION 109

Evaluate: $\displaystyle\int_0^{\pi/2} \cos^4 x \sin^6 x \, dx$

SOLUTION

$$I = \int_0^{\pi/2} \cos^4 x \sin^6 x \, dx = \frac{(3\cdot 1)(5\cdot 3\cdot 1)}{10\cdot 8\cdot 6\cdot 4\cdot 2} \times \left(\frac{\pi}{2}\right) = \frac{3\pi}{512}$$

ILLUSTRATION 110

Evaluate: $\displaystyle\int_0^{\pi} \cos^{10} x \, dx$

SOLUTION

$$I = \int_0^{\pi} \cos^{10} x \, dx = \int_0^{\pi/2} \cos^{10} x + \cos^{10}(\pi - x)\, dx$$

$$I = \int_0^{\pi/2} \cos^{10} x + \cos^{10} x \, dx = 2\int_0^{\pi/2} \cos^{10} x \, dx$$

$$I = 2\left[\frac{9\cdot 7\cdot 5\cdot 3\cdot 1}{10\cdot 8\cdot 6\cdot 4\cdot 2} \times \frac{\pi}{2}\right] = \frac{63\pi}{256}$$

ILLUSTRATION 111

Evaluate: $\displaystyle\int_0^{\pi} \sin^5 x \cos^4 x \, dx$

SOLUTION

$$I = \int_0^{\pi} \sin^5 x \cos^4 x \, dx$$

$$I = \int_0^{\pi/2} \sin^5 x\cos^4 x + \sin^5(\pi - x)\cos^4(\pi - x)\;dx$$

$$I = \int_0^{\pi/2} \sin^5 x\cos^4 x + \sin^5 x\cos^4 x\;dx$$

$$I = 2\int_0^{\pi/2} \sin^5 x\cos^4 x\;dx = 2\left[\frac{(4\cdot 2)(3\cdot 1)}{9\cdot 7\cdot 5\cdot 3\cdot 1}1\right] = \frac{16}{315}$$

ILLUSTRATION 112

Evaluate: $\int_0^{\pi} x\sin^5 x\cos^6 x\;dx$

SOLUTION

$$I = \int_0^{\pi} x\sin^5 x\cos^6 x\;dx$$

$$I = \int_0^{\pi/2} x\sin^5 x\cos^6 x + (\pi - x)\sin^5(\pi - x)\cos^6(\pi - x)\;dx$$

$$I = \int_0^{\pi/2} x\sin^5 x\cos^6 x + (\pi - x)\sin^5 x\cos^6 x\;dx$$

$$= \pi\int_0^{\pi/2} \sin^5 x\cos^6 x\;dx$$

$$I = \pi\left(\frac{(4\cdot 2)(5\cdot 3\cdot 1)}{11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1}\times 1\right) = \frac{8\pi}{693}$$

ILLUSTRATION 113

If $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x\operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$ then

evaluate $\int_0^{\pi/2} f(x)\;dx$

SOLUTION

$$f(x) = \begin{vmatrix} \sec x & \cos x & \operatorname{cosec} x\cdot\cot x + \sec^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

$R_3 \to \dfrac{1}{\cos x}\,R_3$

$$\therefore\quad f(x) = \cos x\begin{vmatrix} \sec x & \cos x & \operatorname{cosec} x\cdot\cot x + \sec^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ \sec x & \cos x & \cos x \end{vmatrix}$$

$R_1 \to R_1 - R_3$

$$f(x) = \cos x\begin{vmatrix} 0 & 0 & \operatorname{cosec} x\cdot\cot x + \sec^2 x - \cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ \sec x & \cos x & \cos x \end{vmatrix}$$

$$= (\operatorname{cosec} x\cdot\cot x + \sec^2 x - \cos x)\cdot(\cos^3 x - \cos x)\cdot\cos x$$

$$= -\left[\frac{\sin^2 x + \cos^3 x - \cos^3 x\cdot\sin^2 x}{\sin^2 x\cdot\cos^2 x}\right]\cdot\cos^2 x\cdot\sin^2 x.$$

$$= -\sin^2 x - \cos^3 x\,(1 - \sin^2 x) = -\sin^2 x - \cos^5 x$$

$$\therefore\quad \int_0^{\pi/2} f(x)\;dx = -\int_0^{\pi/2}(\sin^2 x + \cos^5 x)\;dx$$

(Using Walli's formula)

$$= -\left\{\frac{1}{2}\cdot\frac{\pi}{2} + \frac{4.2}{5.3.1}\right\} = -\left\{\frac{\pi}{4} + \frac{8}{15}\right\} = -\left(\frac{15\pi + 32}{60}\right)$$

ILLUSTRATION 114

Prove that $\int_0^{a} \dfrac{x^n dx}{\sqrt{ax - x^2}} = \dfrac{\pi a^n\;2n!}{4^n(n!)^2}$.

SOLUTION

$$\text{LHS} = \int_0^{a} \frac{x^n dx}{\sqrt{x}\sqrt{a - x}}$$

Put $x = a\sin^2\theta \Rightarrow dx = a2\sin\theta\cos\theta d\theta$

$$= \int_0^{\pi/2} \frac{(a\sin^2\theta)^n}{\sqrt{a}\sin\theta\sqrt{a}\cos^2\theta}2a\sin\theta\cos\theta d\theta$$

$$= 2a^n\int_0^{\pi/2} \sin^{2n}\theta\;d\theta \qquad \text{(Using Walli's formula)}$$

$$= 2a^n\left[\frac{(2n-1)(2n-3)\ldots 1}{2n\cdot(2n-2)(2n-4)\ldots 2}\right]\cdot\frac{\pi}{2}$$

$$= \frac{\pi a^n}{2^n}\;\frac{[2n(2n-1)(2n-2)\ldots 2\cdot 1]}{n!\,[2n\cdot 2n-2\cdot 2n-4\ldots 2]} = \frac{\pi a^n(2n)!}{2^n n!\,2^n n!}$$

$$= \frac{\pi a^n(2n)!}{4^n(n!)^2} = \text{RHS}$$

ILLUSTRATION 115

If $\int_0^{\pi/2} \dfrac{\sin^2(10)\theta}{\sin^2\theta}\,d\theta = k\pi$, where $k \in N$ then find the value of k.

SOLUTION

Let $I_n = \int_0^{\pi/2} \dfrac{\sin^2 n\theta}{\sin^2\theta}\;d\theta;\; I_1 = \int_0^{\pi/2} \dfrac{\sin^2\theta}{\sin^2\theta}\;d\theta = \dfrac{\pi}{2}$

$$I_n - I_{n-1} = \int_0^{\pi/2} \frac{\sin^2 n\theta - \sin^2(n-1)\theta}{\sin^2\theta}\, d\theta$$

$$\Rightarrow \quad I_n - I_{n-1} = \int_0^{\pi/2} \frac{2\sin(2n-1)\theta\sin\theta}{2\sin^2\theta}\, d\theta = \int_0^{\pi/2} \frac{\sin(2n-1)\theta}{\sin\theta}\, d\theta$$

$$\Rightarrow \quad I_n - I_{n-1} = J_{2n-1} \qquad \ldots(1)$$

Now $J_{2n-1} - J_{2n-3} = \displaystyle\int_0^{\pi/2} \frac{\sin(2n-1)\theta - \sin(2n-3)\theta}{\sin\theta}\, d\theta$

$$= \int_0^{\pi/2} \frac{2\cos(2n-2)\theta\cdot\sin\theta}{\sin\theta}\, d\theta = 2\left.\frac{\sin(2n-2)\theta}{2n-2}\right|_0^{\pi/2} = 0$$

$$\therefore \quad J_{2n-1} = J_{2n-3} = \int_0^{\pi/2} \frac{\sin(2n-1)\theta}{\sin\theta}\, d\theta$$

$$J_1 = \frac{\pi}{2};$$

$$J_3 = \int_0^{\pi/2} \frac{\sin 3\theta}{\sin\theta}\, d\theta = \int_0^{\pi/2} (3 - 4\sin^2\theta)\, d\theta$$

$$= \int_0^{\pi/2} (3 - 2(1-\cos 2)\theta)\, d\theta = \int_0^{\pi/2} (1 + 2\cos 2\theta)\, d\theta$$

$$J_3 = \left(\frac{\pi}{2} + (\sin 2\theta)_0^{\pi/2}\right) = \frac{\pi}{2}$$

$$J_1 = J_3 = J_5 = J_7 = J_9 = J_{11} = \ldots = J_{\text{odd}} = \frac{\pi}{2}$$

From (1) $I_n - I_{n-1} = J_{2n-1}$

$n = 10$ $\quad I_{10} - I_9 = J_{19}$

$n = 9$ $\quad I_9 - I_8 = J_{17}$

$n = 8$ $\quad I_8 - I_7 = J_{15}$

……………………………

$n = 3$ $\quad I_3 - I_2 = J_5$

$n = 2$ $\quad I_2 - I_1 = J_3$

Adding all above, we obtain

$$I_{10} - I_1 = 9J_3 = \frac{9\pi}{2}$$

$$I_{10} - \frac{\pi}{2} = \frac{9\pi}{2} \Rightarrow I_{10} = 5\pi$$

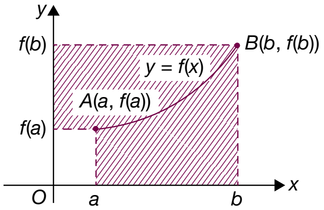
INTEGRATION INVOLVING *F*(*x*) AND *F*$^{-1}$(*x*)

Let *f*(*x*) is increasing and bijective continuous function $f : [a, b] \to [c, d]$ then

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a)$$

PROOF:

Let *f*(*x*) is an increasing function for $x \in [a, b]$.



$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = \text{Shaded area}$$

= Larger rectangle – Smaller rectangle

= $bf(b) - af(a)$

Alternate:

As it is increasing and bijective $f(a) = c, f(b) = d$

$$\int_c^d f^{-1}(x)dx = \int_{f(a)}^{f(b)} f^{-1}(x)dx$$

Put $f^{-1}(x) = t;\ x = f(t);\ dx = f'(t)dt$

$$= \int_a^b tf'(t)dt = tf(t)\Big|_a^b - \int_a^b f(t)dt$$

$$\int_a^b f(t)dt + \int_c^d f^{-1}(x)dx = bf(b) - af(a)$$

ILLUSTRATION 116

If $f(x) = x + \sin x$, then $\displaystyle\int_\pi^{2\pi} f^{-1}(x)dx$ is equal to:

SOLUTION

Put $x = f(t) \Rightarrow dx = f'(t)dt$

$$\int_\pi^{2\pi} f^{-1}(x)dx = \int_\pi^{2\pi} t.f'(t)dt$$

$$(\because f(\pi) = \pi \text{ and } f(2\pi) = 2\pi)$$

$$= \left[t.f(t)\right]_{\pi}^{2\pi} - \int_{\pi}^{2\pi} 1.f(t)\,dt$$

$$= 2\pi f(2\pi) - \pi f(\pi) - \left[\frac{t^2}{2} - \cos t\right]_{\pi}^{2\pi}$$

$$= 4\pi^2 - \pi^2 - \frac{1}{2}\left(4\pi^2 - \pi^2\right) + \left(\cos 2\pi - \cos\pi\right)$$

$$\int_{\pi}^{2\pi} f^{-1}(x)\,dx = \frac{3\pi^2}{2} + 2$$

ILLUSTRATION 117

Let 'f' be a one-to-one continuous function such that $f(2)=3$ and $f(5)=7$. Given $\int_2^5 f(x)\,dx = 17$, then compute $\int_3^7 f^{-1}(x)\,dx$.

SOLUTION

$$y = f(x) \Rightarrow x = f^{-1}(y) \text{ and } dy = f'(x)dx$$

$$I = \int_3^7 f^{-1}(x)\,dx = \int_3^7 f^{-1}(y)\,dy = \int_2^5 xf'(x)\,dx$$

$f(2) = 3, f(5) = 7$

$$I = \left[xf(x)\right]_2^5 - \int_2^5 f(x)\,dx = 5.7 - 2.3 - 17 = 35 - 6 - 17 = 12$$

ILLUSTRATION 118

Evaluate: $\int_0^1 e^{\sqrt{e^x}}\,dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)\,dx$

SOLUTION

$f(x)$ is increasing and bijective continuous function $f:[a,b] \to [c,d]$ then

$$\int_a^b f(x)dx + \int_{f(a)}^{f(b)} f^{-1}(x)dx = bf(b) - af(a) \qquad \ldots(1)$$

$$f(x) = e^{\sqrt{e^x}}\ ; f^{-1}(x) = \ln(\ln x)^2 = 2\ln(\ln x)$$

$$a = 0, b = 1, f(0) = e, f(1) = e^{\sqrt{e}}$$

From (1), $\int_0^1 e^{\sqrt{e^x}}\,dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)\,dx = 1.\ e^{\sqrt{e}} - 0.e = e^{\sqrt{e}}$

Alternate:

$$I = \int_0^1 e^{\sqrt{e^x}}\,dx + 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)\,dx$$

Put $e^x = u^2 \Rightarrow x = 2\ln u \Rightarrow dx = \frac{2}{u}du$

$$\Rightarrow \quad I_1 = \int_0^1 e^{\sqrt{e^x}}\,dx = 2\int_1^{\sqrt{e}} \frac{e^u}{u}\,du$$

Put $\log x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$$\Rightarrow \quad I_2 = 2\int_e^{e^{\sqrt{e}}} \ln(\ln x)dx = 2\int_1^{\sqrt{e}} e^t \log t\,dt = 2\int_1^{\sqrt{e}} e^u \log u\,du$$

$$I = I_1 + I_2 = 2\int_1^{\sqrt{e}} e^u\left(\frac{1}{u} + \log u\right)du$$

$$= 2\left(e^u \ln u\right)_1^{\sqrt{e}} = 2e^{\sqrt{e}}\log\sqrt{e} = e^{\sqrt{e}}$$

ILLUSTRATION 119

If $f(x) = x^3 + 3x + 4$, then find the value of $\int_{-1}^{1} f(x)\,dx + \int_0^4 f^{-1}(x)\,dx$.

SOLUTION

In second integral

Put $x = f(t) \Rightarrow dx = f'(t)dt = (3t^2 + 3)dt$

$$0 = f(t) = t^3 + 3t + 4 \Rightarrow t = -1$$

$$4 = f(t) = t^3 + 3t + 4 \Rightarrow t = 0$$

$$\int_{-1}^{1} f(x)\,dx + \int_0^4 f^{-1}(x)\,dx = 2\int_0^1 4\ dx + \int_{-1}^{0} t\cdot(3t^2 + 3)\,dt$$

$$= 8 + \left.\frac{3t^4}{4} + \frac{3t^2}{2}\right|_{-1}^{0} = 8 - \frac{9}{4} = \frac{23}{4}$$

DEFINITE INTEGRATION WITH TWO VARIABLES

ILLUSTRATION 120

Evaluate: $F(x) = \int_0^x x^2 t\,dt + \int_{x^2}^{\cos x} \operatorname{cosec} xt^2\ dt$

SOLUTION

$$f(x) = \int_0^x x^2 t\,dt = \int_0^x x^2\alpha\,d\alpha = \int_0^x x^2 p\,dp \neq \int_0^x x^2 x\,dx = = \frac{x^4}{4}$$

$$f(x) = x^2\int_0^x p\ dp = x^2\left(\frac{p^2}{2}\right)_0^x = \frac{x^4}{2}$$

$$g(x) = \int_{x^2}^{\cos x} \operatorname{cosec} xt^2\ dt = \operatorname{cosec} x\int_{x^2}^{\cos x} t^2\ dt$$

$$g(x) = \text{cosec}\, x \left(\frac{t^3}{3}\right)_{x^2}^{\cos x} = \frac{\text{cosec}\, x}{3}[\cos^3 x - x^6]$$

Now, $F(x) = f(x) + g(x) = \dfrac{x^4}{2} + \dfrac{\text{cosec}\, x}{3}[\cos^3 x - x^6]$

ILLUSTRATION 121

Evaluate: $F(x) = \displaystyle\int_0^2 \sin x t^2\, dt + \int_0^{\sin x} \ln x t^2\, dt$

SOLUTION

$$F(x) = \sin x \int_0^2 t^2\, dt + \ln x \int_0^{\sin x} t^2\, dt$$

$$F(x) = \sin x \left(\frac{t^3}{3}\right)_0^2 + \ln x \left(\frac{t^3}{3}\right)_0^{\sin x}$$

$$= \frac{8 \sin x}{3} + \frac{\sin^3 x \ln x}{3}$$

ILLUSTRATION 122

Prove that $\displaystyle\int_0^x \left(\int_0^u f(x)dx\right) du = \int_0^x f(u)(x-u)\, du.$

SOLUTION

$$\int_0^x \left(\int_0^u f(x)dx\right) du = \int_0^x 1.\left(\int_0^u f(t)dt\right) du$$

Using integration by parts, we get

$$= \left[u\int_0^u f(t)dt\right]_0^x - \int_0^x uf(u)du = x\int_0^x f(t)\, dt - \int_0^x uf(u)du$$

$$= x\int_0^x f(u)\, du - \int_0^x uf(u)du = \int_0^x f(u)(x-u)\, du = \text{RHS}.$$

Hence proved.

Test Your Concepts-IV

(Solutions on page H.274)

1. $\int_0^1 xt^2\, dt$ is equal to:

(i) $\int_0^1 t^3\, dt$ (ii) $x\int_0^1 t^2\, dt$

(iii) $\int_0^1 x^3\, dx$ (iv) $x\int_0^1 x^2\, dx$

2. $\int_0^x xt\, dt$ equal to:

(i) $\int_0^u xu\, du$ (ii) $x\int_0^x t\, dt$

(iii) $x\int_0^x x\, dx$ (iv) $\int_0^x xu\, du$

3. Mark following as True or False.

(i) $\int_0^x \sin x\, dx = 0$ (ii) $\int_0^x \sin(t-x)\, dx = \int_0^x \sin t\, dx$

(iii) $\int_0^x xt\, dx = 0$ (iv) $\int_0^x xt\, dt = \int_0^x x(x-t)\, dt$

(v) $\int_0^x xt\, dx = \int_0^x t(x-u)\, du$

(vi) $\int_0^x \sin(x-t)\, dt = \int_0^x \sin t\, dt$

4. Evaluate: $\displaystyle\int_{-20\pi}^{20\pi} |\cos x|dx$

5. Evaluate: $\displaystyle\int_0^{50\pi} \sqrt{(1-\cos 2x)}\, dx$

6. Evaluate: $\displaystyle\int_0^{\pi/4} (\cos 2x)^{3/2}.\cos x\, dx$

7. Evaluate $\displaystyle\int_{19}^{37} (\{x\}^2 + 3(\sin 2\pi x))\, dx$ where $\{x\}$ denotes the fractional part function.

8. Evaluate: $\displaystyle\int_{-\pi}^{\pi} \left(x^3 + 3x + 2\right) \sin^4 x \cos^2 x\, dx$

9. Evaluate: $\displaystyle\int_0^a \frac{x^4 dx}{\sqrt{a^2 - x^2}}$

10. If $g(x) = \displaystyle\int_0^x \cos^4 t\, dt$, then show that $g(x+\pi) = g(x) + g(\pi)$.

11. Show that $\displaystyle\int_0^{n\pi+v} |\sin x|\, dx = 2n + 1 - \cos v$ where n is a positive integer and $0 \le v < \pi$.

12. If $f(x+f(y)) = f(x) + y \forall x, y \in R$ and $f(0) = 1$, then show that $\displaystyle\int_0^{10} f(10-x)dx = 10\int_0^1 f(x)dx$.

13. If $f(4-x)=f(4+x)$ and $f(8-x)=f(x+8)$ and $f(x)$ is a function for which $\int_0^8 f(x)dx = 5$, then find $\int_0^{200} f(x)dx$.

14. If the function $\int_0^x f(t)dt \to 5$ as $|x| \to 1$, then find the value of 'a' so that the equation $2x+\int_0^x f(t)dt = a$ has at least two roots of opposite signs in (−1, 1).

15. Find the maximum value of $f(x) = \int_0^1 t\sin(x+\pi t)\,dt$.

16. Let $f(x)$ be *a* continuous and periodic function such that $f(x) = f(x+T)$ for all $x \in R$, $T > 0$.

If $\int_{-2T}^{a+5T} f(x)\,dx = 19$ $(a > 0)$ and $\int_0^T f(x)\,dx = 2$,

then find $\int_0^a f(x)\,dx$.

17. Evaluate $\int_0^1 (e-1)\sqrt{\ln(1+(e-1)x)}\,dx + \int_0^1 e^{x^2}\,dx$.

Answer key

1. (ii), (iv) **2.** (ii), (iii), (iv) **3.** (i) F (ii) F (iii) F (iv) T (v) T (vi) T **4.** 80

5. $100\sqrt{2}$ **6.** $\frac{3\pi}{16\sqrt{2}}$ **7.** 6 **8.** $\frac{\pi}{4}$ **9.** $\frac{3\pi}{16}a^4$ **13.** 125 **14.** $a \in (0,3)$

15. $\frac{1}{\pi^2}\sqrt{\pi^2+4}$ **16.** 5 **17.** e

NEWTON–LEIBNITZ RULE

Let $f(x) = \int_{\psi(x)}^{\phi(x)} g(t)\,dt$, on differentiation with respect to x

$$\frac{df(x)}{dx} = g(\phi(x))\frac{d\phi(x)}{dx} - g(\psi(x))\frac{d\psi(x)}{dx}$$

ILLUSTRATION 123

Let $f(x) = \int_{x^3}^{x^2} \ln\sin t\,dt$, find $f'(x)$

SOLUTION

$$f(x) = \int_{x^3}^{x^2} \ln\sin t\,dt, \text{ on differentiation}$$

$$f'(x) = \ln\sin(x^2)\,(2x) - \ln\sin(x^3)\,(3x^2)$$

$$f'(x) = 2x\ln\sin x^2 - 3x^2\ln\sin x^3$$

ILLUSTRATION 124

Let $f(x) = \int_x^{x^2} \sqrt{\sin t}\,dt - \int_2^4 (\ln t)^{3/2}\,dt$ find $f'(x)$

SOLUTION

$$f'(x) = \sqrt{\sin x^2}\,(2x) - \sqrt{\sin x}\,(1) - (0-0)$$

$$f'(x) = 2x\sqrt{\sin x^2} - \sqrt{\sin x}$$

ILLUSTRATION 125

Evaluate: $\lim_{x\to 0}\int_0^{x^2}\frac{\cos t^2\,dt}{x\sin x}$

SOLUTION

$$\lim_{x\to 0}\int_0^{x^2}\frac{\cos t^2\,dt}{x\sin x} = \lim_{x\to 0}\frac{\int_0^{x^2}\cos t^2\,dt}{x\sin x}$$

It is $\left(\frac{0}{0}\right)$ form, using L'Hospital Rule, we obtain

$$= \lim_{x\to 0}\frac{\cos x^4\cdot(2x)-0}{(\sin x + x\cos x)}$$

$$= \lim_{x\to 0}\frac{2\cos x^4}{\frac{\sin x}{x}+\cos x} = \frac{2(1)}{1+1} = 1$$

ILLUSTRATION 126

Evaluate: $\lim\limits_{x\to 0}\dfrac{xe^{x^2}}{\int\limits_0^x e^{t^2}\,dt}$

SOLUTION

It is $\left(\dfrac{0}{0}\text{ form}\right)$, using L'Hospital Rule, we obtain

Given limit $= \lim\limits_{x\to 0}\dfrac{e^{x^2}+xe^{x^2}(2x)}{e^{x^2}(1)-0} = \lim\limits_{x\to 0}\dfrac{1+2x^2}{1} = \dfrac{1+0}{1} = 1$

ILLUSTRATION 127

Evaluate: $\lim\limits_{x\to 0}\int\limits_0^{x^2}\dfrac{\sec^2 t\,dt}{x\sin x}$

SOLUTION

$$\lim_{x\to 0}\int_0^{x^2}\frac{\sec^2 t\,dt}{x\sin x} = \lim_{x\to 0}\frac{\int\limits_0^{x^2}\sec^2 t\,dt}{x\sin x}$$

It is $\dfrac{0}{0}$ form, so we can use L'Hospital Rule

$$= \lim_{x\to 0}\frac{\sec^2 x^2(2x)-0}{1\sin x + x\cos x}$$

$$= \lim_{x\to 0}\frac{2\sec^2 x^2}{\dfrac{\sin x}{x}+\cos x} = \frac{2(1)}{1+1} = 1$$

Alternate:

$$\lim_{x\to 0}\frac{\int\limits_0^{x^2}\sec^2 t\,dt}{x\sin x} = \lim_{x\to 0}\frac{\tan t\big|_0^{x^2}}{x\sin x}$$

$$= \lim_{x\to 0}\frac{\tan x^2}{x\sin x} = \lim_{x\to 0}\left(\frac{\tan x^2}{x^2}\right)\frac{1}{\left(\dfrac{\sin x}{x}\right)} = 1$$

ILLUSTRATION 128

Evaluate: $\lim\limits_{x\to 0}\dfrac{1}{x^3}\int\limits_0^x\dfrac{t\ln(1+t)}{t^4+4}\,dt$

SOLUTION

$$\lim_{x\to 0}\frac{1}{x^3}\int_0^x\frac{t\ln(1+t)}{t^4+4}\,dt = \lim_{x\to 0}\left(\frac{\int\limits_0^x\dfrac{t\ln(1+t)}{t^4+4}dt}{x^3}\right)$$

It is $\left(\dfrac{0}{0}\text{ form}\right)$, using L'Hospital Rule, we get

$$= \lim_{x\to 0}\frac{\dfrac{x\ln(1+x)}{x^4+4}(1)-0}{3x^2}$$

$$= \lim_{x\to 0}\left(\frac{\ln(1+x)}{x}\right)\frac{1}{3(x^4+4)}$$

$$= 1\left(\frac{1}{3}\right)\left(\frac{1}{0+4}\right) = \frac{1}{12}$$

ILLUSTRATION 129

Evaluate: $\lim\limits_{x\to 0}\dfrac{\dfrac{d}{dx}\int\limits_0^{x^3}\sqrt{\cos t}\,dt}{1-\sqrt{\cos x}}$

SOLUTION

$$\lim_{x\to 0}\frac{\dfrac{d}{dx}\int\limits_0^{x^3}\sqrt{\cos t}\,dt}{1-\sqrt{\cos x}} = \lim_{x\to 0}\frac{\left(\sqrt{\cos x^3}\cdot 3x^2-0\right)}{\left(1-\sqrt{\cos x}\right)}$$

Using Leibnitz Rule, we get

$$= \lim_{x\to 0}\frac{\left(\sqrt{\cos x^3}.3x^2\right)\left(1+\sqrt{\cos x}\right)}{\left(1-\sqrt{\cos x}\right)\left(1+\sqrt{\cos x}\right)} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\frac{3x^2\sqrt{\cos x^3}\left(1+\sqrt{\cos x}\right)}{1-\cos x}$$

$$= \lim_{x\to 0}\left(\frac{x^2}{1-\cos x}\right)3(1)\,(1+1)$$

It is $\dfrac{0}{0}$ form, using L'Hospital Rule, we obtain

$$= 6\lim_{x\to 0}\left(\frac{2x}{0+\sin x}\right) = 6(2) = 12$$

ILLUSTRATION 130

If $x = \int\limits_{c^2}^{\sin t}\text{arc}\sin z\,dz$, and $y = \int\limits_n^{\sqrt{t}}\dfrac{\sin z^2}{z}\,dz$,

then compute $\dfrac{dy}{dx}$.

SOLUTION

$$x = \int_{c^2}^{\sin t}\text{arc}\sin z\;dz$$

On differentiation, $\dfrac{dx}{dt} = \sin^{-1}(\sin t)\cos t - 0 = t\cos t$

$$y=\int_{n}^{\sqrt{t}}\frac{\sin z^2}{z}\,dz,$$

On differentiation, $\dfrac{dy}{dt}=\dfrac{\sin t}{\sqrt{t}}\cdot\dfrac{1}{2\sqrt{t}}-0=\dfrac{\sin t}{2t}$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{\sin t}{2t\cdot t\cos t}=\frac{\tan t}{2t^2}$$

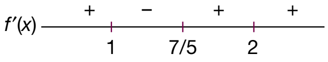
ILLUSTRATION 131

Let $f(x)=\displaystyle\int_{1}^{x}(2(t-1)(t-2)^3+3(t-1)^2(t-2)^2)dx,$

then find extrema

SOLUTION

$$f'(x)=2(x-1)\,(x-2)^3+3(x-1)^2\,(x-2)^2$$
$$=(x-1)\,(x-2)^2\,[2(x-2)+3(x-1)]$$
$$=(x-1)\,(x-2)^2\,[5x-7]$$



At $x=1$ local maxima, at $x=\dfrac{7}{5}$ local minima, at $x=2$ no extrema

ILLUSTRATION 132

Evaluate: $\displaystyle\lim_{x\to\infty}\frac{d}{dx}\left(\int_{\sqrt{3}}^{\sqrt{x}}\frac{r^3dr}{(r^2-1)}\right)$

SOLUTION

$$\lim_{x\to\infty}\frac{d}{dx}\left(\int_{\sqrt{3}}^{\sqrt{x}}\frac{r^3dr}{(r^2-1)}\right)\qquad\text{(using Leibnitz Rule)}$$

$$=\lim_{x\to\infty}\left(\frac{\left(\sqrt{x}\right)^3\left(\frac{1}{2\sqrt{x}}\right)}{\left(\sqrt{x}\right)^2-1}-0\right)=\lim_{x\to\infty}\frac{x}{2(x-1)}$$

$$=\lim_{x\to\infty}\frac{1}{2\left(1-\frac{1}{x}\right)}=\frac{1}{2(1-0)}=\frac{1}{2}$$

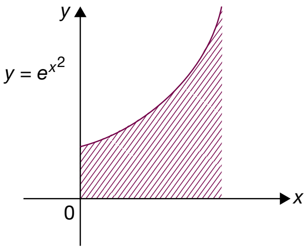
ILLUSTRATION 133

Evaluate: $\displaystyle\lim_{x\to\infty}\frac{\left(\int_0^x e^{t^2}dt\right)^2}{\int_0^x e^{2t^2}dt}$

SOLUTION

As $x\to\infty$; $\displaystyle\int_0^x e^{t^2}dt\to\infty$

As $x\to\infty$; $\displaystyle\int_0^x e^{2t^2}dt\to\infty$



Given limit is $\left(\dfrac{\infty}{\infty}\text{ form}\right)$, using L-Hospital Rule

$$=\lim_{x\to\infty}\frac{2\int_0^x e^{t^2}dt[e^{x^2}(1)-0]}{e^{2x^2}(1)-0}=\lim_{x\to\infty}\frac{2\int_0^x e^{t^2}dt}{e^{x^2}}$$

It is still $\left(\dfrac{\infty}{\infty}\text{ form}\right)$, using L-Hospital Rule again

$$=\lim_{x\to\infty}\frac{2\left(e^{x^2}\cdot 1-0\right)}{e^{x^2}(2x)}=\lim_{x\to\infty}\frac{1}{x}=0$$

ILLUSTRATION 134

Evaluate $f(x)=\sin x+\displaystyle\int_0^x f'(t)(2\sin t-\sin^2 t)\,dt$, then find $f(x)$.

SOLUTION

$$f(x)=\sin x+\int_0^x f'(t)(2\sin t-\sin^2 t)\,dt$$
$$f(0)=0$$
$$f'(x)=\cos x+f'(x)\,(2\sin x-\sin^2 x)$$
$$f'(x)\,(1-2\sin x+\sin^2 x)=\cos x\Rightarrow f'(x)=\frac{\cos x}{(1-\sin x)^2}$$
$$f(x)=\int\frac{\cos x}{(1-\sin x)^2}dx;\ f(x)=\frac{1}{(1-\sin x)}+c$$
$$f(0)=0=1+c\Rightarrow c=-1$$
$$f(x)=\frac{1}{1-\sin x}-1=\frac{1-1+\sin x}{1-\sin x};\ f(x)=\frac{\sin x}{1-\sin x}$$

ILLUSTRATION 135

Let $y=\displaystyle\int_0^x 2|x-1|\,dx$. Find tangents to the curve $y=f(x)$ which are parallel to $y=2x$.

SOLUTION

$$\frac{dy}{dx}=2|x-1|=2$$

$$\Rightarrow\quad |x-1|=1\Rightarrow x-1=\pm1\Rightarrow x=0,\ x=2$$

When $x = 0$, $y = 0$ equation of tangent $y = 2x$

When $x = 2$, $y = \int_0^2 2|x-1|\ dx = 2\left(\int_0^1 (1-x)dx + \int_1^2 (x-1)dx\right)$

$$= 2\left(\left(x-\frac{x^2}{2}\right)_0^1 + \left(\frac{x^2}{2}-x\right)_1^2\right)$$

$$= 2\left(\left(1-\frac{1}{2}\right)+\left(\frac{3}{2}-1\right)\right) = 2$$

At point (2, 2), slope of tangent $= 2$

Required tangent $y - 2 = 2(x - 2)$ or $y = 2x - 2$

ILLUSTRATION 136

Let $f(x) = \int_0^x e^t \sin(x-t)\,dt$. Find range of $g(x) = f''(x) - f(x)$.

SOLUTION

$$f(x) = \int_0^x e^t \sin(x-t)\,dt$$

Replacing x by $x + 0 - t$, (Using King Rule)

$$f(x) = \int_0^x e^{x-t} \sin(x-(x-t))\,dt$$

$$f(x) = \int_0^x e^{x-t} \sin t\,dt = e^x \int_0^x e^{-t} \sin t\,dt$$

$$f'(x) = e^x \int_0^x e^{-t} \sin t\,dt + e^x(e^{-x}\sin x - 0)$$

$$f'(x) = \sin x + e^x \int_0^x e^{-t} \sin t\,dt$$

$$f''(x) = \cos x + e^x \int_0^x e^{-t} \sin t\,dt + e^x(e^{-x}\sin x)$$

$$f''(x) = \sin x + \cos x + e^x \int_0^x e^{-t} \sin t\,dt$$

$$g(x) = f''(x) - f(x) = \sin x + \cos x$$

$\because \quad -\sqrt{a^2+b^2} \le a\sin x + b\cos x \le \sqrt{a^2+b^2}$

Range of $g(x)$ $\left[-\sqrt{2}, \sqrt{2}\right]$

ILLUSTRATION 137

Prove that $f(x) = \int_0^x \ln\left(\frac{1-t}{1+t}\right) dt$ is an even function.

SOLUTION

Given $f(x) = \int_0^x \ln\left(\frac{1-t}{1+t}\right) dt$

$$f(-x) = \int_0^{-x} \ln\left(\frac{1-t}{1+t}\right) dt$$

Put $t = -\alpha$, $dt = -d\alpha$

$t = -x = -\alpha$, we get $\alpha = x$ (upper limit)

$t = 0$; $\alpha = 0$ (lower limit)

$$f(-x) = \int_0^x \ln\left(\frac{1+\alpha}{1-\alpha}\right)(-d\alpha) \qquad \left[\because \ln\frac{b}{a} = -\ln\frac{a}{b}\right]$$

$$= -\int_0^x \left(-\ln\left(\frac{1-\alpha}{1+\alpha}\right)\right) d\alpha = \int_0^x \ln\left(\frac{1-\alpha}{1+\alpha}\right) d\alpha = f(x)$$

$$f(-x) = f(x)$$

$f(x)$ is an even function
Hence Proved

Alternate:

$$f(x) = \int_0^x \ln\left(\frac{1-t}{1+t}\right) dt$$

Put $x = 0$, we get $f(0) = 0$

On differentiation

$$f'(x) = \ln\left(\frac{1-x}{1+x}\right)(1) = 0;$$

$$f'(-x) = \ln\left(\frac{1+x}{1-x}\right) = -f'(-x)$$

$$f(x) = f(-x) + C$$

At $x = 0$

$$f(0) = f(0) + C \Rightarrow C = 0$$

We get $f(x) = f(-x)$, $f(x)$ is an even function.

ILLUSTRATION 138

Let $f(x) = \int_{-1}^x e^{t^2}\,dt$ and $h(x) = f(1+g(x))$ where $g(x)$ is defined $\forall\ x \in R$, $g'(x)$ exists $\forall\ x \in R$ and $g(x) < 0\ \forall\ x > 0$. If $h'(1) = e$ and $g'(1) = 1$, then compute $g(1)$.

SOLUTION

$$h(x) = f(1+g(x)) = \int_{-1}^{1+g(x)} e^{t^2}\,dt$$

$$h'(x) = e^{(1+g(x))^2}\, g'(x)$$

$$h'(1) = e^{(1+g(1))^2}\, g'(1)$$

∴ $(1+g(1))^2 = 1 \Rightarrow g(1) = 0$ or –2.

As given $g(x) < 0\ \forall\ x > 0$, we have $g(1) = -2$

ILLUSTRATION 139

If $f'(x) = f(x) + \int_0^1 f(x)\,dx$ and $f(0) = 1$ find $f(x)$

SOLUTION

Given $f'(x) = f(x) + \int_0^1 f(x)\,dx$

Let $y = f(x)$, $\int_0^1 f(x)\,dx = \text{constant} = \alpha$ (let)

$$\frac{dy}{dx} = y + \alpha$$

$$\Rightarrow \int \frac{dy}{y+\alpha} = \int dx \Rightarrow \ln(y+\alpha) = x + c \Rightarrow y + \alpha = e^{x+c}$$

$$\Rightarrow f(x) + \alpha = e^x \cdot e^c \qquad \ldots(1)$$

Given, $f(0) = 1$

$$1 + \alpha = e^{o} \cdot e^c \Rightarrow e^c = 1 + \alpha$$

From (1) we get:

$$f(x) + \alpha = e^x e^c = e^x(1+\alpha)$$

$$\Rightarrow f(x) = e^x(1+\alpha) - \alpha \qquad \ldots(2)$$

where, $\alpha = \int_0^1 f(x)\,dx = \int_0^1 e^x(1+\alpha) - \alpha\,dx = (1+\alpha)e^x - x\alpha\Big|_0^1$

$$\alpha = (1+\alpha)(e-1) - \alpha(1-0)$$

$$\Rightarrow \alpha(3-e) = e - 1 \Rightarrow \alpha = \frac{e-1}{3-e}$$

$$f(x) = e^x(1+\alpha) - \alpha = e^x\left[1 + \frac{e-1}{3-e}\right] - \left(\frac{e-1}{3-e}\right)$$

$$f(x) = \frac{2e^x}{3-e} + \frac{1-e}{3-e}$$

ILLUSTRATION 140

Let function $f(x)$ satisfies the equation

$f(x) + 3\int_{-1}^{1} (xy - x^2y^2) f(y)\,dy = x^3$, then find the value of $f(5)$.

SOLUTION

$$f(x) + 3x\int_{-1}^{1} yf(y)\,dy - 3x^2\int_{-1}^{1} y^2 f(y) = x^3$$

Let $\alpha = \int_{-1}^{1} yf(y)dy$, $\beta = \int_{-1}^{1} y^2 f(y)dy$

$$f(x) + 3x\alpha - 3x^2\beta = x^3$$

$$f(x) = x^3 + 3x^2\beta - 3x\alpha$$

$$\alpha = \int_{-1}^{1} yf(y)\,dy = \int_{-1}^{1} y^4 + 3y^3\beta - 3y^2\alpha\,dy$$

$$\alpha = \frac{y^5}{5} - \frac{3y^3}{3}\alpha\Big|_{-1}^{1} = \left(\frac{1}{5} - \alpha\right)2$$

$$5\alpha = 2 - 10\alpha \Rightarrow \alpha = \frac{2}{15}$$

$$\beta = \int_{-1}^{1} y^2 f(y)\,dy = \int_{-1}^{1} y^5 + 3y^4\beta - 3y^3\alpha\,dy$$

$$\beta = 2\,\frac{3y^5}{5}\beta\Big|_0^1 = \frac{6}{5}\beta \Rightarrow \beta = 0$$

$f(x) = x^3 + 0 - \frac{2x}{5}$; we have $f(5) = 125 - 2 = 123$

ILLUSTRATION 141

If $f(x) = x + \int_0^1 (xy^2 + yx^2) f(y)\,dy$, then find $f(x)$.

SOLUTION

$$f(x) = x + \int_0^1 (xy^2 + yx^2) f(y)\,dy$$

$$f(x) = x + \int_0^1 xy^2 f(y)\,dy + \int_0^1 yx^2 f(y)\,dy$$

$$f(x) = x + x\int_0^1 y^2 f(y)\,dy + x^2\int_0^1 yf(y)\,dy$$

$$f(x) = x + \alpha x + \beta x^2 = (1+\alpha)x + \beta x^2$$

where $\alpha = \int_0^1 y^2 f(y)\,dy = \int_0^1 y^2[(1+\alpha)y + \beta y^2]\,dy$

$$= (1+\alpha)\int_0^1 y^3\,dy + \beta\int_0^1 y^4\,dy$$

$$\alpha = (1+\alpha)\frac{y^4}{4}\Big|_0^1 + \beta\,\frac{y^5}{5}\Big|_0^1$$

$$\alpha = \frac{(1+\alpha)}{4} + \frac{\beta}{5} \Rightarrow 20\alpha = 5 + 5\alpha + 4\beta$$

$$\Rightarrow \quad 15\alpha = 5 + 4\beta \Rightarrow \alpha = \left(\frac{5+4\beta}{15}\right) \qquad \ldots(1)$$

$$\beta = \int_0^1 y f(y)\, dy = \int_0^1 y\left[(1+\alpha)y + \beta y^2\right] dy$$

$$\beta = (1+\alpha)\int_0^1 y^2\, dy + \beta\int_0^1 y^3\, dy$$

$$\beta = (1+\alpha)\left.\frac{y^3}{3}\right|_0^1 + \beta\left.\frac{y^4}{4}\right|_0^1 = \frac{1+\alpha}{3} + \frac{\beta}{4}$$

$$\Rightarrow \quad 12\beta = 4 + 4\alpha + 3\beta \Rightarrow 9\beta = 4 + 4\alpha \qquad \ldots(2)$$

$$9\beta = 4 + 4\alpha = 4 + 4\left(\frac{5+4\beta}{15}\right) \qquad \text{from (1)}$$

$$\Rightarrow \quad 9\beta \times 15 = 80 + 16\beta$$

$$\Rightarrow \quad \beta(135 - 16) = 80 \Rightarrow \beta = \frac{80}{119}$$

$$4(\alpha + 1) = 9\beta = \frac{9\times 80}{119} \Rightarrow \alpha + 1 = \frac{180}{119}$$

$$f(x) = (1+\alpha)x + \beta x^2$$

$$f(x) = \frac{180x}{119} + \frac{80}{119}x^2$$

EXTENDED NEWTON-LEIBNITZ THEOREM

$$f(x) = \int_{\psi(x)}^{\phi(x)} g(x,t)\, dt$$

$$\frac{df(x)}{dx} = g(x, \phi(x))\frac{d\phi(x)}{dx} - g(x,\psi(x))\frac{d\psi(x)}{dx} + \int_{\psi(x)}^{\phi(x)}\left(\frac{\partial}{\partial x}g(x,t)\right)dt$$

where $\frac{\partial}{\partial x}g(x,t)$ is partial diff. of $g(x,t)$ w.r.t. to x by treating t as constant.

ILLUSTRATION 142

If $\phi(x) = \cos x - \int_0^x (x-t)\phi(t)\, dt$, then find $\phi''(x) + \phi(x)$.

SOLUTION

$$\phi(x) = \cos x - \int_0^x x\phi(t)\, dt + \int_0^x t\phi(t)\, dt;$$

$$\phi(x) = \cos x - x\int_0^x \phi(t)\, dt + \int_0^x t\phi(t)\, dt$$

$$\phi'(x) = -\sin x - \int_0^x \phi(t)\, dt - x(\phi(x) - 0) + x\phi(x) - 0;$$

$$\phi'(x) = -\sin x - \int_0^x \phi(t)\, dt$$

$$\phi''(x) = -\cos x - \phi(x) \Rightarrow \phi''(x) + \phi(x) = -\cos x$$

ILLUSTRATION 143

If $f(x) = \int_{\ln x}^{x} \frac{dt}{x+t}$, then find $f'(1)$.

SOLUTION

$$f(x) = \int_{\ln x}^{x} \frac{dt}{x+t}$$

$$\Rightarrow \quad f'(x) = \frac{1}{x+x}(1) - \frac{1}{x+\ln x}\left(\frac{1}{x}\right) + \int_{\ln x}^{x} \frac{\partial}{\partial x}\left(\frac{1}{x+t}\right)dt$$

$$\Rightarrow \quad f'(x) = \frac{1}{2x} - \frac{1}{x(x+\ln x)} + \int_{\ln x}^{x}\left(-\frac{1}{(x+t)^2}\right)dt$$

$$f'(x) = \frac{1}{2x} - \frac{1}{x(x+\ln x)} - \int_{\ln x}^{x}\frac{dt}{(x+t)^2}$$

$$f'(1) = \frac{1}{2} - \frac{1}{1(1+0)} - \int_0^1 \frac{dt}{(1+t)^2}$$

$$f'(1) = -\frac{1}{2} + \left.\frac{1}{1+t}\right|_0^1 = -\frac{1}{2} + \frac{1}{2} - 1 = -1$$

ILLUSTRATION 144

Evaluate $\int_0^1 \frac{x^b - 1}{\ln x}\, dx$ where b is parameter

SOLUTION

$$f(b) = \int_0^1 \frac{x^b - 1}{\ln x}\, dx \qquad (\because\ f(0) = 0)$$

$$f'(b) = 0 - 0 + \int_0^1 \frac{\partial}{\partial b}\left(\frac{x^b - 1}{\ln x}\right)dx = \int_0^1\left(\frac{x^b \ln x - 0}{\ln x}\right)dx$$

$$f'(b) = \int_0^1 x^b\, dx = \left.\frac{x^{b+1}}{b+1}\right|_0^1 = \frac{1}{b+1}$$

$$f(b) = \int \frac{1}{b+1}\,db = \ln(b+1) + C$$

$$\because \quad f(0) = 0,\ 0 = \ln 1 + C \Rightarrow C = 0$$

$$f(b) = \ln(b+1)$$

ILLUSTRATION 145

Evaluate $\displaystyle\int_0^1 \frac{\tan^{-1} ax}{x\sqrt{1-x^2}}\,dx$, where a is parameter.

SOLUTION

$$f(a) = \int_0^1 \frac{\tan^{-1} ax}{x\sqrt{1-x^2}}\,dx$$

For $a = 0$, we have $f(0) = 0$

$$f'(a) = 0 - 0 + \int_0^1 \frac{\partial}{\partial a}\left(\frac{\tan^{-1} ax}{x\sqrt{1-x^2}}\right)dx = \int_0^1 \frac{\frac{1}{1+a^2x^2}\cdot x}{x\sqrt{1-x^2}}\,dx$$

$$f'(a) = \int_0^1 \frac{dx}{(1+a^2x^2)\sqrt{1-x^2}}$$

Put $x = \sin\theta$; $dx = \cos\theta\, d\theta$

$x = 0,\ \theta = 0$ (lower limit)

$x = 1,\ \theta = \dfrac{\pi}{2}$ (upper limit)

$$f'(a) = \int_0^{\pi/2} \frac{\cos\theta d\theta}{(1+a^2\sin^2\theta)\cos\theta} = \int_0^{\pi/2} \frac{d\theta}{\cos^2\frac{(1+a^2\sin^2\theta)}{\cos^2\theta}}$$

$$f'(a) = \int_0^{\pi/2} \frac{\sec^2\theta d\theta}{\sec^2\theta + a^2\tan^2\theta}$$

Put $\tan\theta = \alpha$; $\sec^2\theta\, d\theta = d\alpha$

$$f'(a) = \int_0^{\infty} \frac{d\alpha}{1+\alpha^2+a^2\alpha^2} = \int_0^{\infty} \frac{d\alpha}{(1+a^2)\alpha^2+1}$$

$$= \frac{1}{(1+a^2)}\int_0^{\infty} \frac{d\alpha}{\alpha^2 + \left(\frac{1}{\sqrt{1+a^2}}\right)^2}$$

$$f'(a) = \frac{1}{(1+a^2)}\,\frac{1}{\frac{1}{\sqrt{1+a^2}}}\tan^{-1}\frac{\alpha}{\frac{1}{\sqrt{1+a^2}}}\Bigg|_0^{\infty}$$

$$f'(a) = \frac{1}{\sqrt{1+a^2}}\left[\frac{\pi}{2} - 0\right] = \frac{\pi}{2\sqrt{1+a^2}}$$

$$f(a) = \frac{\pi}{2}\int \frac{da}{\sqrt{1+a^2}} = \frac{\pi}{2}\ln\left(a+\sqrt{a^2+1}\right) + C$$

$$f(0) = 0 = 0 + C \Rightarrow C = 0$$

We have $f(a) = \dfrac{\pi}{2}\ln\left(a+\sqrt{a^2+1}\right)$.

ILLUSTRATION 146

Let $f : (0, \infty) \to R$ be a continuous function such that $F(x) = \displaystyle\int_0^x t\, f(t)\, dt$. If $F(x^2) = x^4 + x^5$, then compute $\displaystyle\sum_{r=1}^{12} f(r^2)$.

SOLUTION

$$F(x) = \int_0^x tf(t)dt \Rightarrow F'(x) = xf(x) \qquad \ldots(1)$$

$$F(x^2) = x^4 + x^5$$

$$F(x) = x^2 + x^{5/2}$$

$$F'(x) = 2x + \frac{5}{2}x^{3/2} = xf(x) \qquad \text{Using (1)}$$

$$f(x) = 2 + \frac{5}{2}x^{1/2} \Rightarrow f(r^2) = 2 + \frac{5}{2}r$$

$$\sum_{r=1}^{12} f(r^2) = \sum_{r=1}^{12}\left(2+\frac{5}{2}r\right) = 2\times 12 + \frac{5}{2}\cdot\frac{12.13}{2} = 219$$

ILLUSTRATION 147

Let g be a real-valued differentiable function on R such that $g(x) = 3e^{x-2} + 4\displaystyle\int_2^x \sqrt{2t^2+6t+5}\,dt\ \forall\ x \in R$ and let g^{-1} be the inverse function of g. If $(g^{-1})'(3)$ is equal to p/q where p and q are relatively prime, then find $(p + q)$.

SOLUTION

$$g(x) = 3e^{x-2} + 4\int_2^x \sqrt{2t^2+6t+5}\,dt\ \forall\ x \in R$$

when $g(x) = 3$, we get $x = 2$

$$g'(x) = 3e^{x-2} + 4(\sqrt{2x^2+6x+5} - 0)$$

$$g'(2) = 3 + 4\left(\sqrt{8+12+5}\right) = 3 + 4\,(5) = 23$$

Let $g^{-1}(x) = h(x)$

$h(g(x)) = x$ on differentiation

$\Rightarrow \quad h'(g(x))\, g'(x) = 1$, put $x = 2$, $g(2) = 3$

$$\Rightarrow \quad h'(3) = \frac{1}{g'(x)} = \frac{1}{g'(2)}$$

$$h'(3) = \frac{1}{23} = \frac{p}{q}$$

We get, $p + q = 24$.

ILLUSTRATION 148

Let $f(x)$ be a differentiable function satisfying

$$f\left(\frac{x+y}{2}\right)=\frac{f(x)+f(y)}{2}\quad \forall\ x, y \in R \text{ and } f(0)=0.$$

If $\displaystyle\int_0^{2\pi}(f(x)-\sin x)^2\,dx$ is minimised, then find the value of $f(-4\pi^2)$.

SOLUTION

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f\left(\frac{2x+2h}{2}\right)-f\left(\frac{2x+0}{2}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{f(2x)+f(2h)}{2}-\frac{f(2x)+f(0)}{2}}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{f(0+2h)-f(0)}{2h}=f'(0)$$

$$\frac{dy}{dx}=f'(0)\Rightarrow\int dy=\int f'(0)\,dx$$

$y=f'(0)x+c$, as $f(0)=0$, we get $c=0$

$f(x)=f'(0)x=kx$

$$\int_0^{2\pi}(kx-\sin x)^2\,dx=\int_0^{2\pi}k^2x^2+\frac{1-\cos 2x}{2}-2kx\sin x\,dx$$

$$=\frac{k^2}{3}x^3+\frac{x}{2}-\frac{\sin 2x}{4}\bigg|_0^{2\pi}-2k\int_0^{2\pi}(x\sin x)\,dx$$

$$=\frac{8}{3}k^2\pi^3+\pi-0-0-2k\left(x(-\cos x)\big|_0^{2\pi}-\int_0^{2\pi}(-\cos x)dx\right)$$

$$=\frac{8k^2}{3}\pi^3+\pi-2k\left(-2\pi-0+\sin x\big|_0^{2\pi}\right)$$

$$=\pi\left(\frac{8k^2}{3}\pi^2+4k+1\right)\text{ min.}$$

Minimum occurs when $k=-\dfrac{B}{2A}=\dfrac{-4}{2\left(\frac{8}{3}\pi^2\right)}$

$$\Rightarrow\quad k=-\frac{3}{4\pi^2}\Rightarrow f(x)=kx$$

$$f(-4\pi^2)=\left(\frac{-3}{4\pi^2}\right)(-4\pi^2)=3$$

Test Your Concepts-V

(Solutions on page H.276)

1. If $\displaystyle\lim_{t\to a}\frac{\int_a^t f(x)dx-\left(\frac{t-a}{2}\right)\{f(t)+f(a)\}}{(t-a)^3}=3a$ and $f(1)=-6$, $f'(1)=42$, then find $f(0)$.

2. Find $\displaystyle\lim_{x\to 0}\frac{\int_0^{x^2}\cos t^2\,dt}{x\sin x}$.

3. Let $I=\displaystyle\int_x^{xy}f(x)\,dx$ be a function of y, i.e., independent of x. If $f(1)=2$, then find $f(x)$.

4. Find $\displaystyle\lim_{x\to 0}\frac{\int_0^x(1-\cos 2t)dt}{x\int_0^x\tan t\,dt}$

5. Find the range of the function $f(x)=\displaystyle\int_1^x t(t^2-3t+2)dt$ on $[1, 3]$

6. If $f(x)=\displaystyle\int_0^x\log_{0.5}\left(\frac{2t-8}{t-2}\right)dt$, then find the interval in which $f(x)$ is increasing.

7. If f is continuous in $(0,\infty)$ and satisfies $f(x)=1+\dfrac{1}{x}\displaystyle\int_1^x f(t)dt\ \ \forall x>0$, then find $f(e)$.

8. If $\displaystyle\int_0^y\cos t^2\,dt=\int_0^{x^2}\frac{\sin t}{t}\,dt$, then find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

9. Evaluate $\displaystyle\lim_{x\to 0^+}\int_1^{\cos x}\left(\frac{\cos^{-1}t}{2x-\sin 2x}\right)dt$

10. Evaluate $\displaystyle\lim_{x\to a}\frac{1}{x-a}\int_{\sqrt{a}}^{\sqrt{x}}te^t\sin t\,dt$, $(a>0)$

11. Evaluate $\displaystyle\lim_{x\to 0}\frac{1}{x^6}\int_0^{x^2}\sin(t^2)\,dt$

12. Evaluate $f(x) = x^2 + \int_0^x e^{-t} f(x-t)\, dt$, then $\int_0^1 f(x)\, dx$

13. Evaluate $\lim\limits_{t \to \frac{\pi}{3}} \frac{1}{3t - \pi} \int_{\pi/3}^{t} \frac{3\tan x - \tan^3 x}{\cos\left(x + \frac{\pi}{6}\right)} dx$

14. Evaluate $\lim\limits_{x \to 0} \frac{1}{\frac{x^3}{3} - \frac{x^2}{2}} \int_0^{x^2} (t^2 - 1) e^{-x^2}\, dt$

15. Evaluate $\lim\limits_{x \to 1} \frac{\int_1^x (t-1)\sin \pi t\, dt}{(x-1)^3}$

16. Evaluate $\lim\limits_{x \to -\pi} \frac{\int_0^{\sin x} \sin^{-1} t\, dt}{(x+\pi)^2}$

17. Let $f(x) = \int_2^x \frac{dt}{\sqrt{1+t^4}}$ and g be the inverse of f, then find the value of $g'(0)$.

18. Evaluate: $\lim\limits_{x \to \infty} \frac{e^{2x}}{\int_0^x (\sqrt{t + e^{4t}})dt}$

19. If $F(x) = \int_1^x f(t)\, dt$ where $f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u} du$, then find the value of $F''(2)$.

Answer key

1. -60
2. 1
3. $f(x) = \frac{2}{x}$
4. $\frac{4}{3}$
5. $\left[-\frac{1}{4}, 2\right]$
6. $x \in (4,6)$
7. 2
8. $\frac{dy}{dx} = \frac{2\sin x^2}{x} . \sec y^2; \frac{d^2y}{dx^2} = \frac{8\sin^2 x^2}{x^3} . y \sec^2 y^2 \tan y^2 + 2\sec y^2\left(2\cos x^2 - \frac{\sin x^2}{x^2}\right)$
9. $-\frac{1}{4}$
10. $\frac{1}{2} e^{\sqrt{a}} \sin \sqrt{a}$
11. $\frac{1}{3}$
12. $\frac{5}{12}$
13. 8
14. 2
15. $-\frac{\pi}{3}$
16. $\frac{1}{2}$
17. $\sqrt{17}$
18. 2
19. $\sqrt{257}$

DEFINITE INTEGRATION USING LIMIT OF SUM

If $f(x)$ is an integrable function defined on $[a, b]$, let assume $f(x)$ is non-negative and increasing function.

Suppose we are interested in finding the area under the graph between $x = a$ and $x = b$. One way in which this area can be approximated is to divide it into n number of rectangles of equal width, If n rectangles span the distance from a to b the width of each rectangle is h.

where $h = \frac{b-a}{n}$.

(i) When we are taking the left-hand side of the rectangle as height as shown in Figure 10.1, then small areas are dropped.

s = sum of area of rectangles with left side of rectangle as height

$$s = \sum_{r=0}^{n-1} hf(a + rh)$$

$$s = hf(a) + hf(a+h) + hf(a+2h) + \ldots + hf(a+(n-1)h)$$

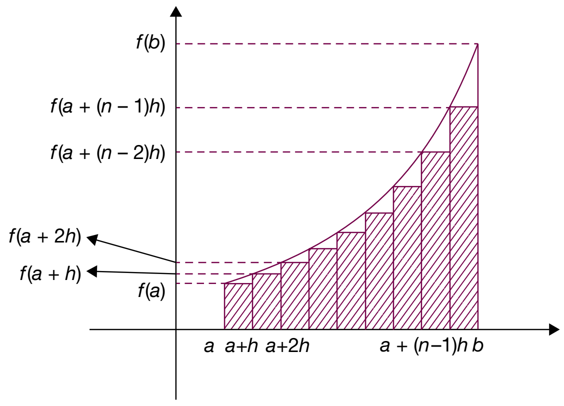


Figure 10.1

(ii) When we are taking right-hand side of rectangle for height as shown in Figure 10.2, then small areas are increased.

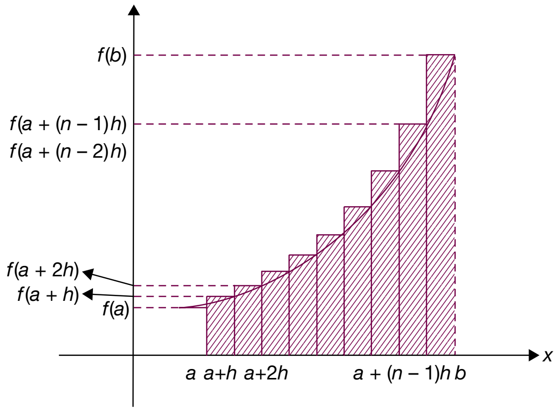


Figure 10.2

S = sum of areas of rectangles with right side of rectangle as height

$S = hf(a + h) + hf(a + 2h) + hf(a + 3h) + \dots + hf(a + nh)$

$$S = \sum_{r=1}^{n} hf(a+rh)$$

Clearly from graph we can conclude that

$$s < \int_a^b f(x)dx < S$$

$$\sum_{r=0}^{n-1} hf(a+rh) < \int_a^b f(x)dx < \sum_{r=1}^{n} hf(a+rh)$$

Marginal area which are decreased or increased tends to zero as we increase number of rectangles,
If $n \to \infty$, then width of rectangle h tends to zero,

$$\lim_{n\to\infty} \sum_{r=0}^{n-1} hf(a+rh) = \int_a^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} hf(a+rh)$$

Working Rule

Solving definite integration using limit of sum,

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} hf(a+rh)$$

$$\int_a^b f(x)dx = \lim_{h\to 0} h\,[f(a+h) + f(a+2h) + \dots + f(a+nh)],$$

where $h = \dfrac{b-a}{n}$; $h \to 0$, when $n \to \infty$

ILLUSTRATION 149

Let $f(x)$ be a continuous monotonic function such that $f(x) > 0$ for $x \in (a, b)$. Let $s = h\sum_{r=0}^{n-1} f(a+rh)$ and $S = h\sum_{r=1}^{n} f(a+rh)$ and where $b - a = nh$ and $I = \int_a^b f(x)dx$

Compare s and S and integral I and $\dfrac{s+S}{2}$ according to the following condition:

(i) $f(x)$ is concave upwards increasing function
(ii) $f(x)$ is concave downwards increasing function
(iii) $f(x)$ is concave upwards decreasing function
(iv) $f(x)$ is concave downwards, decreasing function

SOLUTION

For given $f(x)$ which is a continuous monotonic function such that $f(x) > 0$ for $x \in (a, b)$

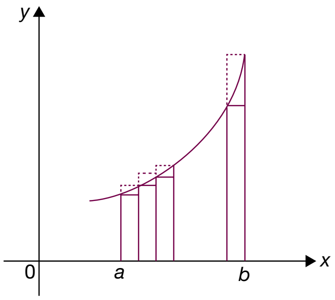
s = sum of area of rectangles with left side of rectangle as height

$$s = \sum_{r=0}^{n-1} hf(a+rh)$$

S = sum of areas of rectangles with right side of rectangle as height

$$S = \sum_{r=1}^{n} hf(a+rh)$$

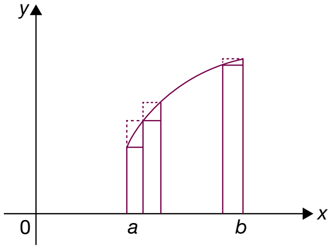
(i) Let $f(x)$ is concave upwards increasing function, as shown in the diagram:



We can conclude from the graph is $s < S$
As it concave upwards, the area included in S for any two consecutive values of r is more than the area excluded in s for the same values of r;

$$I - s < S - I \Rightarrow I < \frac{s+S}{2}.$$

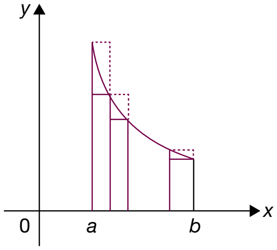
(ii) Let $f(x)$ is concave downwards increasing function, as shown in the diagram:



We can conclude from the graph is $s < S$.
As it concave downwards, the area included in S for any two consecutive values of r is less than the area excluded in s for the same values of r;

$$S - I < I - s \Rightarrow \frac{s+S}{2} < I.$$

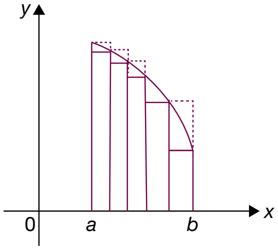
(iii) $f(x)$ is concave upwards decreasing function



We can conclude from the graph is $s > S$.
As it concave downwards, the area included in S for any two consecutive values of r is more than the area excluded in s for the same values of r;

$$I - s < S - I \Rightarrow I < \frac{s+S}{2}.$$

(iv) $f(x)$ is concave downwards, decreasing function



We can conclude from the graph is $s > S$
As it concave downwards, the area included in S for any two consecutive values of r is less than area excluded in s for same values of r;

$$S - I < I - s \Rightarrow \frac{s+S}{2} < I$$

ILLUSTRATION 150

Evaluate $\int_2^3 x^2\,dx$ using limit of sum.

SOLUTION

$$f(x) = x^2;\ a = 2;\ b = 3;\ h = \frac{b-a}{n} = \frac{1}{n}$$

$$\int_2^3 x^2\,dx = \lim_{n\to\infty}\sum_{r=1}^{n} f(a+rh)\,h = \lim_{n\to\infty}\sum_{r=1}^{n}\left(2+\frac{r}{n}\right)^2\frac{1}{n}$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{4+\frac{r^2}{n^2}+\frac{4r}{n}}{n}\right)$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{n}4+\frac{1}{n^3}\sum_{r=1}^{n}r^2+\frac{4}{n^2}\sum_{r=1}^{n}r\right]$$

$$= \lim_{n\to\infty}\left[\frac{4n}{n}+\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}+\frac{4}{n^2}\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[4+\frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{6}+2\left(1+\frac{1}{n}\right)\right]$$

$$= 4+\frac{1(2)}{6}+2 = \frac{19}{3}$$

ILLUSTRATION 151

Evaluate $\int_{-1}^{1} e^x\,dx$ using limit of sum

SOLUTION

$$f(x) = e^x,\ a = -1,\ b = 1;\ h = \frac{b-a}{n} = \frac{2}{n}$$

$$\int_{-1}^{1} e^x\,dx = \lim_{n\to\infty}\sum_{r=1}^{n} f(a+rh)\,h = \lim_{n\to\infty}\sum_{r=1}^{n} e^{-1+\frac{2r}{n}}\left(\frac{2}{n}\right)$$

$$= \frac{1}{e}\cdot\frac{2}{n}\lim_{n\to\infty}\sum_{r=1}^{n} e^{\frac{2r}{n}}$$

$$= \frac{2}{ne}\lim_{n\to\infty}\left[e^{2/n}+e^{4/n}+\ldots\ n\text{ times}\right]$$

$$= \frac{2}{ne}\lim_{n\to\infty}\frac{e^{2/n}\left[(e^{2/n})^n-1\right]}{e^{2/n}-1}\qquad\text{(sum of GP)}$$

$$= \frac{1}{e}(e^2-1)\lim_{n\to\infty}\frac{e^{2/n}}{\left(\frac{e^{2/n}-1}{\frac{2}{n}}\right)}$$

$$= \frac{1}{e}(e^2-1)\frac{e^0}{1} = e-\frac{1}{e}$$

ILLUSTRATION 152

Evaluate $\int\limits_{0}^{4} x+e^{2x}\, dx$ using limit of sum

SOLUTION

$$f(x)=x+e^{2x};\ a=0;\ b=4;\ h=\frac{b-a}{n}=\frac{4}{n};\ a+rh=\frac{4r}{n}$$

$$\int\limits_{0}^{4} x+e^{2x}\, dx=\lim_{n\to\infty}\sum_{r=1}^{n} f(a+rh)\,h$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{4r}{n}+e^{8r/n}\right)\frac{4}{n}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\left[\left(\frac{4}{n}\right)^{2} r+e^{\frac{8r}{n}}\cdot\frac{4}{n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{16}{n^{2}}\sum_{r=1}^{n} r+\frac{4}{n}\sum_{r=1}^{n} e^{8r/n}\right]$$

$$=\lim_{n\to\infty}\left[\frac{16}{n^{2}}\cdot\frac{n(n+1)}{2}+\frac{4}{n}e^{8/n}\,\frac{\left(e^{8/n}\right)^{n}-1}{e^{8/n}-1}\right]$$

$$=\lim_{n\to\infty}\left[8\left(1+\frac{1}{n}\right)+\frac{1}{2}\,\frac{e^{8/n}(e^{8}-1)}{\left(\dfrac{e^{8/n}-1}{\frac{8}{n}}\right)}\right]$$

$$=8(1+0)+\frac{1}{2}\,\frac{e^{o}(e^{8}-1)}{1}=8+\frac{e^{8}-1}{2}=\frac{e^{8}+15}{2}$$

SOLVING LIMIT USING INTEGRATION

$$\lim_{n\to\infty}\sum_{r=r_{\min}}^{r_{\max}} f\left(\frac{r}{n}\right)\frac{1}{n}=\int\limits_{a}^{b} f(x)\,dx$$

Replace $\lim\limits_{n\to\infty}\sum$ by $\int$, $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx

Lower limit $= a = \lim\limits_{n\to\infty}\frac{r_{\min}}{n}$

Upper limit $= b = \lim\limits_{n\to\infty}\frac{r_{\max}}{n}$

If general r^{th} term is not written then write it.

ILLUSTRATION 153

Evaluate: $\lim\limits_{n\to\infty}\left(\frac{n+1}{n^{2}+1^{2}}+\frac{n+2}{n^{2}+2^{2}}+\ldots+\frac{3}{5n}\right)$

SOLUTION

$$L=\lim_{n\to\infty}\left(\frac{n+1}{n^{2}+1^{2}}+\frac{n+2}{n^{2}+2^{2}}+\ldots+\frac{n+2n}{n^{2}+(2n)^{2}}\right)$$

$$L=\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{n+r}{n^{2}+r^{2}}=\lim_{n\to\infty}\sum_{r=1}^{2n}\frac{n\left(1+\frac{r}{n}\right)}{n^{2}\left[1+\frac{r^{2}}{n^{2}}\right]}$$

$$L=\lim_{n\to\infty}\sum_{r=1}^{2n}\left[\frac{1+\frac{r}{n}}{1+\left(\frac{r}{n}\right)^{2}}\right]\frac{1}{n}$$

Replace $\lim\limits_{n\to\infty}\sum$ by $\int$, $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx

$$a=\lim_{n\to\infty}\frac{1}{n}=0;\ b=\lim_{n\to\infty}\frac{2n}{n}=2$$

$$L=\int\limits_{0}^{2}\frac{1+x}{1+x^{2}}\,dx=\int\limits_{0}^{2}\frac{1}{1+x^{2}}\,dx+\frac{1}{2}\int\limits_{0}^{2}\frac{2x\,dx}{1+x^{2}}$$

$$L=\tan^{-1}x\Big|_{0}^{2}+\frac{1}{2}\ln(1+x^{2})\Big|_{0}^{2}$$

$$L=\tan^{-1}2-0+\frac{1}{2}\ln 5-\ln 1=\tan^{-1}2+\ln\sqrt{5}$$

ILLUSTRATION 154

Evaluate: $\lim\limits_{n\to\infty}\left(\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+2\sqrt{n}}{n\sqrt{n}}\right)$

SOLUTION

$$L=\lim_{n\to\infty}\left(\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{4n}}{n\sqrt{n}}\right)$$

$$L=\lim_{n\to\infty}\sum_{r=1}^{4n}\frac{1}{n}\sqrt{\frac{r}{n}}$$

Replace $\lim\limits_{n\to\infty}\sum$ by $\int$, $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx

$$a=\lim_{n\to\infty}\frac{1}{n}=0;\ b=\lim_{n\to\infty}\frac{4n}{n}=4$$

$$L=\int\limits_{0}^{4}\sqrt{x}\,dx=\left.\frac{x^{3/2}}{\frac{3}{2}}\right|_{0}^{4}=\frac{2}{3}(4)^{3/2}=\frac{16}{3}$$

ILLUSTRATION 155

Evaluate: $\lim\limits_{n\to\infty}\left(\frac{1}{n}+\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+2n}}+\ldots+\frac{1}{3n}\right)$

SOLUTION

$$L=\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+2n}}+\ldots+\frac{1}{\sqrt{9n^2}}\right)$$

$$L=\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+n}}+\frac{1}{\sqrt{n^2+2n}}+\ldots+\frac{1}{\sqrt{n^2+(8n)n}}\right)$$

$$L=\lim_{n\to\infty}\left(\sum_{r=0}^{8n}\frac{1}{\sqrt{n^2+rn}}\right)=\lim_{n\to\infty}\sum_{r=0}^{8n}\frac{1}{n\sqrt{1+\frac{r}{n}}}$$

Replace $\lim\limits_{n\to\infty}\sum$ by $\int$, $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx

$$a=\lim_{n\to\infty}\frac{1}{n}=0;\ b=\lim_{n\to\infty}\frac{8n}{n}=8$$

$$L=\int_0^8\frac{dx}{\sqrt{1+x}}=2\sqrt{1+x}\Big|_0^8=2\sqrt{9}-2\sqrt{1}=6-2=4$$

ILLUSTRATION 156

Evaluate: $\lim\limits_{n\to\infty}\left(\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\ldots+\frac{1}{2n}\right)$

SOLUTION

$$L=\lim_{n\to\infty}\left(\frac{1^2}{1^3+n^3}+\frac{2^2}{2^3+n^3}+\ldots+\frac{n^2}{n^3+n^3}\right)$$

$$L=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^2}{r^3+n^3}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\left(\frac{r}{n}\right)^2\frac{1}{n}}{\left(\frac{r}{n}\right)^3+1}$$

Replace $\lim\limits_{n\to\infty}\sum$ by $\int$, $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx

$$a=\lim_{n\to\infty}\frac{1}{n}=0;\ b=\lim_{n\to\infty}\frac{n}{n}=1$$

$$L=\int_0^1\frac{x^2}{x^3+1}\,dx=\frac{1}{3}\int_0^1\frac{3x^2}{x^3+1}\,dx=\frac{1}{3}\ln(x^3+1)\Big|_0^1=\frac{1}{3}\ln 2$$

ILLUSTRATION 157

Evaluate:

$$\lim_{n\to\infty}\left(\frac{1}{2n}+\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\ldots+\frac{1}{\sqrt{3n^2+2n-1}}\right)$$

SOLUTION

$$L=\lim_{n\to\infty}\left(\frac{1}{\sqrt{4n^2}}+\frac{1}{\sqrt{4n^2-1^2}}+\frac{1}{\sqrt{4n^2-2^2}}+\ldots+\frac{1}{\sqrt{4n^2-(n-1)^2}}\right)$$

$$L=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{4n^2-r^2}}=\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{n\sqrt{4-\left(\frac{r}{n}\right)^2}}$$

Replace $\lim\limits_{n\to\infty}\sum$ by $\int$, $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx

$$a=\lim_{n\to\infty}\frac{0}{n}=0;\ b=\lim_{n\to\infty}\frac{n-1}{n}=1$$

$$L=\int_a^1\frac{dx}{\sqrt{4-x^2}}=\sin^{-1}\frac{x}{2}\Big|_0^1=\frac{\pi}{6}$$

ILLUSTRATION 158

If $\lim\limits_{n\to\infty}\frac{1}{n^3}\left[\sqrt{n^2+1}+2\sqrt{n^2+4}+\ldots+\sqrt{2}n^2\right]=\frac{\sqrt{a}-1}{\sqrt{b}}$

Find a and b

SOLUTION

Given limit

$$L=\lim_{n\to\infty}\frac{1}{n^3}\left[\sqrt{n^2+1^2}+2\sqrt{n^2+2^2}+\ldots+n\sqrt{n^2+n^2}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n^3}\sum_{r=1}^{n}r\sqrt{n^2+r^2}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left(\frac{r}{n}\right)\sqrt{1+\frac{r^2}{n^2}}$$

Replace $\lim\limits_{n\to\infty}\sum$ by $\int$, $\frac{r}{n}$ by x, $\frac{1}{n}$ by dx

$$a=\lim_{n\to\infty}\frac{1}{n}=0;\ b=\lim_{n\to\infty}\frac{n}{n}=1$$

$$L=\int_0^1 x\sqrt{1+x^2}\,dx$$

Put $\sqrt{1+x^2}=t\Rightarrow 1+x^2=t^2\Rightarrow 2x\,dx=2t\,dt$

when $x=0$; $t=1$, and $x=1$; $t=\sqrt{2}$

$$L=\int_0^1 x\sqrt{1+x^2}\,dx=\int_1^{\sqrt{2}}t\cdot t\,dt=\frac{t^3}{3}\Big|_1^{\sqrt{2}}$$

$$L=\frac{\sqrt{8}-1}{3}=\frac{\sqrt{a}-1}{\sqrt{b}}\text{; we get } a=8;\ b=9$$

ILLUSTRATION 159

Evaluate: $\lim\limits_{n\to\infty}\dfrac{\pi}{6n}\left[\sec^2\dfrac{\pi}{6n}+\sec^2\dfrac{2\pi}{6n}+\ldots+\dfrac{4}{3}\right]$

SOLUTION

$$L=\lim_{n\to\infty}\frac{\pi}{6n}\left[\sec^2\frac{\pi}{6n}+\sec^2\frac{2\pi}{6n}+\ldots+\sec^2\frac{n\pi}{6\pi}\right]$$

As $\sec^2\dfrac{n\pi}{6\pi}=\dfrac{4}{3}$

$$L=\lim_{n\to\infty}\frac{\pi}{6n}\left(\sum_{r=1}^{n}\sec^2\frac{r\pi}{6n}\right)$$

Replace $\lim\limits_{n\to\infty}\sum$ by $\int$, $\dfrac{r}{n}$ by x, $\dfrac{1}{n}$ by dx

$$a=\lim_{n\to\infty}\frac{1}{n}=0;\ b=\lim_{n\to\infty}\frac{n}{n}=1$$

$$L=\frac{\pi}{6}\int_0^1\sec^2\frac{\pi x}{6}\,dx=\tan\frac{\pi x}{6}\bigg|_0^1=\frac{1}{\sqrt{3}}$$

ILLUSTRATION 160

Evaluate: $\lim\limits_{n\to\infty}\left(\tan\dfrac{\pi}{2n}\cdot\tan\dfrac{2\pi}{2n}\cdot\tan\dfrac{3\pi}{2n}\ldots\tan\dfrac{(n-1)\pi}{2n}\right)^{1/n}$

SOLUTION

$$L=\lim_{n\to\infty}\left(\tan\frac{\pi}{2n}\cdot\tan\frac{2\pi}{2n}\cdot\tan\frac{3\pi}{2n}\ldots\tan\frac{(n-1)\pi}{2n}\right)^{1/n}$$

Taking log both sides,

$$\ln L=\lim_{n\to\infty}\frac{1}{n}$$

$$\left[\ln\tan\frac{\pi}{2n}+\ln\tan\frac{2\pi}{2n}+\ln\tan\frac{3\pi}{2n}+\ldots+\ln\tan\frac{(n-1)\pi}{2n}\right]$$

$$=\lim_{n\to\infty}\frac{1}{n}\left(\sum_{r=1}^{n-1}\ln\tan\frac{r\pi}{2n}\right)$$

Replace $\lim\limits_{n\to\infty}\sum$ by $\int$, $\dfrac{r}{n}$ by x, $\dfrac{1}{n}$ by dx

$$a=\lim_{n\to\infty}\frac{1}{n}=0;\ b=\lim_{n\to\infty}\frac{n-1}{n}=1$$

$$\ln L=\int_0^1\ln\tan\frac{\pi}{2}x\,dx$$

$$\ln L=\int_0^{1/2}\ln\tan\frac{\pi}{2}x\ +\ln\tan\frac{\pi}{2}(1-x)dx$$

$$\ln L=\int_0^{1/2}\ln\left[\tan\frac{\pi}{2}x\cdot\cot\frac{\pi}{2}x\right]dx=\int_0^{1/2}\ln(1)\,dx=0$$

$$\ln L=0\Rightarrow L=e^{0}=1$$

ILLUSTRATION 161

Evaluate: $\lim\limits_{n\to\infty}\left(\dfrac{n!}{n^n}\right)^{1/n}$

SOLUTION

$$L=\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}$$

$$L=\lim_{n\to\infty}\left(\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\left(\frac{3}{n}\right)\ldots\left(\frac{(n-1)}{n}\right)\left(\frac{n}{n}\right)\right)^{1/n}$$

Taking log both sides, we get

$$\ln L=\lim_{n\to\infty}\frac{1}{n}\left[\ln\left(\frac{1}{n}\right)+\ln\left(\frac{2}{n}\right)+\ldots+\ln\left(\frac{(n-1)}{n}\right)+\ln\left(\frac{n}{n}\right)\right]$$

$$\ln L=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(\frac{r}{n}\right)$$

Replace $\lim\limits_{n\to\infty}\sum$ by $\int$, $\dfrac{r}{n}$ by x, $\dfrac{1}{n}$ by dx

$$a=\lim_{n\to\infty}\frac{1}{n}=0;\ b=\lim_{n\to\infty}\frac{n}{n}=1$$

$$\ln L=\int_0^1\ln x\,dx=\int_0^1 1\cdot\ln x\,dx=x\ln x\big|_0^1-\int_0^1 x\left(\frac{1}{x}\right)dx$$

$$\ln L=1\ln 1-\lim_{x\to 0^+}x\ln x-(x)_0^1$$

$$\ln L=-1-\lim_{x\to 0^+}\frac{\ln x}{\dfrac{1}{x}}$$

(It is $\dfrac{\infty}{\infty}$ form using L-Hospital Rule)

$$\ln L=-1-\lim_{x\to 0^+}\frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}}=-1+0=-1;$$

$$L=e^{-1}=\frac{1}{e}$$

ILLUSTRATION 162

If $L=\lim\limits_{n\to\infty}\dfrac{1}{n^4}\prod\limits_{i=1}^{2n}(n^2+i^2)^{1/n}$, then compute $\ln L$

SOLUTION

$$L=\lim_{n\to\infty}\frac{1}{n^4}\left((n^2+1)^{1/n}(n^2+2^2)^{1/n}(n^2+3^2)^{1/n}\ldots(n^2+(2n)^2)^{1/n}\right)$$

$$= \lim_{n\to\infty} \frac{\left((n^2)^{2n}\right)^{1/n}}{n^4}\left(\left(1+\left(\frac{1}{n}\right)^2\right)\left(1+\left(\frac{2}{n}\right)^2\right)\dots\left(1+\left(\frac{2n}{n}\right)^2\right)\right)^{1/n}$$

$$L = \lim_{n\to\infty}\left(\left(1+\left(\frac{1}{n}\right)^2\right)\left(1+\left(\frac{2}{n}\right)^2\right)\dots\left(1+\left(\frac{2n}{n}\right)^2\right)\right)^{1/n}$$

$$\ln L = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\ln\left(1+\left(\frac{r}{n}\right)^2\right) = \int_0^2 \ln(1+x^2)\,dx$$

$$= x\cdot \ln(1+x^2)\Big|_0^2 - \int_0^2 \frac{x}{1+x^2}(2x)\,dx$$

$$= 2\ln 5 - 2\int_0^2 \frac{x^2+1-1}{x^2+1}\,dx$$

$$= 2\ln 5 - 2\left(x-\tan^{-1}x\right)_0^2$$

$$= 2\ln 5 - 2(2-\tan^{-1}2) = 2\ln 5 + 2\tan^{-1}2 - 4$$

ILLUSTRATION 163

Evaluate: $\lim\limits_{n\to\infty}\left((1+n)^{1/n}\left(1+\dfrac{n}{2}\right)^{2/n}\left(1+\dfrac{n}{3}\right)^{3/n}\dots 2\right)^{1/n}$

SOLUTION

$$\lim_{n\to\infty} e^{\ln\left((1+n)^{1/n}\left(1+\frac{n}{2}\right)^{2/n}\left(1+\frac{n}{3}\right)^{3/n}\dots\left(1+\frac{n}{n}\right)^{n/n}\right)^{1/n}}$$

$$= \lim_{n\to\infty} e^{\frac{1}{n}\left[\frac{1}{n}\ln(1+n)+\frac{2}{n}\ln\left(1+\frac{n}{2}\right)+\frac{3}{n}\ln\left(1+\frac{n}{3}\right)+\dots+\frac{n}{n}\ln\left(1+\frac{n}{n}\right)\right]}$$

$$= \lim_{n\to\infty} e^{\frac{1}{n}\sum\limits_{r=1}^{n}\frac{r}{n}\ln\left(1+\frac{n}{r}\right)} = e^{0}{}^{\int\limits^1 x\ln\left(1+\frac{1}{x}\right)dx}$$

$$= e^{\left(\frac{x^2}{2}\ln\left(1+\frac{1}{x}\right)\Big|_0^1 - \int\limits_0^1\frac{x^2}{2}\frac{1}{\left(\frac{x+1}{x}\right)}\frac{-1}{x^2}dx\right)} = e^{\frac{1}{2}\ln 2}\; e^{\frac{1}{2}\int\limits_0^1 1-\frac{1}{1+x}dx}$$

$$= \sqrt{2}\; e^{\frac{1}{2}(x-\ln(1+x))_0^1} = \sqrt{2}\; e^{\frac{1}{2}(1-\ln 2)} = e^{1/2}$$

ILLUSTRATION 164

Given that $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\dfrac{\log(n^2+r^2)-2\log n}{n} = \log 2+\dfrac{\pi}{2}-2,$

then evaluate $\lim\limits_{n\to\infty}\dfrac{1}{n^{2m}}\left[(n^2+1^2)^m(n^2+2^2)^m\dots(2n^2)^m\right]^{1/n}$

SOLUTION

$$\text{Given } \lim_{x\to\infty}\sum_{r=1}^{n}\frac{\ln\left(1+\frac{r^2}{n^2}\right)}{n} = \int_0^1 \ln(1+x^2)\,dx = \frac{\pi}{2} + \ln 2 - 2$$

$$\text{Now, } y = \lim_{n\to\infty}\left[\frac{(n^2+1^2)^m}{n^{2m}}\frac{(n^2+2^2)^m}{n^{2m}}\dots\frac{(2n^2)^m}{2^{2m}}\right]^{\frac{1}{n}}$$

$$\ln y = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r^2}{n^2}\right)^m = \lim_{n\to\infty}\frac{m}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r^2}{n^2}\right)$$

$$\ln y = m\int_0^1 \ln(1+x^2)\,dx = m\left(\ln 2+\frac{\pi}{2}-2\right)$$

$$\Rightarrow\quad y = \left(e^{\ln 2+\frac{\pi}{2}-2}\right)^m\; y = \left(2e^{\frac{\pi}{2}-2}\right)^m = 2^m e^{m\left(\frac{\pi}{2}-2\right)}$$

ILLUSTRATION 165

Let $\lambda = \displaystyle\int_0^1 \frac{dx}{1+x^3}$, $p = \lim\limits_{n\to\infty}\left(\dfrac{\prod\limits_{r=1}^{n}(n^3+r^3)}{n^{3n}}\right)^{1/n}$, then find $\ln p$.

SOLUTION

$$p = \lim_{n\to\infty}\left(\frac{\prod\limits_{r=1}^{n}(n^3+r^3)}{n^{3n}}\right)^{1/n}$$

$$p = \lim_{n\to\infty}\left(\frac{(n^3+1^3)}{n^3}\frac{(n^3+2^3)}{n^3}\dots\frac{(n^3+n^3)}{n^3}\right)^{\frac{1}{n}}$$

$$\ln p = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r^3}{n^3}\right) = \int_0^1 \ln(1+x^3)\,dx$$

$$\ln p = x\,\ln(1+x^3)\Big|_0^1 - \int_0^1\frac{x}{1+x^3}(3x^2)\,dx$$

$$\ln p = \ln 2 - 3\int_0^1\frac{(x^3+1-1)}{1+x^3}\,dx$$

$$\ln p = \ln 2 - 3\int_0^1 dx + 3\int_0^1\frac{dx}{1+x^3} = \ln 2 - 3 + 3\lambda$$

Test Your Concepts-VI

(Solutions on page H.277)

1. Evaluate $\int_a^b (px+q)dx$ using limit of sum

2. Evaluate:

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\frac{1}{\sqrt{4n^2-9}}+\ldots\frac{1}{\sqrt{3n^2}}\right]$$

3. Evaluate: $\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots+\frac{1}{64n}\right]$

4. $\lim_{n\to\infty}\left\{\frac{1}{1-n^2}+\frac{2}{1-n^2}+\frac{3}{1-n^2}+\ldots+\frac{n}{1-n^2}\right\}$

5. If $f(n)=\frac{1}{n}\,[(n+1)(n+2)(n+3)\ldots(n+n)]^{1/n}$, then evaluate $\lim_{n\to\infty} f(n)$.

6. Evaluate $\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\ldots+\frac{1}{2n}$ as $n\to\infty$ using definite integral.

7. Evaluate: $\lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n}\sqrt{\left(\frac{n+r}{n-r}\right)}$

8. Evaluate: $\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\sin\frac{3\pi}{2n}+\ldots+\sin\frac{n\pi}{2n}\right)$

9. Evaluate: $\sum_{k=1}^{n}\frac{2}{n}\left(2+\frac{2k}{n}\right)^5$

10. Evaluate: $\lim_{n\to\infty}\frac{1}{n^2}\sum_{k=1}^{n}\sqrt{n^2-k^2}$

11. Evaluate: $\lim_{n\to\infty}\frac{1}{n^4}\left(\sum_{r=1}^{2n}(3nr^2+2n^2r)\right)$

12. Evaluate:

$$\lim_{n\to\infty}\frac{1}{n^3}\left(\sqrt{n^2+1}+2\sqrt{n^2+2^2}+\ldots+n\sqrt{n^2+n^2}\right)$$

13. If $P_n=\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)^2\left(1+\frac{3^2}{n^2}\right)^3\ldots\left(1+\frac{n^2}{n^2}\right)^n$, then find $\lim_{n\to\infty}(P_n)^{\frac{-1}{n^2}}$

14. Let $f(x)=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{n}{n^2+x^2r^2}$, then find the value of $\sum_{k=1}^{3}kf(k)$ (where $x>0$)

15. Evaluate:

$$\lim_{n\to\infty}\frac{1}{n}\left[1+\sqrt{\frac{n}{n+1}}+\sqrt{\frac{n}{n+2}}+\sqrt{\frac{n}{n+3}}+\ldots+\sqrt{\frac{n}{n+3(n-1)}}\right]$$

Answer key

1. $\frac{p}{2}(b^2-a^2)+q(b-a)$
2. $\frac{\pi}{6}$
3. $\frac{15}{32}$
4. $-\frac{1}{2}$
5. $\frac{4}{e}$
6. $\frac{\pi}{4}$
7. $\frac{\pi}{2}+1$
8. $\frac{2}{\pi}$
9. 672
10. $\frac{\pi}{4}$
11. 12
12. $\frac{2\sqrt{2}-1}{3}$
13. $\frac{\sqrt{e}}{2}$
14. π
15. 2

ESTIMATION OF DEFINITE INTEGRATION

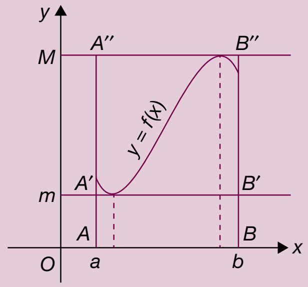
(i) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$

(ii) Let $f(x)$ is a continuous function defined in $x\in[a,b]$ $m\le f(x)\le M$ where M is maximum of $f(x)$ and m is minimum of $f(x)$

$$\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$\Rightarrow m(b-a)\le\int_a^b f(x)\,dx\le M(b-a)$$

We can also deduce it from the figure,



Area of rectangle $ABB'A' \le \int_a^b f(x)\,dx \le$ Area of rectangle $ABB''A''$

$$\Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

For Identify M(maximum) and m(minimum)

(i) Using graph
(ii) Using AOD if $f(x)$ is continuous
(iii) By observing two suitable functions

$g(x) \le f(x) \le h(x)$ where $x \in (a, b)$

$$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx \le \int_a^b h(x)\,dx$$

ILLUSTRATION 166

Estimate $\int_0^1 e^{x^2}\,dx$. Also find $\left[\int_0^1 e^{x^2}\,dx\right]$, where [.] is greatest integer function.

SOLUTION

$$f(x) = e^{x^2};\ x \in (0, 1]$$

$$f'(x) = 2xe^{x^2} \ge 0;\ f(x) \text{ is increasing function}$$

minimum of $f(x) = m = f(0) = 1$;
maximum of $f(x) = M = f(1) = e$

$$1 < f(x) < e$$

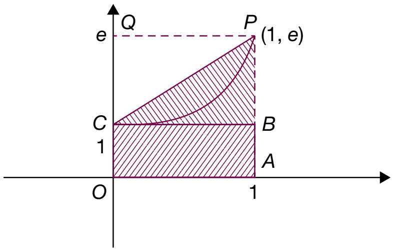
$$\Rightarrow \quad \int_0^1 dx < \int_0^1 f(x)\,dx < \int_0^1 e\,dx$$

$$\Rightarrow \quad 1 < \int_0^1 e^{x^2}\,dx < e \Rightarrow I \in (1, e)$$

Alternate:

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = 2e^{x^2}(1 + 2x^2) > 0$$

$f(x)$ always concave using graph



We can make better estimation as:

Area of square $OABC < \int_0^1 e^{x^2}\,dx <$ area of trapezium $OAPC$

$$1 < \int_0^1 e^{x^2}\,dx < \frac{1}{2}(1)(1+e)$$

$$\Rightarrow \quad 1 < I < \frac{e+1}{2} \approx \frac{2\cdot 71+1}{2} \approx 1.85;$$

$$\Rightarrow \quad \int_0^1 e^{x^2}\,dx \in (1, 1.85)$$

Alternate:

$$0 < x < 1 \Rightarrow 0 < x^2 < x \Rightarrow e^o < e^{x^2} < e^x$$

$$\Rightarrow \quad \int_0^1 1\,dx < \int_0^1 e^{x^2}\,dx < \int_0^1 e^x\,dx$$

$$\Rightarrow \quad x\Big|_0^1 < I < e^x\Big|_0^1 \Rightarrow 1 - 0 < I < e - 1 \Rightarrow 1 < I < 1.71$$

This is better estimation compare to above

$$\therefore \quad \left[\int_0^1 e^{x^2}\,dx\right] = 1$$

ILLUSTRATION 167

Prove that $1 < \int_0^{\pi/2} \frac{\sin x}{x} < \frac{\pi}{2}$

SOLUTION

Let $f(x) = \frac{\sin x}{x}$ where $x \in \left(0, \frac{\pi}{2}\right)$

$$f'(x) = \left(\frac{x\cos x - \sin x}{x^2}\right) = \frac{\cos x(x - \tan x)}{x}$$

We know that $x < \tan x$ where $x \in \left(0, \frac{\pi}{2}\right)$.

$$f'(x) = \frac{\cos x(x - \tan x)}{x} < 0$$

$f(x)$ is strictly monotonically decreasing

$$0 < x < \frac{\pi}{2} \Rightarrow f(0^+) > f(x) > f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \lim_{x \to 0^+} \frac{\sin x}{x} > f(x) > \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}$$

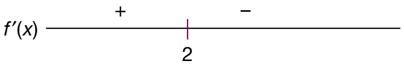
$$\Rightarrow \quad 1 > f(x) > \frac{2}{\pi} \Rightarrow \int_0^{\pi/2} 1\,dx > \int_0^{\pi/2} f(x)\,dx > \int_0^{\pi/2} \frac{2}{\pi}\,dx$$

$$\Rightarrow \quad \frac{\pi}{2} > \int_0^{\pi/2} \frac{\sin x}{x}\,dx > \frac{2}{\pi}\left(\frac{\pi}{2}-0\right) \Rightarrow 1 < \int_0^{\pi/2} \frac{\sin x}{x} < \frac{\pi}{2}$$

ILLUSTRATION 168

Show that $0 < \int_0^2 \frac{x\,dx}{16+x^3} < \frac{1}{6}$.

SOLUTION



Let $f(x) = \frac{x}{16+x^3}$ where $x \in (0, 2]$

$$f'(x) = \frac{(16+x^3)(1) - x(0+3x^2)}{(16+x^3)^2}$$

$$f'(x) = \frac{(16-2x^3)}{(16+x^3)^2} = \frac{2(2-x)(4+2x+x^2)}{(16+x^3)^2}$$

$f(x)$ is increasing where $x \in [0, 2]$

$$0 < x < 2 \Rightarrow f(0) < f(x) < f(2)$$

$$0 < f(x) < \frac{2}{16+8} = \frac{1}{12}$$

$$\Rightarrow \quad \int_0^2 0\,dx < \int_0^2 f(x)\,dx < \int_0^2 \frac{dx}{12}$$

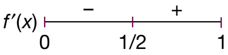
$$\Rightarrow \quad 0 < \int_0^2 f(x)\,dx < \frac{1}{12}\; x\Big|_0^2 \Rightarrow 0 < \int_0^2 \frac{x\,dx}{16+x^3} < \frac{1}{6}$$

ILLUSTRATION 169

Prove that $\frac{1}{e^{1/4}} < \int_0^1 e^{x^2-x}\,dx < 1$

SOLUTION

Let $f(x) = e^{x^2-x}$ where $x \in [0, 1]$



$$f'(x) = e^{x^2-x}(2x-1)$$

$$f(0) = e^0 = 1;\ f(1) = e^0 = 1$$

$x = \frac{1}{2}$ local minima

$$f\left(\frac{1}{2}\right) = e^{\frac{1}{4}-\frac{1}{2}} = e^{-\frac{1}{4}};$$

$$e^{-\frac{1}{4}} \le f(x) < 1$$

$$\Rightarrow \quad \int_0^1 e^{-\frac{1}{4}}\,dx < \int_0^1 f(x)\,dx < \int_0^1 1\,dx$$

$$\Rightarrow \quad e^{-\frac{1}{4}} < \int_0^1 e^{x^2-x} < 1$$

ILLUSTRATION 170

Prove that $0 < \int_0^1 \frac{x^7\,dx}{\sqrt[3]{1+x^3}} < \frac{1}{8}$

SOLUTION

$$\because \quad 0 < \frac{x^7}{\left(1+x^3\right)^{1/3}} < x^7 \text{ where } x \in (0, 1)$$

$$\Rightarrow \quad \int_0^1 0\,dx < \int_0^1 \frac{x^7}{\left(1+x^3\right)^{1/3}}\,dx < \int_0^1 x^7\,dx$$

$$\Rightarrow \quad 0 < \int_0^1 \frac{x^7\,dx}{\sqrt[3]{1+x^3}} < \frac{x^8}{8}\Bigg|_0^1 = \frac{1}{8} \Rightarrow 0 < \int_0^1 \frac{x^7\,dx}{\sqrt[3]{1+x^3}} < \frac{1}{8}$$

ILLUSTRATION 171

Consider the integrals $I_1 = \int_0^1 e^{-x}\cos^2 x\,dx$,

$I_2 = \int_0^1 e^{-x^2}\cos^2 x\,dx$, $I_3 = \int_0^1 e^{-\frac{x^2}{2}}\cos^2 x\,dx$, $I_4 = \int_0^1 e^{-\frac{x^2}{2}}\,dx$,

then compare I_1, I_2, I_3, I_4.

SOLUTION

For $x \in (0, 1)$, where $x > x^2 > \frac{x^2}{2}$

$$\Rightarrow \quad -x < -x^2 < -\frac{x^2}{2} \Rightarrow e^{-x} < e^{-x^2} < e^{-\frac{x^2}{2}}$$

$$\Rightarrow \quad 0 < e^{-x}\cos^2 x < e^{-x^2}\cos^2 x < e^{-\frac{x^2}{2}}\cos^2 x < e^{-\frac{x^2}{2}}$$

$$\Rightarrow \quad \int_0^1 e^{-x}\cos^2 x\,dx < \int_0^1 e^{-x^2}\cos^2 x\,dx < \int_0^1 e^{-x^2/2}\cos^2 x\,dx < \int_0^1 e^{-x^2/2}\,dx$$

$$\Rightarrow \quad I_1 < I_2 < I_3 < I_4$$

ILLUSTRATION 172

If $I_1 = \int_0^{\pi/2} \cos(\sin x)\,dx$, $I_2 = \int_0^{\pi/2} \sin(\cos x)\,dx$ and

$I_3 = \int_0^{\pi/2} \cos x\,dx$, then compare I_1, I_2, I_3.

SOLUTION

We know that for $x > 0$, $\sin x < x$

$\cos(\sin x) > \cos x$ (as $\cos x$ is decreasing in 0 to $\frac{\pi}{2}$)

$$\int_0^{\pi/2}\cos(\sin x)\,dx > \int_0^{\pi/2}\cos x\,dx$$

$\therefore \quad I_1 > I_3 \qquad \ldots(1)$

$$I_2 = \int_0^{\pi/2}\sin(\cos x)\,dx = \int_0^{\pi/2}\sin\left(\cos\left(\frac{\pi}{2}-x\right)\right)dx$$

$$= \int_0^{\pi/2}\sin(\sin x)\,dx$$

$$I_3 = \int_0^{\pi/2}\cos dx = \int_0^{\pi/2}\cos\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi/2}\sin x\,dx$$

For $x > 0$, we have $x > \sin x$

$\sin x > \sin(\sin x)$ (as sin is increasing in 0 to $\frac{\pi}{2}$)

$$\int_0^{\pi/2}\sin x\,dx > \int_0^{\pi/2}\sin(\sin x)\,dx$$

$I_3 > I_2 \qquad \ldots(2)$

From (1) and (2), we have

$\therefore \quad I_1 > I_3 > I_2$

ILLUSTRATION 173

Estimate: $\displaystyle\int_0^2 \sqrt{\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x + 4}\;dx$

SOLUTION

$$f(x) = \frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x + 4,\ \text{where } x \in [0, 2]$$

$$\frac{df(x)}{dx} = x^3 - 6x^2 + 11x - 6$$

$$\frac{df(x)}{dx} = (x-1)(x-2)(x-3) = 0$$

Point $x = 1, 2, 3$ are point of extrema but $x \in [0, 2]$

$$f(0) = 4, f(1) = \frac{7}{4}; f(2) = 2$$

$$\frac{7}{4} \le f(x) \le 4 \Rightarrow \frac{\sqrt{7}}{2} \le \sqrt{f(x)} \le 2$$

$$\Rightarrow \quad \int_0^2 \frac{\sqrt{7}}{2}dx \le \int_0^2 \sqrt{f(x)}dx \le \int_0^2 2dx$$

$$\Rightarrow \quad (2-0)\frac{\sqrt{7}}{2} \le I \le 2\times 2 \Rightarrow \sqrt{7} \le I \le 4.$$

Test Your Concepts-VII

(Solutions on page H.279)

1. Prove that $\dfrac{\pi}{\pi^3 + 10\pi + 5} < \displaystyle\int_0^{\pi}\dfrac{dx}{x^3 + 10x + 9\sin x + 5} < \dfrac{\pi}{5}.$

2. Show that $\dfrac{\pi}{6} < \displaystyle\int_0^1 \dfrac{dx}{\sqrt{4 - x^2 - x^3}} < \dfrac{\pi}{4\sqrt{2}}.$

3. Estimate $I = \displaystyle\int_3^4 \dfrac{1}{\sqrt[3]{\log x}}dx$

4. Let $I_1 = \displaystyle\int_0^{\pi/4} x^{2009}(\tan x)^{2009}\,dx$, $I_2 = \displaystyle\int_0^{\pi/4} x^{2010}(\tan x)^{2010}\,dx$ and $I_3 = \displaystyle\int_0^{\pi/4} x^{2011}(\tan x)^{2011}\,dx$. Then arrange them in ascending order.

Answer key

3. $\frac{1}{2} < I < 1$ **4.** $I_3 < I_2 < I_1$

SOLVED PROBLEMS

PROBLEM 1

If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$

Show that $C_1 - \frac{1}{2}\,C_2 + \frac{1}{3}\,C_3 + \ldots + (-1)^{n-1}\,\frac{1}{n}\,C_n$

$$= 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

SOLUTION

$$(1+x)^n = C_0 + C_1 + C_2\,x^2 + \ldots + C_nx^n$$

$$\frac{(1+x)^n}{x} = \frac{C_0}{x} + C_1 + C_2x + \ldots + C_nx^{n-1}$$

$$\Rightarrow \quad \frac{(1+x)^n}{x} - \frac{C_0}{x} = C_1 + C_2x + \ldots + C_nx^{n-1}$$

$$\Rightarrow \quad \frac{1[(1+x)^n - 1]}{(1+x)-1} = 1+(1+x)+(1+x)^2\ldots+(1+x)^{n-1}$$

$$= C_1 + C_2x + \ldots + C_nx^{n-1}$$

$$\left(\because \text{ Sum of GP} = \frac{a(r^n-1)}{r-1} \text{ where } a = 1,\ r = 1+x\right)$$

$$\Rightarrow \quad \int_{-1}^{0} 1+(1+x)+(1+x)^2\ldots+(1+x)^{n-1}\,dx$$

$$= \int_{-1}^{0} C_1 + C_2x + C_3x^2 + \ldots + C_nx^{n-1}\,dx$$

$$\Rightarrow \quad x + \frac{(1+x)^2}{2} + \frac{(1+x)^3}{3} \ldots + \frac{(1+x)^n}{n}\Bigg|_{-1}^{0}$$

$$= C_1x + \frac{C_2x^2}{2} + \ldots \frac{C_nx^n}{n}\Bigg|_{-1}^{0}$$

$$\Rightarrow \quad \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{n} - (-1) = 0 - \left(-C_1 + \frac{C_2}{2}\ldots\frac{C_n(-1)^n}{n}\right)$$

$$\Rightarrow \quad C_1 - \frac{1}{2}C_2 + \frac{1}{3}C_3 + \ldots (-1)^{n-1}\frac{C_n}{n}$$

$$= 1 + \frac{1}{2} + \frac{1}{3} \ldots + \frac{1}{n}$$

Hence proved

PROBLEM 2

Evaluate the following integrals where $\beta > \alpha$.

(i) $\displaystyle\int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)}\,dx$ **(ii)** $\displaystyle\int_{\alpha}^{\beta} \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$

(iii) $\displaystyle\int_{\alpha}^{\beta} \frac{xdx}{\sqrt{(x-\alpha)(\beta-x)}}$

SOLUTION

(i) $$\int_{\alpha}^{\beta} \sqrt{(x-\alpha)(\beta-x)}\,dx = \int_{\alpha}^{\beta} \sqrt{-x^2 + (\alpha+\beta)x - \alpha\beta}\,dx$$

$$= \int_{\alpha}^{\beta} \sqrt{\left(\frac{\beta-\alpha}{2}\right)^2 - \left(x - \left(\frac{\alpha+\beta}{2}\right)\right)^2}\,dx$$

$$= \frac{\left(x - \left(\frac{\alpha+\beta}{2}\right)\right)}{2}\sqrt{(x-\alpha)(\beta-x)} +$$

$$\frac{\left(\frac{\beta-\alpha}{2}\right)^2}{2}\sin^{-1}\left(\frac{x - \left(\frac{\alpha+\beta}{2}\right)}{\frac{\beta-\alpha}{2}}\right)\Bigg|_{\alpha}^{\beta}$$

$$= 0 - 0 + \frac{1}{2}\left(\frac{\beta-\alpha}{2}\right)^2\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{8}\,(\alpha-\beta)^2$$

(ii) $$I = \int_{\alpha}^{\beta} \sqrt{\frac{x-\alpha}{\beta-x}}\,dx$$

Put $\sqrt{\beta-x} = t \Rightarrow \beta - x = t^2 \Rightarrow -dx = 2t\,dt$

$$I = \int_{\sqrt{\beta-\alpha}}^{0} \frac{\sqrt{\beta - t^2 - \alpha}}{t}\,(-2t)\,dt = 2\int_{0}^{\sqrt{\beta-\alpha}} \sqrt{\beta-\alpha-t^2}\,dt$$

$$= 2\left(\frac{t}{2}\sqrt{\beta-\alpha-t^2} + \frac{\beta-\alpha}{2}\sin^{-1}\frac{t}{\sqrt{\beta-\alpha}}\Bigg|_{0}^{\sqrt{\beta-\alpha}}\right)$$

$$= 2\left(0 + \frac{\beta-\alpha}{2}\cdot\frac{\pi}{2} - 0 - 0\right) = (\beta-\alpha)\,\frac{\pi}{2}$$

(iii) $$I = \int_{\alpha}^{\beta} \frac{x\,dx}{\sqrt{(x-\alpha)(\beta-x)}} \qquad \ldots(1)$$

Replacing x by $\alpha+\beta-x$

$$I = \int_{\alpha}^{\beta} \frac{(\alpha+\beta-x)dx}{\sqrt{(x-\alpha)(\beta-x)}} \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2I = \int_{\alpha}^{\beta} \frac{\alpha+\beta}{\sqrt{(x-\alpha)(\beta-x)}}\,dx$$

$$I = \left(\frac{\alpha+\beta}{2}\right)\int_{\alpha}^{\beta} \frac{dx}{\sqrt{\left(\frac{\beta-\alpha}{2}\right)^2 - \left(x - \left(\frac{\alpha+\beta}{2}\right)\right)^2}}$$

$$= \left(\frac{\alpha+\beta}{2}\right)\sin^{-1}\left(\frac{x-\left(\frac{\alpha+\beta}{2}\right)}{\frac{\beta-\alpha}{2}}\right)\Bigg|_{\alpha}^{\beta}$$

$$= \left(\frac{\alpha+\beta}{2}\right)\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right) = \left(\frac{\alpha+\beta}{2}\right)\pi$$

PROBLEM 3

If $I_{m,n} = \int\limits_0^1 x^m (\ln x)^n \, dx$, then prove that $I_{m,n} = (-1)^n \dfrac{n!}{(m+1)^{n+1}}$ where $m, n \in N$.

SOLUTION

$$I_{m,n} = \int\limits_0^1 x^m (\ln x)^n \, dx$$

$$= \left[\frac{x^{m+1}}{m+1}(\ln x)^n\right]_0^1 - \int_0^1 \frac{x^{m+1}}{m+1}\,\frac{n(\ln x)^{n-1}}{x}\,dx$$

$$I_{m,n} = -\frac{n}{m+1}\int_0^1 x^m (\ln x)^{n-1}\,dx$$

$$I_{m,n} = -\frac{n}{m+1} I_{m,\,n-1}$$

$$\text{Again } I_{m,n} = -\frac{n}{m+1}\left[\left\{\frac{x^{m+1}}{m+1}(\ln x)^{n-1}\right\}_0^1 - \int_0^1 \frac{x^{m+1}}{m+1}\,\frac{(n-1)(\ln x)^{n-2}}{x}\,dx\right]$$

$$I_{m,n} = -\frac{n}{m+1}\left[0 - \frac{n-1}{m+1}\int_0^1 x^m (\ln x)^{n-2}\,dx\right]$$

$$I_{m,n} = (-1)^2\,\frac{n(n-1)}{(m+1)^2}\,I_{m,\,n-2}$$

Continue we get,

$$I_{m,n} = (-1)^n\,\frac{n(n-1)\ldots3.2.1}{(m+1)^n}\,I_{m,0}$$

$$= (-1)^n\,\frac{n!}{(m+1)^n}\int_0^1 x^m\,dx$$

$$= (-1)^n\,\frac{n!}{(m+1)^n}\cdot\left[\frac{x^{m+1}}{m+1}\right]_0^1$$

$$I_{m,n} = (-1)^n\,\frac{n!}{(m+1)^{n+1}} \quad \text{R.H.S.}$$

Hence proved

PROBLEM 4

Evaluate $\int\limits_0^1 |k-x|\cos\pi x \, dx$; where k is any real number.

SOLUTION

$$I_1 = \int (x-k)\cos(\pi x)\,dx = (x-k)\,\frac{\sin(\pi x)}{\pi} - \int \frac{\sin(\pi x)}{\pi}\,dx$$

$$= (x-k)\,\frac{\sin(\pi x)}{\pi} - \left(-\frac{\cos\pi x}{\pi^2}\right) + C$$

$$I_1 = (x-k)\,\frac{\sin(\pi x)}{\pi} + \frac{\cos\pi x}{\pi^2} + C$$

CASE 1: $k \le 0$

$$I = \int\limits_0^1 |k-x|\cos\pi x\,dx = \int\limits_0^1 (x-k)\cos\pi x\,dx$$

$$I = [I_1]_0^1 = -\frac{2}{\pi^2}$$

CASE 2: $k \ge 1$,

$$I = \int\limits_0^1 |k-x|\cos\pi x\,dx = -\int\limits_0^1 (x-k)\cos\pi x\,dx$$

$$I = -[I_1]_0^1 = \frac{2}{\pi^2}$$

CASE 3: $0 < k < 1$,

$$I = \int\limits_0^1 |k-x|\cos\pi x\,dx$$

$$= -\int\limits_0^k (x-k)\cos\pi x\,dx + \int\limits_k^1 (x-k)\cos\pi x\,dx$$

$$I = -\left((x-k)\frac{\sin(\pi x)}{\pi} + \frac{\cos\pi x}{\pi^2}\right)_0^k + \left((x-k)\frac{\sin(\pi x)}{\pi} + \frac{\cos\pi x}{\pi^2}\right)_k^1$$

$$I = -\frac{\cos\pi k}{\pi^2} + \frac{1}{\pi^2} - \frac{1}{\pi^2} - \frac{\cos\pi k}{\pi^2} = -\frac{2}{\pi^2}\cos\pi k$$

PROBLEM 5

Prove that $\int\limits_0^\pi \dfrac{dx}{1-2a\cos x + a^2} = \begin{cases} \dfrac{\pi}{a^2-1} & \text{if } |a| > 1 \\ \dfrac{\pi}{1-a^2} & \text{if } |a| < 1 \end{cases}$.

SOLUTION

$$I = \int_0^\pi \frac{1}{a^2 - 2a\left(\dfrac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right) + 1}\,dx$$

$$I=\int_0^{\pi}\frac{\sec^2\frac{x}{2}}{(1-a)^2+(1+a)^2\tan^2\frac{x}{2}}\,dx$$

Let $\tan\frac{x}{2}=t\Rightarrow\frac{1}{2}\sec^2\frac{x}{2}\,dx=dt$

$$I=2\int_0^{\infty}\frac{1}{(1-a)^2+t^2(1+a)^2}\,dt$$

$$I=\frac{2}{(1+a)^2}\int_0^{\infty}\frac{1}{t^2+\left(\frac{1-a}{1+a}\right)^2}\,dt$$

$$=\frac{2}{(1+a)^2}\left|\frac{1+a}{1-a}\right|\left[\tan^{-1}\left|\frac{1+a}{1-a}\right|t\right]_0^{\infty}$$

$$I=\frac{2}{|1+a|\,|1-a|}\left(\frac{\pi}{2}-0\right)$$

$$I=\frac{\pi}{|1-a^2|}=\begin{cases}\dfrac{\pi}{a^2-1} & \text{if } |a|>1\\[2mm] \dfrac{\pi}{1-a^2} & \text{if } |a|<1\end{cases}$$

PROBLEM 6

If $m, n\in N$, evaluate: $\int_0^b (x-a)^m(b-x)^n\,dx$.

SOLUTION

Put $x=a\cos^2\theta+b\sin^2\theta$, $dx=(b-a)2\sin\theta\cos\theta\,d\theta$

$$I=\int_0^b (x-a)^m(b-x)^n\,dx$$

$$=\int_0^{\pi/2}\left[(b-a)\sin^2\theta\right]^m\,((b-a)\cos^2\theta)^n\times(b-a)\sin2\theta\,d\theta$$

$$I=2(b-a)^{m+n+1}\int_0^{\pi/2}\sin^{2m+1}\theta\cos^{2n+1}\theta\,d\theta$$

Using Walli's formula

$$I=2(b-a)^{m+n+1}\frac{(2m.2m-2.2m-4\ldots4.2)(2n.2n-2.2n-4\ldots4.2)}{(2m+2n+2).(2m+2n).(2m+2n-2)\ldots6.4.2}$$

$$I=2(b-a)^{m+n+1}\frac{2^m m!\,2^n n!}{2^{m+n+1}(m+n+1)!}$$

$$=(b-a)^{m+n+1}\frac{m!n!}{(m+n+1)!}$$

PROBLEM 7

Prove that $\int_0^{\infty}f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{\ln x}{x}\,dx=\ln a\int_0^{\infty}f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{dx}{x}$.

SOLUTION

$$\text{LHS}=\int_0^{\infty}f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{\ln x}{x}\,dx$$

Put $x=\frac{a}{t}$; $dx=-\frac{a}{t^2}\,dt$

$$=\int_{\infty}^{0}f\left(t+\frac{1}{t}\right)\frac{\ln\left(\frac{a}{t}\right)}{\frac{a}{t}}\cdot\frac{-a}{t^2}\,dt$$

$$=\int_0^{\infty}f\left(t+\frac{1}{t}\right)\frac{(\ln a-\ln t)}{t}\,dt$$

Put $t=\frac{z}{a}$; $dt=\frac{dz}{a}$

$$=\int_0^{\infty}f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{\left(\ln a-\ln\left(\frac{z}{a}\right)\right)}{\frac{z}{a}}\frac{dz}{a}$$

$$=\int_0^{\infty}f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{(2\ln a-\ln z)}{z}\,dz$$

$$\text{LHS}=\int_0^{\infty}f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{2\ln a}{z}\,dz-\int_0^{\infty}f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{\ln z}{z}\,dz$$

$$=2\int_0^{\infty}f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{\ln a}{z}\,dz-\text{LHS}$$

$$2\text{LHS}=2\int_0^{\infty}f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{\ln a}{z}\,dz;$$

$$\int_0^{\infty}f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{\ln x}{x}\,dx=\ln a\int_0^{\infty}f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{dx}{x}$$

Hence proved

PROBLEM 8

Prove that for any positive integer k,

$$\frac{\sin 2kx}{\sin x}=2.\,[\cos x+\cos3x+\ldots+\cos(2k-1)x]$$

Hence, prove that $\int_0^{\pi/2}\sin 2kx\cot x\,dx=\frac{\pi}{2}$.

SOLUTION

We know,

$$2\sin x\,\{\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x\}$$

$$= 2\sin x\cos x + 2\sin x\cos 3x + 2\sin x\cos 5x + \ldots + 2\sin x\cos(2k-1)x$$

$$= \sin 2x + (\sin 4x - \sin 2x) + (\sin 6x - \sin 4x) + \ldots + \{\sin 2kx - \sin(2k-2)x\} = \sin 2kx$$

$$\because\quad 2\{\cos x + \cos 3x + \cos 5x + \ldots \cos(2k-1)x\} = \frac{\sin 2kx}{\sin x} \qquad \ldots(1)$$

$$\text{Now, } \sin 2kx \cdot \cot x = \frac{\sin 2kx}{\sin x}\cdot\cos x$$

$$= 2\cos x\,\{\cos x + \cos 3x + \cos 5x + \ldots + \cos(2k-1)x\} \quad \{\text{using (1)}\}$$

$$= \{2\cos^2 x + 2\cos x\cos 3x + 2\cos x\cos 5x + \ldots + 2\cos x\cos(2k-1)x\}$$

$$= (1+\cos 2x) + (\cos 4x + \cos 2x) + (\cos 6x + \cos 4x) + \ldots + (\cos 2kx + \cos(2k-2)x)$$

$$= 1 + 2\{\cos 2x + \cos 4x + \cos 6x + \ldots + \cos(2k-2)x\} + \cos 2kx$$

$$\therefore\quad \int_0^{\pi/2} (\sin 2kx)\cdot\cot x\,dx$$

$$= \int_0^{\pi/2} 1\cdot dx + 2\int_0^{\pi/2} (\cos 2x + \cos 4x \ldots \cos(2k-2)x) + \int_0^{\pi/2} \cos(2k)x\,dx$$

$$= \frac{\pi}{2} + 2\left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \ldots + \frac{\sin(2k-2)x}{(2k-2)}\right]_0^{\pi/2} + \left[\frac{\sin(2k)x}{2k}\right]_0^{\pi/2} = \frac{\pi}{2}$$

PROBLEM 9

Show that $\displaystyle\int_0^{\pi/2} f(\sin 2x)\sin x\,dx = \sqrt{2}\int_0^{\pi/4} f(\cos 2x)\cos x\,dx.$

SOLUTION

$$\text{Let, } I = \int_0^{\pi/2} f(\sin 2x).\sin x\,dx \qquad \ldots(1)$$

$$\text{Then, } I = \int_0^{\pi/2} f\left\{\sin 2\left(\frac{\pi}{2}-x\right)\cdot\sin\left(\frac{\pi}{2}-x\right)\right\}dx$$

$$I = \int_0^{\pi/2} f\{\sin 2x\}.\cos x\,dx \qquad \ldots(2)$$

Adding (1) and (2), we get:

$$2I = \int_0^{\pi/2} f\{\sin 2x\}.(\sin x + \cos x)\,dx$$

$$2I = 2\int_0^{\pi/4} f(\sin 2x).(\sin x + \cos x)\,dx$$

$$I = \sqrt{2}\int_0^{\pi/4} f(\sin 2x)\sin\left(x + \frac{\pi}{4}\right)dx$$

$$I = \sqrt{2}\int_0^{\pi/4} f\left(\sin 2\left(\frac{\pi}{4}-x\right)\right)\cdot\sin\left(\frac{\pi}{4}-x+\frac{\pi}{4}\right)dx$$

$$I = \sqrt{2}\int_0^{\pi/4} f(\cos 2x).\cos x\,dx = \text{RHS}$$

Hence Proved

PROBLEM 10

If $U_n = \displaystyle\int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\,dx$ where n is a whole number then show that $U_{n+2} + U_n = 2U_{n+1}$.

Also, show that $\displaystyle\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta}\,d\theta = \frac{n\pi}{2}$

SOLUTION

$$\because\quad U_n = \int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\,dx$$

$$\therefore\quad U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\{(1-\cos(n+2)x\} - \{1-\cos(n+1)x\}}{(1-\cos x)}\,dx$$

$$U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\cos(n+1)x - \cos(n+2)x}{(1-\cos x)}\,dx$$

$$U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{2\sin\left(n+\frac{3}{2}\right)x\sin\frac{x}{2}}{2\sin^2\frac{x}{2}}\,dx$$

$$\Rightarrow\quad U_{n+2} - U_{n+1} = \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x}{\sin\frac{x}{2}}\,dx \qquad \ldots(1)$$

Similarly,

$$\Rightarrow\quad U_{n+1} - U_n = \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx \qquad \ldots(2)$$

From (1) and (2), we get

$$(U_{n+2} - U_{n+1}) - (U_{n+1} - U_n)$$

$$= \int_0^{\pi} \frac{\sin\left(n+\frac{3}{2}\right)x - \sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}\,dx$$

$$= \int_0^{\pi} \frac{2\cos(n+1)x \sin\frac{x}{2}}{\sin\frac{x}{2}}\, dx = 2\left\{\frac{\sin(n+1)x}{(n+1)}\right\}_0^{\pi} = 0$$

∴ $U_{n+2} + U_n = 2U_{n+1}$

Hence proved

Now $U_{n+2} - U_{n+1} = U_{n+1} - U_n$.

Similarly implies

$$U_{n+2} - U_{n+1} = U_{n+1} - U_n = U_n - U_{n-1} = \dots = U_1 - U_0$$

It mean $U_0, U_1, U_2, U_3, U_4, \dots, U_{n-1}, U_n$ are in A.P.

$U_1 = \pi,\ U_0 = 0$

$\Rightarrow \quad U_n = U_0 + n\pi \Rightarrow U_n = n\pi$ ...(1)

Hence,

$$\therefore \quad \int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta}\, d\theta = \int_0^{\pi/2} \frac{1-\cos 2n\theta}{1-\cos 2\theta}\, d\theta$$

Put $2\theta = x$, then $d\theta = \frac{dx}{2}$

Hence, $\int_0^{\pi/2} \frac{\sin^2 n\theta}{\sin^2\theta}\, d\theta = \frac{1}{2}\int_0^{\pi} \frac{1-\cos nx}{1-\cos x}\, dx$

$= \frac{1}{2} U_n = \frac{n\pi}{2}$ {from (1)}

Hence proved

PROBLEM 11

If $f(x)$ is a monotonic and differentiable real-valued function and a, b are two real numbers. Show that $\int_a^b \{f(x) + f(a)\}\{f(x) - f(a)\}dx = 2\int_{f(a)}^{f(b)} x\{b - f^{-1}(x)dx$.

SOLUTION

As $f(x)$ is monotonic, $f^{-1}(x)$ exists.

Also, if $x = f(z)$ then $f^{-1}(x) = z$ and $x = f(a) \Rightarrow z = f^{-1}\{f(a)\} = a$

$x = f(b) \Rightarrow z = f^{-1}\{f(b)\} = b$. Then

$$\text{RHS} = 2\int_a^b f(z)\,(b - z)\, f'(z)dz$$

$\{\because x = f(z) \Rightarrow dx = f'(z)dz\}$

$$= \int_a^b (b - z)\left(\frac{d(f(z))^2}{dz}\right)dz$$

$= [(b-z)\{f(z)\}^2]_a^b - \int_a^b \{f(z)\}^2(-1)dz,$ using by parts

$$= -(b-a)\{f(a)\}^2 + \int_a^b \{f(z)\}^2 dz$$

$$= -\int_a^b \{f(a)\}^2 dz + \int_a^b \{f(z)\}^2 dz$$

$$= \int_a^b [\{f(z)\}^2 - \{f(a)\}^2]dz = \int_a^b \{f(x)\}^2 - \{f(a)\}^2]dx,$$

$$= \int_a^b \{f(x) + f(a)\}\{f(x) - f(a)\}dx = \text{LHS}.$$

Hence proved

PROBLEM 12

If $\lim\limits_{a\to\infty} \frac{1}{a}\int_0^{\infty} \frac{x^2 + ax + 1}{1 + x^4} \cdot \tan^{-1}\left(\frac{1}{x}\right) dx$ is equal to $\frac{\pi^2}{k}$ where $k \in N$ equals, find k.

SOLUTION

$$I = \lim_{a\to\infty} \frac{1}{a}\int_0^{\infty} \frac{x^2 + ax + 1}{1 + x^4} \cdot \tan^{-1}\left(\frac{1}{x}\right) dx \qquad \text{...(1)}$$

Put $x = \frac{1}{t}$, $dx = -\frac{1}{t^2}dt$;

$$I = \lim_{a\to\infty} \frac{1}{a}\int_{\infty}^{0} \frac{1 + at + t^2}{\left(\frac{t^4+1}{t^2}\right)} \tan^{-1}(t)\left(-\frac{1}{t^2}\right)dt$$

$$I = \lim_{a\to\infty} \frac{1}{a}\int_0^{\infty} \frac{1 + at + t^2}{t^4 + 1} \tan^{-1} t\,dt \qquad \text{...(2)}$$

Adding (1) and (2)

$$2I = \lim_{a\to\infty} \frac{1}{a}\int_0^{\infty} \frac{1 + at + t^2}{t^4 + 1}(\cot^{-1} t + \tan^{-1} t)dt$$

$$I = \lim_{a\to\infty} \frac{1}{2a}\int_0^{\infty} \frac{1 + at + t^2}{t^4 + 1} \cdot \frac{\pi}{2}\,dt = \lim_{a\to\infty} \frac{\pi}{4}\,\frac{\int_0^{\infty} \frac{1 + at + t^2}{t^4 + 1}dt}{a}$$

$$= \frac{\pi}{4}\,\frac{\int_0^{\infty} \frac{t}{t^4 + 1}dt}{1} = \frac{\pi}{8}\tan^{-1} t^2 \Big|_0^{\infty} = \frac{\pi^2}{16} = \frac{\pi^2}{k}$$

$\Rightarrow \quad k = 16$

PROBLEM 13

Let $f(x)$ be a continuously differentiable function then prove that, $\int_1^x [t] f'(t)\,dt = [x].\ f(x) - \sum_{k=1}^{[x]} f(k)$ where $[\cdot]$ denotes the greatest integer function and $x > 1$.

SOLUTION

$$\text{LHS} = \int_1^x [t].f'(t)\cdot dt$$

$$= \int_1^2 1 f'(t)\,dt + \int_2^3 2.f'(t)\,dt + \int_3^4 3f'(t)\,dt \ldots + \int_{[x]-1}^{[x]} ([x]-1)f'(t)\,dt + \int_{[x]}^{x} [x]f'(t)\,dt$$

$$= [f(2) - f(1)] + 2[f(3) - f(2)] + 3[f(4) - f(3)] + \ldots + ([x] - 1)[f([x]) - f([x] - 1)] + [x][f(x) - f([x])]$$

$$= -f(1) - f(2) - f(3) \ldots - f([x]) + [x]f(x)$$

$$= [x]f(x) - \sum_{k=0}^{[x]} f(k)$$

Hence proved.

PROBLEM 14

Evaluate: $\displaystyle\int_0^{\pi/2} \frac{\sin^6 x dx}{\sin x + \cos x}$

SOLUTION

$$f(x) = \int_0^{\pi/2} \frac{\sin^6 x dx}{\sin x + \cos x} \quad \ldots(1)$$

Using king Rule, replace x by $\frac{\pi}{2} - x$

$$f(x) = \int_0^{\pi/2} \frac{\cos^6 x dx}{\sin x + \cos x} \quad \ldots(2)$$

Adding (1) + (2), we get

$$2f(x) = \int_0^{\pi/2} \frac{(\sin^6 x + \cos^6 x)dx}{(\sin x + \cos x)}$$

$$2f(x) = \int_0^{\pi/2} \frac{(1)(\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x)}{(\sin x + \cos x)}dx$$

$$2f(x) = \int_0^{\pi/2} \frac{(1)\left(1 - \frac{3(\sin 2x)^2}{4}\right)dx}{(\sin x + \cos x)}$$

$$= \int_0^{\pi/2} \frac{4 - 3(\sin 2x)^2}{4(\sin x + \cos x)}dx = \frac{2}{4}\int_0^{\pi/4} \frac{4 - 3(\sin 2x)^2}{\sin x + \cos x}dx$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{4 - 3(\cos^2 2x)}{\sin\left(\frac{\pi}{4} - x\right) + \cos x\left(\frac{\pi}{4} - x\right)}dx$$

$$2f(x) = \frac{1}{2}\int_0^{\pi/4} \frac{(4 - 3\cos^2 2x)\,dx}{\left(\frac{1}{\sqrt{2}}\right)[\cos x - \sin x + \cos x + \sin x]}$$

$$2f(x) = \frac{1}{2\sqrt{2}}\int_0^{\pi/4} \frac{(4 - 3\cos^2 2x)dx}{\cos x}$$

$$(\because \cos 2x = 2\cos^2 x - 1)$$

$$2f(x) = \frac{1}{2\sqrt{2}}\int_0^{\pi/4} \frac{[1 - 12\cos^4 x + 12\cos^2 x]}{\cos x}dx$$

$$\Rightarrow \quad f(x) = \frac{1}{4\sqrt{2}}\int_0^{\pi/4} (\sec x - 12\cos^3 x + 12\cos x)\,dx$$

Now, $\displaystyle\int \cos^3 x\ dx = \int (\cos x)(1 - \sin^2 x)\ dx$

$$= \int \cos x\ dx - \int \sin^2 x\cos x\ dx = \sin x - \frac{\sin^3 x}{3}$$

$$f(x) = \frac{1}{4\sqrt{2}}$$

$$\left(\ln(|\sec x + \tan x|) - 12\sin x + \frac{12\sin^3 x}{3} + 12\sin x\right)_0^{\pi/4}$$

$$= \frac{1}{4\sqrt{2}}\left[\ln(\sqrt{2} + 1) + 4\left[\left(\frac{1}{\sqrt{2}}\right)^3\right]\right]$$

$$= \frac{1}{4\sqrt{2}}[\ln(\sqrt{2} + 1) + \sqrt{2}]$$

PROBLEM 15

For $a \geq 2$, if the value of the definite integral $\displaystyle\int_0^\infty \frac{dx}{a^2 + \left(x - \frac{1}{x}\right)^2}$ equals $\frac{\pi}{5050}$. Find the value of a.

SOLUTION

$$f(x) = \int_0^\infty \frac{dx}{a^2 + \left(x - \frac{1}{x}\right)^2} \quad \ldots(1)$$

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\cdot dt$

$$f(x) = \int_0^\infty \frac{\left(\frac{1}{t^2}\right)dt}{a^2 + \left(\frac{1}{t} - t\right)^2} = \int_0^\infty \frac{dt}{a^2t^2 + (t^2 - 1)^2}$$

$$f(x) = \int_0^\infty \frac{dx}{a^2x^2 + (x^2 - 1)^2} \quad \ldots(2)$$

Eq. (1) + (2), we obtain

$$2f(x) = \int_0^\infty \frac{x^2\,dx}{a^2x^2+(x^2-1)^2} + \int_0^\infty \frac{dx}{a^2x^2+(x^2-1)^2}$$

$$= \int_0^\infty \frac{(x^2+1)dx}{x^4+1+(a^2-2)x^2} = \int_0^\infty \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+a^2}$$

$$2f(x) = \left(\frac{1}{a}\right)\left[\tan^{-1}\left(\frac{x-\frac{1}{x}}{a}\right)\right]_0^\infty = \left(\frac{1}{a}\right)\left[\frac{\pi}{2}+\frac{\pi}{2}\right]$$

$$\Rightarrow \quad f(x) = \frac{\pi}{2a} = \frac{\pi}{5050}$$

We get $a = 2525$.

PROBLEM 16

If a_1, a_2 and a_3 are the three values of a which satisfy the equation

$$\int_0^{\pi/2} (\sin x + a\cos x)^3\,dx - \frac{4a}{\pi-2}\int_0^{\pi/2} x\cos x\,dx = 2,$$

then find the value of $1000(a_1^2 + a_2^2 + a_3^2)$.

SOLUTION

$$f(x) = \int_0^{\pi/2} (\sin x + a\cos x)^3\,dx$$

$$= \int_0^{\pi/2} \sin^3 x\,dx + \int_0^{\pi/2} a^3\cos^3 x dx + 3a\int_0^{\pi/2} \sin x.\cos x(\sin x + a\cos x)\,dx$$

$$= \int_0^{\pi/2} (1-\cos^2 x)\sin x\,dx + a^3\int_0^{\pi/2} (1-\sin^2 x)\cos x\,dx + 3a^2\int_0^{\pi/2} \cos^2 x\sin x + a\int_0^{\pi/2} 3\sin^2 x\cos x\,dx$$

$$= \left[\frac{\cos^3 x}{3} - \cos x\right]_0^{\pi/2} + a^3\left[\sin x - \frac{\sin^3 x}{3}\right]_0^{\pi/2} + a^2 3\left(-\frac{\cos^3 x}{3}\right)_0^{\pi/2} + 3a\left(\frac{\sin^3 x}{3}\right)_0^{\pi/2}$$

$$= \left[0-\left(\frac{1}{3}-1\right)+a^3\left[1-\frac{1}{3}\right]-a^2[0-1]+a[1]\right]$$

$$f(x) = \left(\frac{1}{3}\right)[2a^3+3a^2+3a+2] \qquad \ldots(1)$$

$$\text{Also, } g(x) = \frac{4a}{\pi-2}\int_0^{\pi/2} x\cos x\,dx$$

$$= \frac{4a}{\pi-2}[x\sin x + \cos x]_0^{\pi/2} = \left(\frac{4a}{\pi-2}\right)\left[\frac{\pi}{2}-1\right] = 2a$$

$$\therefore \quad f(x) - g(x) = 2 \Rightarrow \frac{2}{3}a^3 + a^2 + a + \frac{2}{3} - 2a = 2$$

$$\Rightarrow \quad \left(\frac{2}{3}\right)(a^3+1) + a^2 - a = 2 \Rightarrow 2a^3 + 2 + 3a^2 - 3a - 6 = 0$$

$\Rightarrow \quad 2a^3 + 3a^2 - 3a - 4 = 0$ it has roots a_1, a_2 and a_3

Sum of roots $= a_1 + a_2 + a_3 = -\dfrac{3}{2}$

$$a_1a_2 + a_2a_3 + a_3a_1 = -\frac{3}{2}$$

$$\therefore \quad a_1{}^2 + a_2{}^2 + a_3{}^2 = (a_1+a_2+a_3)^2 - 2(a_1a_2+a_2a_3+a_3a_1)$$

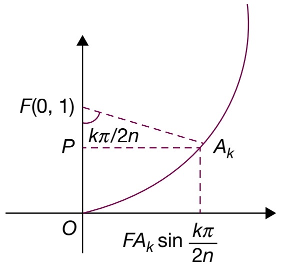
$$= \left(\frac{3}{2}\right)^2 - 2\left[-\frac{3}{2}\right] = \frac{9}{4} + 3 = \frac{21}{4}$$

$$\therefore \quad 1000(a_1{}^2+a_2{}^2+a_3)^2 = 1000 \times \frac{21}{4} = 250 \times 21 = 5250$$

PROBLEM 17

Consider a parabola $y = \dfrac{x^2}{4}$ and the point $F(0, 1)$. Let $A_1(x_1, y_1)$, $A_2(x_2, y_2)$, $A_3(x_3, y_3)$, …, $A_n(x_n, y_n)$ are 'n' points on the parabola such $x_k > 0$ and $\angle OFA_k = \dfrac{k\pi}{2n}$ where $k = 1, 2, 3, \ldots n$. then evaluate $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} FA_k$.

SOLUTION



On given parabola $x^2 = 4y$

We can assume points $A_1(2t_1, t_1{}^2)$ $A_2(2t_2, t_2{}^2)$ $A_3(2t_3, t_3{}^2)$ … $A_n(2t_n, t_n{}^2)$ on parabola.

$$FA_k = 1 + t_k^2 = \text{focal distance}$$

$$FA_k \sin\frac{k\pi}{2n} = 2t_k = PA_k$$

$$FP + OP = OF \qquad (OP = t_k^2 = \text{ordinate of point } A_k)$$

$$\Rightarrow \quad FA_k \cos\frac{k\pi}{2x} + t_k^2 = 1$$

$$\Rightarrow \quad FA_k \cos\frac{k\pi}{2x} + FA_k - 1 = 1$$

$$FA_k = \frac{2}{1+\cos\frac{k\pi}{2n}} = \frac{2}{2}\sec^2\frac{k\pi}{4n}$$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} FA_k = \lim_{n\to\infty}\frac{1}{n}\left(\sec^2\frac{k\pi}{4n}\right)$$

$$= \int_0^1 \sec^2\frac{\pi}{4}x\,dx = \left.\frac{\tan\frac{\pi}{4}x}{\frac{\pi}{4}}\right|_0^1 = \frac{4}{\pi}(1-0) = \frac{4}{\pi}$$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. The value of $\int_e^{\pi^2}\left[\log_\pi x\right]d\left(\log_e x\right)$ (where [.] denotes greatest integer function) is:

(A) $2\log_e \pi$

(B) $\log_e \pi$

(C) 1

(D) 0

2. $\int\limits_0^{\pi/4}(\pi x-4x^2)\ln(1+\tan x)\,dx$

(A) $\frac{\pi^3}{192}$ (B) $\frac{\pi^3}{192}\ln 2$

(C) $\frac{\pi^2}{96}$ (D) $\frac{\pi^2}{96}\ln 2$

3. $\int\limits_0^{\pi/2}\sin x\sin 2x\sin 3x\sin 4x\,dx=$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{8}$

(C) $\frac{\pi}{16}$ (D) $\frac{\pi}{32}$

4. $\int\limits_0^{\pi/2}\frac{\sin x}{1+\cos x+\sin x}dx$

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{4}+\log\sqrt{2}$

(C) $\frac{\pi}{4}-\log\sqrt{2}$ (D) $\frac{\pi}{4}-\log 2$

5. If $f(x)=\int_0^1\frac{dt}{1+|x-t|}$, then the value of $f'\left(\frac{1}{2}\right)$ is:

(A) 0 (B) 1

(C) 2 (D) 3

6. Let $f(x)$ be maximum and $g(x)$ be minimum of $\{x|x|, x^2|x|\}$, then $\int\limits_{-1}^{1}[f(x)-g(x)]dx=$

(A) $\frac{1}{12}$ (B) $\frac{1}{3}$

(C) $\frac{2}{3}$ (D) $\frac{7}{12}$

7. Let $T>0$ be a fixed real number. Suppose f is continuous function such that for all $x\in R$, $f(x+T)=f(x)$. If $I=\int\limits_0^T f(x)\,dx$, then $\int\limits_3^{3+3T}f(2x)\,dx=KI$ where K is:

(A) 1 (B) 2

(C) 3 (D) 4

8. $\lim\limits_{n\to\infty}\left(\left(\cos\frac{\pi}{2n}\right)\left(\cos\frac{2\pi}{2n}\right)\left(\cos\frac{3\pi}{2n}\right)\ldots\left(\cos\frac{(n-1)\pi}{2n}\right)\right)^{1/n}=$

(A) 1 (B) 2

(C) 0 (D) $\frac{1}{2}$

9. If [.] denotes G.I.F, $\int\limits_0^{2\pi}\left[|\sin x|-|\cos x|\right]dx=$

(A) -2π (B) -4π

(C) $-\pi$ (D) 0

10. The value of $\int\limits_0^{n^2}\left[\sqrt{x}\right]dx$, ([.] denotes the greatest integer function), $n\in N$ is:

(A) $\frac{n^2(n+1)^2}{4}$ (B) $\frac{1}{6}n(n-1)(4n+1)$

(C) $\sum n^2$ (D) $\frac{n(n+1)(n+2)}{6}$

11. $\int\limits_{-2}^{2}\max(x+|x|, x-[x])dx$ (where $[x]$ denotes the greatest integer $\le x$) is equal to:

(A) 4 (B) 5

(C) 6 (D) 8

12. $\int\limits_0^{2\pi}x\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx=$ ______.

(A) $\frac{\pi}{2}\ln 3$ (B) $\frac{\pi}{6}\ln 3$

(C) $\frac{\pi}{12}\ln 3$ (D) 0

13. The range of the function $f(x) = \int_{1}^{x} |t|\, dt,\ x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$ is:

(A) $\left[\frac{3}{8}, \frac{5}{8}\right]$ (B) $\left[-\frac{5}{8}, \frac{3}{8}\right]$

(C) $\left[-\frac{3}{8}, \frac{5}{8}\right]$ (D) $\left[-\frac{5}{8}, -\frac{3}{8}\right]$

14. Value of the definite integral

$$\int_{-1/2}^{1/2} \left(\sin^{-1}(3x - 4x^3) - \cos^{-1}(4x^3 - 3x)\right) dx$$

(A) 0 (B) $-\frac{\pi}{2}$

(C) $\frac{7\pi}{2}$ (D) $\frac{\pi}{2}$

15. A function $g(x)$ is given by $g(x) = \frac{x}{1 + (\log_e x)(\log_e x)\ldots\infty}$ where $x \in [1, \infty)$, then $\int_{1}^{2e} g(x)dx$ is equal to:

(A) $\frac{1}{2}\left(e - \frac{1}{e}\right)$ (B) $\frac{1}{2}\left(e + \frac{1}{e}\right)$

(C) $\frac{1}{2}\left(e^2 - 1\right)$ (D) $\frac{1}{2}\left(e^2 + 1\right)$

16. Suppose $f(x) = \int_{0}^{x} \sqrt{1 + t^4}\, dt$ for all real x and let $f(1) = c$, then the value of $(f^{-1})'(c)$ is:

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{\sqrt{3}}{2}$

(C) $\frac{1}{2}$ (D) None of these

17. If x satisfies the equation

$$\left(\int_{0}^{1} \frac{dt}{t^2 + 2t\cos\alpha + 1}\right)x^2 - \left(\int_{-3}^{3} \frac{t^2 \sin 2t}{t^2 + 1} dt\right)x - 2 = 0 \ (0 < \alpha < \pi),$$

then the value x is:

(A) $\pm\sqrt{\frac{\alpha}{2\sin\alpha}}$ (B) $\pm\sqrt{\frac{2\sin\alpha}{\alpha}}$

(C) $\pm\sqrt{\frac{\alpha}{\sin\alpha}}$ (D) $\pm 2\sqrt{\frac{\sin\alpha}{\alpha}}$

18. A cubic $f(x)$ vanishes at $x = -2$ and has local minimum/maximum at $x = -1$ and $x = \frac{1}{3}$. If $\int_{-1}^{1} f(x)dx = \frac{14}{3}$, then $f(1) =$

(A) 1 (B) 2

(C) 3 (D) 4

19. If $A = \int_{1}^{\sin\theta} \frac{t\ dt}{1+t^2} dt$, $B = \int_{1}^{\operatorname{cosec}\theta} \frac{1}{t(1+t^2)} dt$, then $\begin{vmatrix} A & A^2 & B \\ e^{A+B} & B^2 & -1 \\ 1 & A^2 + B^2 & -1 \end{vmatrix}$:

(A) sin θ (B) cosec θ

(C) 0 (D) 1

20. The value of $\int_{-1}^{1} \{x^2 + x - 3\} dx$, where $\{.\}$ denotes the fractional part of 'x', is equal to:

(A) $\frac{1}{3}\left(1 + 3\sqrt{5}\right)$ (B) $\frac{1}{6}\left(1 + 3\sqrt{5}\right)$

(C) $\frac{1}{6}\left(3\sqrt{5} - 1\right)$ (D) $\frac{1}{3}\left(3\sqrt{5} - 1\right)$

21. If $f(x)$ is monotonic and differentiable function, then $\int_{f(a)}^{f(b)} 2x(b - f^{-1}(x))\, dx =$

(A) $\int_{a}^{b} f^2(x)\, dx$ (B) $\int_{a}^{b} (f^2(x) - f^2(a))\, dx$

(C) $\int_{a}^{b} (f^2(x) - f^2(b))\, dx$ (D) $\int_{a}^{b} (f^2(x) + f^2(b))\, dx$

22. The value of $\int_{-2n}^{2n+\frac{1}{2}} \left\{\frac{x}{2}\right\}(\sin \pi x) \cdot dx$ (where $\{x\}$ denotes the fractional part of x) is:

(A) $\frac{n}{\pi}$ (B) $\frac{-4n\pi + 1}{2\pi^2}$

(C) $\frac{1 - 2n\pi}{\pi^2}$ (D) $\frac{2n\pi - 1}{\pi^2}$

23. Let $I_n = \int_{0}^{\infty} e^{-x}(\sin x)^n\, dx,\ n \in N,\ n > 1$, then $\frac{I_{2008}}{I_{2006}}$ equals:

(A) $\frac{2007 \times 2006}{2008^2 + 1}$ (B) $\frac{2008 \times 2007}{2008^2 + 1}$

(C) $\frac{2006 \times 2004}{2008^2 - 1}$ (D) $\frac{2008 \times 2007}{2008^2 - 1}$

24. If $f(x) = \int_{1/x}^{\sqrt{x}} \cos t^2\ dt (x > 0)$, then $\frac{df(x)}{dx}$ is:

(A) $\frac{\sqrt{x}\cos x + 2\cos(x^{-2})}{2x\sqrt{x}}$ (B) $\frac{x\sqrt{x}\cos x + 2\cos(x^{-2})}{2x^2}$

(C) $2\sqrt{2}\ \cos x - \frac{2}{x}\cos\frac{1}{x}$ (D) none of these

25. If $A = \int_0^{\pi} \frac{\cos x}{(x+2)^2}\, dx$, then $\int_0^{\pi/2} \frac{\sin 2x}{x+1}\, dx$ is equal to:

(A) $A - \frac{1}{2} - \frac{1}{\pi + 2}$
(B) $\frac{1}{2} + \frac{1}{\pi + 2} - A$
(C) $\frac{1}{\pi + 2} - A$
(D) $1 + \frac{1}{\pi + 2} - A$

26. The value of the integral $\int_0^{n\pi + t} (|\cos x| + |\sin x|)\, dx$, where $0 < t < \frac{\pi}{2}$, is:

(A) n
(B) $2n + \sin t + \cot$
(C) $\cos t$
(D) $4n + \sin t - \cos t + 1$

27. If $\int_0^{\pi/2} \sin^{10} x \cos^{10} x\, dx = 2^{-n} \int_0^{\pi/2} (\sin^{10} x + \cos^{10} x)\, dx$, $(n \in N)$, then the value of n is equal to:

(A) 7
(B) 9
(C) 10
(D) 11

28. The value of the integral $\left| \int_0^{2\pi} [2 \sin x] dx \right|$ is ([.] denotes the greatest integer function):

(A) π
(B) 2π
(C) 3π
(D) 4π

29. If $\int_0^1 e^{5x^6} (x - \alpha)\, dx = 0$, then:

(A) $1 < \alpha < 2$
(B) $\alpha < 0$
(C) $0 < \alpha < 1$
(D) $\alpha = 0$

30. If $\int_{\sin x}^{1} t^2 . f(t)\, dt = 1 - \sin x$, $\forall\ x \in \left(0, \frac{\pi}{2}\right)$, then the value of $f\left(\frac{1}{\sqrt{3}}\right)$ is:

(A) $\frac{1}{\sqrt{3}}$
(B) $\sqrt{3}$
(C) $\frac{1}{3}$
(D) 3

31. The value of the integral $\int_0^{1/\sqrt{3}} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ must be:

(A) $\frac{\pi}{2\sqrt{2}}$
(B) $\frac{\pi}{4\sqrt{2}}$
(C) $\frac{\pi}{8\sqrt{2}}$
(D) None of these

32. $\int_0^{\infty} f(x^n + x^{-n}) \ln x \frac{dx}{x}$ equals:

(A) 0
(B) $f(x^n + x^{-n})$
(C) function of x
(D) can not be determined

33. If $I = \int_0^{\pi/2} e^{-\alpha \sin x}\, dx$, $\alpha \in (0, \infty)$, then:

(A) $I < \frac{\pi}{2} e^{-\alpha}$
(B) $I > \frac{\pi}{2}$
(C) $\frac{\pi}{2} e^{-\alpha} < I < \frac{\pi}{2}$
(D) $I = \frac{\pi}{2} e^{-\alpha}$

34. The value of the integral $\int_{-1}^{3} \left(\tan^{-1} \frac{x}{x^2 + 1} + \tan^{-1} \frac{x^2 + 1}{x} \right) dx$ is equal to:

(A) π
(B) 2π
(C) 4π
(D) None of these

35. If $f(x) = \int_1^x \frac{dt}{2 + t^4}$, then:

(A) $f(2) < \frac{1}{3}$
(B) $f(2) > \frac{1}{3}$
(C) $f(2) = \frac{1}{3}$
(D) $f(2) > 1$

36. $\int_0^{\pi/4} \left(\frac{x}{x \sin x + \cos x} \right)^2 dx =$

(A) $\frac{4 + \pi}{4 - \pi}$
(B) $\frac{5 - \pi}{5 + \pi}$
(C) $\frac{4 - \pi}{4 + \pi}$
(D) $\frac{5 + \pi}{5 - \pi}$

37. The value of $\int_{-\pi/4}^{\pi/4} \frac{1 + \sin x}{\cos x \sqrt{\cos 2x}} dx$ is:

(A) 0
(B) $\frac{\pi}{4}$
(C) $\frac{\pi}{2}$
(D) π

38. Let $f: (0, \infty) \to R$ and $F(x) = \int_1^x f(t)\, dt$. If $F(x^2) = x^2 (1 + x)$, then $f(4)$ equals:

(A) 5/4
(B) 7
(C) 4
(D) 2

39. A function f is defined by

$$f(x) = \frac{1}{2^{r-1}}, \frac{1}{2^r} < x \le \frac{1}{2^{r-1}}, r = 1, 2, 3, \ldots, \text{ then } \int_0^1 f(x) dx = :$$

(A) 1/3
(B) 1/4
(C) 2/3
(D) 1/3

40. If *a* function $f(x)$ satisfies the relation $f(x) = e^x + \int_0^1 e^x f(t) dt$, then $f(x) =$ ____.

(A) $\frac{e^x}{2 - e}$
(B) $\frac{e^x}{2}$
(C) $2e^x$
(D) $(e - 2)e^x$

41. Let $f(x)=\int_1^x \frac{e^t}{t^2}dt\ \forall x>0$. Then value of $\int_1^x \frac{e^t}{(t+a)^2}dt$ is:

(A) $e^{-a}\left(f(x+a)+f(1+a)\right)$ (B) $e^{-a}\left(f(x+a)+f(a)\right)$

(C) $e^{-a}\left(f(x+a)-f(1+a)\right)$ (D) none of these

42. If $f(x)=\int_0^x (1+t^3)^{-1/2}\,dt$ and $g(x)$ is the inverse of f, then the value of $\frac{g''(x)}{g^2(x)}$ is:

(A) $3/2$ (B) $2/3$
(C) $1/3$ (D) $1/2$

43. Consider the real-valued function $y=f(x)=\int_0^1 \frac{x^2+t^2}{2-t}dt$. It represents:
(A) a parabola with latus rectum ln 2
(B) an ellipse with latus rectum ln 4
(C) a parabola with latus rectum $\frac{1}{\ln 2}$
(D) an ellipse with latus rectum $\frac{1}{\ln 2}$

44. The value of $\int_3^6 \left(\sqrt{x+\sqrt{12x-36}}+\sqrt{x-\sqrt{12x-36}}\right)dx$ is equal to:

(A) $6\sqrt{3}$ (B) $4\sqrt{3}$

(C) $12\sqrt{3}$ (D) $2\sqrt{3}$

45. If $z\neq 0$, then $\int_{x=0}^{100} [\arg|z|]\,dx$ is (where [.] denotes the greatest integer function):
(A) 0 (B) 10
(C) 100 (D) not defined

46. $I_1=\int_0^1 \frac{x^2}{\left(2-x^3\right)e^{x^3}}dx$, $I_2=\int_0^1 \frac{e^x}{1+x}dx$ then $I_1 : I_2 =$ ____.

(A) 1 : 3 (B) 3 : 1
(C) 1 : 3*e* (D) 3*e* : 1

47. If $f(x)=\int_1^x \frac{\tan^{-1}t}{t}dt;\ x>0$, then the value of $f\left(e^2\right)-f\left(\frac{1}{e^2}\right)$ is:

(A) 0 (B) $\frac{\pi}{2}$

(C) π (D) 2π

48. Let $f(x)$ be a function satisfying $f'(x)=f(x)$ with $f(0)=1$ and g be the function satisfying $f(x)+g(x)=x^2$. The value of the integral $\int_0^1 f(x)g(x)\,dx$ is:

(A) $e-\frac{1}{2}e^2-\frac{5}{2}$ (B) $e-e^2-3$

(C) $\frac{1}{2}(e-3)$ (D) $e-\frac{1}{2}e^2-\frac{3}{2}$

49. If $y=\int_1^x x^2\sqrt{\ln t}\,dt$ then the value of $\frac{d^3y}{dx^3}$ at $x=e$:

(A) 33 (B) 11

(C) $\frac{33}{4}$ (D) None of these

50. If $f(x)$ is a differentiable function and $\int_0^{x^3} t^2 f(t)\,dt=\frac{3}{13}x^{13}+5$, then $f\left(\frac{8}{27}\right)=$

(A) 8/27 (B) 16/27
(C) 16/81 (D) 8/9

51. $\int_{-2\pi}^{2\pi} \frac{\sin^6 x}{(\sin^6 x+\cos^6 x)(1+e^{-x})}dx=$

(A) 2π (B) π
(C) $\pi/2$ (D) 4π

52. Let $f(x)$, $g(x)$, $h(x)$ be continuous in [0, 2*a*] and satisfies: $f(2a-x)=f(x)$, $g(2a-x)=g(x)$, $h(x)+h(2a-x)=3$, $f(2a-x)g(2a-x)=f(x)g(x)$, then $\int_0^{2a} f(x)g(x)h(x)\,dx=$

(A) $\int_0^{2a} f(x)g(x)$ (B) $3\int_0^a f(x)g(x)\,dx$

(C) $2\int_0^a f(x)g(x)\,dx$ (D) $\int_0^a f(x)g(x)\,dx$

53. $I_n=\int_0^{\pi/2} \cos^n x\cos(nx)\,dx$, $n\in N$ then $\sqrt{I_{2001}:I_{2002}}$ can be the eccentricity of:
(A) Parabola (B) Ellipse
(C) Circle (D) Hyperbola

54. $I_n=\int_0^1 x^n\tan^{-1}x\,dx$. If $a_nI_{n+2}+b_nI_n=c_n\ \forall\ n\in N,\ n\geq 1$ and $a_n I_{n+2}+b_nI_n=c_n\ \forall\ n\in N,\ n\geq 1$ then:
(A) a_1, a_2, a_3 … are in A.P.
(B) b_1, b_2, b_3 … are in G.P.
(C) c_1, c_2, c_3 ….. are in H.P.
(D) a_1, a_2, a_3 … are in H.P.

55. Let $a_n=\int_0^{\frac{\pi}{2}} \frac{1-\cos(2nx)}{1-\cos(2x)}dx$, then the value of $\begin{vmatrix} \frac{\pi}{2} & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$ equals to:
(A) 1 (B) 2
(C) 3 (D) 0

56. Let $I_1 = \int\limits_{\sec^2 z}^{2-\tan^2 z} xf(x(3-x))\,dx$ and let $I_2 = \int\limits_{\sec^2 z}^{2-\tan^2 z} f(x(3-x))$ dx Where 'f' is a continuous function and 'z' is any real number, then $\frac{I_1}{I_2}$ =

(A) $\frac{3}{2}$ (B) $\frac{1}{2}$

(C) 1 (D) $\frac{2}{3}$

57. The integral $\int\limits_{\pi/4}^{\pi/2} e^x$ $(\ln(\cos x))(\cos x \sin x - 1)\,\text{cosec}^2\, xdx$ is:

(A) $e^{\frac{\pi}{4}}\left(e^{\frac{\pi}{4}}+\frac{1}{2}\ln 2-1\right)$ (B) $e^{\frac{\pi}{4}}\left(e^{\frac{\pi}{2}}+\frac{1}{2}\ln 2-2\right)$

(C) $e^{\frac{\pi}{4}}\left(e^{\frac{\pi}{4}}-\frac{1}{2}\ln 2-1\right)$ (D) $e\,e^{\frac{\pi}{4}}\left(e^{\frac{\pi}{4}}+\ln 2-1\right)$

58. If $\int\limits_{-2}^{-1}\frac{\lambda^2 d\lambda}{\sqrt{\lambda^4-\lambda^2+1}} = I,$ then the value of $\int\limits_{-2}^{-1}\frac{xdx}{\sqrt{\left(x-x^{-1}\right)^2+1}}+\int\limits_{-1}^{-1/2}\frac{dx}{x^3\sqrt{x^2+x^{-2}-1}}$ is equal to:

(A) zero (B) I

(C) $-I$ (D) $-2I$

59. If $\sum\limits_{i=1}^{6}(\sin^{-1}x_i+\cos^{-1}y_i) = 9\pi$, then $\int\limits_{\sum_{i=1}^{6}x_i}^{\sum_{i=1}^{6}y_i} x\ln(1+x^2)\left(\frac{e^x}{1+e^{2x}}\right)$ dx is equal to:

(A) 0 (B) e^6+e^{-6}

(C) $\ln\left(\frac{37}{12}\right)$ (D) e^6-e^{-6}

60. If $f(x)=\begin{vmatrix} x & \cos x & e^{x^2} \\ \sin x & x^2 & \sec x \\ \tan x & 1 & 2 \end{vmatrix}$, then the value of $\int_{-\pi/2}^{\pi/2} f(x)dx$ is equal to:

(A) 5 (B) 3

(C) 1 (D) 0

61. $\lim\limits_{x\to 0}\frac{1}{x}\left(\int_y^c e^{\sin^2 t}dt-\int_{x+y}^c e^{\sin^2 t}dt\right)$ is equal to (where c is a constant):

(A) $e^{\sin^2 y}$ (B) $\sin 2y\, e^{\sin^2 y}$

(C) 0 (D) none of these

62. The value of $\int\limits_{-2011}^{\infty}\left[\cot^{-1}x\right]dx$, where [.] greatest integer function:

(A) $\cot 3+\cot 2+\cot 1+2011$

(B) $\cot 3+\cot 2+\cot 1+4022$

(C) $\cot 3+\cot 2+\cot 1+6033$

(D) 0

63. Let $f: R^+ \to R$ be a differentiable function with $f(A) = 3$ and satisfying $\int\limits_1^{xy} f(t)dt = y\int\limits_1^{x} f(t)dt + x\int\limits_1^{y} f(t)\,dt\ \forall\, x, y \in R^+$, then $f(x) =$

(A) $3\ln x$ (B) $\ln x+3$

(C) $3+3\ln x$ (D) None of these

64. $\lim\limits_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}\ldots+\sqrt{n}}{8n+10n^{3/2}}$ is equal to:

(A) $\frac{2}{15}$ (B) $\frac{1}{15}$

(C) $\frac{4}{15}$ (D) $\frac{7}{15}$

65. If $[x]$ denotes the greatest integer function, then $\int_{-2011}^{2012}[x+[x+[x]]]dx$ =

(A) 3×4023 (B) 0

(C) 1 (D) -1

66. $\int\limits_0^{\pi}\frac{dx}{(5+4\cos x)^2} =$

(A) $\frac{3\pi}{7}$ (B) $\frac{5\pi}{17}$

(C) $\frac{5\pi}{27}$ (D) $\frac{5\pi}{26}$

67. If z_1, z_2, z_3 are the affixes of vertices of an equilateral triangle, then $\int\limits_2^4\left|Arg\left(\frac{2z_1-z_2-z_3}{2(z_3-z_2)}\right)\right|dx =$

(A) π (B) $\frac{2\pi}{3}$

(C) $\frac{\pi}{2}$ (D) 0

68. If $I_1 = \int\limits_1^{\sin\theta}\frac{x}{1+x^2}dx$ and $I_2 = \int\limits_1^{\text{cosec}\,\theta}\frac{dx}{x\left(x^2+1\right)}$, then the value of $\begin{vmatrix} I_1 & I_1^2 & I_2 \\ e^{I_1+I_2} & I_2^2 & -1 \\ 1 & I_1^2+I_2^2 & -1 \end{vmatrix}$ is:

(A) 1 (B) $\frac{-11}{2}$

(C) 9 (D) 0

69. $\int_{-\pi}^{\pi} \left(1-x^3\right) \sin^2 x.\cos^4 x\, dx$

(A) 0 (B) $\pi/16$

(C) $\pi/8$ (D) $\pi^2/8$

70. If p, q, r, s are in A.P and

$$f(x) = \begin{vmatrix} p+\sin x & q+\sin x & p-r+\sin x \\ q+\sin x & r+\sin x & -1+\sin x \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix} \text{ such that}$$

$\int_0^2 f(x)dx = -4$ then the common difference of A.P is:

(A) ± 2 (B) ± 4

(C) $\pm 1/2$ (D) 1

71. Let 'f' be a differentiable function such that $f(x) = x^2 + \int_0^x e^{-t} f(x-t)dt$, , then $f(1) + f(2) + f(3) + \ldots + f(9) =$

(A) 960 (B) 1000
(C) 1024 (D) 1126

72. If $f(x)$ is continuous function such that $f(x) > 0$, $\forall\, x \ge 0$ and $(f(x))^{2014} = 1 + \int_0^x f(t)dt$, then the value of $(f(2014))^{2013}$ is equal to:

(A) 2013 (B) 2014
(C) 2015 (D) 2016

73. If $f : [0, 1] \to [0, \infty)$ is differentiable function with decreasing first derivative, such that $f(0) = 0$ and $f'(x) > 0$, then:

(A) $\int_0^1 \frac{dx}{f^2(x)+1} > \frac{f(1)}{f'(1)}$

(B) $\int_0^1 \frac{dx}{f^2(x)+1} < \frac{f(1)}{f'(1)}$

(C) $\int_0^1 \frac{dx}{f^2(x)+1} > \frac{\tan^{-1}(f(1))}{f'(1)}$

(D) $\int_0^1 \frac{dx}{f^2(x)+1} = \frac{f(1)}{f'(1)}$

74. Let $I(n) = \int_{2016}^{2016+\frac{1}{n}} x \cos^2(x-2016)dx$,

(A) $I\left(\frac{1}{\pi}\right) = \frac{(\pi+2016)\pi}{2}$ (B) $I\left(\frac{1}{\pi}\right) = \frac{(\pi+4032)\pi}{4}$

(C) $\lim_{n\to\infty} nI(n) = 2016$ (D) $\lim_{n\to\infty} n.I(n) = 2017$

Then which of the above are correct?

(A) A, C (B) A, D
(C) B, C (D) B, D

75. Let $f : D \to y$, $f(x) = \ln\left[\cos|x| + \frac{1}{2}\right]$, (where [.] represents greatest integer function), then $\int_{x_1}^{x_2} \left(\lim_{n\to\infty}\left(\frac{(f(x))^n}{x^2+\tan^2 x}\right)\right)dx$ where $(x_1, x_2, \in D)$ is:

(A) 0 (B) 1

(C) $\frac{3}{2}$ (D) 2

76. Let f be a non - negative function defined on the interval $[0, 1]$. If $\int_0^x \sqrt{1-\left(f'(t)^2\right)}dt = \int_0^x f(t)dt.\ 0 \le x \le 1$ and $f(0) = 0$, then:

(A) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(B) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(C) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(D) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

77. Let $f(x) = \tan x$ and $g(x) = \sin x$ where $[.], \{.\}$ denotes greatest integer function, fractional part function, then $\int_{2\pi}^{2\pi+\pi/4} \sqrt{\left[\cos x + \left\{\left[f(x)\right] + \left\{g(x)\right\}\right\}\right]}\, dx$:

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$

(C) $\frac{\sqrt{2}\pi}{4}$ (D) $\frac{\pi}{2}$

78. $\int_0^{\pi/2} \frac{\sin^2 9x}{\sin x} dx =$

(A) $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{9}$ (B) $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{18}$

(C) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots + \frac{1}{19}$ (D) $1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{17}$

79. If ([.] denotes G.I.F), then $I = \int_1^5 \frac{dx}{[x][1+x]}$, then $[I] =$ ____.

(A) 0 (B) 2
(C) 1 (D) does not exist

80. Let $I_1 = \int_0^\infty \frac{x^2\sqrt{x}}{(1+x)^6}dx$, $I_2 = \int_0^\infty \frac{x\sqrt{x}}{(1+x)^6}dx$, then:

(A) $I_1 = 2I_2$ (B) $I_2 = 2I_1$

(C) $I_1 = I_2^2$ (D) $I_1 = I_2$

81. If $I_1 = \int\limits_0^1 \dfrac{x^{5/2}(1-x)^{7/2}}{12}dx$, $I_2 = \int\limits_0^1 \dfrac{x^{5/2}(1-x)^{7/2}}{(3+x)^8}dx$, then:

(A) $I_2 = 144\sqrt{3}\,I_1$ (B) $I_1 = 864\sqrt{3}\,I_2$

(C) $I_1 = I_2$ (D) $I_1 I_2 = 864\sqrt{3}$

82. If $f(x) = (p\beta - \beta^2 - 2)x - \int\limits_0^x (\cos^4 t)dt$ is decreasing function of $x\ \forall\ x \in R$, and $\forall\ \beta \in R$ where β being independent of x, then:

(A) $p \in (-\infty, 1)$ (B) $p \in (-2\sqrt{2}, 2\sqrt{2})$

(C) $p \in (1, \infty)$ (D) $p \in R - \left(-\sqrt{3}, \sqrt{3}\right)$

83. Let $f(x) = \sec^{-1}x - \tan^{-1}\sqrt{x^2+1}$ for $|x| \geq 1$ and $f(x) = 0$ for $|x| < 1$. Then the value of $\int\limits_{-y}^{y} f(x)dx$ Given that $\int\limits_1^y \sec^{-1} x dx = \lambda$ where $y > 1$

(A) $\pi y - 2\lambda$ (B) $\pi(y-1) - 2\lambda$

(C) $\pi(y-1) - \lambda$ (D) 0

84. The value of constant $a > 0$ such that $\int\limits_0^a \left[\tan^{-1}\sqrt{x}\right]dx = \int\limits_0^a \left[\cot^{-1}\sqrt{x}\right]dx$ where [.] denotes G.I.F is ____.

(A) $\dfrac{2(3+\cos 4)}{1-\cos 4}$ (B) $\dfrac{(3-\cos 4)}{1+\cos 4}$

(C) $\dfrac{2(3+\cos 4)}{1+\cos 4}$ (D) $\dfrac{(3+\cos 4)}{1-\cos 4}$

85. $\int\limits_{-\pi/2}^{\pi/2} \sqrt{\cos^{2n-1} x - \cos^{2n+1} x}\ dx$

(A) $\dfrac{2n+1}{4}$ (B) $\dfrac{2n-1}{4}$

(C) $\dfrac{4}{2n+1}$ (D) $\dfrac{4}{2n-1}$

86. If $\int\limits_0^1 e^{2\{x\}}e^{e^{\{x\}}}dx = ke^e$, then $[k]$ is (where {.} denotes the fractional part and [.] denotes greatest integer function)

(A) 0 (B) 1

(C) 2 (D) none of these

87. If $\int_a^b \dfrac{x^n}{x^n + (16-x)^n}dx = 6, (n \in N)$ and $a + b = 16$, then:

(A) $a = 4, b = 12$ (B) $a = 2, b = 14$

(C) $a = -4, b = 20$ (D) $a = 2, b = 8$

88. The line $y = k$ intersects the curve $y = f(x)$ at least at two points. If $\int_2^x f(t)\,dt = \dfrac{x^2}{2} + \int_2^x t^2 f(t)\,dt$ then $k \in$

(A) $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$ (B) $(-1, 1)$

(C) $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right) - \{0\}$ (D) $\left(-\dfrac{1}{2}, 1\right)$

89. Let $f(x)$ be twice differentiable function such that $f'(1) = 1/2$ and $f(0) = 0$ and satisfies $\int\limits_0^x (f''(t) + 2tf'(t) + t^2 f''(t))\,dt = k$, where k is constant, then $\lim\limits_{x\to\pi/4^+} [f(x)]$ equals, where $[\cdot]$ denotes G.I.F.

(A) 0 (B) 1

(C) –1 (D) $\pi/4$

90. The value of $\int\limits_{\frac{3\pi}{4}}^{\pi} \left[\sin x + \left[\dfrac{4x}{\pi}\right]\right]dx$, where [.] denotes greatest integer function.

(A) 0 (B) $\dfrac{4}{\pi}$

(C) $\dfrac{7\pi}{4}$ (D) $\dfrac{3\pi}{4}$

91. The value of $\int\limits_0^{\pi/2} \log(\sin^2\theta + k^2\cos^2\theta)\,d\theta$ is equal to $(k \geq 0)$

(A) $\pi \ln\left(\dfrac{k^2+1}{2}\right)$ (B) $\pi \ln\left(\dfrac{k^3+1}{2}\right)$

(C) $\pi \ln\left(\dfrac{1}{2}\right)$ (D) $\pi \ln\left(\dfrac{k+1}{2}\right)$

92. If [.] denotes G.I.F and $\int\limits_0^{\pi/3} \left[\sqrt{3}\tan x\right]dx = \sin^{-1}\left(\sqrt{k}\right)(k > 0)$ then $\int\limits_0^{2k} [\tan x]\,dx =$

(A) 1 (B) –1

(C) $\dfrac{6}{7} - \dfrac{\pi}{4}$ (D) $\dfrac{6}{7} - \dfrac{\pi}{6}$

93. $\int\limits_{-\pi/4}^{3\pi/4} \dfrac{e^{\pi/4}\,dx}{(e^x + e^{\pi/4})(\sin x + \cos x)} = \lambda \int\limits_{-\pi/2}^{\pi/2} \sec x\ dx$, then λ =

(A) $\dfrac{1}{2}$ (B) $\dfrac{1}{\sqrt{2}}$

(C) $\dfrac{1}{2\sqrt{2}}$ (D) $2\sqrt{2}$

94. The value of the definite integral $\int_0^{\pi^2/4} \frac{dx}{1+\sin\sqrt{x}+\cos\sqrt{x}}$ is:

(A) $\pi \ln 2$ (B) $\frac{\pi \ln 2}{2}$

(C) $\frac{\pi \ln 2}{4}$ (D) $2\pi \ln 2$

95. If $\int_0^{2\pi} \frac{dx}{2+\sin 2x} = k \int_0^{\pi^2/2} \frac{dt}{7+\cos 2t}$, then the value of k, is:

(A) 4 (B) 8

(C) 12 (D) 16

96. If the value of the definite integral $\int_0^{\pi/2} (\tan^{-1}(\cot x)+\cot^{-1}(\tan x))\, dx$ is equal to:

(A) $\frac{\pi^2}{2}$ (B) $\frac{\pi^2}{4}$

(C) $\frac{\pi^2}{8}$ (D) π^2

97. Let $f(x) = \begin{cases} \frac{1}{x^3}\int_0^x \sin(t^2)dt \; ; & \text{for } x \neq 0 \\ a \; ; & \text{for } x = 0 \end{cases}$ $(a \in R)$. If $f(x)$ is continuous at $x = 0$, then the value of a, is:

(A) 1 (B) $\frac{1}{2}$

(C) $\frac{1}{3}$ (D) $\frac{1}{6}$

98. The value of definite integral $\int_{\frac{1}{3}}^{3} \frac{1}{x}\ln\left(\left|\frac{x+x^2-1}{x-x^2+1}\right|\right) dx$ is equal to:

(A) $\frac{1}{8}$ (B) 0

(C) $\frac{-8}{3}$ (D) 3

99. Let $f : R \to [4, \infty)$ be an onto quadratic function whose leading coefficient is 1, such that $f'(x) + f'(2-x) = 0$. Then the value of $\int_1^3 \frac{dx}{f(x)}$ is equal to:

(A) $\frac{1}{4}$ (B) $\frac{1}{8}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{8}$

100. The value of definite integral $\int_{-\pi}^{\pi} \frac{x^2}{1+\sin x+\sqrt{1+\sin^2 x}}\, dx$, is:

(A) $\frac{3\pi^2}{2}$ (B) $\frac{\pi^3}{3}$

(C) $\frac{2\pi^3}{3}$ (D) $\frac{\pi^3}{6}$

101. $\lim\limits_{n\to\infty}\left(\sin\frac{1}{\sqrt{n}}\right)\left(\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\frac{1}{\sqrt{n+3}}+\ldots+\frac{1}{2\sqrt{n}}\right)$ has the value equal to:

(A) $2\sqrt{2}-2$ (B) $2\sqrt{2}-1$

(C) $2\sqrt{2}$ (D) 2

102. The value of definite integral $\int_{2\pi}^{4\pi} \frac{\sin^{-1}(\sin x)}{\sin^4 x+\cos^2 x}\, dx$ is equal to:

(A) 0 (B) π

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

103. The value of definite integral $\int_{\pi/2}^{2\pi/3} \frac{e^{\frac{-\theta}{2}}\sqrt{1-\sin\theta}}{1+\cos\theta}\, d\theta$ is equal to:

(A) $\frac{2}{\sqrt{3}}\, e^{\frac{-\pi}{2}} - \sqrt{2}e^{\frac{-\pi}{4}}$ (B) $\sqrt{2}e^{\frac{-\pi}{3}}\left(\frac{e^{\frac{\pi}{6}}}{\sqrt{3}}-1\right)$

(C) $2e^{\frac{-\pi}{2}}\left(\frac{e^{\frac{\pi}{3}}}{\sqrt{3}}-\sqrt{2}e^{\frac{\pi}{4}}+e^{\frac{\pi}{6}}\right)$ (D) $2e^{\frac{-\pi}{3}} - \sqrt{2}e^{\frac{-\pi}{4}}$

104. The value of definite integral $\int_0^1 \left(\sqrt{1-x^2}\cdot\ln x\right) dx$ is equal to:

(A) $-\frac{\pi}{4}\ln 2$ (B) $-\frac{\pi}{8}\ln 2$

(C) $-\frac{\pi}{4}\ln 2-\frac{\pi}{8}$ (D) $-\frac{\pi}{8}\ln 2-\frac{\pi}{4}$

105. Let $f(x)$ be a twice differentiable function on R satisfying $f(x) = \int_0^x e^t \sin(x-t)\, dt$ and $g(x) = f''(x) - f(x)$ for all $x \in$ R. Then the number of integers in the range of $g(x)$, is:

(A) 0 (B) 1

(C) 2 (D) 3

106. Let $f : R \to R$ be a twice differentiable function satisfying $f(2) = -1, f'(2) = 4$ and $\int_2^3 (3-x) f''(x)\, dx = 7$, then the value of $f(3)$ lies in the interval:

(A) $(0, e)$ (B) (e, e^2)

(C) (e^2, e^3) (D) (e^3, e^4)

107. Let $f : [0, 1] \to R$ be a continuous function then the maximum value of $\int_0^1 f(x)\cdot x^2\,dx - \int_0^1 x\cdot(f(x))^2\,dx$ for all such function(s) is:

(A) $\frac{1}{8}$ (B) $\frac{1}{20}$

(C) $\frac{1}{12}$ (D) $\frac{1}{16}$

108. $\int_0^{\pi/2} \frac{\sin^3 x}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}\,dx$ is equal to:

(A) $\frac{\pi}{2} - \tan^{-1} 2$ (B) $\ln\frac{\pi}{2} - \ln(\tan^{-1}2)$

(C) $\ln(\tan^{-1}2)$ (D) $\ln\frac{\pi}{2}$

109. The value of definite integral

$$\int_{-1}^{1} (\tan^{-1}(\sin(\cos^{-1} x))) + \cot^{-1}(\cos(\sin^{-1} x))\,dx \text{ is equal to:}$$

(A) 0 (B) -1

(C) 1 (D) π

110. The value of the definite integral $\int_0^{\pi/6} \frac{\sqrt{1+\sin x}}{\cos x}\,dx$ is equal to:

(A) $\sqrt{2}\ln(\sqrt{8} - \sqrt{6} + \sqrt{4} - \sqrt{3})$

(B) $\sqrt{2}\ln(\sqrt{8} + \sqrt{6} - \sqrt{4} - \sqrt{3})$

(C) $\frac{1}{\sqrt{2}}\ln 2$

(D) $\sqrt{2}\ln 2$

111. The value of the definite integral

$$2\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{x}\,dx - \int_0^{1} \frac{\tan^{-1} x}{x}\,dx \text{ is equal to:}$$

(A) $\frac{\pi}{8}\ln 2$ (B) $\frac{\pi}{4}\ln 2$

(C) $\frac{\pi}{2\sqrt{2}}\ln 2$ (D) $\frac{\pi}{2}\ln 2$

112. Let $V_n = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x}\,dx$. If $\sum_{r=1}^{100} V_r = \frac{k\pi}{2}$, where $k \in N$, then k is equal to:

(A) 100 (B) 2525

(C) 5050 (D) 4950

113. If $f(x) = \frac{4}{3}(1 - \cos^6 x - \sin^6 x)$, $x \in R$, then

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{f\left(\frac{1}{n}\right)} + \sqrt{f\left(\frac{2}{n}\right)} + \sqrt{f\left(\frac{3}{n}\right)} + \dots + \sqrt{f\left(\frac{n}{n}\right)}\right) \text{ is}$$

equal to:

(A) $\frac{1-\cos 1}{2}$ (B) $\frac{1-\sin 1}{2}$

(C) $\frac{1-\sin 2}{2}$ (D) $\frac{1-\cos 2}{2}$

114. If $f(x) = \cos^{-1}x$ and $g(x) = [\cos(\cos x)] + [\sin(\sin x)]$ where [.] is greatest integer function then the value of $\int_{-\pi}^{\pi} fog(x)\,dx$, is equal to:

(A) $\frac{3\pi^2}{2}$ (B) $\frac{3\pi^2}{4}$

(C) $\frac{5\pi^2}{2}$ (D) $\frac{\pi^2}{4}$

115. The value of the definite integral

$$\int_0^{\pi} ((1 - x\sin 2x)e^{\cos^2 x} + (1 + x\sin 2x)e^{\sin^2 x})\,dx \text{ is equal to:}$$

(A) π (B) πe

(C) $\pi(e - 1)$ (D) $\pi(e + 1)$

116. The value of the definite integral $\int_0^{\frac{\pi}{2}} \frac{\sin 8x}{\sin x}\,dx$ is equal to:

(A) 0

(B) $2\left(1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8}\right)$

(C) $2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\right)$

(D) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8}$

117. If x and y are independent variables and

$f(x) = \left(\int_0^x e^{x-y} f'(y)\,dy\right) - (x^2 - x + 1)e^x$, where $f(x)$ is a differentiable function, then $f\left(-\frac{1}{2}\right)$ equals:

(A) $2\sqrt{e}$ (B) $-2\sqrt{e}$

(C) $-\frac{2}{\sqrt{e}}$ (D) $\frac{2}{\sqrt{e}}$

118. Let $f : R \to R$ be defined as $f(x) = \int_0^x (\sin^2 t + 1)\,dt$ and g is the inverse function of f, then $g'\left(\frac{3\pi}{4}\right)$ is equal to:

(A) $\frac{4}{3\pi}$ (B) $\frac{4\pi}{3}$

(C) $\frac{1}{2}$ (D) 1

119. The value of the definite integral

$$\int_2^4 (x(3-x)(4+x)(6-x)(10-x)+\sin x)\,dx \text{ equals:}$$

(A) $\cos 2+\cos 4$ (B) $\cos 2-\cos 4$
(C) $\sin 2+\sin 4$ (D) $\sin 2-\sin 4$

120. Let $f(x) = \cos^4 x + 3\sin^4 x$ then the value of $\int_0^{\pi/2} |f'(x)|\,dx$ equals:

(A) $\frac{7}{2}$ (B) $\frac{5}{2}$
(C) 3 (D) 4

121. The value of

$$\lim_{n\to\infty} \frac{\pi}{n}\left(\frac{1}{\sin\frac{\pi(n+1)}{4n}}+\frac{1}{\sin\frac{\pi(n+2)}{4n}}+\ldots+\frac{1}{\sin\frac{\pi(n+n)}{4n}}\right)$$

is equal to:

(A) $4\ln(\sqrt{2}+1)$ (B) $2\ln(\sqrt{2}+1)$
(C) $2\ln(\sqrt{2}-1)$ (D) $4\ln(\sqrt{2}-1)$

122. The value of

$$\lim_{n\to\infty}\frac{(1^2+2^2+\ldots+n^2)(1^3+2^3+\ldots+n^3)(1^4+2^4+\ldots+n^4)}{(1^5+2^5+\ldots+n^5)^2}$$

is equal to:

(A) $\frac{1}{5}$ (B) $\frac{2}{5}$
(C) $\frac{3}{5}$ (D) $\frac{4}{5}$

123. The value of definite integral

$$\int_0^{2\pi} (\max.(\sin x,\cos x)-\min.(\sin x,\cos x))\,dx \text{ is equal to:}$$

(A) $2\sqrt{2}$ (B) $4\sqrt{2}$
(C) $6\sqrt{2}$ (D) $8\sqrt{2}$

124. If $f(x) = x^5 + 5x - 1$ then $\int_5^{41} \frac{dx}{\left(f^{-1}(x)\right)^5 + 5\cdot f^{-1}(x)}$ equals:

(A) 0 (B) ln 2
(C) ln 7 (D) $\ln\left(\frac{41}{5}\right)$

125. The value of $\lim_{h\to 0}\frac{\int_a^{x+h}\frac{(\sin t)^{2014}}{t^{2013}+1}dt-\int_a^{x}\frac{(\sin t)^{2014}}{t^{2013}+1}dt}{h}$ is equal to:

(A) $\frac{2013(\sin x)^{2013}}{2014(x)^{2013}+1}$ (B) $\frac{(\sin x)^{2015}}{(x)^{2014}+1}$
(C) $\frac{(\sin x)^{2014}}{(x)^{2013}+1}$ (D) $\frac{2014(\sin x)^{2013}}{2013(x^{2012}+1)}$

126. If $f(t) = \int_{2t}^{t^2} \tan^{-1}\left|\frac{(1+t)^2-x}{1+x}\right| dx$ then minimum value of $f(x)$ is:

(A) $\frac{-\pi}{4}$ (B) $\frac{-\pi}{2}$
(C) $\frac{\pi}{2}$ (D) $\frac{3\pi}{4}$

127. Consider, $f(x) = \begin{vmatrix} x^3-4x & \sin x-\tan x & \sec^2 x \\ 3a^2+1 & 2a & a^6+6 \\ a+k & 1 & 2 \end{vmatrix}$.

If $\int_{-\pi/4}^{\pi/4} f(x)\,dx \ge 0\ \forall\ a \in R$ then number of integral values of k is(are):

(A) 0 (B) 1
(C) 2 (D) 3

128. Let $f(x) = \cos x + \sin x$, $x \in R$ and $g(x) = \begin{cases} \frac{|x|}{x} & ; \quad x \ne 0 \\ 2 & ; \quad x = 0 \end{cases}$.

Then $\int_{-\pi/4}^{3\pi/4} gof(x)\,dx$ is equal to:

(A) 0 (B) $\frac{\pi}{4}$
(C) $\frac{3\pi}{4}$ (D) π

129. If the locus of the centre of the variable circle which intersects the circles $x^2 + y^2 = 4$ and $x^2 + y^2 + 2x + 4y = 6$ orthogonally has the equation $ax + by = 1$, then the value of $\int_a^b \left(\ln\left|\sin\frac{\pi x}{2}\right|\right) dx$ is equal to:

(A) ln 2 (B) – ln 2
(C) $\frac{\pi}{2}\ln 2$ (D) $\frac{-\pi}{2}\ln 2$

130. Let f be a positive continuous function defined on the set of real numbers satisfying $f(x) = 3 + \int_0^x \frac{tf(t)}{1+t^2}dt\ \forall\ x \in R$. Then the value of $f\left(\frac{4}{3}\right)$ equals:

(A) 2 (B) 3
(C) 4 (D) 5

131. Let c_1 and c_2 be positive real numbers. Consider the function $f(x) = \begin{cases} c_1 x & ; \quad 0 \le x < \frac{1}{3} \\ c_2(1-x) & ; \quad \frac{1}{3} \le x \le 1 \end{cases}$. If f is continuous and $\int_0^1 f(x)\,dx = 1$, then the value of c_2 is:

(A) $\frac{1}{2}$ (B) 1
(C) 2 (D) 3

132. The value of

$$\lim_{x\to 0}\frac{1}{(\tan x-\cos x-x+1)}\int_0^x\left(\sqrt[3]{1+\sin^2 t}-\sqrt[4]{1-2\tan t}\right)dt$$

is equal to:

(A) – 1 (B) 1

(C) $\frac{1}{2}$ (D) $\frac{-1}{2}$

133. $\dfrac{\int_0^\infty x^9\cdot e^{-x^2}dx}{\int_0^\infty x^7\cdot e^{-x^2}dx}$ is equal to:

(A) 2 (B) 4

(C) 6 (D) 8

134. The value of

$$\lim_{n\to\infty}\left(\frac{1-\cos\frac{2}{n}}{1+\cos\frac{2}{n}}\right)\left(\sum_{r=1}^{2n}\frac{n}{\sqrt{8n^2-r^2}}\right)\left(\sum_{r=1}^{3n}\frac{n}{\sqrt{18n^2-r^2}}\right) \text{ is}$$

equal to:

(A) $\frac{\pi^2}{4}$ (B) $\frac{\pi^2}{8}$

(C) $\frac{\pi^2}{16}$ (D) $\frac{3\pi^2}{16}$

135. The value of definite integral $\int_{-1}^{1}\frac{dx}{x^2+x+1+\sqrt{x^4+3x^2+1}}$, is equal to:

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$

(C) π (D) 0

136. Let $I_n=\int_0^1 x^n\sqrt{1-x^2}\,dx$ then value of $\lim_{n\to\infty}\frac{I_n}{I_{n-2}}$ is equal to:

(A) – 1 (B) 0

(C) $\frac{1}{3}$ (D) 1

137. Let P_k be a point in xy plane whose x coordinate is $1+\frac{k}{n}$ (k = 1, 2, 3, …., n) on the curve $y=\ln x$. If A is (1, 0), then $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(AP_k)^2$ equals:

(A) $\frac{1}{3}+2\ln^2 2$ (B) $\frac{1}{3}+2\ln^2\left(\frac{2}{e}\right)$

(C) $\frac{1}{3}+\ln^2\left(\frac{2}{e}\right)$ (D) $\frac{1}{3}+2\ln\left(\frac{2}{e}\right)$

138. The value of $\lim_{x\to\infty}\sum_{k=1}^{n}\frac{n-k}{n^2}\cos\frac{4k}{n}$ equals:

(A) $\frac{1}{4}\sin 4+\frac{1}{16}\cos 4-\frac{1}{16}$

(B) $\frac{1}{4}\sin 4-\frac{1}{16}\cos 4+\frac{1}{16}$

(C) $\frac{1}{16}(1-\sin 4)$

(D) $\frac{1}{16}(1-\cos 4)$

139. The value of $\lim_{x\to 0}\frac{1}{x^3(x-\sin x)}\int_0^{x^2}t^2\cos^5 t\,dt$ is equal to:

(A) 0 (B) 1

(C) 2 (D) non-existent

140. Let a, b R$(a<b)$ and $f, g : R$ R be two continuous function such that $f(a+b-x)=f(x)$ and $g(a+b-x)=-g(x)\ \forall\ x\in R$. If $c>1$ and $\int_a^b f(x)\,dx=k\int_a^b\frac{f(x)}{c^{g(x)}+1}dx$, then the value of k, is:

(A) 1 (B) $\frac{1}{2}$

(C) 2 (D) – 1

141. Let S(t) be the area of the DOAB with O(0, 0, 0), A(2, 2, 1) and B(t, 1, t + 1). The value of the definite integral $\int_1^e (S(t))^2\ln t\ \,dt$ is equal to:

(A) $\frac{2e^3+5}{2}$ (B) $\frac{e^3+5}{2}$

(C) $\frac{2e^3+15}{2}$ (D) $\frac{e^3+15}{2}$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. Let $f(x)$ be a function by $f(x) = \int\limits_1^x x(x^2 - 3x + 2)dx, 1 \leq x \leq 4$. Then the:

(A) maximum value of $f(x)$ is $\frac{53}{4}$

(B) maximum value of $f(x)$ is $\frac{63}{4}$

(C) minimum value of $f(x)$ is $-\frac{1}{2}$

(D) minimum value of $f(x)$ is $-\frac{1}{4}$

2. Let $y = f(x)$ and $y = g(x)$ are symmetrical about the line $x = \frac{\alpha + \beta}{2}$, then $\int\limits_\alpha^\beta f(x)g'(x)dx$ is equal to:

(A) $\int\limits_\alpha^\beta f'(x)g(x)dx$

(B) $-\int\limits_\alpha^\beta f'(x)g(x)dx$

(C) $\frac{1}{2}\int\limits_\alpha^\beta (f(x)g'(x) - f'(x)g(x))dx$

(D) $\frac{1}{2}\int\limits_\alpha^\beta (f(x)g'(x) + f'(x)g(x))dx$

3. Suppose $I_1 = \int\limits_0^{\pi/2} \cos(\pi \sin^2 x)dx$, $I_2 = \int\limits_0^{\pi/2} \cos(2\pi \sin^2 x)dx$, $I_3 = \int\limits_0^{\pi/2} \cos(\pi \sin x)dx$, then:

(A) $I_1 = 0$ (B) $I_2 + I_3 = 0$

(C) $I_1 + I_2 + I_3 = 0$ (D) $I_2 = I_3$

4. Let $f(x) = \begin{cases} (x-2)^2 + 3 & ; \quad x \geq 2 \\ k + x^2 & ; \quad x < 2 \end{cases}$, then which of the following is/are true:

(A) $f(x)$ is non differentiable for all real values of k

(B) The range of $f(x)$ is $[3, \infty)$ $\forall$ $k \in [3, \infty)$

(C) $f(x)$ has no local extrema at $x = 2$, for $k \leq -1$

(D) $\int\limits_2^3 f(x)dx = \frac{11}{3}$

5. If $I = \int_0^\pi \frac{(a - \cos x)dx}{1 - 2a\cos x + a^2}$, then:

(A) $I = \frac{\pi}{a}$, if $a > 1$ (B) $I = \frac{\pi}{a}$, if $a < 1$

(C) $I = 0$, if $a < 1$ (D) $I = 0$, if $a > 1$

6. If $\int_0^\alpha \frac{dx}{1 - \cos\alpha \cos x} = \frac{A}{\sin\alpha} + B$ where $a \in (0, \pi/2)$ then possible values of A and B are:

(A) $A = \pi/2, B = 0$ (B) $A = \pi/4, B = \frac{\pi}{4\sin\alpha}$

(C) $A = \pi/6, B = \frac{\pi}{\sin\alpha}$ (D) $A = \pi, B = \frac{\pi}{\sin\alpha}$

7. Let $f(x) = \begin{cases} 2 & ; \quad 0 \leq x \leq 1 \\ 3 & ; \quad 1 < x \leq 2 \end{cases}$. Define $g(x) = \int\limits_0^x f(t)dt$ for $0 \leq x \leq 2$, then:

(A) g is continuous at $x = 1$

(B) g is not differentiable at $x = 1$

(C) g is discontinuous at $x = 1$

(D) g is differentiable at $x = 1$

8. Let $I_n = \int\limits_0^{\pi/4} \tan^n x \, dx$, $n \in N$

Then the true statements among the following are:

(A) $I_{n+1} < I_n$

(B) $I_{n+1} > I_n$

(C) $I_n + I_{n-2} = \frac{1}{n-1}$ for $n > 2$

(D) $\frac{1}{n+1} < 2I_n < \frac{1}{n-1}$ for $n > 2$

9. If $\int\limits_0^x f(t)dt = x^2 - 2x + \int\limits_0^x tf(t)dt$, where $x \in (0, \infty)$. Then $f(x)$ is:

(A) Periodic

(B) Periodic but fundamental period does not exist

(C) Periodic but fundamental period exists

(D) nothing can be said

10. If $f(2 - x) = f(2 + x)$ and $f(4 - x) = f(4 + x)$ and $f(x)$ is a function for which $\int\limits_0^2 f(x)dx = 5$, then $\int\limits_0^{50} f(x)dx$ is equal to:

(A) 125 (B) $\int\limits_{-4}^{46} f(x)dx$

(C) $\int\limits_1^{51} f(x)dx$ (D) $\int\limits_2^{52} f(x)dx$

11. If $F(x) = \dfrac{1}{x^2}\displaystyle\int_4^x (4t^2 - 2F'(t))\,dt$, then $F'(4)$ equals:

(A) $\dfrac{32}{9}$ (B) $\dfrac{64}{9}$

(C) $\dfrac{F(8)}{28}$ (D) $\dfrac{11F(8)}{28}$

12. Let $I_n = \displaystyle\int_0^{\pi/2} \dfrac{\sec^2 x}{(\sec x + \tan x)^n}\,dx$; $n > 1$, then which of the following statement is/are correct?

(A) $I_3 = \dfrac{3}{8}$ (B) $I_2 + I_4 = \dfrac{14}{15}$

(C) $I_3 + I_5 = \dfrac{5}{12}$ (D) $I_2 + I_3 = \dfrac{25}{24}$

13. If $f : (0, 1) \to R$ is defined by $f(x) = \displaystyle\sum_{n=0}^{\infty} \dfrac{x^{2^n}}{1 - x^{2^{n+1}}}$, then:

(A) $f^{-1}(x) = \dfrac{x}{x+1}$ (B) $\displaystyle\int_0^{1/2} f(x)dx = \ln 2 - \dfrac{1}{2}$

(C) $\displaystyle\int_0^1 f(x)dx = \ln 2 - 1$ (D) $\displaystyle\int_0^1 f(x)dx = \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \ldots.\infty$

14. If $f(x) = \dfrac{\sin x}{x}$, $I_1 = \displaystyle\int_0^{\pi} f(x)dx$ $I_2 = \displaystyle\int_0^{\pi/2} f(x).f\left(\dfrac{\pi}{2} - x\right)dx$, then:

(A) $I_1 > \pi$ (B) $I_1 < \pi$

(C) $\dfrac{I_1}{I_2} = \pi$ (D) $\dfrac{I_1}{I_2} = \dfrac{\pi}{2}$

15. Let $f(x) = \displaystyle\int_x^x (t \sin at + bt + c)\,dt$ where a, b, c are non zero real numbers, then $\displaystyle\lim_{x\to 0} \dfrac{f(x)}{x}$ is:

(A) Independent of a
(B) Independent of a and b has the value equal to $2c$
(C) Independent of a, b and c
(D) Dependent only on c

16. Let $f : R \to \mathrm{R}$ be a continuous function such that $\displaystyle\int_0^1 (f(xt)\,dt = 0$ for all $x \in \mathrm{R}$, then:

(A) There exists three integral values of P such that the graph of $y = f(x)$ and $y = x^3 - 3x^2 + P$ intersects at three distinct points.
(B) $y = f(x)$ is a periodic function.
(C) Area bounded by $y = f(x)$, $y = x^2 - 2x$ and ordinates $x = 2$ and $x = 4$ is 20/3.
(D) $f(x)$ is neither even function nor odd function.

17. Consider the function $f(k) = \displaystyle\int_0^{\pi/2} \dfrac{dx}{\sqrt{1 - k\cos^2 x}}$ where $k \in [0, 1)$, then which of the following is/are correct:

(A) $f\left(\dfrac{1}{2}\right) < f\left(\dfrac{1}{3}\right) < f\left(\dfrac{1}{4}\right)$

(B) $f\left(\dfrac{1}{4}\right) < f\left(\dfrac{1}{3}\right) < f\left(\dfrac{1}{2}\right)$

(C) $f\left(\dfrac{1}{4}\right) > f\left(\dfrac{1}{2}\right) > f\left(\dfrac{3}{4}\right)$

(D) $f\left(\dfrac{1}{4}\right) < f\left(\dfrac{1}{2}\right) < f\left(\dfrac{3}{4}\right)$

18. Let $A = \displaystyle\int_1^{e^2} \dfrac{\ln x}{\sqrt{x}}\,dx$. Identify which of the following statement(s) is(are) correct?

(A) $A < 2\left(e - \dfrac{1}{e}\right)$ (B) $A < (e - 1)\left(2 + \dfrac{1}{\sqrt{e}}\right)$

(C) $A = \displaystyle\int_0^2 t\sqrt{e^t}\,dt$ (D) $3 < A < 5$

19. If $I_{m,n} = \displaystyle\int_0^1 x^{n-1}(\log x)^m\,dx$, then $\displaystyle\sum_{r=1}^{51} \dfrac{\mathrm{I}_{r,52}}{\mathrm{I}_{r-1,52}} = k$:

(A) $k = \dfrac{-51}{2}$ (B) $[k] = -26$

(C) $k = -26$ (D) $[k] = -25$

20. Let $I_1 = \displaystyle\int_2^3 \dfrac{x^2}{2x^2 - 10x + 25}\,dx$ and $I_2 = \displaystyle\int_0^1 \ln\left(\cos\dfrac{\pi x}{2}\right)dx$, then:

(A) $I_1 - I_2 = \ln(2\sqrt{e})$ (B) $I_1 - I_2 = \ln(\sqrt{2}\,e)$

(C) $I_2 = \ln(I_1)$ (D) $I_1 I_2 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi/2} \ln(\sin 2x)\,dx$

21. If f is a differentiable function such that $f(x + y) = f(x) \,.\, f(y)$ $\forall\, x, y \in R$, $f(0) \neq 0$ and $f'(0) = 1$, then:

(A) $\displaystyle\int_{-1}^{1} \dfrac{(\ln f(x))^2}{1 + (f(x))^2}\,dx = \dfrac{1}{3}$

(B) $\displaystyle\int_{-\pi/2}^{\pi/2} \dfrac{\cos(\ln f(x))}{1 + f(x)}\,dx = 1$

(C) $\displaystyle\int_{-2015}^{2015} \ln\left(\ln f(x) + \sqrt{1 + (\ln f(x))^2}\right)dx = 0$

(D) $\displaystyle\int_0^2 (f(x) - \ln f(x))\,dx = e^2 - 1$

22. Let $L = \displaystyle\lim_{y\to 0} \dfrac{1}{y}\int_0^{\pi} \tan(y \sin x)\,dx$ and $l = \displaystyle\lim_{k\to\infty} \int_0^k \left(1 - \dfrac{x}{k}\right)^k \cdot e^{x/3}\,dx$, then:

(A) $3l = 4L$ (B) $3L = 4l$

(C) $L + 2l = 5$ (D) $2(l + L) = 7$

23. Let $f(x)$ be a continuous function and c is a constant satisfying $\int_0^x f(t)\,dt = e^x - ce^{2x}\int_0^1 f(t)e^{-t}\,dt$, then:

(A) $f(x) = e^x + 2e^x$ (B) $f(x) = e^x - 2e^x$

(C) $c = -\dfrac{1}{1+2e}$ (D) $c = \dfrac{1}{3-2e}$

24. The function f is continuous and has the property $f(f(x)) = 1 - x$ for all $x \in [0, 1]$ and $J = \int_0^1 f(x)\,dx$, then:

(A) $f\left(\dfrac{1}{4}\right) + f\left(\dfrac{3}{4}\right) = 1$

(B) the value of J equal to $\dfrac{1}{2}$

(C) $f\left(\dfrac{1}{3}\right) f\left(\dfrac{2}{3}\right) = 1$

(D) $\displaystyle\int_0^{\pi/2} \frac{\sin x\,dx}{(\sin x + \cos x)^3}$ has the same value as J

25. Let $f(x) = \tan x - \tan^3 x + \tan^5 x - \tan^7 x + \ldots\ \infty$, where $x \in \left(0, \dfrac{\pi}{4}\right)$, then which of the following is/are correct?

(A) $\displaystyle\int_0^{\pi/6} f(x)\,dx = \frac{1}{8}$ (B) $f'\left(\dfrac{\pi}{12}\right) = \dfrac{1}{2}$

(C) $\displaystyle\lim_{x\to 0^+} \frac{f(x)}{x} = 1$ (D) $f(x)$ is an odd function

26. Let $l_1 = \displaystyle\lim_{x\to 0} \frac{1}{x^3}\int_0^x \frac{t(e^{3t} - 1)\ln(1 + 2t)}{(t^3 + 3)(1 - \cos\sqrt{t})}\,dt$ and

$$l_2 = \lim_{x\to 0} \frac{1}{x}\left(\int_{\pi/4}^{v} \log_{\frac{1}{2}} \sin^2 t\,dt - \int_{\left(\frac{\pi}{4}+x\right)}^{v} \log_{\frac{1}{2}} \sin^2 t\,dt\right)$$ where

$\pi < v < 2\pi$. Then which of the following is(are) correct?

(A) $9\,l_1^2 + l_2^2 = 18$ (B) $3l_1 + 4l_2 = 8$

(C) $l_1 > 0$ and $l_2 < 0$ (D) $l_1 > l_2$

27. For $0 \le x < 1$, let $f(x) = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}}$, then which of the following hold(s) good?

(A) $f'\left(\dfrac{\sqrt{3}}{2}\right) = 2$ (B) $f\left(\dfrac{2}{3}\right) > \dfrac{2}{3}$

(C) $\displaystyle\lim_{x\to 0} \frac{f(x)}{x} = 1$ (D) $f(x) + f\left(\sqrt{1-x^2}\right) = \dfrac{\pi}{2}$

28. If $I_n = \displaystyle\int_0^1 \frac{dx}{(1+x^2)^n}$; $n \in N$, then which of the following statements hold good?

(A) $2n\,I_{n+1} = 2^{-n} + (2n-1)I_n$ (B) $I_2 = \dfrac{\pi}{8} + \dfrac{1}{4}$

(C) $I_2 = \dfrac{\pi}{8} - \dfrac{1}{4}$ (D) $I_3 = \dfrac{\pi}{16} - \dfrac{5}{48}$

29. Let $f(x) = \begin{cases} x + 1 & ;\ 0 \le x \le 1 \\ 2x^2 - 6x + 6 & ;\ 1 < x \le 2 \end{cases}$ and $g(t) = \displaystyle\int_{t-1}^{t} f(x)\,dx$ for $t \in [1, 2]$. Which of the following hold(s) good?

(A) $f(x)$ is continuous and differentiable in $[0, 2]$

(B) $g'(t)$ vanishes for $t = 3/2$ and 2

(C) $g(t)$ is maximum at $t = 3/2$

(D) $g(t)$ is minimum at $t = 1$

30. If $f: R \to R$ be a continuous function such that $f(x) = \displaystyle\int_1^x 2t\,f(t)\,dt$, then which of the following does not hold(s) good?

(A) $f(\pi) = e^{\pi^2}$ (B) $f(1) = e$

(C) $f(0) = 1$ (D) $f(2) = 2$

31. Which of the following statement(s) is(are) correct?

(A) If $f: [0, 1] \to R$ is continuous and $\displaystyle\int_0^x f(x)\,dx = \int_x^1 f(x)\,dx$ for all $x \in [0, 1]$ then $f(x) = 0$ for all $x \in [0, 1]$

(B) Let f be a function defined in $[a, b]$ such that $f(x) \ge 0\ \forall\ x \in [a, b]$ and $\displaystyle\int_a^b f(x)\,dx = 0$ then $f(x) = 0\ \forall\ x \in [a, b]$

(C) If $f(x)$ is continuous on R then $\displaystyle\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f(\sin 2x)\,dx$

(D) For $a > 0$, if f is continuous in $[-a, a]$ then $\displaystyle\int_{-a}^{a} f(x^2)\,dx = 2\int_0^a f(x^2)\,dx$

32. $T_n = \displaystyle\sum_{r=2n}^{3n-1} \frac{rn}{r^2 + n^2}$, $S_n = \displaystyle\sum_{r=2n+1}^{3n} \frac{rn}{r^2 + n^2}$, then $\forall\ n \in \{1, 2, 3, \ldots\}$

(A) $T_n > \dfrac{1}{2}\ln 2$ (B) $S_n < \dfrac{1}{2}\ln 2$

(C) $T_n < \dfrac{1}{2}\ln 2$ (D) $S_n > \dfrac{1}{2}\ln 2$

33. Let $S_n = \displaystyle\lim_{n\to\infty}\left(\frac{1}{n^6} + \frac{32}{n^6} + \ldots + \frac{1}{n}\right)$ and $T_n = \displaystyle\lim_{n\to\infty}\left(\frac{1}{n^6} + \frac{32}{n^6} + \ldots + \frac{(n-1)^5}{n^6}\right)$, then which of the following is/are true?

(A) $S_n > \frac{1}{6}$ (B) $(S_n + T_n) < \frac{1}{3}$

(C) $(S_n + T_n) > \frac{1}{3}$ (D) $T_n < \frac{1}{6}$

34. Let $f(x)$ be a non-constant differentiable function satisfying

$$f(x) = x^2 - \int_0^1 (f(t) + x)^2\, dt\,.$$

Then which of the following is/are correct?

(A) $f(x)$ is monotonic.

(B) $\lim\limits_{x \to \frac{4}{3}} (f(x))^{\text{cosec}(3\pi x)}$ is equal to $e^{\frac{1}{\pi}}$

(C) Derivative of $f(x)$ w.r.t. $\ln x$ at $x = 1$ is equal to 3.

(D) Area bounded by $f(x)$ and co-ordinate axes is $\frac{2}{3}$.

35. Let $f(x) = \int_0^x (3t^2 + 2t + 2)dt$, where x satisfies the inequality $\log_2\left(1 + \sqrt{6x - x^2 - 5}\right) \geq 0$. If minimum and maximum values of $f(x)$ are L and M respectively, then:

(A) $L + M = 94$ (B) $L + M = 164$

(C) $M - L = 156$ (D) $L - M = 80$

36. If $\int_0^1 \left(\sum_{k=1}^{2014} \frac{x^2}{x^3 + k^3}\right)\left(\prod_{k=1}^{2014} (x^3 + k^3)\right) dx = \frac{1}{\lambda}\left(\prod_{k=1}^{2014} (1 + k^3) - \mu^3\right)$, then:

(A) $\lambda = \frac{1}{3}$ (B) $\lambda = 3$

(C) $\mu = 2014$ (D) $\mu = 2014!$

37. Consider the function $f(\theta) = \int_0^1 \frac{\left|\sqrt{1-x^2} - \sin\theta\right|}{\sqrt{1-x^2}}dx$, where $0 \leq x \leq \frac{\pi}{2}$, then:

(A) $\max\limits_{0 \leq x \leq \frac{\pi}{2}} \cdot f(\theta) = 1$ (B) $\min\limits_{0 \leq x \leq \frac{\pi}{2}} \cdot f(\theta) = \sqrt{2} - 1$

(C) $\max\limits_{0 \leq x \leq \frac{\pi}{2}} \cdot f(\theta) = \frac{\pi}{2} - 1$ (D) $\int_0^{\frac{\pi}{2}} f(\theta)d\theta = 2 - \frac{\pi}{2}$

38. Let $f : [0, \infty] \to R$ be a continuous strictly increasing function, such that $f^3(x) = \int_0^x t \cdot f^2(t)dt$ for every $x \geq 0$. Which of the following is/are correct?

(A) $f(6)$ is equal to 6

(B) $f(x)$ is surjective

(C) $\int_0^1 f(x)dx$ is equal to $\frac{1}{18}$

(D) Number of solutions of equation $f(x) = 6$ are two.

39. Given, $f(x) - 2\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^5 x} \cdot \cos t \cdot f(t)dt = \frac{\sin^2 x}{\cos^5 x}$, then which of the following is (are) correct?

(A) $\lim\limits_{x \to \frac{\pi}{3}} f(x) = 72$.

(B) $f(x)$ is periodic with period π.

(C) The equation of normal to the graph of $f(x)$ at point M whose abscissa is π is given by $x - \pi = 0$.

(D) Number of solutions of the equation $f(x) = 0$ in $(0, 3)$ is one.

40. If complete set of real values of x satisfying $|\cos^{-1} x| + |\sin^{-1} x| + |\tan^{-1} x| = |\mu - \cot^{-1} x|$ is $[\alpha, \beta]$ then:

(A) $\int_\alpha^\beta \frac{\tan^{-1} x}{\cot^{-1}(1 - x + x^2)}dx$ equal $\frac{1}{2}$

(B) $\int_0^{\frac{\pi}{2}} \sin(2(\alpha + \beta)x) \cdot \sin(\sin\beta x)dx$ equals $2(\sin 1 - \cos 1)$

(C) $2|\sin x| + |\cos x| = |2 \sin x + \cos x|$ is true for all $x \in [\alpha, \beta]$

(D) $\alpha^\beta + \beta^\alpha + \beta^2 = 0$

41. Let $I = \int_0^1 \sqrt{\frac{1 + \sqrt{x}}{1 - \sqrt{x}}}\, dx$ and $J = \int_0^1 \sqrt{\frac{1 - \sqrt{x}}{1 + \sqrt{x}}}\, dx$, then the correct statement is:

(A) $I + J = 2$ (B) $I - J = \pi$

(C) $I = \frac{2 + \pi}{2}$ (D) $J = \frac{4 - \pi}{2}$

42. If $f(\theta) = \left|\,\left|\,|3\cos\theta - \sin\theta| + 2\cos\theta\right| + 2\cos\theta\,\right|$ where $\theta \in \left(\frac{\pi}{2}, \pi\right)$, then:

(A) $\int_{\pi/2}^{\pi} f(\theta)\, d\theta = 1$ (B) $f'\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2}$

(C) $\int_{\pi/2}^{\pi} f^2(\theta)\, d\theta = \frac{\pi}{4}$ (D) $f'\left(\frac{2\pi}{3}\right) = \frac{-1}{2}$

43. If $A_n = \int_0^{\pi/2} \frac{\sin(2n - 1)x}{\sin x}dx$; $B_n = \int_0^{\pi/2} \left(\frac{\sin nx}{\sin x}\right)^2 dx$; for $n \in N$, then:

(A) $A_{n+1} = A_n$ (B) $B_{n+1} = B_n$

(C) $A_{n+1} - A_n = B_{n+1}$ (D) $B_{n+1} - B_n = A_{n+1}$

44. Let $f : R^+ \to R$ be a differentiable function with $f(1) = 3$ and satisfying:

$$\int_1^{xy} f(t)dt = y\int_1^x f(t)dt + x\int_1^y f(t)dt \;\; \forall x, y \in R^+ \text{ then}$$

(A) $f(x) = 3\log_e(ex)$ (B) $f(x) = \log_e ex$

(C) $f'(y) = \dfrac{3}{y}$ (D) $f'(x) = \dfrac{1}{x}$

45. Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(2 - x)$ and $f'\left(\frac{1}{2}\right) = 0$, then:

(A) $f'(1) = 0$

(B) $f''(x)$ vanishes at least twice on [0, 2]

(C) $\displaystyle\int_{-1}^{1} f(1+x)\left(\frac{e^x - 1}{e^x + 1}\right)dx = 0$

(D) $\displaystyle\int_{0}^{1} f(x)e^{\sin \pi x}\,dx = \int_{1}^{2} f(2-x)e^{\sin \pi x}dx$

46. Consider the integral $I_1 = \displaystyle\int_1^e (1+x)(x+\ln x)^{100}\,dx$,

$$I_2 = \int_{\sin^{-1}(1/e)}^{\pi/2} (1 + e\sin x + \ln\sin x)^{101}\cos x\,dx\ .$$

If $I_1 + \dfrac{e}{101}I_2$ is $\dfrac{e(1+e)^{101} - k}{101}$, then k is greater than or equal to:

(A) 0 (B) 1

(C) 2 (D) –1

47. Given $f(x) - \mu \displaystyle\int_0^{\pi/4} \sin^2 x\cos^{-5} x\cos t f(t)\,dt = \sin^2 x\cos^{-5} x$, where μ is a constant then:

(A) $f(x) = \dfrac{3}{3-\mu}\sin^2 x\cos^{-5} x$

(B) $f(x) = \dfrac{3}{\mu}$ sin x cos x

(C) $f(x)$ is periodic with period π

(D) $f(x)$ is periodic with period 2π

48. Let 'e' be the eccentricity of a hyperbola and $f(e)$ be the eccentricity of its conjugate hyperbola, then $\displaystyle\int_1^3 \underbrace{f(f(f\ldots f(e)))}_{(n\text{ times})}de =$

(A) $2\sqrt{2}$ if 'n' is even (B) $2\sqrt{2}$ if 'n' is odd

(C) 4, if 'n' is even (D) 4, if 'n' is odd

49. Let $f(x)$ be strictly monotonic differentiable function on [0,1] such that $f(0) = 1, f(1) = -1$, then $2\displaystyle\int_{-1}^{1} x(f^{-1}(x) - 1)\,dx =$

(A) $\displaystyle\int_0^1 f^2(x)dx - 1$ (B) $\displaystyle\int_0^1 f(x)\,dx$ exists

(C) $1 + \displaystyle\int_0^1 f^2(x)\,dx$ (D) is positive

50. Let f be a differentiable function such that

$f'(x) = f(x) + \displaystyle\int_0^2 f(x)\,dx, f(0) = \dfrac{4 - e^2}{3}$, then $f(x)$ is:

(A) $e^x - \left(\dfrac{e^2 - 1}{3}\right)$ (B) $e^x - \dfrac{(e^2 - 2)}{3}$

(C) $e^x + \dfrac{e - [\arg|z-1|]}{3}$ (D) $e^x - \dfrac{e^2 - 1}{[\arg(-|z|)]}$

51. If $f(x)$ is monotonic and differentiable function, then $\displaystyle\int_{f(a)}^{f(b)} 2x(b - f^{-1}(x))\,dx$ equals:

(A) $\displaystyle\int_a^b (f^2(x) - f^2(a))\,dx$ (B) $\displaystyle\int_a^b (f^2(x) - f^2(b))\,dx$

(C) $\displaystyle\int_a^b f^2(x)dx + (a-b)f^2(b)$ (D) $\displaystyle\int_a^b f^2(x)dx + (a-b)f^2(a)$

52. If $f(x)$ be a periodic function with period T so that $f(x + 13) + f(x + 630) = 0$ and $\displaystyle\sum_{p=1}^{777}\int_p^{p+T} f(x)dx = \lambda\int_0^T f(x)dx$, then:

(A) $T = 1234$ (B) $\lambda + T = 2011$

(C) λ, T both are prime (D) $\lambda + T = 2012$

53. If $A(x + y) = A(x)A(y)\ \forall x, y$ and $A(0) \neq 0$ and $B(x) = \dfrac{A(x)}{1 + (A(x))^2}$, then:

(A) $\displaystyle\int_{-2010}^{2010} B(x)dx = \int_0^{2011} B(x)dx$

(B) $\displaystyle\int_{-2010}^{2011} B(x)dx - \int_0^{2010} B(x)dx = \int_0^{2011} B(x)dx$

(C) $\displaystyle\int_{-2010}^{2011} B(x)dx = 0$

(D) $\displaystyle\int_{-2010}^{2010} (2B(-x) - B(x))dx = 2\int_0^{2010} B(x)dx$

54. Let $I(n) = \displaystyle\int_{2016}^{2016+\frac{1}{n}} x\cos^2(x - 2016)dx$, then which of the following is/are correct?

(A) $I\left(\dfrac{1}{\pi}\right) = \dfrac{(\pi + 2016)\pi}{2}$ (B) $I\left(\dfrac{1}{\pi}\right) = \dfrac{(\pi + 4032)\pi}{4}$

(C) $\lim\limits_{n\to\infty} n \cdot I(n) = 2016$ (D) $\lim\limits_{n\to\infty} n \cdot I(n) = 2017$

55. If $I_n = \int_0^{\pi/2} x(\sin x + \cos x)^n dx$; then:

(A) $\dfrac{101 I_{101} - \dfrac{\pi}{2}}{I_{99}} = 200$ (B) $\dfrac{4 I_4 - \dfrac{\pi}{2}}{I_2} = 6$

(C) $\dfrac{101 I_{101} - \dfrac{\pi}{2}}{I_{99}} = 6$ (D) $\dfrac{4 I_4 - \dfrac{\pi}{2}}{I_2} = 8$

56. If $\int_0^x f(t)d = e^x - ae^{2x}\int_0^1 f(t)e^{-t}\,dt$ then:

(A) $a = \dfrac{1}{3-2e}$ (B) $f(x) = e^x - 2e^{2x}$

(C) $a = \dfrac{1}{e}$ (D) $f(x) = e^x - e^{-x}$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Let $I_n = \int \sin nx \cdot \sec x\, dx$ and $S_n = \int_0^{\pi/2} \sin nx \cdot \sec x\, dx$ where $n \in N$, then:

1. $I_n + I_{n-2}$ is

(A) $-\dfrac{2\sin(n-1)x}{n-1} + C$ (B) $\dfrac{2\cos(n-1)x}{n-1} + C$

(C) $-\dfrac{2\cos(n-1)x}{n-1} + C$ (D) $-\dfrac{2\sin(n-1)x}{n-1} + C$

2. The value of $\int_0^{\pi/2} \dfrac{\cos 5\theta \cdot \sin 3\theta}{\cos\theta}\, d\theta$ is:

(A) $\dfrac{181}{105}$ (B) $-\dfrac{181}{105}$

(C) $-\dfrac{181}{210}$ (D) $\dfrac{181}{210}$

Comprehension 2

Let $f(x)$ is a differentiable function such that

$f'(x) = f(x) + \int_0^2 f(x)\,dx$ if $f(0) = \dfrac{4-e^2}{3}$, then

3. The number of solutions of $x + f(x) = 0$, is:

(A) 0
(B) 1
(C) 2
(D) None of these

4. $\int_0^1 f(x)\,dx$ equal to:

(A) $\dfrac{3e - e^2 - 2}{3}$ (B) $\dfrac{3e + e^2 - 4}{3}$

(C) $\dfrac{3e - e^2 - 4}{3}$ (D) None of these

Comprehension 3

Consider a sequence $I_1, I_2, I_3, \ldots. I_n$ where the n^{th} term I_n is given by $I_n = \int_0^{\pi/2} \cos^n x \cos nx\,dx$ and another sequence $T_1, T_2, T_3 \ldots. T_n$ has general terms given by $T_n = (n^2+1)n!$ and $a_n = \dfrac{(T_1 + T_2 + T_3 \ldots + T_n)}{n!}$

5. The sequence $I_1, I_2, I_3 \ldots. I_n$ represents:

(A) A.P. (B) G.P.
(C) A.P. and G.P. both (D) Neither A.P. nor G.P.

6. $\sum_{n=1}^{5} \dfrac{1}{a_n}$ equals to:

(A) $\dfrac{5}{6}$ (B) $\dfrac{3}{5}$

(C) $\dfrac{1}{6}$ (D) $\dfrac{4}{5}$

Comprehension 4

Let f be a monic polynomial satisfying

$f(f(x)) = (f(x))^2 - 4\ \forall x \in$ R and $l = \lim\limits_{t\to\sqrt{2}} \left(\dfrac{\int_2^{t^2} \dfrac{f(x)dx}{\ln(5-x^2)}}{\sin(t-\sqrt{2})} \right)$ where 'l' is a non-zero finite quantity.

7. The value of l^2 is:

(A) 4 (B) $4\sqrt{2}$
(C) 8 (D) 16

8. The value of $\int_0^4 f(f(x))\,d(x^2)$ is:

(A) $-\dfrac{8}{3}$ (B) $-\dfrac{32}{3}$

(C) $\dfrac{16}{3}$ (D) $\dfrac{32}{3}$

Comprehension 5

If $m > 0, n > 0$, the definite integral $I = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ depends upon the values of m and n and is denoted by $\beta(m,n)$, called the beta function. E.g.: $\int_0^1 x^4(1-x)^5\,dx = \int_0^1 x^{5-1}(1-x)^{6-1}\,dx = \beta(5,6)$ and $\int_0^1 x^{5/2}(1-x)^{-1/2}\,dx = \int_0^1 x^{7/2-1}(1-x)^{1/2-1}\,dx = \beta\left(\frac{7}{2},\frac{1}{2}\right)$.

Obviously, $\beta\ (n, m) = \beta\ (m, n)$

9. The integral $\int_0^{\pi/2} \cos^{2m}\theta \sin^{2n}\theta\, d\theta$ is equal to:

(A) $\frac{1}{2}\beta\left(m+\frac{1}{2}, n+\frac{1}{2}\right)$ (B) $2\beta(2m,2n)$

(C) $\beta\,(2m+1, 2n+1)$ (D) none of these

10. If $\int_0^n \left(1-\frac{x}{n}\right)^n x^{k-1}\,dx = R\beta(k, n+1)$, then R is equal to:

(A) n (B) n^{kn}

(C) n^k (D) none of these

11. If $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = k\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$, then k is equal to:

(A) $\frac{m}{n}$ (B) 1

(C) $\frac{n}{m}$ (D) none of these

Comprehension 6

A series of the form $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ is called Fourier series where $a_0, a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n$ are constants and the coefficients are evaluated for $x \in [\alpha, \alpha + 2\pi]$ using the formula $a_0 = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\,dx$, $a_n = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\cos nx\,dx$ and $b_n = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\sin nx\,dx$. Let us consider $f(x) = x + x^2$ for $-\pi \le x \le \pi$, then:

12. The value of a_0 is:

(A) $\frac{2\pi^2}{3}$ (B) $\frac{4\pi^3}{5}$

(C) π (D) $\frac{3\pi^2}{2}$

13. The value of a_n is:

(A) $\frac{4(-1)^n}{n^2}$ (B) $\frac{4}{n^2}$

(C) $\frac{2(-1)^n}{n^2}$ (D) $\frac{2}{n^2}$

14. The approximate value of $\frac{\pi^2}{6}$ is given by:

(A) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \infty$

(B) $2 - \frac{1}{2^2} + \frac{2}{3} - \frac{2}{4^3} + \ldots \infty$

(C) $2 + \frac{1}{2} + \frac{2}{3} + \frac{2}{4^2} + \ldots \infty$

(D) $2 - \frac{2}{3^2} + \frac{2}{5^2} - \frac{2}{7^2} + \ldots \infty$

Comprehension 7

Given: $\int_{x=a}^{x=b} f(x)\,d\alpha(x) + \int_{x=a}^{x=b} \alpha(x)\,df(x) = \alpha(b)f(b) - \alpha(a)f(a)$

15. $\int_0^3 (x^2+1)\,d[x]$ (where [.] denotes the greatest integer function) is equal to:

(A) 3 (B) $\frac{9}{2}$

(C) 17 (D) $\frac{27}{2}$

16. $\int_0^6 (x^2+[x])\,d\,|3-x|$ (where [.] denotes the greatest integer function) is equal to:

(A) 36
(B) 72
(C) 63
(D) 126

17. $\int_{-2}^{3} [\,|x|\,]\,d\,|x|$ (where [.] denotes the greatest integer function) is equal to:

(A) 0
(B) 1
(C) 2
(D) None of these

Comprehension 8

Let f be a function defined by $f(k) = \int_0^{\frac{\pi}{2}} |k - 2\sin t|\cos t\,dt$ where k is any real number.

18. Which one of the following statement is correct?

(A) $f(k)$ is continuous $\forall\, k$ but not derivable at two values of k.
(B) $f(k)$ is differentiable $\forall\, k$.
(C) $f(k) = 1 - k\ \forall\, k \ge 2$.
(D) $f(k) = k - 1\ \forall\, k \ge 0$.

19. Minimum value of $f(k)$, is:

(A) 0 (B) $\frac{1}{2}$

(C) 1 (D) $\frac{3}{2}$

Comprehension 9

Let $g(x)$ be a differentiable function on R and satisfying $\int_0^x (x-t+1)g(t)dt = x^2 + x^4$ for all $x \in R$.

Given, $h(x) = g(x) + g'(x)$ for all $x \in R$ and $f(n) = h(n+1) - h(n)$ for all $n \in N$.

20. The value of definite integral $\int_0^1 g''(x)dx$ is equal to:

(A) $26(1 - e^{-2})$ (B) $26(1 - e^{-1})$

(C) $26(e^{-2} - 1)$ (D) $26(e^{-1} - 1)$

21. If tangents are drawn to $y = h(x)$ from origin then the area bounded by tangent lines and the curve $y = h(x)$ is equal to:

(A) $\frac{3\sqrt{6}}{4}$ (B) $\frac{2\sqrt{6}}{5}$

(C) $\frac{3\sqrt{6}}{5}$ (D) $\frac{2\sqrt{6}}{9}$

22. The value of sum $\sum_{r=1}^{10} f(r)$ is equal to:

(A) 1440 (B) 1320

(C) 1280 (D) 1160

Comprehension 10

Let $y = g(x)$ be a function of the graph of broken lines connected by points $(-2, 0)$, $(0, 3)$ and $(2, 4)$ in the x-y plane.

Also $f(x) = \int_{-2}^{x} |g(t) - t|\,dt,\ x \in [-2, 2]$ and $h(x)$ is the inverse of $f(x)$.

23. The value of $f'(0)$ is:

(A) 3 (B) –3

(C) $\frac{1}{3}$ (D) does not exist

24. The value of $h\left(\frac{31}{4}\right) + h'\left(\frac{31}{4}\right)$ is:

(A) $\frac{17}{5}$ (B) $\frac{7}{5}$

(C) $\frac{9}{5}$ (D) $\frac{13}{5}$

25. The sum of roots of $f(x) = 2x^2 + 5$ is:

(A) 3 (B) $\frac{1}{3}$

(C) $\frac{4}{3}$ (D) $\frac{3}{4}$

Comprehension 11

Let $I_1 = \int_0^x \frac{t^2 dt}{(x - \sin x)\sqrt{\lambda^2 + 2\lambda t}}$ (where λ is a non zero constant

and $I_2 = \int_0^{x^2} \frac{e^t - 1 - t}{(x^6)} dt$.

26. If $\lim_{x \to 0} I_1 = 1$, then sum of value(s) of λ is:

(A) 0 (B) $\frac{1}{2}$

(C) 1 (D) 2

27. If $\lambda = 1$, the value of $\lim_{x \to 0} (I_1 + 6I_2)$ is equal to:

(A) 2 (B) 3

(C) –1 (D) 0

Comprehension 12

Let f and g be two real-valued differentiable functions on R satisfying. $\int_0^x g(t)dt = 3x + \int_0^{-x} \cos^2 t\, g(t)dt$ and

$$f(x) = \lim_{\alpha \to 0} \frac{1}{\alpha^4} \int_0^{\alpha} \frac{(e^{x+t} - e^x)\ln^2(1+t)}{2t^3 + 3} dt$$

28. $f(\ln 2)$ is equal to:

(A) 0 (B) $\frac{1}{6}$

(C) $\frac{1}{2}$ (D) ln 2

29. Range of $g(x)$ is equal to:

(A) $\left[\frac{3}{2}, 3\right]$ (B) $\left[\frac{1}{2}, 3\right]$

(C) $\left[\frac{3}{2}, \infty\right)$ (D) $[-3, 3]$

30. The value of definite integral $\int_0^{\frac{\pi}{2}} g(x)dx$ is equal to:

(A) 0 (B) $\frac{\pi}{2\sqrt{2}}$

(C) $\frac{3\pi}{\sqrt{2}}$ (D) $\frac{3\pi}{2\sqrt{2}}$

Comprehension 13

Let $f(x)$ be a twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(2 - x)$ and $f'\left(\frac{1}{2}\right) = f'\left(\frac{1}{4}\right) = 0$. Then:

31. The minimum number of values where $f''(x)$ vanishes on $[0, 2]$ is:

(A) 2 (B) 3

(C) 4 (D) 5

32. $\int_{-1}^{1} f'(1+x)x^2 e^{x^2} dx$ is equal to:

(A) 1 (B) π

(C) 2 (D) 0

33. $\int_0^1 f(1-t)e^{-\cos \pi t} dt - \int_1^2 f(2-t)e^{-\cos \pi t} dt$ is equal to:

(A) $\int_0^2 f'(t)e^{\cos \pi t} dt$ (B) 1

(C) 2 (D) π

Comprehension 14

Consider $I_1 = \int_0^{\frac{\pi}{2}} \frac{x^3 \cos x\, dx}{(3\sin x - \sin 3x)}$, $I_2 = \int_0^{\frac{\pi}{2}} x^2 \operatorname{cosec}^2 x\, dx$ and $I_3 = \int_0^{\frac{\pi}{2}} x \cot x\, dx$

34. If $I_1 = \frac{\pi^3}{k_1} + k_2\, \pi \ln 2$, where $k_1, k_2 \in Q$ then $(k_1 + k_2)$ equals:

(A) $\frac{-509}{8}$ (B) $\frac{509}{8}$

(C) $\frac{511}{8}$ (D) $\frac{-511}{8}$

35. If $I_2 = k_1 + k_2\, \pi \ln 2$, where $k_1, k_2 \in Q$, then $(k_1^2 + k_2^2)$ equals:

(A) $\frac{1}{4}$ (B) $\frac{5}{32}$

(C) 1 (D) 9

36. If $I_3 = k_1 + k_2\, \pi \ln 2$, where $k_1, k_2 \in$ Q, then $(k_1^3 + k_2^3)$ equals:

(A) $\frac{1}{8}$ (B) $\frac{1}{27}$

(C) $\frac{1}{64}$ (D) $\frac{3}{32}$

Comprehension 15

Let f be a real-valued continuous function on R(the set of all real numbers) and satisfying $f(x) = x^2 + \int_{-1}^{1} (x+y) f(y) dy$ where x and y are independent variables.

37. If $f(x) > k$ for every real x, then the largest integral value of k is:

(A) – 4 (B) – 3

(C) – 2 (D) 1

38. If $g(t) = f(-t) - f(t)$ for every $t \in R$, then the value of definite integral $\int_0^{\frac{\pi}{2}} g(t) \sin t\, dt$ is equal to:

(A) 0 (B) 2

(C) 3 (D) 4

39. If the range of function $k(x) = 3 f(x) + b + 4$ for every real x is $[0, \infty)$ then b equals:

(A) – 1 (B) 0

(C) 3 (D) 6

Comprehension 16

Let f and g be two real-valued functions defined on R. Given $f(x) = \int_0^1 |x - t| \cos \pi t\, dt$ and $g(x) = 7 + 2x \ln 25 - 5^{x-1} - 5^{2-x}$.

Suppose α be the value of x for which the function $g(x)$ has the greatest value and β be the value of $\lim\limits_{x \to 0} \frac{x^2 \sin(x - 2\pi)}{\int_0^x t^2\, dt}$.

40. If the function $h(x) = f(x)$ in the interval [0, 1] and $k(x) = h^{-1}(x)$, then the value of $k'(0)$ is equal to:

(A) 0 (B) $\frac{\pi}{2}$

(C) $\frac{-\pi}{4}$ (D) $\frac{\pi}{6}$

41. The value of $\sum_{n=0}^{\infty} \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n$ is equal to:

(A) 4 (B) 5

(C) 6 (D) 8

Comprehension 17

Let g be a continuous function on R such that $g(1) = 0$. Also for all $x \in R$.

$\int_0^x g(t)dt - \int_x^1 g(t)dt = 2g(x) + 3x + k$, where k is some real constant.

42. The y-intercept of normal drawn to $g(x)$ at $M(1, 0)$ is equal to:

(A) $\frac{2}{3}$ (B) $\frac{-2}{3}$

(C) $\frac{3}{2}$ (D) $\frac{-3}{2}$

43. Which of the following statement is correct?

(A) There exist some $c \in R$ such that $g(x) \geq 2$ for every $x \geq$ c.

(B) $g(x)$ is strictly increasing on R.

(C) Number of real solutions of the equation $g(x) = 3$ are two

(D) There exist some $x_0 \in R$ such that $g(x_0) = \frac{1}{2}$.

44. If $\int_0^1 g(x)dx = \lambda(k+3)$ then λ is equal to:

(A) $\frac{1}{2}$ (B) 1

(C) $\frac{1}{3}$ (D) 2

Comprehension 18

Let $g : R \to R$ be a differentiable function which satisfies $g(x) = 1 + \int_0^x g(t)dt$ and $g'(0) = 1$

45. The value of $g(\ln 10) + g'(\ln 10) + g''(\ln 10)$ is equal to:

(A) 0 (B) $\frac{1}{10}$

(C) 30 (D) $\frac{1}{30}$

46. The value of definite integral $\int_{-3}^{-1} \left(\sum_{r=1}^{\infty} g(rx)\right) dx$ is equal to:

(A) $\ln(1 + e + e^{-1})$ (B) $\ln(1 + e^{-1} + e^{-2})$

(C) $\ln(1 + e + e^2)$ (D) $(1 + e^{-1} + e^2)$

47. Number of solution of the equation $g(-x) = g(x)$ is equal to:

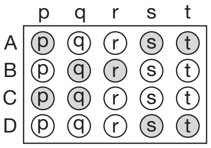
(A) 0 (B) 1

(C) 2 (D) 3

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



1. Consider $f(x) = \int_0^1 t\,|t - x|\,dt$, then:

COLUMN-I	COLUMN-II
(A) $\min\limits_{x \in R} f(x)$ is equal to	(p) $\frac{1}{3}$
(B) $\max\limits_{x \in [0,1]} f(x)$ is equal to	(q) $\frac{1}{\sqrt{2}}$
(C) $\min\limits_{x \geq 1} f(x)$ is equal to	(r) $-\frac{1}{\sqrt{2}}$
(D) $f''(x) > 0$ at $x =$	(s) $\frac{1}{6}$
	(t) $\frac{1}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$

2. Match the following:

COLUMN-I	COLUMN-II
(A) Let $J = \int_0^{\infty} \frac{\log_e(1+t^2)}{1+t^2}\,dt$ and $K = \int_0^{\frac{\pi}{2}} \log_e(\sin t)\,dt$. If $J + \alpha K = 0$, then α is equal to:	(p) 1
(B) The value of $4\left(\lim\limits_{t \to 0} \frac{\int_0^{\tan^{-1} t} x.\tan^2 x\,dx}{\int_1^{\sec t} (x^2 - 1)\,dx}\right)$ is equal to:	(q) 2
(C) The slope of tangent drawn to the curve $f(x) = \sin x - \int_0^x (x - t) f(t)\;dt$ at $x = 0$, is equal to:	(r) 3
(D) The number of integral values of k for which the equation $e^x = \frac{k}{x-3}$ has exactly two solutions is	(s) 4
	(t) 7

(Continued)

3. Let u(x), v(x) and w(x) are three differentiable functions satisfying:

$$\int (u(x) + v(x))\,dx = \frac{x^4}{4} + C_1\,; \quad \int (u(x) - v(x))\,dx = x^2 - \frac{x^4}{4} + C_2$$

and $\int \frac{2u(x)}{w(x)}\,dx = -\frac{1}{x^2} + C_3$ where C_1, C_2, C_3 are constant of integration. Then:

COLUMN-I	COLUMN-II
(A) $\int_{-\pi/4}^{\pi/4} v(x)\,dx$ is equal to:	(p) 0
(B) Number of points of non-differentiability of function $h(x) = \max.(u(x), u(x) + v(x), w(x))$ is equal: to	(q) 1
(C) Number of extremum points of function $h(x) = \max.(u(x), u(x) + v(x), w(x))$ is equal to:	(r) 2
(D) Number of integral values of x satisfying the inequality $w(x) < u(x)$ is equal to:	(s) 3
	(t) 4

4. Match the following:

COLUMN-I	COLUMN-II
(A) If $\int_0^1 \frac{\cos x}{1+x}dx = k$ and $\int_{6\pi-3}^{6\pi} \frac{\cos\left(\frac{x}{3}\right)}{6\pi+3-x}dx = mk$, then m is:	(p) 4
(B) $\int_{-1}^{1}\left(\sin^{-1}\left[x+\frac{3}{4}\right]\right)dx = \frac{\pi}{k}$, then k is { where [.] denotes greatest integer function}	(q) 1
(C) If $f(x) = \max(x-\lvert x\rvert, x+[x])$ and $\int_{-3}^{3} f(x)dx = -\frac{3}{k}$, then l is { where [.] denotes greatest integer function}	(r) 3
(D) If $\int_0^{20}\sqrt{1-\cos\pi x}dx = \frac{10k\sqrt{2}}{\pi}$, then k is:	(s) 2

5. Match the following:

COLUMN-I	COLUMN-II
(A) If $I = \int_2^3 ((x-1)^3+(4-x)^3+x)\cos\pi x$ dx, then $\lvert 50\pi^2 I\rvert$ is equal to:	(p) 0
(B) If $J = \int_0^{10} \text{sgn}(\sin\pi x)\ dx$, then $10J$ is equal to, where sgn x denotes signum function of x:	(q) 100
(C) If $K = \int_0^{102} [\cot^{-1} x]\,dx$, then $[K]$ is equal to, where $[y]$ denotes largest integer less than or equal to y:	(r) 50
(D) If $L = \dfrac{\int_0^{51}[x+25]dx}{\int_0^{51}\{x+25\}dx}$, then $\frac{L}{2}$ is equal to, where $[y]$ and $\{y\}$ denote greatest integer function and fractional part function respectively:	(s) 70

6. Match the definite integrals in column I with their values in column II.

COLUMN-I	COLUMN-II
(A) $\int_0^{\pi} x\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$	(p) $\frac{\pi^2}{16}$
(B) $\int_0^{\pi^2/4}\left(2\sin\sqrt{x}+\sqrt{x}\cos\sqrt{x}\right)dx$	(q) $\frac{\pi^2}{2}$
(C) $\left\lvert \frac{\pi}{\ln 2}\int_{-\pi/4}^{\pi/4}\ln\left(\sqrt{1+\sin 2x}\right)dx\right\rvert$	(r) $\frac{\pi^2}{4}$
(D) $\int_0^{\pi}\frac{8x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}dx$	(s) $\frac{\pi^2}{8}$

7. Match the following, where [.] denotes G.I.F.

COLUMN-I	COLUMN-II
(A) $\int_0^{\pi/2}\left[\frac{1+2\sin^2 x}{1+\sin^2 x}\right]dx =$	(p) $\frac{\pi}{4}$
(B) $\int_{-\pi/2}^{\pi/2}\left[\cos x-\cos^2 x\right]dx =$	(q) 0
(C) $\int_{\pi/6}^{\pi/4}\left[\frac{\sec x+\text{cosec}\,x-\sec x\,\text{cosec}\,x}{2}\right]dx =$	(r) π
(D) $\int_{\pi/6}^{\pi/4}\left[\frac{\cot x-\tan x}{2}\right]dx =$	(s) $\frac{\pi}{2}$

8. Match the following:

COLUMN-I	COLUMN-II
(A) $\int_{\pi/4}^{\pi/3}\left(\frac{\sin^2 3x}{\sin^2 x}-\frac{\cos^2 3x}{\cos^2 x}\right)dx$	(p) $\frac{-1}{2}$
(B) $\int_{\pi/2}^{\pi/4}\frac{1}{1+\sin 2x}\sqrt[2009]{\frac{\tan(x)-1}{\tan(x)+1}}\,dx$	(q) $\frac{2\sqrt{2}-2}{3}$
(C) $\int_0^{\pi/2}\frac{x^2}{(x\sin x+\cos x)^2}dx$	(r) $2\sqrt{3}-4$

(Continued)

COLUMN-I	COLUMN-II
(D) $\int_1^e \frac{\ln x}{x(\sqrt{1-\ln x}+\sqrt{\ln(x)+1})}$	(s) $\frac{2}{\pi}$
	(t) None

9. Match the following:

COLUMN-I	COLUMN-II
(A) The value of definite integral $\int_0^{\pi/2} \frac{(\sin x)}{(\sin x)^{\cos x}+(\cos x)^{\sin x}}dx$	(p) π
(B) The value of definite integral $\int_1^2 \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$ is less than:	(q) $\frac{\pi}{4}$
(C) The value of definite integral $\int_{-2}^{2} \frac{x^3-x+1}{x^3\sqrt{4-x^2}}$ is greater than:	(r) $\frac{1}{8}$
(D) Let $I_1 = \int_0^1 \frac{5^x}{(x+1)}dx$ and $I_2 = \int_0^1 \frac{x^2}{5^{x^3}(2-x^3)}dx$. Then the value of $\frac{I_1}{I_2}$:	(s) 15

(*Continued*)

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. The value of the definite integral $\int_{\pi/2}^{5\pi/2} \frac{e^{\tan^{-1}(\sin x)}}{e^{\tan^{-1}(\sin x)}+e^{\tan^{-1}(\cos x)}}dx$ is ____.

2. Let $I_n = \int_0^1 x^n\sqrt{1-x^2}\,dx$ where 'n' is an integer ≥ 2, then $\lim_{n\to\infty}\frac{I_n}{I_{n-2}}$ equals ____.

3. A function $f: R \to R$ satisfies the equation $f(x+y)=f(x)+f(y)$ $\forall\, x, y \in$ R and is continuous throughout the domain.

If $I_1+I_2+I_3+I_4+I_5=450$ where $I_n = n\int_0^n f(x)dx$, find $f'(0)$.

4. If $\int_0^1 {}^{207}C_7\, x^{200}(1-x)^7\,dx = P$, , then $416\,P = \ldots$

5. Let f be a function defined by $f(x) = \frac{\lambda^x}{\lambda^x+\sqrt{\lambda}}$ ($\lambda > 0$) and

$I_1 = \int_{f(1-k)}^{f(k)} x\, f(x(1-x))dx$ and $I_2 = \int_{f(1-k)}^{f(k)} f(x(1-x))dx$, where $2k-1>0$, if $\frac{I_1}{I_2}=\frac{a}{2}$, then a is equal to ____.

6. If $f(x)$ is twice differentiable function for $x \in [0, \pi]$ such that $f(0)=f(\pi)=0$ and $\int_0^{\pi/2}(f(2x)+f''(2x))\,\sin x\cos x\,dx = k\pi$. then k is equal to ____.

7. Given that $\lim_{n\to\infty}\int_0^\infty \cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}....\cos\frac{x}{2^{n-1}}dx = \frac{\pi}{2}$, then the value of $\int_0^\infty \frac{\sin^3 x}{x}dx = \frac{\pi}{\lambda}$. Find λ.

8. Let $f: [0, \infty) \to R$ be a continuous, strictly increasing function such that $f^3(x) = \int_0^x t\, f^2(t)dt$. If a normal is drawn to curve $y = f(x)$ with gradient $-\frac{1}{2}$, then find the intercept made by it on y-axis.

9. For any real number x, let $[x]$ denotes the largest integer less than or equal to x and let f be a real-valued function defined on the interval $[-20, 20]$ by

$$f(x) = \begin{cases} \left[x+\frac{1}{2}\right]-x \;\; ; & \text{if } \left[x+\frac{1}{2}\right] \text{is odd} \\ x-\left[x+\frac{1}{2}\right] \;\; ; & \text{if } \left[x+\frac{1}{2}\right] \text{is even} \end{cases}$$

. Then find the value of $\frac{\pi^2}{5}\int_{-20}^{20} f(x)\cos(\pi x)\,dx$.

10. Let $f(x)$ be periodic function with fundamental period 12. If $\int_{-6}^{12} f(x)\,dx = 9$ and $\int_{0}^{12} f(x)\,dx = 12$, then find the value of $\left|\int_0^6 f(x)dx\right|$

11. If the value of the definite integral $\int_{-4}^{\frac{9}{2}} (\sin \pi x)\left\{\frac{x}{2}\right\} dx = \frac{1-a\pi}{b\pi^2}$, then find the value of $(a + b)$ (where $\{x\}$ represent fractional part function and $a, b \in$ N)

12. $\int_{\pi/4}^{\pi/2} \frac{dx}{(\sin x + \cos x + 2\sqrt{\sin x.\cos x})\sqrt{\sin x.\cos x}}$ equals

13. If $f:[0,2] \rightarrow R, f(x) = \sqrt{x^3+2-2\sqrt{x^3+1}} + \sqrt{x^3+10-6\sqrt{x^3+1}}$, then $\int_0^2 f(x)dx$ is equal to

14. Let $f(x) = x^3 + 3x^2 - 6x + 11$, $g(x) = \ln|x|$. If $A = f(g'(-1))$, $B = g(f'(-1))$, $C = \int_{-2}^{1} f'(x)g'(f(x))dx$, $D = \frac{d}{dt}\int_1^{t^2} g(x)dx$ evaluated at $t = 3$ and $A(B + C) - D = k\ln(3)$ then the value of k is

15. Let $g : R \rightarrow \{4\}$ be a function given by $g(x) = x^3 (f'(t) - 2) + x^2 f''(t) + 4x(f(0) + 6) + 4$ and $h(x)$ is defined as

$$h(x) = \begin{cases} \int_0^x |f(t)-2|\,dt & ; \quad 0 \le x \le 6 \\ (x-6)^2 + 20 & ; \quad 6 < x \le 12 \end{cases}$$

. If number of integers in the range of $h(x)$ is N then find the value of $\frac{N}{19}$.

16. If $p = \int_{\sqrt[3]{\ln 4}}^{\sqrt[3]{\ln 5}} \frac{u^2 \cot^{-1}(u^3)}{\cot^{-1}(u^3) + \cot^{-1}(\ln 20 - u^3)} du$, then find the value of $(4e^{6p})$.

17. Let f be a twice differentiable function defined in $[-3, 3]$ such that $f(0) = -4, f'(3) = 0, f'(-3) = 12$ and $f''(x) \ge -2\int_0^\infty \left(\frac{\sin x}{x}\right)^3 dx = A$ and $\int_0^\infty \left(\frac{x - \sin x}{x^3}\right) dx = \frac{aA}{b}$ $x \in [-3, 3]$. If $g(x) = \int_0^x f(t)\,dt$, then find maximum value of $\frac{g(x)}{12}$.

18. If $\int_0^1 \frac{1}{(5+2x-2x^2)(1+e^{2-4x})} dx = \frac{1}{\sqrt{11}} \log \frac{\sqrt{11}+a}{\sqrt{10}}$, then a is ____.

19. Let $g(x) = \int e^x \left(\int_0^x f(t)dt + f(x)\right) dx$, where $f(x)$ is continuous on R. If $f(0) = 1, g(0) = 0$ then $g'(0) =$ ____.

20. If $f(x)$ is non-constant differentiable function, $f(x) \in (0, \pi)$ and $x\int_0^x \sin(f(t))dt = (x+2)\int_0^x t\sin(f(t))dt$, $x > 0$, then find the value of $f'(x)\cot(f(x)) + \frac{3}{1+x} \forall x > 0.$.

21. Consider $f(x) = 3x^2 + 2x + a$, where 'a' is parameter such that $\frac{da}{dt} = 3$. Let $a = 0$, when $t = 0$ and $A(t) = \int_0^t \{f(x)\}dx$ (where $\{.\}$ denotes fractional part function). If A(2) can be expressed as sum of 'n' integrals, then find the minimum value of '$n/2$'.

22. If $f : [0, 1] \rightarrow R$ is a continuous function satisfying $\int_0^1 f(x)dx = \frac{1}{3} + \int_0^1 \left(f(x^2)\right)^2 dx$, then find the reciprocal of $f\left(\frac{1}{4}\right)$.

23. If the value of the definite integral $\int_{\frac{1}{2}}^{1} \frac{dx}{x\sqrt{(3+4x)(3-2x)}}$ can be the expressed in the form as $\frac{1}{3}\ln\left(\frac{p+2\sqrt{q}}{r+\sqrt{p}}\right)$ where $p, q, r \in$ N, then find the least value of $(p + q + r)$.

24. Let $L_1 = \lim_{x\to 0} \int_0^x \frac{(1+\cos t)^2}{x} dt$ and $L_2 = \lim_{x\to\infty} \frac{1}{x}\int_0^x (1+\cos t)^2\,dt$, then find the value of $2(L_1 + L_2)$.

25. Let $g : R \rightarrow R$ be a differentiable function satisfying $g(x) = \int_0^x (g(t)\cos t - \cos(t - x))\,dt$, for all $x \in R$. Find number of integers in the range of $g(x)$.

26. Find the value of definite integral $\int_0^4 \frac{(x^2-4x+5)\sin(x-2)}{(2x^2-8x+11)} dx$.

27. The value of $\lim_{n\to\infty}\left\{\frac{(n^3+1)(n^3+2^3)(n^3+3^3).....(n^3+n^3)}{n^{3n}}\right\}^{\frac{1}{n}}$ is equal $\alpha.e^{\beta\int_0^1 \frac{x^3}{1+x^3}dx}$, then $(\alpha \in N, \beta \in R)$; then $\alpha - \beta$ equals _____.

28. Let $f(x)$ be a function defined by $f(x) = \int_1^x t\left(t^2 - 3t + 2\right)dt, 1 \le x \le 3$, then the number of positive integers in the range of $f(x)$ is ____.

29. If $\int_0^a \frac{dx}{\sqrt{x+a}+\sqrt{x}} = \int_0^{\pi/8} \frac{2\tan\theta}{\sin 2\theta}d\theta$; $a > 0$,then find the value of $\left[\frac{1}{a}\right]$ where [.] is greatest integer function.

30. If $\int_0^{2\pi} \frac{dx}{a+b\cos x+c\sin x} = \frac{k\pi}{\sqrt{a^2-b^2-c^2}}$ where $a > \sqrt{b^2+c^2} > 0$, then value of k is ____.

31. If $\int_0^{\pi} x\, f\left(\cos^2 x+\tan^4 x\right)dx = \frac{\pi k}{1997}\int_0^{\pi/2} f\left(\cos^2 x+\tan^4 x\right)dx$, then the value of $\left[\frac{k}{2000}\right]$ is ____. Where [.] is G.I.F

32. If $\int_0^{\pi} \frac{x^2\cos x}{(1+\sin x)^2}dx = \frac{A}{498}\pi - \pi^2$ then the digit in the units place of A is ___.

33. If $\lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n}\left(1+\frac{k}{n}\right)}$ has the value of equal to ke^{-1} where $k \in N$, find k.

34. If $\int_0^{\frac{\pi}{16}} \frac{64x\cot^2(8x)dx}{\cot(x)-\tan(x)-2\tan(2x)-4\tan(4x)} = \frac{\pi}{a}\ln b$, where a, $b \in N$, then find the greatest value of $\frac{a}{b}$.

35. Consider, $P(x) = x^3 + ax^2 + 9x + b$, $a, b \in R$. If $P(2 + x) + P(2 - x) = 2P(2)\ \forall\ x \in R$ and $P(x) + P(x + 3) + P(4 - x) + P(1 - x) = 20$ $\forall\ x \in$ R, then find the value of $\int_{-a}^{a} P(x)\,dx$.

36. If $\lim_{n\to\infty}\left(\prod_{r=1}^{n}\sin\left(\frac{r\pi}{4n}\right)\cos\left(\frac{r\pi}{4n}\right)\right)^{\frac{1}{n}} = \frac{a}{b}$, a, $b \in N$, then find the least value of $(a + b)$.

37. Consider, $f(x) = 3\int_0^1 x^2 f(t)\,dt + 2\int_0^1 xf(|1-t|)\,dt + 3$. If maximum value of $\int_{-1}^{a} f(x)\,dx$ $(a > -1)$ is S, then find the integral part of S.

38. If $f(x) = \lim_{t\to 0}\int_0^x\left(\frac{tx^2+x\left((t+1)^{2t+2}-1\right)}{e^{-x}t}\right)dx$. Then find the number of solution(s) of the equation $2f(-|x|) - 1 = 0$.

39. If the value of the definite integral $\int_{-1}^{1}\cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right).\left(\cot^{-1}\frac{x}{\sqrt{1-\left(x^2\right)^{|x|}}}\right)dx = \frac{\pi^2\left(\sqrt{a}-\sqrt{b}\right)}{\sqrt{c}}$ where $a, b, c \in N$ in their lowest form, then find the value of $(a + b + c)$.

40. If $l = \lim_{x\to 1^+} 2^{-2^{\frac{1}{1-x}}}$ and $m = \lim_{x\to 1^+}\frac{x\sin\left(x-[x]\right)}{x-1}$ where [.] denotes greatest integer function, then $\int_l^m \frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx =$ ____.

41. $\int_1^e\left(\frac{1}{\sqrt{x\ln x}}+\sqrt{\frac{\ln x}{x}}\right)dx = L$, then the value of $3L/\sqrt{e}$ is ____.

42. Let $I_n = \int_0^{\pi/2}(\sin x+\cos x)^n dx\ (n \geq 2)$. Then the value of $nI_n - 2(n-1)I_{n-2}$ is ____.

43. If $f(x) = \begin{cases}\tan^{-1}\alpha - \beta\tan^{-1}x & ; & x < -1\\ \cot^{-1}\beta - \alpha\cot^{-1}x & ; & x \geq -1\end{cases}$ is differentiable in R (set of all real numbers) and $\sqrt{\int_{2\tan^{-1}\beta}^{2\tan^{-1}\alpha} f(x)d(\tan^{-1}x)} = \frac{a}{b}\pi$ where $a, b \in N$ then find the least value of $(a + b)$.

44. Let $g(x)$ be a polynomial function satisfying $g(x)\,g(y) = g(x) + g(y) + g(xy) - 2\ \forall\ x, y \to R$. Also $g(1) \neq 1$ and $g(2) = 5$. If the value of the integral $\int_1^{\infty}\frac{dx}{xg(x)}$ is $\frac{p}{q}\ln 2$ where p and q are co-prime, then find the value of $(p^2 + q^2)$.

45. $\frac{\int_0^1\left(1-x^5\right)^6 dx}{\int_0^1\left(1-x^5\right)^7 dx} = \frac{12}{35}k$ then k is ____.

46. If $I_n = \int_{\pi/2}^{\infty} e^{-x}\cos^n x dx = \frac{720}{3145}e^{-\pi/2}$, $\forall n = 2m, m \in N$, then the value of n is ____________ .

47. If $f(x) = \max\left(\frac{1}{\pi}\cos^{-1}(\cos \pi x), \{x\}\right)$ and $g(x) = \min\left(\frac{1}{\pi}\cos^{-1}(\cos\pi x), \{x\}\right)$ (where {.} represents fractional part of x). Then find the value of $\frac{\int_{1}^{2} f(x)dx}{\int_1^{2} g(x)dx}$.

48. If $I = \int_{-3/2}^{4} f(x)dx$, where $f(x)$ is given by $f(x) = \max_{-\frac{3}{2}\leq t\leq x}\{|t-1|-|t|+|t+1|\}$ then the value of $I - \frac{51}{8} =$ is ____.

49. Let $f(x)=\begin{cases}1-|x| & ; \quad |x|\le 1\\ 0 & ; \quad |x|>1\end{cases}$ and $g(x) = f(x - 1) + f(x + 1)$ $\forall\, x \in R$, then the value of $\int_{-2011}^{2011} g(x)\,dx$ is equal to ____.

50. Let $f: R \to R$ be a differentiable function satisfying $f(x) = f(y)f(x - y)$ $\forall\, x, y \in R$ and $f'(0)=\int_0^4\{2x\}dx$, where {.} denotes the fractional part function and $f'(-3)=\alpha e^{\beta}$. Then $|\alpha+\beta|$ is equal to ______________.

51. A function $f(x)$ satisfies $f(x) = f\left(\frac{c}{x}\right)$ for some real number $c > 1$ and all positive number 'x'. If $\int_1^{\sqrt{c}}\frac{f(x)}{x}dx=3$, then $\int_1^{c}\frac{f(x)}{x}dx$:

52. Let $f: R\to R$ be defined by $f(x)=\int_1^3\frac{dt}{1+|t-x|}$.

If $\int_1^3 f(t)dt=\left(6\log_e k\right)-(k+1)$, then $k=$ ____.

53. Let $f(x)$ be differentiable function such that $f(x)=x^2+\int_0^x e^{-t}f(x-t)dt$, then $6f(1)=:$ ________.

54. If $I=\int_0^{\pi}x\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$, then $[I]=$ ________, where $[.]$ denotes the greatest integer function.

55. The integral $\int_{\pi/4}^{5\pi/4}(|\cos t|\sin t+|\sin t|\cos t)dt$ has the value equal to ____.

56. If $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{r=1}^{n}\sqrt{r}\sum\limits_{r=1}^{n}\frac{1}{\sqrt{r}}}{\sum\limits_{r=1}^{n}r}=\dfrac{k}{3}$ then find k.

57. $\lim\limits_{n\to\infty}\sum\limits_{k=1}^{n}\dfrac{k^{1/a}\left\{n^{a-\frac{1}{a}}+k^{a-\frac{1}{a}}\right\}}{n^{a+1}}$ is equal to _________.

58. If $\lim\limits_{x\to 0}\int_0^x\dfrac{t^2dt}{(x-\sin x)\sqrt{a+t}}=1$, then the value of a is ________ .

59. Let x_1 and $x_2 (x_1 < x_2)$ be two value of x satisfying $||x|^2 - 4\ |x|\ + 5\ |\ +\ ||\ x^3 - 4x\ |\ + 3\ |\ = 4$ and $f(x)$ is a continuous periodic function with period $\frac{1}{2}$ such that $\int_0^{3/2} f(x)dx = 6$. Find the value of $\int_{x_1}^{x_2} f(x)dx$.

60. If value of definite integral $\int_0^{\ln 2}\dfrac{(x-\ln 2)e^{x+\ln 2}}{e^{\ln(1+e^x)^2}}dx$ can be the expressed in the form $\ln\left(\frac{a}{b}\right)$ where $a, b \in N$ and are relatively prime, then find the value of $(a + b)$.

61. If the value of definite integral $\dfrac{\int_0^{\frac{\pi}{2}}(\cos x)^{\sqrt{2}+1}dx}{\int_0^{\frac{\pi}{2}}(\cos x)^{\sqrt{2}-1}dx}$ is equal to $\left(n-\sqrt{n}\right)$, where $n \in N$, then find the value of n.

62. Let $I_1=\int_{1/2}^{2}\dfrac{x^{2012}-1}{x^{2014}+1}dx$ and $I_2=\int_2^4\left(\log_x 2-\dfrac{(\log_x 2)^2}{\ln 2}\right)dx$ then find $I_1 + I_2$.

63. Evaluate $\left[\int_0^{\pi}\dfrac{dx}{1+2\sin^2 x}\right]$ where [.] denotes greatest integer function.

64. Let $F(x)$ be a non-negative continuous function defined on R such that $F(x)+F\left(x+\frac{1}{2}\right)=3$ and the value of $\int_0^{1500}F(x)dx$ is $\dfrac{9000}{\lambda}$. Then the numerical value of λ is ___________.

65. If $I=\int_{-1}^{1}(1+x)^{1/2}(1-x)^{3/2}dx$, then the value of $\sec^6\left(\frac{1}{2}\right)$ is _____ .

66. If $\int_0^{\pi/2}x^n\sin x\,dx=\left(\frac{3}{4}\right)(\pi^2-8)$, then the value of n is _____ .

67. If the value of the definite integral $\int_1^{\sqrt{3}}\frac{1}{x^2}\ln\left(\sqrt{1+x^2}\right)dx$ is equal to $\dfrac{(\sqrt{3}-2)\ln 2}{\sqrt{3}k_1}+\dfrac{\pi}{k_2}$, then find the value of $\dfrac{k_2}{k_1}$.

68. Find the value of k satisfying the equation,

$$3\int_{-3}^{1}(x-k)(x+3)(x-1)\,dx=64$$

69. Find the value of definite integral

$$\frac{\int\limits_0^{\pi}\left(\sqrt[2013]{\cos x}+\sqrt[2013]{\sin x}+\sqrt[2013]{\tan x}\right)dx}{\int\limits_0^{\pi/2}\left(\sqrt[2013]{\sin x}\right)dx}.$$

70. Find the value of the definite integral $\int\limits_0^1 (2x^3-3x^2-x+1)^{1/3}dx$.

71. Let y = P(x) be a quadratic function with P(2) = 1. Find the value of the integral $\int\limits_{2-\pi}^{2+\pi} f(x)\cdot\sin\left(\frac{x-2}{2}\right)dx$.

72. Let $f:[0,\infty)\rightarrow R$ be a continuous, strictly increasing function such that $f^3(x) = \int\limits_0^x tf^2(t)\,dt$. If a normal is drawn to the curve $y = f(x)$ with gradient $\frac{-1}{2}$, then find the intercept made by it on the y-axis.

73. Let $f(x)=x^3-\frac{3x^2}{2}+x+\frac{1}{4}$. Find the value of $\left(\int\limits_{1/4}^{3/4} f(f(x))dx\right)^{-1}$.

74. Consider a function of the form $f(x) = \alpha e^{2x} + \beta e^{x} - \gamma x$, where α, β, γ are independent of x and $f(x)$ satisfies the following conditions $f(0) = -1$. $f'(\ln 2) = 30$ and $\int\limits_0^{\ell n\,4}(f(x)+\gamma x) = 24$. Find the value of $(\alpha+\beta+\gamma)$.

75. Let the function $f(x)$ and $g(x)$ are differentiable and strictly increasing $\forall\ x \in R$. Also $J = \int \frac{f(x)\cdot g'(x) - f'(x)\cdot g(x)}{f^2(x)g(x)}dx$, then the value of $(6J + 1)$ is equal to $\ln\frac{a}{b}$, where $a, b \in N$. Find the minimum value of $(a + b)$.

76. If the value of definite integral

$$\int\limits_{-4}^{4}\frac{x^3-2x^2-\operatorname{cosec}^3 x+\tan x}{5-|x|}dx = a + b\ln 5, \text{ where } a, b \in I,$$

then find the value of $(a + b)$.

77. If the value of definite integral $\int\limits_{1/3}^{1/2}\left\{x\left[\frac{2}{x}\right]\right\}dx = \frac{m}{n}$ where m and n are coprime, then find the absolute value of $(m - n)$.

78. Let $p(x)$ be fifth degree polynomial such that $p(x) + 1$ is divisible by $(x - 1)^3$ and $p(x) - 1$ is divisible by $(x + 1)^3$. Then find the value of definite integral $\int\limits_{-10}^{10} p(x)dx$.

79. If a_1, a_2, a_3 are 3 values of 'a' which satisfy the equation $\int\limits_0^{\pi/2}(\sin x + a\cos x)^3\,dx = 2 + \frac{4a}{\pi - 2}\int\limits_0^{\pi/2} x\cos x\ dx$, then find the value of $\left[a_1^2 + a_2^2 + a_3^2\right]$.

80. Let $I_n = \int\limits_0^{\frac{\pi}{2}} x\sin^n x\,dx$, then find the value of $\left(\frac{2561_{16}-1}{1_{14}}\right)$.

81. Let $F(n) = \int\frac{x^n}{e^x}dx$, then find the value of definite integral $\int\limits_2^5\left(e^{\int\frac{F(1)}{F(2)}dx}\right)^2 dx$, where $F(1)$ and $F(2)$ contains no constant term. Also $\int\frac{F(1)}{F(2)}dx$ contains no constant term.

82. If the value of the definite integral $\int\limits_0^{2\pi}\frac{dx}{\sin^2 x + (100)^2\cdot\cos^2 x}$ $= \frac{a\pi}{b}$ where a and b are coprime. Find $(a + b)$.

83. Let $I = \int\limits_0^1\left(1-x^{50}\right)^{99}x^{100}dx$ and $J = \int\limits_0^1\left(1-x^{50}\right)^{100}x^{100}dx$. If $\frac{I}{J}$ is equal to $\frac{m}{n}$ where m and n are coprime, then find the value of $\left(\frac{m-n-1}{20}\right)$.

84. If $\int\limits_0^{\pi}\left(\frac{8\sin^2 x}{g(x)}-1\right)(g(x))^2\,dx = 8\pi$; where $g(x)$ is a continuous function, then the maximum value of the function $g(x)$ in $[0, \pi]$ is____.

ARCHIVE: JEE MAIN

1. [JEE Mains 2021]

$$\lim_{x\to 0^+}\frac{\int_0^{x^2}\sin\sqrt{t}\,dt}{x^3}$$ is equal to:

(A) $\frac{1}{15}$ (B) $\frac{3}{2}$

(C) 0 (D) $\frac{2}{3}$

2. [JEE Mains 2021]

The value of $\int_1^3[x^2-2x-2]dx$ ([.] denotes greatest integers function)

(A) –4 (B) –5

(C) $-\sqrt{2}-\sqrt{3}-1$ (D) $1-\sqrt{2}-\sqrt{3}$

3. [JEE Mains 2021]

Let $f(x)$ be a differentiable function defined on [0, 2] such that $f'(x)=f'(2-x)$ for all $x\in(0,2)$, $f(0)=1$ and $f(2)=e^2$. Then the value of $\int_0^2 f(x)\,dx$ is:

(A) $2(1-e^2)$ (B) $2(1+e^2)$

(C) $1-e^2$ (D) $1+e^2$

4. [JEE Mains 2021]

The value of $\int_{-1}^{1}x^2e^{[x^3]}dx$, where $[t]$ denotes the greatest integer $\le t$, is:

(A) $\frac{e-1}{3e}$ (B) $\frac{e+1}{3}$

(C) $\frac{e+1}{3e}$ (D) $\frac{1}{3e}$

5. [JEE Mains 2021]

If $I_n=\int_{\pi/4}^{\pi/2}\cot^n x\,dx$, then:

(A) $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$ are in A.P.

(B) I_2+I_4, I_3+I_5, I_4+I_6 are in A.P.

(C) I_2+I_4, $(I_3+I_5)^2$, I_4+I_6 are in G.P.

(D) $\frac{1}{I_2+I_4}, \frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}$ are in G.P.

6. [JEE Mains 2021]

The value of $\sum_{n=1}^{100}\int_{n-1}^{n}e^{x-[x]}dx$, where $[x]$ is the greatest integer $\le x$, is:

(A) $100(e-1)$ (B) $100(1+e)$

(C) $100e$ (D) $100(1-e)$

7. [JEE Mains 2021]

The value of $\int_{-\pi/2}^{\pi/2}\frac{\cos^2 x}{1+3^x}dx$ is:

(A) $\frac{\pi}{4}$ (B) 2π

(C) $\frac{\pi}{2}$ (D) 4π

8. [JEE Mains 2021]

For $x>0$, if $f(x)=\int_1^x\frac{\log_e(t)}{(1+t)}dt$, then $f(e)+f\left(\frac{1}{e}\right)$ is equal to:

(A) $\frac{1}{2}$ (B) –1

(C) 1 (D) 0

9. [JEE Mains 2021]

If $\int_{-a}^{a}(|x|+|x-2|)\,dx=22$, $(a>2)$ and $[x]$ denotes the greatest integer $\le x$, then $\int_{a}^{-a}(x+[x])\,dx$ is equal to ______.

10. [JEE Mains 2021]

The value of: $\int_{-2}^{2}|3x^2-3x-6|\,dx$ is ______.

11. [JEE Mains 2021]

The value of the integral $\int_0^{\pi}|\sin 2x|\,dx$ is ______.

12. [JEE Mains 2021]

If $I_{m,n}=\int_0^1 x^{m-1}.(1-x)^{n-1}\,dx$, for $m,n\ge 1$ and $\int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$ $=\alpha I_{m,n}$, $\alpha\in R$, then find 'α' equals ______.

13. [JEE Mains 2021]

Let $f: R\to R$ be a continuous function such that $f(x)+f(x+1)=2$, for all $x\in R$. If $I_1=\int_0^8 f(x)\,dx$ and $I_2=\int_{-1}^{3}f(x)\,dx$, then the value of I_1+2I_2 is equal to ________.

14. [JEE Mains 2020]

The value of $\int_0^{2\pi}\frac{x\sin^8 x}{\sin^8 x+\cos^8 x}dx$ is equal to:

(A) π^2 (B) 2π

(C) 4π (D) $2\pi^2$

15. [JEE Mains 2020]

If the value of the integral $\int_0^{1/2} \frac{x^2}{(1-x^2)^{3/2}}\,dx$ is $\frac{k}{6}$, then k is equal to:

(A) $2\sqrt{3} - \pi$ (B) $3\sqrt{2} + \pi$

(C) $3\sqrt{2} - \pi$ (D) $2\sqrt{3} + \pi$

16. [JEE Mains 2020]

$\int_{-\pi}^{\pi} |\pi - |x||\,dx$ is equal to:

(A) π^2 (B) $2\pi^2$

(C) $\sqrt{2}\pi^2$ (D) $\frac{\pi^2}{2}$

17. [JEE Mains 2020]

$\int_{\pi/6}^{\pi/3} \tan^3 x \cdot \sin^2 3x(2\sec^2 x \cdot \sin^2 3x + 3\tan x \cdot \sin 6x)\,dx$ is equal to:

(A) $\frac{9}{2}$ (B) $-\frac{1}{9}$

(C) $-\frac{1}{18}$ (D) $\frac{7}{18}$

18. [JEE Mains 2020]

Let $f(x) = |x - 2|$ and $g(x) = f(f(x))$, $x \in [0, 4]$. Then $\int_0^3 (g(x)) - f(x))\,dx$ is equal to:

(A) $\frac{3}{2}$ (B) 0

(C) $\frac{1}{2}$ (D) 1

19. [JEE Mains 2020]

The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{1+e^{\sin x}}\,dx$ is:

(A) π (B) $\frac{3\pi}{2}$

(C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$

20. [JEE Mains 2020]

The integral $\int_1^2 e^x . x^x (2 + \log_e x)\,dx$ equal:

(A) $e(4e + 1)$ (B) $e(2e - 1)$

(C) $4e^2 - 1$ (D) $e(4e - 1)$

21. [JEE Mains 2020]

If $I_1 = \int_0^1 (1-x^{50})^{100}\,dx$ and $I_2 = \int_0^1 (1-x^{50})^{101}\,dx$ such that $I_2 = \alpha I_1$ then α equals to:

(A) $\frac{5050}{5051}$ (B) $\frac{5050}{5049}$

(C) $\frac{5049}{5050}$ (D) $\frac{5051}{5050}$

22. [JEE Mains 2020]

Let a function $f : [0, 5] \to R$ be continuous, $f(1) = 3$ and F be defined as:

$F(x) = \int_1^x t^2 g(t)\,dt$, where $g(t) = \int_1^t f(u)\,du$. Then for the function F, the point $x = 1$ is:

(A) not a critical point.

(B) a point of inflection.

(C) a point of local maxima.

(D) a point of local minima.

23. [JEE Mains 2020]

The integral $\int_0^2 ||x-1| - x|\,dx$ is equal to _____.

24. [JEE Mains 2020]

Let $[t]$ denote the greatest integer less than or equal to t. Then the value of $\int_1^2 |2x - [3x]|\,dx$ is _____

25. [JEE Mains 2020]

Let $\{x\}$ and $[x]$ denote the fractional part of x and the greatest integer $\le x$ respectively of *a* real number x. If $\int_0^n \{x\}dx$, $\int_0^n [x]dx$ and $10(n^2 - n)$, $(n \in N, n > 1)$ are three consecutive terms of *a* G.P., then n is equal to ____.

26. [JEE Mains 2019]

The value of $\int_0^{\pi} |\cos x|^3\,dx$ is:

(A) $-\frac{4}{3}$ (B) 0

(C) $\frac{4}{3}$ (D) $\frac{2}{3}$

27. [JEE Mains 2019]

Let f be a differentiable function from R to R such that $|f(x) - f(y)| \le 2|x - y|^{3/2}$, for all $x, y \in R$. If $f(0) = 1$, then $\int_0^1 f^2(x)\,dx$ is equal to:

(A) 2 (B) $\frac{1}{2}$

(C) 0 (D) 1

28. [JEE Mains 2019]

If $\int_0^{\pi/3} \frac{\tan\theta}{\sqrt{2k\sec\theta}}\,d\theta = 1 - \frac{1}{\sqrt{2}}$ $(k > 0)$, then the value of k is:

(A) 2 (B) 1

(C) 4 (D) $\frac{1}{2}$

29. [JEE Mains 2019]

Let $I = \int_a^b (x^4 - 2x^2)\,dx$. If I is minimum, then the ordered pair (a, b) is:

(A) $(\sqrt{2}, -\sqrt{2})$
(B) $(0, \sqrt{2})$
(C) $(-\sqrt{2}, \sqrt{2})$
(D) $(-\sqrt{2}, 0)$

30. [JEE Mains 2019]

The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x]+[\sin x]+4}$, where $[t]$ denotes the greatest integer less than or equal to t, is:

(A) $\frac{1}{12}(7\pi - 5)$
(B) $\frac{1}{12}(7\pi + 5)$
(C) $\frac{3}{10}(4\pi - 3)$
(D) $\frac{3}{20}(4\pi - 3)$

31. [JEE Mains 2019]

If $\int_0^x f(t)dt = x^2 + \int_x^1 t^2 f(t)dt$ then $f'\left(\frac{1}{2}\right)$ is:

(A) $\frac{18}{25}$
(B) $\frac{6}{25}$
(C) $\frac{24}{25}$
(D) $\frac{4}{5}$

32. [JEE Mains 2019]

The value of the integral $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}\,dx$ (where $[x]$ denotes the greatest integer less than or equal to x) is:

(A) 0
(B) 4
(C) 4 – sin 4
(D) sin 4

33. [JEE Mains 2019]

The integral $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x(\tan^5 x + \cot^5 x)}$ equals:

(A) $\frac{\pi}{40}$
(B) $\frac{1}{20}\tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)$
(C) $\frac{1}{10}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{9\sqrt{3}}\right)\right)$
(D) $\frac{1}{5}\left(\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3\sqrt{3}}\right)\right)$

34. [JEE Mains 2019]

Let f and g be continuous functions on $[0, a]$ such that $f(x) = f(a - x)$ and $g(x) + g(a - x) = 4$, then $\int_0^a f(x)g(x)dx$ is equal to:

(A) $4\int_0^a f(x)dx$
(B) $-3\int_0^a f(x)dx$
(C) $\int_0^a f(x)dx$
(D) $2\int_0^a f(x)dx$

35. [JEE Mains 2019]

The integral $\int_1^e \left\{\left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^{x}\right\}\log_e x\,dx$ is equal to:

(A) $-\frac{1}{2} + \frac{1}{e} - \frac{1}{2e^2}$
(B) $\frac{3}{2} - e - \frac{1}{2e^2}$
(C) $\frac{1}{2} - e - \frac{1}{e^2}$
(D) $\frac{3}{2} - \frac{1}{e} - \frac{1}{2e^2}$

36. [JEE Mains 2019]

$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \ldots + \frac{1}{5n}\right)$ is equal to:

(A) $\tan^{-1}(2)$
(B) $\tan^{-1}(3)$
(C) $\frac{\pi}{4}$
(D) $\frac{\pi}{2}$

37. [JEE Mains 2018]

The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x}\,dx$:

(A) $\frac{\pi}{2}$
(B) 4π
(C) $\frac{\pi}{4}$
(D) $\frac{\pi}{8}$

38. [JEE Mains 2017]

The integral $\int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x}$ is equal to:

(A) –1
(B) –2
(C) 2
(D) 4

39. [JEE Mains 2016]

$\lim_{n\to\infty}\left(\frac{(n+1)(n+2)\ldots 3n}{n^{2n}}\right)^{1/n}$ is equal to:

(A) 3 log 3 – 2
(B) $\frac{18}{e^4}$
(C) $\frac{27}{e^2}$
(D) $\frac{9}{e^2}$

40. [JEE Mains 2015]

The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)}\,dx$ is equal to:

(A) 1
(B) 6
(C) 2
(D) 4

41. [JEE Mains 2014]

The integral $\displaystyle\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals:

(A) $\pi-4$

(B) $\dfrac{2\pi}{3}-4-4\sqrt{3}$

(C) $4\sqrt{3}-4$

(D) $4\sqrt{3}-4-\dfrac{\pi}{3}$

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. [JEE Adv 2017]

If $g(x)=\displaystyle\int_{\sin x}^{\sin(2x)}\sin^{-1}(t)dt$ then:

(A) $g'\left(\dfrac{\pi}{2}\right)=-2\pi$

(B) $g'\left(-\dfrac{\pi}{2}\right)=2\pi$

(C) $g'\left(\dfrac{\pi}{2}\right)=2\pi$

(D) $g'\left(-\dfrac{\pi}{2}\right)=-2\pi$

2. [JEE Adv 2016]

The value of $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{x^2\cos x}{1+e^x}dx$ is equal to:

(A) $\dfrac{\pi^2}{4}-2$

(B) $\dfrac{\pi^2}{4}+2$

(C) $\pi^2-e^{\frac{\pi}{2}}$

(D) $\pi^2+e^{\frac{\pi}{2}}$

3. [JEE Adv 2015]

Let $f'(x)=\dfrac{192x^3}{2+\sin^4\pi x}$ for all $x\in R$ with $f\left(\dfrac{1}{2}\right)=0$. If $m\le\displaystyle\int_{1/2}^{1}f(x)dx\le M$, then the possible values of m and M are:

(A) $m=13, M=24$

(B) $m=\dfrac{1}{4}, M=\dfrac{1}{2}$

(C) $m=-11, M=0$

(D) $m=1, M=12$

4. [JEE Adv 2014]

The following integral $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\text{cosec}x)^{17}dx$ is equal to:

(A) $\displaystyle\int_0^{\log(1+\sqrt{2})}2(e^u+e^{-u})^{16}du$

(B) $\displaystyle\int_0^{\log(1+\sqrt{2})}(e^u+e^{-u})^{17}du$

(C) $\displaystyle\int_0^{\log(1+\sqrt{2})}(e^u-e^{-u})^{17}du$

(D) $\displaystyle\int_0^{\log(1+\sqrt{2})}2(e^u-e^{-u})^{16}du$

5. [JEE Adv 2014]

Let $f:[0,2]\to R$ be a function which is continuous on $[0,2]$ and is differentiable on (0, 2) with $f(0)=1$. Let $F(x)=\displaystyle\int_0^{x^2}f(\sqrt{t})dt$ for $x\in[0,2]$. If $F'(x)=f'(x)$ for all $x\in(0,2)$, then $F(2)$ equals:

(A) e^2-1

(B) e^4-1

(C) $e-1$

(D) e^4

Multiple Correct Choice Type Questions

6. [JEE Adv 2021]

Let $f:\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\to$ R be a continuous function such that $f(0)=1$ and $\displaystyle\int_0^{\pi/3}f(t)\,dt=0$

Then which of the following statements is (are) TRUE?

(A) The equation $f(x)-3\cos 3x=0$ has at least one solution in $\left(0,\dfrac{\pi}{3}\right)$

(B) The equation $f(x)-3\sin 3x=-\dfrac{6}{\pi}$ has at least one solution in $\left(0,\dfrac{\pi}{3}\right)$

(C) $\displaystyle\lim_{x\to 0}\frac{x\int_0^x f(t)dt}{1-e^{x^2}}=-1$

(D) $\displaystyle\lim_{x\to 0}\frac{\sin x\int_0^x f(t)dt}{x^2}=-1$

7. [JEE Adv 2020]

Which of the following inequalities is/are TRUE?

(A) $\displaystyle\int_0^1 x\cos x\,dx\ge\frac{3}{8}$

(B) $\displaystyle\int_0^1 x\sin x\,dx\ge\frac{3}{10}$

(C) $\displaystyle\int_0^1 x^2\cos x\ge\frac{1}{2}$

(D) $\displaystyle\int_0^1 x^2\sin x\ge\frac{2}{9}$

8. [JEE Adv 2019]

For $a \in R$, $|a| > 1$, let

$$\lim_{n\to\infty}\left(\frac{1+\sqrt[3]{2}+\ldots+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(an+1)^2}+\frac{1}{(an+2)^2}+\ldots+\frac{1}{(an+n)^2}\right)}\right)=54$$

Then the possible value(s) of a is/are:

(A) –9 (B) –6

(C) 7 (D) 8

9. [JEE Adv 2019]

Let $f : R \to R$ be given by $f(x) = (x - 1)(x - 2)(x - 5)$. Define

$$F(x)=\int_0^x f(t)dt, x>0$$

Then which of the following option is/are correct?

(A) F has a local minimum at $x = 1$

(B) F has a local maximum at $x = 2$

(C) F has two local maximum and one local minimum in $(0, \infty)$

(D) $F(x) \neq 0$ for all $x \in (0, 5)$

10. [JEE Adv 2017]

If $I=\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx$ then

(A) $I > \log_e 99$ (B) $I < \log_e 99$

(C) $I < \frac{49}{50}$ (D) $I > \frac{49}{50}$

11. [JEE Adv 2017]

Let $f : R \to (0, 1)$ be a continuous function. Then, which of the following functions (s) has (have) the value zero at some point is the interval (0, 1) ?

(A) $x^9 - f(x)$ (B) $x-\int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

(C) $e^x-\int_0^x f(t)\sin t\,dt$ (D) $f(x)+\int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$

12. [JEE Adv 2016]

Let $f(x) = \lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right)....\left(x+\frac{n}{n}\right)}{n!\left(x^2+n^2\right)\left(x^2+\frac{n^2}{4}\right)....\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}}$, for all $x > 0$. Then:

(A) $f\left(\frac{1}{2}\right) \ge f(1)$ (B) $f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$

(C) $f'(2) \le 0$ (D) $\frac{f'(3)}{f(3)} \ge \frac{f'(2)}{f(2)}$

13. [JEE Adv 2015]

The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at+\cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at+\cos^4 at)dt}=L?$$

(A) $a=2, L=\frac{e^{4\pi}-1}{e^{\pi}-1}$ (B) $a=2, L=\frac{e^{4\pi}+1}{e^{\pi}+1}$

(C) $a=4, L=\frac{e^{4\pi}-1}{e^{\pi}-1}$ (D) $a=4, L=\frac{e^{4\pi}+1}{e^{\pi}+1}$

14. [JEE Adv 2015]

Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the correct expression(s) is(are):

(A) $\int_0^{\pi/4} xf(x)dx=\frac{1}{12}$ (B) $\int_0^{\pi/4} f(x)dx=0$

(C) $\int_0^{\pi/4} xf(x)dx=\frac{1}{6}$ (D) $\int_0^{\pi/4} f(x)dx=1$

15. [JEE Adv 2014]

Let f: $(0, \infty) \to R$ be given by $f(x)=\int_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t}$. Then:

(A) (x) is monotonically increasing on $[1, \infty)$

(B) (x) is monotonically decreasing on $(0, 1)$

(C) $f(x)+f\left(\frac{1}{x}\right)=0$, for all $x \in (0, \infty)$

(D) $f(2^x)$ is an odd function of $x$on $\mathbb{R}$

16. [JEE Adv 2014]

Let f: $[a, b] \to [1, \infty)$ be a continuous function and let g: $\mathbb{R} \to \mathbb{R}$ be defined as

$$g(x)=\begin{cases} 0 & \text{if } x<a \\ \int_a^x f(t)dt & \text{if } a\le x\le b \text{, then:} \\ \int_a^b f(t)dt & \text{if } x>b \end{cases}$$

(A) $g(x)$ is continuous but not differentiable at a

(B) $g(x)$ is differentiable on $\mathbb{R}$

(C) $g(x)$ is continuous but not differentiable at b

(D) $g(x)$ is continuous and differentiable at either a or b but not both

Linked Comprehension Type Questions

Comprehension 1

[JEE Adv 2021]

Let $f_1 : (0, \infty) \to R$ and $f_2 : (0, \infty) \to R$ be defined by

$$f_1(x) = \int_0^x \prod_{j=1}^{21} (t-j)^j \, dt, \; x > 0 \text{ and } f_2(x) = 98(x-1)^{50} - 600(x-1)^{49} +$$

2450, $x > 0$, where, for any positive integer n and real numbers $a_1, a_2, \dots, a_n$, $\prod_{i=1}^{n} a_i$ denotes the product of $a_1, a_2, \dots, a_n$. Let m_i and n_i, respectively, denote the number of points of local minima and the number of points of local maxima of function f_i, $i = 1, 2$, in the interval $(0, \infty)$

17. The value of $2m_1 + 3n_1 + m_1 n_1$ is _____.

18. The value of $6m_2 + 4n_2 + 8m_2 n_2$ is _____.

Comprehension 2

[JEE Adv 2021]

Let $g_i : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R$, $i = 1, 2$, and $f : \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \to R$, be functions such that $g_1(x) = 1$, $g_2(x) = |4x - \pi|$ and $f(x) = \sin^2 x$, for all $\in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$ Define $S_i = \int_{\pi/8}^{3\pi/8} f(x) \cdot g_i(x) \, dx, \; i = 1, 2$

19. The value of $\frac{16 S_1}{\pi}$ is _____.

20. The value of $\frac{48 S_2}{\pi^2}$ is _____.

Comprehension 3

[JEE Adv 2021]

Let $y_1 : [0, \infty) \to R$, $y_2 : [0, \infty) \to R$, $f : [0, \infty) \to R$ and $g : [0, \infty) \to R$ be functions such that $f(0) = g(0) = 0$, $y_1(x) = e^{-x} + x$, $x \geq 0$, $y_2(x) = x^2 - 2x - 2e^{-x} + 2$, $x \geq 0$, $f(x) = \int_{-x}^{x} (t - t^2) e^{-t^2} \, dt$, $x > 0$ and $g(x) = \int_0^{x^2} \sqrt{t} e^{-t} \, dt$, $x > 0$

21. Which of the following statements is TRUE?

(A) $f(\sqrt{\ln 3})| + g(\sqrt{\ln 3}) = \frac{1}{3}$

(B) For every $x > 1$, there exists an $a \in (1, x)$ such that $y_1(x) = 1 + ax$

(C) For every $x > 0$, there exists an a $\beta \in (0, x)$ such that $y_2(x) = 2x(y_1(\beta) - 1)$

(D) f is an increasing function on the interval $\left[0, \frac{3}{2}\right]$

22. Which of the following statements is TRUE?

(A) $y_1(x) \leq 1$, for all $x > 0$

(B) $y_2(x) \leq 0$, for all $x > 0$

(C) $f(x) \geq 1 - e^{-x^2} - \frac{2}{3} x^5$, for all $x \in \left(0, \frac{1}{2}\right)$

(D) $g(x) \leq \frac{2}{3} x^3 - \frac{2}{5} x^5 +$ for all $x \in \left(0, \frac{1}{2}\right)$

23. For any real number x, let $[x]$ denote the largest integer less than or equal to x. If $I = \int_0^{10} \left[\sqrt{\frac{10x}{x+1}}\right] dx$, then the value of $9I$ is _____.

Comprehension 4

[JEE Adv 2015]

Let $F : R \to R$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$, Let $f(x) = xF(x)$ for all $x \in R$

24. The correct statement(s) is(are)

(A) $f'(1) < 0$

(B) $f(2) < 0$

(C) $f'(x) \neq 0$ for any $x \in (1, 3)$

(D) $f'(x) = 0$ for some $x \in (1, 3)$

25. If $\int_1^3 x^2 F'(x) dx = -12$ and $\int_1^3 x^3 F''(x) dx = 40$, then the correct expression(s) is(are):

(A) $9f'(3) + f'(1) - 32 = 0$

(B) $\int_1^3 f(x) dx = 12$

(C) $9f'(3) - f'(1) + 32 = 0$

(D) $\int_1^3 f(x) dx = -12$

Comprehension 5

[JEE Adv 2014]

Given that for each $a \in (0, 1)$, $\lim_{h \to 0^+} \int_h^{1-h} t^{-a} (1-t)^{a-1} dt$ exists.

Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$.

26. The value of $g\left(\frac{1}{2}\right)$ is:

(A) π

(B) 2π

(C) $\frac{\pi}{2}$

(D) $\frac{\pi}{4}$

27. The value of $g'\left(\frac{1}{2}\right)$ is:

(A) $\frac{\pi}{2}$

(B) π

(C) $-\frac{\pi}{2}$

(D) 0

Matrix Match/Column Match Type Questions

28. [JEE Adv 2014]

COLUMN-I	COLUMN-II
(A) The number of polynomials $f(x)$ with non-negative integer coefficients of degree ≤ 2, satisfying $f = (0)$ and $\int_0^1 f(x)dx = 1$, is:	(p) 8
(B) The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is:	(q) 2
(C) $\int_{-2}^{2} \frac{3x^2}{(1+e^x)} dx$ equal	(r) 4
(D) $\frac{\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}{\left(\int_{0}^{\frac{1}{2}} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}$ equals	(t) 0

	A	B	C	D
(A)	(r),	(q),	(s),	(p)
(B)	(q),	(r),	(s),	(p)
(C)	(r),	(q),	(p),	(s)
(D)	(q),	(r),	(p),	(s)

Integer/Numerical Answer Type Questions

29. [JEE Adv 2015]

Let $f : R \to R$ be a function defined by $f(x) = \begin{cases} [x] & ; \quad x \leq 2 \\ 0 & ; \quad x > 2 \end{cases}$, Where $[x]$ is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2 + f(x+1)} dx$, then the value of $(4I - 1)$ is _____.

30. [JEE Adv 2015]

If $\alpha = \int_0^1 (e^{9x+3\tan^{-1}x}) \left(\frac{12+9x^2}{1+x^2}\right) dx$ where $\tan^{-1} x$ takes only principal values, then the value of $\left(\log_e |1+\alpha| - \frac{3\pi}{4}\right)$ is _____.

31. [JEE Adv 2015]

Let $f : R \to$ R be continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t\,|f(f(t))|\,dt$ for all $x \in [-1, 2]$. If $\lim_{x \to 1} \frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is _____.

32. [JEE Adv 2020]

Let $f : R \to R$ be a differentiable function such that its derivative f' is continuous and $f(\pi) = -6$.
If $F : [0, \pi] \to R$ is defined by $F(x) = \int_0^x f(t)dt$ and if $\int_0^{\pi} (f'(x) + F(x)) \cos x\, dx = 2$, then the value of $f(0)$ is ____.

33. [JEE Adv 2019]

If $I = \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$, then $27I^2$ equals ________ .

34. [JEE Adv 2019]

The value of the integral $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{(\sqrt{\cos\theta} + \sqrt{\sin\theta})^5} d\theta$ equals ________.

35. [JEE Adv 2018]

The value of the integral $\int_0^{1/2} \frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{1/4}} dx$ is _______.

36. [JEE Adv 2018]

For each positive integer, n let $y_n = \frac{1}{n}((n+1)(n+2)...(n+n))^{\frac{1}{n}}$.

For $x \in R$, let $[x]$ be the greatest integer less than or equal to x. If $\lim_{n \to \infty} y_n = L$, , then the value of $[L]$ is _______.

37. [JEE Adv 2017]

Let f: $R \to R$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$.

If $g(x) = \int_x^{\pi/2} [f'(t) \operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)]dt$ for

$x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x \to 0} g(x) =$

38. [JEE Adv 2016]

The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1$ is _____.

39. [JEE Adv 2014]

The value of $\int_0^1 4x^3 \left\{\frac{d^2}{dx^2}(1-x^2)^5\right\} dx$ is _____.

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. B	**2.** B	**3.** C	**4.** C	**5.** A	**6.** C	**7.** C	**8.** D	**9.** C	**10.** B
11. B	**12.** D	**13.** D	**14.** B	**15.** C	**16.** A	**17.** D	**18.** C	**19.** C	**20.** B
21. B	**22.** B	**23.** B	**24.** B	**25.** B	**26.** D	**27.** D	**28.** A	**29.** C	**30.** D
31. B	**32.** A	**33.** C	**34.** A	**35.** A	**36.** C	**37.** D	**38.** C	**39.** C	**40.** A
41. C	**42.** A	**43.** C	**44.** A	**45.** A	**46.** C	**47.** D	**48.** D	**49.** C	**50.** C
51. B	**52.** B	**53.** D	**54.** A	**55.** D	**56.** A	**57.** A	**58.** D	**59.** A	**60.** D
61. A	**62.** C	**63.** C	**64.** B	**65.** B	**66.** C	**67.** A	**68.** D	**69.** C	**70.** D
71. A	**72.** B	**73.** B	**74.** C	**75.** A	**76.** C	**77.** A	**78.** D	**79.** A	**80.** D
81. B	**82.** B	**83.** B	**84.** A	**85.** C	**86.** B	**87.** B	**88.** C	**89.** A	**90.** D
91. D	**92.** C	**93.** C	**94.** B	**95.** D	**96.** B	**97.** C	**98.** B	**99.** D	**100.** B
101. D	**102.** A	**103.** D	**104.** C	**105.** D	**106.** C	**107.** D	**108.** B	**109.** D	**110.** B
111. B	**112.** C	**113.** D	**114.** A	**115.** D	**116.** C	**117.** C	**118.** C	**119.** B	**120.** B
121. A	**122.** C	**123.** B	**124.** C	**125.** C	**126.** A	**127.** D	**128.** D	**129.** B	**130.** D
131. D	**132.** C	**133.** B	**134.** C	**135.** B	**136.** D	**137.** B	**138.** D	**139.** C	**140.** C
141. B									

Multiple Correct Choice Type Questions

1. B, D	**2.** B, C	**3.** A, B, C	**4.** A, B, C	**5.** A, C
6. A, B	**7.** A, B	**8.** A, C, D	**9.** A, B	**10.** A, B
11. A, D	**12.** A, B, D	**13.** A, B, D	**14.** B, D	**15.** A, B, D
16. ABC	**17.** B, D	**18.** A, B, C, D	**19.** A, B	**20.** A, C, D
21. A, B, C	**22.** B, C, D	**23.** B, D	**24.** A, B, D	**25.** A, C
26. B, D	**27.** A, B, C, D	**28.** A, B	**29.** B, C, D	**30.** A, B, C, D
31. A, C, D	**32.** A, B	**33.** A, C, D	**34.** A, B, C	**35.** B, C
36. B, D	**37.** A, B	**38.** A, C	**39.** A, C	**40.** A, B, C
41. B, D	**42.** A, B, C, D	**43.** A, D	**44.** A, C	**45.** A, B, C
46. A, B, D	**47.** A, D	**48.** B, C	**49.** A, B	**50.** A, D
51. A, D	**52.** A, B	**53.** B, D	**54.** B, C	**55.** A, B
56. A, B				

Linked Comprehension Type Questions

1. C	**2.** B	**3.** B	**4.** A	**5.** B	**6.** A	**7.** C	**8.** C	**9.** A	**10.** C
11. B	**12.** A	**13.** A	**14.** A	**15.** C	**16.** C	**17.** C	**18.** **B**	**19.** **B**	**20.** B
21. D	**22.** A	**23.** A	**24.** B	**25.** C	**26.** A	**27.** B	**28.** B	**29.** A	**30.** D
31. C	**32.** D	**33.** A	**34.** A	**35.** C	**36.** A	**37.** B	**38.** D	**39.** C	**40.** B
41. C	**42.** B	**43.** D	**44.** B	**45.** C	**46.** B	**47.** B			

Matrix Match/Column Match Type Questions

1. A → (*t*)	B → (p)	C → (s)	D → (p), (q), (s), (t)
2. A → (q)	B → (s)	C → (p)	D → (*t*)
3. A → (p)	B → (r)	C → (q)	D → (p)
4. A → (q)	B → (p)	C → (s)	D → (p)
5. A → (q)	B → (p)	C → (p)	D → (r)

6. A → (q)	B → (q)	C → (r)	D → (r)
7. A → (s)	B → (q)	C → (q)	D → (q)
8. A → (r)	B → (*t*)	C → (s)	D → (q)
9. A → (q)	B → (r)	C → (p)	D → (s)

Integer/Numerical Answer Type Questions

1. 1	**2.** 1	**3.** 4	**4.** 2	**5.** 1	**6.** 0	**7.** 4	**8.** 9	**9.** 0	**10.** 15
11. 10	**12.** 1	**13.** 4	**14.** 7	**15.** 3	**16.** 5	**17.** 4	**18.** 1	**19.** 1	**20.** 0
21. 8	**22.** 2	**23.** 21	**24.** 11	**25.** 2	**26.** 0	**27.** 5	**28.** 3	**29.** 1	**30.** 2
31. 0	**32.** 6	**33.** 4	**34.** 8	**35.** 828	**36.** 5	**37.** 3	**38.** 4	**39.** 7	**40.** 0
41. 6	**42.** 2	**43.** 3	**44.** 5	**45.** 3	**46.** 6	**47.** 3	**48.** 6	**49.** 2	**50.** 4
51. 6	**52.** 3	**53.** 8	**54.** 4	**55.** 0	**56.** 8	**57.** 1	**58.** 4	**59.** 16	**60.** 17
61. 2	**62.** 0	**63.** 1	**64.** 4	**65.** 8	**66.** 3	**67.** 6	**68.** 1	**69.** 2	**70.** 0
71. 1	**72.** 9	**73.** 4	**74.** 3	**75.** 35	**76.** 12	**77.** 389	**78.** 0	**79.** 5	**80.** 240
81. 66	**82.** 51	**83.** 5	**84.** 4						

ARCHIVE: JEE MAIN

1. D	**2.** C	**3.** D	**4.** C	**5.** A	**6.** A	**7.** A	**8.** A	**9.** 3	**10.** 19
11. 2	**12.** 1	**13.** 16	**14.** A	**15.** A	**16.** A	**17.** C	**18.** D	**19.** D	**20.** D
21. A	**22.** D	**23.** 1.50	**24.** 1.0	**25.** 21	**26.** C	**27.** D	**28.** A	**29.** C	**30.** D
31. C	**32.** A	**33.** C	**34.** D	**35.** B	**36.** A	**37.** C	**38.** C	**39.** C	**40.** A
41. D									

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. Bonus	**2.** A	**3.** D	**4.** A	**5.** B

Multiple Correct Choice Type Questions

6. A, B, C	**7.** A, B, D	**8.** A, D	**9.** A, B, D	**10.** B, D
11. A, B	**12.** B, C	**13.** A, C	**14.** A, B	**15.** A, C, D
16. A, C				

Linked Comprehension Type Questions

17. 57	**18.** 6	**19.** 2	**20.** 1.5	**21.** C	**22.** D	**23.** 182	**24.** A, B, C	**25.** C, D	**26.** A
27. D									

Matrix Match/Column Match Type Questions

28. D

Integer/Numerical Answer Type Questions

29. 0	**30.** 9	**31.** 7	**32.** 4	**33.** 4	**34.** 0.5	**35.** 2	**36.** 1	**37.** 2	**38.** 1
39. 2									

CHAPTER

10 Area Under the Curve

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Curve Tracing and Asymptotes
(b) Area Enclosed by Curve $y = f(x)$ with x-axis
(c) Area Enclosed by Curve $y = f(x)$ with y-axis
(d) Area Between Two Curves
(e) Area Computation for Parametric Curves
(f) Concept based on variable Area

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

CURVE TRACING

Let consider a curve $f(x, y) = 0$. For making a graph of given curve following steps are essential to understand the behaviour of curve.

(i) **Identifying type of symmetry:** By following transformation in curve's equation if curve's equation remains unchanged then curve has mentioned symmetry

Table 10.1 Types of symmetry based on the transformation

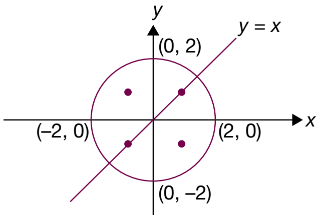
Transformation	Type of Symmetry
Replace x by $-x$	Symmetric about y-axis
Replace y by $-y$	Symmetric about x-axis
Replace x by $-x$ and y by $-y$	Symmetric about origin (opposite quadrant symmetry)
Replace x by y and y by x	Symmetric about line $y = x$

(ii) Find points where curve intersect co-ordinate axes.
Put $y = 0$ in curve's equation to get points where curve cuts x-axis
Put $x = 0$ in curve's equation to get points where curve cuts y-axis

(iii) Analyse curve, using calculus such as continuity, maxima/minima, monotonicity (increasing/decreasing)

EXAMPLE:

Consider circle $x^2 + y^2 = 4$



Replace $x \to -x$, curve's equation remains unchanged, so curve is symmetric about y-axis.

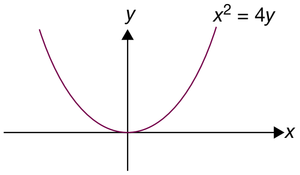
Replace $y \to -y$, curve's equation remains unchanged, so curve is symmetric about x-axis.

Replace $x \to -x$, $y \to -y$, curve's equation remains unchanged, so curve is symmetric about origin

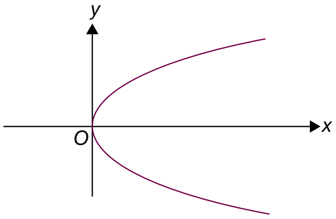
Interchange x by y, curve's equation remains unchanged, so curve is symmetric about line $y = x$

It cuts y-axis when $x = 0$ then $y = +2, -2$

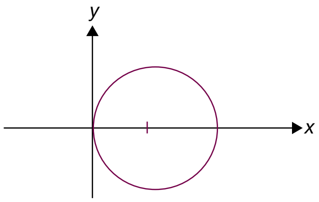
It cuts x-axis when $y = 0$ then $x = +2, -2$



Curve $x^2 = 4y$ is symmetric about y-axis



Curve $y^2 = 4x$ is symmetric about x-axis



Curve $x^2 + y^2 = 2x$ is
symmetric about x-axis

Students are advised to revise graph transformation topic from chapter Function and its Graph.

ILLUSTRATION 1

Identify the types of symmetry and plot following curves

(i) $xy = 4$ **(ii)** $|xy| = 1$ **(iii)** $y = x^2 - 4|x| + 3$

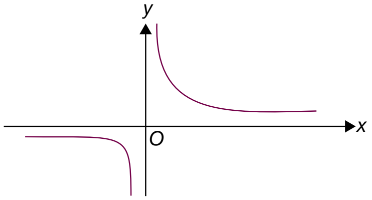
(iv) $|y| = x^2 - 4x + 3$ **(v)** $y = \sqrt{1-|x|}$

SOLUTION

(i) $xy = 4$

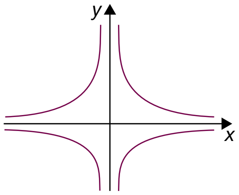
Symmetric about line $x = y$ and origin

It is not symmetric about x-axis and y-axis



(ii) $|xy| = 1$

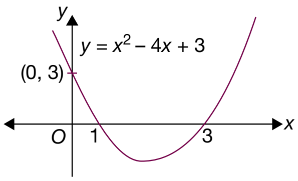
Symmetric about x-axis, y-axis, origin, line $y = x$



(iii) $y = x^2 - 4|x| + 3$

Symmetric about y-axis

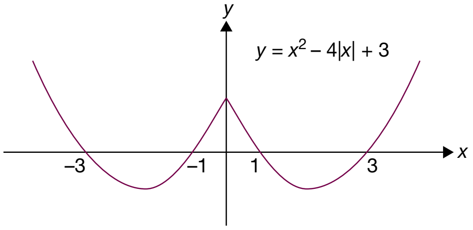
$f(x) = x^2 - 4x + 3 = (x-1)(x-3)$



$y = f(|x|) = x^2 - 4|x| + 3$

It cuts y-axis at point (0, 3)

and x-axis at point (–3, 0) and (3, 0)

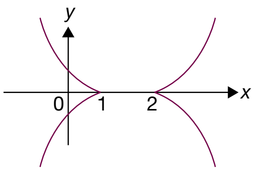


(iv) $|y| = x^2 - 4x + 3$

Symmetric about x-axis

$|y| = (x-1)(x-3)$

It cuts x-axis at point (1, 0) and (3, 0) and y-axis at point (0, 3)

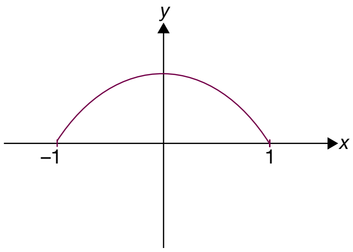


(v) $y = \sqrt{1-|x|}$

By replacing $x \to -x$, we obtain $y = \sqrt{1-|x|}$

Curve is symmetric about y-axis . Therefore,

$1 - |x| \geq 0 \Rightarrow |x| \leq 1 \Rightarrow -1 \leq x \leq 1$, Domain of curve



ASYMPTOTES OF CURVE

A straight line whose distance from a curve tends to zero as it moves towards infinity.

(i) $\lim\limits_{x\to\pm\infty} y = \alpha(\text{finite}) \Rightarrow y = \alpha$

Asymptote is x-axis ($\alpha = 0$) or parallel to x-axis ($\alpha \neq 0$)

(ii) $\lim\limits_{x\to\alpha} y \to \pm\infty \Rightarrow x = \alpha$

Asymptote is y-axis ($\alpha = 0$) or parallel to y-axis ($\alpha \neq 0$)

(iii) oblique asymptote

$\lim\limits_{x\to\pm\infty} \left(\frac{y}{x}\right) = m$ (finite) and $\lim\limits_{x\to\pm\infty} (y - mx) = c$ (finite)

$\therefore$ $y = mx + c$ is an asymptote of given curve

ILLUSTRATION 2

Identify asymptotes if exists and plot following curves

(i) $xy = 1$ **(ii)** $y = x + \frac{1}{x}$

(iii) $y = \frac{3x^2}{x-1}$

SOLUTION

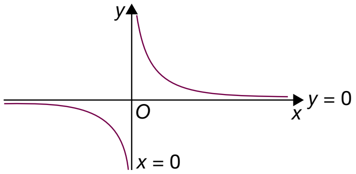
(i) Curve $xy = 1 \Rightarrow y = \dfrac{1}{x}$

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{x} = 0, \text{ we get } y = 0 \text{ as asymptote}$$

$$\lim_{x\to 0} y = \lim_{x\to 0}\left(\frac{1}{x}\right) \text{ (RHL} + \infty\text{), (LHL} - \infty\text{),}$$

We get $x = 0$ as asymptote

Line $y = 0$, $x = 0$ are asymptotes



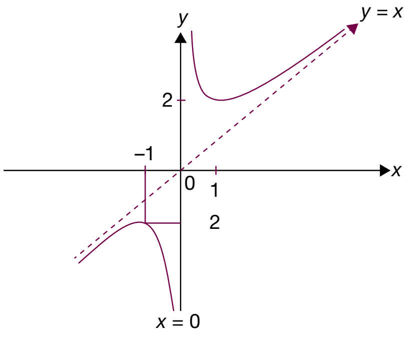
(ii) Curve $y = x + \dfrac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

$f(-1) = -2$, $\qquad f(1) = 2$

$$\left.\begin{aligned} \lim_{x\to 0^+} y &= \lim_{x\to 0^+} x + \frac{1}{x} \to +\infty \\ \lim_{x\to 0^-} y &= \lim_{x\to 0^-} x + \frac{1}{x} \to -\infty \end{aligned}\right] \text{Line } x = 0 \text{ is an asymptote}$$

$$\Rightarrow m = \lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty}\left(\frac{x + \frac{1}{x}}{x}\right) = \lim_{x\to\infty} 1 + \frac{1}{x^2} = 1$$



$$C = \lim_{x\to\infty} (y - mx) = \lim_{x\to\infty}\left(\frac{1}{x}\right) = 0$$

Line $y = x$ is also an asymptote. Therefore, given curve has two asymptotes $x = 0$, $y = x$

(iii) Curve $y = \dfrac{x+3}{x-3}$

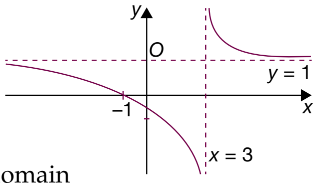
$$\left.\begin{aligned} \lim_{x\to 3^+} y &= \lim_{x\to 3^+} \frac{x+3}{x-3} \to +\infty \\ \lim_{x\to 3^-} y &= \lim_{x\to 3^-} \frac{x+3}{x-3} \to -\infty \end{aligned}\right] x = 3 \text{ is an asymptote}$$

$$\Rightarrow \lim_{x\to\infty} y = \lim_{x\to\infty} \frac{x+3}{x-3} = \lim_{x\to\infty} \frac{1+\frac{3}{x}}{1-\frac{3}{x}} = \frac{1+0}{1-0} = 1$$

Lines $x = 3$ and $y = 1$ are asymptotes

$$y = 1 + \frac{6}{x-3}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{6}{(x-3)^2} < 0$$

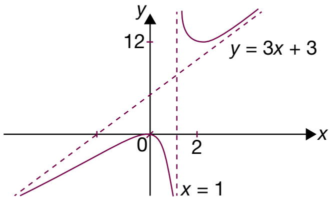


Curve is decreasing in domain

ILLUSTRATION 3

Identify asymptotes if exists and plot curve $y = \dfrac{3x^2}{x-1}$

SOLUTION



Curve $y = \dfrac{3x^2}{x-1}$

As $x \to 1^+$; $y \to \infty$ and as $x \to 1^-$; $y \to -\infty$

Line $x = 1$ asymptote

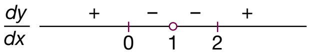
$$m = \lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty} \frac{3x}{x-1} = \lim_{x\to\infty} \frac{3}{1-\frac{1}{x}} = 3$$

$$C = \lim_{x\to\infty} (y - 3x) = \lim_{x\to\infty}\left(\frac{3x^2}{x-1} - 3x\right) = \lim_{x\to\infty} \frac{3x}{x-1} = 3$$

Asymptotes are $y = 3x + 3$ and $x = 1$

Curve $y = \dfrac{3x^2}{x-1}$;

$$\Rightarrow \frac{dy}{dx} = \frac{(x-1)6x - 3x^2(1-0)}{(x-1)^2} = \frac{3x(x-2)}{(x-1)^2}$$



Hence, at $x = 2$; $y = 12$ local minima

ILLUSTRATION 4

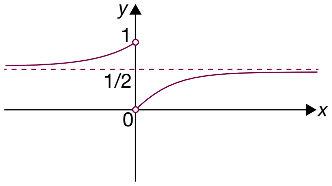
Identify asymptotes if exists and plot curve $y = \dfrac{1}{1+e^{2/x}}$

SOLUTION

Given, $y = \dfrac{1}{1+e^{2/x}}$,

As $x \to 0^+$; $y \to \dfrac{1}{\infty} \to 0$

As $x \to 0^-$; $y \to \dfrac{1}{1+0} \to 1$



So $x = 0$ is not an asymptote

$$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{1}{1+e^{2/x}} = \frac{1}{1+1} = \frac{1}{2}$$

$$\lim_{x\to-\infty} y = \frac{1}{2}$$

Thus, $y = \frac{1}{2}$ is an asymptote

$$\frac{dy}{dx} = -\frac{1}{(1+e^{2/x})^2}e^{2/x}\left(\frac{-2}{x^2}\right) = \frac{2e^{2/x}}{(1+e^{2/x})^2}\frac{1}{x^2} > 0$$

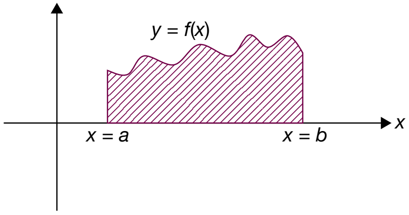
Hence, curve is increasing in domain.

AREA ENCLOSED BY CURVE $y = f(x)$ WITH X-AXIS

$\int_a^b f(x)\,dx =$ Algebraic sum of area enclosed by curve $y = f(x)$ and x-axis from ordinates $x = a$ to $x = b$

Area enclosed by curve $y = f(x)$ and x-axis from ordinates $x = a$ to $x = b$ is given by $= \int_a^b |f(x)|\,dx$

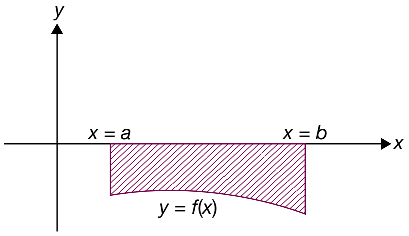
(i)



For $a < x < b$, $f(x)$ is above the x-axis , It means $f(x) > 0$,

then required shaded area $= \int_a^b f(x)\,dx$

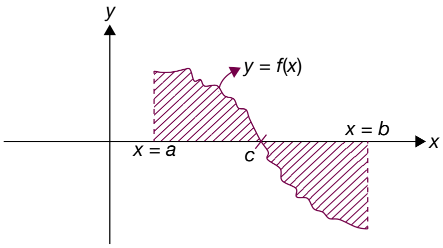
(ii)



For $a < x < b$, $f(x)$ is below the x-axis , It means $f(x) < 0$,

then required shaded area $= -\int_a^b f(x)\,dx$

(iii)



For $a < x < b$, $f(x)$ crosses the x-axis,

For $a < x < c$, $f(x)$ is above the x-axis, which means $f(x) > 0$,

For $c < x < b$, $f(x)$ is below the x-axis, which means $f(x) < 0$,

Then required shaded area $= \int_a^c f(x)\,dx - \int_c^b f(x)\,dx$

ILLUSTRATION 5

Find area enclosed by curve $y = x^2 - 3x + 2$ and x-axis from ordinates

(i) $x = 1$ to $x = 2$ **(ii)** $x = 0$ to $x = 2$

SOLUTION

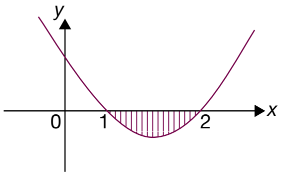
(i) $y = x^2 - 3x + 2 = (x - 1)(x - 2)$

When $1 < x < 2$, $f(x)$ lies below x-axis

So, required shaded area

$$= -\int_1^2 (x^2 - 3x + 2)\,dx$$

$$= -\left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2 = \frac{1}{6} \text{ sq. units}$$



(ii) When $0 < x < 1$, $f(x)$ lies above x-axis and for $1 < x < 2$, $f(x)$ lies below x-axis

So, required shaded area

$$= \int_0^1 x^2 - 3x + 2\,dx - \int_1^2 (x^2 - 3x + 2)\,dx$$

$$= \frac{x^3}{3} - \frac{3x^2}{2} + 2x\Big|_0^1 + \frac{1}{6}$$

$$= \frac{1}{3} - \frac{3}{2} + 2 + \frac{1}{6}$$

$= 1$ sq. units

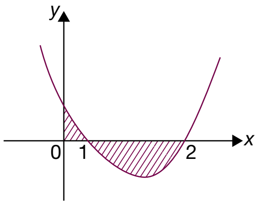
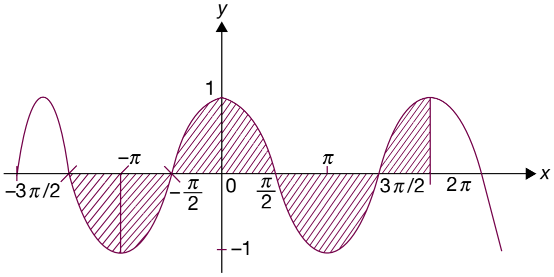


ILLUSTRATION 6

Find area bounded by curve $y = \cos x$ and x-axis from $x = -\frac{3\pi}{2}$ to $x = 2\pi$

SOLUTION

Area bounded by curve $y = \cos x$ and x-axis from $x = -\frac{3\pi}{2}$ to $x = 2\pi$ is shaded area in figure.



Required Shaded Area $= \int_{-3\pi/2}^{2\pi} |\cos x|\,dx = 7\int_0^{\pi/2} \cos x\,dx$

$= 7(\sin x)_0^{\pi/2} = 7$ sq. units

ILLUSTRATION 7

Find area bounded by curve $y = -3x^2 - |x| + 2$ and $y = 0$.

SOLUTION

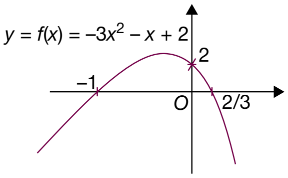
$y = -3x^2 - x + 2$

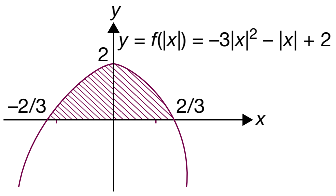
$\Rightarrow \quad y = -(3x - 2)(x + 1) = f(x)$

It is shown as diagram given curve is

$y = -3|x|^2 - |x| + 2 = f(|x|) \qquad (\because x^2 = |x|^2)$

It can be obtained, by removing left side part of $f(x)$ and by taking image of right side of $f(x)$ about y-axis





Now required Area is same as shaded region shown in figure $= 2\int_0^{2/3} (-3x^2 - x + 2)\,dx$

$= 2\left(-x^3 - \frac{x^2}{2} + 2x\right)_0^{2/3} = \frac{44}{27}$ sq. units

ILLUSTRATION 8

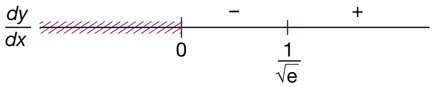
Find area bounded by x-axis and curve $y = x^2 \ln x$

SOLUTION

$y = x^2 \ln x \qquad$ (domain $x > 0$)

$\Rightarrow \quad \frac{dy}{dx} = 2x \ln x + x^2\left(\frac{1}{x}\right) = x(2 \ln x + 1)$

$\Rightarrow \quad \frac{dy}{dx} = 0 \Rightarrow \ln x = -\frac{1}{2} \Rightarrow x = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

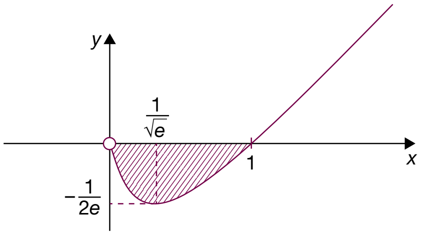


$\lim_{x \to 0^+} x^2 \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x^2}} \qquad \left(\infty/\infty\right)$ form

$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x \to 0^+} \left(-\frac{x^2}{2}\right) = 0$

$f\left(\frac{1}{\sqrt{e}}\right) = \left(\frac{1}{\sqrt{e}}\right)^2 \ln\left(\frac{1}{\sqrt{e}}\right) = -\frac{1}{2e}$

Graph of curve $y = x^2 \ln x$ is as shown in figure



Now required area is same as shaded area in figure

$= -\int_0^1 x^2 \ln x\,dx = -\int_0^1 \underset{\overset{\downarrow}{\text{II}}}{x^2}\, \underset{\overset{\downarrow}{\text{I}}}{\ln x}\,dx$

$= -\left[\frac{x^3}{3} \ln x\Big|_0^1 - \int_0^1 \frac{x^3}{3} \cdot \frac{1}{x}\,dx\right]$

$= -\left[0 - \lim_{x \to 0^+} \frac{x^3}{3} \ln x - \frac{x^3}{9}\Big|_0^1\right] = \frac{1}{9}$ sq. units

ILLUSTRATION 9

Find the area bounded by the curve $y = xe^{-x}$; $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point

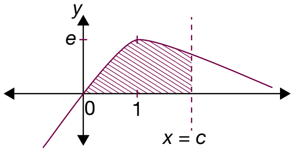
SOLUTION

$y = xe^{-x}$

$\frac{dy}{dx} = e^{-x} + x(-e^{-x}) = e^{-x}(1 - x)$

$\frac{d^2y}{dx^2} = e^{-x}(-1) + (-e^{-x})(1 - x) = e^{-x}(x - 2)$

Point of inflection $x = c = 2$



Required area $= \int_0^2 xe^{-x}\,dx$

$= x(-e^{-x})\Big|_0^2 - \int_0^2 1(-e^{-x})\,dx$

$$= 2(-e^{-2}) - 0 + (-e^{-x})\Big|_0^2$$

$$= -2e^{-2} + (-e^{-2} + 1) = 1 - 3e^{-2} \text{ sq. units}$$

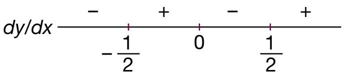
ILLUSTRATION 10

Find the area between the curve $y = 2x^4 - x^2$, the x-axis and the ordinates of two minima of the curve.

SOLUTION

$$y = f(x) = 2x^4 - x^2 = x^2(2x^2 - 1) = x^2(\sqrt{2}\,x - 1)(\sqrt{2}\,x + 1)$$

$$\frac{dy}{dx} = 8x^3 - 2x = 2x(4x^2 - 1) = 2x(2x - 1)(2x + 1)$$



Here, $f(0) = 0;\ f\left(\frac{1}{2}\right) = -\frac{1}{8} = f\left(-\frac{1}{2}\right)$

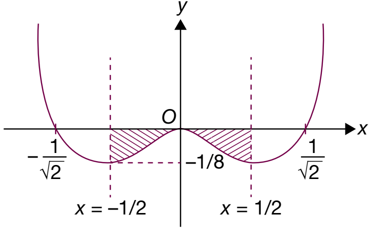
Thus, $x = \frac{1}{2}$ and $-\frac{1}{2}$ are point of minima

$$\text{Required area} = \int_{-1/2}^{1/2} -(2x^4 - x^2)\,dx$$

$$= 2\int_0^{1/2} x^2 - 2x^4\,dx$$

$$= 2\left(\frac{x^3}{3} - \frac{2x^5}{5}\right)_0^{1/2}$$

$$= 2\left(\frac{1}{24} - \frac{2}{32(5)}\right) = \frac{1}{12} - \frac{1}{40} = \frac{7}{120} \text{ sq. units}$$



Test Your Concepts-I

(Solutions on page H.343)

1. The area bounded by $y = x^3$; x-axis and $x = -2$ and $x = 1$ is
2. Find the area of the figure bounded by $y = -3x^2 - |x| + 2$ and $y = 0$.
3. Find the area bounded by the curve $y = x\,e^{-x}$; $xy = 0$ and $x = c$ where C is the x-coordinate of the curve's inflection point
4. The area bounded by the curve $y = f(x)$ (where $f(x) \geq 0$), the co-ordinate axes and the line $x = x_1$ is given by $x_1 . e^{x_1}$. Then compute $f(x)$
5. Show that the area bounded by the curve $y = \frac{\ln x - c}{x}$, the x-axis and the vertical line through the maximum point of the curve is independent of the constant c.
6. If the area of the loop of the curve, $ay^2 = x^2(a - x)$ is $\frac{ka^2}{15}$ where $a > 0$ then find k
7. Find area enclosed by the curve $y = (x^2 + 2x)e^{-x}$ and the positive x-axis.
8. Find the area bounded by $y = \frac{|x|}{x}$ $x \neq 0$, and the lines $y(x - 1)(x - 3) = 0$
9. Find the area bounded by the curve $|x| = \cos^{-1} y$ and the line $|x| = 1$ and the x-axis
10. Find area bounded by the curve
$$f(x) = \begin{cases} \log_e |x| & ; \ |x| \geq \frac{1}{e} \\ |x| - 1 - \frac{1}{e} & ; \ |x| < \frac{1}{e} \end{cases} \text{ and } x\text{-axis}$$
11. If area bounded by non-negative curve $y = f(x)$ and x-axis, $x = 1$ to $x = b$ is $(b - 1)\sin(3b + 4)$, then find $f(x)$

Answer key

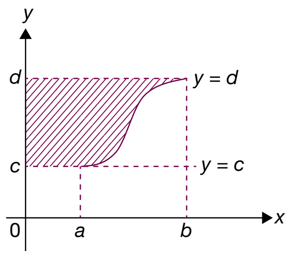
1. $\frac{17}{4}$ sq. units **2.** $\frac{44}{27}$ sq. units **3.** $1 - 3e^{-2}$ sq units **4.** $xe^x + e^x$ **6.** 8 **7.** 4 sq. units **8.** 2 sq. units

9. 2 sin1 sq units **10.** $\frac{1}{e^2} + 2 - \frac{2}{e}$ sq. units **11.** $3(x - 1)\cos(3x + 4) + \sin(3x + 4)$

AREA ENCLOSED BY CURVE *y* = *f*(*x*) WITH *Y*-AXIS

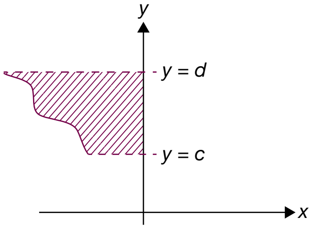
Area enclosed by curve $y = f(x)$ and y-axis from $y = c$ to $y = d$ is $= \int_c^d |g(y)|\,dy$ $\quad [\because\ y = f(x) \Rightarrow x = g(y)]$

(i) For $c < y < d$, curve $y = f(x)$ is right side of y-axis then required area equals to shaded region in given figure $= \int_c^d g(y)dy$



(ii) For $c < y < d$, curve $y = f(x)$ is left side of y-axis then required area equals to shaded region in given figure

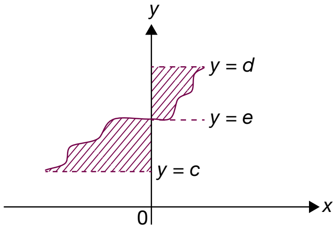
$$= -\int_c^d g(y)dy$$



(iii) For $c < y < d$, curve $y = f(x)$ crosses y-axis.

When $c < y < e$, curve $y = f(x)$ is left side of y-axis

When $e < y < d$, curve $y = f(x)$ is right side of $f(x)$

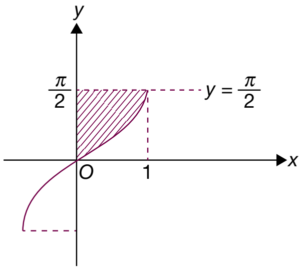


Now required area equals to shaded region in given figure $= -\int_c^e g(y)dy + \int_e^d g(y)dy$

ILLUSTRATION 11

Find area bounded by $y = \sin^{-1}x$ and y-axis from $y = 0$ to $y = \frac{\pi}{2}$.

SOLUTION



Curve $y = \sin^{-1}x$

$\Rightarrow \quad x = \sin y$

Required area bounded by curve and y-axis from $y = 0$ to $y = \frac{\pi}{2}$.

$$= \int_0^{\pi/2} \sin y\, dy = -\cos y\Big|_0^{\pi/2} = 1 \text{ sq. units}$$

Alternate:

Required shaded area = area of rectangle – area of curve enclosed with x-axis

$$= 1.\frac{\pi}{2} - \int_0^1 \sin^{-1} x\, dx = \frac{\pi}{2} - \left(x\sin^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx \right)$$

$$= \frac{\pi}{2} - \left[1\cdot\frac{\pi}{2} - 0 - \int_0^1 \frac{x}{\sqrt{1-x^2}}dx\right] = \int_0^1 \frac{xdx}{\sqrt{1-x^2}}$$

Put $\sqrt{1-x^2} = t \Rightarrow 1 - x^2 = t^2 \Rightarrow -x\, dx = t\, dt$

$$= \int_1^0 \frac{-t\, dt}{t} = \int_0^1 dt = 1 \text{ sq. units}$$

ILLUSTRATION 12

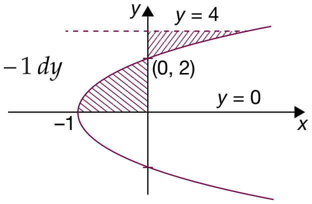
Find area between $y^2 = 4(x+1)$ and y-axis from $y = 0$ to $y = 4$.

SOLUTION

Curve $y^2 = 4(x+1) \quad \Rightarrow \quad x = \frac{y^2}{4} - 1 = g(y)$

Curve is symmetric about x-axis

Required shaded area



$$= -\int_0^2 \left(\frac{y^2}{4} - 1\right) dy + \int_2^4 \frac{y^2}{4} - 1\, dy$$

$$= y - \frac{y^3}{12}\Big|_0^2 + \frac{y^3}{12} - y\Big|_2^4$$

$$= 2 - \frac{8}{12} + \frac{64}{12} - 4 - \frac{8}{12} + 2 = 4 \text{ sq units}$$

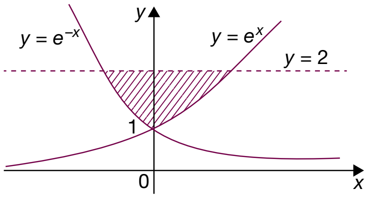
ILLUSTRATION 13

Find area bounded by curves $y = e^x$, $y = e^{-x}$ and $y = 2$.

SOLUTION

$y = e^x \Rightarrow x = \ln y$

Required shaded area



$$= 2\int_1^2 \ln y\, dy$$

$$= 2\left[y \ln y\Big|_1^2 - \int_1^2 y\left(\frac{1}{y}\right)dy \right]$$

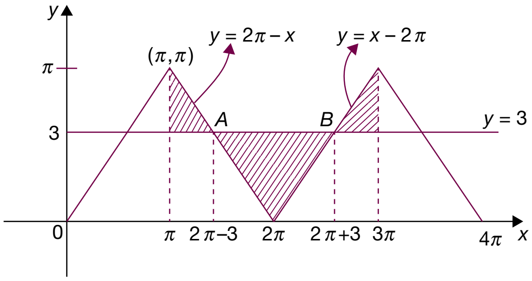
$$= 2\left[2 \ln 2 - 0 - y\Big|_1^2\right] = 2 \ln\left(\frac{4}{e}\right) \text{ sq. units}$$

ILLUSTRATION 14

Find area bounded by curve $y = \cos^{-1}(\cos x)$ and line $y = 3$ from ordinates $x = \pi$ to $x = 3\pi$.

SOLUTION

Curve $y = \cos^{-1}(\cos x)$ is drawn as shown



Curve intersects line $y = 3$ at points $A(2\pi - 3, 3)$ and $B(2\pi + 3, 3)$

Required shaded area

$$= \frac{1}{2}(\pi - 3)(\pi - 3) + \frac{1}{2}(6)\,3 + \frac{1}{2}(\pi - 3)(\pi - 3)$$

$$= (\pi - 3)^2 + 9 \text{ sq units}$$

ILLUSTRATION 15

Find area bounded by curves $x = y^2 - y^3$ and y-axis

SOLUTION

Curve $x = y^2 - y^3 = y^2(1 - y)$ cuts y-axis at $y = 0$ and 1

Required shaded area

$$= \int_0^1 y^2 - y^3\, dy$$

$$= \left.\frac{y^3}{3} - \frac{y^4}{4}\right|_0^1 = \frac{1}{12} \text{ sq. units}$$

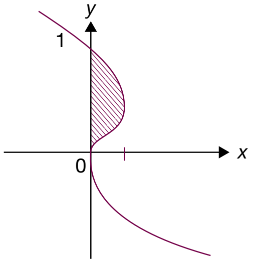


ILLUSTRATION 16

Plot the following region.

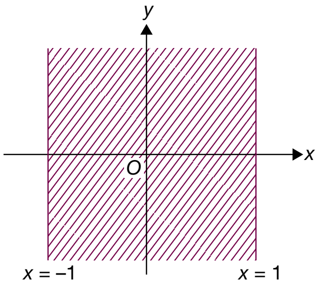
(i) $|x| \le 1$ **(ii)** $|x| \le 1$ and $|y| \le 1$

SOLUTION

(i) $|x| \le 1 \Rightarrow -1 \le x \le 1$

Region enclosed in between ordinates $x = -1$ and $x = 1$

As we can observe from graph that area enclosed by this region is infinite

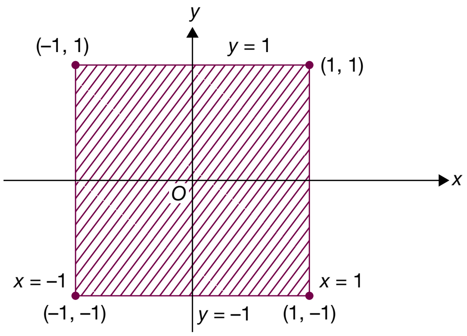


(ii) $|x| \le 1$ and $|y| \le 1$

$-1 \le x \le 1$ and $-1 \le y \le 1$

Region enclosed in between straight lines $x = -1$ and $x = 1$ and $y = 1, y = -1$.

It is a square shown in figure



Required shaded area = 4 sq. units

ILLUSTRATION 17

Find area enclosed by region $|x| + |y| \le 1$

SOLUTION

Curve $|x| + |y| = 1$

It is symmetric about x-axis, y-axis and origin

Required area enclosed by region is shown as shaded area

$$= 4\left(\frac{1}{2} \times 1 \times 1\right) = 2 \text{ sq. units}$$

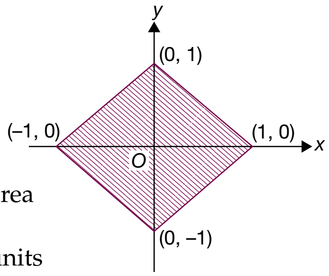


ILLUSTRATION 18

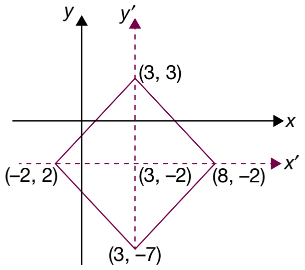
Find area enclosed by region $|x - 3| + |y + 2| \le 5$

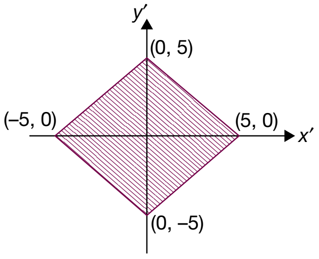
SOLUTION

Let $x' = x - 3$ and $y' = y + 2$

$\Rightarrow \quad |x'| + |y'| \le 5$

We have made shifting of origin to point (3, –2) as area is independent of origin.





Required shaded area $= 4\left(\frac{1}{2} \times 5 \times 5\right) = 50$ sq. units

ILLUSTRATION 19

Find area bounded by region $1 \le |x + 2| + |y - 1| \le 2$

SOLUTION

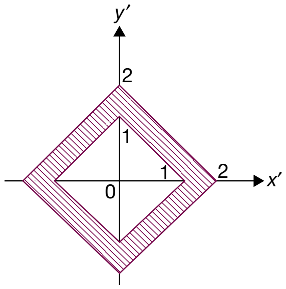
Given region

$1 \le |x + 2| + |y - 1| \le 2$

Let $x + 2 = x'$ and $y - 1 = y'$

$1 \le |x'| + |y'| \le 2$

Required area is area enclosed between curves $|x'| + |y'| = 1$ and $|x'| + |y'| = 2$



which is shown as shaded region in figure.

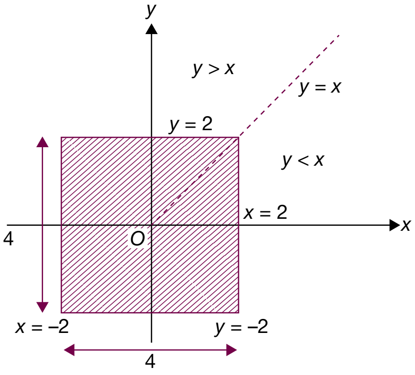
$$\text{Required area} = 4\left[\frac{1}{2}2(2) - \frac{1}{2}1(1)\right] = 4\left(\frac{3}{2}\right) = 6 \text{ sq. units}$$

ILLUSTRATION 20

Find area bounded by region $|x+y| + |x-y| \le 4$

SOLUTION

As replacing $y \to -y$, inequality remains unchanged, so it is symmetric about x-axis



As replacing $x \to -x$, inequality remains unchanged, so it is symmetric about y-axis.

As replacing $x \to -x$, $y \to -y$, inequality remains same, so it is symmetric about origin.

We can conclude that given region is symmetric about x-axis, y-axis, origin and line $y = x$

Lets discuss curve in Ist quadrant $x > 0$; $y > 0$

$$x + y + |x - y| \le 4$$

If $x \ge y$	**If $x < y$**
$x + y + x - y \le 4$	$x + y - x + y \le 4$
$\Rightarrow x \le 2$	$\Rightarrow y \le 2$

Required area = area of square as shown in diagram

$$= 4 \times 4 = 16 \text{ sq. units}$$

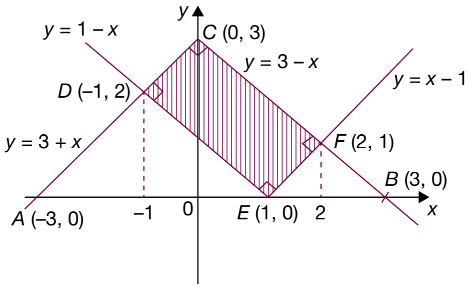
ILLUSTRATION 21

Find area bounded by curves $y = |x - 1|$ and $y = 3 - |x|$

SOLUTION

$$y = |x - 1| \text{ and } y = 3 - |x|$$

Point of intersection of curves (1, 0), (0, 3), (–1, 2), (2, 1)



Required shaded area = area of ΔABC – area of ΔADE – Area of ΔBEF

$$= \frac{1}{2}(6)(3) - \frac{1}{2}(4)\,2 - \frac{1}{2}\;2 \times 1 = 4 \text{ sq. units}$$

ILLUSTRATION 22

Find area bounded by curve $[x] + [y] = 4$ in Ist quadrant where [.] is GIF.

SOLUTION

If $0 \le x < 1 : [x] = 0$;
$[y] = 4 \Rightarrow 4 \le y < 5$

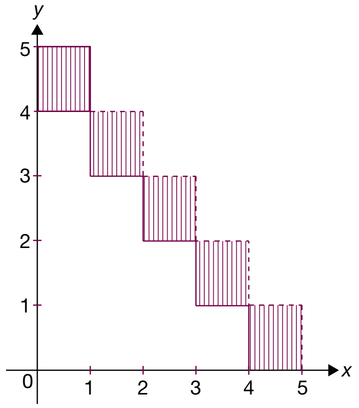
If $1 \le x < 2 : [x] = 1$;
$[y] = 3 \Rightarrow 3 \le y < 4$

If $2 \le x < 3 : [x] = 2$;
$[y] = 2 \Rightarrow 2 \le y < 3$

If $3 \le x < 4 : [x] = 3$;
$[y] = 1 \Rightarrow 1 \le y < 2$

If $4 \le x < 5 : [x] = 4$;
$[y] = 0 \Rightarrow 0 \le y < 1$

Required bounded area is shown as shaded area = 5 sq. units.



Test Your Concepts-II

(Solutions on page H.344)

1. Find the area enclosed by the solution set of $[x].\,[y] = 2$. Where [.] represent greatest integer function of x.
2. Area bounded by the straight lines $x^2y - y^3 - x^2 + 5y^2 - 8y + 4 = 0$ is equal to
3. The area bounded by the curves $x^2 + y^2 = 1$ and $|x| + y^2 = 1$ is $\pi - k$. Then find integral part of k.
4. If the area bounded by $y \le 3 - |3 - x|$ and $y \ge |x - 3|$ is A then find $2A$
5. Find the area of the region bounded by the curves $2x + y = 4$, $3x - 2y = 6$ and $x - 3y + 5 = 0$.
6. Find the area of quadrilateral, combined equation of whose sides are $(x^2 - y^2)(x^2 - y^2 - 8x + 16) = 0$
7. Calculate the area bounded by the curve $x = |y - 1|$ and $y = x - 5$
8. Let $f(x, y) = \cos^{-1}(3|x| + 4|y|)$, if area enclosed by the points (x, y) such that $f(x, y)$ is defined, is μ, then 6μ is

9. The area bounded by parabola $y^2 = 2x$ and line $y = 4x - 1$ is $\frac{k}{32}$ square units then find k

10. Let $f(x) = \begin{cases} -2 & -3 \le x \le 0 \\ x - 2 & 0 < x \le 3 \end{cases}$ where $g(x) = \min\{f|x| + |f(x)|, f|x| - |f(x)|\}$, then find area bounded by curve $g(x)$, x-axis, $x = 3$ and $x = -3$

11. Let the area bounded by non-positive curve $y = f(x)$, x-axis and $x = 0$ and $x = \beta$ is $(\beta \sin\beta + 2)$ (where β is variable and positive), then find $\left|f\left(\frac{\pi}{2}\right)\right|$

12. Find area of the region satisfying $|y| \le |x| \le 4$

13. Let $f(x)$ be a continuous function such that $f(x) \ge 0$ and the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = 0$ and $x = a$ is $\left(\frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a\right)$ sq. units. Then find $f(\pi) + f(-\pi) + f\left(\frac{\pi}{2}\right)$

Answer key

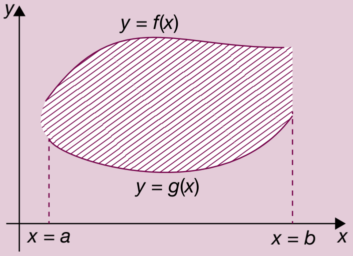
1. 4 sq. units
2. 1 sq. units
3. 2
4. 9
5. $\frac{7}{2}$ Sq. units
6. 8 sq units
7. $\frac{109}{6}$ sq. units
8. 1
9. 9
10. $\frac{23}{2}$ sq. units
11. 1
12. 32 sq. units
13. $\frac{1}{2}$

AREA BETWEEN TWO CURVES

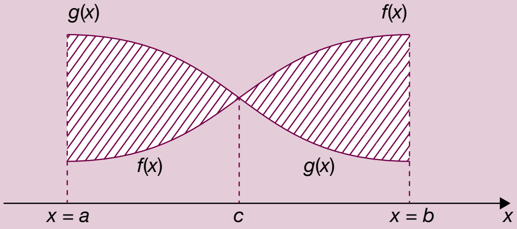
Area enclosed by curves $y = f(x)$ and $y = g(x)$ from ordinates $x = a$ to $x = b$ is $= \int_a^b |f(x) - g(x)|\, dx$

(i) Curve $y = f(x)$ is always above the other curve $y = g(x)$ for $a < x < b$

Required area is shaded area $= \int_a^b f(x) - g(x) dx$



(ii) Curve $y = f(x)$ and curve $y = g(x)$ intersect at $x = c$ as shown in figure



Required shaded area

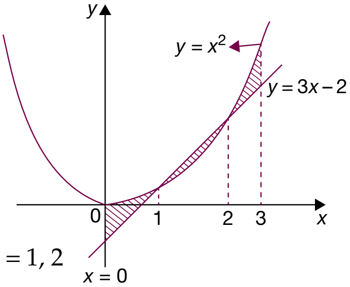
$$= \int_a^c (g(x) - f(x))\, dx + \int_c^b f(x) - g(x)\, dx$$

ILLUSTRATION 23

Find area between curves $y = x^2$ and $y = 3x - 2$ from lines $x = 0$ to $x = 3$.

SOLUTION

Lets first get point of intersections of given curves $y = x^2$ and $y = 3x - 2$



$$y = x^2 = 3x - 2$$

$$\Rightarrow \quad x^2 - 3x + 2 = 0$$

$$\Rightarrow \quad (x-1)(x-2) = 0 \Rightarrow x = 1, 2$$

Required shaded area

$$= \int_0^1 x^2 - (3x - 2)\, dx + \int_1^2 3x - 2 - x^2\, dx + \int_2^3 x^2 - 3x + 2\, dx$$

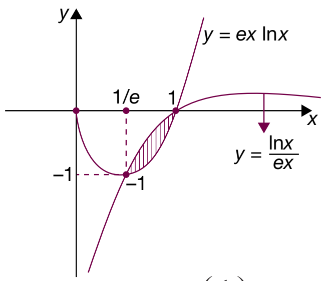
$$= \frac{x^3}{3} - \frac{3x^2}{2} + 2x\Big|_0^1 + \frac{3x^2}{2} - 2x - \frac{x^3}{3}\Bigg|_1^2 + \frac{x^3}{3} - \frac{3x^2}{2} + 2x\Big|_2^3$$

$$= \frac{11}{6} \text{ sq. units}$$

ILLUSTRATION 24

Compute the area of the region bounded by the curves $y = ex\ln x$ and $y = \dfrac{\ln x}{ex}$

SOLUTION



$y = f(x) = ex.\ln x$ and

$y = g(x) = \dfrac{\ln x}{ex}$

$f(x) = g(x)$

$\Rightarrow \quad ex\ln x - \dfrac{\ln x}{ex} = 0$

$\Rightarrow \quad x = 1, \dfrac{1}{e}$

For $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} ex\ln x = \lim\limits_{x\to 0} e\dfrac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0}\dfrac{(e)\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = 0$

For $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0}\dfrac{\ln x}{ex} \to -\infty$

$\therefore \quad$ Required area $= \displaystyle\int_{1/e}^{1}\left(\frac{\ln x}{ex} - ex\ln x\right)\cdot dx$

$$= \int_{1/e}^{1}\frac{\ln x}{ex}dx - \int_{1/e}^{1} ex.\ln x\,dx$$

$$= \left[\frac{(\ln x)^2}{2e}\right]_{1/e}^{1} - e\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_{1/e}^{1}$$

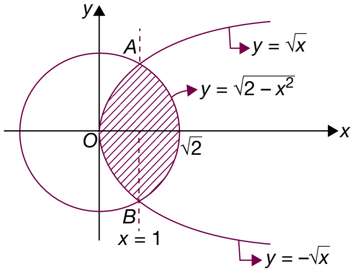
$$= \frac{1}{2e}[0-1] - e\left[\left(0-\frac{1}{4}\right) - \left(\frac{-1}{2e^2} - \frac{1}{4e^2}\right)\right]$$

$$= \frac{e}{4} - \frac{5}{4e} \text{ sq. units}$$

ILLUSTRATION 25

Find area enclosed by circle $x^2 + y^2 = 2$ and parabola $y^2 = x$.

SOLUTION



Lets first get point of intersections of given curves, circle $x^2 + y^2 = 2$ and parabola $y^2 = x$

$x^2 + x = 2$

$\Rightarrow \quad x^2 + x - 2 = 0$

$\Rightarrow \quad (x+2)(x-1) = 0$

when $x = -2$, $y^2 = x = -2$ not possible

When $x = 1$, $y^2 = x = 1$ or $y = \pm 1$

Point of intersection $A(1, 1)$ and $B(1, -1)$

Required shaded area $= 2\left(\displaystyle\int_0^1 \sqrt{x}dx + \int_1^{\sqrt{2}}\sqrt{2-x^2}dx\right)$

$$= 2\left[\frac{x^{3/2}}{\frac{3}{2}}\Bigg|_0^1 + \frac{x}{2}\sqrt{2-x^2} + \frac{2}{2}\sin^{-1}\frac{x}{\sqrt{2}}\Bigg|_1^{\sqrt{2}}\right]$$

$$= 2\left[\frac{2}{3} + \left(0 + \frac{\pi}{2}\right) - \frac{1}{2} - \frac{\pi}{4}\right]$$

$$= 2\left[\frac{1}{6} + \frac{\pi}{4}\right] = \frac{1}{3} + \frac{\pi}{2} \text{ sq. units}$$

ILLUSTRATION 26

Find area enclosed by curve $y = x^2 + x + 1$ and its tangent drawn at point (1, 3) from $x = -1$ to $x = 1$.

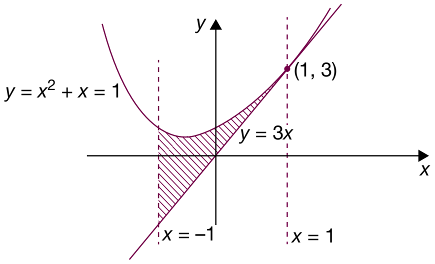
SOLUTION

Curve $y = x^2 + x + 1$

Tangent at point (1, 3)

Slope of tangent $= m_T = \dfrac{dy}{dx} = 2x + 1\big|_{x=1} = 3$

Equation of Tangent $y - 3 = 3(x - 1) \Rightarrow y = 3x$



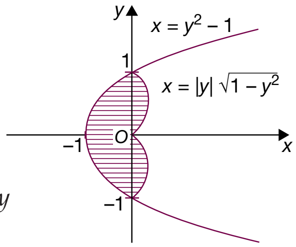
Required shaded area $= \displaystyle\int_{-1}^{1} x^2 + x + 1 - (3x)\,dx = \int_{-1}^{1}(x-1)^2\,dx$

$$= \frac{(x-1)^3}{3}\Bigg|_{-1}^{1} = 0 + \frac{8}{3} = \frac{8}{3} \text{ sq. units}$$

ILLUSTRATION 27

Find area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1-y^2}$

SOLUTION



Given curves $x = y^2 - 1$

and $x = |y|\sqrt{1-y^2}$

Required area

$$= 2\int_0^1 \left(y\sqrt{1-y^2} - (y^2 - 1)\right)dy$$

$$=2\left[\int_0^1 y\sqrt{1-y^2}\,dy\right]-2\left[\frac{y^3}{3}-y\right]_0^1$$

$$=2\left[\int_0^1 y\sqrt{1-y^2}\,dy\right]-2\left(\frac{1}{3}-1\right)=\frac{4}{3}+\left[\int_0^1\sqrt{1-t}\,dt\right]$$

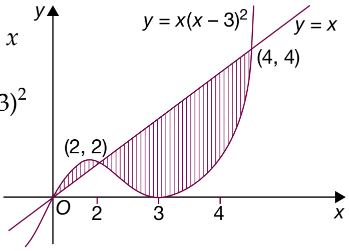
Put $y^2=t \Rightarrow 2y\,dy=dt$

$$=\frac{4}{3}+\left[-\frac{(1-t)^{3/2}}{\frac{3}{2}}\right]_0^1=\frac{4}{3}-\frac{2}{3}(0-1)=2 \text{ sq. units}$$

ILLUSTRATION 28

Find the area bounded by the curves $y=x(x-3)^2$ and $y=x$ is (in sq. units)

SOLUTION



Curve $y=x(x-3)^2$ and $y=x$

$$\frac{dy}{dx}=x^2(x-3)+(x-3)^2$$

$$=3(x-3)(x-1)$$

Solving given curves

$$y=x(x-3)^2=x$$

So, $x=0, 2, 4$ are point of intersection

$$\text{Area}=\int_0^2 x(x-3)^2-x\,dx+\int_2^4 x-x(x-3)^2\,dx$$

$$=\int_0^2 x^3-6x^2+9x-x\,dx+\int_2^4 -x^3+6x^2-8x\,dx$$

$$=\left[\frac{x^4}{4}-2x^3+4x^2\right]_0^2+\left[-\frac{x^4}{4}+2x^3-4x^2\right]_2^4$$

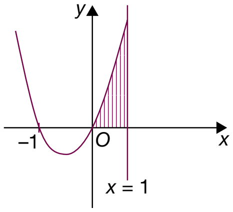
$$=(4-16+16)+(-64+128-64)-(-4+16-16)$$

$$=4+4=8 \text{ sq. units}$$

ILLUSTRATION 29

The slope of the tangent to a curve $y=f(x)$ at $(x, f(x))$ is $2x+1$. If the curve passes through the point $(1, 2)$ then find the area of the region bounded by the curve, the x-axis and the line $x=1$ in first quadrant

SOLUTION



$$\text{Slope of tangent}=\frac{dy}{dx}=2x+1$$

$$\int dy=\int(2x+1)\,dx$$

$$y=x^2+x+C$$

Curve passes though point (1, 2)

$$2=1+1+C \Rightarrow C=0$$

$\therefore$ Curve $y=x^2+x$

$$\text{Area}=\int_0^1(x^2+x)\,dx=\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1=\frac{5}{6} \text{ sq. units}$$

ILLUSTRATION 30

The line $3x+2y=13$ divides the area enclosed by the curve $9x^2+4y^2-18x-16y-11=0$ into two parts. Find the ratio of the larger area to the smaller area.

SOLUTION

Curve $4y^2-16y+9x^2-18x-11=0$

$$\Rightarrow \frac{(x-1)^2}{4}+\frac{(y-2)^2}{9}=1 \text{ is an ellipse}$$

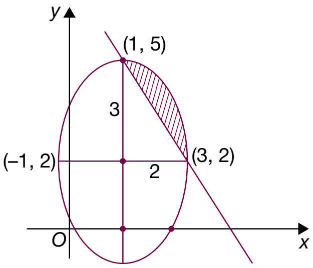
where $a^2=4$; $b^2=9$

$\therefore$ Total area $=\pi ab=6\pi$

$$\text{Ellipse } \frac{(x-1)^2}{4}+\frac{(y-2)^2}{9}=1$$

Line $3x+2y=13$

$\therefore$ Point of intersection (1, 5), (3, 2)



Required shaded area

$$=\frac{\pi(6)}{4}-\left(\frac{1}{2}\right)(3)(2)$$

$$=\frac{3\pi}{2}-3=3\frac{(\pi-2)}{2} \text{ sq. units}$$

$$\text{Remaining area}=6\pi-\frac{3\pi}{2}+3=\frac{9\pi}{2}+3$$

$$=(3\pi+2)\frac{3}{2} \text{ sq. units}$$

$$\therefore \text{ Required ratio}=\frac{(3\pi+2)\left(\frac{3}{2}\right)}{(\pi-2)\left(\frac{3}{2}\right)}=\frac{3\pi+2}{\pi-2}$$

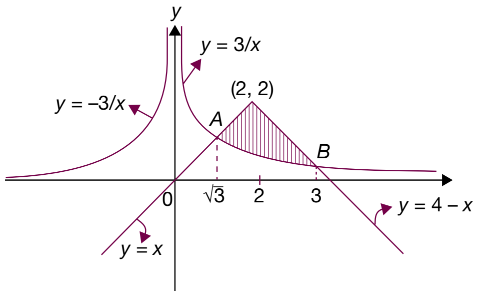
ILLUSTRATION 31

Find area bounded by curves $y=2-|2-x|$ and $y=\dfrac{3}{|x|}$

SOLUTION

$$y=2-|2-x|=\begin{cases}2-(2-x); & x<2\\ 2-(x-2); & x\geq 2\end{cases}$$

$$y=\begin{cases}x\ ; & x<2\\ 4-x; & x\geq 2\end{cases}$$



If $0 < x < 2$; $y = x = \frac{3}{x} \Rightarrow x = \sqrt{3}$ Point A

If $x \ge 2 \Rightarrow y = 4 - x = \frac{3}{x} \Rightarrow x^2 - 4x + 3 = 0$

$\Rightarrow (x - 1)(x - 3) = 0 \Rightarrow x = 1, 2$ acceptable $x = 3$ Point B

Required shaded area $= \int_{\sqrt{3}}^{2} x - \frac{3}{x}\,dx + \int_{2}^{3} 4 - x - \frac{3}{x}\,dx$

$= \frac{4 - 3\ln 3}{2}$ sq. units

ILLUSTRATION 32

Find the area bounded by curves $y = \ln x$ and $y = \ln^2 x$.

SOLUTION

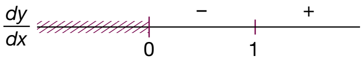
First we should find point of intersection of given curves

$y = \ln x = \ln^2 x \Rightarrow \ln x\,(1 - \ln x) = 0$

$\Rightarrow \ln x = 0$ and $\ln x = 1$

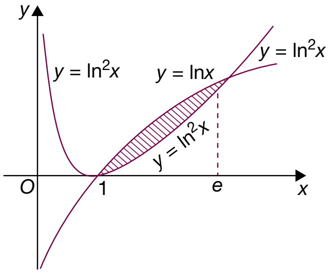
We get $x = 1$; $x = e$ as point of intersection

$y = (\ln x)^2$



$\Rightarrow \frac{dy}{dx} = 2\ln x\left(\frac{1}{x}\right) = \frac{2\ln x}{x}$

Required area



$= \int_1^e \ln x - \ln^2 x\,dx$

$= \int_1^e 1\left(\ln x - \ln^2 x\right)dx$

$= x(\ln x - \ln^2 x)\Big|_1^e - \int_1^e x\left(\frac{1}{x} - \frac{2\ln x}{x}\right)dx$

$= 0 + 0 - \int_1^e (1 - 2\ln x)\,dx = \int_1^e 2\ln x - 1\,dx$

$= 2x\ln x\Big|_1^e - \int_1^e (2x\left(\frac{1}{x}\right)dx - x\Big|_1^e$

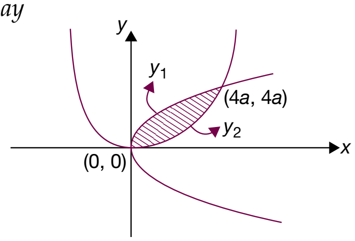
$= 2e - 0 - 2(e - 1) - (e - 1) = 2e - 3e + 3 = 3 - e$ sq. units

ILLUSTRATION 33

Find 'a' for which area bounded by curves $y^2 = 4ax$ and $x^2 = 4ay$ is 1 sq. units where $(a > 0)$

SOLUTION

Curves $y^2 = 4ax$ and $x^2 = 4ay$



Using $y = \frac{x^2}{4a}$ in equation

$y^2 = 4ax$

We get $\frac{x^4}{16a^2} = 4ax$

$\Rightarrow x^3 = 64a^3 \Rightarrow x = 4a$

Both curves meet at points (0, 0) and $(4a, 4a)$

Curve $y_1 = \sqrt{4ax}$ and $y_2 = \frac{x^2}{4a}$ in first quadrant

Required shaded area $= \int_0^{4a} \sqrt{4a}\sqrt{x} - \frac{x^2}{4a}\,dx = 1$

$\Rightarrow \sqrt{4a}\frac{x^{3/2}}{3/2} - \frac{x^3}{12a}\Big|_0^{4a} = 1$

$\Rightarrow \frac{4}{3}\sqrt{a}\,(4a)^{3/2} - \frac{64a^3}{12a} = 1$

$\Rightarrow \frac{32}{3}a^2 - \frac{16}{3}a^2 = 1 \Rightarrow a = \frac{\sqrt{3}}{4}$

ILLUSTRATION 34

Find the area enclosed by the parabola $y = 1 + x^2$ and a normal drawn to its with gradient –1.

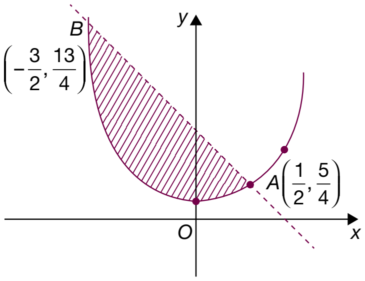
SOLUTION

Curve $y = 1 + x^2 \Rightarrow \frac{dy}{dx} = 2x$

Slope of normal $= m_{\text{normal}} = -\frac{1}{2x} = -1 \Rightarrow x = \frac{1}{2}; y = \frac{5}{4}$

Equation of normal $y - \frac{5}{4} = -x + \frac{1}{2} \Rightarrow x + y = \frac{7}{4}$

Now let get point of intersection of Normal and given parabola



$y = \frac{7}{4} - x = 1 + x^2$

$\Rightarrow x^2 + x - \frac{3}{4}$

$\Rightarrow 4x^2 + 4x - 3 = 0$

$\Rightarrow x = -\frac{3}{2}, x = \frac{1}{2}$

For, $x = \frac{3}{2}, y = 1 + \frac{9}{4} \Rightarrow \frac{13}{4}$

Point of intersection $A\left(\frac{1}{2}, \frac{5}{4}\right)$; $B\left(-\frac{3}{2}, \frac{13}{4}\right)$

$$\therefore \quad \text{Required area} = \int_{-3/2}^{1/2} \left[\left(\frac{7}{4} - x\right) - (1 + x^2)\right] dx$$

$$= \frac{3}{4}(2) - \left(\frac{x^2}{2}\right)_{-3/2}^{1/2} - \left(\frac{x^3}{3}\right)_{-3/2}^{1/2}$$

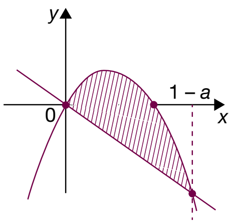
$$= \frac{3}{2} - \frac{1}{2}\left[\frac{1}{4} - \frac{9}{4}\right] - \frac{1}{3}\left[\frac{1}{8} + \frac{27}{8}\right] = \frac{4}{3} \text{ sq. units}$$

ILLUSTRATION 35

Let $f(x) = x - x^2$ and $g(x) = ax$. If the area bounded by $f(x)$ and $g(x)$ is equal to the area bounded by the curves $x = 3y - y^2$ and $x + y = 3$, then find the value of a.

SOLUTION

$$f(x) = x - x^2;\ g(x) = ax \Rightarrow ax = x - x^2 \Rightarrow x = 1 - a,\ x = 0$$



$$A = \int_0^{1-a} (x - x^2) - ax\, dx$$

$$A = \left[\frac{x^2}{2} - \frac{x^3}{3} - \frac{ax^2}{2}\right]_0^{1-a}$$

$$\Rightarrow \quad A = \frac{(1-a)^2}{2} - \frac{(1-a)^3}{3} - \frac{a(1-a)^2}{2}$$

$$= \frac{(1-a)^3}{2} - \frac{(1-a)^3}{3} = \frac{(1-a)^3}{6} \qquad \ldots(1)$$

Also curves $x = 3y - y^2$ and $x + y = 3$

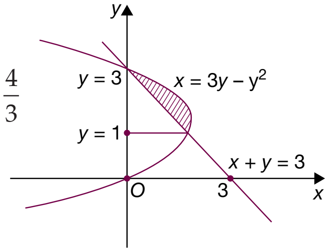
Point of intersection $3 - y = 3y - y^2 \Rightarrow y^2 - 4y + 3 = 0$

$\Rightarrow \quad y = 1, y = 3$

$$\text{Required area} = \int_1^3 (3y - y^2) - (3 - y)\, dy = \int_1^3 (4y - y^2 - 3)\, dy$$

$$= \left[2y^2 - \frac{y^3}{3} - 3y\right]_1^3$$

$$= 2[8] - \frac{1}{3}[26] - 3[2] = \frac{4}{3}$$



As both areas are equal

then $\frac{(1-a)^3}{6} = \frac{4}{3}$

$\Rightarrow \quad (1 - a)^3 = 8 \Rightarrow 1 - a = 2 \Rightarrow a = -1$

ILLUSTRATION 36

Find area bounded by curve $xy^2 = a^2(a - x)$ and y-axis where $a > 0$

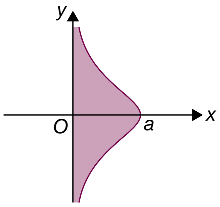
SOLUTION

Given curve $xy^2 = a^2(a - x)$

Replace y by $-y$, it is symmetric about x-axis.

$$\Rightarrow \quad y^2 = \frac{a^2(a-x)}{x} \geq 0 \qquad \text{Domain } x \in (0, a]$$

$$x \to 0^+;\ y = \pm a\sqrt{\frac{a-x}{x}} \to \pm\infty$$



Curve $xy^2 + a^3x = a^3$

$$\text{or} \quad x = \frac{a^3}{a^2 + y^2}$$

$$\text{Required area} = \int_{-\infty}^{\infty} \frac{a^3}{a^2 + y^2}\, dy = \frac{a^3}{a}\tan^{-1}\frac{y}{a}\Bigg|_{-\infty}^{\infty}$$

$$= a^2\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \pi a^2 \text{ sq. units}$$

ILLUSTRATION 37

Find area enclosed by $|x| + |y| \leq 3$ and $xy \geq 2$

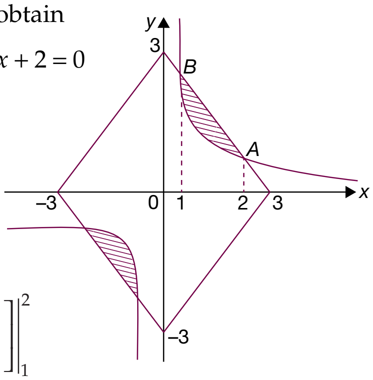
SOLUTION

solving for point of intersection of curves $x + y = 3$ and $xy = 2$, we obtain

$$x + \frac{2}{x} = 3 \Rightarrow x^2 - 3x + 2 = 0$$

$\Rightarrow \quad (x - 1)(x - 2) = 0$

$\Rightarrow \quad x = 1, 2$



$$\text{Area} = 2\int_1^2 3 - x - \frac{2}{x}\, dx$$

$$= 2\left[3x - \frac{x^2}{2} - 2\ln x\right]_1^2$$

$$= 2\left[3(2-1) - \frac{(4-1)}{2} - 2(\ln 2 - \ln 1)\right]$$

$$= 2\left[\frac{3}{2} - 2\ln 2\right] = 3 - 4\ln 2 \text{ sq. units}$$

ILLUSTRATION 38

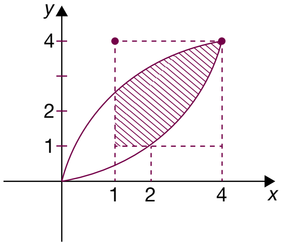
Let $y^2 = 4[\sqrt{y}]\, x$ and $x^2 = 4[\sqrt{x}]\, y$ be two curves, where [ ] represents greatest integer function. Find area bounded by these two curves within the square formed by the lines $x = 1, y = 1, x = 4, y = 4$.

SOLUTION

$$1 < x < 4 \Rightarrow 1 < \sqrt{x} < 2; [\sqrt{x}] = 1$$

$$1 < y < 4 \Rightarrow 1 < \sqrt{y} < 2; [\sqrt{y}] = 1$$

It means we need to calculate area enclosed by $y^2 = 4x$ and $x^2 = 4y$ from $x = 1$ to 4



$$A = \int_1^4 2\sqrt{x}\,dx - 1 - \int_2^4 \frac{x^2}{4}\,dx$$

$$= \frac{4}{3}\left(x^{3/2}\right)_1^4 - 1 - \frac{1}{12}\left(x^3\right)_2^4$$

$$= \frac{28}{3} - 1 - \frac{56}{12} = \frac{11}{3} \text{ sq. units}$$

ILLUSTRATION 39

Find out the area enclosed $|z| = 2$ and parabola $y = x^2 + x + 1$ and curve $y = \left[\sin^2\frac{x}{4} + \cos\frac{x}{4}\right]$ and x-axis, where [.] denotes the greatest integer function.

SOLUTION

$\because \quad |z| = 2 \quad$ Let $z = x + iy$

$\Rightarrow \quad x^2 + y^2 = 4 \Rightarrow y^2 = 4 - x^2$ where $x \in [-2, 2]$

$$y = x^2 + x + 1 \Rightarrow \left(y - \frac{3}{4}\right) = \left(x + \frac{1}{2}\right)^2$$

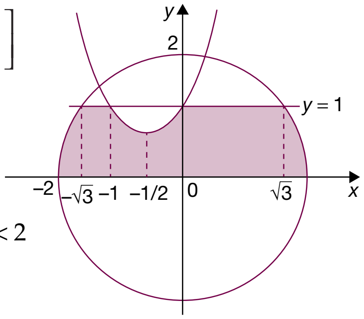
Curve $y = \left[\sin^2\frac{x}{4} + \cos\frac{x}{4}\right]$

As $\quad -2 \le x \le 2$

$$-\frac{1}{2} \le \frac{x}{4} \le \frac{1}{2}$$

$\because \quad 1 < \sin^2\frac{x}{4} + \cos\frac{x}{4} < 2$

$\Rightarrow \quad y = 1$



Required area $= (\sqrt{3} - 1) \times 1 +$

$$\int_{-1}^{0}(x^2 + x + 1)\,dx + \sqrt{3} \times 1 + 2\int_{\sqrt{3}}^{2}\sqrt{4 - x^2}\,dx$$

$$= (2\sqrt{3} - 1) + \left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{0} +$$

$$2\left(\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\frac{x}{2}\right)_{\sqrt{3}}^{2}$$

$$= (2\sqrt{3} - 1) + 0 + \frac{1}{3} - \frac{1}{2} + 1 + 2\pi - \sqrt{3} - \frac{4\pi}{3}$$

$$= \left(\sqrt{3} + \frac{2\pi}{3} - \frac{1}{6}\right) \text{ sq. units}$$

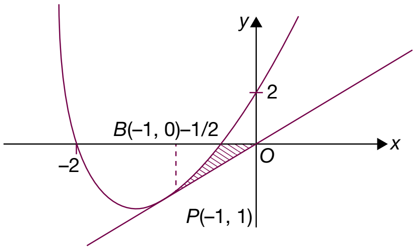
ILLUSTRATION 40

The tangent drawn from the origin to the curve, $y = 2x^2 + 5x + 2$ meets the curve at a point whose y-coordinate is negative. Find the area of the figure bounded by the tangent between the point of contact and origin, the x-axis and the parabola.

SOLUTION

Curve $y = 2x^2 + 5x + 2 = (2x + 1)(x + 2)$

$P(x_1, y_1)$ tangent at P is $\dfrac{y + y_1}{2} = 2x\,x_1 + \dfrac{5}{2}(x + x_1) + 2$



Since, origin lies on it, we obtain $y_1 = 5x_1 + 4$

$5x_1 + 4 = 2x_1^{\,2} + 5x_1 + 2;\ x_1 = \pm 1;\ x_1 = -1,\ y_1 = -1$ point $(-1, -1)$; tangent $y = x$;

Required area = area of triangle – area enclosed by curve with x-axis

$$= \frac{1}{2}.1.1 - \left(-\int_{-1}^{-1/2}(2x^2 + 5x + 2)dx\right) = \frac{5}{24} \text{ sq. units}$$

ILLUSTRATION 41

Let f be a real valued function satisfying $f\left(\dfrac{x}{y}\right) = f(x) - f(y)$ and $\displaystyle\lim_{x\to 0}\frac{f(1+x)}{x} = 2$. The area bounded by curve $y = f(x)$, y-axis and the line $y = 2$ is $\dfrac{\lambda e}{3}$, then $\lambda =$

SOLUTION

Let $f(x) = n \ln x$

$$\lim_{x\to 0}\frac{f(1+x)}{x} = \lim_{x\to 0}\frac{n\ln(x+1)}{x} = 2$$

$\Rightarrow \quad n = 2;\ y = f(x) = 2\ln x \Rightarrow x = e^{y/2}$

$$A = \int_{-\infty}^{2} e^{y/2}\,dy = 2\left[e^{y/2}\right]_{-\infty}^{2}$$

$$= 2.e = \frac{\lambda e}{3} \Rightarrow \lambda = 6$$

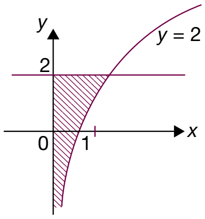


ILLUSTRATION 42

Let $f : R \to R$ be a differentiable function such that $f(0) = 1$ and $f(xy + 1) = f(x)f(y) - f(y) - x + 2\ \forall\ x, y \in R$ then area bounded by $f(x)$ and $g(x) = x^2 + 1$ can be expressed as p/q are relatively prime. Find $(p + q)$.

SOLUTION

$$f(xy+1) = f(x)\,f(y) - f(y) - x + 2$$

Differentiation

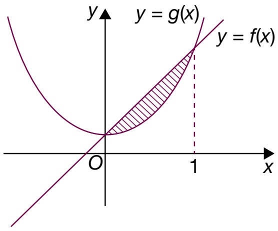
$$f'(xy+1)y = f(y)f'(x) - 1$$

Since $f(0) = 1$, we put $y = 0$

$$f'(x) = 1;\ f(x) = x + c$$

$\Rightarrow \quad c = 1$

$$f(x) = x + 1;\ g(x) = x^2 + 1$$



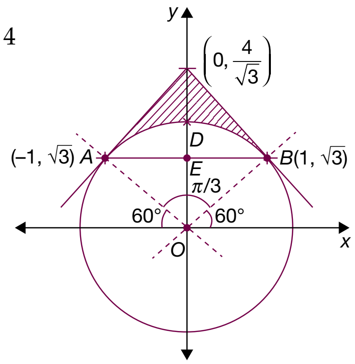
Required Area $= \displaystyle\int_0^1 (x+1-x^2-1)\,dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \frac{p}{q}$

Therefore, $p + q = 7$

ILLUSTRATION 43

If the area bounded between the tangents, drawn to the circle $x^2 + y^2 = 4$ at its point of intersection with the curve $y = \sqrt{3|x|}$ and the circle is $\left(\frac{a}{\sqrt{b}} - \frac{2\pi}{3}\right)$, where $a, b \in N$, then find the least value of $(a + b)$.

SOLUTION



Given curves are $x^2 + y^2 = 4$

and $y = \sqrt{3|x|}$

$\Rightarrow \quad |x|^2 + 3|x| - 4 = 0;$

$\Rightarrow \quad x = \pm 1$ and $y = \sqrt{3}$

Now, equation intersects on y-axis at $\left(0, \frac{4}{\sqrt{3}}\right)$

Now, $\angle AOB = \frac{\pi}{3}$

So, required area $= \left(\frac{4}{\sqrt{3}} - \frac{2\pi}{3}\right) \equiv \left(\frac{a}{\sqrt{b}} - \frac{2\pi}{3}\right)$

$\therefore \quad a = 4, b = 3; (a + b)_{\text{least}} = 7.$

ILLUSTRATION 44

Let $f(x)$ be polynomial of degree 3. If the curve $y = f(x)$ has relative extrema at $x = \pm\frac{2}{\sqrt{3}}$ and passes through (0, 0) and (1, –2). If area enclosed by the circle $x^2 + y^2 = 4$ and $y \geq f(x)$ is $\frac{k\pi}{2}$. Find the value k.

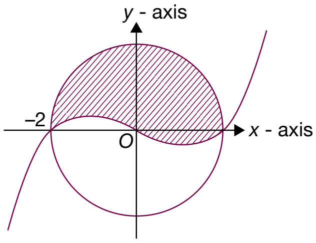
SOLUTION

Using given information, we can write

$$f'(x) = a\left(x - \frac{2}{\sqrt{3}}\right)\left(x + \frac{2}{\sqrt{3}}\right) = a\left(x^2 - \frac{4}{3}\right)$$

$\Rightarrow \quad f(x) = a\left(\frac{x^3}{3} - \frac{4x}{3}\right) + b,$

passes through (0, 0) and (1, – 2)



$\therefore \quad b = 0, a = 2$

$\Rightarrow \quad f(x) = \frac{2x}{3}(x^2 - 4)$

Required area $= \frac{\pi(2)^2}{2} = 2\pi = 4\frac{\pi}{2}$

We get $k = 4$.

ILLUSTRATION 45

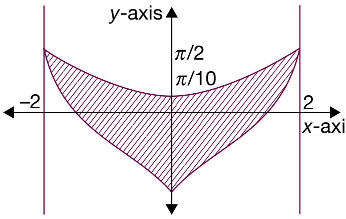
If area enclosed by the curves $y = \sin^{-1}(|x| - 1)$ and $y = \frac{\pi}{10}(x^2 + 1)$ is $\frac{p}{q}\pi$ where $p, q \in N$, then find the least value of $(q - p)$

SOLUTION

Required area $= 2\displaystyle\int_0^2 \left(\frac{\pi}{10}(x^2+1) - \sin^{-1}(x-1)\right)dx$

$$= 2\left(\left(\frac{\pi}{10}\left(\frac{x^3}{3} + x\right)\right)_0^2 - \int_0^2 \sin^{-1}(x-1)dx\right)$$

Put $x - 1 = t$



$$= \frac{14\pi}{15} - 2\int_{-1}^{1} \sin^{-1} t\,dt$$

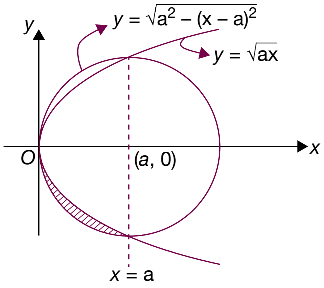
$$= \frac{14\pi}{15} - 0 = \frac{14\pi}{15} = \frac{p}{q}\pi$$

Hence, least value of $q - p = 1$

ILLUSTRATION 46

Find area bounded by regions $x^2 + y^2 \leq 2ax;\ x \geq 0;\ y^2 \geq ax,\ x \geq 0;\ y \leq 0$

SOLUTION



As $x \geq 0;\ y \leq 0$ region lies in fourth quadrant

Curve $x^2 + y^2 \leq 2ax$

or $(x - a)^2 + y^2 \leq a^2$

It is an interior part of circle with centre $C(a, 0)$ $r = a$

Parabola $y^2 \geq ax$

Solving circle and parabola, we get

$$x^2 - 2ax + ax = 0 \text{ or } x(x - a) = 0$$

or $\quad x = 0;\ x = a$

Shaded area = quarter area of circle – area enclosed by parabola

$$= \frac{\pi a^2}{4} - \int_0^a \sqrt{a}\sqrt{x}\,dx = \frac{\pi a^2}{4} - \sqrt{a}\left.\frac{x^{3/2}}{\frac{3}{2}}\right|_0^a$$

$$= \frac{\pi a^2}{4} - \frac{2}{3}a^2 = \left(\frac{\pi}{4}-\frac{2}{3}\right)a^2 \text{ sq. units}$$

ILLUSTRATION 47

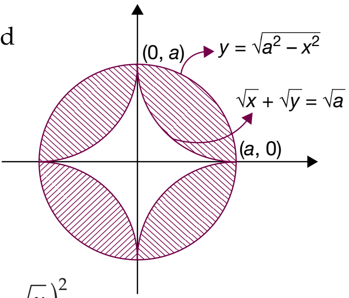
Find region included between curve $x^2 + y^2 = a^2$ and $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}\ (a > 0)$

SOLUTION

Both curves $x^2 + y^2 = a^2$ and $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$ are symmetric about x-axis, y-axis and origin

So, we can compute area bounded in first quadrant by curves

$y = \sqrt{a^2 - x^2}$ and $y = \left(\sqrt{a} - \sqrt{x}\right)^2$



$$\text{Required shaded area} = 4\int_0^a \sqrt{a^2 - x^2} - \left(\sqrt{a} - \sqrt{x}\right)^2 dx$$

$$= 4\int_0^a \sqrt{a^2 - x^2} - (a + x - 2\sqrt{a}\ \sqrt{x})dx$$

$$= 4\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} - \left(ax + \frac{x^2}{2} - 2\sqrt{a}\,\frac{x^{3/2}}{3/2}\right)\right]_0^a$$

$$= 4\left[0 + \frac{a^2}{2}\left(\frac{\pi}{2}\right) - 0 - \left[\left(a^2 + \frac{a^2}{2} - \frac{4}{3}a^2\right) - (0)\right]\right]$$

$$= 4\left[\frac{\pi a^2}{4} + \left(\frac{4}{3} - \frac{3}{2}\right)a^2\right] = \left(\pi - \frac{2}{3}\right)a^2 \text{ sq. units}$$

ILLUSTRATION 48

Consider the following regions in the plane

$R_1 = \{(x, y) : 0 \le x \le 1 \text{ and } 0 \le y \le 1\}$ and

$R_2 = \{(x, y) : x^2 + y^2 \le \frac{4}{3}\}$ The area of the region $R_1 \cap R_2$ can be expressed as $\frac{a\sqrt{3} + b\pi}{9}$, where a and b are integers.

Then find the value of $(a + b)$

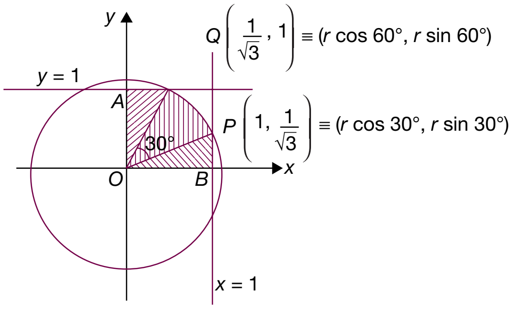
SOLUTION

$R_1 : \{(x, y) : 0 \le x \le 1, \text{ and } 0 \le y \le 1\}$

$R_2 : \{(x, y) : x^2 + y^2 \le \frac{4}{3}\}$ is interior region of circle with radius $r = \frac{2}{\sqrt{3}}$

$x = 1$ and circle cuts at point $P\left(1, \frac{1}{\sqrt{3}}\right)$

$y = 1$ and circle cuts at point $Q\left(\frac{1}{\sqrt{3}}, 1\right)$



y
$Q\left(\frac{1}{\sqrt{3}}, 1\right) \equiv (r\cos 60°, r\sin 60°)$
y = 1
A
$P\left(1, \frac{1}{\sqrt{3}}\right) \equiv (r\cos 30°, r\sin 30°)$
30°
x
O
B
x = 1

Required shaded area = area of ΔOPM + Area of ΔOAQ + Area of arc OPQ

$$= \frac{1}{2}(1)\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{2}(1)\left(\frac{1}{\sqrt{3}}\right) + \pi\left(\frac{4}{3}\right)\left(\frac{30°}{360°}\right)$$

$$= \frac{1}{\sqrt{3}} + \frac{4\pi}{3}\left(\frac{1}{12}\right) = \frac{3\sqrt{3} + \pi}{9} = \frac{a\sqrt{3} + b\pi}{9}$$

We have $a = 3$, $b = 1$

Now required term $= a + b = 4$

ILLUSTRATION 49

Find the area bounded by the curves $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and above the x-axis.

SOLUTION

Both curves are symmetric about y-axis

Point of intersection of given curve.

when $|x| \ge 2$

$$y = \frac{x^2 - 4}{4} \Rightarrow x^2 = 4y + 4$$

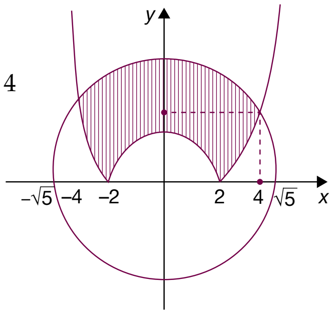
$$\Rightarrow \quad y^2 + 4y - 21 = 0$$

$$\Rightarrow \quad y^2 + 7y - 3y - 21 = 0$$

$$\Rightarrow \quad (y - 3)(y + 7) = 0$$

$$y = 3 : x = \pm 4$$

$$y = -7 : x \in \phi$$



$$\text{Required area} = 2\int_0^4 \sqrt{25 - x^2}\,dx - 2\int_0^2 \frac{4 - x^2}{4}dx - 2\int_2^4 \frac{x^2 - 4}{4}dx$$

$$= \left[x\sqrt{25-x^2}+25\sin^{-1}\frac{x}{5}\right]_0^4 - 2\left[x-\frac{x^3}{12}\right]_0^2 - 2\left[\frac{x^3}{12}-x\right]_2^4$$

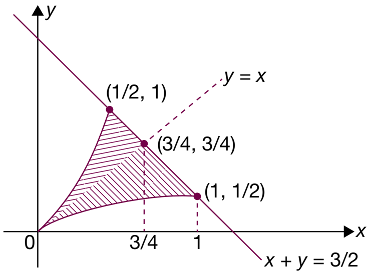
$$= \left[12+25\sin^{-1}\frac{4}{5}\right] - 4 + \frac{1}{12}\,.\,16 - \frac{1}{12}\,.\,112 + 4$$

$$= 25.\sin^{-1}\left(\frac{4}{5}\right) + 4 \text{ sq. units}$$

ILLUSTRATION 50

Let $f : [0, 1] \rightarrow \left[0, \frac{1}{2}\right]$ be a function such that $f(x)$ is a polynomial of 2nd degree satisfy the following conditions $f(0) = 0$, and has vertex $\left(1, \frac{1}{2}\right)$. Find the area bounded by $y = f(x)$, $y = f^{-1}(x)$ and the line $2x + 2y - 3 = 0$ in first quadrant.

SOLUTION



$f : [0, 1] \rightarrow \left[0, \frac{1}{2}\right]$

Let $f(x) = ax^2 + bx + c$

Given, $f(0) = 0$ then $c = 0$

Vertex $\left(1, \frac{1}{2}\right)$; $a + b = \frac{1}{2}$

$$x = 1 = -\frac{b}{2a} \Rightarrow b = -2a$$

we get $f(x) = -\frac{x^2}{2} + x$ and line $2x + 2y = 3$

$$\text{Required area} = 2\left[\int_0^{3/4} x\,dx + \int_{3/4}^{1} \frac{3}{2} - x\,dx - \int_0^1 \left(-\frac{x^2}{2} + x\right)dx\right]$$

$$= 2\left[\frac{x^2}{2}\bigg|_0^{3/4} + \frac{3x}{2} - \frac{x^2}{2}\bigg|_{3/4}^{1} - \left(-\frac{x^2}{6} + \frac{x^2}{2}\right)\bigg|_0^1\right]$$

$$= 2\left[\frac{9}{32} - 0 + \frac{3}{2} - \frac{1}{2} - \left(\frac{9}{8} - \frac{9}{32}\right) - \left(-\frac{1}{6} + \frac{1}{2}\right)\right]$$

$$= 2\left[\frac{9}{16} - \frac{1}{8} - \frac{1}{3}\right] = \frac{5}{24} \text{ sq. units}$$

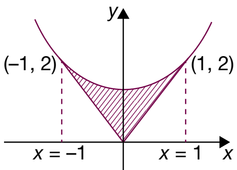
ILLUSTRATION 51

A polynomial function $f(x)$ satisfies the condition $f(x + 1) = f(x) + 2x + 1$ and $f(0) = 1$. Find the equations of the pair of tangents from the origin on the curve $y = f(x)$ and compute the area enclosed by the curve and the pair of tangents.

SOLUTION

$$f(x + 1) = f(x) + 2x + 1$$

$$f(x + 1) - f(x) = (x + 1)^2 - x^2$$



$\therefore \quad f(x) = x^2 + K$

$f(0) = 1 \Rightarrow K = 1$

$\Rightarrow \quad y = f(x) = x^2 + 1$

Point (x_1, y_1) on curve $y_1 = x_1^2 + 1$

Tangent $y - y_1 = 2x_1(x - x_1)$ point (0, 0) lies on it

$\Rightarrow \quad 0 - y_1 = 2x_1(0 - x_1)$

$\Rightarrow \quad y_1 = 2x_1^2 = x_1^2 + 1$

we get $x_1 = 1, -1$

Point (1, 2) and (–1, 2)

$\therefore$ Pair of tangents $y = 2x$ and $y = -2x$

$$\text{Required area } A = 2\int_0^1 (1 + x^2) - 2x\,dx = 2\int_0^1 (x - 1)^2\,dx$$

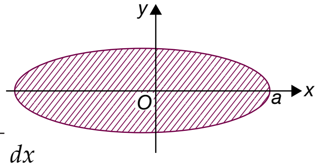
$$= 2\left(\frac{(x-1)^3}{3}\right)_0^1 = (2)\left(0 + \frac{1}{3}\right) = \frac{2}{3} \text{ sq. units}$$

ILLUSTRATION 52

Find area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

SOLUTION

Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, It is symmetric about x-axis, y-axis, and origin



$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\text{Required area} = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx$$

$$= \frac{4b}{a}\left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\bigg|_0^a\right]$$

$$= \frac{4b}{a}\left[0 + \frac{a^2}{2}\left(\frac{\pi}{2}\right) + 0 + 0\right] = \pi ab \text{ sq. units}$$

ILLUSTRATION 53

Find area of ellipse $2x^2 + 6xy + 5y^2 = 1$.

SOLUTION

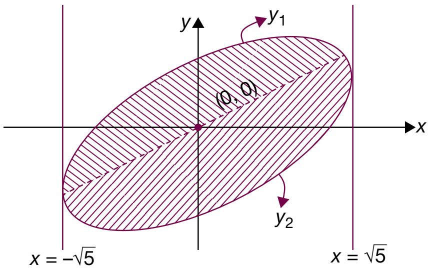
Given ellipse $5y^2 + y(6x) + 2x^2 - 1 = 0$

$$y = -\frac{6x \pm \sqrt{36x^2 - 4(5)(2x^2 - 1)}}{10}$$

We can take two curves

$$y_1 = \frac{-3x + \sqrt{5 - x^2}}{5} \text{ and } y_2 = \frac{-3x - \sqrt{5 - x^2}}{5}$$

$$5 - x^2 \ge 0 \Rightarrow (x - \sqrt{5})(x + \sqrt{5}) \le 0 \Rightarrow -\sqrt{5} \le x \le \sqrt{5}$$



$$\text{Required enclosed area} = \int_{-\sqrt{5}}^{\sqrt{5}} y_1 - y_2 \, dy$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \frac{-3x + \sqrt{5 - x^2}}{5} - \left(\frac{-3x - \sqrt{5 - x^2}}{5}\right) dx$$

$$= \int_{-\sqrt{5}}^{\sqrt{5}} \frac{2}{5}\sqrt{5 - x^2}\, dx = \frac{4}{5}\int_{0}^{\sqrt{5}} \sqrt{5 - x^2}\, dx$$

$$= \frac{4}{5}\left[\frac{x}{2}\sqrt{5 - x^2} + \frac{5}{2}\sin^{-1}\frac{x}{\sqrt{5}}\Big|_0^{\sqrt{5}}\right]$$

$$= \frac{4}{5}\left[0 + \frac{5}{2}\left(\frac{\pi}{2}\right) - 0 - 0\right] = \pi \text{ sq. units}$$

ILLUSTRATION 54

Compute the area of the figure which lies in the first quadrant inside the curve $x^2 + y^2 = 3a^2$ and is bounded by the parabola $x^2 = 2ay$ and $y^2 = 2ax$ $(a > 0)$.

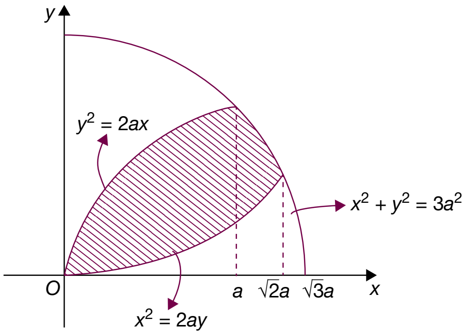
SOLUTION

Curve $x^2 + y^2 = 3a^2$ and $y^2 = 2ax$

Solving the equation

$\Rightarrow \quad x^2 + 2ax - 3a^2 = 0 \Rightarrow (x - a)(x + 3a) = 0$

$\Rightarrow \quad x = a, y = \sqrt{2}a; x = -3a$ (not acceptable)



Similarly, curves $x^2 + y^2 = 3a^2$ and $x^2 = 2ay$

$\Rightarrow \quad y = a, x = \sqrt{2}a$ Point of intersection

$$\text{Required Area} = \int_0^a \sqrt{2ax}\, dx + \int_a^{2a} \sqrt{3a^2 - x^2}\, dx - \int_0^{\sqrt{2}a} \frac{x^2}{2a} dx$$

$$= \left[\sqrt{2a}\frac{(x^{3/2})}{3/2}\right]_0^a + \left[\frac{x}{2}\sqrt{3a^2 - x^2} + \frac{3a^2}{2}\sin^{-1}\left(\frac{x}{\sqrt{3}a}\right)\right]_a^{\sqrt{2}a} - \frac{1}{2a}\left(\frac{x^3}{3}\right)_0^{\sqrt{2}a}$$

$$= \frac{2\sqrt{2}}{3}\sqrt{a}(a\sqrt{a}) + \left[\frac{\sqrt{2}a}{2}a + \frac{3a^2}{2}\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) - \left(\frac{a}{2}\right)\sqrt{2}a - \frac{3a^2}{2}\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right] - \frac{1}{2a}\left(\frac{1}{3}\right)(2\sqrt{2})(a)^3$$

$$= \frac{2\sqrt{2}}{3}a^2 + \frac{3}{2}a^2\left(\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right) - \sin^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) - \frac{\sqrt{2}}{3}a^2$$

$$= \frac{\sqrt{2}}{3}a^2 + \frac{3}{2}a^2\sin^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\cdot\frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right)$$

$$= \left[\frac{\sqrt{2}}{3} + \frac{3}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]a^2 \text{ sq. unit}$$

ILLUSTRATION 55

Given $f(x) = \int_0^x e^t(\ln\sec t - \sec^2 t)\,dt$ and $g(x) = -2e^x \tan x$.

Find the area bounded by the curves $y = f(x)$ and $y = g(x)$ between the ordinates $x = 0$ and $x = \frac{\pi}{3}$.

SOLUTION

$$f(x) = \int_0^x e^t(\ln\sec t - \sec^2 t)\,dt$$

$$\Rightarrow \quad f(x) = \int_0^x e^t(\ln\sec t + \tan t) - e^t(\tan t + \sec^2 t)\,dt$$

Using $\int e^x(f(x) + f'(x))dx = e^x f(x) + c$

$\Rightarrow \quad f(x) = \left[e^t(\ln\sec t)\right]_0^x - \left[e^t(\tan t)\right]_0^x$

$\Rightarrow \quad f(x) = e^x \ln(\sec x) - e^x \tan x$

Point of intersection of curves $f(x) = g(x)$

$\Rightarrow \quad e^x \ln(\sec x) - e^x \tan x = -2e^x \tan x$

$\Rightarrow \quad \ln(|\sec x|) + \tan x = 0 \Rightarrow \ln(|\sec x|) = -\tan x$

Only $x = 0$;

Required area bounded by both curves

$$A = \int_0^{\pi/3} e^x (\ln|\sec x| - \tan x\, dx) \; + e^x(2\tan x)\, dx$$

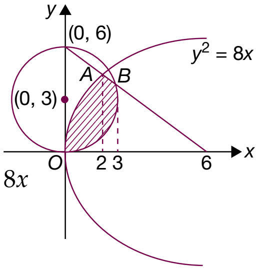
$$A = \int_0^{\pi/3} e^x (\ln(\sec x) + \tan x)\, dx$$

$$A = [e^x \cdot \ln(\sec x)]_0^{\pi/3} \; = e^{\pi/3} \ln 2 \text{ sq. units}$$

ILLUSTRATION 56

Find the area given by $x + y \le 6$, $x^2 + y^2 \le 6y$ and $y^2 \le 8x$.

SOLUTION



Given regions

$x + y \le 6$, $x^2 + (y - 3)^2 \le 9$, $y^2 \le 8x$

We take

$$x + y = 6,\ x^2 + (y - 3)^2 = 9,\ y^2 = 8x$$

Solving $y^2 = 8x$ and $x + y = 6$, we get $A(2, 4)$ and solving $x^2 + (y - 3)^2 = 9$ and $x + y = 6$, we get $B(3, 3)$

Required area

$$= \int_0^2 \left(2\sqrt{2x} - (3 - \sqrt{9 - x^2}\right) dx + \int_2^3 \left[(6 - x) - (3 - \sqrt{9 - x^2})\right] dx$$

$$= 2\sqrt{2} \int_0^2 \sqrt{x}\, dx + \int_2^3 (6 - x)\, dx - \int_0^3 (3 - \sqrt{9 - x^2})\, dx$$

$$= 2\sqrt{2} \,.\, \frac{2}{3} \left[x^{3/2}\right]_0^2 + \left[6x - \frac{x^2}{2}\right]_2^3 - \left[3x - \frac{x}{2}\sqrt{9 - x^2} - \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_0^3$$

Solving, we get required area $= \dfrac{1}{12}(27\pi - 2)$ sq units

ILLUSTRATION 57

Let $f(x)$ = maximum $\{x^2, (1 - x)^2, 2x(1 - x)\}$ where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$ and $x = 1$.

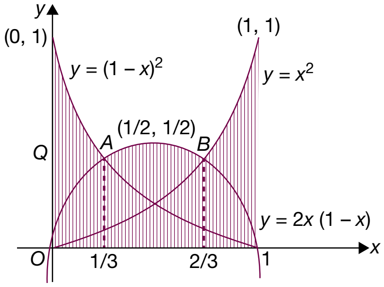
SOLUTION

We can draw the graph of $y = x^2$, $y = (1 - x)^2$ and $y = 2x(1 - x)$ in following fig. Now, to get the point of intersection of $y = x^2$ and $y = 2x(1 - x)$, we solve both the equations.

We get $x^2 = 2x(1 - x)$

$$\Rightarrow \quad x^2 = 2x - 2x^2 \Rightarrow 3x^2 = 2x \Rightarrow 3x^2 - 2x = 0$$

$$\Rightarrow \quad x(3x - 2) = 0 \Rightarrow x = 0,\ \frac{2}{3}$$



Similarly, we can find the coordinate of the points of intersection of $y = (1 - x)^2)$ and $y = 2x(1 - x)$ are $x = \dfrac{1}{3}$ and $x = 1$.

From the figure it is clear that

$$f(x) = \begin{cases} (1 - x)^2 & ; \quad 0 \le x \le \dfrac{1}{3} \\ 2x(1 - x) & ; \quad \dfrac{1}{3} \le x \le \dfrac{2}{3} \\ x^2 & ; \quad \dfrac{2}{3} \le x \le 1 \end{cases}$$

The required area A is given by $A = \displaystyle\int_0^1 f(x)\, dx$

$$= \int_0^{1/3} (1 - x)^2\, dx + \int_{1/3}^{2/3} 2x(1 - x)\, dx + \int_{2/3}^{1} x^2\, dx$$

$$= \left[-\frac{1}{3}(1 - x)^3\right]_0^{1/3} + \left[x^2 - \frac{2x^3}{3}\right]_{1/3}^{2/3} + \left[\frac{1}{3}x^3\right]_{2/3}^{1}$$

$$= -\frac{1}{3}\left(\frac{2}{3}\right)^3 + \frac{1}{3} + \left(\frac{2}{3}\right)^2 - \frac{2}{3}\left(\frac{2}{3}\right)^3 - \left(\frac{1}{3}\right)^2 + \frac{2}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{3}(1) - \frac{1}{3}\left(\frac{2}{3}\right)^3$$

$$= \frac{17}{27} \text{ sq. units}$$

Test Your Concepts-III

(Solutions on page H.345)

1. Find the area bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$.

2. Find the area of the region(s) enclosed by the curves $y = x^2$ and $y = \sqrt{|x|}$.

3. Find the area of the region bounded by the curve $C: y = \tan x$, tangent drawn to C at $x = \dfrac{\pi}{4}$ and x-axis.

4. Find the area of the triangle formed by joining the origin to the points of intersection of the line $x\sqrt{5} + 2y = 3\sqrt{5}$ and circle $x^2 + y^2 = 10$.

5. Find the area of the region bounded by the axis of x, $0 \le x \le 1$, $y = \text{arc}(\cos x)$ and $y = \text{arc}(\sin x)$.

6. Find the area of the region formed by

$x^2 + y^2 - 6x - 4y + 12 \le 0$, $y \le x$ and $x \le \frac{5}{2}$.

7. Find the area bounded by circle $x^2 + y^2 - 2x = 0$ and curve $y = \sin\left(\frac{\pi x}{2}\right)$ in the upper half of the circle.

8. Find the area of the region bounded by the parabola $y^2 = 4x$ and the normal to it at one of the ends of its latus rectum.

9. Find the ratio of the areas of two regions of the curve $C_1 : 4x^2 + \pi^2 y^2 = 4\pi^2$ divided by the curve

$C_2 : y = -\text{sgn}\left(x - \frac{2}{\pi}\right).\cos x$ (where sgn(x) denotes signum function).

10. Let $P(x, y)$ lies in the region satisfying $\sin^{-1}.\left(\frac{x^2}{9} + \frac{y^2}{4}\right) \le \frac{\pi}{2}$ and $\cot^{-1}(x^2 - y^2) > \frac{\pi}{2}$ such that area bounded by locus of P is, $a \sin^{-1}\frac{b}{\sqrt{c}}$, where b and c are co-primes, then find $a + b + c$.

11. The curve $y = ax^2 + bx + c$ passes through the point (1, 2) and its tangent at origin is the line $y = x$. find the area bounded by the curve, the ordinate of the curve at minima and the tangent line.

12. Find the area of the region enclosed between the two circles $x^2 + y^2 = 1$ and $(x - 1)^2 + y^2 = 1$.

13. Find the equation of the line passing through the origin and dividing the curvilinear triangle with vertex at the origin, bounded by the curves $y = 2x - x^2$, $y = 0$ and $x = 1$ into two parts of equal area.

14. Find the ratio in which the area bounded by the curves $y^2 = x$ and $x^2 = y$ is divided by the line $x = \frac{1}{2}$

15. Find the area bounded by the curve $y = e^{|x|}$, $y = e^{-|x|}$, $x \ge 0$ and $x \le 5$

16. Prove that the area of the region bounded by the curve $a^4 y^2 = x^5(2a - x)$, is to that of the circle whose radius is a, is 5 to 4.

17. Find the area of the region bounded by the x-axis and the curve $y = \frac{1}{2}(2 - 3x - 2x^2)$.

18. If A be the area bounded by the curves $|y| \ge e^{-|x|} - \frac{1}{2}$ and $\frac{|x|+|y|}{2} + \left|\frac{|x|-|y|}{2}\right| \le 2$ then find $A - \ln 4 - 7$

Answer key

1. $\left(\frac{9}{2}\right)$ sq. units
2. $\frac{2}{3}$ sq. units
3. $\left(\frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2\right)$ sq. units
4. 5 sq. units
5. $(\sqrt{2} - 1)$sq units
6. $\frac{1}{8} - \frac{\sqrt{3}}{8} + \frac{\pi}{6}$ sq units
7. $\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$ sq. units
8. $\frac{64}{3}$ sq. units
9. $\frac{\pi^2 - 2}{\pi^2 + 2}$ sq. units
10. 27
11. $\frac{1}{24}$ sq. units
12. $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$ sq. units
13. $y = \frac{2x}{3}$
14. $\frac{4\sqrt{2} - 1}{9 - 4\sqrt{2}}$
15. $e^5 + e^{-5} - 2$ sq. unit
17. $\frac{125}{48}$ sq. units
18. 7

AREA COMPUTATION FOR PARAMETRIC CURVES

If a curve is given by its parametric equation in the form $x = f(t)$, $y = g(t)$ and suppose the derivatives $f'(t)$ and $g'(t)$ are continuous functions on the interval $[t_1, t_2]$. If t_1 and t_2 are the values of parameter 't' corresponding respectively to the initial and final position in which the curve can be described as a contour in the positive direction then the area described by the curve

Given $y = f(x)$ is nonnegative on the interval $a \le x \le b$

$$\text{Area} = \int_a^b f(x)\,dx = \int_a^b y\,dx = \int_{t_1}^{t_2} y(t)\left(\frac{dx}{dt}\right)dt$$

ILLUSTRATION 58

Determine the area of the region enclosed by the parametric curve given by $x = t^2$ and $y = t^3 - 3t$

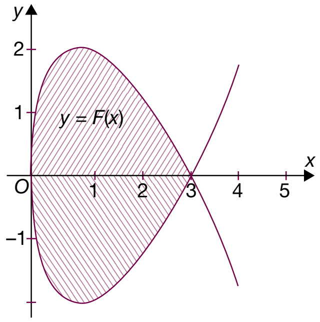
SOLUTION

$x = f(t) = t^2$;

$y = g(t) = t^3 - 3t$

At $t=0$; $x=0$, $y=0$

At $t=\pm\sqrt{3}$; $x=3$, $y=0$



$$\text{Required area} = \int_{t_1}^{t_2} y\left(\frac{dx}{dt}\right)dt$$

$$= \int_{-\sqrt{3}}^{\sqrt{3}} y\left(\frac{dx}{dt}\right)dt$$

$$= 2\int_{0}^{\sqrt{3}} (t^3-3t)\cdot 2t\,dt$$

$$= 2\int_{0}^{\sqrt{3}} (2t^4-6t^2)\,dt = \frac{4}{5}t^5 - 4t^3\Big|_0^{\sqrt{3}}$$

$$= \frac{4}{5}3^{5/2} - 4\cdot 3^{3/2}\text{ sq units}$$

ILLUSTRATION 59

Determine the area of the region enclosed by the parametric curve given by $x = 3t-2$; $y=\frac{1}{2}t^2+1$ where $0\le t\le 3$ and x-axis.

SOLUTION

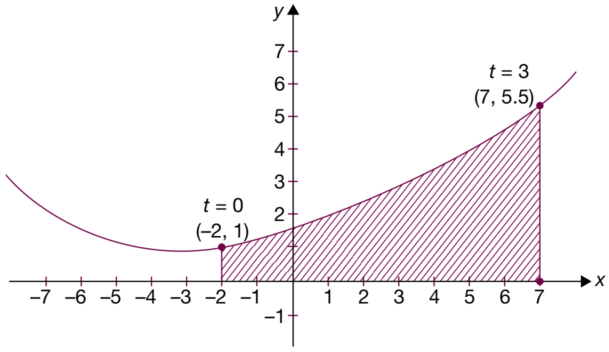
At $t=0$, $x=-2$, $y=1$

At $t=3$, $x=7$, $y=5.5$

$$t=\frac{x+2}{3};\ y=\frac{t^2}{2}+1$$

$$\Rightarrow\quad y=f(x)=1+\frac{1}{2}\left(\frac{x+2}{3}\right)^2$$

Given $f(x)$ is non-negative on the interval $a\le x\le b$



$$\text{Required Area} = \int_a^b y\,dx = \int_{t_1}^{t_2} y\left(\frac{dx}{dt}\right)dt$$

$$= \int_0^3\left(\frac{1}{2}t^2+1\right)(3)\,dt = \int_0^3\left(\frac{3}{2}t^2+3\right)dt$$

$$= \left[\frac{3}{2}\cdot\frac{t^3}{3}+3t\right]_0^3 = \left[\left(\frac{(3)^3}{2}+3(3)-(0)\right)\right] = \frac{45}{2}\text{ sq. units}$$

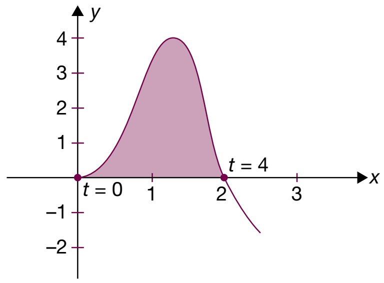
ILLUSTRATION 60

Determine the area of the region bounded by the parametric curve $x=\sqrt{t}$ and $y=4t-t^2$ and the x-axis

SOLUTION

$$x=\sqrt{t}\Rightarrow t=x^2;\ y=4x^2-x^4$$

Given $f(x)$ is non-negative on the interval $a\le x\le b$



$$\text{Required area} = \int_a^b f(x)\,dx = \int_a^b y\,dx = \int_{t_1}^{t_2} y(t)\left(\frac{dx}{dt}\right)dt$$

$t=0$ to $t=4$

$$A=\int_0^4 (4t-t^2)\left(\frac{1}{2}t^{-1/2}\right)dt = \int_0^4\left(2t^{1/2}-\frac{1}{2}t^{3/2}\right)dt$$

$$= \left[2\cdot\frac{2}{3}t^{3/2}-\frac{1}{2}\cdot\frac{2}{5}t^{5/2}\right]_0^4 = \frac{64}{15}\text{ sq. units}$$

ILLUSTRATION 61

Find area enclosed by cure $x=3\cos t+\cos 3t$ and $y=3\sin t-\sin 3t$ where $0\le t\le\pi$

SOLUTION

Given $y=f(x)$ is nonnegative on the interval $a\le x\le b$; where x is decreasing as t increases

$$\text{Required area} = -\int_a^b y\,dx = -\int_{t_1}^{t_2} y(t)\left(\frac{dx}{dt}\right)dt$$

$$\text{Required area} = -\int_0^{\pi} (3\sin t - \sin 3t)(-3\sin t - 3\sin 3t)dt$$

$$= -\int_0^{\pi} 4\sin^3 t(-3)(\sin t + \sin 3t)dt$$

$$= 12\int_0^{\pi} \sin^3 t(2\sin 2t\cos t)\,dt \; = 48\int_0^{\pi} \sin^4 t\cos^2 t\,dt$$

$$= 96\int_0^{\pi/2} \sin^4 t\cos^2 t\,dt \qquad \text{[using Walli's formula]}$$

$$= 96\left(\frac{3\cdot 1\times 1}{6\times 4\times 2}\,\frac{\pi}{2}\right) = 3\pi \text{ sq. units}$$

ILLUSTRATION 62

Find area of the loop of the curve $x = a(1 - t^2)$, $y = at(1 - t^2)$, $-1 \le t \le 1$

SOLUTION

$$y = tx \Rightarrow t = \frac{y}{x}$$

$$x = a\left(1 - \frac{y^2}{x^2}\right) \Rightarrow x^3 = a(x^2 - y^2) \Rightarrow ay^2 = ax^2 - x^3$$

The curve is described when t varies from -1 to 1. Indeed when $t = -1$ both x and y are zero. When $t = 0$, $x = a$, $y = 0$. Thus is the adjoining figure upper portion is described when $0 \le t \le 1$ and the lower portion is described when $-1 < t < 0$

$$\text{Required area} = -\int_{t_1}^{t_2} y(t)\left(\frac{dx}{dt}\right)dt$$

$$= -\int_{-1}^{1} at(1 - t^2)(-2at)\,dt = 2a^2\int_{-1}^{1} (t^2 - t^4)dt$$

$$= 4a^2\int_0^1 t^2 - t^4 dt \; = 4a^2\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{8a^2}{15} \text{ sq. units}$$

CONCEPT BASED ON VARIABLE AREA

ILLUSTRATION 63

Find the value (s) of the parameter 'a' ($a > 0$) for each of which the area of the figure bounded by the straight line, $y = \dfrac{a^2 - ax}{1 + a^4}$ and the parabola $y = \dfrac{x^2 + 2ax + 3a^2}{1 + a^4}$ is the greatest.

SOLUTION

given curves $y = \dfrac{a^2 - ax}{1 + a^4}$; $y = \dfrac{x^2 + 2ax + 3a^2}{1 + a^4}$

Solving them to get point of intersection

$$\Rightarrow \quad a^2 - ax = x^2 + 2ax + 3a^2 \Rightarrow x^2 + 3ax + 2a^2 = 0$$

$$\Rightarrow \quad (x + 2a)\,(x + a) = 0 \Rightarrow x = -a,\ x = -2a$$

$$\text{Required area} = \left|\int_{-2a}^{-a}\left(\frac{x^2 + 2ax + 3a^2}{1 + a^4} - \frac{a^2 - ax}{1 + a^4}\right)dx\right|$$

$$= \int_{-2a}^{-a} \frac{(x^2 + 3ax + 2a^2)}{1 + a^4}\,\frac{1}{a^4 + 1}\left[\frac{x^3}{3} + \frac{3ax^2}{2} + 2a^2x\right]_{-2a}^{-a}$$

$$= \left|\frac{1}{a^4 + 1}\left[\left(\frac{1}{3}\right)[-a^3 + 8a^3] + \frac{3a}{2}[a^2 - 4a^2] + 2a^2[-a + 2a]\right]\right|$$

$$A = \frac{a^3}{6(a^4 + 1)}$$

For greatest Area, we have $\dfrac{dA}{da} = 0$

$$\Rightarrow \quad \frac{dA}{da} = \frac{(a^4 + 1)3a^2 - 4a^3(a^3)}{6(a^4 + 1)^2} = 0$$

$$\Rightarrow \quad 3a^6 + 3a^2 - 4a^6 = 0 \Rightarrow a^4 = 3 \Rightarrow a = 3^{1/4}$$

$$\frac{d^2A}{da^2} < 0 \text{ at } a = 3^{1/4} \qquad \text{(Point of maxima)}$$

ILLUSTRATION 64

Find the value of 'c' for which the area of the figure bounded by the curve, $y = 8x^2 - x^5$, the straight lines $x = 1$ and $x = c$ and the abscissa axis is equal to $\dfrac{16}{3}$.

SOLUTION

$$y = 8x^2 - x^5 = x^2[8 - x^3]$$

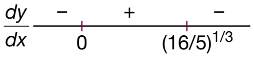
$$\Rightarrow \quad y = -x^3(x - 2)\,[x^2 + 2x + 4]$$

$y = 0$ at $x = 0, 2$ points where it cuts x-axis

For $\dfrac{dy}{dx} = 16x - 5x^4 = x[16 - 5x^3] = -5x(x^3 - 16/5)$

Here, $x = 0$ point of minima

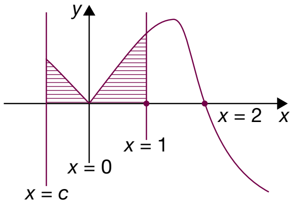
$x = (16/5)^{1/3}$ point of maxima



CASE 1: $c < 0$

$$\text{Required area} = \int_c^1 (8x^2 - x^5)\,dx$$

$$= \left(\frac{8x^3}{3} - \frac{x^6}{6}\right)_c^1 = \frac{16}{3}$$



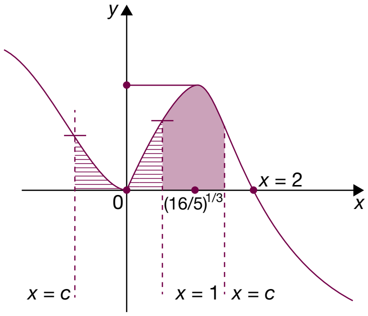
$$\Rightarrow \quad \left(\frac{8}{3}-\frac{1}{6}\right)-\left(\frac{8c^3}{3}-\frac{c^6}{6}\right)=\frac{16}{3}$$

$$\Rightarrow \quad \frac{5}{2}-\frac{8}{3c^3}+\frac{c^6}{6}=\frac{16}{3} \Rightarrow c^6-16c^3-17=0$$

For, $c^3 = 17, -1$

Acceptable $c = -1$

CASE 2: $0 < c < 2$



$$\text{Required area} = \int_1^c (8x^2 - x^5) = \left(\frac{8}{3}x^3 - \frac{x^6}{6}\right)_1^c = \frac{16}{3}$$

$$\Rightarrow \quad \left(\frac{8}{3}c^3-\frac{c^6}{6}\right)-\left(\frac{8}{3}-\frac{1}{6}\right)=\frac{16}{3}$$

$$\Rightarrow \quad c^6 - 16c^3 + 47 = 0$$

$$\Rightarrow \quad c^3 = 8 - \sqrt{17} \Rightarrow c = (8 - \sqrt{17})^{1/3}$$

$\therefore$ Two values of c, $c = -1$ and $c = (8 - \sqrt{17})^{1/3}$

ILLUSTRATION 65

For what value of 'a' is the area of the figure bounded by the lines, $y=\frac{1}{x}$, $y=\frac{1}{2x-1}$, $x = 2$ and $x = a$ equal to $\ln\frac{4}{\sqrt{5}}$?

SOLUTION

Given curves $y = \frac{1}{x}$ and $y = \frac{1}{2x-1}$; $x = 2$, $x = a$

$$\Rightarrow \quad \left(x-\frac{1}{2}\right)y=\frac{1}{2} \Rightarrow \left(x-\frac{1}{2}\right)\frac{1}{x}=\frac{1}{2}$$

$$\Rightarrow \quad \frac{x}{x}-\frac{1}{2x}=\frac{1}{2} \Rightarrow x = 1 \quad \text{Points of intersection}$$

(Position of $x = a$ is not fixed)
(Area between the curves from $x = 1$ to $x = 2$)

$$A = \int_1^2 \left(\frac{1}{x}-\frac{1}{2x-1}\right)dx$$

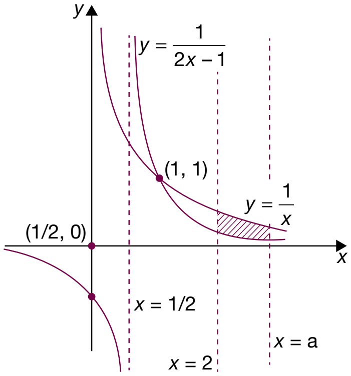
$$A = \ln 2 - \ln\sqrt{3} = \ln\left(\frac{2}{\sqrt{3}}\right) < \ln\left(\frac{4}{\sqrt{5}}\right)$$

We want $A = \ln\frac{4}{\sqrt{5}}$

$\therefore$ Two possibilities $a > 2$ or $a < 1$

CASE 1: $a > 2$

Graph of curves is as shown



$$\text{Required area } \ln\left(\frac{4}{\sqrt{5}}\right) = \int_2^a \left(\frac{1}{x}-\frac{1}{2x-1}\right)dx$$

$$= (\ln a - \ln 2) - \frac{1}{2}(\ln(2a-1) - \ln 3)$$

$$\Rightarrow \quad \ln\left(\frac{4}{\sqrt{5}}\right) = \ln a - \frac{\ln(2a-1)}{2} - \ln 2 + \frac{\ln 3}{2}$$

$$\Rightarrow \quad \ln\left(\frac{4}{\sqrt{5}}\right) = \ln a - \ln\sqrt{2a-1} - \ln\left(\frac{2}{\sqrt{3}}\right)$$

$$\Rightarrow \quad \ln\frac{8}{\sqrt{15}} = \ln\left(\frac{a}{\sqrt{2a-1}}\right) \Rightarrow \frac{a}{\sqrt{2a-1}} = \frac{8}{\sqrt{15}}$$

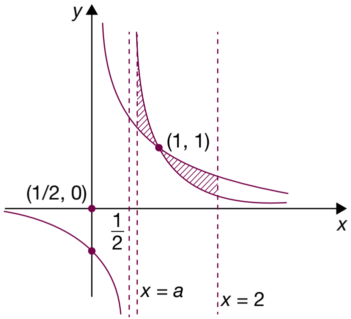
$$\Rightarrow \quad 15a^2 = 64(2a-1) \Rightarrow 15a^2 - 128a + 64 = 0$$

$$\Rightarrow \quad (15a-8)(a-8) = 0 \Rightarrow a = 8 \text{ and } a = \frac{8}{15}$$

We have acceptable $a = 8$.

CASE 2: $\frac{1}{2} < a < 1$

Graph of given curves is as shown



Required Area

$$\int_a^1\left(\frac{1}{2x-1}-\frac{1}{x}\right)dx+\int_1^2\left(\frac{1}{x}-\frac{1}{2x-1}\right)dx=\ln\left(\frac{4}{\sqrt5}\right)$$

$$\Rightarrow\quad\left[\frac{\ln(1)-\ln(2a-1)}{2}\right]-[\ln 1-\ln a]+\ln\frac{2}{\sqrt3}=\ln\frac{4}{\sqrt5}$$

$$\Rightarrow\quad \ln a-\ln\sqrt{2a-1}=\ln\left(\frac{4}{\sqrt5}\right)-\ln\left(\frac{2}{\sqrt3}\right)$$

$$\Rightarrow\quad \ln\left(\frac{a}{\sqrt{2a-1}}\right)=\ln\left(\frac{2\sqrt3}{\sqrt5}\right)\Rightarrow\frac{a}{\sqrt{2a-1}}=\frac{2\sqrt3}{\sqrt5}$$

$$\Rightarrow\quad 5a^2=(12)\,(2a-1)\Rightarrow 5a^2-24a+12=0$$

$$\Rightarrow\quad a=\frac{2}{5}\left(6-\sqrt{21}\right))\text{ acceptable}$$

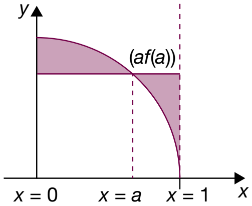
Final set of a, we have $a = 8$ or $\frac{2}{5}\left(6-\sqrt{21}\right)$

ILLUSTRATION 66

For what values of $a \in [0, 1]$ does the area of the figure bounded by the graph of the function $y = f(x)$ and the straight lines $x = 0$, $x = 1$ and $y = f(a)$ is at a minimum and for what values it is at a maximum if $f(x)=\sqrt{1-x^2}$. Find also the maximum and the minimum areas.

SOLUTION

$f(x)=\sqrt{1-x^2}$ is shown in diagram with given information



Required area

$$A=\int_0^a\left(\sqrt{1-x^2}-f(a)\right)dx+\int_a^1\left(f(a)-\sqrt{1-x^2}\right)dx$$

$$\Rightarrow\quad A=\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)\right]_0^a-\sqrt{1-a^2}\ \ (x)_0^a+$$

$$\sqrt{1-a^2}\ \ (x)_a^1-\left[\frac{x}{2}\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}(x)\right]_a^1$$

$$\Rightarrow\quad A=\frac{a}{2}\sqrt{1-a^2}+\frac{1}{2}\sin^{-1}(\mathrm{a})-0-\sqrt{1-a^2}\,(a)+$$

$$\sqrt{1-a^2}\,(1-a)-\left(\left(0+\frac{\pi}{4}\right)-\left(\frac{a}{2}\sqrt{1-a^2}+\frac{1}{2}\sin^{-1}(a)\right)\right]$$

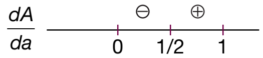
$$A=\sin^{-1}(a)-a\sqrt{1-a^2}+\sqrt{1-a^2}-\frac{\pi}{4}$$

$$\Rightarrow\quad \frac{dA}{da}=\frac{1}{\sqrt{1-a^2}}-\left(\frac{a(-a)}{\sqrt{1-a^2}}+\sqrt{1-a^2}\right)+\frac{1(-a)}{\sqrt{1-a^2}}$$

$$\Rightarrow\quad \frac{dA}{da}=\frac{1-(-a^2+1-a^2)-a}{\sqrt{1-a^2}}=\frac{2a^2-a}{\sqrt{1-a^2}}=\frac{2a\left(a-\frac{1}{2}\right)}{\sqrt{1-a^2}}$$

From first derivative test

We say that $a=\frac{1}{2}$ (local minima)



$a = 0, 1$ (local maxima)

$$\therefore\quad A=\sin^{-1}(a)-a\sqrt{1-a^2}+\sqrt{1-a^2}-\frac{\pi}{4}$$

For, $a = 0$; $A_0=0-0+1-\frac{\pi}{4}=1-\frac{\pi}{4}$

For, $a=\frac{1}{2}$; $A_{1/2}\,(\text{Min})=\frac{\pi}{6}-\frac{1}{2}\frac{\sqrt3}{2}+\frac{\sqrt3}{2}-\frac{\pi}{4}$

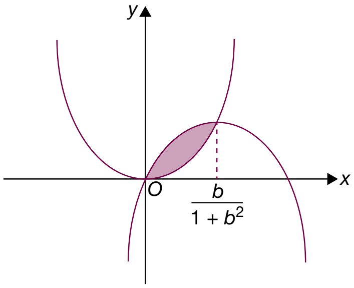
$$=\frac{3\sqrt3-\pi}{12}\text{ sq. units}$$

For $a = 1$, $A_1(\max)=\frac{\pi}{2}-0-0-\frac{\pi}{4}=\frac{\pi}{4}$ sq. units

ILLUSTRATION 67

Find all possible values of $b > 0$, so that the area of the bounded region enclosed between the parabola $y = x - bx^2$ and $y=\frac{x^2}{b}$ is maximum.

SOLUTION



Eliminating y from

$y=\frac{x^2}{b}$ and $y = x - bx^2$,

we get $x^2 = bx - b^2x^2$

$$\Rightarrow\quad x=0,\ \frac{b}{1+b^2}$$

Thus, the area enclosed between the parabolas,

$$A=\int_0^{b/1+b^2}\left(x-bx^2-\frac{x^2}{b}\right)dx$$

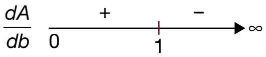
$$A=\left\{\frac{x^2}{2}-\frac{x^3}{3}\cdot\frac{1+b^2}{b}\right\}_0^{b/1+b^2}=\frac{1}{6}\cdot\frac{b^2}{(1+b^2)^2}$$

For maximum value of A, $\frac{dA}{db}=0$

But $\frac{dA}{db}=\frac{1}{6}\cdot\frac{(1+b^2)^2.2b-2b^2.(1+b^2).2b}{(1+b^2)^4}=\frac{1}{3}\cdot\frac{b(1-b^2)}{(1+b^2)^3}$

Hence, $\frac{dA}{db}=0$ gives $b = -1, 0, 1$ since $b > 0$

$\therefore$ We consider only $b = 1$



Sign scheme for $\frac{dA}{db}$ around $b = 1$ is as below

From first derivative test, it is clear, A is maximum at $b = 1$.

Test Your Concepts-IV

(Solutions on page H.347)

1. Find the area bounded by the curve $x = a\cos^3 t$, $y = a\sin^3 t$ is
2. Find the area bounded by the curves $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$, is
3. Find area enclosed by the curve $y = f(x)$ defined parametrically as $x = \frac{1-t^2}{1+t^2}$, $y = \frac{2t}{1+t^2}$
4. For what value of 'a' is the area bounded by the curve $y = a^2x^2 + ax + 1$ and the straight line $y = 0$, $x = 0$ and $x = 1$ the least?
5. If the area enclosed by the parabolas $y = a - x^2$ and $y = x^2$ is $18\sqrt{2}$ sq. units. Find the value of 'a'.
6. Consider the curve $y = x^n$ where $n > 1$ in the 1st quadrant. If the area bounded by the curve, the x-axis and the tangent line to the graph of $y = x^n$ at the point (1, 1) is maximum then find the value of n.
7. Find the area bounded by the curves $\{(x, y) : y \ge x^2, y \le |x|\}$
8. Find the area of the loop of the curve, $xy^2 + (x + a)^2 (x + 2a) = 0$
9. Find the smaller area enclosed by the curve $y = f(x)$ where $f(x)$ is polynomial of least degree satisfying $\lim_{x\to 0}\left\{1+\frac{f(x)}{x^3}\right\}^{1/x} = e$ and the circle $x^2 + y^2 = 2$ above the x-axis.
10. Find, by using integration process, the area of the ellipse $ax^2 + 2hxy + by^2 = 1$.
11. Let A_n be the area enclosed by the curve $y = x^n$ $(n \ge 1)$, ordinate at $x = \frac{1}{2}$ and $y = 0$. If $\sum_{n=1}^{n}\frac{2^n A_n}{n} = \frac{1}{3}$ then find the value of n.
12. Find positive value of the parameter 'a' for which the area of the figure bounded by the curve $y = \cos ax$, $y = 0$, $x = \frac{\pi}{6a}$ and $x = \frac{5\pi}{6a}$ is greater than 3

Answer key

1. $\frac{3\pi a^2}{8}$ sq. units

2. $3\pi a^2$ sq. units

3. π sq. units

4. $\frac{3}{4}$

5. 9

6. $\sqrt{2}+1$

7. $\frac{1}{3}$ sq. units

8. $\frac{1}{2}a^2(4 - \pi)$ sq. units

9. $\left(\frac{\pi}{2}+\frac{3}{5}\right)$ sq. units

10. $\frac{\pi}{\sqrt{(ab-h^2)}}$ Sq. units

11. 2

12. $\left(0, \frac{1}{3}\right)$

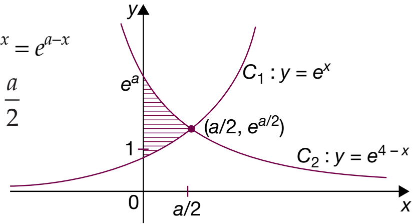
SOLVED PROBLEMS

PROBLEM 1

Let 'a' be a positive constant number. Consider two curves $C_1 : y = e^x$, $C_2 : y = e^{a-x}$. Let S be the area of the part surrounding by C_1, C_2 and the y-axis, then find $\lim\limits_{a\to 0} \dfrac{S}{a^2}$

SOLUTION

Point of intersection of given curves $y = e^x = e^{a-x}$



$$x = a - x \Rightarrow x = \frac{a}{2}$$

$$y = e^x$$

$$\Rightarrow \quad x = \ln y$$

$$y = e^{a-x} \Rightarrow x = a - \ln y$$

Required Area $S = \displaystyle\int_1^{e^{a/2}} \ln y\, dy + \int_{e^{a/2}}^{e^a} a - \ln y\, dy$

$$S = \left[y\ln(y)\right]_1^{e^{a/2}} - \int_1^{e^{a/2}} 1dy + \left[ay\right]_{e^{a/2}}^{e^a} - \int_{e^{a/2}}^{e^a} \ln y\, dy$$

$$S = \frac{a}{2}e^{a/2} - 0 - (e^{a/2} - 1) + a(e^a - e^{a/2}) - \left[\left[y\ln(y)\right]_{e^{a/2}}^{e^a} - \int_{e^{a/2}}^{e^a} 1dy\right]$$

$$S = \frac{a}{2}e^{a/2} - e^{a/2} + 1 + ae^a - ae^{a/2} - [ae^a - \frac{a}{2}e^{a/2} - (e^a - e^{a/2})]$$

$$S = 1 + e^a - 2e^{a/2} = (e^{a/2} - 1)^2$$

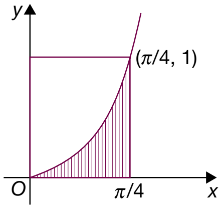
Now $\lim\limits_{a\to 0}\dfrac{S}{a^2} = \dfrac{1}{4}\lim\limits_{x\to 0}\left(\dfrac{e^{a/2} - 1}{\frac{a}{2}}\right)^2 = \dfrac{1}{4}$

PROBLEM 2

Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \dfrac{\pi}{4}$.

Prove that for $n > 2$, $A_n + A_{n\text{-}2} = \dfrac{1}{(n-1)}$ and deduce that $\dfrac{1}{(2n+2)} < A_n < \dfrac{1}{(2n-2)}$.

SOLUTION



$$y = (\tan x)^n;\ x = 0;\ y = 0;\ x = \frac{\pi}{4}$$

Required Area $A_n = \displaystyle\int_0^{\pi/4} (\tan x)^n\, dx$

$$= \int_0^{\pi/4} (\tan x)^{n-2}\,(\sec^2 x - 1)\, dx$$

$$\Rightarrow \quad A_n = \int_0^{\pi/4} (\tan x)^{n-2}\sec^2 x\, dx - A_{n-2}$$

$$\Rightarrow \quad A_n + A_{n-2} = \frac{1}{n-1} \qquad \ldots(1)$$

As $0 < x < \dfrac{\pi}{4}$, $0 < \tan x < 1$

$$\Rightarrow \quad (\tan x)^{n+2} < (\tan x)^n < (\tan x)^{n-2}$$

$$\Rightarrow \quad A_{n+2} < A_n < A_{n-2}$$

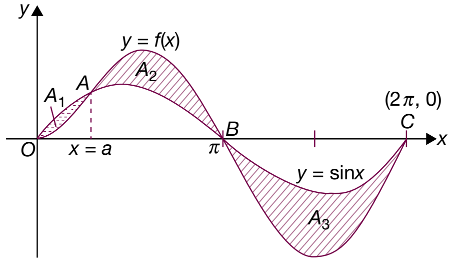
$$\Rightarrow \quad A_{n+2} + A_n < 2A_n < A_n + A_{n-2}$$

$$\Rightarrow \quad \frac{1}{(n+1)} < 2A_n < \frac{1}{(n-1)} \qquad \text{Using (1)}$$

$$\Rightarrow \quad \frac{1}{(2n+2)} < A_n < \frac{1}{(2n-2)} \qquad \text{(Hence Proved)}$$

PROBLEM 3

In the adjacent figure, graphs of two functions $y = f(x)$ and $y = \sin x$ are given. $y = \sin x$ intersects, $y = f(x)$ at $A\ (a, f(a))$; $B(\pi, 0)$ and $C(2\pi, 0)$. A_i $(i = 1, 2, 3)$ is the area bounded by the curves $y = f(x)$ and $y = \sin x$ between $x = 0$ and $x = a$; $i = 1$, between $x = a$ and $x = \pi$; $i = 2$, between $x = \pi$ and $x = 2\pi$; $i = 3$. If $A_1 = 1 - \sin a + (a - 1)\cos a$, determine the function $f(x)$. Hence determine 'a' and A_1. Also calculate A_2 and A_3.



SOLUTION

Given, $A_1 = 1 - \sin a + (a - 1)\cos a$

$$A_1 = \int_0^a [\sin x - f(x)]dx = (-\cos x)_0^a - \int_0^a f(x)dx$$

$$A_1 = -[\cos a - 1] - \int_0^a f(x)dx$$

$$\therefore \quad 1 - \sin a + a\cos a - \cos a = -\cos a + 1 - \int_0^a f(x)dx$$

$\Rightarrow \quad a\cos a - \sin a = -\int_0^a f(x)\,dx$ on differentiation

$\Rightarrow \quad -a\sin a + \cos a - \cos a = -f(a)$

$\Rightarrow \quad f(a) = a\sin a \Rightarrow f(x) = x\sin x$

At $x = a$ both curves $y = x\sin x$ and $y = \sin x$ meet

$a\sin a = \sin a \Rightarrow a = 1$

Also, $A_2 = \int_1^{\pi} x\sin x - \sin x\,dx = \int_1^{\pi} (x-1)\sin x dx$

$$A_2 = \left[(x-1)(-\cos x)\right]_1^{\pi} - \int_1^{\pi} -\cos x\,dx$$

$$= \left[-(x-1)\cos x + \sin x\right]_1^{\pi}$$

$$= \left(\sin x - (x-1)\cos x\right)_1^{\pi} = [0 - (\pi - 1)\,(-1)] - [\sin 1 - 0]$$

$$A_2 = \pi - 1 - \sin 1$$

Also, $A_3 = \int_{\pi}^{2\pi} \sin x - x\sin x dx = -\int_{\pi}^{2\pi} (x-1)\sin x dx$

$$= \left[(x-1)\cos x - \sin x\right]_{\pi}^{2\pi}$$

$$= [(2\pi - 1)(1) - 0] - [(\pi - 1)(-1) - 0]$$

$$= 2\pi - 1 + \pi - 1 = 3\pi - 2 \text{ sq. units}$$

PROBLEM 4

A figure is bounded by the curves $y = \left|\sqrt{2}\sin\dfrac{\pi x}{4}\right|$, $y = 0$, $x = 2$ and $x = 4$. At what angles to the positive x-axis straight lines must be drawn through $(4, 0)$ so that these lines partition the figure into three parts of the same size.

SOLUTION

Total area $= \int_2^4 \sqrt{2}\sin\dfrac{\pi x}{4}\,dx = \sqrt{2}\left(\dfrac{4}{\pi}\right)\left[-\cos\dfrac{\pi x}{4}\right]_2^4$

$$A = 4\sqrt{2}/\pi$$

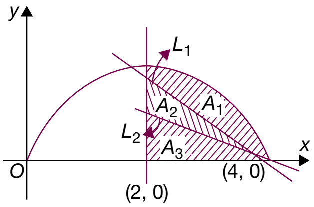
Lines from point (4, 0)

$$y = m(x - 4)$$

Line L_1: $y = m_1(x - 4)$; L_2 : $y = m_2(x - 4)$

Area enclosed by L_2 with x-axis is $\dfrac{1}{3}$ of total area

$$A_3 = \frac{4\sqrt{2}}{3\pi} = \frac{1}{2}2(-2m_2) = -2m_2$$



$$m_2 = \tan\theta_2 = -\frac{2\sqrt{2}}{3\pi}$$

$\Rightarrow \quad \theta_2 = \pi - \tan^{-1}\left(\dfrac{2\sqrt{2}}{3\pi}\right)$

Now computing area
$A_2 =$ area enclosed by both lines between $x = 2$ and $x = 4$

$\therefore \quad \int_2^4 [m_1(x-4) - m_2(x-4)]\,dx = \dfrac{4\sqrt{2}}{3\pi}$

$\Rightarrow \quad (m_1 - m_2)\int_2^4 (x-4)\,dx = (m_1 - m_2)\left[\dfrac{x^2}{2} - 4x\right]_2^4 = \dfrac{4\sqrt{2}}{3\pi}$

$\Rightarrow \quad (m_1 - m_2)\,[(8 - 16) - (2 - 8)] = \dfrac{4\sqrt{2}}{3\pi}$

$\Rightarrow \quad m_2 - m_1 = \dfrac{2\sqrt{2}}{3\pi} \qquad \ldots(1)$

As $m_2 = -\dfrac{2\sqrt{2}}{3\pi} \qquad \therefore\ m_1 = -\dfrac{4\sqrt{2}}{3\pi}$

$\Rightarrow \quad \tan\theta_1 = -\dfrac{4\sqrt{2}}{3\pi} \Rightarrow \theta_1 = \pi - \tan\left(\dfrac{4\sqrt{2}}{3\pi}\right)$

Required angles of lines made with x-axis are

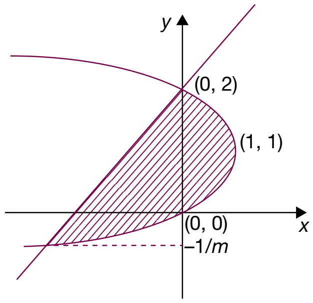
$$\pi - \tan^{-1}\frac{2\sqrt{2}}{3\pi};\ \pi - \tan^{-1}\frac{4\sqrt{2}}{3\pi}$$

PROBLEM 5

Find the values of m $(m > 0)$ for which the area bounded by the line $y = mx + 2$ and $x = 2y - y^2$ is, (i) $\dfrac{9}{2}$ square units and (iii) minimum. Also find the minimum area.

SOLUTION

Given line $y = mx + 2 \Rightarrow x = y - \dfrac{2}{m}$



Curve $x = y(2 - y) = \dfrac{(y-2)}{m}$

$\Rightarrow \quad y = -\dfrac{1}{m}$ and $y = 2$

are point of intersection

Curve $x = 2y - y^2$

$\Rightarrow \quad x - 1 = -(y^2 - 2y + 1)$

$\Rightarrow \quad x = 1 - (y - 1)^2$

$\therefore \quad$ Required area $A = \int_{-1/m}^{2} \left[1 - (y-1)^2 - \dfrac{(y-2)}{m}\right]dy$

$$A=\left[y-\frac{(y-1)^3}{3}-\frac{(y-2)^2}{2m}\right]_{-1/m}^{2}$$

$$A=\left(2+\frac{1}{m}\right)-\frac{1}{3}\left(1+\frac{(1+m)^3}{m^3}\right)-\frac{1}{2m}\left[0-\left(\frac{(1+2m)}{m}\right)^2\right]$$

$$A=\frac{4}{3}+\frac{1}{m^2}+\frac{1}{6m^3}+\frac{2}{m}$$

(i) $A=\frac{4}{3}+\frac{1}{m^2}+\frac{1}{6m^3}+\frac{2}{m}=\frac{9}{2}$

$$\Rightarrow \frac{1}{m^2}+\frac{1}{6m^3}+\frac{2}{m}=\frac{19}{6}\Rightarrow m=1$$

(ii) For $A_{\min}=\frac{dA}{dm}=0\Rightarrow -\frac{2}{m^3}+\left(\frac{1}{6}\right)\left(-\frac{3}{m^4}\right)-\frac{2}{m^2}=0$

$$\Rightarrow 4m^2+4m+1=0\Rightarrow(2m+1)^2=0\Rightarrow m=-\frac{1}{2}\text{ or } m=\infty$$

Not accepted, as $m\neq-\frac{1}{2}$ as $m>0$

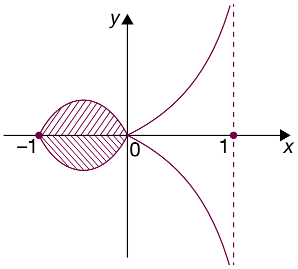
$\therefore\ m=\infty$

$$A_{\min}=\frac{4}{3}+0=\frac{4}{3}$$

PROBLEM 6

Compute the area of the loop of the curve $y^2=x^2\left(\frac{1+x}{1-x}\right)$.

SOLUTION



curve $y^2=x^2\left(\frac{1+x}{1-x}\right)$ is symmetric about x-axis and $x=1$ is an asymptote

Required Area

$$A=2\int_{-1}^{0}\left(x\sqrt{\frac{1+x}{1-x}}\right)dx$$

$$A=-2\int_{-1}^{0}\left(\frac{x}{1-x}\right)\sqrt{1-x^2}\,dx$$

Let $x=\sin\theta$; $dx=\cos\theta\,d\theta$

$$=-2\int_{-\pi/2}^{0}\frac{(\sin\theta)}{(1-\sin\theta)}\cos\theta\cdot\cos\theta\,d\theta$$

$$=2\int_{-\pi/2}^{0}(\sin\theta+\sin^2\theta)\,d\theta$$

$$=-2\left(\left(-\cos\theta\right)_{-\pi/2}^{0}+\int_{-\pi/2}^{0}\frac{(1-\cos 2\theta)}{2}d\theta\right)$$

$$=-2\left(-\cos\theta+\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right)_{-\pi/2}^{0}=2-\frac{\pi}{2}\text{ sq. unit}$$

PROBLEM 7

Consider the collection of all curves of the form $y=a-bx^2$ that pass through the the point (2, 1), where a and b are positive constants. Determine the value of a and b that will minimize the area of the region bounded by $y=a-bx^2$ and x-axis. Also find the minimum area.

SOLUTION

$y=a-bx^2$ point (2, 1) lies on it then $1=a-4b$ …(1)

$\Rightarrow\quad b=\frac{a-1}{4}$ …(1)

As $a>0$, $b>0$

Curve cuts x-axis at points $x=-\sqrt{\frac{a}{b}},\ \sqrt{\frac{a}{b}}$

Required area $A=\int_{-\sqrt{a/b}}^{\sqrt{a/b}}(a-bx^2)\,dx$

$$\Rightarrow\quad A=2\int_{0}^{\sqrt{a/b}}(a-bx^2)\,dx\ \Rightarrow a=1+4b$$

$$A=2\left(ax-b\frac{x^3}{3}\right)_0^{\sqrt{a/b}}=2\left[a\left(\sqrt{\frac{a}{b}}\right)-\frac{b}{3}\left(\frac{a}{b}\sqrt{\frac{a}{b}}\right)\right]$$

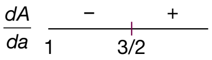
$$A=2\sqrt{\frac{a}{b}}\left[a-\frac{a}{3}\right]=\frac{4a}{3}\sqrt{\frac{a}{b}}\qquad\text{From (1)}$$

$$A=\frac{4a}{3}\sqrt{\frac{4a}{a-1}}=\frac{8a}{3}\sqrt{\frac{a}{a-1}}$$

$$\because\quad A=\frac{8a^{3/2}}{3(a-1)^{1/2}}$$

$$\frac{dA}{da}=\frac{8\left((a-1)^{1/2}\frac{3}{2}a^{1/2}-a^{3/2}\left(\frac{1}{2}\right)(a-1)^{-1/2}\right)}{3(a-1)}=0$$

$$\frac{dA}{da}=\frac{8\sqrt{a}(2a-3)}{3(a-1)}$$

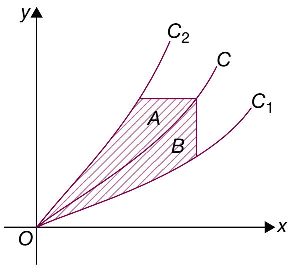


$$\Rightarrow\quad 3a-3-a=0\Rightarrow a=\frac{3}{2},\ b=\frac{1}{8}$$

$$\therefore\quad A_{\min}=\frac{4a}{3}\sqrt{\frac{a}{b}}=\frac{4}{3}\left(\frac{3}{2}\right)\sqrt{\frac{3/2}{1/8}}=4\sqrt{3}\text{ sq. unit}$$

PROBLEM 8

Let C_1 and C_2 be two curves passing through the origin as shown in the figure. A curve C is said to "bisect the area" the region between C_1 and C_2, if for each point P of C, the two shaded regions A and B shown in the figure have equal areas. Determine the upper curve C_2, given that the bisecting curve C has the equation $y = x^2$ and that the lower curve C_1 has the equation $y = \frac{x^2}{2}$.



SOLUTION

Let curve C_2, $y = f(x)$

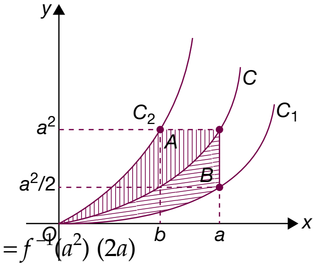
Given, $A = B$

$$\int_0^{a^2} \sqrt{y} - f^{-1}(y)\,dy = \int_0^a \left(x^2 - \frac{x^2}{2}\right) dx$$

$$\Rightarrow \quad \frac{2}{3} y^{3/2}\Big|_0^{a^2} - \int_0^{a^2} f^{-1}(y)\,dy = \int_0^a \frac{x^2}{2}\,dx = \frac{a^3}{6}$$

$$\Rightarrow \quad \frac{2}{3} a^3 - \int_0^{a^2} f^{-1}(y)\,dy = \frac{a^3}{6}$$

$$\Rightarrow \quad \frac{a^3}{2} = \int_0^{a^2} f^{-1}(y)\,dy$$



Using Leibnitz theorem, $\frac{3}{2} a^2 = f^{-1}(a^2)\,(2a)$

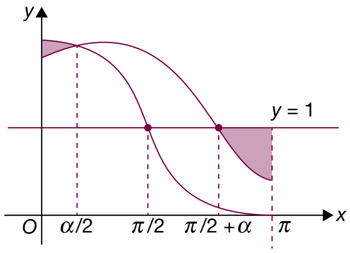
$$\Rightarrow \quad f^{-1}(a^2) = \frac{3}{4} a \Rightarrow f\left(\frac{3a}{4}\right) = a^2 \Rightarrow f(a) = \left(\frac{4}{3}\right)^2 a^2$$

$$\Rightarrow \quad f(a) = \frac{16}{9} a^2 \Rightarrow f(x) = \frac{16}{3} x^2$$

PROBLEM 9

Consider the two curves C_1: $y = 1 + \cos x$ and C_2: $y = 1 + \cos(x - \alpha)$ for $\alpha \in \left(0, \frac{\pi}{2}\right)$; $x \in [0, \pi]$. Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 and $x = 0$ is same as that of the figure bounded by C_2, $y = 1$ and $x = \pi$. For this value of α, find the ratio in which the line $y = 1$ divides the area of the figure by the curves C_1, C_2 and $x = \pi$.

SOLUTION



Given curves

$y = 1 + \cos x$ and

$y = 1 + \cos(x - \alpha)$

Point of intersection of curves

$$y = 1 + \cos(x - \alpha)$$
$$= 1 + \cos x$$

$$\Rightarrow \quad x - \alpha = -x \Rightarrow x = \frac{\alpha}{2}$$

Area of both shaded region is equal

$$\therefore \quad \int_0^{\alpha/2} [\cos x - \cos(x - \alpha)]\,dx = \int_{\pi/2+\alpha}^{\pi} [1 - 1 - \cos(x - \alpha)]\,dx$$

$$\Rightarrow \quad [\sin x - \sin(x - \alpha)]_0^{\alpha/2} = -\left(\sin(x - \alpha)\right)_{\pi/2+\alpha}^{\pi}$$

$$\Rightarrow \quad \left(\sin\frac{\alpha}{2} + \sin\frac{\alpha}{2}\right) - (\sin\alpha) = -[\sin\alpha - 1] \qquad \ldots(1)$$

$$\Rightarrow \quad \sin\frac{\alpha}{2} = \frac{1}{2} \Rightarrow \frac{\alpha}{2} = \frac{\pi}{6} \Rightarrow \alpha = \frac{\pi}{3}.$$

$\therefore$ Curves $y = \cos x$ and $y = \cos\left(x - \frac{\pi}{3}\right)$

Total area from $x = \frac{\pi}{6}$ to $x = \pi$ bounded by C_1 and C_2

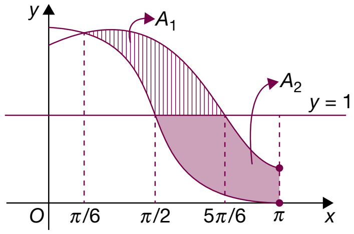
$$\Rightarrow \quad A = \int_{\pi/6}^{\pi} \left(\cos\left(x - \frac{\pi}{3}\right) - \cos x\right) dx$$

$$= \left(\sin\left(x - \frac{\pi}{3}\right) - \sin x\right)_{\pi/6}^{\pi}$$

$$= \left(\sin\frac{2\pi}{3} - 0\right) - \left(-\sin\frac{\pi}{6} - \sin\frac{\pi}{6}\right)$$

$$\therefore \quad A = \sin\frac{2\pi}{3} + 2\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + 1$$

The line $y = 1$ divides the area of the figure by the curves C_1, C_2 and $x = \pi$.



Shaded area = A_1

$$= \int_{\pi/6}^{\pi/2} [\cos\left(x - \frac{\pi}{3}\right) - \cos x]\,dx + \int_{\pi/2}^{5\pi/6} [1 + \cos\left(x - \frac{\pi}{3}\right) - 1]\,dx$$

$$= \left(\sin\left(x - \frac{\pi}{3}\right) - \sin x\right)_{\pi/6}^{\pi/2} + \left(\sin\left(x - \frac{\pi}{3}\right)\right)_{\pi/2}^{5\pi/6}$$

$$= \left(\frac{1}{2} - 1 + \frac{1}{2} + \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right) = 1$$

$$A = A_1 + A_2$$

$$\Rightarrow \quad \frac{\sqrt{3}}{2} + 1 = 1 + A_2 \Rightarrow A_2 = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \text{Required ratio } \frac{A_1}{A_2} = \frac{2}{\sqrt{3}}$$

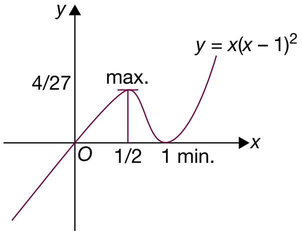
PROBLEM 10

Find all maxima and minima of the function $y = x(x - 1)^2$, $0 \le x \le 2$. Also determine the area bounded by the curve $y = x(x - 1)^2$, the x-axis and the line $x = 2$.

SOLUTION

Given curve $y = x(x - 1)^2$

$$\Rightarrow \quad \frac{dy}{dx} = x.2(x - 1) + (x - 1)^2$$



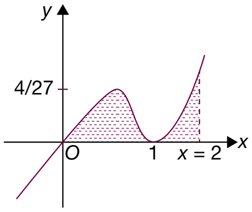
$$= (x - 1).\{2x + x - 1\}$$

$$= (x - 1)\,(3x - 1)$$

$$\therefore \quad \text{Maximum at } x = 1/3 \Rightarrow y_{\max} = \frac{1}{3}\left(-\frac{2}{3}\right)^2 = \frac{4}{27}$$

Minimum at $x = 1 \Rightarrow y_{\min} = 0$

Now, to find the area bounded by the curve $y = x\,(x - 1)^2$, the y-axis and line $x = 2$.



$$\text{Required area} = \int_0^2 x(x - 1)^2\,.dx$$

$$= \int_0^2 (x^3 - 2x^2 + x)\,.dx$$

$$= \left(\frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2}\right)_0^2$$

$$= \left(4 - \frac{16}{3} + 2\right) = \frac{2}{3} \text{ sq. units.}$$

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. A point P lying inside the curve $y = \sqrt{2ax - x^2}$ in first quadrant moving such that its shortest distance from the curve at any position is greater than its distance from x-axis. The point P enclose a region whose area is equal to $(a > 0)$

(A) $\frac{\pi a^2}{2}$ (B) $\frac{a^2}{3}$

(C) $\frac{2a^2}{3}$ (D) $\frac{3a^2}{2}$

2. Let $0 \le a \le 4$, if the maximum area bounded by the curves $y = 1 - |x - 1|$ and $y = |2x - a|$ is 'A' then -

(A) $A = \frac{1}{3}$ (B) $A = \frac{1}{4}$

(C) $A = \frac{1}{2}$ (D) None of these

3. The area bounded by curve $f(x) = ||\tan x + \cot x| - |\tan x - \cot x||$ between the lines $x = 0$, $x = \frac{\pi}{2}$ and x-axis is -

(A) $\ln 4$ (B) $\ln\sqrt{2}$

(C) $2 \ln \sqrt{2}$ (D) None of these

4. The area bounded by $f(x) = \tan^{-1}(\tan x) + \cos^{-1}(\cos x)$ and x-axis when $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ will be –

(A) π^2 (B) $\frac{3\pi^2}{4}$

(C) $\frac{\pi^2}{4}$ (D) $2\pi^2$

5. $f : [0, \infty) \to [0, \infty)$, $f(x)$ is a function such that area bounded between the curve, x-axis and $x = t$ is $\frac{1}{2}$ of the area bounded by triangle having $(0, 0)$, $(t, 0)$ and $(t, f(t))$ as its vertices. $f(x)$ can be

(A) x (B) x^2

(C) x^3 (D) $\sqrt{x}$

6. A point P moves inside a square of area 4 square units such that it is nearer to point of intersection of its diagonal than any of the vertex. Area of the region traced by P is -

(A) 4 (B) 2

(C) 3 (D) 1

7. If $(a, 0)$; $a > 0$ is the point when the curve $y = \sin 2x - \sqrt{3} \sin x$ cuts the x-axis first, A is the area bounded by this part of the curve the origin and the positive x-axis, then -

(A) $4A + 8 \cos a = 7$ (B) $4A + 8 \sin a = 7$

(C) $4A - 8 \sin a = 7$ (D) $4A - 8 \cos a = 7$

8. The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and its tangent at origin is the line $y = x$. The area bounded by the curve the ordinate of the curve at minima and the tangent line is -

(A) $\frac{1}{24}$ (B) $\frac{1}{12}$

(C) $\frac{1}{8}$ (D) $\frac{1}{6}$

9. The area bounded by the curve $y = f(x)$, the co-ordinate axes and the line $x = x_1$ is given by $x_1 . e^{x_1}$. Therefore $f(x)$ equals :

(A) e^x (B) $x e^x$

(C) $xe^x - e^x$ (D) $x e^x + e^x$

10. The slope of the tangent to a cure $y = f(x)$ at $(x, f(x))$ is $2x + 1$. If the curve passes through the point $(1, 2)$ then the area of the region bounded by the curve, the x-axis and between the line $x = 0$ to $x = 1$ is -

(A) $\frac{5}{6}$ (B) $\frac{6}{5}$

(C) $\frac{1}{6}$ (D) 1

11. The area bounded by the x-axis and the part of graph of $y = \cos x$ between $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ is separated into two regions by the line $x = k$. If the area of the region for $-\frac{\pi}{2} \le x \le k$ is three times the area of the region for $k \le x \le \frac{\pi}{2}$, then k is equal to

(A) $\arcsin\left(\frac{1}{4}\right)$ (B) $\arcsin\left(\frac{1}{3}\right)$

(C) $\frac{\pi}{6}$ (D) $\frac{\pi}{3}$

12. The line $y = mx$ bisects the area enclosed by the curve $y = 1 + 4x - x^2$ and the lines $x = 0$, $x = \frac{3}{2}$ and $y = 0$. Then the value of m is -

(A) $\frac{13}{6}$ (B) $\frac{6}{13}$

(C) $\frac{3}{2}$ (D) 4

13. The area defined by $|y| \le e^{-|x|} - \frac{1}{2}$ in Cartesian co-ordinate system, is -

(A) $(4 - 2\ln 2)$ (B) $(4 - \ln 2)$

(C) $(2 - \ln 2)$ (D) $(2 - 2\ln 2)$

14. Area remains invariant under translation of co-ordinate axes. Using it find the area of the region described by the inequality $|3x - 18| + |2y + 7| \le 3$

(A) 3 (B) 2
(C) 4 (D) none

15. The area enclosed between the curves $y = \sin^4 x \cos^3 x$, $y = \sin^2 x \cos^3 x$ between $x = 0$ and $x = \dfrac{\pi}{2}$ is

(A) $\displaystyle\int_0^1 t^2(1-t^2)dt$ (B) $\displaystyle\int_0^1 t^2(1-t^2)^2 dt$
(C) $\displaystyle\int_0^1 t(1-t^2)^2 dt$ (D) $\displaystyle\frac{\pi}{2}\int_0^1 t(1-t^2)dt$

16. The values of the parameter $a(a \ge 1)$ for which the area of the figure bounded by the pair of straight lines $y^2 - 3y + 2 = 0$ and the curves $y = [a]\,x^2$, $y = \dfrac{1}{2}[a]x^2$ is greatest is. (Here [.] denotes the greatest integer function).

(A) [0, 1) (B) [1, 2)
(C) [2, 3) (D) [3, 4)

17. The area of figure bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$ is

(A) $\dfrac{9}{2}$ (B) 9
(C) $\dfrac{7}{2}$ (D) 7

18. The area bounded by the curve $y = \dfrac{e^x + e^{-x}}{2}$ and the lines $y = 0$, $x = \ln 2$ and $x = \ln \dfrac{1}{2}$ is

(A) $\dfrac{3}{4}$ (B) $\dfrac{5}{4}$
(C) $\dfrac{3}{2}$ (D) $\dfrac{5}{2}$

19. If $f(x) = \begin{cases} \sqrt{\{x\}}, & x \notin I \\ 1, & x \in I \end{cases}$ and $g(x) = \{x\}^2$ where $\{x\}$ is fractional part of x, then area bounded between $f(x)$ and $g(x)$ for $x \in [0, 10]$ is

(A) $\dfrac{5}{3}$ (B) 5
(C) $\dfrac{10}{3}$ (D) $\dfrac{8}{3}$

20. Let $f : [0, \infty) \to R$ be a continuous and strictly increasing function such that $f^3(x) = \displaystyle\int_0^x t f^2(t)dt$, $\forall\ x > 0$. The area enclosed by $y = f(x)$, the x-axis and the ordinate at $x = 3$, is

(A) 1 (B) 3/2
(C) 2 (D) 3

21. The area bounded by the curve $y = x^2$ and $y = \dfrac{2}{1+x^2}$ is λ sq. units, then the value of $[\lambda]$ where $[\cdot]$ is GIF is

(A) 2 (B) 3
(C) 4 (D) 5

22. The area of the region in the xy plane defined by $|y| < (|x| - 2)^2$ is

(A) 16/3 (B) 32/3
(C) 29/4 (D) 40/7

23. The area bounded by the curve $(y - \arcsin x)^2 = x - x^2$, is

(A) $\dfrac{\pi}{4}$ (B) $\dfrac{\pi}{2}$
(C) π (D) $\dfrac{\pi}{3}$

24. Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f(x)$, the x-axis and the two ordinates $x = 0$ and $x = a$ is $\dfrac{a^2}{2} + \dfrac{a}{2}\sin a + \dfrac{\pi}{2}\cos a$ then $f\left(\dfrac{\pi}{2}\right)$ is

(A) $\dfrac{1}{2}$ (B) $\dfrac{\pi^2}{8} + \dfrac{\pi}{4}$
(C) $\dfrac{\pi + 1}{2}$ (D) $\dfrac{\pi^2}{8} - \dfrac{\pi}{4}$

25. The area bounded by the curves $y = \sin^{-1}|\sin x|$ and $y = (\sin^{-1}|\sin x|)^2$, $0 < x < 2\pi$, is

(A) $\dfrac{1}{3} + \dfrac{\pi^2}{4}$ (B) $\dfrac{1}{6} + \dfrac{\pi^3}{8}$
(C) 2 (D) none of these

26. The area of the region $\left\{(x, y) : x^2 + y^2 \le 5, \left||x| - |y|\right| \ge 1\right\}$ is ________ (in sq. units)

(A) $4\left(\pi - \tan^{-1}\left(\dfrac{24}{7}\right)\right) - 4$ (B) $5\left(\pi - \tan^{-1}\left(\dfrac{24}{7}\right)\right) - 4$
(C) $3\left(\pi - \tan^{-1}\left(\dfrac{24}{7}\right)\right) - 4$ (D) $2\left(\pi - \tan^{-1}\left(\dfrac{24}{7}\right)\right) - 1$

27. The area of the region whose boundaries are defined by the curves $y = 2\cos x$, $y = 3\tan x$ and the y-axis, is

(A) $1 + 3\ln\left(\dfrac{2}{\sqrt{3}}\right)$ (B) $1 + \dfrac{3}{2}\ln 3 - 3\ln 2$
(C) $1 + \dfrac{3}{2}\ln 3 - \ln 2$ (D) $\ln 3 - \ln 2$

28. The area of the region bounded by the curve $y = x^2$ and $y = \sec^{-1}[-\sin^2 x]$, (where [.] denotes the greatest integer function), is

(A) $\pi\sqrt{\pi}$ (B) $\dfrac{4}{3}\pi\sqrt{\pi}$
(C) $\dfrac{2}{3}\pi\sqrt{\pi}$ (D) $\dfrac{1}{3}\pi\sqrt{\pi}$

29. The area of the loop of the curve $y^2 = x^4(x + 2)$ is [in square units]

(A) $\dfrac{32\sqrt{2}}{105}$ (B) $\dfrac{64\sqrt{2}}{105}$
(C) $\dfrac{128\sqrt{2}}{105}$ (D) $\dfrac{256\sqrt{2}}{105}$

30. The area of the smaller portion enclosed by the curves $x^2 + y^2 = 9$ and $y^2 = 8x$ is

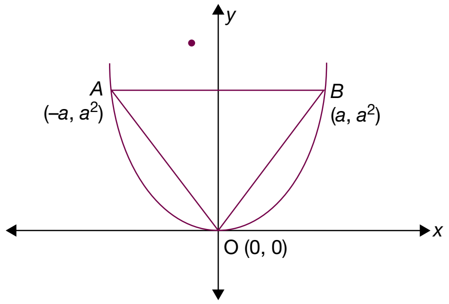
(A) $\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$

(B) $2\left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$

(C) $2\left(\frac{\sqrt{2}}{3} + \frac{9\pi}{4} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right)$

(D) $\frac{\sqrt{2}}{3} + \frac{9\pi}{4} + \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)$

31. The figure shown a ΔAOB and the parabola $y = x^2$. The ratio of the area of the ΔAOB to the area of the region AOB of the parabola $y = x^2$, is equal to



(A) $\frac{3}{4}$ (B) $\frac{3}{5}$

(C) $\frac{5}{6}$ (D) $\frac{2}{3}$

32. The area of the domain of the function $f(x, y) = \sqrt{16 - x^2 - y^2} - \sqrt{|x| - y}$ is $k\pi$ where k equals

(A) 8 (B) 9
(C) 10 (D) 12

33. The area bounded by the curve $y = |\cos x - \sin x|$, $0 \le x \le \frac{\pi}{2}$ and above x-axis is

(A) $2\sqrt{2}$ (B) $2\sqrt{2} - 2$

(C) $2\sqrt{2} + 1$ (D) $2\sqrt{2} + 2$

34. Area of region bounded by $x = 0, y = 0, x = 2, y = 2, y\ e^x$ and $y \ge \ln x$ is

(A) $6 - 4 \ln 2$ (B) $4 \ln 2 - 2$
(C) $2 \ln 2 - 4$ (D) $6\ 2 - \ln 2$

35. Area enclosed by the curve $y = (x^2 + 2x)e^{-x}$ and the positive x-axis is

(A) 1 (B) 2
(C) 4 (D) 6

36. Let A_n be the area of region bounded by a curve $y = x^3(1 - x^2)^n$, $0 \le x \le 1$ and the x-axis, then the value of $\sum_{n=1}^{\infty} A_n$ is equal to

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) $\frac{1}{4}$ (D) 1

37. The area bounded by curves $y = \text{cosec}\ x$, $y = \sec x$, $y = \cos x$ and $y = \sin x$ on interval $\left(0, \frac{\pi}{2}\right)$ is equal to

(A) $1 + \ln(\sqrt{2} + 1)$ (B) $2 \ln(\sqrt{2} + 1)$

(C) $2 \ln(\sqrt{2} + 1) - \sqrt{2}$ (D) $\ln(\sqrt{2} + 1) - \frac{1}{\sqrt{2}}$

38. The area of the region in the xy-plane defined by the inequalities $x - 2y^2 \ge 0$, $1 - x - |y| \ge 0$ is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$

(C) $\frac{1}{4}$ (D) $\frac{7}{12}$

39. Area of the region in which point $p(x, y)$, $\{x > 0\}$ lies; such that $y \le \sqrt{16 - x^2}$ and $\left|\tan^{-1}\left(\frac{y}{x}\right)\right| \le \frac{\pi}{3}$ is

(A) $\left(\frac{16}{3}\pi\right)$ (B) $\left(\frac{8\pi}{3} + 8\sqrt{3}\right)$

(C) $\left(4\sqrt{3} - \pi\right)$ (D) $\left(\sqrt{3} - \pi\right)$

40. Area bounded between the curves $y = \sqrt{4 - x^2}$ and $y^2 = 3|x|$ is/are

(A) $\frac{\pi - 1}{\sqrt{3}}$ (B) $\frac{2\pi - 1}{3\sqrt{3}}$

(C) $\frac{2\pi - \sqrt{3}}{3}$ (D) $\frac{2\pi - \sqrt{3}}{3\sqrt{3}}$

41. Let $f : [0, \infty) \to R$ be a continuous and strictly increasing function such that $f^3(x) = \int_0^x t.f^2(t)\ dt$, $\forall\ x > 0$. The area enclosed by $y = f(x)$, the x-axis and the ordinate at $x = 3$, is

(A) 1 (B) $\frac{3}{2}$

(C) 2 (D) 3

42. The area of the region bounded by the curves $|x + y| \le 2$, $|x - y| \le 2$ and $2x^2 + 6y^2 \ge 3$ is

(A) $\left(8 + \frac{\sqrt{3}}{2}\pi\right)$ sq. units (B) $\left(8 - \frac{\sqrt{3}}{2}\pi\right)$ sq. units

(C) $\left(4 - \frac{3\sqrt{3}}{2}\pi\right)$ sq. units (D) $\left(8 - \frac{3\sqrt{3}}{2}\pi\right)$ sq. units

43. Let $F(x) = \sin x \int_0^x \cos t\, dt + 2\int_0^x t\, dt + \cos^2 x - x^2$. Then area bounded by $xF(x)$ and ordinate $x = 0$ and $x = 5$ with x-axis is

(A) 16
(B) $\frac{25}{2}$
(C) $\frac{35}{2}$
(D) 25

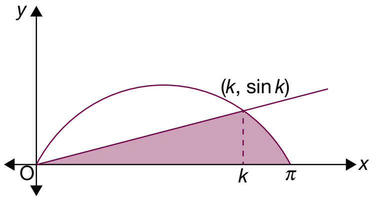
44. The curve $y = (|x| - 1)\text{sgn}(x - 1)$ divides $\frac{9x^2}{64} + \frac{4}{25}y^2 = \frac{1}{\pi}$ in two parts having area A_1 and A_2 (where $A_1 < A_2$), then

(A) $\frac{A_1}{A_2} = \frac{7}{13}$
(B) $\frac{A_1}{A_2} = \frac{3}{7}$
(C) $A_1 = \frac{7}{3}$
(D) $A_2 = \frac{13}{7}$

45. Area bounded by the circle which is concentric with the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ and which passes through $\left(4, -\frac{9}{5}\right)$, the vertical chord common to both circle and ellipse on the positive side of x-axis is

(A) $\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right) - \frac{36}{5}$
(B) $2\tan^{-1}\left(\frac{9}{20}\right)$
(C) $\frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right)$
(D) none of these

46. In the shown figure, half a period of sin x from 0 to π is split into two regions (light and dark shaded) of equal area by a line through the origin. If the line and the sine function intersect at a point whose x co-ordinate is k, then k satisfies the equation



(A) $k\cos k + 2\sin k = 0$
(B) $k\cos k + 2\cos k = 0$
(C) $k\cos k - 2\cos k = 0$
(D) $k\cos k - 2\sin k = 0$

47. The area bounded by the lines $y = 2$, $x = 1$, $x = a$ and the curve $y = f(x)$, which cuts the last two lines above the first line for all $a \geq 1$, is $\frac{2}{3}\left[(2a)^{\frac{3}{2}} - 3a + 3 - 2\sqrt{2}\right]$ then $f(x)$ is

(A) $2\sqrt{2x}; x \geq 1$
(B) $\sqrt{2x}; x \geq 1$
(C) $2\sqrt{x}; x \geq 1$
(D) $\frac{\sqrt{x}}{2}; x \geq 1$

48. If the line $x - 1 = 0$ divides the area bounded by the curve $2x + 1 = \sqrt{4y+1}$, $y = x$ and $y = 2$ in two regions of area A_1 and A_2 $(A_1 < A_2)$, then $(A_1^{-2} - A_2^{-2})$ is equal to

(A) 4
(B) 5
(C) 6
(D) 8

49. The area of the closed figure bounded by $x = -1$, $x = 2$ and $y = \begin{cases} -x^2 + 2, & x \leq 1 \\ 2x - 1, & x > 1 \end{cases}$ and the x-axis is

(A) 16 /3 sq. units
(B) 10/3 sq. units
(C) 13/3 sq. units
(D) 7/3 sq. units

50. The value of the parameter 'a' such that the area bounded by $y = a^2x^2 + ax + 1$, positive coordinate axes and the line $x = 1$ attains its least value, is equal to

(A) $-\frac{1}{4}$
(B) $-\frac{1}{2}$
(C) $-\frac{3}{4}$
(D) -1

51. The ratio of the areas in which the curve $y = \left[\frac{x^3}{100} + \frac{x}{35}\right]$ divides the circle $x^2 + y^2 - 4x + 2y + 1 = 0$, where $[x]$ denotes the greatest integer less than or equal to x is (where [.] denotes the greatest integer function).

(A) $\frac{4\pi - 3\sqrt{3}}{8\pi + 3\sqrt{3}}$
(B) $\frac{4\pi - 2\sqrt{3}}{8\pi + 2\sqrt{3}}$
(C) $\frac{4\pi + 2\sqrt{3}}{8\pi - 2\sqrt{3}}$
(D) $\frac{4\pi + 3\sqrt{3}}{8\pi - 3\sqrt{3}}$

52. A point P moves in xy – plane in such a way that $[|x|] + [|y|] = 1$, where [.] denotes the greatest integer function. Area of the region representing all possible positions of the point P is equal to

(A) 4 sq unit
(B) 16 sq unit
(C) $2\sqrt{2}$ sq unit
(D) 8 sq unit

53. Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$ is $\beta\sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta$. Then $f'\left(\frac{\pi}{2}\right)$ is

(A) $\left(\frac{\pi}{2} - \sqrt{2} - 1\right)$
(B) $\left(\frac{\pi}{2} + \sqrt{2} - 1\right)$
(C) $-\frac{\pi}{2}$
(D) $\left(1 - \frac{\pi}{4} + \sqrt{2}\right)$

54. Let A_r be the area bounded by the line $\frac{x}{r} + \frac{y}{(r+1)} = 1$ with coordinate axes, then $\sum_{r=1}^{n} A_r$ is equal to

(A) $\frac{n(n+1)(n+2)}{3}$
(B) $\frac{n(n+1)(n+2)}{4}$
(C) $\frac{n(n+1)(n+2)}{2}$
(D) $\frac{n(n+1)(n+2)}{6}$

55. Two lines drawn from the point $P(4, 0)$ divide the area bounded by the curve $y = \sqrt{2}\sin\frac{\pi x}{4}$ and x-axis between the lines $x = 2$ and $x = 4$ into there equal parts . sum of the slopes of the drawn lines is equal to

(A) $\frac{-2\sqrt{2}}{\pi}$
(B) $\frac{-\sqrt{2}}{\pi}$
(C) $\frac{-2}{\pi}$
(D) $\frac{-4\sqrt{2}}{\pi}$

56. The area bounded by min $(|x|,|y|)=2$ and max $(|x|,|y|)=4$ is:

(A) 4
(B) 8
(C) 16
(D) 32

57. If $f(x)=\begin{cases}\sqrt{\{x\}} & \text{for} \quad x\notin z \\ 1 & \text{for} \quad x\in z\end{cases}$ and $g(x)=\{x\}^2$ where {.} denotes fractional part of x then area bounded by $f(x)$ and $g(x)$ for $x\in[0,6]$ is

(A) $\frac{2}{3}$
(B) 2
(C) $\frac{10}{3}$
(D) 6

58. For $b > 0$. Let A_1 be the area bounded by $x = 0$, $x + y = 1$. $y = bx^2$ and A_2 be the area bounded by $y = 0$. $x + y = 1$. $y = bx^2$ such that $A_1 : A_2 = 11 : 16$, then the value of b is

(A) 2
(B) 3
(C) 4
(D) 6

59. Let $f(a) = \dfrac{1}{\ln\sqrt{1+a^2}}\displaystyle\int_0^a \frac{\ln(1+ax)}{1+x^2}dx$. Then area bounded by the curves $y = f(x)$ and $y = \dfrac{\pi}{2} - f(x)$ and the y-axis is

(A) ln 2
(B) $\frac{1}{2}$ ln 4
(C) ln 5
(D) ln 3

60. A polynomial $P(x)$ is positive for $x > 0$ and the area of the region bounded by $P(x)$, the x-axis and the vertical lines $x = 0$ and $x = \alpha(\alpha > 0)$ is $\dfrac{\alpha^2}{3}(\alpha + 3)$ then the value of $P(1)$ is equal to

(A) 1
(B) 2
(C) 3
(D) 4

61. Let $2a > -1$. If the area of the region of the plane defined by $\{(x, y) : x \ge 0, 2 - x \ge 0, ax + y - 3 \le 0\}$ is equal to 3, then the value of a, lies in

(A) $\left(\frac{1}{2},\frac{3}{4}\right)$
(B) $\left(\frac{3}{4},\frac{3}{2}\right)$
(C) $\left(\frac{3}{2},\frac{5}{2}\right)$
(D) $\left(\frac{5}{2},\frac{10}{3}\right)$

62. If $f(x)=x^2, g(x)=\cos x$ and $\alpha,\beta(\alpha<\beta)$ are the roots of the equation $18x^2-9\pi x+\pi^2=0$, then the area bounded by the curve $y=(fog)(x)$ and x-axis between the Lines $x=\alpha$ and $x=\beta$ is

(A) $\frac{1}{2}(\pi-3)$
(B) $\frac{\pi}{3}$
(C) $\frac{\pi}{12}$
(D) $\frac{\pi}{6}$

63. Area of the loop of the curve $x=a(1-t^2), y=at(1-t^2)$ for $-1\le t\le 1$ is

(A) $\frac{2}{15}a^2$
(B) $\frac{4}{15}a^2$
(C) $\frac{8}{15}a^2$
(D) $\frac{a^2}{5}$

64. Let $S_1=\left\{(x,y) \text{ where } \log_{4-x}(16x-y^2) \text{ is defined}\right\}$,

$S_2=\left\{(x,y) \text{ where } \dfrac{xy(x^2-3x+2)}{x^2-7x+12}>0\right\}$, then area bounded by $S_1\cap S_2$ is

(A) $\frac{8}{3}$
(B) $\frac{64}{3}$
(C) $\frac{32}{3}$
(D) none of these

65. If A denotes the area bounded by $f(x)=\left|\dfrac{\sin x+\cos x}{x}\right|$, x – axis, $x=\pi$ and $x=3\pi$, then

(A) $1 < A < 2$
(B) $0 < A < 2$
(C) $2 < A < 3$
(D) none of these

66. Let $P(x)=x^3-3b^2x+16$. If $P(|x|)$ has exactly two real roots, then the area bounded by $y = P(x)$ and x -axis is

(A) 81 sq. units
(B) 108 sq. units
(C) 168 sq. units
(D) infinite

67. The area bounded by $f(x) = \tan^{-1}(\tan x) + \cos^{-1}(\cos x)$ and x-axis, when $x\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ will be

(A) π^2
(B) $\frac{3\pi^2}{4}$
(C) $\frac{\pi^2}{4}$
(D) none of these

68. If area of the region bounded by the straight lines $x = 0$, $x = 2$ and curves $y = 2^x$, $y = 2x - x^2$ is 'm' square units then 'm' is equal to

(A) $\frac{3}{\log 2}-\frac{1}{3}$
(B) $\frac{3}{\log 2}-\frac{2}{3}$
(C) $\frac{3}{\log 2}-1$
(D) $\frac{3}{\log 2}-\frac{4}{3}$

69. If a curve $y=a\sqrt{x}+bx$ passes through the points (1, 2) and the area bounded by the curve, line $x=4$, x-axis and y-axis is 8 square units, then

(A) $a=3, b=-1$
(B) $a=3, b=1$
(C) $a=-3, b=1$
(D) $a=-3, b=-1$

70. The area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x=1$ and $x=b$ is $(b-1)\cos(3b+4)$ square units. Then $f(x)$ is given by

(A) $(x-1)\sin(3x+4)$

(B) $3(x-1)\sin(3x+4)+\cos(3x+4)$

(C) $\cos(3x+4)-3(x-1)\sin(3x+4)$

(D) $\sin(3x+4)$

71. If the curve given by parametric equation $x = t - t^3$, $y = 1 - t^4$ forms a loop for all values of $t \in [-1,1]$, then the area of the loop is

(A) 1/ 7 sq .units (B) 3/ 5 sq. units

(C) 16/ 35 sq .units (D) 8/35 sq. units

72. $f(x)$ is an increasing function such that $f''(x) > 0 \;\forall\; x \in R$; $f(0) = 0$; if $f(1) = 1$, $\int_0^1 f^{-1}(x)\,dx = k$ then area bounded by $f(x)$ and $f^{-1}(x)$ can be

(A) $2k - 2$ (B) $2k - 1$

(C) $1 - 2k$ (D) k

73. The area enclosed by the curves $y = \sqrt{4-x^2}$, $y \geq \sqrt{2}\sin\left(\frac{x\pi}{2\sqrt{2}}\right)$ and x-axis is divided by the y-axis in the ratio

(A) $\frac{\pi^2-8}{\pi^2+8}$ (B) $\frac{\pi^2-4}{\pi^2+4}$

(C) $\frac{\pi-4}{\pi+4}$ (D) $\frac{2\pi^2}{2\pi+\pi^2-8}$

74. Let $y = g(x)$ be the inverse of a bijective mapping $f : R \to R, f(x) = 3x^3 + 2x$. The area bounded by the graph of $g(x)$, the x-axis and the ordinate at $x = 5$ is

(A) $\frac{5}{4}$ (B) $\frac{7}{4}$

(C) $\frac{9}{4}$ (D) $\frac{13}{4}$

75. Area of the figure contained between the parabola $x^2 = 4y$ and the curve $y = \frac{8}{x^2+4}$ is $2\pi - K$ then $K =$

(A) $\frac{2}{3}$ (B) $\frac{4}{3}$

(C) $\frac{8}{3}$ (D) $\frac{1}{3}$

76. The area bounded by the curves $y = 2 - |x-1|$, $y = \sin x$; $x = 0$ and $x = 2$ is

(A) $1+2\cos^2 1$ (B) $2+\sin^2 1$

(C) $\frac{\pi}{2}$ (D) $1+\log 2$

77. The area of the region bounded by the curves $y = \frac{4-x^2}{4+x^2}$, $25y^2 = 9x$ and $y = \frac{3}{5}|x| - \frac{6}{5}$ which contains (1, 0) point in its interior is

(A) $\pi - 4\tan^{-1}\frac{1}{2} - \frac{1}{10}$ (B) $\pi - 4\tan^{-1}\frac{1}{2} + \frac{1}{10}$

(C) $\pi - 4\tan^{-1}\frac{1}{2} + \frac{3}{10}$ (D) $\pi - 4\tan^{-1}\frac{1}{2} - \frac{3}{10}$

78. The area the region bounded by $\max\{|x-1|, |y-1|\} \leq 1$ and $(x-1)(y-1) \leq \frac{1}{2}$

(A) 1+ ln 2 (B) 3 – ln 2

(C) 3 + ln 2 (D) 2 – ln 2

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. The area enclosed between the curves, $x^2 = y$ and $y^2 = x$ is equal to

(A) $\frac{1}{3}$ sq. unit

(B) $2\int_0^1 (x - x^2)\,dx$

(C) area of the region $\{(x, y) : x^2 \leq y \leq |x|\}$

(D) None

2. Let f and g be continuous function on $a \leq x \leq b$ and set $p(x) = \max\{f(x), g(x)\}$ and $q(x) = \min.\{f(x), g(x)\}$, the area bounded by the curves $y = p(x)$, $y = q(x)$ and the ordinates $x = a$ and $x = b$ is given by

(A) $\int_a^b (f(x) - g(x))\,dx$ (B) $\int_a^b (p(x) - q(x))\,dx$

(C) $\int_a^b |p(x) - q(x)|\,dx$ (D) $\int_a^b |f(x) - g(x)|\,dx$

3. If A be the area bounded by $f(x) = \frac{\sin x}{x}$, $x = 0$, $x = \frac{\pi}{2}$ and x-axis, then which of following is/are true

(A) $1 < A < \frac{\pi}{2}$ (B) $\left(\frac{1+\sqrt{2}}{2}\right) < A < \left(\frac{\pi+2\sqrt{2}}{4}\right)$

(C) $A > \frac{\pi+2\sqrt{2}}{4}$ (D) $\frac{2}{\pi} < A < 1$

4. Consider the function $f(x)$ and $g(x)$, both defined from $R \to R$ and are defined as $f(x) = 2x - x^2$ and $g(x) = x^n$ where $n \in N$. If the area between $f(x)$ and $g(x)$ is $\frac{1}{2}$ then n is a divisor of

(A) 12 (B) 15

(C) 20 (D) 30

5. Let $f(x) = 2 - |x - 1|$ and $g(x) = (x - 1)^2$, then

(A) area bounded by $f(x)$ and $g(x)$ is $\frac{7}{6}$

(B) area bounded by $f(x)$ and $g(x)$ is $\frac{7}{3}$

(C) area bounded by $f(x)$, $g(x)$ and x-axis is $\frac{5}{3}$

(D) area bounded by $f(x)$, $g(x)$ and x-axis is $\frac{5}{6}$

6. If A_i is the area bounded by $|x - a_i| + |y| = b_i$, $i \in N$, where $a_{i+1} = a_i + \frac{3}{2}b_i$ and $b_{i+1} = \frac{b_i}{2}$, $a_1 = 0$, $b_1 = 32$, then

(A) $A_3 = 128$ (B) $A_3 = 256$

(C) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} A_i = \frac{8}{3}(32)^2$ (D) $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} A_i = \frac{4}{3}(16)^2$

7. Which of the following statement(s) is/are True for the function $f(x) = (x - 1)^2 (x - 2) + 1$ defined on [0, 2]?

(A) Range of f is $\left[\frac{23}{27}, 1\right]$

(B) The coordinates of the turning point of the graph of $y = f(x)$ occur at (1, 1) and $\left(\frac{5}{3}, \frac{23}{27}\right)$

(C) The value of p for which the equation $f(x) = p$ has 3 distinct solutions lies in interval $\left(\frac{23}{27}, 1\right)$

(D) The area enclosed by $y = f(x)$, the lines $x = 0$ and $y = 1$ as x varies from 0 to 1 is $\frac{7}{12}$.

8. The area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is

(A) $e - 1$ (B) $\int_1^e \ln(e + 1 - y)\,dy$

(C) $e - \int_0^1 e^x\,dx$ (D) $\int_1^e \ln y\,dy$

9. Area bounded by curve $y = \cot^{-1}x$, $y = \tan^{-1}x$ and y-axis is equal to

(A) $\int_0^{\pi/4} \tan x\,dx + \int_{\pi/4}^{\pi/2} \cot x\,dx$

(B) $\int_0^{\pi/4} \tan^{-1} x\,dx + \int_{\pi/4}^{\pi/2} \cot^{-1} x\,dx$

(C) $\frac{-2}{\pi}\int_0^{\pi/2} \log(\sin x)\,dx$

(D) $\int_{\pi/4}^{\pi/2} \tan^{-1} x\,dx + \int_0^{\pi/4} \cot - x\,dx$

10. A tangent line with positive slope to a parabola $y = x^2 + 1$ is drawn such that the area of the figure bounded by the parabola, the tangent line and the coordinate axes is $\frac{11}{3}$. Given that the y-intercept of the tangent is negative, then the x-coordinate of the point of tangency is greater than

(A) 1 (B) 2

(C) $\frac{7}{3}$ (D) 3

11. Let T be the triangle with vertices (0, 0), $(0, c^2)$ and (c, c^2) and let R be the region between $y = cx$ and $y = x^2$ where $c > 0$ then

(A) Area $(R) = \frac{c^3}{6}$

(B) Area $(R) = \frac{c^3}{3}$

(C) $\lim\limits_{c\to 0^+} \frac{\text{Area}(T)}{\text{Area}(R)} = 3$

(D) $\lim\limits_{c\to 0^+} \frac{\text{Area}(T)}{\text{Area}(R)} = \frac{3}{2}$

12. Suppose f is defined from $R \to [-1, 1]$ as $f(x) = \frac{x^2 - 1}{x^2 + 1}$ where R is the set of real number. Then the statement which does not hold is -

(A) f is many one onto

(B) f increases for $x > 0$ and decreases for $x < 0$

(C) minimum value is not attained even though f is bounded

(D) the area included by the curve $y = f(x)$ and the line $y = 1$ is π sq. units

13. Consider $f(x) = \begin{cases} \cos x & 0 \le x < \frac{\pi}{2} \\ \left(\frac{\pi}{2} - x\right)^2 & \frac{\pi}{2} \le x < \pi \end{cases}$ such that f is periodic with period π, then

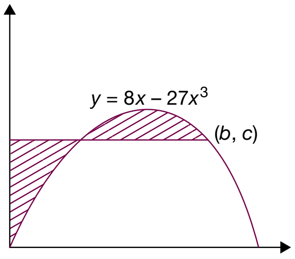
(A) The range of f is $\left[0, \frac{\pi^2}{4}\right)$

(B) f is continuous for all real x, but not differentiable for some real x

(C) f is continuous for all real x

(D) The area bounded by $y = f(x)$ and the x-axis from $x = -n\pi$ to $x = n\pi$ is $2n\left(1 + \frac{\pi^3}{24}\right)$ for a given $n \in N$

14. The figure shows a horizontal line $y = c$ passing through (b, c) intersecting the curve $y = 8x - 27x^3$. If the shaded areas are equal, then



(A) $b = \dfrac{1}{9}$

(B) $b = \dfrac{4}{9}$

(C) $c = \dfrac{32}{27}$

(D) $c = \dfrac{23}{27}$

15. The function $y = f(x)$ is continuous over [0, 10]. If the area bounded by $f(x)$ and the x-axis between $x = 0$ and $x = 10$ is S where $-10 \le f(x) \le 10$, then which of the following is not always correct?

(A) $S \le 100$ (B) $\left|\int_0^{10} f(x)dx\right| = S$

(C) $\int_0^{10} f(x)dx = S$ (D) $S \le \int_0^{10} f(x)dx$

16. Area bounded by $y = \sec^{-1}x$, $y = \cot^{-1}x$ and line $x = 1$ is given by

(A) $\displaystyle\int_1^{\sqrt{\frac{1+\sqrt{5}}{2}}} (\cot^{-1}x - \sec^{-1}x)\, dx$

(B) $\displaystyle\int_0^{\alpha} \sec x\, dx + \int_{\alpha}^{\frac{\pi}{4}} \cot x\, dx - \frac{\pi}{4}$, where $\sin\alpha = \cos^2\alpha$

(C) $\displaystyle\int_0^{\alpha} \sec x\, dx + \int_{\alpha}^{\frac{\pi}{4}} \cot x\, dx - \frac{\pi}{4}$, where $\sin\alpha = \cos^2\alpha$

(D) $\displaystyle\int_1^{\frac{1+\sqrt{5}}{2}} (\cot^{-1}x - \sec^{-1}x)\, dx$

17. If A_1 denotes area of the region bounded by the curves C_1: $y = (x - 1)e^x$, tangent to C_1 at (1, 0) and y-axis and A_2 denotes the area of the region bounded by C_1 and co-ordinate axes in fourth quadrant, then

(A) $A_1 > A_2$ (B) $A_1 < A_2$

(C) $2A_1 + A_2 = 2$ (D) $A_1 + 2A_2 = 4$

18. Let A_r be the area of the region bounded between the curves $y^2 = (e^{-kr})x$ (where $k > 0$, $r \in N$) and the line $y = mx$ (where $m \neq 0$), k and m are some constants. Then which of the following is(are) correct?

(A) $A_1, A_2, A_3, \ldots$ are in $G.P.$ with common ratio equals e^{-2k}

(B) $A_1, A_2, A_3, \ldots$ are in $G.P.$ with common ratio equals e^{-k}

(C) If $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} A_i = \frac{1}{48(e^{2k}-1)}$, then m equals 2.

(D) If $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} A_i = \frac{1}{48(e^{2k}-1)}$, then m equals 3.

19. Consider the areas S_0, S_1, S_2, … bounded by the x-axis and half-waves of the curve $y = e^{-x}\sin x$, where $x \ge 0$. Then which of the following is(are) correct?

(A) The value of S_0 equals $\dfrac{1}{2}(1 + e^{\pi})$ sq. units.

(B) The value of S_0 equals $\dfrac{1}{2}(1 + e^{-\pi})$ sq. units.

(C) The sequence $S_0, S_1, S_2, \ldots$, forms a $G.P.$ with common ratio equals $e^{-\pi}$.

(D) The sequence $S_0, S_1, S_2, \ldots$, forms a $G.P.$ with common ratio equals e^{π}.

20. Let $A(k)$ be the area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$.

(A) The range of $A(k)$ is $\left[\dfrac{10\sqrt{5}}{3}, \infty\right)$

(B) The range of $A(k)$ is $\left[\dfrac{20\sqrt{5}}{3}, \infty\right)$

(C) The value of k for which area is minimum is 0.

(D) The value of k for which area is minimum is 1.

21. If the curve $y = a\sqrt{x} + bx$ passes through the point (1, 2) and lies above the x-axis in $0 \le x \le 9$ and the area enclosed by the curve, the x-axis and the line $x = 4$ is 8 sq.

(A) $a = 1$ (B) $b = 1$

(C) $a = 3$ (D) $b = -1$

22. The area bounded by the non-negative curve $y = f(x)$, x-axis and the ordinates $x = 1$ and $x = b$ is $(b - 1)\sin(3b + 4)$, then

(A) $f(x) = (x - 1)\cos(3x + 4)$

(B) $f(x) = 0$ has atleast one root (1/2, 2)

(C) $f(x) = \sin(3x + 4) + 3(x - 1)\cos(3x + 4)$

(D) $f(x) = 0$ has atleast one root in (1, 2)

23. If the Area bounded by $y = (\sin x)^{\operatorname{cosec} x}$, $y = (\operatorname{cosec} x)^{\sin x}$, $y = (\sin x)^{\sin x}$ and $y = (\operatorname{cosec} x)^{\operatorname{cosec} x}$ between the ordinates $x = 0$, $x = \dfrac{\pi}{2}$ denoted by A_1, A_2, A_3 and A_4 then

(A) A_4 is the greatest (B) A_1 is the least

(C) $A_2 > A_3$ (D) $A_2 < A_3$

24. The parabolas $y^2 = 4x$, $x^2 = 4y$ divide the square region bounded by the lines $x = 4$ and $y = 4$ and the coordinate axes. If S_1, S_2, S_3 are the areas of these parts numbered from top to bottom, respectively then.

(A) $S_1 : S_2 = 1 : 1$ (B) $S_2 : S_3 = 1 : 2$

(C) $S_1 : S_3 = 1 : 1$ (D) $S_1 : (S_1 + S_2) = 1 : 2$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

Let C be a curve $y = f(x)$ passing through $M(-\sqrt{3}, 1)$ such that the y-intercept of the normal at any point $P(x, y)$ on the curve C is equal to the distance of P from the origin.

1. Which one of the following statement is correct?
(A) $f(x)$ has exactly two points of inflection
(B) $f(x)$ has maxima but no minima
(C) The equation $f(x) = -2$ has two distinct real solutions
(D) Inclination of the tangent line drawn to $f(x)$ at $(1, 0)$ is $\tan^{-1} 1$

2. The area enclosed by the curve C and the co-ordinate axes is equal to
(A) $\frac{1}{6}$ (B) $\frac{1}{2}$
(C) $\frac{2}{3}$ (D) $\frac{1}{3}$

3. The shortest distance between the curve C and the circle $x^2 + y^2 - 2y = 0$ is equal to -
(A) $\frac{1}{2}$ (B) $\sqrt{2} - 1$
(C) $\sqrt{6} - 1$ (D) $\sqrt{5} - 1$

Comprehension 2

Consider three real-valued functions f, g and h defined on R (the set of real numbers). Let

$f(x) = 2x^3 + 3\left(1 - \frac{3a}{2}\right)x^2 + 3(a^2 - a)\,x + b$ where $a, b \in R$ and $g(x) = \frac{f'(x)}{6}$. Also $h(x)$ is such that $h''(x) = 6x - 4$ and $h(x)$ has a local minimum value 5 at $x = 1$.

4. The true set of values of a for which $f(x)$ has negative point of local minimum, is
(A) $(-\infty, 0)$ (B) $(1, \infty)$
(C) $(0, 1)$ (D) $(1, \infty) - \{2\}$

5. The complete set of values of a for which vertex of parabola $y = g(x)$ has negative ordinate, is
(A) $\left(\frac{1}{2}, 1\right)$ (B) $(0, \infty)$
(C) $R - \{2\}$ (D) $(1, \infty) - \{2\}$

6. The area bounded by $y = h(x)$ between $x = 0$ and $x = 2$, is -
(A) $\frac{23}{3}$ (B) $\frac{20}{3}$
(C) $\frac{40}{3}$ (D) $\frac{32}{3}$

Comprehension 3

Let $f(x)$ be a continuous and positive function in $[0, 5]$ and the area bounded by the graph of $y = f(x)$ and x-axis for $0 \le x \le 5$ is 8. $A(c)$ denotes the area between the graph of $y = f(x)$ and x-axis for $0 \le x \le c$ and $B(c)$ denotes the area between the graph of $y = f(x)$ and x-axis for $c \times 5$. Also $R(c) = \frac{A(c)}{B(c)}$, $R(3) = 1$ and $\frac{dR}{dc}$ is equal to 7 at $c = 3$.

7. The value of $A(3)$ is equal to
(A) 2 (B) 4
(C) 8 (D) 16

8. The value of $A'(c) + B'(c)$ at $c = 10$, is
(A) 0 (B) 8
(C) 10 (D) 16

9. The value of $f(3)$ is equal to
(A) 7 (B) 14
(C) 21 (D) 28

Comprehension 4

Consider the parabola $C : 2y = x^2$ in xy plane. Let l_1 be the tangent line of C with slope 2 l_2 be the tangent line of C with negative slope which makes an angle $\sec^{-1}\sqrt{2}$ with l_2.

10. The coordinates of the point of intersection of the lines l_1 and l_2 are
(A) $\left(\frac{-1}{2}, -3\right)$ (B) $\left(\frac{-1}{4}, \frac{-5}{2}\right)$
(C) $\left(\frac{-1}{4}, -3\right)$ (D) $\left(\frac{-1}{2}, \frac{-5}{2}\right)$

11. Area bounded by C, l_1 and l_2 is equal to
(A) $\frac{116}{2}$ (B) $\frac{125}{48}$
(C) $\frac{125}{24}$ (D) $\frac{116}{15}$

Comprehension 5

Let $h(x) = f(x) - g(x)$ where $f(x) = \sin^4 \pi x$ $g(x) = \ln x$. Let $x_0, x_1, x_2, \ldots, x_{n+1}$ be the roots of $f(x) = g(x)$ in increasing order

12. Then the absolute area enclosed by $y = f(x)$ and $y = g(x)$ is given by
(A) $\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r h(x)\,dx$ (B) $\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^{r+1} h(x)\,dx$
(C) $2\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^r h(x)\,dx$ (D) $\frac{1}{2}\sum_{r=0}^{n} \int_{x_r}^{x_{r+1}} (-1)^{r+1} h(x)\,dx$

13. In the above question the value of n is

(A) 1 (B) 2

(C) 3 (D) 4

14. The whole area bounded by $y = f(x)$, $y = g(x)$ and $x = 0$ is

(A) $\frac{11}{8}$ (B) $\frac{8}{3}$

(C) 2 (D) $\frac{13}{3}$

Comprehension 6

Let $f(x) = 3 + \int_0^x \frac{1+\sin t}{t^2+2}\,dt\ \forall x \in R$. $P(x)$ is a second degree polynomial function of x such that $P(0) = f(0)$; $P'(0) = f'(0)$ and $P''(0) = f''(0)$.

15. Least value of $P(x)$ is

(A) $\frac{7}{2}$ (B) $\frac{9}{5}$

(C) $\frac{12}{5}$ (D) $\frac{11}{4}$

16. Slope of the tangent line at the point $(-2, P(-2))$ on the curve $y = P(x)$ is

(A) $\frac{-1}{2}$ (B) $\frac{-2}{3}$

(C) $\frac{-3}{2}$ (D) $\frac{-5}{4}$

17. Area enclosed by $y = P(x)$ and the y-axis and the tangent at point $(-2, P(-2))$ is

(A) $\frac{3}{4}$ (B) $\frac{8}{3}$

(C) $\frac{12}{7}$ (D) $\frac{2}{3}$

Comprehension 7

Let $f(x) = x^2 - 3x + 2$ be a function, $\forall x \in R$

18. The area bounded by the curve $f(|x|)$ and x-axis is

(A) 2 sq. units (B) $\frac{5}{2}$sq.units

(C) $\frac{3}{5}$sq.units (D) $\frac{5}{3}$sq.units

19. The area bounded by the curve $|f(|x|)|$ between $1 \le |x| \le 2$, and the x-axis is

(A) $\frac{1}{5}$sq.units (B) $\frac{1}{4}$sq.units

(C) $\frac{1}{3}$sq.units (D) $\frac{1}{2}$sq.units

20. If $|f(|x|)| = a$ has 6 real roots, then

(A) $a \in \left(0, \frac{1}{4}\right)$

(B) $a \in \left(\frac{1}{4}, 2\right)$

(C) $a \in \left(-\frac{1}{4}, 0\right)$

(D) None of these

Comprehension 8

Let $A_r (r \in N)$ be the area of the bounded region whose boundary is defined by $(6\pi^3 r y^2 - x)(6e^2 y - x) = 0$ then

21. $\sqrt{A_1}, \sqrt{A_2}, \sqrt{A_3}, \ldots$ are in

(A) $A.P.$ (B) $G.P.$

(C) $H.P.$ (D) $A.G.P.$

22. $\lim\limits_{n\to\infty}\left[\sqrt{A_1 A_2 A_3} + \sqrt{A_2 A_3 A_4} + \sqrt{A_3 A_4 A_5} \ldots n \text{ terms}\right]$ is

(A) $\left(\frac{e}{\pi}\right)^9$ (B) $\frac{1}{2}\left(\frac{e}{\pi}\right)^9$

(C) $\frac{1}{3}\left(\frac{e}{\pi}\right)^9$ (D) $\frac{1}{4}\left(\frac{e}{\pi}\right)^9$

23. $\lim\limits_{n\to\infty}\left[e^{\frac{-1}{\sqrt{A_1}}} + e^{\frac{-1}{\sqrt{A_2}}} + e^{\frac{-1}{\sqrt{A_3}}} + \ldots n \text{ terms}\right]$ is

(A) $e^{\frac{-\pi^3}{e^3}}$ (B) $\left(e^{\frac{\pi^3}{e^3}} - 1\right)^{-1}$

(C) $e^{\frac{2\pi^3}{e^3}}$ (D) $\left(e^{\frac{\pi^3}{e^3}} - 1\right)^{-2}$

Comprehension 9

Let the curves $S_1 : y = x^2$, $S_2 : y = -x^2$, $S_3 : y^2 = 4x - 3$

24. Area bounded by the curves S_1, S_2, S_3 is

(A) $\frac{4}{3}$ sq. units (B) $\frac{8}{3}$ sq. units

(C) $\frac{1}{6}$ sq. units (D) $\frac{1}{3}$ sq. units

25. Area bounded by the curves S_1, S_3 and the line $x = 3$ is

(A) $\frac{13}{3}$ sq. units (B) $\frac{5}{4}$ sq. units

(C) $\frac{8}{3}$ sq. units (D) $\frac{7}{4}$ sq. units

26. Area bounded by the curve S_3, $y \le -1$ and the line $x = 3$ is

(A) $\frac{7}{3}$ sq. units (B) $\frac{11}{3}$ sq. units

(C) $\frac{9}{2}$ sq. units (D) $\frac{13}{4}$ sq. units

Comprehension 10

Consider a polynomial $y = P(x)$ of the least degree passing through $A(-1, 1)$ and whose graph has two points of inflexion $B(1, 2)$ and C with abscissa 0 at which the curve is inclined to the positive axis of abscissas at an angle of $\sec^{-1}\left(\sqrt{2}\right)$.

27. If $\int_{-2}^{2} P(x)\,dx = \frac{k}{5}, k \in N$ then k is greater than -

(A) 17 (B) 24
(C) 32 (D) 41

28. The area of ΔABC is greater than -

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$
(C) $\frac{1}{8}$ (D) $\frac{1}{12}$

Comprehension 11

Let $f(x) = \min\{x - [x], -x - [-x]\}, -2 \le x \le 2$. (where [.] denotes greatest integer function).

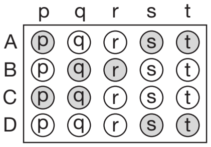
29. The number of solutions of the equation $x^2 + 2f(x) = 1$, is
(A) 2
(B) 4
(C) 5
(D) 6

30. The area bounded by $y = f(x)$ and $y = 1 - x^2$, is (in sq. units)
(A) 5/6
(B) 2/3
(C) 7/6
(D) none of these

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:



	p	q	r	s	t
A	●	ⓠ	ⓡ	●	●
B	ⓟ	●	●	ⓢ	ⓣ
C	●	●	ⓡ	ⓢ	ⓣ
D	ⓟ	ⓠ	ⓡ	●	●

1. Match the following:

COLUMN-I	COLUMN-II
(A) Area of region formed by points (x, y) satisfying $[x]^2 = [y]^2$ for $0 \le x \le 4$ is equal to (where [ ] denotes greatest integer function)	(p) 48
(B) The area of region formed by points (x, y) satisfying $x + y \le 6$, $x^2 + y^2 \le 6y$ and $y^2 \le 8x$ is $\frac{k\pi - 2}{12}$, then $k =$	(q) 27
(C) The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes, lies in the first quadrant. If its area is 2, then the value of $-b$ is	(r) 7
(D) If the area bounded by the graph of $y = xe^{-ax}$ $(a > 0)$ and the abscissa axis is 1/9 then the value of 'a' is equal to	(s) 4
	(t) 3

2. Match the following Column – I with Column – II

COLUMN-I	COLUMN-II
(A) If the area of the region bounded by the curves $y = x^2$, $y = \left\|2 - x^2\right\|$ and $y = 2$ which lies to the right of the line $x = 1$ is A then $\frac{[A]}{3}$ is ([.] denotes G.I.F)	(p) $\frac{4}{3}$
(B) The area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 + 3 = 4x$ is	(q) $\frac{9}{8}$
(C) The area bounded by the curves $x^2 = 4y$ and straight line $x = 4y - 2$ is	(r) $\frac{5}{3}$
(D) A curve passes through (2, 0) and the slope of tangent at point $P(x, y)$ equals $\frac{(x+1)^2 + y - 3}{x+1}$. The area enclosed by the curve and the x-axis in the fourth quadrant is	(s) $\frac{1}{3}$

3. Match the following Column – I with Column – II

COLUMN-I	COLUMN-II
(A) Area bounded between curve $f(x) = \cos^{-1}(\cos x)$ $0 < x < 2\pi$ with the tangent to the curve $f(x) = \|\cos x\|$ at $x = \pi$ is	(p) $\frac{\pi^2}{2}$
(B) $y = f(x)$ be a function such that $f(x) = \min\left\{\sqrt{x(2-x)}, 2-x\right\}$, then area a bounded by $y = f(x)$ and x-axis is	(q) $(\pi - 1)^2$
(C) Area bounded by the curve $\|y\| = \sin^{-1}(\sin x)$, $2\pi < x < 3\pi$ is	(r) $\frac{\pi^2}{4}$
(D) Area bounded between curves $y = \tan^{-1}(\tan x)$, $y = \pi - x$, $y = -x + 2\pi$, $\pi < x < 2\pi$ is	(s) $\frac{\pi}{4} + \frac{1}{2}$

4. Match the following Column – I with Column – II

COLUMN-I	COLUMN-II
(A) The area bounded by curve $y = x\|x\|$, $x =$ axis and the ordinates $x = 1$, $x = -1$ is	(p) $\frac{10}{3}$ sq.units
(B) The area of the region lying between the line $x - y + 2 = 0$, the curve $x = \sqrt{y}$. and y – axis is	(q) $\frac{64}{3}$ sq.units
(C) The area enclosed between the curve $y^2 = x$ and $y = \|x\|$ is	(r) $\frac{2}{3}$ sq.units
(D) The area bounded by parabola $y^2 = x$ the straight line $y = 4$ and the y-axis is	(s) $\frac{1}{6}$ sq.units

(Continued)

5. Match the following Column – I with Column – II

COLUMN-I	COLUMN-II
(A) The curve $y^2 = 4a(x + 2)$ $(a > 0)$ and $x^2 + y^2 = 4$ intersect each other in points A and B then value of a for which area of the region bounded by the parabola and chord AB is maximum is given by λ then $8\lambda =$	(p) $\frac{8}{3}$
(B) The area bounded by $y = xe^{\|x\|}$ and lines $\|x\| = 1$ and $y = 0$ is	(q) $\frac{3}{2}$
(C) The area of the figure bounded by $\|y\| = 1 - x^2$ is	(r) 2
(D) If the area of the region described by complex number z satisfying $\sin\|z\| > 0$ (for $\|z\| < 10$) is $k\pi^3$ then $K =$	(s) 6

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. If A be the area of the region bounded by $y = 2011^x \log 2011$, $y = 2011^{-x} \log 2011$ and $x = 1$, then $[A + 2]\{A\} = ?$ (where $[x]$ denotes integral part and $\{x\}$ denotes fractional part of x)

2. Let $f(x, y) = \cos^{-1}(3|x| + 4|y|)$, if area enclosed by the points (x, y) such that $f(x, y)$ is defined, is μ, then 6μ is -

3. The area of the domain of the function

$f(x, y) = \sqrt{16 - x^2 - y^2} - \sqrt{|x| - y}$ is $k\pi$, where $k/2$ equals

4. If the area bounded by the curve

$f(x) = \cos^{-1}\left(\sin\left(\left\{\frac{\pi}{2}\right\} - [\sin x]\right)\right)$ and x-axis from $x = 0$ to

$x = 2\pi$ is $k\pi$, then find the value of k. where [.] is GIF and $\{\cdot\}$ is fpf.

5. Let S_k be the area bounded by the curve $y = x^2(1 - x)^k$ and the lines $x = 0$, $y = 0$ and $x = 1$. If $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} S_k$ equals $\frac{p}{q}$, where $p, q \in N$, then find the least value of $(p + q)$.

6. If the area bounded by the curve

$y = |\cos^{-1}(\sin x)| - |\sin^{-1}(\cos x)|$ and x-axis from $\frac{3\pi}{2} \le x \le 2\pi$, is equal to $\frac{\pi^2}{k}$, where $k \in N$, then find k.

7. Let $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, $g(x) = \cot^{-1}\left(\frac{2x}{x^2-1}\right)$ where $x \in (-1, 1)$.

If area bounded by the curves $y = f(x) + g(x)$ and $y = \pi x^2$ is A then find the value of $[A]$.

8. Let $f(x) = x - x^2$ and $g(x) = ax$. If the area bounded by $f(x)$ and $g(x)$ is equal to the area bounded by the curves $x = 3y - y^2$ and $x + y = 3$, then find the value of $|[a]|$.

9. The parabola $y^2 = x - y$ divides the triangle formed by the tangent to it at the origin and the straight lines $x + y = 0$ and $y = -1$ into two parts whose areas are A_1 and A_2. If $\frac{A_1}{A_2} = \frac{p}{q}$, $p, q \in N$ then find the least value of $(p + q)$.

10. Tangent lines are drawn at the points P and Q where $f''(x)$ vanishes for the function $f(x) = \cos x$ on $[0, 2\pi]$. The tangent lines at P and Q intersect each other at R as to form a triangle PQR. If area of triangle PQR is kp^2, then find the value of $36k$.

11. The equation of asymptotes of the curve $y^2 = \dfrac{x^3}{(2-x)}$ is $ax + by + c = 0$, where $a \in N$ then the least value of $|a + b + c|$ is -

12. Area enclosed by the curve $|x + y - 1| + |2x + y - 1| = 1$ is

13. If area enclosed by the curves $f(x) = \sin^{-1}\left(\dfrac{2|x|}{1+x^2}\right) + \sin^{-1}\left(\dfrac{|x|}{\sqrt{1+x^2}}\right)$ and $g(x) = |\tan^{-1}x|$ from $x = -1$ to $x = 1$ is $\pi - \ln(p)$, then find the value of 'p'.

14. Let S_1 and S_2 be the set of points in x-y plane (where $x \geq 0$ and $y \in R$) defined as $S_1 = \{(x, y) : |x + |x|| + |y + |y|| \leq 2\}$ and $S_2 = \{(x, y) : |x - |x|| + |y - |y|| \leq 2\}$. If the area of intersection of S_1 and S_2 is A, then find the value of $2A$.

15. Let $f(x) = \begin{cases} 2x, & 0 \leq x < 1 \\ 2, & 1 \leq x \leq 3 \\ 8 - 2x, & 3 < x \leq 4 \end{cases}$. If the area of region bounded by the curve $|y + 1| = f(x + 4)$ is S then find the value of $\left(\dfrac{S}{3}\right)$.

16. Consider an ellipse $E : \dfrac{(3x - 4y + 5)^2}{100} + \dfrac{(4x + 3y - 10)^2}{50} = 1$. M and m are lengths of major and minor axes respectively. If p_1 and p_2 are perpendicular distances on major and minor axes from a point P in the x-y plane such that $m \leq p_1 + p_2 \leq M$, then P lies in the region whose area is

17. Let $f(x) = x^3 + 3x + 2$ and $g(x)$ is the inverse of it. The area bounded by $g(x)$, the x-axis and the ordinates at $x = -2$ and $x = 6$ is $\dfrac{m}{n}$ where $m, n \in N$ & $G.C.D$ of $(m, n) = 1$ then $m - 2 =$

18. If the area bounded by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ and the line $y = 3x - 15$ is $\dfrac{73}{\lambda}$ (when $x \geq 1$), then the value of l is

19. The minimum area bounded by the function $y = f(x)$ and $y = \alpha x + 9$ ($\alpha \in R$) where f satisfies the relation $f(x + y) = f(x) + f(y) + y\sqrt{f(x)}$ $\forall x, y \in R$ and $f'(0) = 0$ is $9A$, value of A is

20. Let L: be the value of $c > 0$ such that the area of the region bounded by the parabolas $y = x^2 - c^2$ and $y = c^2 - x^2$ is 576, and M: be the sum of abscissa and ordinate of the point on the hyperbola $xy = 8$ which is closest to the point $(3, 0)$. Find the value of $(L + M)$.

21. If the maximum area of the region enclosed by the curves $y = |x|e^{|x|}$ and the line $y = a(0 \leq a \leq e)$ in $x \in [-1, 1]$ is A, then find the value of $[A]$.

22. If the area bounded by the curves $f(x) = (\cos^{-1}|\cos x|)^2$, $g(x) = (\cos^{-1}|\cos x|)$ and $|x| = \dfrac{\pi}{2}$ is $a\pi^3 + b\pi^2 + c$, then find the minimum value of $(|a| + |b| + |c|)$.

23. Find the area enclosed by the graph of $y^2 + 2xy + 40|x| = 400$.

24. Let $R = \{x, y : x^2 + y^2 \leq 144$ and $\sin(x + y) \geq 0\}$. And S be the area of region given by R, then find $S/9\pi$.

25. If the area bounded by $[x] + [y] = n$ and $y = k$; $n, k \in N$ and $k \leq (n+1)$ and [.] is greatest integer function, in the first quadrant, is $n + r$, then find r.

26. Let α, β be roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$, where $\alpha < \beta$. Also $f(x) = x^2$ and $g(x) = \cos x$. If the area bounded by the curve $y = (fog)(x)$, the vertical lines $x = \alpha$, $x = \beta$ and x-axis is $\dfrac{\pi}{\lambda}$, then find the sum of the digit in λ

27. Let $y = g(x)$ be the inverse of a bijective mapping $f : R \to R$ defined as $f(x) = 3x^3 + 2x$. The area bounded by the graph of $g(x)$, the x-axis and the coordinate at $x = 5$ is 'A' then the value of $(4A - 7)$ is

28. A point $P(x, y)$ moves is such a way that $[x + y + 1] = [x]$ (where [.] greatest integer function) and $x \in (0, 2)$. Then the area representing all the possible positions of P equals

29. Let $f(x) = 1 + \cos x$ and $g(x) = \dfrac{a}{bx^2 + cx + 1}$. If $f(0) = g(0)$, $f'(0) = g'(0)$, $f''(0)$ and the area bounded by the graph of $g(x)$ and x-axis is $k\pi$ then find the value of k.

30. If the area enclosed by the curves $f(x) = \cos^{-1}(\cos x)$ and $g(x) = \sin^{-1}(\cos x)$ in $x \in \left[\dfrac{9\pi}{4}, \dfrac{15\pi}{4}\right]$ is $\dfrac{a\pi^2}{b}$ (where a and b are coprime), then find $(a + b)$.

31. Given a line segment PQ with end points on the parabola $y = x^2$ such that the area bounded by PQ and the parabola always equal to $\dfrac{4}{3}$. If the equation to the locus of the middle point of PQ is $y = ax^2 + bx + c$, then find the value of $(a + b + c)$.

32. Find the area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to it at the point where the ordinate is 3, and the x-axis.

33. The area between the curves $\max\{|x|, |y|\} = 1$ and $\pi\left(x^2 + y^2\right) = 3$ is equal to.

34. Area bounded by $2 \geq \max.\{|x - y|, |x + y|\}$ is k sq. units then $k =$

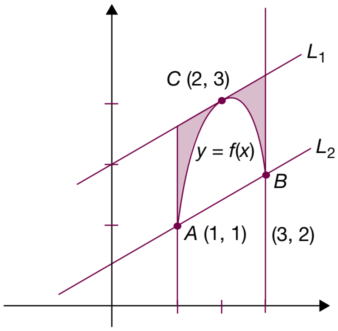
35. If A is the area formed by the positive x-axis and the normal and tangent to the curve $x^2 + y^2 = 4$ at $\left(1, \sqrt{3}\right)$ then $\dfrac{A}{\sqrt{3}} =$

36. If the area bounded by $y = f(x)$, y-axis and line $2y = \pi(x+1)$ where $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}\frac{1}{x} + \tan^{-1}x$ is $\frac{\pi}{k}$, then value of k is ______________

37. The area bounded by the curve $f(x) = x + \sin x$ and its inverse between the ordinates $x = 0$ to, $x = 2\pi$ is 2^k, $k \in N$ then $k =$ ______________

38. Area bounded by the minimum of $\{|x|,|y|\} = 1$ and the maximum of $\{|x|,|y|\} = 2$ is

39. The following figure shows the graph of a continuous function $y = f(x)$ on the interval [1, 3]. The point A, B, C have co-ordinates (1, 1), (3, 2), (2, 3) respectively and the lines L_1 and L_2 are parallel with L_1 being tangent to the curve at C. If the area under the graph of $y = f(x)$ from $x = 1$ to $x = 3$ is 4 square units, then find the area (in square units) of shaded region.



40. Consider a curve, $x = \begin{cases} \frac{4-y}{2}, & y > 2 \\ (y-2)^{\frac{1}{3}} + 1, & 2 \le y \le 10 \\ \frac{19-y}{3}, & y < 10 \end{cases}$. If area enclosed by the curve, tangent to it at $x = 2$ and the x-axis is $\frac{p}{q}$ where p and q are relatively prime numbers, then find the value of $(p - 19q)$

41. If $S_0, S_1, S_2 \ldots$ are the areas bounded by the x-axis and half-waves of the curve $y = e^{-2x}\sin 3x$ for $x \ge 0$ then $\frac{S_{n+1}}{S_n} = e^{\frac{-2\pi}{k}}$ where $K =$ ____

42. If area of the region bounded by the curves $y = |1 + \{|x|\}|$ and $3y = |-2x^2 + 5x + 3|$ for $-\frac{1}{2} < x \le 1$, (where {.} denotes fractional part of x) is A, then $\frac{72}{17}A$ equals ____________

43. Find the value of $k > 0$ so that the area of the bounded region enclosed between the parabolas $y = x - kx^2$ and $y = \frac{x^2}{k}$ is maximum.

44. Let A_n be the area enclosed by the curve $y = x^n$ $(n \ge 1)$, ordinate at $x = \frac{1}{2}$ and $y = 0$. If $\sum_{n=1}^{n} \frac{2^n A_n}{n} = \frac{1}{3}$ then find the value of n.

45. Let f: [0, 1] → [0, 1/2] be a function such that $f(x)$ is a polynomial of 2nd degree, satisfy the following conditions

(a) $f(0) = 0$

(b) has a maximum value of $\frac{1}{2}$ at $x = 1$

If A be the area bounded by $y = f(x)$; $y = f^{-1}(x)$ and the line $2x + 2y - 3 = 0$ in 1st quadrant, then value of $24A$ is equal to ____________

46. If A is the area of the figure bounded by the straight lines $x = 0$ and $x = 2$, and the curves $y = 2^x$ and $y = 2x - x^2$ then the value of $\left(\frac{3}{\log 2} - \frac{A}{3}\right)$ is

47. Let $y = g(x)$ be the image of $f(x) = x + \sin x$ about the line $x + y = 0$. If the area bounded by $y = g(x)$, x-axis, $x = 0$ and $x = 2\pi$ is A, then $\frac{A}{\pi^2}$ is _________

48. The area of the figure enclosed by the curve $5x^2 + 6xy + 2y^2 + 7x + 6y + 6 = 0$ is 'K' sq. units. then $[k]$, (where [.] denotes G.I.F) is equal to

49. If the area included between the two parabolas $y^2 = 4a(x+a)$ and $y^2 = 4b(b-x)$ is $\frac{16}{3}$sq. units, then the product of $A.M$ and $G.M$ of a, b is

50. If area bounded by the curve $y = \left[\frac{|x|}{\sqrt{2011}}\right]$ and $x^2 + y = 2011$ be expressed as $2\lambda(2011)^{1/\lambda}$ then 3λ is where [.] denotes G.I.F

51. If 'C' is a constant such that the line joining the points (0, 3), (5, –2) is a tangent to $y = \frac{c}{x+1}$ then the area of the region bounded by the curve and x-axis and $x = 1$ and $x = 2$ is $\lambda\left\{1 - \frac{1}{2.2} + \frac{1}{2^2.3} - \frac{1}{2^3.4} + \ldots\ldots\infty\right\}$ then λ equals.

ARCHIVE: JEE MAIN

1. **[IIT JEE Mains 2021]**
The area (in sq. units) of part of the circle $x^2 + y^2 = 36$, which is outside the parabola $y^2 = 9x$, is

(A) $24\pi + 3\sqrt{3}$
(B) $12\pi + 3\sqrt{3}$
(C) $24\pi - 3\sqrt{3}$
(D) $12\pi - 3\sqrt{3}$

2. **[IIT JEE Mains 2021]**
The area of the region : $R = \{(x, y) : 5x^2 \le y \le 2x^2 + 9\}$ is

(A) $9\sqrt{3}$ square units
(B) $6\sqrt{3}$ square units
(C) $12\sqrt{3}$ square units
(D) $11\sqrt{3}$ square units

3. **[IIT JEE Mains 2021]**
Let A_1 be the area of the region bounded by the curves $y = \sin x$, $y = \cos x$ and y–axis in the first quadrant. Also let A_2 be the are area of the region bounded by the curve $y = \sin x$, $y = \cos x$, x-axis and $x = \frac{\pi}{2}$ in the first quadrant. Then,
(A) $2A_1 = A_2$ and $A_1 + A_2 = 1 + \sqrt{2}$
(B) $A_1 : A_2 = 1 : \sqrt{2}$ and $A_1 + A_2 = 1$
(C) $A_1 = A_2$ and $A_1 + A_2 = \sqrt{2}$
(D) $A_1 : A_2 = 1 : 2$ and $A_1 + A_2 = 1$

4. **[IIT JEE Mains 2021]**
The area bounded by the curve $4y^2 = x^2(4 - x)(x - 2)$ is equal to:
(A) $\frac{\pi}{8}$
(B) $\frac{3\pi}{8}$
(C) $\frac{3\pi}{2}$
(D) $\frac{\pi}{16}$

5. **[IIT JEE Mains 2021]**
The graphs of sine and cosine functions, intersect each other at a number of points and between two consecutive points of intersection, the two graphs enclose the same area A. Then A^4 is equal to ______.

6. **[IIT JEE Mains 2021]**
The area bounded by the line $y = ||x - 1| - 2|$ and $y = 2$ is

7. **[IIT JEE Mains 2021]**
Let $f : [-3, 1] \rightarrow R$ be given as $f(x) = \begin{cases} \min\{(x+6), x^2\}, & -3 \le x < 0 \\ \max\{\sqrt{x}, x^2\}, & 0 \le x \le 1 \end{cases}$.
If the area bounded by $y = f(x)$ and x-axis is A, then the value of $6A$ is equal to ____.

8. **[IIT JEE Mains 2021]**
Let T be the tangent to the ellipse $E : x^2 + 4y^2 = 5$ at the point $P(1, 1)$. If the area of the region bounded by the tangent T, ellipse E, lines $x = 1$ and $x = \sqrt{5}$ is $\alpha\sqrt{5} + \beta + \gamma\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$, then $|\alpha + \beta + \gamma|$ is equal to _________.

9. **[IIT JEE Mains 2020]**
Consider a region $R = \{(x, y) \in R^2 : x^2 \le y \le 2x\}$. If a line $y = \alpha$ divides the area of region R into two equal parts, then which of the following is true?
(A) $\alpha^3 - 6\alpha^2 + 16 = 0$
(B) $3\alpha^3 - 8\alpha^2 + 8 = 0$
(C) $\alpha^3 - 6\alpha^{3/2} - 16 = 0$
(D) $3\alpha^3 - 8\alpha^{3/2} + 8 = 0$

10. **[IIT JEE Mains 2020]**
The area (in sq. units) of the region
$\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, \frac{1}{2} \le x \le 2\}$ is:
(A) $\frac{79}{16}$
(B) $\frac{23}{6}$
(C) $\frac{79}{24}$
(D) $\frac{23}{16}$

11. **[IIT JEE Mains 2020]**
The area (in sq. units) of the region
$A = \{(x, y) : (x - 1)[x] \le y \le 2\sqrt{x}, 0 \le x \le 2\}$, where $[t]$ denotes the greatest integer function, is
(A) $\frac{8}{3}\sqrt{2} - \frac{1}{2}$
(B) $\frac{8}{3}\sqrt{2} - 1$
(C) $\frac{4}{3}\sqrt{2} - \frac{1}{2}$
(D) $\frac{4}{3}\sqrt{2} + 1$

12. **[IIT JEE Mains 2020]**
The area (in sq. units) of the region enclosed by the curves $y = x^2 - 1$ and $y = 1 - x^2$ is equal to:
(A) $\frac{4}{3}$
(B) $\frac{8}{3}$
(C) $\frac{16}{3}$
(D) $\frac{7}{2}$

13. **[IIT JEE Mains 2020]**
The area (in sq. units) of the region
$A = \{(x, y) : |x| + |y| \le 1, 2y^2 \ge |x|\}$ is:
(A) $\frac{1}{6}$
(B) $\frac{1}{3}$
(C) $\frac{7}{6}$
(D) $\frac{5}{6}$

14. **[IIT JEE Mains 2019]**
The area (in sq. units) bounded by the parabola $y = x^2 - 1$, the tangent at the point $(2, 3)$ to it and the y-axis is:
(A) $\frac{56}{3}$
(B) $\frac{32}{3}$
(C) $\frac{8}{3}$
(D) $\frac{14}{3}$

15. **[IIT JEE Mains 2019]**
The area of the region $A = \{(x, y) : 0 \le y \le x|x| + 1$ and $-1 \le x \le 1\}$ in sq. units, is:
(A) $\frac{4}{3}$
(B) $\frac{1}{3}$
(C) $\frac{2}{3}$
(D) 2

16. **[IIT JEE Mains 2019]**
If the area enclosed between the curves $y = kx^2$ and $x = ky^2$, $(k > 0)$, is 1 square unit. Then k is:
(A) $\sqrt{3}$
(B) $\frac{\sqrt{3}}{2}$
(C) $\frac{2}{\sqrt{3}}$
(D) $\frac{1}{\sqrt{3}}$

17. **[IIT JEE Mains 2019]**
The area (in sq. units) of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is:
(A) $\frac{3}{4}$
(B) $\frac{5}{4}$
(C) $\frac{7}{8}$
(D) $\frac{9}{8}$

18. **[IIT JEE Mains 2019]**
The area (in sq. units) in the first quadrant bounded by the parabola, $y = x^2 + 1$, the tangent to it at the point (2, 5) and the coordinate axes is:
(A) $\frac{8}{3}$ (B) $\frac{14}{3}$
(C) $\frac{187}{24}$ (D) $\frac{37}{24}$

19. **[JEE Mains 2018]**
Let $g(x) = \cos x^2, f(x) = \sqrt{x}$, and $\alpha, \beta (\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (is sq. units) bounded by the curve $y = (gof)(x)$ and the lines $x = \alpha$, $x = \beta$ and $y = 0$, is:
(A) $\frac{1}{2}(\sqrt{3}+1)$ (B) $\frac{1}{2}(\sqrt{3}-\sqrt{2})$
(C) $\frac{1}{2}(\sqrt{2}-1)$ (D) $\frac{1}{2}(\sqrt{3}-1)$

20. **[JEE Mains 2017]**
The are (in sq. units) of the region $\{(x, y): x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is
(A) $\frac{5}{2}$ (B) $\frac{59}{12}$
(C) $\frac{3}{2}$ (D) $\frac{7}{3}$

21. **[JEE Mains 2016]**
The area (in sq. units) of the region $\{(x, y) : y^2 \geq 2x$ and $x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is:
(A) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$ (B) $\pi - \frac{4}{3}$
(C) $\pi - \frac{8}{3}$ (D) $\pi - \frac{4\sqrt{2}}{3}$

22. **[JEE Mains 2015]**
The area (in sq. units) of the region described by $\{(x, y); y^2 \leq 2x$ and $y \geq 4x - 1\}$ is
(A) 7/32 (B) 5/34
(C) 15/64 (D) 9/32

23. **[JEE Mains 2014]**
The area of the region describe by
$A = \{(x, y) : x^2 + y^2 \leq 1$ and $y^2 \leq 1 - x\}$ is
(A) $\frac{\pi}{2} + \frac{4}{3}$ (B) $\frac{\pi}{2} - \frac{4}{3}$
(C) $\frac{\pi}{2} - \frac{2}{3}$ (D) $\frac{\pi}{2} + \frac{2}{3}$

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. **[JEE Adv 2020]**
The area of the region
$$\left\{(x, y): 0 \leq x \leq \frac{9}{4}, \quad 0 \leq y \leq 1, \quad x \geq 3y, \quad x + y \geq 2\right\} \text{ is}$$
(A) $\frac{11}{32}$ (B) $\frac{35}{96}$
(C) $\frac{37}{96}$ (D) $\frac{13}{32}$

2. **[JEE Adv 2020]**
Let the functions $f : R \to R$ and $g: R \to R$ be defined by
$f(x) = e^{x-1} - e^{-|x-1|}$ and $g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$.
Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is
(A) $\left(2-\sqrt{3}\right) + \frac{1}{2}(e - e^{-1})$ (B) $\left(2+\sqrt{3}\right) + \frac{1}{2}(e - e^{-1})$
(C) $\left(2-\sqrt{3}\right) + \frac{1}{2}(e + e^{-1})$ (D) $\left(2+\sqrt{3}\right) + \frac{1}{2}(e + e^{-1})$

3. **[JEE Adv 2019]**
The area of the region $\{(x, y) : xy \leq 8, 1 \leq y \leq x^2\}$ is
(A) $16\log_e 2 - \frac{14}{3}$ (B) $8\log_e 2 - \frac{14}{3}$
(C) $16\log_e 2 - 6$ (D) $8\log_e 2 - \frac{7}{3}$

4. **[JEE Adv 2016]**
Area of the region $\{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq x + 9 \leq 15\}$ is equal to
(A) $\frac{1}{6}$ (B) $\frac{4}{3}$
(C) $\frac{3}{2}$ (D) $\frac{5}{3}$

5. **[JEE Adv 2015]**
Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x \in R$ and $f : \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $\alpha \in \left[0, \frac{1}{2}\right]$, $F'(\alpha) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = \alpha$, then $f(0)$ is

6. **[JEE Adv 2013]**
The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

(A) $4\left(\sqrt{2}-1\right)$ (B) $2\sqrt{2}\left(\sqrt{2}-1\right)$

(C) $2\left(\sqrt{2}+1\right)$ (D) $2\sqrt{2}\left(\sqrt{2}+1\right)$

Multiple Correct Choice Type Questions

7. **[JEE Adv 2018]**
Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin $O\,(0, 0)$ and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then which of the following statement(s) is (are) True ?

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the all ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

8. **[JEE Adv 2018]**
Let $f : [0, \infty] \to R$ be a continuous function such that $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)\,dt$ for all $x \in (0, \infty)$. Then, which of the following statement (s) is (are) TRUE ?

(A) The curve $y = f(x)$ passes through the point (1, 2)

(B) The curve $y = f(x)$ passes through the point (2, –1)

(C) The area of the region
$\left\{(x, y) \in [0,1] \times R : f(x) \le y \le \sqrt{1-x^2}\right\}$ is $\frac{\pi - 2}{4}$

(D) The area of the region
$\left\{(x, y) \in [0,1] \times R : f(x) \le y \le \sqrt{1-x^2}\right\}$ is $\frac{\pi - 1}{4}$

9. **[JEE Adv 2017]**
If the line $x = \alpha$ divides the area of region $R = \left\{(x, y) \in R^2 : x^3 \le y \le x, 0 \le x \le 1\right\}$ into two equal parts, then

(A) $0 < \alpha \le \frac{1}{2}$ (B) $\frac{1}{2} < \alpha < 1$

(C) $2\alpha^4 - 4\alpha^2 + 1 = 0$ (D) $\alpha^4 + 4\alpha^2 - 1 = 0$

Integer/Numerical Answer Type Questions

10. **[JEE Adv 2018]**
A farmer F_1 ha a land in the shape of a triangle with vertices at $P(0, 0)$, $Q(1, 1)$ and $R(2, 0)$. From this land, a neighbouring farmer F_2 takes away the region which lines between the side PQ and a curve of the form $y = x^n (n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of DPQR, then the value of n is __________.

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. C	**2.** A	**3.** A	**4.** B	**5.** C	**6.** B	**7.** A	**8.** A	**9.** D	**10.** A
11. C	**12.** A	**13.** D	**14.** A	**15.** B	**16.** B	**17.** A	**18.** C	**19.** C	**20.** B
21. A	**22.** B	**23.** A	**24.** A	**25.** D	**26.** B	**27.** B	**28.** B	**29.** D	**30.** D
31. A	**32.** D	**33.** B	**34.** A	**35.** C	**36.** C	**37.** C	**38.** D	**39.** B	**40.** C
41. B	**42.** B	**43.** B	**44.** A	**45.** A	**46.** B	**47.** A	**48.** B	**49.** A	**50.** C
51. A	**52.** D	**53.** C	**54.** D	**55.** A	**56.** C	**57.** B	**58.** D	**59.** A	**60.** C
61. B	**62.** C	**63.** C	**64.** B	**65.** B	**66.** B	**67.** B	**68.** D	**69.** A	**70.** C
71. C	**72.** B	**73.** D	**74.** D	**75.** B	**76.** A	**77.** B	**78.** C		

Multiple Correct Choice Type Questions

1. A, B, C	**2.** B, C, D	**3.** A, B	**4.** B, C, D	**5.** B, C
6. A, C	**7.** B, C, D	**8.** B, C, D	**9.** A, C	**10.** A, B, C
11. A, C	**12.** A, C, D	**13.** A, D	**14.** B, C	**15.** B, C, D
16. A, B	**17.** B, C	**18.** A, C	**19.** A, C	**20.** B, C
21. C, D	**22.** B, C, D	**23.** A, B, C	**24.** A, C, D	

Linked Comprehension Type Questions

1. D	**2.** D	**3.** B	**4.** A	**5.** C	**6.** D	**7.** B	**8.** B	**9.** B	**10.** A
11. C	**12.** A	**13.** B	**14.** A	**15.** D	**16.** A	**17.** D	**18.** A	**19.** C	**20.** D
21. C	**22.** D	**23.** B	**24.** D	**25.** A	**26.** A	**27.** A, B, C	**28.** C, D	**29.** B	**30.** A

Matrix Match/Column Match Type Questions

1.	$A \to r$	$B \to q$	$C \to t$	$D \to t$
2.	$A \to r$	$B \to s$	$C \to q$	$D \to p$
3.	$A \to q$	$B \to s$	$C \to p$	$D \to p$
4.	$A \to r$	$B \to p$	$C \to s$	$D \to q$
5.	$A \to r$	$B \to r$	$C \to p$	$D \to s$

Integer/Numerical Answer Type Questions

1. 1	**2.** 1	**3.** 6	**4.** 1	**5.** 7	**6.** 4	**7.** 4	**8.** 1	**9.** 3	**10.** 9
11. 1	**12.** 2	**13.** 4	**14.** 3	**15.** 4	**16.** 7	**17.** 7	**18.** 6	**19.** 1	**20.** 12
21. 3	**22.** 1	**23.** 800	**24.** 8	**25.** 1	**26.** 3	**27.** 6	**28.** 2	**29.** 4	**30.** 17
31. 2	**32.** 9	**33.** 1	**34.** 8	**35.** 2	**36.** 2	**37.** 3	**38.** 4	**39.** 2	**40.** 7
41. 3	**42.** 2	**43.** 1	**44.** 2	**45.** 5	**46.** 4	**47.** 2	**48.** 1	**49.** 1	**50.** 2
51. 2									

ARCHIVE: JEE MAIN

1. C	**2.** C	**3.** B	**4.** C	**5.** 64	**6.** 8	**7.** 41	**8.** 1.25	**9.** D	**10.** C
11. A	**12.** B	**13.** D	**14.** C	**15.** D	**16.** D	**17.** D	**18.** D	**19.** D	**20.** A
21. C	**22.** D	**23.** D							

ARCHIVE: JEE ADVANCED

1. A	**2.** A	**3.** A	**4.** C	**5.** 3	**6.** B	**7.** A, C	**8.** B, C	**9.** B, C	**10.** 4

CHAPTER

11 Differential Equation

Learning Objectives

After reading this chapter, you will be able to understand concepts and problems based on:

(a) Formation of Differential Equation
(b) Solution of Differential Equation
(c) Variable Separable Form
(d) Solving Homogenous Differential equation
(e) Solving Exact Differential Equation
(f) Solving Differential Equation Using Polar Coordinates
(g) Linear Differential Equation
(h) Bernoulli's Differential Equation
(i) Clairaut's Differential Equation
(j) Orthogonal Trajectory
(k) Geometrical and Physical Application of Differential Equation

All this is followed by a variety of Exercise Sets (fully solved) which contain questions as per the latest JEE pattern. At the end of Exercise Sets, a collection of problems asked previously in JEE Main are also given.

An equation consisting of independent variable and dependent variables and their derivatives is known as differential equation.

E.g., $\frac{d^2y}{dx^2} - 4xy \sin x = 0$; $x^2 \frac{d^3y}{dx^3} - 10xy^2 + \frac{y}{x}\left(\frac{dy}{dx}\right) = 0$; $e^x \frac{dy}{dx} + \frac{\ln x}{\cos x} = 0$

Order of Differential Equation = Highest derivative present in differential equation

Degree of Differential Equation = Power of highest derivative when differential equation is written in polynomial form of derivatives.

The degree of a differential equation which can be written as a polynomial in the derivatives is the degree of the derivative of the highest order occurring in it, after it has been expressed in a form which is free from radicals and fractions so far as derivatives are concerned, thus the differential equation

$$f(x,y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + \ldots = 0 \text{ is order } m$$

and degree p.

Note

Order of differential equation always exists but degree of differential equation may/may not exist.

EXAMPLE:

Let a differential equation $\frac{d^2y}{dx^2} = \left[3 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

By squaring both side, we get

$$\left(\frac{d^2y}{dx^2}\right)^2 = \left[3 + \left(\frac{dy}{dx}\right)^2\right]^3$$

Hence, degree is 2 and order is also 2.

ILLUSTRATION 1

Write order and degree of following differential equation

(i) $\left(\frac{d^2y}{dx^2}\right) = \left(y + \left(\frac{dy}{dx}\right)^8\right)^{1/4}$ **(ii)** $y = e^{\left(\frac{dy}{dx}\right)^3 + \frac{d^4y}{dx^4}}$

(iii) $y = \sin\left(\left(\frac{dy}{dx}\right)^7 - 5\left(\frac{d^5y}{dx^5}\right)^2\right)$

(iv) $e^{\frac{d^3y}{dx^3}} - 2x\frac{d^2y}{dx^2} + (\sin x)y = 0$

(v) $y + \frac{d^4y}{dx^4} = \ln\left(\frac{d^5y}{dx^5} + 1\right)$

(vi) $y = 1 + x\frac{dy}{dx} + \frac{x^2\left(\frac{dy}{dx}\right)^2}{2!} + \frac{x^3\left(\frac{dy}{dx}\right)^3}{3!} + \ldots$

SOLUTION

(i) $\left(\frac{d^2y}{dx^2}\right) = \left(y + \left(\frac{dy}{dx}\right)^8\right)^{1/4}$

$\Rightarrow \left(\frac{d^2y}{dx^2}\right)^4 = y + \left(\frac{dy}{dx}\right)^8$ (Order = 2, Degree = 4)

(ii) $y = e^{\left(\frac{dy}{dx}\right)^3 + \frac{d^4y}{dx^4}}$

$\Rightarrow \ln y = \left(\frac{dy}{dx}\right)^3 + \frac{d^4y}{dx^4}$ (Order = 4, Degree = 1)

(iii) $y = \sin\left(\left(\frac{dy}{dx}\right)^7 - 5\left(\frac{d^5y}{dx^5}\right)^2\right)$

$\Rightarrow \sin^{-1}y = \left(\frac{dy}{dx}\right)^7 - 5\left(\frac{d^5y}{dx^5}\right)^2$ (Order = 5, Degree = 2)

(iv) $e^{\frac{d^3y}{dx^3}} - 2x\frac{d^2y}{dx^2} + (\sin x)y = 0$

$\Rightarrow e^{\frac{d^3y}{dx^3}} = 2x\frac{d^2y}{dx^2} - (\sin x)y$

$\Rightarrow \frac{d^3y}{dx^3} = \ln\left(2x\frac{d^2y}{dx^2} - (\sin x)y\right)$

Order = 3, Degree does not exist as it can not be written in polynomial form of derivatives

(v) $y + \frac{d^4y}{dx^4} = \ln\left(\frac{d^5y}{dx^5} + 1\right)$

$\Rightarrow e^{y + \frac{d^4y}{dx^4}} = \frac{d^5y}{dx^5} + 1$

Hence, order = 5, Degree does not exist as it can not be written in polynomial form of derivatives

(vi) $y = 1 + x\frac{dy}{dx} + \frac{x^2\left(\frac{dy}{dx}\right)^2}{2!} + \frac{x^3\left(\frac{dy}{dx}\right)^3}{3!} + \ldots$

$\left[\because e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]$

$\Rightarrow y = e^{x\frac{dy}{dx}} \Rightarrow \ln y = \frac{xdy}{dx}$

Hence, order and degree of given differential equation are 1

ILLUSTRATION 2

Let λ_1 and λ_2 are order and degree of differential equation $\left(\frac{dy}{dx}\right)^{1/3} + \left(\frac{dy}{dx}\right)^{2/3} = x$, then find $|(\lambda_1 - \lambda_2)|$

SOLUTION

Order of differential equation = $\lambda_1 = 1$

$$\left(\frac{dy}{dx}\right)^{1/3} + \left(\frac{dy}{dx}\right)^{2/3} = x,$$

Cube both sides, we get

$$\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} + 3\frac{dy}{dx}(x) = x^3$$

Now differential equation is second degree in polynomial form of derivatives

Hence, degree of differential equation = $\lambda_2 = 2$

Now $|(\lambda_1 - \lambda_2)| = 1$

FORMATION OF DIFFERENTIAL EQUATION

To form differential equation from given curve's equation we can do following thing

(i) Count arbitrary constant present in curve equation

Arbitrary Constant: in curve's equation where x and y are variable and other parameters are arbitrary constants.

Straight line is represented by equation $y = mx + c$, where x and y are variables and m and c are arbitrary constants.

For all possible real values of m and c, equation $y = mx + c$ represents all possible straight lines.

Equation $\frac{x}{a} + \frac{y}{b} = 1$ also represent straight line where a and b are arbitrary constant

Equation $x\cos\alpha + y\sin\alpha = p$ is normalized form of straight line which also has two arbitrary constant α and p

Equation $ax + by + c = 0$ is general form of straight line and a, b and c are arbitrary constants.

Rewriting it $x + \frac{b}{a}y + \frac{c}{a} = 0$, it has b/a and c/a as two effective arbitrary constants so straight line $ax + by + c = 0$ has only two effective arbitrary constants

Let consider equation

$y = \left(\frac{m_1 + m_2 + m_3}{m_4}\right)^{m_5} x + c_1 e^{c_2 + \ln c_3}$ where we can assume $\left(\frac{m_1 + m_2 + m_3}{m_4}\right)^{m_5} = k$ and $c_1 e^{c_2 + \ln c_3} = t$, then it equation becomes $y = kx + t$ so, finally it has only two effective arbitrary constants

Let consider straight line $y = 2x + c$, It has only one arbitrary constant c and 2 is a fixed constant

Conceptual Note(s)

(i) To identify actual number of effective arbitrary constants, we should simplify curve equation as much as we can.

(ii) Order of differential equation = number of effective arbitrary constant (let *n*)

(iii) Now differentiate curves equation *n* times. The relation in *x*, *y* and their derivatives after eliminating all arbitrary constants is known as differential equation.

Arbitrary constants are never present in differential equation

E.g. $y = mx + c$ is a straight line with two arbitrary constant m and c

$$\frac{dy}{dx} = m, \text{ differentiating it one more time,}$$

we get $\frac{d^2y}{dx^2} = 0$

Differential equation of straight line (Order = 2, Degree = 1)

ILLUSTRATION 3

Identify order of differential equation of following curves

(i) Circle $x^2 + y^2 + 2gx + 2fy + c = 0$

(ii) $y = c_1x + abe^{c(x+d)}$

(iii) $y = Ax - 2Ax^2 + e^{Ax}$

(iv) $y = \left(\frac{c_1+c_2}{c_3}\right) e^{c_4x+c_5} \left(\ln c_6\right)^{c_7}$

SOLUTION

(i) Circle $x^2 + y^2 + 2gx + 2fy + c = 0$ has 3 arbitrary constant g, f and c

Let discuss another form of circle whose centre is (α, β) with radius r

Circle $(x - \alpha)^2 + (y - \beta)^2 = r^2$ has three arbitrary constant α, β and r.

Equation of circle whose end point of diameter are (x_1, y_1) and (x_2, y_2)

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

$\Rightarrow x^2 - (x_1 + x_2)x + y^2 - (y_1 + y_2)y + (x_1x_2 + y_1y_2) = 0$ has three arbitrary constant $(x_1 + x_2)$, $(y_1 + y_2)$ and $(x_1x_2 + y_1y_2)$

$\therefore$ Order of differential equation = 3

(ii) $y = c_1x + abe^{c(x+d)}$

$y = c_1x + abe^{cd}e^{cx}$ has three arbitrary constant c_1, abe^{cd} and c

$\therefore$ Order of differential equation = 3

(iii) $y = Ax - 2Ax^2 + e^{Ax}$ has only one arbitrary constant A

$\therefore$ Order of differential equation = 1

(iv) $y = \left(\frac{c_1+c_2}{c_3}\right) e^{c_4x+c_5} \left(\ln c_6\right)^{c_7}$

Rewriting it $y = \left(\left(\frac{c_1+c_2}{c_3}\right)e^{c_5}\left(\ln c_6\right)^{c_7}\right)e^{c_4x}$ has only two arbitrary constant $\left(\left(\frac{c_1+c_2}{c_3}\right)e^{c_5}\left(\ln c_6\right)^{c_7}\right)$ and c_4.

$\therefore$ Order of differential equation = 2.

ILLUSTRATION 4

Form differential equation of straight-line passing origin.

SOLUTION

Let straight line is $y = mx + c$

Origin (0, 0) lies on it

$$0 = 0 + c \Rightarrow c = 0 \text{ fixed constant}$$

Now straight line $y = mx$ has one arbitrary constant, so order of differential equation is 1

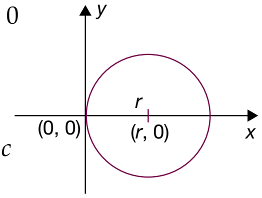
$$\frac{dy}{dx} = m \text{ and } m = \frac{y}{x}$$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$ is a differential equation of given curve with degree and order one.

ILLUSTRATION 5

Form differential Equation of circle which touches y-axis at origin.

SOLUTION



Let circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Origin (0, 0) lies on circle, $0 + c = 0$

$\Rightarrow \quad c = 0$

Circle touches circle at y-axis, $f^2 = c$

$\Rightarrow \quad f = 0$

Now circle $x^2 + y^2 + 2gx = 0$ has 1 arbitrary constant

$$2x + 2y\frac{dy}{dx} + 2g = 0 \qquad (\because\ x^2 + y^2 + 2gx = 0)$$

$$\Rightarrow \quad 2x + 2y\frac{dy}{dx} - \frac{(x^2 + y^2)}{x} = 0$$

$$\Rightarrow \quad 2x^2 + 2xy\frac{dy}{dx} - x^2 - y^2 = 0$$

$\Rightarrow \quad x^2 - y^2 + 2xy\frac{dy}{dx} = 0$ is required differential equation of order 1 and degree 1.

ILLUSTRATION 6

Form differential equation of family of Parabola $y^2 = 4a(x - b)$

SOLUTION

Given curve Parabola $y^2 = 4a(x - b)$ has 2 arbitrary constants a and b.

Thus, $2y\dfrac{dy}{dx} = 4a - 0$

$y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0$ is required differential equation of order 2 and degree 1.

ILLUSTRATION 7

Find differential equation of curve $y = a \sin (bx + c)$ where a and c are arbitrary constant.

SOLUTION

Given curve curve $y = a \sin (bx + c)$ where a and c are two arbitrary constant then b is considered as fixed constant

$$\frac{dy}{dx} = ab\cos(bx + c)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -ab^2\sin(bx + c) = -b^2(a\sin(bx + c))$$

$\dfrac{d^2y}{dx^2} = -b^2y$ is required differential equation with order 2 and degree 1

ILLUSTRATION 8

Write differential equation of following curves

(i) Curve $y = c_1 e^x + c_2 e^{-x}$

(ii) Curve $y = c_1x + \dfrac{c_2}{x}$ where c_1 and c_2 are arbitrary constants

SOLUTION

(i) Curve $y = c_1 e^x + c_2 e^{-x}$

given curve has 2 arbitrary constant c_1 and c_2

$$\frac{dy}{dx} = c_1 e^x - c_2 e^{-x};\ \frac{d^2y}{dx^2} = c_1 e^x + c_2 e^{-x}$$

$\dfrac{d^2y}{dx^2} = y$ is required differential equation with order 2 and degree 1

(ii) Curve $y = c_1x + \dfrac{c_2}{x}$ …(1)

has two arbitrary constants c_1 and c_2

$$\frac{dy}{dx} = c_1 - \frac{c_2}{x^2} \Rightarrow x\frac{dy}{dx} = c_1x - \frac{c_2}{x} \qquad \ldots(2)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 0 + \frac{2c_2}{x^3} \qquad \ldots(3)$$

Subtracting (1) and (2), we get

$$y - x\frac{dy}{dx} = \frac{2c_2}{x}$$

From (3), we get $\dfrac{d^2y}{dx^2} = \left(\dfrac{2c_2}{x}\right)\dfrac{1}{x^2} = \dfrac{y - x\dfrac{dy}{dx}}{x^2}$

$\Rightarrow x^2\dfrac{d^2y}{dx^2} = y - x\dfrac{dy}{dx}$ is required differential equation.

ILLUSTRATION 9

Let circle $(x - h)^2 + (y - k)^2 = a^2$ where a is a constant find differential equation

SOLUTION

Given circle has two arbitrary constant h and k

$$(x - h)^2 + (y - k)^2 = a^2 \qquad \ldots(1)$$

On differentiation

$$\Rightarrow \quad 2(x - h) + 2(y - k)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad (x - h) + (y - k)\frac{dy}{dx} = 0 \qquad \ldots(2)$$

$$\Rightarrow \quad (x - h) = -(y - k)\frac{dy}{dx}$$

From relation (1), we get $(y - k)^2\left(\dfrac{dy}{dx}\right)^2 + (y - k)^2 = a^2$

$$\Rightarrow \quad (y - k)^2 = \frac{a^2}{\left(\dfrac{dy}{dx}\right)^2 + 1} \qquad \ldots(3)$$

Differentiating again relation (2), we get

$$\Rightarrow \quad 1 - 0 + \left(\frac{dy}{dx} - 0\right)\frac{dy}{dx} + (y - k)\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 + (y - k)\frac{d^2y}{dx^2} = 0 \qquad \ldots(1)$$

$$\Rightarrow \quad 1 + \left(\frac{dy}{dx}\right)^2 = -(y - k)\frac{d^2y}{dx^2}$$

By squaring both sides, we get

$$\Rightarrow \quad \left(1 + \left(\frac{dy}{dx}\right)^2\right)^2 = (y - k)^2\frac{d^2y}{dx^2} = \frac{a^2}{\left(\dfrac{dy}{dx}\right)^2 + 1}\frac{d^2y}{dx^2}$$

$\Rightarrow \quad \left(1 + \left(\dfrac{dy}{dx}\right)^2\right)^3 = a^2\dfrac{d^2y}{dx^2}$ is required differential equation of order 2 and degree 1

ILLUSTRATION 10

Find the differential equation of the family of curves $y = e^x(A\cos x + B\sin x)$ where A and B are arbitrary constant.

SOLUTION

Given curve $y = e^x(A\cos x + B\sin x)$ has two arbitrary constant A and B

$y' = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$

$y' = e^x(-A\sin x + B\cos x) + y$

$y'' = e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x) + y'$

$y'' = (y' - y) - y + y'$

$y'' = 2(y' - y)$ is required differential equation

ILLUSTRATION 11

If $y = \dfrac{x}{\ln|cx|}$ (where c is arbitrary constant) is a general solution of differential equation $\dfrac{dy}{dx} = \dfrac{y}{x} + \phi\left(\dfrac{x}{y}\right)$ find $\phi\left(\dfrac{x}{y}\right)$

SOLUTION

$$y = \frac{x}{\ln|cx|} \Rightarrow y\ln|cx| = x$$

On differentiation

$$\frac{dy}{dx}\ln|cx| + y\left(\frac{c}{cx}\right) = 1$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{x}{y}\right) + \frac{y}{x} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2 = \frac{y}{x} + \phi\left(\frac{x}{y}\right)$$

We get $\phi\left(\dfrac{x}{y}\right) = -\left(\dfrac{y}{x}\right)^2$

Test Your Concepts-I

(Solutions on page H.378)

1. find order and degree of the differential equation $xy\left(\dfrac{d^2y}{dx^2}\right)^{\frac{1}{2}} + \left(x^2\dfrac{dy}{dx} - 1\right)^{\frac{3}{4}} = 0$ are

2. Find the degree of the differential equation $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{5}{3}} = \dfrac{d^2y}{dx^2}$

3. Find the order of the differential equation whose general solution is given by $y = (c_1 + c_2)\cos(x + c_3) - c_4 e^{x+c_5}$, where c_1, c_2, c_3, c_4, c_5 are arbitrary constants

4. Find the differential equation representing by family of curve $y^2 = 2c(x + \sqrt{c})$, where c is a positive parameter

5. Obtain a differential equation satisfied by the hyperbola with coordinate axes as transverse and conjugate axis.

6. Form the differential equations of the following curves.
(i) $x^2 - y^2 = c(x^2 + y^2)^2$ (ii) $a(y + a)^2 = x^3$

7. Find the differential equation of the family of curve $y = A\cdot e^{2x} + B\cdot e^{-2x}$ for different values of A and B.

8. Form the differential equation representing the family of curves $y = A\cos 2x + B\sin 2x$, where A and B are arbitrary constants.

9. Find the differential equation satisfied by $y = A\sin(\omega t + \omega_0) + B\cos(\omega t + \omega_0)$ where ω, ω_0 are known constants.

10. Form the differential equation of the family of curves represented by, $c(y + c)^2 = x^3$; where c is any arbitrary constant.

11. Find differential equation of curve $(y + c)^2 = cx$, where c is an arbitrary constant

12. Find the differential equation of the family of curves $y^2 = 4a(x + a)$ is

13. Find the differential equation of the family of curves $cy^2 = 2x + c$, where c is an arbitrary constant.

14. Obtain the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$; where g, f and c are arbitary constants.

15. Find the differential equation of all parabolas whose axes is parallel to the x-axis and having latus rectum a.

Answer key

1. 2, 2 **2.** 3 **3.** 3 **4.** $\left(y - 2x\dfrac{dy}{dx}\right)^2 = 4y\left(\dfrac{dy}{dx}\right)^3$ **5.** $\left(\dfrac{dy}{dx}\right)^2 + y\dfrac{d^2y}{dx^2} = \dfrac{y}{x}\dfrac{dy}{dx}$

6. (i) $\dfrac{dy}{dx} = \dfrac{x(3y^2 - x^2)}{y(3x^2 - y^2)}$ (ii) $8x\left(\dfrac{dy}{dx}\right)^3 - 12y\left(\dfrac{dy}{dx}\right)^2 - 27x = 0$ **7.** $y'' - 4y = 0$ **8.** $\dfrac{d^2y}{dx^2} + 4y = 0$

9. $\dfrac{d^2y}{dt^2} + \omega^2 y = 0$ **10.** $12y(y')^2 = x[3(y')^3 - 27]$ **11.** $y = 2x\dfrac{dy}{dx}\left(1 - 2\dfrac{dy}{dx}\right)$ **12.** $y^2\left(\dfrac{dy}{dx}\right)^2 + 2xy\dfrac{dy}{dx} - y^2 = 0$

13. $y^2 = 2xy\dfrac{dy}{dx} + 1$ **14.** $[1 + (y')^2]\,.\,y''' - 3y'(y'')^2 = 0$ **15.** $a\cdot\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^3 = 0.$

SOLUTION OF DIFFERENTIAL EQUATION

General and Particular Solutions

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equations called the general solution (or integral or primitive).
By satisfying some given condition to the general solution, we get particular solution.

Note

The general solution of a differential equation of the n^{th} order contains 'n' and only 'n' independent arbitrary constants.

Using Variable Separable Approach

Let differential equation $\frac{dy}{dx} = f(x)\, g(y)$

Separating variables x and y terms, to integrate them

$$\Rightarrow \quad \int \frac{dy}{g(y)} = \int f(x)\,dx$$

ILLUSTRATION 12

Solve $\frac{dy}{dx} = (e^x + 1)\,(1 + y^2)$. Also find particular solution when $x = 0$ then $y = 0$

SOLUTION

$$\frac{dy}{dx} = (e^x + 1)\,(1 + y^2) \Rightarrow \int \frac{dy}{1+y^2} = \int e^x + 1\,dx$$

$\Rightarrow$ $\tan^{-1} y = e^x + x + c$ is general solution with c as arbitrary constant

When $x = y = 0$, we have $0 = \tan(1 + c)$; $c = -1$

Hence, $\tan^{-1} y = e^x + x - 1$ is particular solution

ILLUSTRATION 13

Solve $(1 + x)y\,dx = (y - 1)x\,dy$

SOLUTION

$$(1 + x)y\,dx = (y - 1)x\,dy \Rightarrow \int \frac{(1+x)}{x}\,dx = \int \frac{y-1}{y}\,dy$$

$$\Rightarrow \quad \int \frac{1}{x} + 1\,dx = \int 1 - \frac{1}{y}\,dy \Rightarrow \ln x + x = y - \ln y + \lambda$$

ILLUSTRATION 14

Solve $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

SOLUTION

$$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right) \Rightarrow y - ay^2 = (x + a)\,\frac{dy}{dx}$$

$$\Rightarrow \quad \int \frac{dx}{x+a} = \int \frac{dy}{y(1-ay)} = \int \frac{1}{y} + \frac{a}{1-ay}\,dy$$

$$\Rightarrow \quad \ln(x + a) = \ln y - \int \frac{a\,dy}{ay-1} = \ln y - \int \frac{dy}{y - \frac{1}{a}}$$

$$\Rightarrow \quad \ln(x + a) = \ln y - \ln\left(y - \frac{1}{a}\right) + \ln k$$

$$\Rightarrow \quad \ln(x + a) = \ln \frac{ky}{y - \frac{1}{a}} \Rightarrow (x + a) = \frac{ky}{y - \frac{1}{a}}$$

$$\Rightarrow \quad (x + a)\,(ay - 1) = kay$$

ILLUSTRATION 15

Solve $\cos x \cos y \frac{dy}{dx} = -\sin x \sin y$

SOLUTION

$$\cos x \cos y \frac{dy}{dx} = -\sin x \sin y \Rightarrow \int \frac{\cos y\,dy}{\sin y} = \int \frac{-\sin x}{\cos x}dx$$

$$\Rightarrow \quad \int \cot y\,dy = -\int \tan x\,dx$$

$$\Rightarrow \quad \ln \sin y = -\ln \sec x + \ln k$$

$$\Rightarrow \quad \ln \sin y + \ln \sec x = \ln k$$

$$\Rightarrow \quad \ln(\sin y \sec x) = \ln k \Rightarrow \frac{\sin y}{\cos x} = k$$

$$\Rightarrow \quad \sin y = k \cos x$$

ILLUSTRATION 16

Solve $\operatorname{cosec} x \ln y\,dy + x^2 y\,dx = 0$

SOLUTION

$$\operatorname{cosec} x \ln y\,dy + x^2 y\,dx = 0$$

$$\Rightarrow \quad \operatorname{cosec} x \ln y\,dy = -x^2 y\,dx$$

$$\Rightarrow \quad \int \frac{\ln y}{y}\,dy = \int \frac{-x^2\,dx}{\operatorname{cosec} x} = -\int x^2 \sin x\,dx$$

$$\Rightarrow \quad \int \frac{\ln y}{y}\,dy = -\int \underset{\text{I}}{x^2}\,\underset{\text{II}}{\sin x}\,dx$$

Put $\ln y = t \Rightarrow \frac{1}{y}dy = dt$ in first integral and using by part integration in RHS integral

$$\Rightarrow \quad \int t\,dt = x^2(\cos x) - \int \underset{\text{I}}{2x}\,\underset{\text{II}}{\cos x}\,dx$$

$$\Rightarrow \quad \frac{t^2}{2} = x^2 \cos x - 2\left[x \sin x - \int 1\,.\,\sin x\,dx\right]$$

$$\Rightarrow \quad \frac{\ln^2 y}{2} = x^2\cos x - 2x\sin x + 2(-\cos x) + c$$

$$\Rightarrow \quad \frac{\ln^2 y}{2} = x^2\cos x - 2x\sin x - 2\cos x + c$$

ILLUSTRATION 17

Solve $xy\dfrac{dy}{dx} = \dfrac{(1+y^2)(1+x+x^2)}{(1+x^2)}$

SOLUTION

$$xy\frac{dy}{dx} = \frac{(1+y^2)(1+x+x^2)}{(1+x^2)}$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{2y\,dy}{1+y^2} = \int\frac{1+x+x^2}{x(1+x^2)}dx = \int\frac{1}{x}+\frac{1}{1+x^2}dx$$

$$\Rightarrow \quad \frac{1}{2}\ln(1+y^2) = \ln x + \tan^{-1}x + c$$

ILLUSTRATION 18

Solve $1 + x + y + xy = \dfrac{dy}{dx}$

SOLUTION

$$1 + x + y + xy = \frac{dy}{dx} \Rightarrow (1+x)\,(1+y) = \frac{dy}{dx}$$

$$\Rightarrow \quad \int(1+x)\,dx = \int\frac{dy}{1+y} \Rightarrow x + \frac{x^2}{2} + \lambda = \ln(1+y)$$

ILLUSTRATION 19

Solve $\left(\dfrac{dy}{dx}\right)^2 - (x+y)\dfrac{dy}{dx} + xy = 0$

SOLUTION

It is a quadratic in $\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{(x+y)\pm\sqrt{(x+y)^2-4xy}}{2} = \frac{(x+y)\pm\sqrt{(x-y)^2}}{2}$$

$$\frac{dy}{dx} = \frac{(x+y)\pm(x-y)}{2}$$

By taking positive sign $\dfrac{dy}{dx} = \dfrac{(x+y)+(x-y)}{2} = x$

$$\Rightarrow \quad dy = xdx \Rightarrow y = \frac{x^2}{2} + c$$

By taking negative sign $\dfrac{dy}{dx} = \dfrac{(x+y)-(x-y)}{2} = y$

$$\Rightarrow \quad \frac{dy}{dx} = y \quad \Rightarrow \quad \frac{dy}{y} = dx \Rightarrow \ln y = x + c$$

Solution of given differential equation are

$$y = \frac{x^2}{2} + c \text{ and } \ln y = x + c$$

REDUCIBLE TO VARIABLE SEPARABLE FORM

$$\frac{dy}{dx} = f(ax+by+c),\ b \neq 0.$$

To solve this, substitute $t = ax + by + c$. Then the equation reduces to separable type in the variable t and x which can be solved.

ILLUSTRATION 20

Solve $\dfrac{dy}{dx} = (x+y+1)^2$

SOLUTION

Put $x + y + 1 = t \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\frac{dy}{dx} = (x+y+1)^2$$

$$\Rightarrow \quad \frac{dt}{dx} - 1 = t^2 \Rightarrow \frac{dt}{dx} = 1 + t^2$$

$$\Rightarrow \quad \int\frac{dt}{1+t^2} = \int dx \Rightarrow \tan^{-1}t = x + c$$

$$\Rightarrow \quad \tan^{-1}(x+y+1) = x + c$$

ILLUSTRATION 21

Solve $\dfrac{dy}{dx} = 1 + e^{x-y}$

SOLUTION

Put $x - y = t \Rightarrow 1 - \dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\Rightarrow \quad 1 - \frac{dt}{dx} = 1 + e^t$$

$$\Rightarrow \quad -\frac{dt}{e^t} = dx \Rightarrow \int -e^{-t}\,dt = \int dx \Rightarrow e^{-t} = x + c$$

$$\Rightarrow \quad e^{y-x} = x + c \Rightarrow e^y = e^x\,(x+c)$$

ILLUSTRATION 22

Solve $\sin^{-1}\left(\dfrac{dy}{dx}\right) = x + y$

SOLUTION

$$\sin^{-1}\left(\frac{dy}{dx}\right) = x + y \Rightarrow \frac{dy}{dx} = \sin(x+y)$$

Put $x + y = t \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\Rightarrow \quad \frac{dt}{dx} - 1 = \sin t \Rightarrow dt = (1+\sin t)dx$$

$$\Rightarrow \quad \int\frac{dt}{1+\sin t} = \int dx \qquad \left(\because \sin t = \frac{2\tan\frac{t}{2}}{1+\tan^2\frac{t}{2}}\right)$$

$$\Rightarrow \int \frac{\sec^2 \frac{t}{2}\, dt}{\left(1+\tan^2 \frac{t}{2}\right)+2\tan \frac{t}{2}} = x+c \Rightarrow \int \frac{\sec^2 \frac{t}{2}\, dt}{\left(1+\tan \frac{t}{2}\right)^2} = x+c$$

$$\Rightarrow -\frac{2}{\left(1+\tan \frac{t}{2}\right)} = x+c \quad \text{where } t = x+y$$

$$\Rightarrow -2 = (x+c)\left(1+\tan\left(\frac{x+y}{2}\right)\right)$$

ILLUSTRATION 23

Solve $\dfrac{dy}{dx} = \sin(x+y)+\cos(x+y)$

SOLUTION

Put $x+y=t \Rightarrow 1+\dfrac{dy}{dx} = \dfrac{dt}{dx}$

Given differential equation becomes

$$\frac{dt}{dx} - 1 = \sin t + \cos t$$

$$\Rightarrow \frac{dt}{dx} = 1 + \sin t + \cos t$$

$$\Rightarrow \frac{dt}{dx} = 1 + \frac{2\tan\frac{t}{2}}{1+\tan^2\frac{t}{2}} + \frac{1-\tan^2\frac{t}{2}}{1+\tan^2\frac{t}{2}} = \frac{2\left(1+\tan\frac{t}{2}\right)}{\sec^2\frac{t}{2}}$$

$$\Rightarrow \int \frac{\sec^2\frac{t}{2}}{2\left(1+\tan\frac{t}{2}\right)}\, dt = \int dx \Rightarrow \ln\left(1+\tan\frac{t}{2}\right) = x+c$$

$$\Rightarrow \ln\left[1+\tan\left(\frac{x+y}{2}\right)\right] = x+c$$

ILLUSTRATION 24

Solve $(x^2+4y^2+4xy)\, dy = (2x+4y+1)\, dx$

SOLUTION

Rewriting given differential equation $\dfrac{dy}{dx} = \dfrac{2(x+2y)+1}{(x+2y)^2}$

Let $x+2y=t \Rightarrow 1+2\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}\left(\frac{dt}{dx}-1\right) = \frac{2t+1}{t^2}$$

$$\Rightarrow \frac{dt}{dx} = \frac{t^2+4t+2}{t^2} \Rightarrow \int \frac{t^2}{t^2+4t+2}\, dt = \int dx$$

$$\Rightarrow \int 1 - \frac{2(2t+4)}{t^2+4t+2} + \frac{6}{t^2+4t+2}\, dt = x+c$$

$$\Rightarrow t - 2\ln(t^2+4t+2) + \frac{3}{\sqrt{2}} \ln\frac{t+2-\sqrt{2}}{t+2+\sqrt{2}} = x+c$$

where $t = x+2y$

Test Your Concepts-II

(Solutions on page H.379)

1. Solve $\dfrac{dy}{dx} = \sec(x+y)$
2. Solve $\dfrac{dy}{dx} + yf'(x) = f(x)\cdot f'(x)$ where $f(x)$ is a given function of x.
3. Find solution of the differential equation $\ln\left(\dfrac{dy}{dx}\right) = ax+by$
4. Find the solution of the differential equation $x\dfrac{d^2y}{dx^2} = 1$, given that $y = 1$, $\dfrac{dy}{dx} = 0$, when $x = 1$
5. Find equation of curve through point (1, 0) which satisfies the differential equation $(1+y^2)\, dx - xy\, dy = 0$
6. Show that the curve passing through the point (0, 1) and satisfying the equation $\sin\left(\dfrac{dy}{dx}\right) = a$ is $\sin\left(\dfrac{y-1}{x}\right) = a$
7. Solve $\dfrac{\ln(\sec x + \tan x)}{\cos x}\, dx = \dfrac{\ln(\sec y + \tan y)}{\cos y}\, dy$
8. Solve $\dfrac{dy}{dx} + \dfrac{\sqrt{(x^2-1)(y^2-1)}}{xy} = 0$
9. Find the general solution of the differential equation $x(1+y^2)dx + y(1+x^2)dy = 0$ is
10. If $e^{(dy/dx)} = x+1$ given that when $x = 0$, $y = 3$
11. Solve: $(x^3 - y^2x^3)\dfrac{dy}{dx} + y^3 + x^2y^3 = 0$.
12. Solve: $\dfrac{dy}{dx} = \cos(x+y) - \sin(x+y)$.

Answer key

1. $(x+y)-\tan\dfrac{(x+y)}{2}=x+c$

2. $f(x)+\log\{1+y-f(x)\}=c$

3. $-\dfrac{1}{b}e^{-by}=\dfrac{1}{a}e^{ax}+c$

4. $y=x\log x-x+2$

5. $x^2-y^2=1$

7. $\ln^2(\sec x+\tan x)-\ln^2(\sec y+\tan y)=c$

8. $\sqrt{x^2-1}-\sec^{-1}x+\sqrt{y^2-1}=c$

9. $\left(1+x^2\right)\left(1+y^2\right)=c$

10. $y=(x+1).\ln(x+1)-x+3$

11. $\log\left(\dfrac{x}{y}\right)=\dfrac{1}{2}\left(\dfrac{1}{y^2}-\dfrac{1}{x^2}\right)+c$

12. $-\ln\left|1-\tan\dfrac{x+y}{2}\right|=x+c$

SOLVING HOMOGENOUS DIFFERENTIAL EQUATION

$\dfrac{dy}{dx}=\dfrac{f(x,y)}{g(x,y)}$ is known as homogenous differential equation if power of x and y are same in each term of numerator and denominator.

Put $\dfrac{y}{x}=t\Rightarrow y=tx$

ILLUSTRATION 25

Solve $x^2\dfrac{dy}{dx}=x^2+xy+y^2$

SOLUTION

$$\frac{dy}{dx}=\frac{x^2+xy+y^2}{x^2}=1+\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2 \qquad \ldots(1)$$

Put $\dfrac{y}{x}=t\Rightarrow y=xt\Rightarrow\dfrac{dy}{dx}=t+\dfrac{x\,dt}{dx}$

From (1), we have $t+x\dfrac{dt}{dx}=1+t+t^2$

$$\Rightarrow\quad \frac{x\,dt}{dx}=1+t^2\Rightarrow\int\frac{dt}{1+t^2}=\int\frac{dx}{x}$$

$$\Rightarrow\quad \tan^{-1}t=\ln x+c\Rightarrow\frac{y}{x}=\tan(\ln x+c)$$

ILLUSTRATION 26

Solve $x\dfrac{dy}{dx}=y(\ln y-\ln x+1)$

SOLUTION

$$\frac{dy}{dx}=\frac{y}{x}\left[\ln\left(\frac{y}{x}\right)+1\right]$$

Put $\dfrac{y}{x}=t\Rightarrow y=xt\Rightarrow\dfrac{dy}{dx}=t+\dfrac{x\,dt}{dx}$

From (1), we have $t+x\dfrac{dt}{dx}=t(\ln t+1)$

$$\Rightarrow\quad x\frac{dt}{dx}=t\ln t\Rightarrow\int\frac{dt}{t\ln t}=\int\frac{dx}{x}$$

$$\Rightarrow\quad \int\frac{dz}{z}=\int\frac{dx}{x}\qquad\text{Put }\ln t=z\Rightarrow\frac{1}{t}dt=dz$$

$$\Rightarrow\quad \ln z=\ln x+\ln k=\ln kx$$

$\Rightarrow\quad z=kx$ substituting values of z and t

$$\Rightarrow\quad \ln t=kx\Rightarrow\ln\frac{y}{x}=kx\Rightarrow\frac{y}{x}=e^{kx}$$

$\Rightarrow\quad y=xe^{kx}$ is solution of given differential equation

ILLUSTRATION 27

Solve $\dfrac{2y}{x}+\left(\left(\dfrac{y}{x}\right)^2-1\right)\dfrac{dy}{dx}=0$

SOLUTION

Put $\dfrac{y}{x}=t\Rightarrow y=tx\Rightarrow\dfrac{dy}{dx}=t+x\dfrac{dt}{dx}$

Given $\dfrac{2y}{x}+\left(\left(\dfrac{y}{x}\right)^2-1\right)\dfrac{dy}{dx}=0$

$$\Rightarrow\quad 2t+(t^2-1)\left(t+x\frac{dt}{dx}\right)=0$$

$$\Rightarrow\quad 2t+t^3-t+(t^2-1)\,x\frac{dt}{dx}=0$$

$$\Rightarrow\quad t(1+t^2)=(1-t^2)\,x\frac{dt}{dx}\Rightarrow\frac{(1-t^2)}{t(1+t^2)}dt=\frac{dx}{x}$$

$$\Rightarrow\quad \int\frac{1}{t}-\frac{2t}{1+t^2}dt=\int\frac{dx}{x}$$

$$\Rightarrow\quad \ln t-\ln(1+t^2)=\ln x+\ln k\Rightarrow\frac{t}{1+t^2}=kx$$

$$\Rightarrow\quad \frac{\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}=kx\Rightarrow\frac{yx}{x^2+y^2}=kx$$

$$\Rightarrow\quad (x^2+y^2)\,kx=xy$$

So, circle $x^2+y^2=\dfrac{y}{k}$ or y-axis $x=0$

ILLUSTRATION 28

Solve it $(x^2 - y^2)dx + 2xy\,dy = 0$; if $y = 1$ when $x = 1$.

SOLUTION

Rewritting given differential equation

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 1}{2\left(\frac{y}{x}\right)}$$

Put $\frac{y}{x} = t \Rightarrow y = tx \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$$\Rightarrow \quad t + x\frac{dt}{dx} = \frac{t^2 - 1}{2t} \Rightarrow x\frac{dt}{dx} = \frac{-1 - t^2}{2t}$$

$$\Rightarrow \quad \int \frac{2t\,dt}{t^2 + 1} = -\int \frac{dx}{x}$$

$$\Rightarrow \quad \ln(t^2 + 1) = -\ln x + \ln k = \ln\left(\frac{k}{x}\right)$$

$$\Rightarrow \quad t^2 + 1 = \frac{k}{x} \Rightarrow \left(\frac{y}{x}\right)^2 + 1 = \frac{k}{x}$$

$$\Rightarrow \quad y^2 + x^2 = kx$$

It is **general solution** of differential equation.
Given $y = 1$ when $x = 1$, $1 + 1 - k = 0 \Rightarrow k = 2$. Hence,
$x^2 + y^2 - 2x = 0$ is **particular solution** of differential equation

ILLUSTRATION 29

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)} \quad \text{find } \phi\left(\frac{y}{x}\right)$$

SOLUTION

Put $\frac{y}{x} = t \Rightarrow y = tx \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$$\frac{dy}{dx} = \frac{y}{x} + \frac{\phi\left(\frac{y}{x}\right)}{\phi'\left(\frac{y}{x}\right)}$$

$$\Rightarrow \quad t + x\frac{dt}{dx} = t + \frac{\phi(t)}{\phi'(t)} \Rightarrow x\,\phi'(t)\frac{dt}{dx} = \phi(t)$$

$$\Rightarrow \quad \int \frac{\phi'(t)}{\phi(t)}dt = \int \frac{dx}{x} \Rightarrow \ln \phi(t) = \ln x + \ln k$$

$$\Rightarrow \quad \ln \phi(t) = \ln kx \Rightarrow \phi(t) = kx \Rightarrow \phi\left(\frac{y}{x}\right) = kx$$

ILLUSTRATION 30

If $x \sin\left(\frac{y}{x}\right)dy = \left(y \sin\frac{y}{x} - x\right)dx$ and $y(1) = \frac{\pi}{2}$,

find value of $\cos\left(\frac{y}{x}\right)$.

SOLUTION

$$\sin\frac{y}{x}\frac{dy}{dx} = \frac{y}{x}\sin\frac{y}{x} - 1$$

Put $\frac{y}{x} = t \Rightarrow y = tx \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

$$\Rightarrow \quad \sin t\left(t + x\frac{dt}{dx}\right) = t\sin t - 1$$

$$\Rightarrow \quad x\sin t\frac{dt}{dx} = -1 \Rightarrow \int \sin t\,dt = -\int \frac{dx}{x}$$

$$\Rightarrow \quad -\cos t = -\ln x + c \Rightarrow \cos\frac{y}{x} = \ln x - c$$

Given $y(1) = \frac{\pi}{2}$; $\cos\left(\frac{\pi}{2}\right) = \ln 1 - c$

$$\Rightarrow \quad 0 = 0 - c \Rightarrow c = 0$$

Hence, $\cos\frac{y}{x} = \ln x$

ILLUSTRATION 31

Solve: $x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$

SOLUTION

$$\frac{dy}{dx} = \frac{\sqrt{x^2 + y^2} + y}{x} = \sqrt{1 + \left(\frac{y}{x}\right)^2} + \frac{y}{x}$$

Let $y = vx \Rightarrow \frac{dy}{dx} = v + \frac{xdv}{dx}$

$$\Rightarrow \quad v + \frac{xdv}{dx} = \sqrt{1 + v^2} + v \Rightarrow \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}$$

On integrating, we get

$$\ln(v + \sqrt{1 + v^2}) = \ln x + \ln C$$

$$\Rightarrow \quad v + \sqrt{1 + v^2} = cx \Rightarrow y + \sqrt{x^2 + y^2} = cx^2$$

ILLUSTRATION 32

Solve $\frac{xdy}{x^2 + y^2} = \left(\frac{y}{x^2 + y^2} - 1\right)dx$

SOLUTION

$$\frac{dy}{dx} = \left(\frac{y}{x^2 + y^2} - 1\right)\left(\frac{x^2 + y^2}{x}\right) = \left(\frac{y}{x} - \frac{x^2 + y^2}{x}\right)$$

Putting $y = vx \Rightarrow \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$

$$\Rightarrow \quad v + x\cdot\frac{dv}{dx} = \left(v - \frac{x^2 + v^2x^2}{x}\right)$$

$$\Rightarrow \quad v + x\cdot\frac{dv}{dx} = v - x\left(1 + v^2\right) \Rightarrow \frac{dv}{1 + v^2} = -dx$$

$$\Rightarrow \quad \int \frac{dv}{1 + v^2} = -\int dx \Rightarrow \tan^{-1} v = -x + c$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) = -x + c$$

ILLUSTRATION 33

Solve the differential equation

$y\cos\frac{y}{x}\,(xdy-ydx)+x\sin\frac{y}{x}\,(xdy+ydx)=0$. When $y(1)=\frac{\pi}{2}$

SOLUTION

Given equation can be written as

$$\frac{y}{x}\cos x\frac{y}{x}\left(\frac{dy}{dx}-\frac{y}{x}\right)+\sin\frac{y}{x}\left(\frac{dy}{dx}+\frac{y}{x}\right)=0$$

$$\Rightarrow\quad \frac{dy}{dx}=\frac{\left(\frac{y}{x}\right)^2\cos\frac{y}{x}-\frac{y}{x}\sin\frac{y}{x}}{\frac{y}{x}\cos\frac{y}{x}+\sin\frac{y}{x}}$$

Let $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}$

$$\Rightarrow\quad v+x\frac{dv}{dx}=\frac{v^2\cos v-v\sin v}{v\cos v+\sin v}\Rightarrow x\frac{dv}{dx}=-\frac{2v\sin v}{v\cos v+\sin v}$$

$$\Rightarrow\quad \int\left(\frac{v\cos v+\sin v}{v\sin v}\right)dv=-\int\frac{2}{x}dx$$

$$\Rightarrow\quad \ln\sin v+\ln v=-\ln x^2+\ln c$$

$$\Rightarrow\quad (v\sin v)\,x^2=c \qquad \text{Given } y(1)=\frac{\pi}{2}\Rightarrow c=\frac{\pi}{2}$$

$$\Rightarrow\quad (v\sin v)\,x^2=\frac{\pi}{2}\Rightarrow xy\sin\frac{y}{x}=\frac{\pi}{2}$$

ILLUSTRATION 34

Solve

$$\left(x\cos\left(\frac{y}{x}\right)+y\sin\left(\frac{y}{x}\right)\right)y-\left(y\sin\frac{y}{x}-x\cos\frac{y}{x}\right)x\frac{dy}{dx}=0$$

SOLUTION

Divide it by x^2, we get

$$\left(\cos\left(\frac{y}{x}\right)+\frac{y}{x}\sin\left(\frac{y}{x}\right)\right)\frac{y}{x}-\left(\frac{y}{x}\sin\frac{y}{x}-\cos\frac{y}{x}\right)\frac{dy}{dx}=0$$

Put $\frac{y}{x}=v\Rightarrow y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dy}{dx}$

Then given differential equation reduces to

$$(\cos v+v\sin v)v-(v\sin v-\cos v)\left(v+x\frac{dy}{dx}\right)=0$$

$$\Rightarrow\quad (v\cos v+v^2\sin v-v^2\sin v+v\cos v)-x(v\sin v-\cos v)\frac{dv}{dx}=0$$

$$\Rightarrow\quad x\,(v\sin v-\cos v)\frac{dv}{dx}=2v\cos v$$

$$\Rightarrow\quad \frac{2}{x}dx=\left(\tan v-\frac{1}{v}\right)dv$$

$$\Rightarrow\quad 2\log x+\log c=\log(\sec v)-\log v$$

$$\Rightarrow\quad \log x^2c=\log\frac{\sec v}{v}\Rightarrow x^2c=\frac{\sec\frac{y}{x}}{\frac{y}{x}}\Rightarrow xyc=\sec\frac{y}{x}$$

ILLUSTRATION 35

Solve $(1+e^{x/y})\,dx+e^{x/y}\left(1-\frac{x}{y}\right)dy=0$

SOLUTION

Putting $x=vy$, we have $\frac{dx}{dy}=v+y\frac{dv}{dy}$

The given equation reduces to

$$(1+e^v)\left(v+y\frac{dv}{dy}\right)+e^v(1-v)=0$$

$$\Rightarrow\quad (v+ve^v+e^v-ve^v)=-(1+e^v)\,y\frac{dv}{dy}\Rightarrow\frac{dy}{y}=-\frac{1+e^v}{v+e^v}\,dv$$

Integrating, we have

$\log y+\log(v+e^v)=\text{constant}\Rightarrow\log y\left(\frac{x}{y}+e^{x/y}\right)=\text{constant}$

$$\Rightarrow\quad x+ye^{x/y}=C$$

REDUCIBLE TO HOMOGENEOUS DIFFERENTIAL EQUATION

Consider differential equation $\frac{dy}{dx}=\frac{ax+by+c}{\alpha x+\beta y+\gamma}$

CASE 1:

If $\frac{a}{\alpha}=\frac{b}{\beta}$ then put $ax+by=t$ and solve it

CASE 2:

$b+\alpha=0$ then cross multiply and solve it

CASE 3:

If $\frac{a}{\alpha}\neq\frac{b}{\beta}$ and $b+\alpha\neq0$

Put $x=X+h;\ y=Y+k$

$$\frac{dy}{dx}=\frac{dY}{dX}=\frac{aX+bY+(ah+bk+c)}{\alpha X+\beta Y+(\alpha h+\beta k+\gamma)}$$

$$\left.\begin{array}{l}ah+bk+c=0\\ \alpha h+\beta k+\gamma=0\end{array}\right]\text{get } h \text{ and } k$$

Now put $\frac{Y}{X}=t$ and further solve it.

ILLUSTRATION 36

Solve $\frac{dy}{dx}=\frac{2x+3y-1}{4x+6y+5}$

SOLUTION

Ratios of coefficients of x and y in numerator and denominator are equal

$$\frac{a}{\alpha} = \frac{b}{\beta} \Rightarrow \frac{2}{4} = \frac{3}{6}$$

Put $2x + 3y = t \Rightarrow 2 + 3\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\frac{1}{3}\left(\frac{dt}{dx} - 2\right) = \frac{t-1}{2t+5} \Rightarrow \frac{dt}{dx} - 2 = \frac{3t-3}{2t+5}$$

$$\Rightarrow \quad \frac{dt}{dx} = 2 + \frac{3t-3}{2t+5} \Rightarrow \frac{dt}{dx} = \frac{7t+7}{2t+5}$$

$$\Rightarrow \quad \int \frac{2t+5}{t+1}\,dt = \int 7\,dx$$

$$\Rightarrow \quad \int 2 + \frac{3}{t+1}\,dt = 7x + c \Rightarrow 2t + 3\ln(t+1) = 7x + c$$

Where $t = 2x + 3y$

$$\Rightarrow \quad 2(2x + 3y) + 3\ln(2x + 3y + 1) = 7x + c$$

$$\Rightarrow \quad \ln(2x + 3y + 1) = x - 2y + c'$$

ILLUSTRATION 37

Solve $\dfrac{dy}{dx} = \dfrac{x - 2y + 5}{2x + y - 1}$

SOLUTION

Here, $b + \alpha = -2 + 2 = 0$,

Cross multiply, we obtain

$$(2x + y - 1)dy = (x - 2y + 5)dx$$

$$\Rightarrow \quad 2x\,dy + y\,dy - dy = x\,dx - 2\,y\,dx + 5\,dx$$

$$\Rightarrow \quad \int 2(xdy + ydx) + \int ydy - \int dy = \int xdx + \int 5dx$$

$$\Rightarrow \quad 2\int d(xy) + \frac{y^2}{2} - y = \frac{x^2}{2} + 5x + c$$

$$\Rightarrow \quad 2xy + \frac{y^2}{2} - y = \frac{x^2}{2} + 5x + c$$

ILLUSTRATION 38

Solve $\dfrac{dy}{dx} = \dfrac{x + 2y - 3}{2x + y - 3}$

SOLUTION

Put $x = X + h$ and $y = Y + k$.

$$\frac{dY}{dX} = \frac{X + 2Y + h + 2k - 3}{2X + Y + 2h + k - 3} \qquad \ldots(1)$$

Now $h + 2k - 3 = 0$ and $2h + k - 3 = 0$ we get $h = 1, k = 1$

Now (1) becomes, $\dfrac{dY}{dX} = \dfrac{X + 2Y}{2X + Y} = \dfrac{1 + 2Y/X}{2 + Y/X}$

Put $Y = vX \Rightarrow \dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

Then, $v + X\dfrac{dv}{dX} = \dfrac{1 + 2v}{2 + v} \Rightarrow X\dfrac{dv}{dX} = \dfrac{1 - v^2}{2 + v}$

$$\Rightarrow \quad \frac{2+v}{1-v^2}\,dv = \frac{1}{X}\,dX$$

$$\Rightarrow \quad \int \frac{2}{1-v^2}\,dv + \int \frac{v}{1-v^2}\,dv = \int \frac{1}{X}\,dX$$

$$\Rightarrow \quad \log\left|\frac{1+v}{1-v}\right| - \frac{1}{2}\log|1-v^2| = \log(X) + \log c$$

$$\Rightarrow \quad \log\left|\frac{1+v}{1-v}\right| - \frac{1}{2}\log|1+v| - \frac{1}{2}\log|1-v| = \log(cX)$$

$$\Rightarrow \quad \log\left(\frac{|1+v|^{1/2}}{|1-v|^{3/2}}\right) = \log(cX) \Rightarrow \frac{|Y+X|^{1/2}}{|X-Y|^{3/2}} = c$$

$$x = X + 1 \Rightarrow X = x - 1$$

$$y = Y + 1 \Rightarrow Y = y - 1$$

$$\Rightarrow \quad |x + y - 2|^{1/2} = c\,|y - x|^{3/2}$$

$\Rightarrow \quad |x + y - 2| = c\,|y - x|^3$ is required solution

ILLUSTRATION 39

Solve $(3y - 7x + 7)dx + (7y - 3x + 3)dy = 0$

SOLUTION

$$\frac{dy}{dx} = \frac{7x - 3y - 7}{-3x + 7y + 3}$$

Put $x = X + h;\ y = Y + k$

$$\frac{dy}{dx} = \frac{dY}{dX} = \frac{7X + 7h - 3(Y + k) - 7}{-3(X + h) + 7(Y + k) + 3}$$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{7X - 3Y + 7h - 3k - 7}{-3X + 7Y - 3h + 7k + 3}$$

$7h - 3k - 7 = 0$ and $-3h + 7k + 3 = 0$ we get $h = 1, k = 0$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{7X - 3Y}{-3X + 7Y} = \frac{7 - 3Y/X}{-3 + 7Y/X} \qquad \ldots(1)$$

Put $\dfrac{Y}{X} = t \Rightarrow Y = tX \Rightarrow \dfrac{dY}{dX} = t + X\dfrac{dt}{dX}$

From (1),

$$t + X\frac{dt}{dX} = \frac{7-3t}{-3+7t} \Rightarrow X\frac{dt}{dX} = \frac{7-3t}{-3+7t} - t = \frac{7-7t^2}{7t-3}$$

$$\Rightarrow \quad \int \frac{7t-3}{t^2-1}\,dt = -\int \frac{7dX}{X}$$

$$\Rightarrow \quad \frac{7}{2}\int \frac{2t}{t^2-1}\,dt - 3\int \frac{dt}{t^2-1} = -7\ln X + c$$

$$\Rightarrow \quad \frac{7}{2}\ln(t^2-1) - \frac{3}{2}\ln\left(\frac{t-1}{t+1}\right) = -7\ln(X-1) + c$$

Where $x = X + 1;\ y = Y + 0$; where $t = \dfrac{Y}{X} = \dfrac{y}{x-1}$

$\dfrac{7}{2}\ln\left(\left(\dfrac{y}{x-1}\right)^2 - 1\right) - \dfrac{3}{2}\ln\left(\dfrac{y-x+1}{y+x-1}\right) = -7\ln(x-1) + c$ is solution of given differential equation.

ILLUSTRATION 40

Find the solution of the differential equation

$$\left(2x^2+3y^2-7\right)xdx-\left(3x^2+2y^2-8\right)ydy=0$$

SOLUTION

Rewriting differential equation $\dfrac{2ydy}{2xdx}=\dfrac{2x^2+3y^2-7}{3x^2+2y^2-8}$

Put $x^2=X$ and $y^2=Y$

$$\Rightarrow \quad \frac{dY}{dX}=\frac{2X+3Y-7}{3X+2Y-8}$$

Put $X=u+h, Y=v+k$

$$\Rightarrow \quad \frac{dy}{dx}=\frac{dv}{du}=\frac{2u+3v+2h+3k-7}{3u+2v+3h+2k-8}$$

Solving $2h+3k-7=0$ and $3h+2k-8=0$

Here, $h=2; k=1$

$$\Rightarrow \quad \frac{dv}{du}=\frac{2u+3v}{3u+2v}=\frac{2+3v/u}{3+2v/u} \qquad \ldots(2)$$

Put $\dfrac{v}{u}=t\Rightarrow v=tu\Rightarrow \dfrac{dv}{du}=t+u\dfrac{dt}{du}$

From (2)

$$t+u\frac{dt}{du}=\frac{2+3t}{3+2t}\Rightarrow \frac{u\,dt}{du}=\frac{2+3t}{3+2t}-t=\frac{2-2t^2}{3+2t}$$

$$\Rightarrow \quad \frac{3+2t}{2-2t^2}dt=\frac{du}{u}\Rightarrow \int-\frac{dt}{2(t^2-1)}-\int\frac{dt}{t-1}=\int\frac{du}{u}$$

$$\Rightarrow \quad -\frac{1}{4}\ln\left|\frac{t-1}{t+1}\right|-\ln|t-1|=\ln u+\ln k$$

$$\Rightarrow \quad -\frac{5}{4}\ln|t-1|+\frac{1}{4}\ln|t+1|=\ln k\,u$$

$$\Rightarrow \quad \ln\left|\frac{t+1}{(t-1)^5}\right|=\ln(ku)^4 \text{ where } t=\frac{v}{u}$$

$$\Rightarrow \quad \frac{v+u}{u\left(\dfrac{v-u}{u}\right)^5}=k^4u^4\Rightarrow \frac{v+u}{(v-u)^5}=k^4$$

$$\Rightarrow \quad (v+u)=k^4(v-u)^5$$

We have $X=x^2=u+2;\ y=y^2=v+1$

$$u=x^2-2;\ v=y^2-1$$

Hence, curve $(x^2+y^2-3)=c(y^2-x^2+1)^5$

ILLUSTRATION 41

Solve $\dfrac{dy}{dx}=\dfrac{\sin y+x}{\sin 2y-x\cos y}$

SOLUTION

Here, $\dfrac{dy}{dx}=\dfrac{\sin y+x}{\sin 2y-x\cos y}$

$$\Rightarrow \quad \cos y\frac{dy}{dx}=\frac{\sin y+x}{2\sin y-x},$$

$$\left(\text{put }\sin y=t\Rightarrow \cos y\frac{dy}{dx}=\frac{dt}{dx}\right)$$

$$\Rightarrow \quad \frac{dt}{dx}=\frac{t+x}{2t-x}, \quad (\text{put } t=vx)$$

$$\Rightarrow \quad \frac{xdv}{dx}+v=\frac{vx+x}{2vx-x}=\frac{v+1}{2v-1}$$

$$\therefore \quad x\frac{dv}{dx}=\frac{v+1}{2v-1}-v=\frac{v+1-2v^2+v}{2v-1}$$

$$\text{or} \quad \int\frac{2v-1}{-2v^2+2v+1}dv=\int\frac{dx}{x}$$

On solving, we get

$$-\frac{1}{2}\ln(2u^2-2u-1)=\ln x+\ln k=\ln kx$$

$$\Rightarrow \quad \ln(2u^2-2u-1)=\ln\frac{1}{(kx)^2}$$

$$\Rightarrow \quad 2u^2-2u-1=\frac{1}{k^2x^2} \text{ where } v=\frac{\sin y}{x}$$

$$\frac{\sin^2 y}{x^2}-\frac{\sin y}{x}-\frac{1}{2}=\frac{1}{2k^2x^2}$$

$$\sin^2 y=x\sin y+\frac{x^2}{2}+C.$$

Test Your Concepts-III

(Solutions on page H.380)

1. Solve the differential equation:
   $(6x-5y+4)dy+(y-2x-1)dx=0$
2. Solve differential equation $2xyy'=x^2+y^2$ and $y(1)=2$
3. Solve $(x^3+y^3)\,dx-3xy^2\,dy=0$
4. The solution of primitive integral equation $(x^2+y^2)\,dy=xy\,dx$, is $y=y\,(x)$. If $y(1)=1$ and $y(x_0)=e$, then find x_0
5. Solve the differential equation
   $(1+2e^{x/y})\,dx+2e^{x/y}\,(1-x/y)\,dy=0$.
6. Show that the differential equation $y^3\,dy+(x+y^2)\,dx=0$ can be reduced to a homogenous equation. Hence, solve it.
7. Solve $(x^3-3xy^2)\,dx=(y^3-3x^2y)\,dy$

8. Use the substitution $y^2 = a - x$ to reduce the equation $y^3 . \frac{dy}{dx} + x + y^2 = 0$ to homogeneous form and hence solve it.

9. Solve $(x - y)\, dy = (x + y + 1)\, dx$

10. Solve $\frac{dy}{dx} = \frac{x + 2y - 3}{2x + y - 3}$

11. Solve $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 3}$

12. Solve $x\frac{dy}{dx} = y\,(\log y - \log x + 1)$

13. Solve $\frac{dy}{dx} = \frac{x + 2y + 3}{2x + 3y + 4}$.

Answer key

1. $\left(y - x - \frac{3}{4}\right) = k^3\,(5y - 2x - 3)^4$ **2.** $y^2 - x^2 = 3x$ **3.** $(x^3 - 2y^3) = kx$ **4.** $\sqrt{3}\,e.$

5. $x + 2ye^{x/y} = C$ **6.** $\frac{1}{2}\log\,(y^4 + 2y^2x + 2x^2) - \tan^{-1}\frac{y^2 + x}{x} = c$ **7.** $(y^2 - x^2) = k(x^2 + y^2)^2$

8. $\frac{1}{2}\,\ln|x^2 + a^2| - \tan^{-1}\left(\frac{a}{x}\right) = c$, where $a = x + y^2$ **9.** arc tan $\frac{2y+1}{2x+1} = \ln c\sqrt{x^2 + y^2 + x + y + \frac{1}{2}}$

10. $(x + y - 2) = c'(y - x)^3$ **11.** $x + y + \frac{4}{3} = ce^{3(x-2y)}$ **12.** $y = xe^{Cx}$

13. $\frac{2+\sqrt{3}}{2\sqrt{3}}\log\,(\sqrt{3}y - x + 1 + 2\sqrt{3}) - \frac{2-\sqrt{3}}{2\sqrt{3}}\log\,(\sqrt{3}y + x + 2\sqrt{3} - 1) = A$

SOLVING EXACT DIFFERENTIAL EQUATION

$Mdx + Ndy = 0$ is exact differential equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Partial differentiation of *M* about *y*, keeping *x* as constant = Partial differentiation of *M* about *x*, keeping *y* as constant

(i) $xdx = \frac{1}{2}d(x^2)$

(ii) $xdy + y\,dx = d(xy)$

(iii) $dx + dy = d(x + y)$

(iv) $xdx + ydy = \frac{1}{2}d(x^2 + y^2)$

(v) $\frac{1}{x}\,dx = d(\ln x)$

(vi) $t^m\,dt = \frac{1}{m+1}d(t^{m+1})$

(vii) $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$

(viii) $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$

(ix) $\frac{ydx + xdy}{xy} = d(\ln xy)$

(x) $\frac{xdy - ydx}{xy} = d\left(\ln\frac{y}{x}\right)$

(xi) $\frac{xdy - ydx}{x^2 + y^2} = \frac{d\left(\frac{y}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} = d\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

(xii) $\frac{xdy + ydx}{1 + x^2y^2} = \frac{d(xy)}{1 + (xy)^2} = d(\tan^{-1}(xy))$

(xiii) $\frac{xdx + ydy}{x^2 + y^2} = d\left[\ln\sqrt{x^2 + y^2}\right]$

(xiv) $\frac{ye^x dx - e^x dy}{y^2} = d\left(\frac{e^x}{y}\right)$

(xv) $\frac{xe^y dy - e^y dx}{x^2} = d\left(\frac{e^y}{x}\right)$

ILLUSTRATION 42

Solve: $2x\ln y\,dx + \left(\frac{x^2}{y} + 3y^2\right)dy = 0$

SOLUTION

$$(2x\ln y)dx + \left(\frac{x^2}{y} + 3y^2\right)dy = 0$$

Comparing with $Mdx + Ndy = 0$, we have

$$M = 2x\ln y,\ N = \left(\frac{x^2}{y} + 3y^2\right)$$

$$\frac{\partial M}{\partial y} = 2x\left(\frac{1}{y}\right) = \frac{2x}{y}$$

$$\frac{\partial N}{\partial x} = \frac{2x}{y} + 0 \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

It is exact differential equation.

Rewriting given diff. equation, we get

$$(2x \ln y \, dx + \frac{x^2}{y} dy) + 3y^2 dy = 0$$

$$\Rightarrow \quad \int d(x^2 \ln y) + \int 3y^2 \, dy = 0 \quad \Rightarrow \quad x^2 \ln y + y^3 = k$$

ILLUSTRATION 43

Solve $xdy + ydx = \sqrt{1 - x^2y^2} \; dx$

SOLUTION

$$x \, dy + y \, dx = \sqrt{1 - x^2y^2} \; dx$$

$$\Rightarrow \quad \int \frac{d(xy)}{\sqrt{1 - (xy)^2}} = \int dx$$

$$\Rightarrow \quad \sin^{-1}(xy) = x + c \Rightarrow xy = \sin(x + c)$$

ILLUSTRATION 44

Solve $xdy + ydx + xye^y dy = 0$

SOLUTION

$$xdy + ydx + xye^y \, dy = 0$$

$$\Rightarrow \quad d(xy) + xye^y \, dy = 0 \quad \Rightarrow \quad \int \frac{d(xy)}{xy} + \int e^y \, dy = 0$$

$$\Rightarrow \quad \ln(xy) + e^y + k = 0$$

ILLUSTRATION 45

Solve $\left(\frac{y + \sin x \cos^2 xy}{\cos^2 xy}\right) dx + \left(\frac{x}{\cos^2 xy} + \sin y\right) dy = 0$

SOLUTION

$$\left(\frac{y + \sin x \cos^2 xy}{\cos^2 xy}\right) dx + \left(\frac{x}{\cos^2 xy} + \sin y\right) dy = 0$$

$$\Rightarrow \quad \frac{ydx}{\cos^2 xy} + \sin x \, dx + \frac{xdy}{\cos^2 xy} + \sin y \, dy = 0$$

$$\Rightarrow \quad \frac{ydx + xdy}{\cos^2 xy} + \sin x \, dx + \sin y \, dy = 0$$

$$\Rightarrow \quad \int \sec^2 xyd(xy) + \int \sin xdx + \int \sin ydy = 0$$

After integration, we get

$$\Rightarrow \quad \tan(xy) - \cos x - \cos y = k$$

$$\Rightarrow \quad \tan xy = k + \cos x + \cos y$$

ILLUSTRATION 46

Solve $e^{x\left(\frac{y^2-1}{y}\right)}(xy^2dy + y^3dx) + (ydx - xdy) = 0.$

SOLUTION

Rewriting given differential equation, we get

$$e^{xy} . e^{-x/y} y^2(xdy + ydx) + ydx - xdy = 0$$

$$\Rightarrow \quad \frac{e^{xy}(xdy + ydx)}{e^{x/y}} + \frac{ydx - xdy}{y^2} = 0$$

$$\Rightarrow \quad \frac{e^{xy} d(xy)}{e^{x/y}} + d\left(\frac{x}{y}\right) = 0$$

$$\Rightarrow \quad \int e^{xy} d(xy) + \int e^{x/y} d\left(\frac{x}{y}\right) = 0$$

After integration, we obtain $e^{xy} + e^{x/y} = k$

ILLUSTRATION 47

Solve: $2x^2y \, dy = (\tan x^2y^2 - 2xy^2) \, dx$ where $y(1) = \sqrt{\frac{\pi}{2}}$

SOLUTION

$$2x^2y \, dy + 2 \, xy^2 \, dx = \tan x^2y^2 \, dx$$

$$\Rightarrow \quad 2xy(xdy + ydx) = \tan (xy)^2 \, dx$$

$$\Rightarrow \quad \frac{2xyd(xy)}{\tan(xy)^2} = dx \Rightarrow \int \cot(xy)^2 d(xy)^2 = \int dx$$

$$\Rightarrow \quad \ln \sin(xy)^2 = x + c$$

Given when $x = 1, y = \sqrt{\frac{\pi}{2}}$

$$\ln \sin\left(1 \cdot \frac{\pi}{2}\right) = 1 + c \Rightarrow 0 = 1 + c \Rightarrow c = -1$$

$$\Rightarrow \quad \ln \sin(xy)^2 = x - 1 \Rightarrow \sin x^2y^2 = e^{x-1}$$

ILLUSTRATION 48

Solve $\dfrac{x + y\dfrac{dy}{dx}}{y - x\dfrac{dy}{dx}} = x^2 + 2y^2 + \dfrac{y^4}{x^2}.$

SOLUTION

The given equation can be written as

$$\frac{xdx + ydy}{(x^2 + y^2)^2} = \frac{ydx - xdy}{x^2}$$

$$\Rightarrow \quad \frac{1}{2}\int \frac{d(x^2 + y^2)}{(x^2 + y^2)^2} = -\int d\left(\frac{y}{x}\right)$$

Integrating both sides, we get

$$-\frac{1}{2(x^2 + y^2)} = -\frac{y}{x} + C \quad \Rightarrow \quad \frac{y}{x} - \frac{1}{2(x^2 + y^2)} = C.$$

ILLUSTRATION 49

Solve $\frac{y\,dx - x\,dy}{(x-y)^2} = \frac{d\,x}{2\sqrt{1-x^2}}$, given that $y=2$ when $x=1$

SOLUTION

$$\frac{ydx - xdy}{(x-y)^2} = \frac{dx}{2\sqrt{1-x^2}} \Rightarrow \frac{ydx-xdy}{y^2\left[\left(\frac{x}{y}\right)-1\right]^2} = \frac{dx}{2\sqrt{1-x^2}}$$

$$\Rightarrow \int \frac{d\left(\frac{x}{y}\right)}{\left(\frac{x}{y}-1\right)^2} = \int\frac{dx}{2\sqrt{1-x^2}} \Rightarrow \frac{-1}{\left(\frac{x}{y}-1\right)} = \frac{1}{2}\sin^{-1}(x)+c$$

$$\Rightarrow \frac{y}{y-x} = \frac{1}{2}\sin^{-1}(x)+c$$

Given $x=1, y=2$, we get $c = 2-\frac{\pi}{4}$

Required solution is $\frac{\sin^{-1}x}{2}+\frac{y}{x-y} = \frac{\pi}{4}-2$

SOLVING DIFFERENTIAL EQUATION USING POLAR COORDINATES

Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection, it is convenient to remember the following differentials.

If $x = r\cos\theta;\ y = r\sin\theta$ then,

(i) $x^2+y^2=r^2$ on differentiation, we get

$$2xdx+2ydy = 2rdr \Rightarrow xdx+ydy = rdr$$

(ii) $\tan\theta = \frac{y}{x}$

On differentiation $\sec^2\theta\,d\theta = \frac{xdy-ydx}{x^2}$

$$\Rightarrow xdy - ydx = x^2\sec^2\theta\,d\theta = (r\cos\theta)^2\sec^2\theta\,d\theta$$

$$\Rightarrow xdy-ydx = r^2\,d\theta$$

If $x = r\sec\theta$ and $y = r\tan\theta$, then

(i) $x^2-y^2=r^2$ on differentiation,

we get $x\,dx - y\,dy = rdr$

(ii) $\frac{y}{x} = \sin\theta \Rightarrow \frac{xdy-ydx}{x^2} = \cos\theta\,d\theta$

$$\Rightarrow x\,dy - ydx = x^2\cos\theta\,d\theta = (r\sec\theta)^2\cos\theta\,d\theta$$

$$\Rightarrow x\,dy - ydx = r^2\sec\theta\,d\theta$$

ILLUSTRATION 50

Solve $\frac{xdx+ydy}{xdy-ydx} = \sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}}$

SOLUTION

Put $x = r\cos\theta,\ y = r\sin\theta$

$$\Rightarrow \quad x^2+y^2 = r^2 \Rightarrow x\,dx+y\,dy = r\,dr$$

$$\tan\theta = \frac{y}{x} \Rightarrow \frac{x\,dy-y\,dx}{x^2} = \sec^2\theta d\theta \Rightarrow xdy-ydx = r^2d\theta$$

$\therefore$ Given differential equation $\frac{xdx+ydy}{xdy-ydx} = \sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}}$

Transform to $\frac{r\,dr}{r^2d\theta} = \sqrt{\frac{a^2-r^2}{r^2}}$

$$\Rightarrow \frac{dr}{\sqrt{a^2-r^2}} = d\theta \Rightarrow \sin^{-1}\left(\frac{r}{a}\right) = \theta+C$$

$$\Rightarrow r = a\sin(\theta+C)$$

$\Rightarrow \sqrt{x^2+y^2} = a\sin\left\{c+\tan^{-1}\frac{y}{x}\right\}$ is required solution

ILLUSTRATION 51

Find solution of the differential equation

$$(x\,dx+ydy)\ \sqrt{x^2+y^2} = (x\,dy-y\,dx)\ \left(\sqrt{1-x^2-y^2}\right)$$

SOLUTION

Let $x = r\cos\theta, y = r\sin\theta$ then $xdx+ydy = rdr$ and

$$xdy-ydx = r^2\,d\theta$$

Now given differential equation is

$$(x\,dx+ydy)\ \sqrt{x^2+y^2} = (x\,dy-y\,dx)\ \left(\sqrt{1-x^2-y^2}\right)$$

$$\Rightarrow \quad rdr\,(r) = r^2d\theta\,\sqrt{1-r^2} \quad \Rightarrow \quad \frac{dr}{\sqrt{1-r^2}} = d\theta$$

After integration we have $\sin^{-1}r = \theta+c$

$$r = \sin(\theta+c)$$

$$\Rightarrow \quad r = \sin\theta\cos c+\cos\theta\sin c$$

$$\Rightarrow \quad r^2 = (r\sin\theta)\cos c+(r\cos\theta)\sin c$$

$$\Rightarrow \quad x^2+y^2-x\sin c-y\cos c = 0$$

which is a circle of radius $= \frac{1}{2}$

Test Your Concepts-IV

(Solutions on page H.382)

1. Solve $y\,dx - x\,dy + (1 + x^2)\,dx + x^2 \sin y\,dy = 0$.
2. Solve $y(2xy + e^x)dx = e^x dy$
3. Solve the differential equation $3\sec^2(3y)\,e^{2x}\,dy + 2(1 - e^{2x})\tan 3y\,dx = 0$.
4. For the primitive integral equation $ydx + y^2\,dy = x\,dy$; $x \in R$, $y > 0$, $y = y(x)$, $y(1) = 1$, then compute $y(-3)$
5. Solve $xdy - ydx = \sqrt{x^2 + y^2}\;dx$
6. Solve $\dfrac{dy}{dx} = \dfrac{x(2\ln x + 1)}{\sin y + y\cos y}$
7. Solve $x\,dy + y\,dx + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0$

Answer key

1. $y = x^2 - 1 - x\cos y - cx$

2. $yx^2 + e^x = cy$

3. $\log|\tan 3y| = 2x + e^{-2x} + c$

4. 3

5. $y + \sqrt{x^2 + y^2} = cx^2$

6. $y\sin y = x^2 \ln x + c$

7. $xy + \tan^{-1}\dfrac{y}{x} = c$

LINEAR DIFFERENTIAL EQUATION

$\dfrac{dy}{dx} + P(x)y = Q(x)$ is known as linear differential equation where $P(x)$ and $Q(x)$ are function of x

Integrating factor (I.F.) $= e^{\int P(x)dx}$

Multiplying by Integrating Factor both sides

$$\Rightarrow \quad e^{\int P(x)dx}\frac{dy}{dx} + \left(e^{\int P(x)dx}P(x)\right)y = e^{\int P(x)dx}Q(x)$$

$$\Rightarrow \quad \frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}Q(x)$$

$$\Rightarrow \quad \int d\big((\text{I.F.})y\big) = \int(\text{I.F.})Q(x)\,dx$$

$$\Rightarrow \quad y(\text{I.F.}) = \int(\text{I.F.})Q(x)\,dx$$

Working Rule

Linear Differential Equation $\dfrac{dy}{dx} + P(x)y = Q(x)$

Integrating Factor (I.F.) $= e^{\int P(x)dx}$

Solution of linear differential equation is

$$y(\text{I.F.}) = \int(\text{I.F.})Q(x)\,dx$$

ILLUSTRATION 52

Solve: $\dfrac{dy}{dx}(1 + x^3) + 3x^2y = \sin^2 x$

SOLUTION

Rewriting given differential equation

$$\frac{dy}{dx} + \left(\frac{3x^2}{1+x^3}\right)y = \frac{\sin^2 x}{1+x^3}$$

Integrating Factor $= e^{\int P(x)dx} = e^{\int\left(\frac{3x^2}{1+x^3}\right)dx} = e^{\ln(1+x^3)} = 1 + x^3$

$$\Rightarrow \quad y(\text{I.F.}) = \int\frac{\sin^2 x}{1+x^3}(1+x^3)dx$$

$$\Rightarrow \quad \int d(y(1+x^3)) = \int \sin^2 x\,dx$$

$$\Rightarrow \quad y(1+x^3) = \int\frac{1-\cos 2x}{2} = \frac{x}{2} - \frac{\sin 2x}{4} + c$$

ILLUSTRATION 53

Solve it: $\dfrac{dy}{dx} + (\tan x)\,y = x^m \cos x$

SOLUTION

Integrating Factor $= e^{\int \tan x dx} = e^{\ell n \sec x} = \sec x$

$$y(\sec x) = \int(\sec x)Q(x)\,dx = \int \sec x(x^m \cos x)\,dx$$

$$y(\sec x) = \int x^m\,dx \Rightarrow y(\sec x) = \frac{x^{m+1}}{m+1} + c$$

ILLUSTRATION 54

Solve: $x\ln x\dfrac{dy}{dx} + y = 2\ln x$

SOLUTION

$$x\ln x\frac{dy}{dx} + y = 2\ln x \quad\Rightarrow\quad \frac{dy}{dx} + \left(\frac{1}{x\ln x}\right)y = \frac{2}{x}$$

Integrating Factor $= e^{\int\frac{1}{x\ln x}dx}$ Put $\ln x = t$; $\dfrac{dx}{x} = dt$

$$\text{I.F.} = e^{\int\frac{dt}{t}} = e^{\ln t} = t = \ln x$$

$$\Rightarrow \quad y(\text{I.F.}) = \int \ln x . \frac{2}{x}\,dx \quad \text{Put } \ln x = t;\ \frac{dx}{x} = dt$$

$$\Rightarrow \quad y(\ln x) = 2\int t\,dt = t^2 + c \quad \Rightarrow \quad y(\ln x) = (\ln x)^2 + c$$

ILLUSTRATION 55

Solve $(x + 2y^3)\dfrac{dy}{dx} = y$

SOLUTION

The equation which involves y^3 is not linear, since this can be written as

$$y\frac{dx}{dy} = x + 2y^3 \Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2,$$

We see that the equation is linear differential equation, if we take x as the dependent variable

Integrating factor = $e^{-\int\frac{dy}{y}} = e^{-\log y} = y^{-1} = \dfrac{1}{y}.$

Multiplying by this integrating factor, we obtained

$$x\left(\frac{1}{y}\right) = \int 2y^2\left(\frac{1}{y}\right)dy = y^2 + c$$

Therefore the solution is $\dfrac{x}{y} = y^2 + c$ or $x = y(c + y^2)$

ILLUSTRATION 56

Solve it: $t(1 + t^2)dx = (x + xt^2 - t^2)dt$ if $t = 1$ when $x = -\dfrac{\pi}{4}$

SOLUTION

Rewriting given diff. equation

$$t(1 + t^2)\frac{dx}{dt} = x(1 + t^2) - t^2$$

$$\Rightarrow \quad \frac{dx}{dt} - \frac{x}{t} = -\frac{t}{1+t^2}$$

Integrating Factor = $e^{-\int\frac{1}{t}dt} = e^{-\ln t} = e^{\ln\left(\frac{1}{t}\right)} = \dfrac{1}{t}$

$$\Rightarrow \quad x\left(\frac{1}{t}\right) = -\int\frac{dt}{1+t^2} = -\tan^{-1}t + c$$

It is general solution.

Given $t = 1$ when $x = -\dfrac{\pi}{4}$

$$\Rightarrow \quad -\frac{\pi}{4} = -\tan^{-1}1 + c \Rightarrow c = 0$$

$\dfrac{x}{t} = -\tan^{-1}t \Rightarrow x = -t\tan^{-1}t$ is particular solution

ILLUSTRATION 57

If $y(t)$ is a solution of differential equation $(1 + t)\dfrac{dy}{dt} - ty = 1$ and $y(0) = -1$ find $y(1)$

SOLUTION

Rewriting given differential equation

$$\frac{dy}{dt} - \left(\frac{t}{1+t}\right)y = \frac{1}{1+t}$$

Integrating Factor = $e^{-\int\frac{t}{1+t}dt} = e^{-\int\left(1-\frac{1}{1+t}\right)dt}$

$$= e^{-(t-\ln(t+1))} = e^{-t}e^{\ln(t+1)} = (t+1)e^{-t}$$

$$y(\text{I.F.}) = \int\frac{1}{1+t}(t+1)e^{-t}\,dt$$

$$\Rightarrow \quad y(t+1)e^{-t} = \int e^{-t}\,dt = -e^{-t} + c$$

Given $y(0) = -1$

$$(-1)(0 + 1)\,e^{-0} = -e^{-0} + c \Rightarrow -1 = -1 + c \Rightarrow c = 0$$

$$\Rightarrow \quad y(t+1)\,e^{-t} = -e^{-t} \Rightarrow y = -\frac{1}{t+1}$$

Now $y(1) = -\dfrac{1}{2}$

ILLUSTRATION 58

Solve $(1+y^2)dx = (\tan^{-1}y - x)\,dy$

SOLUTION

Given $\dfrac{dx}{dy} = \dfrac{\tan^{-1}y}{1+y^2} - \dfrac{x}{1+y^2} \Rightarrow \dfrac{dx}{dy} + \dfrac{1}{1+y^2}x = \dfrac{\tan^{-1}y}{1+y^2}$

It is linear differential equation

Now, Integrating Factor $= e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

Therefore, solution is given by,

$$x\,e^{\tan^{-1}y} = \int e^{\tan^{-1}y}\,\frac{\tan^{-1}y}{1+y^2}\,dy + c$$

Put $t = \tan^{-1}y \Rightarrow dt = \dfrac{dy}{1+y^2}$

$$\Rightarrow \quad x\,e^{\tan^{-1}y} = \int t\,e^t\,dt + c = t\,e^t - e^t + c$$

$$\Rightarrow \quad x\,e^{\tan^{-1}y} = (\tan^{-1}y)\,e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$$

$$\Rightarrow \quad x\,e^{\tan^{-1}y} = e^{\tan^{-1}y}\left(\tan^{-1}y - 1\right) + c$$

ILLUSTRATION 59

Solve: $\dfrac{dy}{dx} + \dfrac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$ where $0 < x < 1$

SOLUTION

$$\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$$

Integrating factor = $e^{\int \frac{dx}{(1-x)\sqrt{x}}} = e^{\int \frac{2dt}{1-t^2}}$

$(\because$ Put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2tdt)$

$$\text{I.F.} = e^{\ln\frac{1+t}{1-t}} = \frac{1+t}{1-t} = \frac{1+\sqrt{x}}{1-\sqrt{x}}$$

$$y \,.\, \frac{1+\sqrt{x}}{1-\sqrt{x}} = \int \frac{1+\sqrt{x}}{1-\sqrt{x}}.(1-\sqrt{x})dx = \int (1+\sqrt{x})\,dx$$

$$\Rightarrow \quad y\,\frac{1+\sqrt{x}}{1-\sqrt{x}} = x + \frac{2}{3}x^{3/2} + c$$

ILLUSTRATION 60

Find a curve passing through point (2, 0) and having a slope of tangent at any point (x, y) as $\frac{(x+1)^2 + y - 1}{(x+1)}$

SOLUTION

Slope of tangent = $\frac{dy}{dx} = \frac{(x+1)^2 + y - 1}{(x+1)}$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x+1} = x + 1 - \frac{1}{x+1}$$

Integrating Factor = $e^{\int -\frac{1}{x+1}dx} = e^{-\ln(x+1)} = e^{\ln\left(\frac{1}{x+1}\right)} = \frac{1}{x+1}$

$$\Rightarrow \quad y\left(\frac{1}{x+1}\right) = \int 1 - \frac{1}{(x+1)^2}\,dx$$

$$\Rightarrow \quad \frac{y}{x+1} = x + \frac{1}{x+1} + c$$

$$\Rightarrow \quad y = x(x+1) + 1 + c(x+1)$$

It passes through point (2, 0), we get

$$0 = 2(3) + 1 + c(3) \Rightarrow c = -\frac{7}{3}$$

$$y = x^2 + x + 1 - \frac{7}{3}(x+1) \Rightarrow y = \frac{3x^2 - 4x - 4}{3}$$

ILLUSTRATION 61

Solve: $(x + \tan y)\, dy = \sin 2y\, dx$

SOLUTION

$$(x + \tan y)\, dy = \sin 2y\, dx$$

$$\Rightarrow \quad \sin 2y \frac{dx}{dy} = x + \tan y \Rightarrow \frac{dx}{dy} - \frac{x}{\sin 2y} = \frac{\tan y}{\sin 2y}$$

$$\Rightarrow \quad \frac{dx}{dy} - \operatorname{cosec} 2y\, x = \frac{1}{2}\sec^2 y$$

Integrating factor = $e^{-\int \operatorname{cosec} 2y dy} = e^{-\frac{1}{2}\ln(\tan y)} = \frac{1}{\sqrt{\tan y}}$

$$x\left(\frac{1}{\sqrt{\tan y}}\right) = \frac{1}{2}\int \frac{1}{\sqrt{\tan y}} \sec^2 y\, dy = \sqrt{\tan y} + c;$$

$$\Rightarrow \quad x\sqrt{\cot y} = \sqrt{\tan y} + c$$

ILLUSTRATION 62

The tangent at any point (x, y) of a curve makes an angle $\tan^{-1}(2x + 3y)$ with x-axis. Find the equation of the curve if it passes through (1, 2)

SOLUTION

Slope of tangent = $\tan\theta = \frac{dy}{dx} = 2x + 3y$

$$\Rightarrow \quad \frac{dy}{dx} - 3y = 2x$$

Integrating factor = $e^{-\int 3dx} = e^{-3x}$

$$ye^{-3x} = \int 2xe^{-3x}\,dx \qquad \text{[using by parts integration]}$$

$$\Rightarrow \quad ye^{-3x} = -\frac{2}{3}xe^{-3x} + \int \frac{2}{3}e^{-3x}\,dx$$

$$\Rightarrow \quad ye^{-3x} = -\frac{2}{3}xe^{-3x} - \frac{2}{9}e^{-3x} + C$$

$$\Rightarrow \quad y = -\frac{2}{3}x - \frac{2}{9} + ce^{3x}$$

This curve passes through (1, 2)

$$\Rightarrow \quad c = \frac{26}{9}e^{-3} \Rightarrow y = \frac{1}{9}[26\,e^{3x-3} - 6x - 2]$$

ILLUSTRATION 63

Solve: $\frac{dy}{dx} - y \ln 2 = 2^{\sin x}.(\cos x - 1) \ln 2$ where y being bounded when $x \to +\infty$.

SOLUTION

Given differential equation is linear diff equation. Thus,

$$\frac{dy}{dx} - y\ln 2 = 2^{\sin x}.(\cos x - 1)\ln 2$$

Integrating Factor = $e^{-\ln 2\int dx} = e^{-x\ln 2} = e^{\ln 2^{-x}} = 2^{-x} = \frac{1}{2^x}$

$$\frac{y}{2^x} = \int \frac{2^{\sin x}(\cos x - 1)\ln 2}{2^x}\,dx$$

Put $2^{\sin x - x} = t$; $2^{\sin x - x} \ln 2.(\cos x - 1)\, dx = dt$

$$\frac{y}{2^x} = 2^{\sin x - x} + c$$

y being bounded when $x \to +\infty$. We get $c = 0$; $y = 2^{\sin x}$

ILLUSTRATION 64

Solve the differential equation $(1 + \tan y)(dx - dy) + 2x\, dy = 0$

SOLUTION

Given differential equation $(1 + \tan y)(dx - dy) + 2x\, dy = 0$

$$\frac{dx}{dy} - 1 + \frac{2x}{1+\tan y} = 0 \Rightarrow \frac{dx}{dy} + \frac{2x}{1+\tan y} = 1$$

It is linear differential equation

Integrating Factor = $e^{\int \frac{2}{1+\tan y}dy}$

$$\because \quad \int \frac{2}{1+\tan y}\,dy \quad \text{Let } \tan y = t \Rightarrow dy = \frac{dt}{1+t^2}$$

$$= \int \frac{2}{(1+t)(1+t^2)}dt = \int \frac{1}{(1+t)} + \frac{(1-t)}{(1+t^2)}dt\ dt$$

$$= \ln(1+t) + \tan^{-1} t - \frac{1}{2}\ln(1+t^2)$$

$$= \ln(\sin y + \cos y) + y$$

Integrating factor = $e^{\ln(\sin y + \cos y) + y} = (\sin y + \cos y)\,e^y$

Linear differential equation $\frac{dx}{dy} + \frac{2x}{1+\tan y} = 1$

$$\Rightarrow \quad x(\sin y + \cos y)\,e^y = \int (\sin y + \cos y)e^y\,dy$$

$$\Rightarrow \quad x(\sin y + \cos y)\,e^y = e^y \sin y + c$$

$$\Rightarrow \quad x(\sin y + \cos y) = \sin y + ce^{-y}$$

ILLUSTRATION 65

If $y + \frac{d}{dx}(xy) = x(\sin x + \log x)$, find $y(x)$

SOLUTION

$$y + \frac{d}{dx}(xy) = x(\sin x + \log x)$$

$$\Rightarrow \quad y + \frac{xdy}{dx} + y = x(\sin x + \log x);$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x}y = (\sin x + \log x)$$

Integrating Factor = $e^{\int \frac{2}{x}dx} = x^2$

$$\Rightarrow \quad yx^2 = \int x^2(\sin x + \log x)\,dx$$

$$\Rightarrow \quad yx^2 = -x^2\cos x + \int 2x\cos x\,dx + \frac{x^3}{3}\log x - \int \frac{x^2}{3}dx$$

$$\Rightarrow \quad yx^2 = -x^2\cos x + \frac{x^3}{3}\log x + 2x\sin x + 2\cos x - \frac{x^3}{9} + c$$

$$\Rightarrow \quad y = \frac{x}{3}\log x - \frac{x}{9} - \cos x + \frac{2}{x}\sin x + \frac{2}{x^2}\cos x + \frac{c}{x^2}$$

ILLUSTRATION 66

If $g(x)$ is a differentiable real valued function satisfying $g''(x) - 3g'(x) > 3 \;\forall\; x \geq 0$ and $g'(0) = -1$, then show that $g(x) + x$ is an increasing function for $x > 0$.

SOLUTION

$$g''(x) - 3g'(x) > 3 \quad \forall\, x \geq 0$$

$$\Rightarrow \quad e^{-3x}g''(x) - 3e^{-3x}g'(x) > 3e^{-3x}$$

$$\Rightarrow \quad \frac{d}{dx}(g'(x)e^{-3x} + e^{-3x}) > 0$$

Let $h(x) = e^{-3x}(g'(x) + 1)$ is increasing function for $\forall\, x \geq 0$

When $x \geq 0$; $h(x) \geq h(0)$

$$\Rightarrow \quad e^{-3x}(g'(x) + 1) \geq (g'(0) + 1)$$

$$\Rightarrow \quad e^{-3x}(g'(x) + 1) \geq 0$$

$$\Rightarrow \quad g'(x) + 1 \geq 0,$$

we can also write it as $\frac{d}{dx}(g(x) + x) \geq 0$ for $\forall\, x \geq 0$

It means $g(x) + x$ is an Increasing function

ILLUSTRATION 67

Let $P(x)$ be a polynomial such that $\frac{dP(x)}{dx} > P(x)$ for $x > 1$ and $P(1) = 0$ show that $P(x) > 0$ for $x > 1$.

SOLUTION

$$\frac{dP(x)}{dx} > P(x) \Rightarrow \frac{dP(x)}{dx} - P(x) > 0$$

Integrating factor = $e^{\int -dx} = e^{-x} > 0$ when $x > 1$

Multiplying by integrating factor both side

$$e^{-x}\frac{dP(x)}{dx} - e^{-x}P(x) > 0\;(e^{-x})$$

$$\Rightarrow \quad \frac{d}{dx}[e^{-x}P(x)] > 0 \quad \text{Let } g(x) = e^{-x}P(x)$$

$$\Rightarrow \quad \frac{dg(x)}{dx} > 0,\ g(x) \text{ is an increasing function for } x > 1$$

$$\Rightarrow \quad g(x) > g(1) \qquad [\because\ P(1) = 0]$$

$$\Rightarrow \quad e^{-x}P(x) > e^{-1}P(1) = 0$$

$$\Rightarrow \quad e^{-x}P(x) > 0 \quad \Rightarrow \quad P(x) > 0 \text{ for } x > 1$$

Hence Proved

ILLUSTRATION 68

The general solution of the differential equation of $(1+y^2) + \left(x - e^{\tan^{-1} y}\right)\frac{dy}{dx} = 0$ is $2xe^{f(y)} = e^{2f(y)} + c$ then find the area of the region bounded by the curves $x = f(y)$, $y = \pm\sqrt{3}$ and y-axis

SOLUTION

Rewriting differential equation

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1} y}}{1+y^2}$$

Integrating factor = $e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$

General solution $x.e^{\tan^{-1}y}=\int\frac{\left(e^{\tan^{-1}y}\right)^2}{1+y^2}dy$

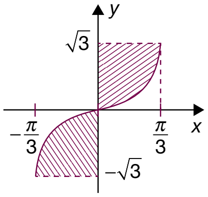
$$\Rightarrow \quad x\,e^{\tan^{-1}y}=\int e^{\tan^{-1}y}d\left(e^{\tan^{-1}y}\right)$$

$$\Rightarrow \quad xe^{\tan^{-1}y}=\frac{e^{2\tan^{-1}y}}{2}+c \Rightarrow 2xe^{\tan^{-1}y}=e^{2\tan^{-1}y}+2c$$

$$\Rightarrow \quad 2xe^{f(y)}=e^{2f(y)}+c$$

Hence, $f(y)=\tan^{-1}y$

Required area of the region bounded by the curves $x=f(y)=\tan^{-1}y$, $y=\pm\sqrt{3}$ and y-axis



$$=\int_{-\sqrt{3}}^{\sqrt{3}}\left|\tan^{-1}y\right|dy=2\int_{0}^{\sqrt{3}}\tan^{-1}y\,dy$$

$$=2\left[\left(y\tan^{-1}y\right)_0^{\sqrt{3}}-\int_0^{\sqrt{3}}\frac{y}{1+y^2}dy\right]$$

$$=2\left[\sqrt{3}\frac{\pi}{3}-0-\left[\frac{1}{2}\log(1+y^2)\right]_0^{\sqrt{3}}\right]$$

$$=\frac{2\pi}{\sqrt{3}}-\log 4=\frac{2\pi}{\sqrt{3}}-2\log 2$$

DIFFERENTIAL EQUATION REDUCIBLE TO LINEAR DIFFERENTIAL EQUATION

Type I

If equation is of the form $f'(y)\frac{dy}{dx}+P\,f(y)=Q$ where P and Q are constant or function of x then put $f(y)=t$, $f'(y)\frac{dy}{dx}=\frac{dt}{dx}$ and equation reduces to $\frac{dt}{dx}+Pt=Q$ which is a Linear Differential Equation in t and x.

Type II: Bernoulli's Differential Equation

Equation of the form $\frac{dy}{dx}+Py=Qy^n$ where P and Q are functions of x.

Divide it by y^n,

we get $\frac{1}{y^n}\frac{dy}{dx}+\frac{1}{y^{n-1}}P(x)=Q(x)$

Put $\frac{1}{y^{n-1}}=t\Rightarrow\frac{-n+1}{y^n}\frac{dy}{dx}=\frac{dt}{dx}$

$$\Rightarrow \quad -\frac{1}{n-1}\frac{dt}{dx}+tP(x)=Q(x)$$

which is a linear differential equation

ILLUSTRATION 69

Solve It: $\frac{dy}{dx}+\frac{y}{x}=y^3$

SOLUTION

It is a Bernoulli's Differential Equation

$$\Rightarrow \quad \frac{1}{y^3}\frac{dy}{dx}+\frac{1}{xy^2}=1$$

Put $\frac{1}{y^2}=t\Rightarrow-\frac{2}{y^3}\frac{dy}{dx}=\frac{dt}{dx}$

$$\Rightarrow \quad -\frac{1}{2}\frac{dt}{dx}+\frac{t}{x}=1\Rightarrow\frac{dt}{dx}-\frac{2t}{x}=-2$$

Integrating Factor $=e^{-\int\frac{2}{x}dx}=e^{-2\ln x}=e^{\ln x^{-2}}=\frac{1}{x^2}$

$$t(\text{I.F.})=\int-2(\text{I.F.})\,dx=-\int\frac{2}{x^2}dx$$

$$\Rightarrow \quad t\left(\frac{1}{x^2}\right)=\frac{2}{x}+c$$

$$\Rightarrow \quad \left(\frac{1}{y^2}\right)\frac{1}{x^2}=\frac{2}{x}+c\Rightarrow\frac{1}{x^2y^2}=\frac{2}{x}+c$$

ILLUSTRATION 70

Solve: $y\sin x\,\frac{dy}{dx}=\cos x\,(\sin x-y^2)$

SOLUTION

Put $y^2=t$; $2y\frac{dy}{dx}=\frac{dt}{dx}$

Given differential equation becomes

$$\sin x\left(\frac{1}{2}\frac{dt}{dx}\right)=\cos x(\sin x-t)$$

$$\Rightarrow \quad \frac{dt}{dx}=\frac{2\cos x}{\sin x}(\sin x-t)=2\cos x-2t\cot x$$

$$\Rightarrow \quad \frac{dt}{dx}+(2\cot x)t=2\cos x$$

Integrating Factor $=e^{\int 2\cot x\,dx}=e^{2\ln\sin x}=e^{\ln\sin^2 x}=\sin^2 x$

$$t(\sin^2 x)=\int\sin^2 x\cdot 2\cos x\,dx$$

$$\Rightarrow \quad y^2\sin^2 x=\frac{2\sin^3 x}{3}+C$$

ILLUSTRATION 71

Solve it: $x\frac{dy}{dx}+y=y^2\ln x$

SOLUTION

$$x\frac{dy}{dx}+y=y^2\ln x\Rightarrow\frac{x}{y^2}\frac{dy}{dx}+\frac{1}{y}=\ln x$$

Put $\frac{1}{y} = t \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \quad \frac{dt}{dx} - \frac{t}{x} = -\frac{\ln x}{x}$$

Integrating Factor = $e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln\left(\frac{1}{x}\right)} = \frac{1}{x}$

$$t\left(\frac{1}{x}\right) = \int \frac{-\ln x}{x^2}\,dx = \int \underset{\text{I}}{\ln x}\underset{\text{II}}{\left(\frac{-1}{x^2}\right)}dx$$

Using by parts integration, we obtain

$$\frac{1}{xy} = \ln x\left(\frac{1}{x}\right) - \int \frac{1}{x}\cdot\frac{1}{x}\,dx$$

$$\Rightarrow \quad \frac{1}{xy} = \frac{\ln x}{x} + \frac{1}{x} + c \Rightarrow \frac{1}{y} = \ln x + 1 + cx$$

ILLUSTRATION 72

Solve the differential equation $(x^3 + y^2 + 2)dx + 2y\,dy = 0$

SOLUTION

Rewriting given differential equation

$$(x^3 + y^2 + 2) + 2y\frac{dy}{dx} = 0$$

Put $y^2 = t;\ 2y\frac{dy}{dx} = \frac{dt}{dx}$, we get $x^3 + t + 2 + \frac{dt}{dx} = 0$

Or $\frac{dt}{dx} + t = -x^3 - 2$

Integrating Factor = e^x

$$t.e^x = -\int (x^3 + 2)e^x\,dx$$

$$t.e^x = -[(x^3 + 2)e^x - 3x^2e^x + 6x\,e^x - 6e^x + c]$$

$$t = -(x^3 + 2) + 3x^2 - 6x + 6 - ce^{-x}$$

Required curve is $y^2 = 3x^2 - 6x - x^3 - ce^{-x} + 4$

ILLUSTRATION 73

Solve the differential equations $y\frac{dy}{dx}\sin x = \cos x\,(\sin x - y^2)$

SOLUTION

Given differential equation $y\frac{dy}{dx}\sin x = \cos x\,(\sin x - y^2)$

Put $y^2 = t;\ 2y\frac{dy}{dx} = \frac{dt}{dx}$;

$$\frac{1}{2}\frac{dt}{dx}\sin x = \cos x\,(\sin x - t)$$

$\Rightarrow \quad \frac{dt}{dx} + 2\cot x\; t = 2\cos x$ is linear differential equation

Integrating factor = $e^{2\int \cot x\,dx} = e^{2\ln\sin x} = \sin^2 x$

$$t.\sin^2 x = 2\int \cos x \sin^2 x\,dx$$

Put $\sin x = u;\ \cos x\,dx = du$

$$\Rightarrow \quad t.\sin^2 x = 2\int u^2\,du = \frac{2u^3}{3} + c$$

$$\Rightarrow \quad y^2\sin^2 x = \frac{2}{3}(\sin x)^3 + c$$

ILLUSTRATION 74

Solve $\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2$

SOLUTION

$$\frac{dz}{dx} + \frac{z}{x}\log z = \frac{z}{x^2}(\log z)^2 \qquad \ldots(1)$$

Divide given differential equation by $z\,(\log z)^2$

$$\frac{1}{z\,(\log z)^2}\frac{dz}{dx} + \frac{1}{x\log z} = \frac{1}{x^2}$$

Let $\frac{1}{\log z} = y \Rightarrow -\frac{1}{z\,(\log z)^2}\frac{dz}{dx} = \frac{dy}{dx}$

Then (1) is transformed to

$$-\frac{dy}{dx} + \frac{1}{x}y = \frac{1}{x^2} \Rightarrow \frac{dy}{dx} - \frac{1}{x}y = -\frac{1}{x^2}$$

It is linear differential equation

Integrating Factor $= e^{-\int \frac{1}{x}dx} = \frac{1}{x}$

Solution is given by

$$\frac{y}{x} = -\int \frac{1}{x^3}\,dx \quad \Rightarrow \frac{y}{x} = \frac{1}{2x^2} + c$$

$$\Rightarrow \quad \frac{x}{\log z} = \frac{1}{2} + c\,x^2$$

ILLUSTRATION 75

Find the solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{xy(x^2\sin y^2 + 1)}$$

SOLUTION

Given differential equation $y\frac{dy}{dx} = \frac{1}{x(x^2\sin y^2 + 1)}$

Put $y^2 = t \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{1}{2}\frac{dt}{dx} = \frac{1}{x(x^2\sin t + 1)}$$

$$\Rightarrow \quad \frac{x^3\sin t + x}{2} = \frac{dx}{dt} \quad \Rightarrow \quad \frac{dx}{dt} - \frac{x}{2} = \frac{x^3\sin t}{2}$$

$$\Rightarrow \quad \frac{1}{x^3}\frac{dx}{dt} - \frac{1}{2x^2} = \frac{\sin t}{2} \qquad \ldots(1)$$

Put $\frac{-1}{2x^2} = \alpha \Rightarrow \frac{1}{x^3}\frac{dx}{dt} = \frac{d\alpha}{dt}$

From (1), we have $\frac{d\alpha}{dt} + \alpha = \frac{\sin t}{2}$

Integrating Factor $= e^t$

$$\Rightarrow \quad \alpha e^t = \frac{1}{2}\int e^t \sin t\, dt$$

$$\Rightarrow \quad 2\alpha e^t = e^t \sin t - \int e^t \cos t\, dt$$

$$\Rightarrow \quad 2\alpha e^t = e^t(\sin t - \cos t) - \int e^t \sin t\, dt$$

$$\Rightarrow \quad 2\alpha e^t = e^t(\sin t - \cos t) - 2\alpha e^t + C$$

$$\Rightarrow \quad 4\alpha e^t = e^t(\sin t - \cos t) + C$$

$$\Rightarrow \quad \left(\frac{-2}{x^2}\right) e^{y^2} = e^{y^2}(\sin y^2 - \cos y^2) + C$$

Hence, $x^2\left(\cos y^2 - \sin y^2 - ce^{-y^2}\right) = 2$ is required solution.

Test Your Concepts-V

(Solutions on page H.383)

1. Solve $\frac{dy}{dx} + (\cot x)\, y = \sin x$

2. Solve $x \cos x \frac{dy}{dx} + y\,(x \sin x + \cos x) = 1$

3. Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$.

4. Solve $(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$

5. Solve $(1 + y + x^2y)dx + (x + x^3)dy = 0$

6. Solve $x\frac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$

7. Solve $\frac{dy}{dx} + xy = y^2 e^{x^2/2} \sin x$

8. Solve $x^2y - x^3\frac{dy}{dx} = y^4 \cos x$

9. Solve $2xy\, dy - (x^2 + y^2 + 1)dx = 0$

10. Solve $x\frac{dy}{dx} + y = y^3\frac{dy}{dx}$

11. Solve $x\frac{dy}{dx} = y - \cos\left(\frac{1}{x}\right)$

12. Solve $x\frac{dy}{dx} + y = x^3 y^6$.

13. Solve $(y \log x - 1)\, ydx = xdy$

14. Solve $(2x - 10y^3)\frac{dy}{dx} + y = 0$

Answer key

1. $y \sin x = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$

2. $yx \sec x = \tan x + c$

3. $\tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2}$

4. $y\,(x^2 - 1) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$

5. $xy = c - \text{arc} \tan x$

6. $y = cx + x \ln \tan x$

7. $e^{-x^2/2} = y(c + \cos x)$

8. $x^3 y^{-3} = 3 \sin x + c$

9. $y^2 = x\,(c + x) - 1$

10. $xy = \left(\frac{1}{4}\right) y^4 + c$

11. $y - x \sin\left(\frac{1}{x}\right) = cx$

12. $y^{-5} x^{-5} = \frac{5}{2}x^{-2} + c$

13. $y\,(\log ex + cx) = 1$

14. $xy^2 = 2y^5 + c$

SINGULAR SOLUTION

A solution of differential equation which does not have any arbitrary constant in it.

It is not obtained from general solution.

Generally singular solutions are envelope to general solutions.

CLAIRAUT'S DIFFERENTIAL EQUATION

Let consider differential equation $y = mx + f(m)$ ...(1)

where $m = \frac{dy}{dx}$

differentiating it one more time

$$\frac{dy}{dx} = m + \frac{x\,dm}{dx} + f'(m)\left(\frac{dm}{dx}\right)$$

$$\Rightarrow \quad \frac{dm}{dx}(x + f'(m)) = 0$$

$$\frac{dm}{dx} = 0 \Rightarrow m = c \qquad \text{...(2)}$$

$$\text{or} \quad x + f'(m) = 0 \qquad \text{...(3)}$$

Solving (1) and (2), we obtain

$$\left.\begin{array}{l} y = mx + f(m) \\ m = c \end{array}\right] \rightarrow \text{we get general solution}$$

Solving (1) and (3), we obtain

$$\left.\begin{array}{l} y = mx + f(m) \\ x + f'(m) = 0 \end{array}\right] \rightarrow \text{we get singular solution}$$

ILLUSTRATION 76

Solve $y = mx + \frac{2}{m}$ where $m = \frac{dy}{dx}$

SOLUTION

$$y = mx + \frac{2}{m} \qquad \ldots(1)$$

On further differentiating, we get

$$\frac{dy}{dx} = m + x\frac{dm}{dx} - \frac{2}{m^2}\frac{dm}{dx} \Rightarrow \frac{dm}{dx}\left(x - \frac{2}{m^2}\right) = 0$$

$$\frac{dm}{dx} = 0 \Rightarrow m = c \qquad \ldots(2)$$

or $$x - \frac{2}{m^2} = 0 \Rightarrow m^2 = \frac{2}{x} \qquad \ldots(3)$$

Solving (1) and (2), we get $y = cx + \frac{2}{c}$

(General Solution with one arbitrary constant c) solving (1) and (3)

$$y = mx + \frac{2}{m}$$

$$\Rightarrow \quad my = m^2x + 2 = \left(\frac{2}{x}\right)x + 2 = 4$$

$$\Rightarrow \quad m^2y^2 = 16 \Rightarrow \frac{2}{x} \cdot y^2 = 16 \Rightarrow y^2 = 8x$$

Singular solution is Parabola $y^2 = 8x$ with no arbitrary constant

General solution $y = cx + \frac{2}{c}$ is tangent to parabola

Clearly we can see that singular solutions are envelope for general solution

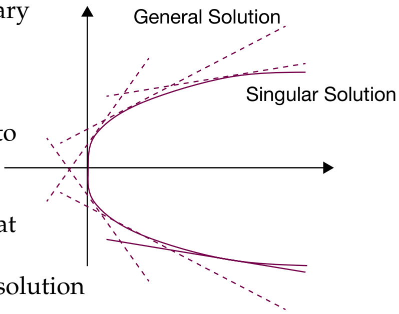


ILLUSTRATION 77

Solve $y = mx + m - m^3$ where $m = \frac{dy}{dx}$

SOLUTION

$$y = mx + m - m^3 \qquad \ldots(1)$$

$$\frac{dy}{dx} = m + \frac{x\,dm}{dx} + \frac{dm}{dx} - 3m^2\frac{dm}{dx}$$

$$\Rightarrow \quad \frac{dm}{dx}(x + 1 - 3m^2) = 0$$

$$\frac{dm}{dx} = 0 \Rightarrow m = c \qquad \ldots(2)$$

or $$x + 1 = 3m^2 \qquad \ldots(3)$$

Solving (1) and (2), we get

$y = cx + c - c^3$ is general solution

By solving (1) and (3), we get

$$y = m[x + 1 - m^2] \Rightarrow y^2 = m^2\,[x + 1 - m^2]^2$$

$$\because \quad x + 1 = 3m^2$$

$$y^2 = \left(\frac{x+1}{3}\right)\left[x+1-\left(\frac{x+1}{3}\right)\right]^2$$

$\Rightarrow \quad 27y^2 = 4(x + 1)^3$ in singular Solution

ILLUSTRATION 78

Solve the following differential equation; $\left(\text{let } p = \frac{dy}{dx}\right)$

(i) $(y - px)\,(p - 1) = p$

(ii) $p^2\,(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$

SOLUTION

(i) $(y - px)\,(p - 1) = p \qquad \ldots(1)$

On differentiation

$$\left(p - p - x\frac{dp}{dx}\right)(p - 1) + (y - px)\,\frac{dp}{dx} = \frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}\,[(1 - p)x + (y - px) - 1] = 0$$

CASE 1: $\frac{dp}{dx} = 0;\ p = k$ (constant)

From (1), we get $(y - kx)\,(k - 1) = k$ is general solution

CASE 2: $2px = x + y - 1 \Rightarrow p = \frac{x+y-1}{2x}$

From (1), $(y - px)\,(p - 1) = p$

$$\left(y - \frac{(x+y-1)}{2}\right)\left(\frac{x+y-1}{2x} - 1\right) = \frac{x+y-1}{2x}$$

$$\Rightarrow \left(\frac{y-x+1}{2}\right)\left(\frac{y-x-1}{2x}\right) = \frac{x+y-1}{2x}$$

$\Rightarrow (y - x)^2 - 1 = 2(x + y - 1)$ singular solution

Required solution are $(y - kx)\,(k - 1) = k$ and $(y - x)^2 - 1 = 2(x + y - 1)$

(ii) $p^2(x^2-a^2)-2pxy+y^2-b^2=0$ …(1)

On differentiation, we get

$$2p\frac{dp}{dx}(x^2-a^2)+p^2 2x-2\frac{dp}{dx}xy-2py-2p^2x+2yp=0$$

$$2\frac{dp}{dx}[p(x^2-a^2)-xy]=0;$$

CASE 1: $\frac{dp}{dx}=0;\ p=k=$ constant

From (1), we have $k^2(x^2-a^2)-2kxy+y^2-b^2=0$ is general solution

CASE 2: $p(x^2-a^2)=xy$ or $p=\frac{xy}{x^2-a^2}$

From (1), we get

$$(px-y)^2=p^2a^2+b^2$$

$$\left(\frac{x^2y}{x^2-a^2}-y\right)^2=\frac{x^2y^2}{\left(x^2-a^2\right)^2}a^2+b^2$$

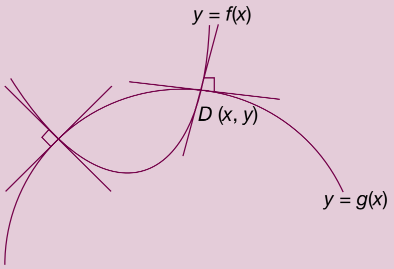
$\Rightarrow \left(\frac{a^2y}{x^2-a^2}\right)^2=\frac{x^2y^2a^2}{\left(x^2-a^2\right)^2}+b^2$ is singular solution

Required solutions are $k^2(x^2-a^2)-2kxy+y^2-b^2=0$

and $\left(\frac{a^2y}{x^2-a^2}\right)^2=\frac{x^2y^2a^2}{\left(x^2-a^2\right)^2}+b^2$

ORTHOGONAL TRAJECTORY

For a given curve, orthogonal trajectory is a curve which is perpendicular to given curve at each point of intersection Let given curve is $y=f(x)$



(i) First form differential equation of given curve $y=f(x)$

(ii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in above differential equation

(iii) Solution of new differential equation $y=g(x)$ is orthogonal trajectory of given curve or we can say $\left(\frac{df(x)}{dx}\right)\left(\frac{dg(x)}{dx}\right)=-1$

ILLUSTRATION 79

Find orthogonal trajectory of curve $xy=c^2$

SOLUTION

Curve has one arbitrary constant

$$1.y+x\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=-\frac{y}{x}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in above differential equation

$$-\frac{dx}{dy}=-\frac{y}{x}\Rightarrow x\,dx=y\,dy$$

$$\Rightarrow\quad \int x\,dx=\int y\,dy\Rightarrow\frac{x^2}{2}=\frac{y^2}{2}+k\Rightarrow x^2-y^2=2k$$

ILLUSTRATION 80

Find orthogonal curve of family of concentric circles at fixed point $(\alpha, 0)$

SOLUTION

family of such family of circle $(x-\alpha)^2+(y-0)^2=r^2$ where r is arbitrary constant and α is fixed constant

$$\Rightarrow\quad x^2+y^2-2x\alpha+\alpha^2-r^2=0$$

$$\Rightarrow\quad 2x+2y\frac{dy}{dx}-2\alpha+0=0\Rightarrow(x-\alpha)=-y\frac{dy}{dx}$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in above differential equation

We get, $x-\alpha=y\frac{dx}{dy}\Rightarrow\int\frac{dy}{y}=\int\frac{dx}{x-\alpha}$

$$\Rightarrow\quad \ln y=\ln(x-\alpha)+\ln k$$

$\Rightarrow\quad \ln y=\ln k\,(x-\alpha)\Rightarrow y=k\,(x-\alpha)$ straight line is orthogonal trajectory of given family of circle

ILLUSTRATION 81

Find the orthogonal trajectories of the family of semi-cubical parabolas $ay^2=x^3$, where a is a variable parameter.

SOLUTION

$$ay^2=x^3 \quad …(1)$$

Differentiate with respect to x we have $2ay\frac{dy}{dx}=3x^2$

Replace $\frac{dy}{dx}$ by $-\frac{1}{\frac{dy}{dx}}$ we get $2ay=-3x^2\frac{dy}{dx}$

$$\Rightarrow\quad a=-\frac{3}{2}\frac{x^2}{y}\frac{dy}{dx}$$

Put the value of a in equation (1) we get $-\frac{3}{2}\frac{x^2}{y}\cdot\frac{dy}{dx}=\frac{x^3}{y^2}$

$$\Rightarrow\quad 3y\,dy=-2x\,dx$$

On integration we get $3y^2+2x^2=C$, is required orthogonal trajectory of given curve.

Test Your Concepts-VI

(Solutions on page H.385)

1. Find orthogonal trajectory of straight line passing through origin.
2. Solve $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$
3. Find the value of k such that the family of parabolas $y = cx^2 + k$ is the orthogonal trajectory of the family of ellipses $x^2 + 2y^2 - y = c$.
4. Solve the following differential equation; $\left(\text{let } p = \frac{dy}{dx}\right)$

   (i) $x^2p^2 - 2xyp + 2y^2 - x^2 = 0$ (ii) $y = px + p - p^2$

Answer key

1. $x^2 + y^2 = r^2$ **2.** $y = cx - c^2$ or $x^2 = 4y$ **3.** $\frac{1}{4}$ **4.** (i) $\frac{y}{x} = \sin(\pm \ln x + c)$ (ii) $y = kx + k - k^2$ or $4y = (x+1)^2$

GEOMETRICAL AND PHYSICAL APPLICATION OF DIFFERENTIAL EQUATION

Form a differential equation from a given geometrical or physical problem.

For geometrical application following formula are useful to remember

(i) Length of tangent $(L_T) = \left|\frac{y\sqrt{1+m^2}}{m}\right|$

(ii) Length of normal $(\mathrm{L_N}) = \left|y\sqrt{1+m^2}\right|$

(iii) Length of sub-tangent $(\mathrm{L_{ST}}) = \left|\frac{y}{m}\right|$

(iv) Length of subnormal $(\mathrm{L_{SN}}) = |\,my\,|$

where y is the ordinate of the point, m is the slope of the tangent.

ILLUSTRATION 82

Find the equation of the curve for which the Cartesian sub-tangent varies as the reciprocal of the square of the abscissa.

SOLUTION

Given that $\ell_{\text{sub-tangent}} \propto \frac{1}{x^2}$

$$\frac{y}{\frac{dy}{dx}} = \frac{k}{x^2} \Rightarrow \frac{x^2}{k}dx = \frac{dy}{y}$$

$$\Rightarrow \quad \frac{x^3}{3k} = \ln y + \ln c \Rightarrow cy = e^{x^3/3k} \Rightarrow y = \frac{1}{c}e^{x^3/3k}$$

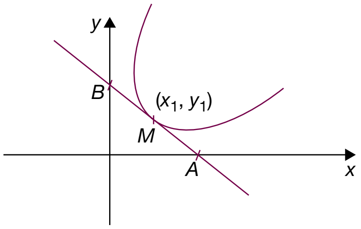
Required curve $y = \lambda e^{x^3/3k}$

ILLUSTRATION 83

Find equation of curve whose segment of any tangent intercepted between co-ordinate axes is bisected at point of contact.

SOLUTION

Any tangent at point (x_1, y_1) is $y - y_1 = m(x - x_1)$



Tangent cut x-axis, then $y = 0$; $x = x_1 - \frac{y_1}{m}$ at point $A\left(x_1 - \frac{y_1}{m}, 0\right)$

Tangent cut y-axis, then $x = 0$; $y = y_1 - mx_1$ at point $B(0, y_1 - mx_1)$

Now mid point of segment of tangent between coordinate axes

$$M(x_1, y_1) = \left(\frac{x_1 - \frac{y_1}{m}}{2}, \frac{y_1 - mx_1}{2}\right)$$

$$x_1 = \frac{x_1 - \frac{y_1}{m}}{2} \Rightarrow 2x_1 = x_1 - \frac{y_1}{m} \Rightarrow x_1 = -\frac{y_1}{m}$$

Replacing x_1 by x and y_1 by y, $x = -\frac{y}{m}$

$$m = \frac{dy}{dx} = -\frac{y}{x} \Rightarrow \int\frac{dy}{y} = -\int\frac{dx}{x}$$

$\Rightarrow$ $\ln y = -\ln x + \ln k \Rightarrow \ln x + \ln y = \ln k$

$\Rightarrow$ $\ln(xy) = \ln k$

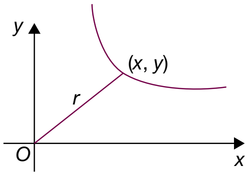
$\Rightarrow$ $xy = k$

It is a hyperbola.

ILLUSTRATION 84

Identify curve whose length of normal at any point is equal to radius vector of that point.

SOLUTION



As given $\ell_{normal} = |y\sec\theta| = r$

$$m = \frac{dy}{dx} = \tan\theta$$

$$|y|\sqrt{1+m^2} = \sqrt{x^2+y^2}$$

$\Rightarrow$ $y^2 + m^2y^2 = x^2 + y^2$

$\Rightarrow$ $m^2y^2 = x^2 \Rightarrow ym = \pm x \Rightarrow y\frac{dy}{dx} = \pm x$

$\Rightarrow$ $\int y\,dy = \int \pm x\,dx \Rightarrow \frac{y^2}{2} = \pm\frac{x^2}{2} + c$

It represents hyperbola $x^2 - y^2 = -2c$ and circle $x^2 + y^2 = 2c$

ILLUSTRATION 85

The rate of disintegration of radium is proportional to the quantity of radius present. Radium disintegrate by one half in 1600yrs. Find the amount of radium that has disintegrated after 100 years

SOLUTION

According to given condition $\frac{dx}{dt} = -kx$ where x is quantity of radium at time t

$$\int\frac{dx}{x} = \int -k\,dt \Rightarrow \ln x = -kt + c$$

$\Rightarrow$ $x = e^{-kt+c} = e^ce^{-kt}$

When $t = 0$; $x = e^c = x_o$

after $t = 1600$ years; $x = \frac{x_0}{2}$

$$\frac{x_0}{2} = x_0e^{-k(1600)}$$

$\Rightarrow$ $\ln 2 = k(1600) \Rightarrow k = \frac{\ln 2}{1600}$

After $t = 100$ years; $x = x_0^{-\frac{\ln 2}{1600}t}$

$$x = x_0e^{-\frac{\ln 2}{1600}.100} = x_0e^{\ln 2^{-1/16}}$$

amount that has disintegrated $= x_0 - \frac{x_0}{2^{1/16}} = \left(1 - \frac{1}{2^{1/16}}\right)$ of initial amount

ILLUSTRATION 86

Show that the curve for which the normal at every point passes through a fixed point is a circle.

SOLUTION

Equation of normal at (x, y) is

$$Y - y = -\frac{dx}{dy}(X - x)$$

Let it is passes through a fixed point (h, k)

$\therefore$ $(k - y) = -\frac{dx}{dy}(h - x) \Rightarrow (x - h)dx = -(y - k)dy$

On integration, we get

$\Rightarrow$ $\frac{(x-h)^2}{2} = -\frac{(y-k)^2}{2} + C$

$\Rightarrow$ $(x - h)^2 + (y - k)^2 = 2C = r^2$

which is a equation of circle.

ILLUSTRATION 87

A particle P moves so that its velocities parallel to the axes of x and y are respectively $-ky$ and kx where k is a constant different from zero. Find the path of the particle, if it passes through the point (3, 4).

SOLUTION

$$v = \frac{ds}{dt} = -ky\hat{i} + kx\hat{j}$$

$$\frac{dx}{dt} = -ky; \quad \frac{dy}{dt} = kx;$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{kx}{-ky} = -\frac{x}{y}$$

$\Rightarrow$ $y\,dy = -x\,dx$

$$\frac{y^2}{2} = -\frac{x^2}{2} + c \Rightarrow x^2 + y^2 = 2c$$

Point (3, 4) lies on it, then $9 + 16 = 2c \Rightarrow 2c = 25$

Hence required curve $x^2 + y^2 = 25$

ILLUSTRATION 88

Population $P(t)$ at time t of a certain species of the Dinosaur is changing by rate as $\frac{dP(t)}{dt} = 0.5\,P(t) - 450$. If $P(0) = 850$ find time by which the Dinosaur disappears from earth.

SOLUTION

Given diff. equation $\frac{dP(t)}{dt} = \frac{P(t) - 900}{2}$

$\Rightarrow$ $\frac{dP(t)}{P(t) - 900} = \frac{dt}{2}$

$\Rightarrow$ $\ln|P(t) - 900| = \frac{t}{2} + c \Rightarrow |P(t) - 900| = e^{\frac{t}{2}+c}$

$\Rightarrow$ $900 - P(t) = e^{\frac{t}{2}} \times e^c$

Given $P(0) = 850$; we have $900 - 850 = e^0 \cdot e^c \Rightarrow e^c = 50$

$\Rightarrow$ $900 - P(t) = 50e^{t/2} \Rightarrow P(t) = 900 - 50e^{t/2}$

Now we are looking for time by which population $P(t) = 0$

$$\Rightarrow \quad P(t) = 900 - 50e^{t/2} = 0$$

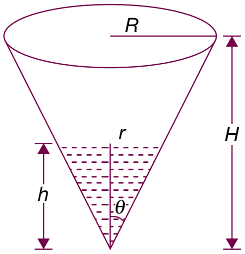
$$\Rightarrow \quad 900 = 50e^{t/2} \Rightarrow e^{t/2} = 18 \Rightarrow \frac{t}{2} = \ln 18$$

Hence, we have $t = 2 \ln 18$

ILLUSTRATION 89

A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant $= K > 0$). Find the time after which the cone is empty.

SOLUTION



Given: liquid evaporates at a rate proportional to its surface area

$$\Rightarrow \quad \frac{dv}{dt} \propto -S$$

We know, volume of cone

$$V = \frac{1}{3}\pi r^2 h$$

Surface area $S = \pi r^2$ (of liquid in contact with air)

$$\Rightarrow \quad V = \frac{1}{3}\pi r^2 h \text{ and } S = \pi r^2 \qquad \ldots(1)$$

Where $\tan\theta = \frac{R}{H}$ and $\frac{r}{h} = \tan\theta$ ...(2)

From (1) and (2), we get

$$\Rightarrow \quad V = \frac{1}{3}\pi r^3 \cot\theta \text{ and } S = \pi r^2 \qquad \ldots(3)$$

Substituting (3) in (1), we get

$$\frac{1}{3}\cot\theta\, 3r^2 \frac{dr}{dt} = -k\pi r^2$$

$$\Rightarrow \quad \cot\theta \int_R^0 dr = -k\int_0^T dt \Rightarrow \cot\theta\,(0 - R) = -k(T - 0)$$

$$\Rightarrow \quad \cot\theta = kT \Rightarrow H = kT \qquad \text{(using 3)}$$

∴ Required time after which the cone is empty $= T = \frac{H}{k}$

ILLUSTRATION 90

Find the equation of the curve passing through the origin if the middle point of the segment of its normal from any point of the curve to the x-axis lies on the parabola $2y^2 = x$.

SOLUTION

Equation of normal $Y - y = -\frac{dx}{dy}(X - x)$ at point $P(x, y)$

Put $Y = 0$ where normal cuts x-axis

$$X = x + y\frac{dy}{dx}, \text{ point } Q\left(x + y\frac{dy}{dx}, 0\right)$$

Mid point of $PQ = \left(\frac{2x + y\frac{dy}{dx}}{2}, \frac{y}{2}\right)$ lies on $y^2 = \frac{x}{2}$

$$\frac{y^2}{4} = \frac{2x + y\frac{dy}{dx}}{4} \Rightarrow y\frac{dy}{dx} - y^2 + 2x = 0$$

Put $y^2 = t$; $2y\frac{dy}{dx} = \frac{dt}{dx}$;

$$\frac{dt}{dx} - 2t = -4x \quad \text{It is linear differential equation}$$

Integrating factor $= e^{-2x}$

$$t.e^{-2x} = -4\int xe^{-2x}\,dx = -4\left[\frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4}\right] + c$$

$$t = 2x + 1 + e^{2x}c \Rightarrow y^2 = 2x + 1 + c.e^{2x}$$

Origin lies on it, put $x = 0$, $y = 0 \Rightarrow c = -1$

Required curve is $y^2 = 2x + 1 - e^{2x}$

ILLUSTRATION 91

Given a curve C. Suppose that the tangent line at $P(x, y)$ on C is perpendicular to the line joining P and $Q(1, 0)$. If the line $2x + 3y - 15 = 0$ is tangent to the curve C then show that the curve C cuts intercept of length $4\sqrt{3}$ on y-axis.

SOLUTION

Tangent at $P(x, y)$ is $Y - y = m(X - x)$

As given tangent is perpendicular to line PQ,

$$\left(\frac{y - 0}{x - 1}\right)m = -1$$

$$\Rightarrow \quad y\frac{dy}{dx} = 1 - x \Rightarrow y\,dy = (1 - x)\,dx$$

After integrating $\frac{y^2}{2} = x - \frac{x^2}{2} + c$

$$\Rightarrow \quad x^2 + y^2 - 2x - c = 0 \text{ is curve } C \text{ (circle)} \qquad \ldots(1)$$

This is the equation of a circle with centre (1, 0)

Given $2x + 3y - 15 = 0$ is tangent to curve C

$$\Rightarrow \quad \left|\frac{2 + 0 - 15}{\sqrt{13}}\right| = \sqrt{1 + c} \Rightarrow c + 1 = 13 \Rightarrow c = 12$$

Hence the curve is $x^2 + y^2 - 2x - 12 = 0$

Length of interception y-axis $= 2\sqrt{f^2 - c}$

$$= 2\sqrt{0 + 12} = 4\sqrt{3}$$

ILLUSTRATION 92

A wet porous substance in the open air loses its moisture at a rate proportional to the moisture content. If a sheet hung in wind loses half its moisture during the first hour when will it have lost 99% of initial moisture content. Assume that the weather conditions remain the same.

SOLUTION

Let x is present amount of moisture.

According to given information $\frac{dx}{dt} = -kx$

$$\Rightarrow \quad \frac{dx}{x} = -k\,dt \Rightarrow \ln x = -kt + c$$

$$\Rightarrow \quad x = e^{-kt+c}$$

Put $t = 0$; $x = e^c = x_0$, we have $x = x_0e^{-kt}$

After $t = 1$ hr, 50% moisture is gone so $\frac{x_0}{2} = x_0e^{-k} \Rightarrow k = \ln 2$

99% of initial amount is lost, 1% is left $\frac{x_0}{100} = x_0e^{-\ln 2t}$

$$\Rightarrow \quad \ln 100 = \ln 2t \quad \Rightarrow \quad t = \frac{\ln 100}{\ln 2} \text{ hrs}$$

ILLUSTRATION 93

A body at a temperature of 50°F is placed outdoors where the temperature is 100°F. If the rate of change of the temperature of body is proportional to the temperature difference between the body and its surroundings. If after 5 min the temperature of body is 60° F, Find

(i) How long will it take to reach a temp of 75° F
(ii) The temp. of body after 20 min.

SOLUTION

According to given information, we can write

$$\frac{dT}{dt} = k(T_s - T) = -k(T - T_s)$$

where T is temperature of body at time T and surrounding temperature is $T_s = 100$°F

$$\int \frac{dT}{T - T_s} = \int -k\,dt \Rightarrow \ln(T - T_s) = -kt + c$$

$$T - T_s = e^{-kt+c} \Rightarrow T = T_s + e^c e^{-kt}$$

At $t = 0$, $50^o = 100^o + e^c \Rightarrow e^c = -50^o$

$$T = 100^o - 50^o e^{-kt}$$

After $t = 5$ min

$$T = 60^o = 100^o - 50^o e^{-5k}$$

$$\Rightarrow \quad 50^o e^{-5k} = 40^o \Rightarrow -5k = \ln\frac{4}{5} \Rightarrow k = \frac{1}{5}\ln\frac{5}{4}$$

(i) $T = 100^o - 50^o e^{-kt} = 75^o$

$$\Rightarrow 25^o = 50^o e^{-kt} \Rightarrow 1 = 2e^{-kt}$$

$$\Rightarrow \ln 2 = kt = \frac{1}{5}\ln\frac{5}{4}t$$

$$\Rightarrow t = \frac{5\ln 2}{\ln\left(\frac{5}{4}\right)} = 5\log_{5/4} 2 \text{ min}$$

(ii) $T = 100^o - 50^o e^{-kt}$

after $t = 20$ min

$$T = 100^o - 50^o e^{-4\ln\frac{5}{4}} = 100^o - 50^o e^{\ln\left(\frac{4}{5}\right)^4}$$

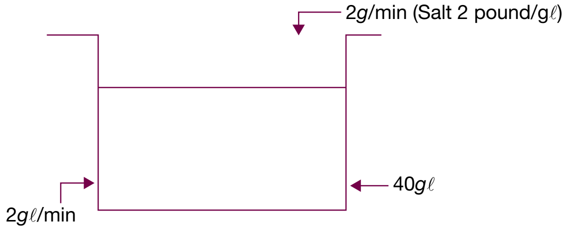
$$T = 100^o - 50^o\left(\frac{4}{5}\right)^4 = 100^o - 20.48^o = 79.52^o \text{ F}$$

ILLUSTRATION 94

A tank initially contains 40 gallons of fresh water. Brine contains 2 pounds per gallons of salt flows into the tank at the rate of 2 gallons of per minutes and the mixture kept uniform by stirring, runs out at the same rate. It takes $k\ln\left(\frac{5}{2}\right)$ min. for the quantity of salt in the tank to increase from 30 to 60 pounds, find k.

SOLUTION

Let x is amount of salt present in tank at time t, 4 pound/min is added per minute and $\frac{x}{40}.2$ pound/min goes out



Rate of change in salt amount $= \frac{dx}{dt} = 4 - \frac{x}{40}2$

$$\Rightarrow \quad \frac{dx}{dt} = -\frac{(x-80)}{20}$$

$$\Rightarrow \quad \int \frac{dx}{x-80} = \int -\frac{1}{20}\,dt$$

$$\Rightarrow \quad \ln|x-80|\Big|_{30}^{60} = -\frac{1}{20}t\Big|_{t_1}^{t_2}$$

$$\Rightarrow \quad \ln 20 - \ln 50 = -\frac{1}{20}(t_2 - t_1)$$

$$\Rightarrow \quad \ln\frac{2}{5} = -\frac{1}{20}.k\ln\frac{5}{2} \Rightarrow k = 20$$

ILLUSTRATION 95

A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from the x-axis. Determine the equation of the curve.

SOLUTION

Equation of normal at point (x, y) is

$$Y - y = -\frac{dx}{dy}(X - x) \qquad \ldots(1)$$

Distance of perpendicular from the origin to normal

$$= \frac{\left|y + \frac{dx}{dy}.x\right|}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}}$$

Also distance between P and x-axis is $|y|$

According to given information $\dfrac{\left|y + \frac{dx}{dy}.x\right|}{\sqrt{1+\left(\frac{dx}{dy}\right)^2}} = |y|$

$$\Rightarrow \quad y^2 + \left(\frac{dx}{dy}\right)^2.x^2 + 2xy\frac{dx}{dy} = y^2\left[1+\left(\frac{dx}{dy}\right)^2\right]$$

$$\Rightarrow \quad \left(\frac{dx}{dy}\right)^2(x^2 - y^2) + 2xy\frac{dx}{dy} = 0$$

$$\Rightarrow \quad \frac{dx}{dy}\left[\left(\frac{dx}{dy}\right)(x^2 - y^2) + 2xy\right] = 0$$

$$\Rightarrow \quad \frac{dx}{dy} = 0 \text{ or } \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

But $\frac{dx}{dy} = 0 \Rightarrow x = c$, where c is a constant.

As curve passes through (1, 1), we get equation of the curve as $x = 1$.

The equation $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ is a homogeneous equation.

Put $y = vx$, so that $v + x\frac{dv}{dx} = \frac{v^2x^2 - x^2}{2x^2v}$

$$x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v} = -\frac{v^2 + 1}{2v}$$

$$\Rightarrow \quad \frac{-2v}{v^2 + 1}dv = \frac{dx}{x} \Rightarrow \log k - \log(v^2 + 1) = \log|x|$$

$$\Rightarrow \quad \log|x|(v^2 + 1) = \log k \Rightarrow |x|\left(\frac{y^2}{x^2} + 1\right) = k$$

$x^2 + y^2 = \pm kx$ is passing through (1, 1) so $1 + 1 = \pm k.1$

$\Rightarrow \quad \pm k = 2$

Therefore, required curve is $x^2 + y^2 = 2x$

ILLUSTRATION 96

A country has a food deficit of 10%. Its population grows continuously at a rate of 3% per year. Its annual food production every year is 4% more than that of the last year. Assuming that the average food requirement per person remains constant. Prove that the country will become self-sufficient in food after n years, where n is the smallest integer bigger than or equal to $\frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$.

SOLUTION

Let X_0 be initial population of the country and Y_0 be its initial food production.

Let the average consumption be a units.

Therefore, food required initially K. It is given

$$Y_0 = K\left(\frac{90}{100}\right) = 0.9\,K \qquad \ldots(1)$$

Let X be the population of the country in year t.

Then $\frac{dX}{dt}$ = rate of change of population = $\frac{3}{100}X$

Integrating, $\int\frac{dX}{x} = \int 0.03\,dt$

$\Rightarrow \quad \log X = 0.03t + c \Rightarrow X = A.e^{0.03t}$ when $A = e^c$

At $t = 0$, $X = X_0$, thus $X_0 = A$

$\therefore \quad X = X_0\,e^{0.03t}$

Let Y be the food production in year t.

Its annual food production every year is 4% more than that of the last year

Then $Y = Y_0\left(1 + \frac{4}{100}\right)^t = 0.9\,K\,(1.04)^t \qquad [\because\; Y_0 = 0.9K]$

Food consumption in the year t is $Ke^{0.03t}$

again $Y - X \geq 0$ (given)

$$\Rightarrow \quad 0.9\,k\,(1.04)^t > k\,e^{0.03t} \Rightarrow \frac{(1.04)^t}{e^{0.03t}} > \frac{1}{0.9} = \frac{10}{9},$$

Taking log on both sides, $t[\ln(1.04) - 0.03] \geq \ln 10 - \ln 9$

$$\Rightarrow \quad t \geq \frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$$

Thus, the least integral values of the year n, when the country becomes self-sufficient, is the smallest integer greater than or equal to $\frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}$

ILLUSTRATION 97

A curve $y = f(x)$ passes through $O(0, 0)$ and slope of tangent line at any point $P(x, y)$ of the curve is $\frac{x^4 + 2xy - 1}{1 + x^2}$, then find $f(-1)$

SOLUTION

According to given information, slope of tangent

$$\Rightarrow \quad \frac{dy}{dx} = x^2 - 1 + \frac{2xy}{1+x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{2x}{1+x^2}\, y = x^2 - 1$$

Integrating factor $= e^{-\int \frac{2x}{1+x^2}dx} = e^{-\ln(1+x^2)} = \frac{1}{1+x^2}$;

$$\Rightarrow \quad \frac{y}{1+x^2} = \int \frac{x^2-1}{x^2+1}dx;$$

$$\Rightarrow \quad \frac{y}{1+x^2} = x - 2\tan^{-1}x + c$$

It passes through origin, $x = 0, y = 0 \Rightarrow c = 0$

$$\Rightarrow \quad y = (x^2+1)x - 2(x^2+1)\tan^{-1}x$$

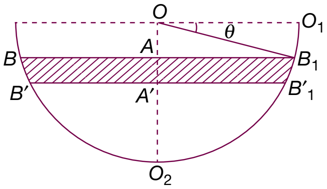
Hence, $f(-1) = -2 + 4\frac{\pi}{4} = \pi - 2$

ILLUSTRATION 98

A hemispherical tank of radius 2 meters is initially full of water and has an outlet of 12 cm^2 cross-sectional area of the bottom. The outlet is opened at some of the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $v(t) = 0.6\sqrt{2gh(t)}$, where $v(t)$ and $h(t)$ are respectively the rate of flow at outlet and water level in tank at time t, and g is the acceleration due to gravity. Find the time takes to empty the tank.

SOLUTION

Let 'O' be the centre of hemispherical tank. Let at any instant 't', water level be BAB_1 and at $t + dt$, water level is $B'A'B_1$.



Let $\angle O_1OB_1 = \theta$

$$\Rightarrow \quad AB_1 = r\cos\theta \text{ and } OA = r\sin\theta$$

Decreases in the water volume in time dt, $= \pi AB_1{}^2.d\,(OA)$

(πr^2 is surface area of water level and $d(OA)$ is depth of water level).

$$= \pi r^2.\cos^2\theta\,.\,r\cos\theta\, d\theta = \pi r^3.\cos^3\theta\, d\theta$$

Also $h(t) = O_2A = r - r\sin\theta = r(1 - \sin\theta)$

Now outflow rate $Q = A.v(t) = A.(0.6)\sqrt{2gr(1-\sin\theta)}$

Where, $\pi r^3\cos^3\theta\, d\theta = A(0.6).\sqrt{2gr}\,.\sqrt{1-\sin\theta}\, dt$

$$\Rightarrow \quad \frac{\pi r^3}{A(0.6)\sqrt{2gr}}\,\frac{\cos^3\theta}{\sqrt{(1-\sin\theta)}}\,d\theta = dt$$

Let the time taken to empty the tank be T.

$$\text{Then } T = \int_0^{\pi/2} \frac{\pi r^2}{A(0.6).\sqrt{2gr}}\,\frac{\cos^3\theta}{\sqrt{(1-\sin\theta)}}\,d\theta$$

$$= -\frac{\pi r^3}{A(0.6)\sqrt{2gr}}\int_0^{\pi/2}\frac{\cos^2\theta(-\cos\theta)}{\sqrt{(1-\sin\theta)}}\,d\theta$$

Let $t_1 = \sqrt{1-\sin\theta} \Rightarrow dt_1 = \frac{-\cos\theta}{\sqrt{1-\sin\theta}}\,d\theta$

$$\Rightarrow \quad T = \frac{-2\pi r^3}{A(0.6)\sqrt{2gr}}\int_1^0 (1-\sin^2\theta)\,dt_1$$

$$\Rightarrow \quad T = \frac{-2\pi r^3}{A(0.6)\sqrt{2gr}}\int_1^0 [1-(\sin\theta)^2]\,dt_1$$

$$\Rightarrow \quad T = \frac{-2\pi r^3}{A(0.6)\sqrt{2gr}}\int_1^0 [1-(1-t_1^2)^2]\,dt_1$$

$$\Rightarrow \quad T = \frac{2\pi r^3}{A(0.6)\sqrt{2gr}}\int_1^0 (t_1^4 - 2t_1^2)\,dt_1$$

$$\Rightarrow \quad T = \frac{2\pi r^3}{A(0.6)\sqrt{2gr}}\left[\frac{t_1^5}{5} - \frac{2t_1^3}{3}\right]_1^0$$

$$\Rightarrow \quad T = \frac{2\pi.2^{5/2}(10^2)^{5/2}}{12.\frac{3}{5}.\sqrt{2}.\sqrt{g}}\left[\frac{2}{3} - \frac{1}{5}\right]$$

$$T = \frac{14\pi\times 10^5}{27\sqrt{g}} \text{ units.}$$

ILLUSTRATION 99

A and B are two separate reservoirs of water. Capacity of reservoir A is double the capacity of reservoir B. Both the reservoirs are filled completely with water, their inlets are closed and then the water is released simultaneously from both the reservoirs. The rate of flow of water out of each reservoir at any instant of time is proportional to the quantity of water in the reservoir at that time. One hour after the water is released, the quantity in reservoirs A is $1\frac{1}{2}$ times the quantity of water is reservoirs B. After how many hours do both the reservoirs have the same quantity of water?

SOLUTION

According to given information $\frac{dv}{dt} \propto v$ for each reservoir where v is volume of reservoir at time t

$$\frac{dv}{dx} \propto -v_A \Rightarrow \frac{dv_A}{dt} = -k_1 v_A$$

(K_1 is the proportional constant).

$$\Rightarrow \int_{v_A}^{v'_A} \frac{dv_A}{v_A} = -k_1 \int_0^t dt \Rightarrow \log \frac{v'_A}{v_A} = -k_1 t$$

$$\Rightarrow \quad v'_A = v_A \,.\, e^{-k_1 t} \qquad \ldots(1)$$

Similarly for B,

$$v_B{}' = v_B \,.\, e^{-k_2 t} \qquad \ldots(2)$$

Dividing (1) by (2), we get

$$\frac{v'_A}{v'_B} = \frac{v_A}{v_B} \,.\, e^{-(k_1 - k_2)t}$$

It is given that at $t = 0$, $v_A = 2v_B$ and at $t = 1$, $v'_A = \frac{3}{2}\, v'_B$.

Thus, $\frac{3}{2} = 2 \,.\, e^{-(k_1-k_2)t} \Rightarrow e^{-(k_1-k_2)} = \frac{3}{4}$

Now, let at $t = t_0$ both the reservoirs have some quantity of water. Then $v'_A = v'_B$.

Hence, $2e^{-(K_1-K_2)} = 1 \Rightarrow 2.\left(\frac{3}{4}\right)^{t_0} = 1 \Rightarrow t_0 = \log_{3/4}\left(\frac{1}{2}\right)$

Test Your Concepts-VII

(Solutions on page H.385)

1. Show that the equation of the curve whose slope at any points is equal to $y + 2x$ and which pass through the origin is
2. Fermentation of a compound is being done in a culture of yeast. The fermentation grows at a rate proportional to the amount of the compound present. If the amount triples in one hour, how much amount would be there at the end of 2.5 hours?
3. Find the curve for which the perpendicular from the foot of the ordinate to the tangent is of constant length.
4. The temperature inside a room is 27°C. A thermometer reading 19°C is brought into this room. One minute later, the thermometer's reading is 23°C. How long will it take for the thermometer to read 26°C?
5. The population of a country increases at a rate proportional to the number of inhabitants. If the population doubles in 30 years, find after how many years the population will triple?
6. It is known that the decay rate of radium is directly proportional to its quantity at each given instant. Find the law of variation of a mass of radium as a function of time if at $t = 0$, the mass of the radius was m_0 and during time t_0 α % of the original mass of radium decay.
7. The population P of a town decreases at a rate proportional to the number by which the population exceeds 1000, proportionality constant being $k > 0$. Find
   (a) Population at any time t, given initial population of the town being 2500.
   (b) If 10 years later the population has fallen to 1900, find the time when the population will be 1500.
   (c) Predict about the population of the town in the long run.
8. If length of tangent at any point on the curve $y = f(x)$ intercepted between the point and the x–axis is of length 1. Find the equation of the curve.
9. A curve C has the property that if the tangent drawn at any point P on C meets the coordinate axes at A and B, then P is the midpoint of A and B, The curve passes through point (1, 1). Determine the equation of the curve.
10. A spherical rain drop evaporates at the rate proportional to its surface area at any instant t. The differential equation giving the rate of change of the radius of the rain drop is
11. Find the curve for which the normal at every point passes through a fixed point (h, k).

Answer key

1. $y = 2\,(e^x - x - 1)$ **2.** 15.6 times **3.** $y^2 - k^2 = (ce^{x/k} - y)^2$ **4.** 3 minutes

5. 48 years **6.** $m = m_0 e^{-kt}$ where $k = -\frac{1}{t_0} \ln\left(1 - \frac{\alpha}{100}\right)$

7. (a) $P = 1000 + 1500e^{-kt}$ where $k = \frac{1}{10}\ln\left(\frac{5}{3}\right)$; (b) $T = 10 \log_{5/3}(3)$; (c) $P = 1000$ as $t \to \infty$

8. $\sqrt{1-y^2} + \ln\left|\frac{1-\sqrt{1-y^2}}{y}\right| = \pm x + c$ **9.** $xy = 1$ **10.** $\frac{dr}{dt} = -k$ **11.** $(x - h)^2 + (y - k)^2 = a^2$

SOLVED PROBLEMS

PROBLEM 1

Solve: $(1-x^2)^2\,dy + \left(y\sqrt{1-x^2} - x - \sqrt{1-x^2}\right)dx = 0.$

SOLUTION

Rewriting given differential equation

$$\frac{dy}{dx} + \frac{y}{(1-x^2)^{3/2}} = \frac{x+\sqrt{1-x^2}}{(1-x^2)^2}$$

Integrating factor $= e^{\int \frac{dx}{(1-x^2)^{3/2}}}$;

Put $x = \sin\theta$; $dx = \cos\theta\, d\theta$

Integrating factor$= e^{\int \frac{\cos\theta d\theta}{\cos^3\theta}} = e^{\int \sec^2\theta d\theta}$

$$= e^{\tan\theta} = e^{\frac{x}{\sqrt{1-x^2}}}$$

$$y.\,e^{\frac{x}{\sqrt{1-x^2}}} = \int e^{\frac{x}{\sqrt{1-x^2}}}.\frac{x+\sqrt{1-x^2}}{(1-x^2)^2}dx$$

Put $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$$y.\,e^{\frac{x}{\sqrt{1-x^2}}} = \int e^{\tan\theta}.\frac{\sin\theta+\cos\theta}{\cos^4\theta}\cos\theta d\theta$$

$$= \int e^{\tan\theta}(\tan\theta+1)\sec^2\theta d\theta$$

Put $\tan\theta = z$; $\sec^2\theta\, d\theta = dz$

$$= \int e^z(z+1)dz = ze^z + c = \tan\theta\, e^{\tan\theta} + c$$

$$\Rightarrow \quad y.\,e^{\frac{x}{\sqrt{1-x^2}}} = \frac{x}{\sqrt{1-x^2}}\,e^{\frac{x}{\sqrt{1-x^2}}} + c$$

$$\Rightarrow \quad y = \frac{x}{\sqrt{1-x^2}} + ce^{-\frac{x}{\sqrt{1-x^2}}}$$

PROBLEM 2

Determine the function $y = f(x)$ satisfying the equation

$$x\int_0^x y(t)\,dt = (x+1)\int_0^x ty(t)\,dt;\ x > 0$$

SOLUTION

$$x\int_0^x y(t)\,dt = (x+1)\int_0^x ty(t)\,dt;\ x > 0$$

On differentiation, we get

$$\int_0^x y(t)\,dt + xy(x) = \int_0^x ty(t)\,dt + (x+1)\,xy(x)$$

On again differentiation, we get

$$y(x) + y(x) + x\frac{dy}{dx}$$
$$= x\,y(x) + x\,y(x) + (x+1)\,y(x) + (x+1)x\frac{dy}{dx}$$

$$\Rightarrow \quad 2y(x) + x\frac{dy}{dx} = 3x\,y(x) + y(x) + (x^2+x)\frac{dy}{dx}$$

$$\Rightarrow \quad x^2\frac{dy}{dx} = y(x) - 3x\,y(x) = y(1-3x)$$

$$\Rightarrow \quad \int\frac{dy}{y} = \int\left(\frac{1}{x^2} - \frac{3}{x}\right)dx \qquad (\because\ y(x) = y)$$

On integration, we get

$$\ln y = -\frac{1}{x} - 3\ln x + c$$

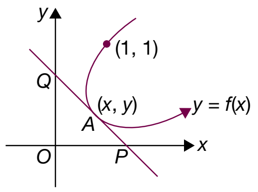
PROBLEM 3

Let $y = f(x)$ be a curve passing through (1, 1) such that the triangle formed by the coordinate the axes and tangent at any point of the curve lies in the first quadrant and has area 2. Form the differential equation and determine all such possible curves.

SOLUTION

Equation of tangent to the curve $y = f(x)$ at point $A(x, y)$ is

$$Y - y = \frac{dy}{dx}(X - x)$$



whose, x-intercept $\left(x - y.\frac{dx}{dy}, 0\right)$; y-intercept $\left(0, y - x\frac{dy}{dx}\right)$

$\Delta OPQ = 2$

$$\Rightarrow \quad \frac{1}{2}\left(x - y\frac{dx}{dy}\right)\left(y - x\frac{dy}{dx}\right) = 2$$

$$\Rightarrow \quad \left(x - y\frac{1}{p}\right)(y - xp) = 4, \text{ where } p = \frac{dy}{dx}$$

$$\Rightarrow \quad p^2x^2 - 2pxy + 4p + y^2 = 0 \Rightarrow (y-px)^2 + 4p = 0$$

$$\therefore \quad y - px = 2\sqrt{-p} \ \Rightarrow y = px + 2\sqrt{-p} \qquad \ldots(1)$$

Differentiating w.r.t. x, we get

$$p = p + \frac{dp}{dx}.x + 2.\left(\frac{1}{2}\right)(-p)^{-1/2}.(-1)\frac{dp}{dx}$$

$$\Rightarrow \quad \frac{dp}{dx}x - (p)^{-1/2}.\frac{dp}{dx} = 0 \Rightarrow \frac{dp}{dx}\{x - (-p)^{-1/2}\} = 0$$

CASE 1:

If $\frac{dp}{dx} = 0 \Rightarrow p = c$

putting this value in (1), we get $y = cx + 2\sqrt{-c}$

This curve passes through (1, 1)

$$\Rightarrow \quad 1 = c + 2\sqrt{-c} \Rightarrow c = -1$$

$$\therefore \quad y = -x + 2 \text{ or } x + y = 2$$

CASE 2:

if $x = (-p)^{-1/2} \Rightarrow -p = \frac{1}{x^2}$ putting in (1)

$$y = \frac{-x}{x^2} + 2\frac{1}{x} = \frac{1}{x} \Rightarrow xy = 1$$

Thus, the two curves are $xy = 1$ and $x + y = 2$.

PROBLEM 4

Let $u(x)$ and $v(x)$ satisfy the differential equations $\frac{du}{dx} + P(x)u = f(x)$ and $\frac{dv}{dx} + P(x)v = g(x)$, where $P(x)$, $f(x)$ and $g(x)$ are continuous functions. If $u(x_1) > v(x_1)$ for some x_1 and $f(x) > g(x)$ for all $x > x_1$. Prove that any point (x, y) where $x > x_1$ does not satisfy the equation $y = u(x)$ and $y = v(x)$

SOLUTION

Let $w(x) = u(x) - v(x)$ and $h(x) = f(x)\ (x) - g(x)$

Differentiating with respect to x $\frac{dw}{dx} = \frac{du}{dx} - \frac{dv}{dx}$.

$$\Rightarrow \quad \frac{dw}{dx} = \{f(x) - p(x)\ .\ u(x)\} - \{g(x) - p(x)\ v(x)\}$$

$$\Rightarrow \quad \frac{dw}{dx} = \{f(x) - g(x)\} - p(x)\ [u(x) - v(x)]$$

$$\Rightarrow \quad \frac{dw}{dx} = h(x) - p(x), w(x) \qquad \ldots(1)$$

$$\Rightarrow \quad \frac{dw}{dx} + p(x)\ w(x) = h(x)$$

which is linear differential equation.

The integrating factor is given by I.F. = $e^{\int p(x)dx} = r(x)$ (let)

Multiplying both sides of (1) of $r(x)$, we get

$$r(x).\frac{dw}{dx} + p(x)\ \{r(x)\}\ w(x) = r(x).h(x)$$

$$\Rightarrow \quad \frac{d}{dx}[r(x)\ w(x)] = r(x).h(x) \qquad \left[\text{since } \frac{dr}{dx} = p(x)\cdot r(x)\right]$$

Now $r(x) = e^{\int p(x)dx} > 0\ \forall\ x$ and $h(x) = f(x) - g(x) > 0$ for $x > x_1$

Thus, $\frac{d}{dx}[r(x)\ w(x)] > 0\ \forall\ x > x_1$

$r(x)\ w(x)$ increasing on the interval $[x, \infty)$

Therefore for all $x > x_1$, $r(x)\ w(x) > r(x_1)\ w(x_1) > 0$

$[\because\ r(x_1) > 0$ and $u(x_1) > v(x_1)]$

$$\Rightarrow \quad w(x) > 0\ \forall\ x > x_1 \Rightarrow u(x) > v(x)\ \forall\ x > x_1 \qquad [\because\ r(x) > 0]$$

Hence, there does not exist a point (x, y) such that $x > x_1$ and $y = u(x)$ and $y = v(x)$.

PROBLEM 5

Solve the differential equation $\cos^2 x\frac{dy}{dx} - (\tan 2x)\ y = \cos^4 x$, where $|x| < \frac{\pi}{4}$. When $y\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8}$

SOLUTION

Rewriting given equation $\frac{dy}{dx} - \frac{\tan 2x}{\cos^2 x}y = \cos^2 x$

Integrating Factor = $e^{-\int\frac{\tan 2x}{\cos^2 x}dx} = e^{-\int\frac{2\sin x\cos x}{(2\cos^2 x - 1)\cos^2 x}dx}$

Put $\cos x = t$; $-\sin x\ dx = dt$

$$\text{If} = e^{\int\frac{2}{(2t^2-1)t}dt} = e^{2\int\left(\frac{2t}{2t^2-1} - \frac{1}{t}\right)dt}$$

$$= e^{2\left(\frac{1}{2}\ln(2t^2-1) - \ln t\right)} = e^{\ln\frac{2t^2-1}{t^2}} = \frac{2t^2-1}{t^2}$$

$$= \frac{2\cos^2 x - 1}{\cos^2 x} = \frac{\cos 2x}{\cos^2 x}$$

$$\Rightarrow \quad y\frac{\cos 2x}{\cos^2 x} = \int \cos 2x\ dx$$

$$\Rightarrow \quad y\frac{\cos 2x}{\cos^2 x} = \frac{\sin 2x}{2} + C$$

$$y\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8} \Rightarrow c = 0$$

So required solution $y = \frac{1}{2}\tan 2x \cos^2 x$

PROBLEM 6

Determine the equation of the curve passing through the origin, in the form $y = f(x)$, which satisfies the differential equation. $\frac{dy}{dx} = \sin(10x + 6y)$

SOLUTION

$$\frac{dy}{dx} = \sin(10x + 6y)$$

Let $10x + 6y = t$ (given) ...(1)

$$\Rightarrow \quad 10 + 6\frac{dy}{dx} = \left(\frac{dt}{dx}\right) \Rightarrow \frac{dy}{dx} = \frac{1}{6}\left(\frac{dt}{dx} - 10\right)$$

Now, the given differential equation becomes

$$\frac{1}{6}\left(\frac{dt}{dx}-10\right)=\sin t \Rightarrow \frac{dt}{dx}=6\sin t+10$$

$$\Rightarrow \quad \int\frac{dt}{6\sin t+10}=\int dx \Rightarrow \frac{1}{2}\int\frac{dt}{3\sin t+5}=x+c \qquad \ldots(2)$$

Let $I_1=\int\frac{dt}{3\sin t+5}$ Put $\tan\frac{t}{2}=u \Rightarrow \frac{1}{2}\sec^2\frac{t}{2}\,dt=du$

$$I_1=\int\frac{dt}{3\left(\frac{2\tan\frac{t}{2}}{1+\tan^2\frac{t}{2}}\right)+5}=\int\frac{\left(1+\tan^2\frac{t}{2}\right)dt}{\left(6\tan\frac{t}{2}+5+5\tan^2\frac{t}{2}\right)}$$

$$I_1=\int\frac{2du}{5u^2+6u+5}=\frac{2}{5}\int\frac{du}{u^2+\frac{6}{5}u+1}$$

$$I_1=\frac{2}{5}\int\frac{du}{\left(u+\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2}$$

$$I_1=\frac{2}{5}\cdot\frac{5}{4}\tan^{-1}\left(\frac{u+\frac{3}{5}}{\frac{4}{5}}\right)=\frac{1}{2}\tan^{-1}\left[\frac{5\tan\frac{t}{2}+3}{4}\right]$$

Putting this in (2),

Now $\frac{1}{2}I_1=x+c. \Rightarrow \frac{1}{4}\tan^{-1}\left[\frac{5\tan\frac{t}{2}+3}{4}\right]=x+c$

$$\Rightarrow \quad \tan^{-1}\left[\frac{5\tan\frac{t}{2}+3}{4}\right]=4x+4c$$

$$\Rightarrow \quad 5\tan(5x+3y)+3=4\tan(4x+4c)$$

When $x=0$, $y=0$, we get $5\tan 0+3=4\tan(4c)$

$$\Rightarrow \quad \frac{3}{4}=\tan 4c \Rightarrow 4c=\tan^{-1}\frac{3}{4}$$

Then, $5\tan(5x+3y)+3=4\tan\left(4x+\tan^{-1}\frac{3}{4}\right)$

$$\Rightarrow \quad \tan(5x+3y)=\frac{4}{5}\tan\left(4x+\tan^{-1}\frac{3}{4}\right)-\frac{3}{5}$$

$$\Rightarrow \quad 5x+3y=\tan^{-1}\left[\frac{4}{5}\left\{\tan\left(4x+\tan^{-1}\frac{3}{4}\right)\right\}-\frac{3}{5}\right]$$

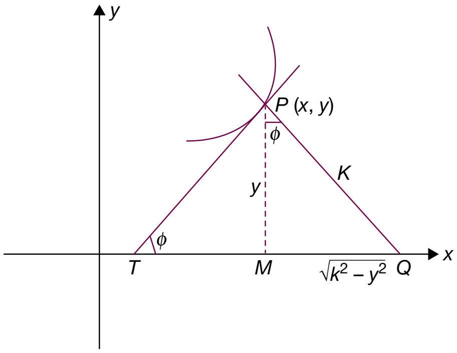
$$\Rightarrow \quad y=\frac{1}{3}\tan^{-1}\left[\frac{4}{5}\left\{\tan\left(4x+\tan^{-1}\frac{3}{4}\right)\right\}-\frac{3}{5}\right]-\frac{5x}{3}$$

PROBLEM 7

A normal is drawn at a point $P(x, y)$ of a curve. It meets x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is $y\frac{dy}{dx}=\pm\sqrt{k^2-y^2}$. Find equation of such a curve passing through $(0, k)$.

SOLUTION

$$\tan\phi=\frac{dy}{dx} \text{ (slope of tangent)}$$



Also $\tan\phi=\frac{\pm\sqrt{k^2-y^2}}{y}$ in Δ PQM

$$\therefore \quad \frac{dy}{dx}=\frac{\pm\sqrt{k^2-y^2}}{y}\Rightarrow y\frac{dy}{dx}=\pm\sqrt{k^2-y^2}$$

Alternatively $PQ^2=y^2+y^2\left(\frac{dy}{dx}\right)^2=k^2$ (given)

$$\therefore \quad y\frac{dy}{dx}=\pm\sqrt{k^2-y^2} \qquad \ldots(1)$$

$\Rightarrow \quad \frac{-2y\,dy}{\sqrt{k^2-y^2}}=\pm 2dx$. Integrating we have

$2\sqrt{k^2-y^2}=C\pm 2x$; If it passes through $(0, k)$

then $0=C\pm 0 \Rightarrow C=0$

Hence, $2\sqrt{k^2-y^2}=2x$ or $x^2+y^2=k^2$, which is equation of a circle.

PRACTICE EXERCISES

SINGLE CORRECT CHOICE TYPE QUESTIONS

This section contains Single Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONLY ONE is correct.

1. The order of differential equation corresponding to $y = c_1 \cos 2x + c_2 \cos^2 x + c_3 \sin^2 x + c_4$ is

(A) 2 (B) 4
(C) 3 (D) 1

2. The degree of the differential equation satisfying the relation

$\sqrt{1+x^2} + \sqrt{1+y^2} = \lambda\left(x\sqrt{1+y^2} - y\sqrt{1+x^2}\right)$ is

(A) 1 (B) 2
(C) 3 (D) none of these

3. If p and q are order and degree of differential equation

$y^2\left(\dfrac{d^2y}{dx^2}\right)^2 + 3x\left(\dfrac{dy}{dx}\right)^{\frac{1}{3}} + x^2y^2 = \sin x$, then

(A) $p > q$ (B) $\dfrac{p}{q} = \dfrac{1}{2}$
(C) $p = q$ (D) $p < q$

4. A curve, whose concavity is directly proportional to the logarithm of its x-coordinates at any point of the curve, is given by

(A) $C_1x^2(2\log x - 3) + C_2x + C_3$
(B) $C_1x^2(2\log x + 3) + C_2x + C_3$
(C) $C_1x^2(2\log x) + C_2$
(D) None of these

5. The differential equation of all parabolas with axis parallel to the y-axis is

(A) $\dfrac{d^2y}{dx^2} = 2y + x$ (B) $\dfrac{d^3y}{dx^3} = 2y$
(C) $\left(\dfrac{d^2y}{dx^2}\right)^3 = \dfrac{dy}{dx}$ (D) $\dfrac{d^3y}{dx^3} = 0$

6. The solution of the differential equation $(x^2\sin^3 y - y^2 \cos x)dx + (x^3 \cos y \sin^2 y - 2y \sin x)\,dy = 0$ is

(A) $x^3\sin^3 y = 3y^2\sin x + C$ (B) $x^3\sin^3 y + 3y^2\sin x = C$
(C) $x^2\sin^3 y + y^3\sin x = C$ (D) $2x^2\sin y + y^2\sin x = C$

7. Solution of differential equation $\dfrac{dx}{dy} - \dfrac{x\log x}{1+\log x} = \dfrac{e^y}{1+\log x}$, if $y(1) = 0$, is -

(A) $x^x = e^{ye^y}$ (B) $e^y = x^{e^y}$
(C) $x^x = ye^y$ (D) None of these

8. If a curve $y = f(x)$ satisfies $y''x^2 + x(y')^2 = 2xy'$, $f(0) = 0$ and $f'(1) = 1$, then $f(x)$ is

(A) $2(x - \tan^{-1}x)$ (B) $2(x - \ln(1 + x)^2)$
(C) $2(\ln(1 + x^2) - x)$ (D) None of these

9. Let $\dfrac{x\,dy}{dx} - y = x^2(xe^x + e^x - 1)$ for all $x \in R - \{0\}$ such that $y(1) = e - 1$. If $y(2) = ky(1)\,(y(1) + 2)$, then the value of k is

(A) 1 (B) 2
(C) 3 (D) 4

10. A continuous function $f : R \to R$ satisfy the differential equation $f(x) = (1 + x^2)\left(1 + \displaystyle\int_0^x \frac{f^2(t)}{1+t^2}dt\right)$ then the value of $f(-2)$ is

(A) 0 (B) $\dfrac{17}{15}$
(C) $\dfrac{-17}{15}$ (D) $\dfrac{15}{17}$

11. The solution of differential equation

$xdx + ydy + \dfrac{xdy - ydx}{x^2 + y^2} = 0$

(A) $y = \tan\left(\dfrac{c - x - y}{2}\right)$ (B) $y = x\tan\left(\dfrac{c - x^2 - y^2}{2}\right)$
(C) $y = x\tan\left(\dfrac{c + x^2 + y^2}{2}\right)$ (D) $y = x\tan\left(\dfrac{c + x^2 - y^2}{2}\right)$

12. Solution of the differential equation $\{y(1 + x^{-1}) + \sin y\}\,dx + (x + \ln x + x\cos y)\,dy = 0$, is

(A) $y + x\ln x + x\sin y = C$
(B) $xy + y\ln x + x\sin y = C$
(C) $xy + x\ln x + y\sin y = C$
(D) $x + y\ln x + y\sin x = C$

13. The equation $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 + 1}{2xy}$, $y(1) = 1$ is the differentiable equation of

(A) a system of parabolas (B) a system of circles
(C) a hyperbola (D) a pair of lines

14. The solution of the differential equation $\dfrac{dy}{dx} = \dfrac{\sin y + x}{\sin 2y - x\cos y}$ is

(A) $\sin^2 y = x\sin y + \dfrac{x^2}{2} + C$ (B) $\sin^2 y = x\sin y - \dfrac{x^2}{2} + C$
(C) $\sin^2 y = x + \sin y + \dfrac{x^2}{2} + C$ (D) $\sin^2 y = x - \sin y + \dfrac{x^2}{2} + C$

15. The primitive of the differential equation $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$ is

(A) $x^2e^y + \dfrac{x^2}{y} - \dfrac{x}{y^3} = K$ (B) $x^2e^y - \dfrac{x^2}{y} + \dfrac{x}{y^3} = K$
(C) $x^2e^y + \dfrac{x^2}{y} + \dfrac{x}{y^3} = K$ (D) $x^2e^y - \dfrac{x^2}{y} - \dfrac{x}{y^3} = K$

16. The differential equation $e^y\,dx + (e^y\,.\,x + 2y)\,dy = 0$ has the particular solution $y(0) = 1$. The value of x, when $y = 0$, is

(A) 1 (B) 0

(C) 2 (D) –1

17. A function $y = f(x)$ satisfies the condition $f'(x)\sin x + f(x)\cos x = 1$, $f(x)$ being bounded when $x \to 0$. If $I = \int\limits_0^{\pi/2} f(x)dx$, then

(A) $\frac{\pi}{2} < I < \frac{\pi^2}{4}$ (B) $\frac{\pi}{4} < I < \frac{\pi^2}{2}$

(C) $1 < I < \frac{\pi}{2}$ (D) $0 < I < 1$

18. Tangent drawn at any point P on a curve meets x-axis at Q such that circumcentre of ΔPOQ has abscissa half that of ordinate. The differential equation to such a curve is

(A) $\frac{dy}{dx} = \frac{x+2y}{2x-y}$ (B) $\frac{dy}{dx} = \frac{2y-x}{2x+y}$

(C) $\frac{dy}{dx} = \frac{2y+y}{2x-y}$ (D) None of these

19. The solution of differential equation

$x\sec\left(\frac{y}{x}\right)(ydx + xdy) = y\operatorname{cosec}\left(\frac{y}{x}\right)(xdy - ydx)$ is.

(A) $xy = c\operatorname{cosec}\left(\frac{y}{x}\right)$ (B) $xy^2\sin\frac{y}{x} = c$

(C) $xy\operatorname{cosec}\frac{y}{x} = c$ (D) $xy = c\sin\left(\frac{x}{y}\right)$

20. If the independent variable x is changed to y, then the differential equation $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \frac{dy}{dx} = 0$ is transformed to $x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 = k$, k being a number. Then k equals

(A) 0 (B) 1

(C) –1 (D) 2

21. If $f''(x) = -f(x)$ then $\frac{[f(2)]^2 + [f'(2)]^2}{[f(4)]^2 + [f'(4)]^2} =$

(A) 2 (B) $\frac{1}{2}$

(C) 1 (D) Data is not enough to find

22. Suppose that the quadratic function $f(x) = ax^2 + bx + c$ is non-negative on the interval [–1, 1]. Then the area under the graph of f over the interval [–1, 1] and the x-axis is given by the formula

(A) $A = f(-1) + f(1)$ (B) $A = f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right)$

(C) $A = \frac{1}{2}[f(-1) + 2f(0) + f(1)]$ (D) $A = \frac{1}{3}[f(-1) + 4f(0) + f(1))]$

23. The solution of $x^2dy - y^2dx + xy^2(x - y)dy = 0$ is

(A) $y = x + cxye^{y^2/2}$ (B) $y = x + cxye^{x^2/2}$

(C) $y = x + cxye^{-y^2/2}$ (D) $y = x + cxye^{-x^2/2}$

24. The solution of the differential equation $y^2dx + \left(x^2 - xy + y^2\right)dy = 0$ is

(A) $\tan^{-1}\left(\frac{x}{y}\right) + \ln y + C = 0$

(B) $2\tan^{-1}\left(\frac{x}{y}\right) + \ln x + C = 0$

(C) $\ln\left(y + \sqrt{x^2 + y^2}\right) + \ln y + C = 0$

(D) $\ln\left(x + \sqrt{x^2 + y^2}\right) + C = 0$

(Where C is arbitrary constant)

25. Let the curve $y = f(x)$ passes through origin and satisfies the differential equation $\frac{dy}{dx} + \int\limits_0^5 ydx = 27$. If a and b are chosen randomly from the set $S = \{1, 2, 3, 4\}$ with replacement. The probability that the above curve passes through (a, b) is

(A) $\frac{1}{2}$ (B) $\frac{1}{6}$

(C) $\frac{1}{8}$ (D) $\frac{1}{12}$

26. Tangent to a curve intersects the y-axis at a point P. A line perpendicular to this tangent through P passes through another point (1, 0). The differential equation of the curve is -

(A) $y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 1$ (B) $\frac{xd^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$

(C) $y\frac{dx}{dy} + x = 1$ (D) None of these

27. If solution of equation $\frac{dy}{dx} = y + \int\limits_0^2 ydx$ is $y(x)$ and $y(0) = 1$. Then $[\,|y(2)|\,]$ is equal to (where [.] denotes greatest integer function)

(A) 2 (B) 3

(C) 4 (D) 5

28. The solutions of the differential equations $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$ is given by

(A) $2\log|x| - \log|y| - \frac{1}{xy} = c$ (B) $2\log|y| - \log|x| - \frac{1}{xy} = c$

(C) $2\log|x| + \log|y| + \frac{1}{xy} = c$ (D) $2\log|y| + \log|x| + \frac{1}{xy} = c$

29. Let $y = y(x)$ and $xdy + y(1 - xy)dx = 0$. If $y(1) = 1$ then the value of $[y(1/e)]$ is ________ (where [.] is greatest integer function)

(A) 0 (B) –1

(C) 1 (D) –2

30. If $y = y(x)$ and $\left(\frac{2 + \sin x}{y + 1}\right)\frac{dy}{dx} = -\cos x$, $y(0) = 0$, then $y\left(\frac{5\pi}{6}\right)$ eq

(A) $\frac{1}{5}$ (B) $-\frac{1}{5}$

(C) $\frac{3}{5}$ (D) $\frac{2}{5}$

31. Let $y'(x) + \frac{g'(x)}{g(x)} \cdot y(x) = \frac{g'(x)}{1+g^2(x)}$ where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function an R, If $g(1) = y(1) = 1$ and $g(e) = \sqrt{(2e-1)}$ then $y(e)$ equals

(A) $\frac{3}{2g(e)}$ (B) $\frac{1}{2g(e)}$

(C) $\frac{1}{g(e)}$ (D) $\frac{2}{3g(e)}$

32. If the substitution $x = \tan^{-1}(t)$ transforms the differential equation $\frac{d^2y}{dx^2} + xy\frac{dy}{dx} + \sec^2 x = 0$ into another differential equation as $(1 + t^2)\frac{d^2y}{dx^2} + (2t + y\tan^{-1}(t))\frac{dy}{dt} = k$ then the value of k is equal to

(A) 1 (B) –1

(C) 2 (D) –2

33. If a function $y = f(x)$ satisfies the equation $x = \frac{dy}{dx} - \frac{1}{2}\left(\frac{dy}{dx}\right)^2 + \frac{1}{3}\left(\frac{dy}{dx}\right)^3 - \ldots. \infty$ and $y(0') = 1$, $x \le \ln 2$ then the value of $f'\left(\ln\frac{1}{2}\right) + f\left(\ln\frac{1}{2}\right)$ is equal to

(A) – ln2 (B) ln2

(C) 1 + ln2 (D) 1 – ln2

34. Let $y = y(x)$ satisfy the equation $y(x) + \int_1^x y(t)dt = x^2$. The value of $y(e)$ is equal to

(A) $2e - 2 + e^{1-e}$ (B) $2e - 1 + e^{1-e}$

(C) $2e + 2 + e^{1-e}$ (D) $2e + 1 + e^{1-e}$

35. If $y_1(x)$ is a solution of the differential equation $\frac{dy}{dx} - f(x)y = 0$, then a solution of the differential equation $\frac{dy}{dx} + f(x)y = r(x)$

(A) $\frac{1}{y_1(x)}\int r(x)y_1(x)\,dx$ (B) $y_1(x)\int \frac{r(x)}{y_1(x)}\,dx$

(C) $\int r(x)y_1(x)\,dx$ (D) none of these

36. Suppose a solutions of the differential equation $(xy^3 + x^2y^7)\frac{dy}{dx} = 1$ satisfies the initial conditions $y\left(\frac{1}{4}\right) = 1$. Then the value of $\frac{dy}{dx}$ when $y = -1$ is _____

(A) $-\frac{3}{20}$ (B) $-\frac{20}{3}$

(C) $-\frac{5}{16}$ (D) $-\frac{16}{5}$

37. The equation of the curve for which the square of the ordinate is twice the rectangle contained by the abscissa and the intercept of the normal on x-axis and passing through (2, 1) is

(A) $x^2 + y^2 - x = 0$ (B) $4x^2 + 2y^2 - 9y = 0$

(C) $2x^2 + 4y^2 - 9x = 0$ (D) $4x^2 + 2y^2 - 9x = 0$

38. The largest value of c such that there exists a differential function $h(x)$ for $-c < x < c$ that is a solution of $y' = 1 + y^2$ with $h(0) = 0$ is

(A) 2π (B) π

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

39. Solution of the equation $\frac{dy}{dx} = \frac{y^2 - y - 2}{x^2 + 2x - 3}$ is __________ (where c is arbitrary constant)

(A) $\frac{1}{3}\log\left(\frac{y-2}{y+1}\right) = \frac{1}{4}\log\left(\frac{x+3}{x-1}\right) + c$

(B) $\frac{1}{3}\log\left(\frac{y+1}{y-2}\right) = \frac{1}{4}\log\left(\frac{x-1}{x+3}\right) + c$

(C) $4\log\left(\frac{y-2}{y+1}\right) = 3\log\left(\frac{x-1}{x+3}\right) + c$

(D) None of these

40. The solution of the differential equation $\frac{dy}{dx} = \frac{x - 2y + 3}{2x - y + 5}$ is $f(x, y, c) = 0$. For suitable values of $c, f(x, y, c) = 0$ represents a pair of lines whose point of intersection is

(A) $\left(-\frac{7}{3}, \frac{2}{3}\right)$ (B) $\left(-\frac{7}{3}, \frac{4}{3}\right)$

(C) $\left(-\frac{7}{3}, \frac{1}{3}\right)$ (D) $\left(\frac{7}{3}, -\frac{1}{3}\right)$

41. If the differential equation of a curve, passing through $\left(0, -\frac{\pi}{4}\right)$ and $(t, 0)$ is $\cos y\left(\frac{dy}{dx} + e^{-x}\right) + \sin y\left(e^{-x} - \frac{dy}{dx}\right) = e^{e^{-x}}$, then the value of $t \cdot e^{e^{-t}}$ is

(A) –1 (B) 1

(C) 2 (D) –2

42. Solution of $\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}$ is

(A) $e^{-2x}y^2 + 2\ln|y| = c$ (B) $e^{2x}y^2 - 2\ln|y| = c$

(C) $e^x + \ln|y| = c$ (D) None of these

43. A normal at any point (x, y) to the curve $y = f(x)$ cuts triangle of unit area with the axes, the equation of the curve is

(A) $y^2 - x^2\left(\frac{dy}{dx}\right)^2 = 4\frac{dy}{dx}$

(B) $x^2 - y^2\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}$

(C) $x + y\frac{dy}{dx} = y$

(D) $x^2 + 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 = 2\frac{dy}{dx}$

44. Solution of differential equation $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2$ is

(A) $\frac{1}{x\log y}=\frac{1}{x^2}+c$ (B) $\frac{1}{x\log y}=\frac{1}{2x^2}+c$

(C) $\frac{1}{x\log y}=\frac{1}{3x^2}+c$ (D) $\frac{1}{\log y}=\frac{1}{x^2}+c$

45. Solution of the differential equation $\frac{dy}{dx}+\frac{1}{x}\tan y=\frac{1}{x^2}\tan y \sin y$ is

(A) $2x = \sin y(1 + 2x)$ (B) $2x = \sin y(1 + cx^2)$

(C) $2x + \sin y\,(1 + cx^2) = 0$ (D) None of these

46. Solution of $ydx - xdy + (1 + x^2)dx + x^2 \sin ydy = 0$

(A) $\frac{y}{x}+\frac{1}{x}-x+\cos y+c=0$

(B) $\frac{y}{x}+\frac{1}{x^2}-x+\cos y+c=0$

(C) $\frac{2y}{x}-\frac{1}{x^2}-x+\cos y+c=0$

(D) None of these

47. The curve whose equation satisfies $x\frac{dy}{dx}-4y=x^2\sqrt{y}$ passes through $(1, (\ln 2)^2)$, then the value of $y(2)$ is

(A) $6 \ln 2$ (B) $(2 \ln 2)^2$

(C) $(6 \ln 2)^2$ (D) None of these

48. A function $y = f(x)$ satisfies the differential equation $\frac{dy}{dx}$ when $x \to \infty$. The area enclosed by $y = f(x)$, $y = \cos x$ and the y-axis in first quadrant is

(A) $\sqrt{2}-1$ (B) $\sqrt{2}$

(C) 1 (D) $\frac{1}{\sqrt{2}}$

49. The differential equation of the family of curves $c(y+c)^2 = x^3$ (where 'c' is an arbitrary constant) is

(A) $8x^2\left(\frac{dy}{dx}\right)^3-12x^2y\left(\frac{dy}{dx}\right)^2-27=0$

(B) $8x\left(\frac{dy}{dx}\right)^3-12y\left(\frac{dy}{dx}\right)^2-27x=0$

(C) $8x\left(\frac{dy}{dx}\right)+12y\left(\frac{dy}{dx}\right)^2+27x=0$

(D) None of these

50. The solution of $ydx - xdy + 3x^2y^2e^{x^3}\,dx = 0$ is

(A) $\frac{y}{x}+e^{x^3}=c$ (B) $\frac{y}{x}-e^{x^3}=c$

(C) $\frac{x}{y}+e^{x^3}=c$ (D) $\frac{x}{y}-e^{x^3}=c$

51. The equation of the trajectories which is orthogonal to the family of curves $\cos y = ae^{-x}$ is

(A) $\sin y = c\,e^x$ (B) $\cos y = c\,e^x$

(C) $\sin y = ce^{-x}$ (D) None of these

52. Let $f(x)$ be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$ and $\lim\limits_{t\to x}\frac{t^2f(x)-x^2f(t)}{t-x}=1$ for each $x > 0$. Then $f(x)$ is

(A) $\frac{1}{3x}+\frac{2x^2}{3}$ (B) $\frac{-1}{3x}+\frac{4x^2}{3}$

(C) $\frac{-1}{x}+\frac{2}{x^2}$ (D) $\frac{1}{x}$

53. A curve in the first quadrant is such that the area of the triangle formed in the first quadrant by the x-axis, a tangent to the curve at any of its point P and radius vector of the point P is 2 sq. units. If the curve passes through (2, 1) then the equation of the curve can be

(A) $x^2 - y^2 = 3$ (B) $xy = 2$

(C) $x^2 = 4y$ (D) $8y = x^3$

54. Let $f(x) = e^x + \sin x + x \;\forall \in [-\pi, 0)$ and $f(0) = 0$. If $f(x)$ satisfies $f(x + y) = f(x) - f(y)\; \forall\, x, y \in R$, then the number of solution of the equation $f(x) = |x|$ in $[-\pi, \pi]$ is

(A) 0 (B) 1

(C) 2 (D) 3

55. The equation of the curve passing through (1, 2), if the intercept of normal at any point of the curve on x-axis is three times the abscissa of point, is

(A) $3x^2 - y^2 + 1 = 0$ (B) $x^2 - 2y^2 + 7 = 0$

(C) $2x^2 - y^2 + 2 = 0$ (D) $x^2 - y^2 + 3 = 0$

56. Number of values of $m \in N$ for which $y = e^{mx}$ is a solution of the differential equation $\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}-4\frac{dy}{dx}+12y=0$, is

(A) 3 (B) 1

(C) 2 (D) zero

57. A function $y = f(x)$ satisfies the differential equation $f(x) \sin 2x - \cos x + (1 + \sin^2 x) f'(x) = 0$ with initial condition $y(0) = 0$. The value of $f\left(\frac{\pi}{6}\right)$ is equal to

(A) $\frac{1}{5}$ (B) $\frac{3}{5}$

(C) $\frac{4}{5}$ (D) $\frac{2}{5}$

58. The solution of the differential equation, $e^x\,(x + 1)dx + (ye^y - xe^x)\,dy = 0$ with initial condition $f(0) = 0$, is

(A) $xe^x + 2y^2e^y = 0$ (B) $2xe^x + y^2e^y = 0$

(C) $xe^x - 2y^2e^y = 0$ (D) $2xe^x - y^2e^y = 0$

59. Water is drained form a vertical cylindrical tank by opening a value at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y, where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k=\frac{1}{15}$ then the time to drain the tank if the water is 4 meter deep to start with is

(A) 30 min (B) 45 min

(C) 60 min (D) 80 min

60. Let a solution $y = y(x)$ of the differential equation, $dy + xy\,dx = x\,dx$, satisfy $y(0) = 2$, then the area enclosed by the curve $y = y(x)$, $x = 0$ and $y = 1$ is

(A) $\int_0^{\infty} e^{\frac{-x^2}{2}}\,dx$ (B) $2\int_0^{\infty} e^{\frac{-x^2}{2}}\,dx$

(C) $\int_0^{\infty}\left(1+e\frac{-x^2}{2}\right)dx$ (D) $2\int_0^{\infty}\left(1+e\frac{-x^2}{2}\right)dx$

61. The solution of differential $e^{-x}(y + 1)dy + (\cos^2 x - \sin 2x)y\,dx = 0$ subject to the condition $y = 1$ when $x = 0$, is

(A) $(y + 1) + e^x \cdot \cos^2 x = 3$

(B) $\ln(y + 1) + e^x \cdot \cos^2 x = 1 + \ln 2$

(C) $y + \ln y = e^x \cdot \cos^2 x$

(D) $y + \ln y + e^x \cdot \cos^2 x = 2$

62. Let f be a twice differentiable function satisfying $\frac{e^x f''(x) - e^x f'(x)}{e^{2x}} = 1$ and $f(0) = 0$, $f'(0) = 1$, then the value of $\int_0^1 \frac{f(x)}{x}dx$ is

(A) e (B) $e - 1$

(C) 1 (D) 0

63. The general solution of differential equation $x^4\frac{dy}{dx} + x^3 y + \operatorname{cosec}(xy) = 0$, is

(A) $2\cos(xy) + \frac{1}{x^2} = C$ (B) $2\cos(xy) + y^{-2} = C$

(C) $2\sin(y) + x^{-2} = C$ (D) $2\sin(xy) + y^{-2} = C$

64. If the differential equation of the family of curve given by $y = Ax + Be^{2x}$ where A and B are arbitrary constant is of the form $(1 - 2x)\frac{d}{dx}\left(\frac{dy}{dx} + \ell y\right) + k\left(\frac{dy}{dx} + \ell y\right) = 0$ then the ordered pair (k, ℓ) is

(A) $(2, -2)$ (B) $(-2, 2)$

(C) $(2, 2)$ (D) $(-2, -2)$

65. Let $y = f(x) \cdot g(x) \cdot h(x)$ be the solution of the differential equation $x\,dy - y\,dx = xe^x(x \ln x + 1)dx$ and $y(1) = 0$.

If $f(x)$ and $h(x)$ are images of each other in the mirror $y = g(x)$ then least distance between $f(x)$ and $h(x)$ is

(A) $\sqrt{2}$ (B) $2\sqrt{2}$

(C) $3\sqrt{2}$ (D) 2

66. The curves satisfying the differential equation $(1 - x^2)y' + xy = ax$ are

(A) ellipses and hyperbolas (B) ellipses and parabola

(C) ellipses and straight lines (D) circles and ellipses

67. Solution of the differential equation: $\frac{dy}{dx} = \frac{3x^2y^4 + 2xy}{x^2 - 2x^3y^3}$ is

(A) $x^2y^2 + \frac{x^2}{y} = c$ (B) $x^3y^2 + \frac{x^2}{y} = c$

(C) $x^3y^2 + \frac{y^2}{x} = c$ (D) $x^2y^3 + \frac{x^2}{y} = c$

68. Solution to the differential equation $\frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots} = \frac{dx - dy}{dx + dy}$ is

(A) $2y\,e^{2x} = C.e^{2x} + 1$ (B) $2y\,e^{2x} = C.e^{2x} - 1$

(C) $y\,e^{2x} = C.e^{2x} + 2$ (D) $2x\,e^{2y} = C.e^{x} - 1$

69. The curve for which the x-intercept of the tangent drawn at any point P on the curve is three times the x-coordinate of the point P, is

(A) $xy = c$ (B) $xy^2 = c$

(C) $xy^3 = c$ (D) $x^2y = c$

70. A curve is such that the mid point of the portion of the tangent intercepted between the point where the tangent is drawn and the point where the tangent meets y-axis, lies on the line $y = x$. If the curve passes through $(1, 0)$, then the curve is

(A) $2y = x^2 - x$ (B) $y = x^2 - x$

(C) $y = x - x^2$ (D) $y = 2(x - x^2)$

71. The solution of the differential equation $\frac{\sqrt{x}dx + \sqrt{y}dy}{\sqrt{x}dx - \sqrt{y}dy} = \sqrt{\frac{y^3}{x^3}}$ is given by

(A) $\frac{3}{2}\log\left(\frac{y}{x}\right) + \log\left|\frac{x^{3/2} + y^{3/2}}{x^{3/2}}\right| + \tan^{-1}\left(\frac{y}{x}\right)^{3/2} + c = 0$

(B) $\frac{2}{3}\log\left(\frac{y}{x}\right) + \log\left|\frac{x^{3/2} + y^{3/2}}{x^{3/2}}\right| + \tan^{-1}\frac{y}{x} + c = 0$

(C) $\frac{2}{3}\left(\frac{y}{x}\right) + \log\left(\frac{x + y}{x}\right) + \tan^{-1}\left(\frac{y^{3/2}}{x^{3/2}}\right) + c = 0$

(D) None of these

72. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x - 1)$. If its graph passes through the point $(2, 1)$ and at that point the tangent to the graph is $y = 3x - 5$, then the function is

(A) $(x - 1)^2$ (B) $(x - 1)^3$

(C) $(x + 1)^2$ (D) $(x + 1)^3$

73. A curve passes through the point $\left(1, \frac{\pi}{4}\right)$ and its slope at any point is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$. Then the curve has the equation

(A) $y = x\tan^{-1}\left(\ln\frac{e}{x}\right)$ (B) $y = x\tan^{-1}(\ln + 2)$

(C) $y = \frac{1}{x}\tan^{-1}\left(\ln\frac{e}{x}\right)$ (D) None of these

74. The x-intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point $(1, 1)$ is

(A) $ye^{x/y} = e$ (B) $xe^{x/y} = e$

(C) $xe^{y/x} = e$ (D) $ye^{y/x} = e$

75. A function $f(x)$ satisfying $\int_0^1 f(tx)\,dt = n\,f(x)$, where $x > 0$, is -

(A) $f(x) = c.\ x^{\frac{1-n}{n}}$ (B) $f(x) = c\ .\ x^{\frac{n}{n-1}}$

(C) $f(x) = c\ .\ x^{\frac{1}{n}}$ (D) $f(x) = c\ .\ x^{(1-n)}$

76. A curve $y = f(x)$ passing through the point $\left(1, \frac{1}{\sqrt{e}}\right)$ satisfies the differential equation $\frac{dy}{dx} + xe^{-x^2/2} = 0$. Then which of the following does not hold good?

(A) $f(x)$ is differentiable at $x = 0$.

(B) $f(x)$ is symmetric w.r.t the origin.

(C) $f(x)$ is increasing for $x < 0$ and decreasing for $x > 0$.

(D) $f(x)$ has two inflection points.

77. Number of straight lines which satisfy the differential equation $\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 - y = 0$ is

(A) 1 (B) 2

(C) 3 (D) 4

78. The solution of the differential equation,

$x^2 \frac{dy}{dx} \cos\frac{1}{x} - y \sin\frac{1}{x} = -1$, where $y \to -1$ as $x \to \infty$ is

(A) $y = \sin\frac{1}{x} - \cos\frac{1}{x}$ (B) $y = \frac{x+1}{x\sin\frac{1}{x}}$

(C) $y = \cos\frac{1}{x} + \sin\frac{1}{x}$ (D) $y = \frac{x+1}{x\cos\frac{1}{x}}$

79. The equation of a curve passing through (1, 0) for which the product of the abscissa of a point P and the intercept made by a normal at P on the x-axis equals twice the square of the radius vector of the point P, is

(A) $x^2 + y^2 = x^4$ (B) $x^2 + y^2 = 2x^4$

(C) $x^2 - y^2 = 4x^4$ (D) $x^2 - y^2 = x^4$

80. The latus rectum of the conic passing through the origin and having the property that normal at each point (x, y) intersects the x-axis at $((x + 1), 0)$ is

(A) 1 (B) 2

(C) 4 (D) None of these

81. The order and the degree of the differential equation whose general solution is, $y = c(x - c)^2$, are respectively

(A) 1, 1 (B) 1, 2

(C) 1, 3 (D) 2, 1

82. In a chemical reaction a substance changes into another such that the rate of decomposition of a chemical substance x present at instant t is proportional to x itself i.e. amount of unchanged substance still present. If half of the substance present initially has been converted at the end of 1 minute then the time t in minutes at the end of which 99% of the substance will have changed lies in the interval

(A) 5 and 6 (B) 6 and 7

(C) 7 and 8 (D) more than 10

83. The equation to the orthogonal trajectories of the system of parabolas $y = ax^2$ is

(A) $\frac{x^2}{2} + y^2 = c$ (B) $x^2 + \frac{y^2}{2} = c$

(C) $\frac{x^2}{2} - y^2 = c$ (D) $x^2 - \frac{y^2}{2} = c$

84. The real value of m for which the substitution, $y = u^m$ will transform the differential equation, $2x^4 y \frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation is

(A) $m = 0$ (B) $m = 1$

(C) $m = 3/2$ (D) no value of m

85. A function $y = f(x)$ satisfies

$(x + 1), f'(x) - 2\,(x^2 + x)\, f(x) = \frac{e^{x^2}}{(x+1)}, \forall x > -1$. If $f(0) = 5$, then $f(x)$ is

(A) $\left(\frac{3x+5}{x+1}\right).e^{x^2}$ (B) $\left(\frac{6x+5}{x+1}\right).e^{x^2}$

(C) $\left(\frac{6x+5}{(x+1)^2}\right).e^{x^2}$ (D) $\left(\frac{5-6x}{x+1}\right).e^{x^2}$

86. If the differentiable equation $\frac{dy}{dx} - y = y^2\,(\sin x + \cos x)$ with $y\,(0) = 1$ then $y(\pi)$ has the value equal to

(A) $e\pi$ (B) $-e^{\pi}$

(C) $e^{-\pi}$ (D) $-e^{-\pi}$

87. A function $y = f(x)$ satisfies the different equation $\frac{dy}{dx} - y = \cos x - \sin x$, with initial condition that y is bounded when $x \to \infty$. The area enclosed by $y = f(x)$, $y = \cos x$ and the y-axis in the 1st quadrant

(A) $\sqrt{2} - 1$ (B) $\sqrt{2}$

(C) 1 (D) $\frac{1}{\sqrt{2}}$

88. A curve passing through (2, 3) and satisfying the differential equation $\int_0^x t\,y(t)\,dt = x^2 y\,(x)$, $(x > 0)$ is

(A) $x^2 + y^2 = 13$ (B) $y^2 = \frac{9}{2}x$

(C) $\frac{x^2}{8} + \frac{y^2}{18} = 1$ (D) $xy = 6$

89. The solution of $(y\,dx - x\,dy)\cot\left(\frac{x}{y}\right) = ny^2\,dx$ is

(A) $\cos\left(\frac{x}{y}\right) = ce^{nx}$ (B) $\sec\left(\frac{x}{y}\right) = ce^{nx}$

(C) $\sin\left(\frac{x}{y}\right) = ce^{nx}$ (D) None of these.

90. If the population of a country doubles in 50 years in how many years will it become thrice the original, assume the rate of increase is proportional to the number of inhabitants

(A) 75 (B) $50 \log_2 3$

(C) $50 \log_3 2$ (D) 100

91. The solution of $\frac{dy}{dx}\sqrt{1+x+y} = x + y - 1$ is

(A) $2\left[\sqrt{1+x+y}+\frac{1}{3}\log\left|\sqrt{1+x+y}-1\right|-\frac{4}{3}\log\left|\sqrt{1+x+y}+2\right|\right]=x+c$

(B) $2\left[\sqrt{1+x+y}+\frac{1}{3}\log\left|\sqrt{1+x+y}\right|-\frac{4}{3}\log\left|\sqrt{1+x+y}+2\right|\right]=x+c$

(C) $2\left[\sqrt{1+x+y}+\frac{1}{3}\log\left|\sqrt{1+x+y}\right|-\frac{4}{3}\log\left|\sqrt{1+x+y}\right|\right]=x+c$

(D) $\left[\sqrt{1+x+y}+\frac{1}{3}\log\left|\sqrt{1+x+y}\right|-\frac{4}{3}\log\left|\sqrt{1+x+y}\right|\right]=x+c$

92. The solution of $(1 + x)\frac{dy}{dx} + 1 = e^{x-y}$ is

(A) $e^y(x + 1) = c$ (B) $e^y(x + 1) = e^x + c$
(C) $e^y(x + 1) = ce^x$ (D) $(x + 1) = e^x + c$

93. If for a curve ratio of the distance between the normal at any of its points and the origin to the distance between the same normal and the point (a, b) is equal to the constant k ($k > 0$, $k \neq 1$), then the curve is a

(A) circle (B) parabola
(C) ellipse (D) hyperbola

94. S_1: The differential equation of parabolas having their vertices at the origin and foci on the x-axis is an equation whose variables are separable

S_2: The differential equation of the straight lines which are at a fixed distance p from the origin is an equation of degree 2

S_3: The differential equation of all conics whose both axes coincide with the axes of coordinates is an equation of order 2

(A) TTT (B) TFT
(C) FFT (D) TTF

95. Solution of $\left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx}=\left(\frac{x+y+1}{x+y+2}\right)$, given that $y = 1$ when $x = 1$ is

(A) $\ln\left|\frac{(x-y)^2-2}{2}\right|=2(x+y)$

(B) $\ln\left|\frac{(x+y)^2-2}{2}\right|=2(x-y)$

(C) $\ln\left|\frac{(x-y)^2+2}{2}\right|=2(x+y)$

(D) $\ln\left|\frac{(x+y)^2+2}{2}\right|=2(x-y)$

96. Solution of the differential equation $\left\{\frac{1}{x}-\frac{y^2}{(x-y)^2}\right\}dx+\left\{\frac{x^2}{(x-y)^2}-\frac{1}{y}\right\}dy=0$ is

(A) $\ln\left|\frac{x}{y}\right|+\frac{xy}{(x-y)}=c$ (B) $\ln\left|xy\right|+\frac{xy}{(x-y)}=c$

(C) $\frac{xy}{(x-y)}=ce^{x/y}$ (D) $\frac{xy}{(x-y)}=ce^{xy}$

97. A conic C satisfies the differential equation $(1 + y^2)dx - xydy = 0$ and passes through the point (1, 0). An ellipse E which is confocal with C has its eccentricity $\sqrt{2/3}$. The angle of intersection of the curves C and E is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

98. The solution of differential equation $(x^5 + x + 2x^2y)\,dy + (3x^4y - y)\,dx = 0$ is

(A) $x^4y + xy^2 + y = cx$ (B) $x^4y^2 + xy + y = cx$
(C) $x^4y + x^2y^2 + xy = c$ (D) $x^4y - xy^2 - y = cx$

99. The equation to the curve which is such that portion of the axis of x cutoff between the origin and the tangent at any point is proportional to the ordinate of that point is (constant of proportionality is K)

(A) $x = y(C - K\log y)$ (B) $\log x = Ky^2 + C$
(C) $x^2 = y(C - K\log y)$ (D) None of these

100. Let a function $f(x)$ be such that $f''(x) = f'(x) + e^x$ and $f(0) = 0$, $f'(0) = 1$, then $\ln\left(\frac{(f(2))^2}{4}\right)$ equal to

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

101. The solution of $\frac{dy}{dx} + x(x + y) = x^3(x + y)^3 - 1$, $y(0) = 1$; is given by $2(1 + x^2) - \frac{1}{(x+y)^2} = f(x)$ where $f(x) =$

(A) e^{-x} (B) e^{-x^2}

(C) e^x (D) e^{x^2}

102. The general solution of $x\frac{dy}{dx} + (\log x)\,y = x^{1-\frac{1}{2}\log x}$ is

(A) $y = x^{1-\frac{1}{2}\log x} + cx^{-\frac{1}{2}\log x}$

(B) $y.x^{\frac{1}{2}\log x} = x^{\frac{1}{2}\log x} + c$

(C) $y = e^{\frac{(\log x)^2}{2}}(x - c)$

(D) $y = e^{\frac{1}{2}(\log x)^2}\left(x^{1-\frac{1}{2}(\log x)} - x^{-\frac{1}{2}(\log x)}\right) + c$

103. Consider the differential equation $x^3dy + x^2ydx + y((xy)^2 + 1)\,dx - x\,((xy)^2 + 1)\,dy = 0$. If the curve passes through (1, 1), then curve is

(A) $\sin^{-1}(xy)+y-x=\frac{\pi}{2}$ (B) $\tan^{-1}(x)+\tan^{-1}(xy)=\frac{\pi}{2}$

(C) $\tan^{-1}(xy)+\frac{y}{x}=\left(\frac{\pi}{4}+1\right)$ (D) $\tan^{-1}(xy)-\frac{y}{x}=\left(\frac{\pi}{4}-1\right)$

104. The solution of

$$\frac{y+\sin x\cos^2(xy)}{\cos^2(xy)}dx+\frac{x}{\cos^2(xy)}dy+\sin y\,dy=0,\text{ is}$$

(where C is constant of integration)

(A) $\sin(xy)-\cos x-\cos y=C$
(B) $\tan xy+\cos x+\cos y=C$
(C) $\cos xy-\sin x-\sin y=C$
(D) $\tan xy-\cos x-\cos y=C$

105. If the curve represented by $x^2\frac{dy}{dx}+xy+\sqrt{1-x^2y^2}=0$ passes through (1, 0), then

(A) $y=-\frac{\sin(\ln x)}{x}$ (B) $y=\frac{\sin(\ln x)}{x}$

(C) $y=\frac{\tan(\ln x)}{x}$ (D) $y=\frac{\sin^{-1}(\ln x)}{x}$

MULTIPLE CORRECT CHOICE TYPE QUESTIONS

This section contains Multiple Correct Choice Type Questions. Each question has four choices (A), (B), (C) and (D), out of which ONE OR MORE is/are correct.

1. The solution of the differential equation

$\left(\frac{dy}{dx}\right)^2-\frac{dy}{dx}(e^x+e^{-x})+1=0$ is/are

(A) $y+e^{-x}=c$ (B) $y-e^{-x}=c$
(C) $y+e^x=c$ (D) $y-e^x=c$

2. Let $y=(A+Bx)e^{3x}$ be a solution of the differential equation $\frac{d^2y}{dx^2}+m\frac{dy}{dx}+ny=0$, $m, n\in I$, then

(A) $m+n=3$ (B) $n^2-m^2=64$
(C) $m=-6$ (D) $n=9$

3. If $y=f(x)$ passes through (1, 0) and satisfying $x-1=xy\frac{dy}{dx}+\frac{x^2y^2}{2!}\left(\frac{dy}{dx}\right)^2+\frac{x^3y^3}{3!}\left(\frac{dy}{dx}\right)^3+\ldots$, then

(A) $f(x)=\ln x$ (B) $f(x)=\ln(1/x)$

(C) $f(x)=(\ln x)^2$ (D) $f(x)=\frac{1}{x}\ln x$

4. The differential equation $\frac{d^2y}{dx^2}+y+\cot^2 x=0$ must be satisfied by

(A) $y=2+c_1\cos x+\sqrt{c_2}\sin x$

(B) $y=\cos x.\ln\left(\tan\frac{x}{2}\right)+2$

(C) $y=2+c_1\cos x+c_2\sin x+\cos x\log\left(\tan\frac{x}{2}\right)$

(D) All the above

5. The equation of the curve such that the distance between the origin and the tangent line at an arbitrary point is equal to the distance between the origin and the normal at the same point is

(A) $x^2+y^2=c\,e^{\text{arc tan}(y/x)}$ (B) $\sqrt{x^2+y^2}=c.\,e^{-\text{arc tan}(y/x)}$

(C) $\sqrt{x^2+y^2}=c.\,e^{\text{arc tan}(y/x)}$ (D) $x^2+y^2=c.\,e^{\text{arc tan}(y/x)}$

6. Identify the statement(s) which is/are True.

(A) $f(x, y)=e^{y/x}+\tan\frac{y}{x}$ is a homogeneous function of degree zero

(B) $x.\ln\frac{y}{x}dx+\frac{y^2}{x}\sin^{-1}\frac{y}{x}dy=0$ is a homogeneous differential equation of degree one

(C) $f(x, y)=x^2+\sin x.\cos y$ is not homogeneous function

(D) $(x^2+y^2)\,dx-(xy^2-y^3)dy=0$ is a homogenous differential equation

7. A function $y=f(x)$ satisfying the differential equation $\frac{dy}{dx}\sin x-y\cos x+\frac{\sin^2 x}{x^2}=0$ is such that $y\to 0$ as $x\to\infty$ then the statement which is correct is

(A) $\lim\limits_{x\to 0}f(x)=1$ (B) $\int_0^{\pi/2}f(x)dx$ is less than $\frac{\pi}{2}$

(C) $\int_0^{\pi/2}f(x)dx$ is greater than unity

(D) $f(x)$ is an odd function

8. Let function $y=f(x)$ satisfies the differential equation $x^2\frac{dy}{dx}=y^2\,e^{\frac{1}{x}}(x\neq 0)$ and $\lim\limits_{x\to 0^-}f(x)=1$. Identify the correct statement(s)?

(A) Range of $f(x)$ is $(0, 1)-\left\{\frac{1}{2}\right\}$ (B) $f(x)$ is bounded

(C) $\lim\limits_{x\to 0^+}f(x)=1$ (D) $\int_0^e f(x)dx>\int_0^1 f(x)dx$

9. Let $\frac{dy}{dx}+y=f(x)$ where y is a continuous function of x with $y(0)=1$ and $f(x)=\begin{cases}e^{-x} & \text{if } 0\le x\le 2\\ e^{-2} & \text{if } x>2\end{cases}$.

Which of the following hold(s) good?

(A) $y(1)=2e^{-1}$ (B) $y'(1)=-e^{-1}$
(C) $y(3)=-2e^{-3}$ (D) $y'(3)=-2e^{-3}$

10. Let C be the family of curves $f(x, y, c)=0$ (no member of C is x-axis) such that length of subnormal at any point $P(x, y)$ on the curve C is equal to four times that of the length of subtangent at the same point. Which of the following statements(s) is (are) correct?

(A) Equation of the line with positive y-intercept passing through (4, 2) and perpendicular to the curve C is $x+2y=8$.

(B) Orthogonal trajectory of C is family of parallel lines having gradient ±2.

(C) Order and degree of the differential equation of family of curves C are 1 and 2 respectively.
(D) Differential equation of family of curves is $2y' \pm x = 0$.

11. The solution of $\frac{dy}{dx} = \frac{ax+h}{by+k}$ represents a parabola if

(A) $a = -2$, $b = 0$
(B) $a = -2$, $b = 2$
(C) $a = 0$, $b = 2$
(D) $a = 0$, $b = 0$

12. Solution of the differential equation:

$(3 \tan x + 4 \cot y - 7) \sin^2 y \, dx - (4 \tan x + 7 \cot y - 5) \cos^2 x \, dy = 0$ is

(A) $\frac{3}{2}\cot^2 x - 7\cot x + \frac{7}{2}\tan^2 y - 5\tan y + 4\cot x.\tan y = c$
(B) $\frac{3}{2}\tan^2 x - 7\tan x + \frac{7}{2}\cot^2 y - 5\cot y + 4\tan x.\cot y = c$
(C) $3 \tan^2 y - 14\cot x . \tan^2 y + 7 \cot^2 x - 10 \tan y \cot^2 x + 8 \cot x . \tan y + 2c \cot^2 x \tan^2 y = 0.$
(D) $3 \cot^2 y - 14\cot x . \cot^2 y + 7 \cot^2 x + 10 \cot y \tan^2 x + 8 \tan x . \cot y = 0.$

13. The coordinates of a point $P(x, y)$ are functions of time t and satisfy the relations $\frac{dx}{dt} + \frac{dy}{dt} = t$ and $\frac{dx}{dt} - 2\frac{dy}{dt} = t^2$ at any instant of time t. The locus of point $P(x, y)$ is a curve given by (assume $x(0) = y(0) = 0$)

(A) $(x + y)^3 = (x - 2y)^2$
(B) $x = \frac{t^2}{3} + \frac{t^3}{9}, y = \frac{t^2}{6} - \frac{t^3}{9}$
(C) $9(x + y)^3 = 8(x - 2y)^2$
(D) $8(x + y)^3 = 9(x - 2y)^2$

14. A curve passes through (2, 0) and the slope of tangent at $P(x, y)$ equals $\frac{(x+1)^2 + y - 3}{x+1}$ then

(A) curve is $y = x^2 - 2x$
(B) curve is $y = x^3 - 8$
(C) area bounded by the curve and x-axis in fourth quadrant is $\frac{2}{3}$ square units
(D) area bounded by the curve and x-axis in fourth quadrant is $\frac{4}{3}$ square units

15. The solution of $p^2 + (2y \cot x)\, p = y^2$ where $p = \frac{dy}{dx}$ is

(A) $y(1 + \cos x) = c$
(B) $y(1 - \cos x) = c$
(C) $x = 2\sin^{-1}\sqrt{c/2y}$
(D) $x = 2 \sin^{-1}\left(\sqrt{2y}\right) + c$

16. Let C be a curve such that the normal at any point P on it meets x-axis and y-axis at A and B respectively. If $BP : PA = 1 : 2$ (internally) and the curve passes through the point (0, 4) then which of the following alternative(s) is/are correct?

(A) The curves passes through $\left(\sqrt{10}, -6\right)$
(B) The equation of tangent at $\left(4, 4\sqrt{3}\right)$ is $2x + \sqrt{3}y = 20$
(C) The differential equation for the curve is $yy' + 2x = 0$
(D) The curve represent a hyperbola

17. The general solution of the differential equation $\frac{dy}{dx} + \frac{y \ln y}{x} = \frac{y}{x^2}(\ln y)^2$ is, (c being the constant of integration)

(A) $2x = (1 - cx^2) \ln y$
(B) $x = (1 + 2cx^2) \ln y$
(C) $2x = (1 + 2cx^2) \ln y$
(D) $x = (1 + cx^2) \ln y$

18. Solution of the differential equation:

$$\frac{x + y\frac{dy}{dx}}{y - x\frac{dy}{dx}} = \frac{x\sin^2(x^2+y^2)}{y^3}$$

(A) $-\cot(x^2 + y^2) = \left(\frac{x}{y}\right)^2 + c$
(B) $\frac{y^2}{x^2 + y^2 c} = -\tan(x^2 + y^2)$
(C) $-\cot(x^2 + y^2) = \left(\frac{y}{x}\right)^2 + c$
(D) $\frac{y^2 + x^2 c}{x^2} = -\tan^2(x^2 + y^2)$

19. The normal at a general point (a, b) on curve makes an angle θ with x-axis which satisfies $b\left(a^2 \tan\theta - \cot\theta\right) = a\left(b^2 + 1\right)$. The equation of curve can be

(A) $y = e^{x^2/2} + c$
(B) $\log\left(ky^2\right) = x^2$
(C) $y = ke^{x^2/2}$
(D) $x^2 - y^2 = k$

20. The differential equation $\frac{dx}{dy} = \frac{3y}{2x}$ represents a family of hyperbolas (except when it represents a pair of lines) with eccentricity

(A) $\sqrt{\frac{3}{5}}$
(B) $\sqrt{\frac{5}{3}}$
(C) $\sqrt{\frac{2}{5}}$
(D) $\sqrt{\frac{5}{2}}$

21. Solution of the differential equation

$xy\left(\frac{dy}{dx}\right)^2 - (x^2 - y^2).\frac{dy}{dx} - xy = 0$ is

(A) A hyperbola whose ecentricity $\sqrt{2}$
(B) $xy = c$
(C) $(xy - c).(y^2 - x^2 - c) = 0$
(D) $(x^2 + y^2 - c).(x + y - c) = 0$

22. The orthogonal trajectories of the system of curves $\left(\frac{dy}{dx}\right)^2 = \frac{a}{x}$ are

(A) $9a(y + (c))^2 = 4x^3$
(B) $y + c = \frac{-2}{3\sqrt{a}}x^{3/2}$
(C) $y + c = \frac{2}{3\sqrt{a}}x^{3/2}$
(D) none of these

23. The graph of the function $y = f(x)$ passing through the point (0, 1) and satisfying the differential equation $\frac{dy}{dx} + y \cos x = \cos x$ is such that

(A) it is a constant function
(B) it is periodic
(C) it is neither an even nor an odd function
(D) it is continuous and differentiable for all x .

24. The equation of the curve passing through (3, 4) and satisfying the differential equation,

$y\left(\frac{dy}{dx}\right)^2 + (x - y)\frac{dy}{dx} - x = 0$ can be

(A) $x - y + 1 = 0$
(B) $x^2 + y^2 = 25$
(C) $x^2 + y^2 - 5x - 10 = 0$
(D) $x + y - 7 = 0$

25. For the differential equation

$y = \ln\left\{1 + y' + \frac{(y')^2}{2!} + \frac{(y')^3}{3!} + \ldots\ldots\infty\right\}$ where $y' = \frac{dy}{dx}$

(A) order is 1
(B) degree is 1
(C) order is not defined
(D) degree is not defined

26. Consider the family is curves C: $y = cx^2$.
(A) The differential equation of C is $x\,dy - 2y dx = 0$
(B) The differential equation of orthogonal trajectory of C is $x\,dx + 2y dy = 0$
(C) The equation of family of orthogonal trajectory of C is $x^2 + 2y^2 = c$.
(D) The equation of family of orthogonal trajectory of C is $2x^2 + y^2 = c$.

27. If $2xy\,dy = (x^2 + y^2 + 1)dx$, $y(1) = 0$ and $y(x_0) = \sqrt{3}$ then x_0 can be
(A) 2
(B) –2
(C) 3
(D) –3

28. If $y(x)$ satisfies the differential equation $\frac{dy}{dx} = \sin 2x + 3y \cot x$ and $y\left(\frac{\pi}{2}\right) = 2$ then which of the following statement(s) is (are) correct?

(A) $y\left(\frac{\pi}{6}\right) = 0$

(B) $y'\left(\frac{\pi}{3}\right) = \frac{9 - 3\sqrt{2}}{2}$

(C) $y(x)$ increases in interval $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

(D) The value of definite integral $\int_{-\pi/2}^{\pi/2} y(x)\,dx$ equals π.

29. Let $y = f(x)$ defined on R satisfies the differential equation $(1 + x^2)\frac{dy}{dx} + 2xy = 2x$ and $f(0) = 2$, then which of the following statement(s) is (are) correct?
(A) $f(x)$ is neither even nor odd function.
(B) $f(x)$ increases on $(-\infty, 0)$ and decreases on $(0, \infty)$.
(C) The x-intercept of normal on the graph of $y = f(x)$ at $x = 1$ equals $\frac{1}{4}$.
(D) The area bounded by $y = f(x)$ with x-axis between ordinates at $x = 0$ and $x = 1$ equals $\left(\frac{\pi + 4}{4}\right)$

30. Let $f(x)$ be differentiable function on the interval $(-\infty, \infty)$ such that $f(1) = 5$, and $\lim_{a \to x} \frac{a\,f(x) - x\,f(a)}{a - x} = 2$, for each $x \in R$. Then which of the alternative(s) is/are correct?

(A) Area bounded by $f(x)$ with co-ordinate axes is $\frac{2}{3}$

(B) $\int_0^2 f(x)dx = 10$

(C) $f'(x) = 3\ \forall\ x \in R$.
(D) $f(x)$ has no inflection point.

31. If $y(x)$ satisfies the differential equation $\frac{dy}{dx} + (2\tan x)y = \sin x$ and $y\left(\frac{\pi}{3}\right) = 0$, then which of the following statement(s) is (are) correct?

(A) $y'\left(\frac{\pi}{6}\right) = \frac{2\sqrt{3} - 1}{2}$

(B) $y(\pi) = 3$

(C) Number of values of x for which $y(x) = -1$ in$(0, 2\pi)$ are three

(D) $\int_0^{\frac{\pi}{2}} y(x)dx = 1 - \frac{\pi}{2}$

32. Let f be a differentiable function on R satisfying $f'(t) = e^t(\cos^2 t - \sin^2 t)$ and $f(0) = 1$, then which of the following is/are correct?
(A) f is bounded in $(-\infty, 0)$
(B) Number of solution satisfying the equation $f(t) - e^t = 0$ in $[0, 2\pi]$ is 3.
(C) The value of $\lim_{t \to 0}\left(f(t)\right)^{\frac{1}{t}} = 1$.
(D) f is neither odd nor even.

33. If a function f satisfies the relation $f(x)\,f''(x) - f(x)\,f'(x) = (f'(x))^2\ \forall\ x \in R$ and $f(0) = f'(0) = 1$, then identify which of the following statement(s) is (are) correct?
(A) Equation $f(x) = e^x$ has two solutions
(B) Equation $f(x) = e^x$ has exactly one solution

(C) $\lim_{x \to -\infty} f(x) = \frac{1}{e}$

(D) $\lim_{x \to 0} \frac{f(x) - 1}{x} = 1$

34. Consider a curve passing through (1, 1) such that perpendicular distance of normal drawn at any point P from origin is equal to ordinate of the point P. Then which of the following statement(s) is/are correct?
(A) The curve passes through origin.
(B) The equation of tangent at (2, 0) is $x = 2$.
(C) The differential equation for the curve is $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$
(D) The differential equation for the curve is $\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$

35. Let $y = f(x)$ and the function satisfying the differential equation $3y^2 \dfrac{dy}{dx} + (\sin x)y^3 = e^{\cos x}$ where $f(0) = 0$ then

(A) $f(x)$ is an even function
(B) $f(x)$ is an odd function
(C) $f(x)$ is neither even nor odd
(D) range of $f(x)$ is R.

36. If $y = y(x)$ is solution of differential equation

$\dfrac{dy}{dx} = -x \sin x$; $y(0) = 0$, then

(A) $y > 0$ if $x \in \left(-\dfrac{\pi}{2}, 0\right)$

(B) $y < 0$ if $x \in \left(-\dfrac{\pi}{2}, 0\right)$

(C) Number of inflection points of $y = y(x)$ in $\left(0, \dfrac{\pi}{2}\right)$ is 0.

(D) $\dfrac{d^2y}{dx^2} + y = -2\sin x$

37. If a curve passing through $P(4, 2)$ and satisfies the differential equation $\dfrac{dy}{dx} = \dfrac{y}{x - y^2}$ is a conic then which of the following is correct?

(A) focus of conic lies on line $y = 2$
(B) equation of directrix of conic is $4x - 17 = 0$
(C) eccentricity of conic is 1
(D) eccentricity of conic is 3

38. The solution of differential equation $2x^3\, y dy + (1 - y^2)(x^2y^2 + y^2 - 1)dx = 0$ is

$x^m y^n = (cx + k)(1 - y^2)$ then

(A) $m + n + k = 3$
(B) $m + n + k = 4$
(C) $m.n + k = 3$
(D) $m + n - k = 5$

LINKED COMPREHENSION TYPE QUESTIONS

This section contains Linked Comprehension Type Questions or Paragraph based Questions. Each set consists of a paragraph followed by questions. Each question has four choices (A), (B), (C) and (D), out of which only one is correct. (For the sake of competitiveness there may be a few questions that may have more than one correct options)

Comprehension 1

A rumour spreads through a population of 5000 people at a rate proportional to the product of the number of people who have heard it and the number who have not. Suppose that 100 people initiate the rumour and that a total of 500 people known the rumour after two days.

Let y (t) denotes the number of people who know the rumour at time t. Take $\dfrac{\log 9}{\log 49} = \dfrac{129}{229}$

On the basis of above information answer the following.

1. The time required for half the population to hear rumour is

(A) 4.22 days
(B) 6.24 days
(C) 4.58 days
(D) 3.32 days

2. The function $y(t)$ is given by

(A) $\dfrac{5000}{1 + 50e^{-5000kt}}$
(B) $\dfrac{5000}{1 + 49e^{-5000kt}}$
(C) $\dfrac{5000}{1 - 50e^{-5000kt}}$
(D) $\dfrac{500}{1 - 49e^{-5000kt}}$

Comprehension 2

Let $f(x)$ and $g(x)$ be two real valued functions satisfying $f(x + y) = f(x) + f(y) - 8xy$ for $\forall\ x, y \in R$ and $g(x + y) = g(x) + g(y) + 3xy(x + y)$ for $\forall\ x, y \in R$, also $f'(0) = 8$ and $g'(0) = -4$.

3. Equation $f(x) = k$ has real roots for

(A) $k \geq -4$
(B) $k \leq 4$
(C) $k \geq 4$
(D) $k \leq -4$

4. If $k \in [-3, 3]$ then the equation $g(x) = k$ has exactly

(A) two roots
(B) three roots
(C) one root
(D) no root

5. The area of the figure bounded by the curves $y = f(x)$ and $y = |g(x)|$ is -

(A) $\dfrac{4}{3}$
(B) $\dfrac{4}{7}$
(C) $\dfrac{28}{3}$
(D) $\dfrac{16}{3}$

6. The area of the figure bounded by $y = f(x)$ and $y = g(x)$ for $x \geq 0$ is divided by x-axis in the ratio

(A) $\dfrac{4}{3}$
(B) $\dfrac{4}{7}$
(C) $\dfrac{28}{3}$
(D) $\dfrac{16}{3}$

Comprehension 3

A curve $y = f(x)$ satisfies the differential equation

$(1 + x^2)\dfrac{dy}{dx} + 2yx = 4x^2$ and passes through the origin.

7. For the function $y = f(x)$ which one of the following does not hold good?

(A) $f(x)$ is strictly increasing $\forall\ x \in R$
(B) $f(x)$ has the same domain and same range
(C) $f(x)$ has a minima but no maxima
(D) $y = f(x)$ is a bijective mapping

8. The area enclosed by $y = f^{-1}(x)$, the x-axis and the ordinate at $x = 2/3$ is -

(A) $2 \ln 2$
(B) $\dfrac{4}{3}\ln 2$
(C) $\dfrac{2}{3}\ln 2$
(D) $\dfrac{1}{3}\ln 2$

Comprehension 4

If $y = f_1(x)$ and $y = f_2(x)$ are two solutions of the equation $ydx + dy = -e^x y^2 dy$ where $f_1(0) = 1$ and $f_2(0) = -1$.

9. The number of solution(s) of the equation $f_1(x) \cdot f_2(x) + x^2 = 0$ is (are)

(A) 0 (B) 1
(C) 2 (D) 3

10. If area enclosed by $y = f_1(x)$, $y = e^x$ and the x-axis is A_1 and $y = f_2(x)$, $y = -e^x$ and the x-axis is A_2 then $A_1 + A_2$ is equal to

(A) 6 ln2 (B) 3
(C) 6 (D) 3 ln2

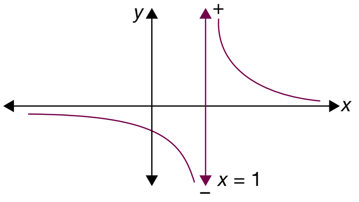
Comprehension 5

Let $C_1 : y = f(x)$ and $C_2 : y = g(x)$ be two curves passing through $M(2, 1)$ such that the length of intercept made by any tangent on the x-axis is equal to square of x-coordinate of the point of tangency.

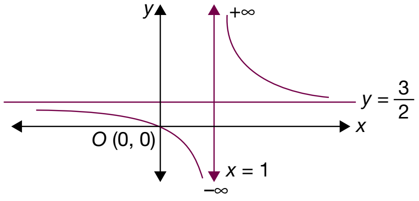
Given that $f(-2) = \frac{1}{3}$ and $g(-2) = 3$.

11. Which of the following best represent the graph of $f(x)$?

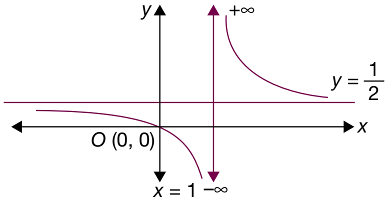
(A)



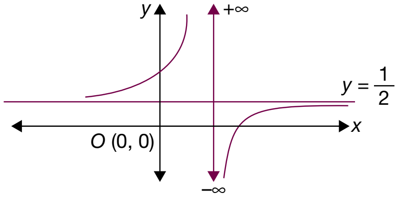
(B)



(C)



(D)



12. The value of definite integral $\int_0^1 g(x)dx$ is equal to

(A) $\frac{3}{2}(1 + \ln 2)$ (B) $\frac{2}{3}(1 - \ln 2)$
(C) $\frac{3}{2}(1 - \ln 2)$ (D) $\frac{2}{3}(1 + \ln 2)$

Comprehension 6

Let $g(x)$ be a polynomial of degree 3 passing through origin and have a local maximum at $x = \frac{-1}{2\sqrt{2}}$. Also $g'(x)$ has a local minimum at $x = 0$, and $g(1) = 5$.

13. Let $f(x) = \text{sgn}(x)$ where sgn (x) denotes signum function of x, then which one of the following statement is incorrect for $f(g(x))$?

(A) Area enclosed by $f(g(x))$ with x-axis between ordinates $x = -\alpha$ to $x = \alpha$ is 2α.
(B) $\int_{-1}^{1} f(g(x))\, dx = 0$
(C) $f(g(x))$ is a many-one function
(D) $f(g(x))$ is periodic function

14. Consider a real-valued function $y = h(x)$ satisfying the differential equation $\frac{dy}{dx} + g'(x)y = (g(x) + 1)\, g'(x)$ such that $h(0) = 1$, then $h(1)$ equals

(A) $5 + e^{-5}$ (B) $11 + e^{11}$
(C) $-5 + e^{-5}$ (D) $11 + e^{-11}$

Comprehension 7

A family of curves has the property that the segment of the tangent between the point of tangency and the x-axis is bisected at the point of intersection with y-axis. A member 'C' of this family passes through the point (9, –6).

15. Which one of the following statement is correct?

(A) Focus of C is (4, 0)
(B) Director circle of C is $x + 4 = 0$
(C) Length of latus-rectum of C is 4
(D) The equation of tangent at vertex of C is $y = 0$

16. The area bounded by curve C and its latus-rectum is equal to

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$
(C) $\frac{4}{3}$ (D) $\frac{8}{3}$

17. The locus of mid points of focal chords of curve C, is

(A) $y^2 = (x - 1)$ (B) $y^2 = 2(x - 1)$
(C) $y^2 = (x + 1)$ (D) $y^2 = (x + 1)$

Comprehension 8

Let f be a real-valued differentiable function on R satisfying $f'(x) + f(x) = x^2 + 3x + 2$ for all $x \in R$ and $f(q) \in Q$ for all $q \in Q$.

18. Which of the following statement is correct?
(A) $f'(x)$ is strictly decreasing in $[1, \infty)$
(B) $f'(x)$ equals 3 when $x = 2$.
(C) There exists $\alpha > 1$ such that $|f(x)| \le \alpha$ for all $x \in R$.
(D) $f(|x|)$ is non-differentiable at $x = 0$.

19. The area bounded by the curve $y = f(x)$ with x-axis between ordinates $x = 0$ and $x = 1$ equals
(A) $\frac{7}{6}$
(B) $\frac{11}{6}$
(C) $\frac{13}{6}$
(D) $\frac{17}{6}$

20. Number of distinct real roots of the equation $f(f(x)) = 0$, is equal to
(A) 6
(B) 4
(C) 2
(D) 0

Comprehension 9

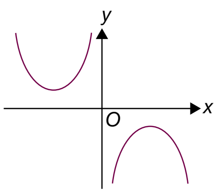
Consider the curves $C_1 : y = f(x)$ and $C_2 : y = g(x)$ such that the tangents to C_1 and C_2 at points with equal abscissae intersect on y-axis and the normals to C_1 and C_2 at points with equal abscissae intersect on x-axis. Also $f(1) = 1$ and $g(2) = 3$.

21. The value of definite integral $\dfrac{\int_{1/2}^{2} f(x)dx}{\int_{1}^{2} g(x)dx}$ is equal to
(A) $\frac{-3+\ln 2}{3+\ln 2}$
(B) $\frac{-3+\ln 4}{3+\ln 4}$
(C) $\frac{-3+\ln 8}{3+\ln 8}$
(D) $\frac{-3+\ln 16}{3+\ln 16}$

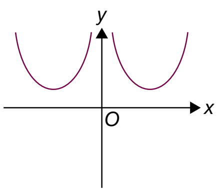
22. Which of the following statement is correct?
(A) $f(x)$ is an even function
(B) $f(x)$ is continuous but non-differentiable at $x = 2$
(C) Range of $f(x)$ is $(-\infty, \infty)$
(D) $f(x)$ decreases in $(0, \infty)$ and increases in $(-\infty, 0)$

23. Which of the following best represent the graph of $y = g(x)$?

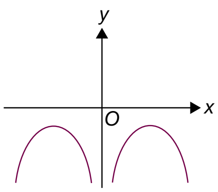
(A)



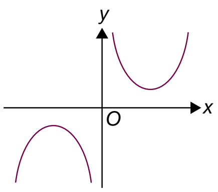
(B)



(C)



(D)



Comprehension 10

Let C be a curve $f(x, y) = 0$ passing through (1, 1) such that it is orthogonal to family of circles $x^2 + y^2 - ax = 0$, where a being parameter.

24. The area enclosed by the curve C and its tangent at (1, 1) and x-axis is equal to
(A) $1 - \cot^{-1}\sqrt{3}$
(B) $\tan^{-1}\sqrt{3} - 1$
(C) $1 - \sec^{-1}\sqrt{2}$
(D) $\sin^{-1}(1) - 1$

25. The shortest distance between the curve C and the parabola $x^2 = 2y + 1$ is equal to
(A) $\sqrt{2} - 1$
(B) $\sqrt{2} + 1$
(C) $\sqrt{6} - 1$
(D) $\sqrt{5} - 1$

Comprehension 11

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$, is called a linear differential equation of the first order. Its solution is given as $ye^{\int Pdx} = \int Qe^{\int Pdx} + k$, k being the constant of integration. A number of differential equations can be reduced to the linear differential equations and then one can solve them.

26. The general solution of the differential equation $\frac{dy}{dx} = e^{x-y}\left(e^x - e^y\right)$ is, (c being the constant of integration
(A) $e^y = e^x - 1 + ce^{-e^x}$
(B) $e^y = 1 - e^x + ce^{-e^x}$
(C) $e^x = e^y - 1 + ce^{-e^x}$
(D) $e^x = 1 - e^y + ce^{-e^x}$

27. The general solution of the differential equation $\frac{dy}{dx} + \frac{y\ln y}{x} = \frac{y}{x^2}(\ln y)^2$ is, (c being the constant of integration)
(A) $2x = \left(1 - cx^2\right)\ln y$
(B) $x = \left(1 + 2cx^2\right)\ln y$
(C) $2x = \left(1 + 2cx^2\right)\ln y$
(D) $x = \left(1 + cx^2\right)\ln y$

28. The general solution of the differential equation $\left(y^2 + e^{2x}\right)dy - y^3dx = 0$, ($c$ being the constant of integration), is
(A) $y^2e^{-2x} + 2\ln y = c$
(B) $y^2e^{-2x} - 2\ln y = c$
(C) $y^2e^{-x} - 2\ln y = c$
(D) $y^2e^{-x} + 2\ln y = c$

MATRIX MATCH/COLUMN MATCH TYPE QUESTIONS

Each question in this section contains statements given in two columns, which have to be matched. The statements in **COLUMN-I** are labelled A, B, C and D, while the statements in **COLUMN-II** are labelled p, q, r, s (and t). Any given statement in **COLUMN-I** can have correct matching with **ONE OR MORE** statement(s) in **COLUMN-II**. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following examples:

If the correct matches are A → p, s and t; B → q and r; C → p and q; and D → s and t; then the correct darkening of bubbles will look like the following:

	p	q	r	s	t
A	●	○	○	●	●
B	○	●	●	○	○
C	●	●	○	○	○
D	○	○	○	●	●

1. The factors which will make the differential equation solvable when they get multiplied to the differential equation are called integrating factor.
Match the differential equations in column I with an integrating factors in column II

Column-I	**Column-II**
(A) $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$	(p) $\frac{1}{x}$
(B) $\left(x + 2y^3\right)\frac{dy}{dx} = y$	(q) $\frac{1}{y}$
(C) $\left(x^2 + y\right)dx - xdy = 0$	(r) $\frac{1}{x^2}$
(D) $y\left(1 + xy\right)dx - xdy = 0$	(s) $\frac{1}{y^2}$

2. The differential equation for which the solution is

Column-I	**Column-II**
(A) $y = ae^{3x} + b\,e^{5x}$	(p) $y_2 + 16y = 0$
(B) $xy = ae^{x} + be^{-x} + x^2$	(q) $y_2 - 8y_1 + 15y = 0$
(C) $y = ax^2 + bx$	(r) $xy_2 + 2y_1 - xy + x^2 - 2 = 0$
(D) $y = a\sin\left(4x + b\right)$	(s) $x^2y_2 - 2xy_1 + 2y = 0$

3. Match the following :

Column-I	**Column-II**
(A) Solution of differential equation $[3x^2y + 2xy - e^x(1 + x)]\,dx + (x^3 + x^2)dy = 0$	(p) $y^2(x^2 + 1 + ce^{x^2}) = 1$
(B) Solution of differential equation $y\,dx - x\,dy - 3xy^2\,e^{x^2}\,dx = 0$	(q) $(x^2 + x^3)y - xe^x = c$
(C) Solution of differential equation $\frac{dy}{dx} = xy\,(x^2y^2 - 1)$	(r) $\frac{x}{y} - \frac{3}{2}e^{x^2} = c$
(D) Solution of differential equation $\frac{dy}{dx}(x^2y^3 + xy) = 1$	(s) $\frac{1}{x} = 2 - y^2 + ce^{-y^2/2}$
	(t) $\frac{2}{x} = 1 - y^2 + ce^{-y/2}$

4. Match the following :

Column-I	**Column-II**
(A) $x\,dy = y(dx + y\,dy)$, $y > 0$, $y(1) = 1$ and $y(x_0) = -3$, then $x_0 = \lambda$ is less than	(p) $\frac{1}{4}$
(B) If $y(t)$ is solution of $(t + 1)\frac{dy}{dt} - ty = 1$, $y(0) = -1$ and $y(1) = \lambda$, then λ is greater than	(q) –15
(C) $(x^2 + y^2)\,dx = xy\,dx$ and $y(1) = 1$ and $y(x_0) = \sqrt{e}$, then $x_0 = \lambda$ then λ is greater than	(r) $-\frac{1}{2}$
(D) $\frac{dy}{dx} + \frac{2y}{x} = 0$, $y(1)$, then $y(2)$ is equal to	(s) $\sqrt{3}\,e$

(*Continued*)

5. Match the following:

Column-I	Column-II
(A) Solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx}(e^x + e^{-x}) + 1 = 0$	(p) $x + y + 2 = ce^y$
(B) Solutions of the differential equation $(x + y + 1)\,dy = dx$ are given by	(q) $y = e^x + cx$
(C) Solution of the differential equation $(x + y + 2)dx + (2x + 2y - 1)dy = 0$ are	(r) $y + e^{-x} = c$
(D) Solution of $x\frac{dy}{dx} - y = (x - 1)e^x$ is	(s) $2(x + y + 2) + 5\ln(x + y - 3) = x + c$

6. Match the following:

Column-I	Column-II
(A) Solution of differential equation $\left(\frac{dy}{dx}\right)^2 - 7\frac{dy}{dx} + 12 = 0$ is	(p) $y^2 - 1 + Cxy = 0$
(B) The curve such that initial ordinate of any tangent is less than the abscissa of the point of tangency by two units, is	(q) $2x^2 + y^2 = C^2$
(C) The solution of $xdy + ydx + y^2(xdy - ydx) = 0$ is	(r) $(y - 4x + C_1)(y - 3x + C_2) = 0$
(D) The orthogonal trajectories of $y^2 = 4ax$ is	(S) $y = Cx - x\log_e \lvert x \rvert - 2$
	(t) $y = x^2 - xy + C$

INTEGER/NUMERICAL ANSWER TYPE QUESTIONS

In this section, the answer to each question is a numerical value obtained after doing series of calculations based on the data given in the question(s).

1. The differential equation of $\frac{y}{x}$ represents a curve given by $f(x, y) = 0$. The point (0, 2) satisfies the equation of the curve and the value of x when $y = 1$ is h. Then $4\lvert h \rvert$ is equal to

2. If $y = f(x)$ satisfies the differential equation $(1 + x^2)\, f'(x) = x(1 - f(x))$, $f(0) = \frac{4}{3}$, then find the value of $f\left(\sqrt{8}\right) + \frac{8}{9}$.

3. Let $y = f(x)$ be a curve C_1 passing through (2, 2) and $\left(8, \frac{1}{2}\right)$ and satisfying a differential equation $y\left(\frac{d^2y}{dx^2}\right) = 2\left(\frac{dy}{dx}\right)^2$. Curve C_2 is the director circle of the circle $x^2 + y^2 = 2$. If the shortest distance between the curves C_1 and C_2 is $\sqrt{p} - q$ where $p, q \in N$, then find the value of $(p^2 - q)$.

4. Let $f(x)$ be a differentiable function satisfying

$$f(x) + \int_0^x f(t)\tan t\,dt - \int_0^x \tan(t - x)dt = 0\left(\lvert x \rvert < \frac{\pi}{2}\right).$$

Find the number of solution of the equation $f(x) = 0$.

5. A function $y = f(x)$ satisfies $xf'(x) - 2f(x) = x^4 f^2(x)$. $\forall\, x > 0$ and $f(1) = -6$. Find the value of $f'\left(3^{\frac{1}{5}}\right)$.

6. Let C_1 and C_2 be two curves which satisfy the differential equation $\left\lvert x - y\cdot\frac{dx}{dy}\right\rvert = 2\left\lvert\frac{dy}{dx}\right\rvert$ and passes through (1, 1). If the area enclosed by curves C_1, C_2 and co-ordinate axes is $\frac{m}{n}$ $(m, n \in N)$ then find the least value of $(m + n)$

7. Let $y = f(x)$ defined in [0, 2] satisfies the differential equation $y^3\, y'' + 1 = 0$ where $f(x) \geq 0\ \forall\ x \in D_f$ and $f'(1) = 0$, $f(1) = 1$ then find the maximum value of $f(x)$.

8. A function $y = f(x)$ satisfies $f''(x) = \frac{-1}{x^2} - \pi^2 \sin(\pi x)$ such that $f'(2) = \pi + \frac{1}{2}$ and $f(1) = 0$. If $J = \int_0^1 f(x)\,dx$ then find the value of $\pi(J + 1)$.

9. If $6 + f''(x) + f'(x) = x^2 + f^2(x)$ be the differential equation of a curve and let P be the point of minima of this curve then the number of tangents which can be drawn from P to the circle $x^2 + y^2 = 4$ is

10. If $\frac{dy}{dx} = xy + x^3y^3$ and $y(0) = 1$ then $3\left(y\left(\frac{1}{2}\right)\right)^2$ is ________

11. If solution of the differential equation $\frac{dy}{dx} + \frac{\cos x(3\cos y - 7\sin x - 3)}{\sin y(3\sin x - 7\cos y + 7)} = 0$ is $(\sin x + \cos y - 1)^{\lambda}(\sin x - \cos y + 1)^{\mu} = c$, where '$c$' is arbitrary constant, then the value of $\lambda + \mu$ is equal to

12. The order and degree of the differential equation satisfying $\sqrt{1 - x^4} + \sqrt{1 - y^4} = a\left(x^2 - y^2\right)$ are P and Q then $P + Q$ is (a is arbitrary constant)

13. The order of differential equations whose general solution is given by $y = C_1\cos(x + C_2) - C_3e^{C_4 - x} + C_5\sin x$ where C_1, C_2, C_3, C_4, C_5 are arbitrary constants is ________

14. If $y = C_1e^{2x} + C_2e^{x} + C_3e^{-x}$ satisfies the differential equations $\dfrac{d^3y}{dx^3} + a\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = 0$ then $a^2 + b^2 + c^2$

15. Let $f : R^+ \to R$ be a differentiable function satisfying $f(x) = e + (1 - x)\ \ln\left(\dfrac{x}{e}\right) + \displaystyle\int_1^x f(t)\ dt\ \forall\ x \in R^+$. If the area enclosed by the curve $g(x) = x(f(x) - e^x)$ lying in the fourth quadrant is A, then find the value of A^{-1}.

16. Let f be a differentiable function on r and satisfying $f(x) = -(x^2 - x + 1)e^x + \displaystyle\int_0^x e^{x-y} \cdot f'(y)dy$. If $f(1) + f'(1) + f''(1) = ke$, where $k \in N$, then find k.

17. If $f : R^+ \to R$ be a differentiable function satisfying $f(x) = x\ln x + x - \displaystyle\int_1^x \dfrac{f(t)}{t \cdot \ell nt} dt\ x\ R^+$, then find the value of $f'(e) + f''\left(\dfrac{1}{4}\right)$?

18. Let C be a curve which has the property that length of the perpendicular from origin upon any of its tangent is equal to the abscissa of the point of tangency. If the curves passes through (2, –2) and the area enclosed by the curve above the x-axis is $k\pi$, then find the value of k.

19. A curve is such that the intercept of its tangent on the x-axis is twice the abscissa and which passes through the point (1, 2). If the ordinate of the point on the curve is $\dfrac{1}{3}$ then the value of the abscissa is _______

20. The slope of the tangent to a curve $y = f(x)$ at $(x, f(x))$ is $2x + 1$. and the curve passes through the point (1, 2). If the area of the region bounded by the curve, the x-axis and the line $x = 1$ in first quadrant is A then $6A =$ _______

21. The order of the differential equation of the family of all circles with one diameter along the line $2x + 3y - 5 = 0$ is

22. If solution of the differential equation $y(2x^4 + y)\dfrac{dy}{dx} = (1 - 4xy^2)x^2$ is given by $\lambda_1x^4y^2 + \lambda_2x^3 + \lambda_3y^3 = c$, where c is an arbitrary constant $\lambda_1, \lambda_2, \lambda_3$ are relatively prime integer then $\lambda_1 + \lambda_2 + \lambda_3 =$

23. If the function $y = e^{4x} + 2e^{-x}$ is a solution of the differential equation $\dfrac{\dfrac{d^3y}{dx^3} - 13\dfrac{dy}{dx}}{y} = 4K$ then the value of K is

24. Let $f(x)$ be the curve passing through (–2, 1) such that slope of the normal line at the point (x, y) on the curve is equal to x^2y. Find the area bounded by the curve $y = x\,f^2(x)$ and coordinates axes.

25. Let k be a positive integer and $f(x)$ be a polynomial with integer coefficients satisfying $2\displaystyle\int_1^x f(t)dt + x^k = xf(x)$, where $x \geq 1$. Find the sum of all possible values of k.

26. If the equation of a curve whose slope of tangent at any point is thrice its abscissa and which passes through (–1, –3) is $2y = ax^2 + bx - c$ (where $a, b, c \in I$), then find the value of $(c + b - a)$.

27. Let f be a differentiable function satisfying $f'(x) = f^2(x)$ and $f(0) = -1$. If $\dfrac{\sqrt{2}}{\pi}\displaystyle\int_0^1 \left(\dfrac{(1-x)}{f(x)(x^4+1)} + \dfrac{2}{x^4+1}\right)dx = \dfrac{p}{q}$ where p, $q \in N$ then find least value of $(p + q)$.

28. The degree of the differential equation of all circles in the first quadrant which touch the coordinate axes is _______.

29. A tangent drawn to the curve $y = f(x)$ at $P(x, y)$ cuts the x and y axes at A and B respectively such that $AP : PB = 1 : 3$. If $f(1) = 1$ then the curve passes through $\left(k.\dfrac{1}{8}\right)$ where $k =$ _________.

30. Slope of the tangent at any point (x, y) on the curve $y = f(x)$ is $\dfrac{y}{x^2}$ and the curve passes through the point $(1, 3)$ then $\lim\limits_{x\to\infty} f(x) = Ke$ where $K =$ ________.

31. The solution of the differential equation $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^6$ is $\dfrac{1}{x^5y^5} = \dfrac{p}{qx^2} + c$ then the value of $p + q$ is ________

32. Let f_1, f_2 and f_3 be three curves satisfying the differential equation $y(1 - y^2)\ dx = x(y^2 + 1)dy$. If f_3 cuts the curves f_1 and f_2 at A and B respectively and one of the curve is passing through $C(2, -1)$, then find the area of ΔABC.

33. Given a curve C. Let the tangent line at $P(x, y)$ on C is perpendicular to the line joining P and $Q(1, 0)$. If the line $2x + 3y - 15 = 0$ is tangent to the curve C then the length of the tangent from the point (5, 0) to the curve C is $\sqrt{n}$ (where $n \in N$). Find the value of n.

34. Let $f(x)$ be a cubic polynomial such that $f''(x) = 12x - 4$. If $f(x)$ has a local minimum value 0 at $x = 1$. Then find the x-intercept of normal to $f(x)$ at point M whose abscissa is 2.

35. The orthogonal trajectory of the family of rectangular hyperbola $xy = c^2$ is $ax^2 + by^2 = k$ then the value of $a + b$ is _______.

36. Let $\dfrac{dy}{dx} - y\dfrac{\phi'(x)}{\phi(x)} = \dfrac{-y^2}{\phi(x)}$, where $\phi(x)$ is a specified function satisfying $\phi(1) = 1$, $\phi(4) = 1296$. If $y(1) = 1$ then $\dfrac{\sqrt{y(4)}}{3} =$

37. The differential equation $dy = x\tan y(\sin y - 1)dx$ has solution $y = \sin^{-1}\left(\dfrac{1}{k + ce^{\frac{x^2}{2}}}\right)$ then $k =$ ___________.

38. If $y = f(x)$ is the solution of the differential equation $x\left(y^3 - x\right)dy = y\left(x + y^3\right)dx$ and $f(1) = (-2)^{1/3}$ and $f^{-1}(-2) = k$. Then $|k| =$ _____________ (Where $|.|$ denotes the modulus function)

39. If the solution of the differential equation $\sec^2 y\left(\frac{dy}{dx}\right)$ + $2x\tan y = x^3$ is $2\tan y = \lambda\left(x^2-1\right)+ce^{-x^2}$ where 'C' is an arbitrary constant then λ = __________.

40. The solution of differential equation $2xy\frac{dy}{dx}=x^2+y^2$ passing through (2, 1) is a conic and e is the eccentricity of the conic then $2e^2$ is equal to ________.

41. If the solution of the differential equation $e^{3x}(p-1)+p^3e^{2y}=0$ is in the form $e^y=ce^x+c^k$ then $k=$ (Where $p=\frac{dy}{dx}$ and 'c' is arbitrary constant)

42. If the solution of the differential equation $y^2(y-xp)=x^4p^2$ is in the form $\frac{1}{y}=\frac{c}{x}+c^k$ then k = _______ (Where $p=\frac{dy}{dx}$ and 'c' is arbitrary constant)

43. The solution of $x^3\frac{dy}{dx}+4x^2\tan y=e^x\sec y$ satisfying $y(1)=0$ is $\sin y=e^x(x-1)x^{-k}$ then $k=$

44. For $y > 0$ and $x\in R$, $ydx+y^2dy=xdy$ where $y = f(x)$. If $f(1) = 1$, then find $f(-3)$.

45. If $\sin y.\frac{dy}{dx} = \cos x(\cos y - 1)$ and $y(0) = \frac{\pi}{2}$, then value of $\cos y + e^{\sin x}$.

46. If the curve satisfying $xdx = \left(\frac{x^2}{y}-y^3\right)dy$ passes through (0, 2) then value of $(y(4))^2\,(4-(y(4))^2)$ is M^2, then $|M|$ is

47. If the orthogonal trajectory of $y = ax^2$ (where a is arbitrary constant) is $y^{\lambda_1} = -\frac{x^{\lambda_2}}{2}+C$ (where C is arbitrary constant), then $\lambda_1 + \lambda_2$ is

48. Let $f(x)$ and $g(x)$ are two differentiable function such that $f'(x) + g(x) = e^x$ and $g'(x) + f(x) = x$, where $f(0) = \frac{1}{2}$ and $g(0) = -1$, then $[f(1) + g(1)]$ is (where [.] represents greatest integer function)

49. Let $f(x)$ be a differentiable function satisfying $f(x)\le 2-f'(x)$, $\forall\ x \in R$. $f(0) = 0$. If the maximum value of $f(2)$ is M, then $[M]$ is (where [.] denotes greatest integer function)

50. Let $f(x)$ be a differentiable function and satisfy $f(0) = 2$, $f'(0) = 3$ and $f''(x) = f(x)$. If the value of $f(x)$ at $x = \ln 2$ is k then $[k]$ is. (where [.] denotes the greatest integer function).

51. Let f be a twice derivable function defined on R. If the equation $f(x) = 0$ has exactly 3 distinct real roots, then minimum number of real roots of the equation $f(x)\,f''(x) = -(f'(x))^2$ is

52. Consider the differential equation of a curve which passes through origin as $x^2dy + 2xydx = 3x^2dx+dy$. If curve passes through $(2, a)$ then $3a$ is equal to

ARCHIVE: JEE MAIN

1. **[JEE Mains 2021]**

The population $P = P(t)$ at time t of a certain species follows the differential equation $\frac{dP}{dt}=0.5P-450$. If $P(0) = 850$, then the time at which population becomes zero is :

(A) $2\log_e 18$ (B) $\log_e 18$

(C) $\log_e 9$ (D) $\frac{1}{2}\log_e 18$

2. **[JEE Mains 2021]**

Let f be a twice differentiable function defined on R such that $f(0) = 1$ and $f'(0) = 2$ and $f'(x) \ne 0$ for all $x \in R$. If $\begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} = 0$, for all $x \in R$, then $f(1)$ lies in the interval:

(A) (3, 6) (B) (0, 3)

(C) (6, 9) (D) (9, 12)

3. **[JEE Mains 2021]**

If a curve $y = f(x)$ passing through the point (1, 2) and satisfies $x\frac{dy}{dx}+y=bx^4$, then for what value of b, $\int_1^2 f(x)\,dx=\frac{62}{5}$?

(A) $\frac{31}{5}$ (B) 10

(C) 5 (D) $\frac{62}{5}$

4. **[JEE Mains 2021]**

If a curve passes through the origin and the slope of the tangent to it at any point (x, y) is $\frac{x^2-4x+y+8}{x-2}$, then this curve also passes through the point:

(A) (5, 4) (B) (4, 5)

(C) (4, 4) (D) (5, 5)

5. **[JEE Mains 2021]**

The rate of growth of bacteria in a culture is proportional to the number of bacteria present and the bacteria count is 1000 at initial time $t = 0$. The number of bacteria is increased by 20% in 2 hours. If the population of bacteria is 2000 after $\frac{k}{\log_e\left(\frac{6}{5}\right)}$ hours, then $\left(\frac{k}{\log_e 2}\right)^2$ is equal to:

(A) 4 (B) 8

(C) 16 (D) 2

6. **[JEE Mains 2021]**

Let slope of the tangent line to a curve at any point $P(x, y)$ be given by $\frac{xy^2+y}{x}$. If the curve intersects the line $x + 2y = 4$ at $x = -2$, then the value of y, for which the point $(3, y)$ lies on the curve, is:

(A) $-\frac{18}{11}$

(B) $-\frac{4}{3}$

(C) $\frac{18}{35}$

(D) $-\frac{18}{19}$

7. **[JEE Mains 2021]**

If $y = y(x)$ is the solution of the differential equation, $\frac{dy}{dx} + 2y\tan x = \sin x$, $y\left(\frac{\pi}{3}\right) = 0$, then the maximum value of the function $y(x)$ over R is equal to:

(A) 8

(B) $\frac{1}{2}$

(C) $-\frac{15}{4}$

(D) $\frac{1}{8}$

8. **[JEE Mains 2021]**

If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} + (\tan x)\,y = \sin x$, $0 \le x \le \frac{\pi}{3}$, with $y(0) = 0$, then $y\left(\frac{\pi}{4}\right)$ equal to:

(A) $\frac{1}{4}\log_c 2$

(B) $\left(\frac{1}{2\sqrt{2}}\right)\log_c 2$

(C) $\log_c 2$

(D) $\frac{1}{2}\log_c 2$

9. **[JEE Mains 2021]**

Let C_1 be the curve obtained by the solution of differential equation $2xy\frac{dy}{dx} = y^2 - x^2$, $x > 0$. Let the curve C_2 be the solution of $\frac{2xy}{x^2 - y^2} = \frac{dy}{dx}$. If both the curves pass through (1, 1), then the area enclosed by the curves C_1 and C_2 is equal to:

(A) $\pi - 1$

(B) $\frac{\pi}{2} - 1$

(C) $\pi + 1$

(D) $\frac{\pi}{4} + 1$

10. **[JEE Mains 2021]**

Which of the following is true for $y(x)$ that satisfies the differential equation $\frac{dy}{dx} = xy - 1 + x - y$; $y(0) = 0$;

(A) $y(1) = e^{-\frac{1}{2}} - 1$

(B) $y(1) = e^{\frac{1}{2}} - e^{-\frac{1}{2}}$

(C) $y(1) = 1$

(D) $y(1) = e^{\frac{1}{2}} - 1$

11. **[JEE Mains 2021]**

If $y = y(x)$ is the solution of the equation $e^{\sin y}\cos y\frac{dy}{dx} + e^{\sin y}\cos x = \cos x$, $y(0) = 0$; then $1 + y\left(\frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}y\left(\frac{\pi}{3}\right) + \frac{1}{\sqrt{2}}y\left(\frac{\pi}{4}\right)$ is equal to __________.

12. **[JEE Mains 2021]**

Let the curve $y = y(x)$ be the solution of the differential equation, $\frac{dy}{dx} = 2(x + 1)$. If the numerical value of area bounded by the curve $y = y(x)$ and x-axis is $\frac{4\sqrt{8}}{3}$, then the value of $y(1)$ is equal to __________.

13. **[JEE Mains 2021]**

Let $y = y(x)$ be the solution of the differential equation $x\,dy - y\,dx = \sqrt{(x^2 - y^2)}\,dx$, $x \ge 1$, with $y(1) = 0$. If the area bounded by the line $x = 1$, $x = e^{\pi}$, $y = 0$ and $y = y(x)$ is $\alpha e^{2\pi} + \beta$, then the value of $10(\alpha + \beta)$ is equal to __________.

14. **[JEE Mains 2021]**

Let a curve $y = y(x)$ be given by the solution of the differential equation $\cos\left(\frac{1}{2}\cos^{-1}(e^{-x})\right)dx = \sqrt{e^{2x} - 1}\,dy$. If it intersects y-axis at $y = -1$, and the intersection point of the curve with x-axis is $(\alpha, 0)$, then e^{α} is equal to _______.

15. **[JEE Mains 2020]**

If a curve $y = f(x)$, passing through the point (1, 2), is the solution of the differential equation, $2x^2dy = (2xy + y^2)dx$, then $f\left(\frac{1}{2}\right)$ is equal to:

(A) $\frac{1}{1 - \log_e 2}$

(B) $\frac{1}{1 + \log_e 2}$

(C) $\frac{-1}{1 + \log_e 2}$

(D) $1 + \log_e 2$

16. **[JEE Mains 2020]**

If $x^3dy + xy\,dx = x^2\,dy + 2y\,dx : y(2) = e$ and $x > 1$, then $y(4)$ is equal to:

(A) $\frac{3}{2} + \sqrt{e}$

(B) $\frac{3}{2}\sqrt{e}$

(C) $\frac{1}{2} + \sqrt{e}$

(D) $\frac{\sqrt{e}}{2}$

17. **[JEE Mains 2020]**

The solution curve of the differential equation, $(1 + e^{-x})(1 + y^2)\frac{dy}{dx} = y^2$, which passes through the point (0, 1), is:

(A) $y^2 = 1 + y\log_e\left(\frac{1 + e^x}{2}\right)$

(B) $y^2 + 1 = y\left(\log_e\left(\frac{1 + e^x}{2}\right) + 2\right)$

(C) $y^2 = 1 + y\log_e\left(\frac{1 + e^{-x}}{2}\right)$

(D) $y^2 + 1 = y\left(\log_e\left(\frac{1 + e^{-x}}{2}\right) + 2\right)$

18. **[JEE Mains 2020]**

The solution of the differential equation $\frac{dy}{dx} - \frac{y + 3x}{\log_e(y + 3x)} + 3 = 0$ is (where C is a constant of integration)

(A) $x - 2\log_e(y + 3x) = C$

(B) $x - \log_e(y + 3x) = C$

(C) $x - \frac{1}{2}(\log_e(y + 3x))^2 = C$

(D) $y + 3x - \frac{1}{2}(\log_e x)^2 = C$

19. **[JEE Mains 2020]**
Let $y = y(x)$ be the solution of the differential equation, $xy' - y = x^2(x \cos x + \sin x)$. $x > 0$. If $y(\pi) = \pi$, then $y''\left(\frac{\pi}{2}\right) + y\left(\frac{\pi}{2}\right)$ is equal to:

(A) $2 + \frac{\pi}{2}$ (B) $1 + \frac{\pi}{2}$

(C) $1 + \frac{\pi}{2} + \frac{\pi^2}{4}$ (D) $2 + \frac{\pi}{2} + \frac{\pi^2}{4}$

20. **[JEE Mains 2020]**
Let $y = y(x)$ be the solution of the differential equation $\cos x \frac{dy}{dx} + 2y \sin x = \sin 2x$, $x \in \left(0, \frac{\pi}{2}\right)$. If $y(\pi/3) = 0$, then $y(\pi/4)$ is equal to:

(A) $\sqrt{2} - 2$ (B) $\frac{1}{\sqrt{2}} - 1$

(C) $2 - \sqrt{2}$ (D) $2 + \sqrt{2}$

21. **[JEE Mains 2020]**
If $y = y(x)$ is the solution of the differential equation $\frac{5 + e^x}{2 + y} \cdot \frac{dy}{dx} + e^x = 0$ satisfying $y(0) = 1$, then a value of $y(\log_e 13)$ is :

(A) 1 (B) –1
(C) 2 (D) 0

22. **[JEE Main 2019]**
If $y = y(x)$ is the solution of the differential equation, $x\frac{dy}{dx} + 2y = x^2$ satisfying $y(1) = 1$, then $y\left(\frac{1}{2}\right)$ is equal to:

(A) $\frac{7}{64}$ (B) $\frac{49}{16}$

(C) $\frac{1}{4}$ (D) $\frac{13}{16}$

23. **[JEE Main 2019]**
Let $f : [0, 1] \to R$ be such that $f(xy) = f(x).f(y)$, for all $x, y \in [0, 1]$, and $f(0) \neq 0$. If $y = y(x)$ satisfies the differential equation, $\frac{dy}{dx} = f(x)$ with $y(0) = 1$, then $y\left(\frac{1}{4}\right) + y\left(\frac{3}{4}\right)$ is equal to:

(A) 2 (B) 3
(C) 5 (D) 4

24. **[JEE Main 2019]**
If $\frac{dy}{dx} + \frac{3}{\cos^2 x} y = \frac{1}{\cos^2 x}$, $x \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$ and $y\left(\frac{\pi}{4}\right) = \frac{4}{3}$, then $y\left(-\frac{\pi}{4}\right)$ equals

(A) $\frac{1}{3} + e^6$ (B) $\frac{1}{3}$

(C) $\frac{1}{3} + e^3$ (D) $-\frac{4}{3}$

25. **[JEE Main 2019]**
The curve amongst the family of curves represented by the differential equation, $(x^2 - y^2)dx + 2xy\,dy = 0$ which passes through (1, 1) is

(A) a circle with centre on the y-axis
(B) an ellipse with major axis along the y-axis
(C) a circle with centre on the x-axis
(D) a hyperbola with transverse axis along the x-axis

26. **[JEE Main 2019]**
Let f be a differentiable function such that $f'(x) = 7 - \frac{3}{4}\frac{f(x)}{x}$, $(x > 0)$ and $f(1) \neq 4$. Then $\lim_{x \to 0^+} xf\left(\frac{1}{x}\right)$

(A) does not exist (B) exists and equals $\frac{4}{7}$

(C) exists and equals 4 (D) exists and equals 0

27. **[JEE Main 2019]**
If $y(x)$ is the solution of the differential equation $\frac{dy}{dx} + \left(\frac{2x + 1}{x}\right)y = e^{-2x}$, $x > 0$, where $y(1) = \frac{1}{2}e^{-2}$, then:

(A) $y(\log_e 2) = \log_e 4$ (B) $y(x)$ is decreasing in (0, 1)

(C) $y(\log_e 2) = \frac{\log_e 2}{4}$ (D) $y(x)$ is decreasing in $\left(\frac{1}{2}, 1\right)$

28. **[JEE Main 2019]**
The solution of the differential equation, $\frac{dy}{dx} = (x - y)^2$, when $y(1) = 1$, is:

(A) $-\log_e\left|\frac{1 + x - y}{1 - x + y}\right| = x + y = 2$

(B) $\log_e\left|\frac{2 - x}{2 - y}\right| = x - y$

(C) $\log_e\left|\frac{2 - y}{2 - x}\right| = 2(y - 1)$

(D) $-\log_e\left|\frac{1 - x + y}{1 + x - y}\right| = 2(x - 1)$

29. **[JEE Main 2018]**
Let $y = y(x)$ be the solution of the differential equation $\sin x\, dy/dx + y \cos x = 4x$, $x \in (0, \pi)$. If $y(\pi/2) = 0$, then $y(\pi/6)$ is equal to

(A) $-\frac{4}{9}\pi^2$ (B) $\frac{4}{9\sqrt{3}}\pi^2$

(C) $\frac{-8}{9\sqrt{3}}\pi^2$ (D) $-\frac{8}{9}\pi^2$

30. **[JEE Main 2017]**
If $(2 + \sin x)\, dy/dx + (y + 1) \cos x = 0$ and $y(0) = 1$ then $y(\pi/2)$ is equal to

(A) 4/3 (B) 1/3
(C) –2/3 (D) –1/3

31. **[JEE Main 2016]**

If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)\,dx = x\,dy$, then $f(-1/2)$ is equal to

(A) $-4/5$ (B) $2/5$

(C) $4/5$ (D) $-2/5$

32. **[JEE Main 2015]**

Let $y(x)$ be the solution of the differential equation $(x \log x)\,dy/dx + y = 2x \log x$, $(x \geq 1)$. Then $y(e)$ is equal to

(A) e (B) 0

(C) 2 (D) $2e$

33. **[JEE Main 2014]**

Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$, then $p(t)$ equals

(A) $400 - 300\,e^{t/2}$ (B) $300 - 200\,e^{-t/2}$

(C) $600 - 500\,e^{t/2}$ (D) $400 - 300\,e^{-t/2}$

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. **[JEE Adv 2017]**

If $y = y(x)$ satisfies the differential equation

$$8\sqrt{x}\left(\sqrt{9\times\sqrt{x}}\right)dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx, x > 0$$

and $y(0) = \sqrt{7}$, then $y\,(256) =$

(A) 3 (B) 9

(C) 16 (D) 80

2. **[JEE Adv 2016]**

Let $f : (0,\infty) \to R$ be a differentiable function such that $f'(x) = 2 - \frac{f(x)}{x}$ for all $x \in (0, \infty)$ and $f(1) \neq 1$. Then

(A) $\lim_{x\to 0^+} f'\left(\frac{1}{x}\right) = 1$ (B) $\lim_{x\to 0^+} xf\left(\frac{1}{x}\right) = 2$

(C) $\lim_{x\to 0^+} x^2 f'(x) = 0$ (D) $|f(x)| \leq 2$ for all $x \in (0,2)$

3. **[JEE Adv 2014]**

The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then

$$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx \text{ is}$$

(A) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

(C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (D) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

4. **[JEE Adv 2013]**

A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right), x > 0$. Then the equation of the curve is

(A) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (B) $\text{cosec}\left(\frac{y}{x}\right) = \log x + 2$

(C) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (D) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$

Multiple Correct Choice Type Questions

5. **[JEE Adv 2021]**

For any real numbers α and β, let $y\ \alpha, \beta(x)$, $x \in \mathbb{R}$, be the solution of the differential equation $\frac{dy}{dx} + \alpha y = xe^{\beta x}$, $y(1) = 1$

Let $S = \{y\ \alpha, \beta(x) : \alpha, \beta \in \mathbb{R}\}$. Then which of the following functions belong(s) to the set S?

(A) $f(x) = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$

(B) $f(x) = \frac{x^2}{2}e^{-x} + \left(e + \frac{1}{2}\right)e^{-x}$

(C) $f(x) = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + \left(e - \frac{e^2}{4}\right)e^{-x}$

(D) $f(x) = \frac{e^x}{2}\left(\frac{1}{2} - x\right) + \left(e + \frac{e^2}{4}\right)e^{-x}$

6. **[JEE Adv 2020]**

Let b be a nonzero real number. Suppose $f : R \to R$ is a differentiable function such that $f(0) = 1$ If the derivative f^{+} of f satisfies the equation $f'(x) = \frac{f(x)}{b^2 + x^2}$ for all $x \in \mathbb{R}$, then which of the following statements is/are TRUE?

(A) If $b > 0$, then f is an increasing function

(B) If $b < 0$, then f is an decreasing function

(C) $f(x)\,f(-x) = 1$ for all $x \in \mathbb{R}$

(D) $f(x) - f(-x) = 0$ for all $x \in \mathbb{R}$

7. **[JEE Adv 2019]**

Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point $(1, 0)$ lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following option is/are correct?

(A) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

(B) $xy' + \sqrt{1-x^2} = 0$

(C) $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

(D) $xy' - \sqrt{1-x^2} = 0$

8. **[JEE Adv 2018]**

Let $f : R \to R$ and $g : R \to R$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x)-g(x))})g'(x)$ for all $x \in R$, and $f(1) = g(2) = 1$, then which of the following statement (s) is (are) TRUE ?

(A) $f(2) < 1 - \log_e 2$ (B) $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$ (D) $g(1) < 1 - \log_e 2$

9. **[JEE Adv 2016]**

A solution curve of the differential equation

$(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0$, $x > 0$, passes through the point (1, 3). Then the solution curve

(A) intersects $y = x + 2$ exactly at one point

(B) intersects $y = x + 2$ exactly at tow point

(C) intersects $y = (x + 2)^2$

(D) does **NOT** intersect $y = (x + 3)^2$

10. **[JEE Adv 2015]**

Consider the family of all circles whose centers lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are functions of x, y and y' $\left(\text{here } y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}\right)$, then which of the following statements is (are) true?

(A) $P = y + x$

(B) $P = y - x$

(C) $P + Q = 1 - x + y + y' + (y')^2$

(D) $P - Q = x + y - y' - (y')^2$

11. **[JEE Adv 2012]**

If $y(x)$ satisfies the differential equation $y' - y\tan x = 2x \sec x$ and $y(0) = 0$, then

(A) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (B) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$

(C) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{9}$ (D) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$

Integer/Numerical Answer Type Questions

12. **[JEE Adv 2018]**

Let f. $R \to R$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = (2 + 5y)(5y - 2)$, then the value of $\lim_{x \to -\infty} f(x)$ is _________.

ANSWER KEY—PRACTICE EXERCISES

Single Correct Choice Type Questions

1. A	**2.** A	**3.** D	**4.** A	**5.** D	**6.** A	**7.** A	**8.** A	**9.** D	**10.** D
11. B	**12.** B	**13.** C	**14.** A	**15.** C	**16.** A	**17.** A	**18.** A	**19.** C	**20.** B
21. C	**22.** D	**23.** A	**24.** A	**25.** C	**26.** A	**27.** C	**28.** A	**29.** C	**30.** B
31. A	**32.** B	**33.** B	**34.** A	**35.** A	**36.** D	**37.** D	**38.** C	**39.** C	**40.** C
41. B	**42.** A	**43.** D	**44.** B	**45.** B	**46.** A	**47.** C	**48.** A	**49.** B	**50.** C
51. C	**52.** A	**53.** B	**54.** D	**55.** C	**56.** C	**57.** D	**58.** B	**59.** C	**60.** A
61. D	**62.** B	**63.** A	**64.** A	**65.** A	**66.** A	**67.** B	**68.** B	**69.** B	**70.** C
71. D	**72.** B	**73.** A	**74.** A	**75.** A	**76.** B	**77.** B	**78.** A	**79.** A	**80.** B
81. C	**82.** B	**83.** A	**84.** C	**85.** B	**86.** A	**87.** A	**88.** D	**89.** C	**90.** B
91. A	**92.** B	**93.** A	**94.** A	**95.** B	**96.** A	**97.** D	**98.** A	**99.** A	**100.** D
101. D	**102.** A	**103.** D	**104.** D	**105.** A					

Multiple Correct Choice Type Questions

1. A, D	**2.** A, C, D	**3.** A, B	**4.** B, C	**5.** B, C
6. A, B, C	**7.** A, B, C	**8.** A, B, D	**9.** A, B, D	**10.** A, C
11. A, C	**12.** B, C	**13.** B, D	**14.** A, D	**15.** A, B, C
16. A, D	**17.** A, C	**18.** A, B	**19.** B, C, D	**20.** B, D
21. A, B, C	**22.** A, B, C	**23.** A, B, D	**24.** A, B	**25.** A, B
26. A, B, C	**27.** A, B	**28.** A, C	**29.** B, C, D	**30.** A, B, C, D
31. A, D	**32.** A, B, C, D	**33.** B, C, D	**34.** A, B, C	**35.** B, D
36. A, C, D	**37.** A, B, C	**38.** A, C, D		

Linked Comprehension Type Questions

1. C	**2.** B	**3.** B	**4.** B	**5.** A	**6.** A	**7.** C	**8.** C	**9.** B	**10.** C
11. C	**12.** C	**13.** A	**14.** A	**15.** C	**16.** D	**17.** B	**18.** D	**19.** B	**20.** D
21. D	**22.** C	**23.** D	**24.** C	**25.** A	**26.** A	**27.** C	**28.** A		

Matrix Match/Column Match Type Questions

1.	A → s	B → q	C → r	D → s
2.	A → q	B → r	C → s	D → p
3.	A → q	B → r	C → p	D → s
4.	A → p, r, s	B → q	C → p, q, r	D → p
5.	A → r	B → p	C → s	D → q
6.	A → r	B → s	C → p	D → q

Integer/Numerical Answer Type Questions

1. 3	**2.** 2	**3.** 62	**4.** 1	**5.** 8	**6.** 9	**7.** 1	**8.** 2	**9.** 2	**10.** 4
11. 7	**12.** 2	**13.** 3	**14.** 9	**15.** 4	**16.** 9	**17.** 6	**18.** 2	**19.** 6	**20.** 5
21. 2	**22.** 3	**23.** 3	**24.** 1	**25.** 8	**26.** 6	**27.** 3	**28.** 2	**29.** 2	**30.** 3
31. 7	**32.** 2	**33.** 3	**34.** 86	**35.** 0	**36.** 6	**37.** 1	**38.** 4	**39.** 1	**40.** 4
41. 3	**42.** 2	**43.** 4	**44.** 3	**45.** 1	**46.** 4	**47.** 4	**48.** 1	**49.** 1	**50.** 4
51. 4	**52.** 8								

ARCHIVE: JEE MAIN

1. A	**2.** C	**3.** B	**4.** D	**5.** A	**6.** D	**7.** D	**8.** B	**9.** B	**10.** A
11. 1	**12.** 2	**13.** 4	**14.** 2	**15.** B	**16.** B	**17.** A	**18.** C	**19.** A	**20.** A
21. B	**22.** B	**23.** B	**24.** A	**25.** C	**26.** C	**27.** D	**28.** D	**29.** D	**30.** B
31. C	**32.** C	**33.** A							

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. A	**2.** A	**3.** B	**4.** A

Multiple Correct Choice Type Questions

5. A, C	**6.** A, C	**7.** A, B	**8.** B, C	**9.** A, D	**10.** B, C	**11.** A, D

Integer/Numerical Answer Type Questions

12. 0.4

HINTS AND EXPLANATIONS

CHAPTER 1: SET AND RELATION

Test Your Concepts-I

1. (i) $A=\left\{\frac{1}{3},\frac{2}{4},\frac{3}{5},\frac{4}{6},\frac{5}{7},\frac{6}{8},\frac{7}{9},\frac{8}{10}\right\}$

A can be written in the set builder form as

$$A=\left\{x : x=\frac{n}{n+2}, n \in N \text{ and } n \le 8\right\}$$

(ii) $B = \{5, 9, 13, 17, 21, \ldots\} = \{n^2 + 1; n \in N\}$

(iii) $C = \{1, 4, 9, 16, \ldots\} = (n^2 : n \in N\}$

(iv) $D = \{5, 7, 11, 13, 17, 19, 23, 29\} = \{x : x$ is a prime number and $5 \le x \le 29\}$

2. $A = \{x : x \in Z$ and $|x| < 5\} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
$B = \{x : x$ is a letter of the word 'upliftmaths'$\} = \{u,p,l,i,f,t,m,a,h,s\}$
$C = \{x : x = 3n - 1 < 20$ and $n \in N\} =\{2, 5, 8, 11, 14,17\}$
$D = \{x : x$ is a month of a year having 30 days$\}=$ {April, June, September, November}

3. (i) $A = (3^n - 2^n : n \in N$ and $1 \le n \le 5\} = \{1, 5, 19, 65, 211\}$

(ii) $B = (a_n : n \in N, a_{n+1} = 2a_n$ and $a_1 = 3) = \{3, 6, 12, 24, \ldots\}$

(iii) $C = \{a_n : n \in N, a_{n+2} = a_{n+1} + a_n, a_1 = 1, a_2 = 2\} = \{1, 2, 3, 5, 8, 13, \ldots\}$

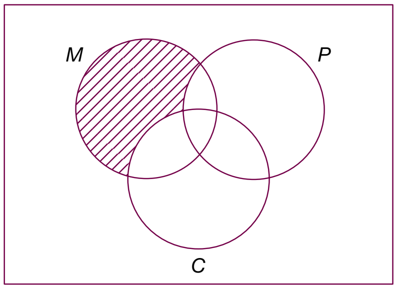
4. $(A \cap B) = \{3, 5\}$
$A - B = A \cap B'$
$A - B = \{1, 6, 7\}, B - A = \{2, 4\}$

5. $A = \{3, 5, 6\}$, $B = \{1, 2, 3\}$, $C = \{2, 4, 6\}$
$A \cap B = \{3\}$, $A \cap C = \{6\}$, $B \cap C = \{2\}$, $B \cup C = \{1, 2, 3, 4, 6\}$
$A \cup B= \{1, 2, 3, 5, 6\}$, $A \cup C = \{2, 3, 4, 5, 6\}$
(i) $(A \cap B) \cup (A \cap C) = \{3, 6\}= A \cap (B \cup C)$
(ii) $(A \cup B) \cap (A \cup C) = \{2, 3, 5, 6\} = A \cup (B \cap C)$

6. $3N = \{3, 6, 9, 12, \ldots\}$
$4N = \{4, 8, 12, 16, \ldots\}$
$\therefore$ $3N \cap 4N = \{12, 24, 36, \ldots\} = 12N$

7. $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A) = (2, 3) \times (2, 3)$
$= \{(2, 2), (2, 3), (3, 2), (3, 3)\}$
$\therefore$ Number of elements = 4.

8. $n(M) = 23, n(P) = 24, n(C) = 19$
$n(M \cap P) = 12, n(M \cap C) = 9, n(P \cap C) = 7$
$n(M \cap P \cap C) = 4.$

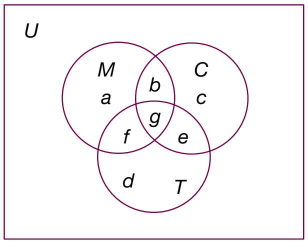


We have to find
$n(M \cap P' \cap C'), n(P \cap M' \cap C'), n(C \cap M' \cap P')$
Now $n(M \cap P' \cap C') = n[M \cap (P' \cap C'),]$
$= n(M) - n(M \cap (P \cup C))$
$= n(M) - n[(M \cap P) \cup (M \cap C)]$
$= n(M) - n(M \cap P) - n(M \cap C) + n(M \cap P \cap C)$
$= 23 - 12 - 9 + 4 = 27 - 21 = 6$
$n(P \cap M' \cap C') = n[P \cap (M \cup C)'] = n(P) - n[P \cap (M \cup C)]$
$= n(P) - n\,[(P \cap M) \cup (P \cap C)]$
$= n(P) - n(P \cap M) - n(P \cap C) + n(P \cap M \cap C)$
$=24 - 12 - 7 + 4 = 9$
$n(C \cap M' \cap P') = n(C) - n(C \cap P) - n(C \cap M) + n(C \cap P \cap M)$
$= 19 - 7 - 9 + 4 = 23 - 16 = 7$

9. Let Total number of students in class $n(U) = 100, n(A) = 45, n(B) = 85$
$\because$ $n(A \cup B) = 100$
$n(B \cap A') = n(B) - n(A \cap B) = n(B) - (n(A) + n(B) - n(A \cup B))$
$= n(A \cup B) - n(A) = 100 - 45 = 55$
Hence, 55% of students offered Biology only.

10. Given $n(U) = 600$
Number of students writing JEE $n(A) = 150$
Number of students writing NEET $n(B) = 225$
$n(A \cup B) = 100 = n(A) + n(B) - n(A \cap B) = 150 + 225 - 100$
$n(A \cup B) = 275$
$n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B) = 600 - 275 = 325$

11.



$n(U) = 100, n(M \cap C \cap T) = 10 = g, n(M \cap C) = 20 = b + g,$
$n(C \cap T) = 30 = e + g, n(M \cap T) = 25 = f + g,$
$n(M \cap C' \cap T') = 12 = a, n(C \cap M' \cap T') = 5 = c,$
$n(M' \cap C' \cap T) = 8 = d$
$g = 10, b = 10, e = 20, f = 15$
$a = 12, c = 5, d = 8$
$a + b + c + d + e + f + g = 12 + 10 + 5 + 8 + 20 + 15 + 10 = 80$
Number of students who did not drink any $= n(U) - n(M \cup C \cup T) = 100 - 80 = 20$

12. $X = \{8^n - 7n - 1 \mid n \in N\} = \{0, 49, 490, \ldots\}$
$Y = \{49n - 49 - 1 \mid n \in N\} = \{0, 49, 98, 147, \ldots\}$
Clearly, every elements of X is in Y but every element of Y is not in X.
$\therefore$ $X \subset Y$

13. $B \cap C = \{4\}$
$\therefore$ $A \cup (B \cap C) = \{1, 2, 3, 4\}.$

Test Your Concepts-II

1. $R \times (P^c \cup Q^c)^c = R \times [(P^c)^c \cap (Q^c)^c] = R \times (P \cap Q) = (R \times P) \cap (R \times Q)$

2. (a) Since, $(q, b) \in R_1$ but $(q, b) \notin A \times B$. So, $R_2 \not\subset A \times B$.
Thus, R_1 is not a relation from A to B.
(b) Clearly, $R_2 \subseteq A \times B$. So R_2 is a relation from A to B.
(c) R_3 is not a relation from A to B, because (q, a) and (s, b) are elements of R_3 but (q, a) and (s, b) are not in $A \times B$. As such $R_3 \not\subset A \times B$.

3. R be a relation on N defined by $x + 2y = 8$.
$\therefore$ $R\{(2,3); (4,2); (6,1)\}$
Hence, Domain of $R = \{2,4,6\}$.

4. Since, $A \subseteq A$
$\therefore$ Relation '$\subseteq$' is reflexive.
Since, $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$
$\therefore$ Relation '$\subseteq$' is transitive.
But $A \subseteq B \Rightarrow B \subseteq C$
$\therefore$ Relation is not symmetric.

5. Since, the two finite sets have m and n elements.

$\therefore$ No. of subsets of these sets will be 2^m and 2^n, respectively.

$\therefore$ $2^m - 2^n = 56$

Putting $m = 6, n = 3$, we get

$2^6 - 2^3 = 64 - 8 = 56$

Hence, $m = 6, n = 3$

6. $|a - a| = 0 < 1$

$\therefore$ $a\,R\,a \;\forall\; a \in R$

$\therefore$ R is symmetric.

Again, $a\,R\,b \Rightarrow |a - b| \leq 1 \Rightarrow |b - a| \leq 1$

$\Rightarrow$ $b\,R\,a$

$\therefore$ R is symmetric.

Again $1R\frac{1}{2}$ and $\frac{1}{2}$ $R1$ but $\frac{1}{2} \neq 1$

$\therefore$ R is not anti-symmetric.

Further, $1R2$ and $2R3$ but $1 \not R 3$ $[\because |1 - 3| = 2 > 1]$

$\therefore$ R is not transitive.

7. Since, $x < y, y < z \Rightarrow x < z \;\forall\; x, y, z \in N$

$\therefore$ $x\,R\,y, y\,R\,z \Rightarrow x\,R\,z$

$\therefore$ Relation is transistive.

$\because$ $x < y$ does not give $y < x$

$\therefore$ Relation is not symmetric and

$\because$ $x < x$ does not hold,

$\therefore$ Relation is not reflexive.

8. $n(A) = 3, n(B) = 2$, and $(x, 1), (y, 2), (z, 1) \in A \times B$

$\because$ $x, y, z \in A$ and $n(A) = 3$

Hence, set $A = \{x, y, z\}$ and set $B = \{1, 2\}$

9. $(a, p), (b, q), (c, p) \in A \times B$, Given that $A = \{a, b, c\}$

$\Rightarrow$ $B = \{p, q\}$

$A \times B = \{a, b, c\} \times \{p, q\}$

$A \times B = \{(a, p), (a, q), (b, p), (b, q), (c, p), (c, q)\}$

Hence, remaining elements in $A \times B$ are $\{(a, q), (b, p), (c, q)\}$

10. $(a, 2), (b, 3), (c, 5) \in A \times B$

$n(A \times B)$ is least and $n(B) = 3$

So, $A = \{a, b, c\}$ and $n(A) = 5$

Hence, remaining elements of $A \times B$ are $\{(a, 3), (a, 5), (b, 2), (b, 5), (c, 2), (c, 3)\}$

11. $xRy \Leftrightarrow x$ is relatively prime to y.

$R = \{(2, 3), (2, 7), (3, 7), (3, 10), (4, 3), (4, 7), (5, 3), (5, 7)\}$

Hence, domain of $R = \{2, 3, 4, 5\}$

12. $A \cap B = \{4\}$ and $A \cup B = \{2, 3, 4, 5\}$

$\therefore$ $(A \cap B) \times (A \cup B) = \{(4, 2), (4, 3), (4, 4), (4, 5)\}$

13. $n(A) = 4, n(B) = 3$

$n(A) \times n(B) \times n(C) = n(A \times B \times C)$

$4 \times 3 \times n(C) = 24 \Rightarrow n(C) = 24/12 = 2.$

Single Correct Choice Type Questions

1. Given $A \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$. Hence, $A = \{3, 5, 9\}$

Hence, the correct answer is (B).

2. $A \cap B \subseteq A$. Hence, $A \cup (A \cap B) = A$.

Hence, the correct answer is (A).

3. If $A = \{a, b, c\}, B = \{b, c, d\}$ and $C = \{a, d, c\}$

$A - B = \{a\}, B \cap C = \{c, d\}$

Then, $(A - B) \times (B \cap C) = \{a\} \times \{c, d\} = \{(a, c), (a, d)\}$

Hence, the correct answer is (A).

4. $(1, 1) \notin R \Rightarrow R$ is not reflexive $(1, 2) \in R$ but $(2, 1) \notin R$

$\Rightarrow$ R is not symmetric. But R is transitive.

Hence, the correct answer is (C).

5. x is sister of y, y is sister of z.

$\Rightarrow$ x is sister of z.

$xRy, yRz \Rightarrow xRz.$

Hence, the correct answer is (C).

6. Let $(a, b) \in R$

Then, $(a, b) \in R \Rightarrow (b, a) \in R^{-1}$, [By def. of R^{-1}]

$\Rightarrow$ $(b, a) \in R$, $[\because R = R^{-1}]$

So, R is symmetric.

Hence, the correct answer is (B).

7. $(1, 1), (2, 2), (3, 3) \in R \Rightarrow R$ is reflexive

$(1, 2) \in R$ but $(2, 1) \notin R$ is not symmetric

$(1, 2) \in R, (2, 3) \in R, (1, 3) \in R \Rightarrow R$ is transitive.

Hence, the correct answer is (A).

8. Let $a = 1$

$\therefore$ $|a^2 - b^2| \leq 5 \Rightarrow |b^2 - 1| \leq 5 \Rightarrow b = 1, 2$

Let $a = 2$

$\therefore$ $|a^2 - b^2| \leq 5 \Rightarrow |4 - b^2| \leq 5 \Rightarrow |b^2 - 4| \leq 5$

$\Rightarrow$ $b = 1, 2, 3$

Let $a = 3$

$\therefore$ $|a^2 - b^2| \leq 5 \Rightarrow |9 - b^2| \leq 5 \Rightarrow |b^2 - 9| \leq 5$

$\Rightarrow$ $b = 2, 3$

$\therefore$ $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$

$R^{-1} = \{(y, x), (x, y) \in R\}$

$= \{(1, 1), (2, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)) = R$

Domain of $R = \{x : (x, y) \in R\} = \{1, 2, 3\}$

Range of $R = \{y : (x, y) \in R\} = \{1, 2, 3\}$

Hence, the correct answer is (D).

9. We have $R = \{(x, y) : |x^2 - y^2| < 16\}$

Let $x = 1$

$\therefore$ $|x^2 - y^2| < 16$

$\Rightarrow$ $|1 - y^2| < 16$

$\Rightarrow$ $|y^2 - 1| < 16$

$\Rightarrow$ $y - 1, 2, 3, 4$

Let $x = 2$

$\therefore$ $|x^2 - y^2| < 16$

$\Rightarrow$ $|4 - y^2| < 16$

$\Rightarrow$ $|y^2 - 4| < 16$

$\Rightarrow$ $y = 1, 2, 3, 4$

Let $x = 3$

$\therefore$ $|x^2 - y^2| < 16$

$\Rightarrow$ $|9 - y^2| < 16$

$\Rightarrow$ $|y^2 - 9| < 16$

$\Rightarrow$ $y = 1, 2, 3, 4$

Let $x = 4$

$\Rightarrow$ $|x^2 - y^2| < 16$

$\therefore$ $|16 - y^2| < 16$

$\Rightarrow$ $|y^2 - 16| < 16$

$\Rightarrow$ $y = 1, 2, 3, 4, 5$

Let $x = 5$

$\Rightarrow$ $|25 - y^2| < 16$

$\therefore$ $|y^2 - 25| < 16$

$\Rightarrow$ $y = 4, 5$

$\therefore$ $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$

Hence, the correct answer is (D).

10. $A = \{2, 4, 6\}; B = \{2, 3, 5\}$

$\therefore$ $A \times B$ contains $3 \times 3 = 9$ elements.

Hence, number of relations from A to $B = 2^9$.

Hence, the correct answer is (A).

11. $R = \{(2, 1), (4, 2), (6, 3), \ldots\}$

So $R^{-1} = \{(1, 2), (2, 4), (3, 6), \ldots\}$

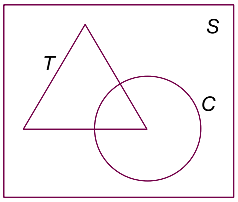
Hence, the correct answer is (B).

12. (A) Clearly $R_1 \subseteq A \times B$. So, it is relation from A to B.
(B) Clearly $R_2 \subseteq A \times B$. So, it is a relation from A to B.
(C) Clearly $R_3 \subseteq A \times B$. So, it is a relation from A to B.
(D) Since, $(4, 2) \in R_4$ but $(4, 2) \notin A \times B$. So, R_4 is not a relation from A to B.
Hence, the correct answer is (D).

13. Let A denote the set of Americans who like cheese.
Let B denote the set of Americans who like apples.
Let Population of America be 100.
Then, $n(A) = 63$, $n(B) = 76$
Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 63 + 76 - n(A \cap B)$
$\therefore \quad n(A \cup B) + n(A \cap B) = 139$
$\Rightarrow \quad n(A \cap B) = 139 - n(A \cup B)$
But $n(A \cup B) \le 100$
$\therefore \quad -n(A \cup B) \ge -100$
$\Rightarrow \quad 139 - n(A \cup B) \ge 139 - 100 = 39$
$\Rightarrow \quad n(A \cap B) \ge 39$ i.e, $39 \le n(A \cap B)$ …(1)
Now, $A \cap B \subseteq A$, $A \cap B \subseteq B$
$\Rightarrow \quad n(A \cap B) \le n(A) = 63$ and $n(A \cap B) \le n(B) = 76$
$\Rightarrow \quad n(A \cap B) \le 63$ …(2)
Then, $39 \le n(A \cap B) \le 63 \Rightarrow 39 \le x \le 63$.
Hence, the correct answer is (C).

14. On the set N of natural numbers.
$R = \{(x, y) : x, y \in N, 2x + y = 41\}$.
Since, $(1, 1) \notin R$ as $2.1 + 1 = 3 \notin 41$. So, R is not reflexive.
$(1, 39) \in R$ but $(39, 1) \notin R$. So R is not symmetric $(20, 1), (1, 39) \notin R$,
So R is not transitive.
Hence, the correct answer is (D).

15. The given sets can be represented in Venn diagram as shown below:



It is clear from the diagram that $S \cup T \cup C = S$
Hence, the correct answer is (C).

16. If elements are not repeated, then number of elements in $A_1 \cup A_2 \cup A_3 \ldots A_{30}$ is 30×5.
But each elements is used 10 times, so $S = \dfrac{30 \times 5}{10} = 15$
If elements in $B_1, B_2, \ldots, B_n$ are not repeated, then total number of elements is $3n$ but each element is repeated 9 times,
So, $S = \dfrac{3n}{9} \Rightarrow 15 = \dfrac{3n}{9}$
Thus, $n = 45$.
Hence, the correct answer is (C).

17. $A' \cup [(A \cup B) \cap B']$ $[\because A \cap (B \cup C) = (A \cap B) \cup (A \cup C)]$
$\Rightarrow \quad A' \cup [(A \cap B') \cup (B \cap B')]$
$\Rightarrow \quad A' \cup [(A \cap B') \cup \phi] = A' \cup (A \cap B')]$ $[\because (A \cap B) = \phi]$
$= A' \cup A$ $(\because A \cap B' = A)$
$= N$
Hence, the correct answer is (B).

18. $A = \{1, 2, 3\}$.
Now, number of equivalence relations as follows:
$R_1 = \{(1, 1), (2, 2), (3, 3)\}$; $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$
$R_3 = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$; $R_4 = \{(1, 1), (2, 2), (3, 3), (2, 3), (3, 2)\}$
$R_5 = \{(1, 2, 3) \Leftrightarrow A, A, A^2\}$
$\therefore$ Maximum number of equivalence relation on the set $A = \{1, 2, 3\} = 5$
Hence, the correct answer is (D).

19. $|a - a| = 0 < 1$
$\therefore \quad aRa \ \forall \in R$
$\therefore \quad R$ is reflexive.
Again, $a \, R \, b \Rightarrow |a - b| \le 1 \Rightarrow |b - a| \le 1 \Rightarrow bRa$
$\therefore \quad R$ is not anti-symmetric.
Further, $1 \, R \, 2$ and $2R \, 3$ but $1 \, R3$, $[\because \ |1 - 3| = 2 > 1]$
$\therefore \quad R$ is not transitive.
Hence, the correct answer is (A).

20. Domain of $R = \{2, 3, 4, 5, 6\}$
Range of R $\{2, 5, 10, 17, 26\}$
$R = \{(2, 2), (3, 5), (4, 10), (5, 17), (6, 26)\}$.
Hence, the correct answer is (A).

21. It is obvious.
Hence, the correct answer is (D).

22. $R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
$R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$
$RoR^{-1} = \{(3, 3), (3, 5), (5, 3), (5, 5)\}$
Hence, the correct answer is (C).

23. $T = \{(x, y) : x - y \in I\}$ as $0 \in I$, T is a reflexive relation.
If $x - y \in I \, y - x \in I$
$\therefore \quad T$ is symmetrical also.
If $x - y = I_1$ and $y - z = I_2$
Then, $x - z = (x - y) + (y - z) = I_1 + I_2 \in I$
$\therefore \quad T$ is also transitive
Hence, T is an equivalence relation.
Clearly $x \ne x + 1 \Rightarrow (x, x) \notin S$
$\therefore \quad S$ is not reflexive.
Hence, the correct answer is (C).

24. $a + b + c = 21$ and $b = \dfrac{a + c}{2}$
$\Rightarrow \quad a + c = 14$ and $b = 7$
So, a can take values from 1 to 6, when c range from 13 to 8, or $a = b = c = 7$
So, 7 triplets.
Hence, the correct answer is (B).

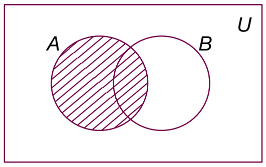
25. $X \cap A = Y \cap A = \phi$
$\therefore \quad X$ and A, Y and A are disjoint sets.
Also, $X \cup A = Y \cup A \Rightarrow X = Y$.
Hence, the correct answer is (C).

26. $\because \quad R = \{(x, y) \mid x, y \in Z, x^2 + y^2 \le 4\}$
$\therefore \quad R = \{(-2, 0), (-1, 0), (-1, 1), (0, -1), (0, 1), (0, 2), (0, -2), (1, -1), (0, 0), (1, 0), (1, 1), (2, 0)\}$
Domain of $R = \{-2, -1, 0, 1, 2\}$.
Hence, the correct answer is (C).

27. $n(A) = 4$
$n(B) = 2$
$n(A \times B) = 8$
So, at least 3 = total – none – one – two digits
$= 2^8 - 1 - {}^8C_1 - {}^8C_2 = 2^8 - 1 - 8 - 28 = 2^8 - 37$
$= 256 - 37 = 219$.
Hence, the correct answer is (A).

28. Every rectangle, rhombus, square in a plane is a parallelogram but every trapezium is not a parallelogram. So, F_1 is either of F_1, F_2, F_3 and F_4.
$\therefore \quad F_1 = F_2 \cup F_3 \cup F_4 \cup F_1$
Hence, the correct answer is (D).

29.



$A \cap (A \cup B)$

$A \cap (A \cup B) = A$

Hence, the correct answer is (A).

30. $X \cap (X \cap Y)' = X \cap (X' \cap Y')$ $[\because\ (A \cup B)' = A' \cap B']$

$\Rightarrow\quad (X \cap X') \cap (X \cap Y')$

$\Rightarrow\quad \phi \cap (X \cap Y') = \phi$ $[\because\ \phi \cap A = \phi.]$

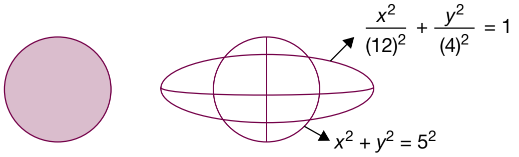
Hence, the correct answer is (C).

31. $A = \{4, 8, 12, 16, 20, 24, \ldots\}$; $B = \{6, 12, 18, 24, 30, \ldots\}$

$\therefore\quad A \subset B = \{12, 24, \ldots\} = \{x : x$ is a multiple of $12\}$.

Hence, the correct answer is (B).

32.



A = Set of all values $(x, y) : x^2 + y^2 = 25 = 5^2$

$$B = \frac{x^2}{114} + \frac{y^2}{16} = 1$$

i.e., $\frac{x^2}{(12)^2} + \frac{y^2}{(4)^2} = 1$

Clearly, $A \cap B$ consists of four points.

Hence, the correct answer is (D).

33. $xRx \Leftrightarrow x - x + \sqrt{2} = \sqrt{2}$ is an irrational number.

R is a reflexive relation.

$xRx \Leftrightarrow x - y + \sqrt{2}$ is an irrational number.

But it does not mean $y - x + \sqrt{2}$ is an irrational number.

For example, $\sqrt{2}\ R - 1 \Leftrightarrow \sqrt{2} - 1 + \sqrt{2} = 2\sqrt{2} = 1 \notin Q$

But $1 - \sqrt{2} + \sqrt{2} = 1 \notin Q$

i.e., $1\ \not R\ \sqrt{2}$

$\Rightarrow\quad R$ is not symmetric.

$xRy \Leftrightarrow x - y + \sqrt{2} \notin Q$

$yRz \Leftrightarrow y - z + \sqrt{2} \notin Q$

$(x - y + \sqrt{2}) + (y - z + \sqrt{2}) = x - z + 2\sqrt{2}$

Addition of two irrational can be rational or irrational number.

$\Rightarrow\quad R$ is not transitive.

Hence, the correct answer is (A).

34. $(6, 12) \in R$ but $(12, 6) \notin R \Rightarrow$ is not symmetric.

Hence, the correct answer is (C).

35. $a \in N$ then, $a \not< a$ R is not reflexive

$A, b \in N$ if $a < b$ then, $b \not< a \Rightarrow R$ is not symmetric.

$a, b, c \in N, a < b, b < c \Rightarrow a < c$

$\Rightarrow\quad a\,R\,b, b\,R\,c \Rightarrow a\,R\,c$

$\Rightarrow\quad R$ is transitive.

Hence, the correct answer is (B).

36. $a \geq a \Rightarrow aRa \Rightarrow = R$ is a reflexive relation.

$2R1 \Rightarrow 1\ \not R\ 2 \Rightarrow$ is not symmetric relation.

$a \geq b, b \geq c \Rightarrow a \geq c$

$aRb, bRc \Rightarrow aRc \Rightarrow R$ is transitive.

Hence, the correct answer is (B).

37. $a \in R \Rightarrow aRa \Rightarrow R$ is reflexive.

$(a, b) \in R \Leftrightarrow (b, a) \in R \Rightarrow R$ is symmetric.

$(2, 3) \in R, (3, 5) \in R$. But $(2, 5) \notin R$

$\Rightarrow\quad R$ is not transitive.

Hence, the correct answer is (B).

38. Given, $A = \{1, 2, 3, 4\}$

$R = \{(1, 3), (4, 2), (2, 4) (2, 3), (3, 1)\}$

$(2, 3) \in R$ but $(3, 2) \notin R$. Hence, R is not symmetric.

R is not reflexive as $(1, 1) \notin R$.

R is not a function as $(2, 4) \in R$ and $(2, 3) \in R$.

R is not transitive as $(1, 3) \in R$ and $(3, 1) \in R$ but $(1, 1) \notin R$.

Hence, the correct answer is (B).

39. $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 12 + 9 - 4 = 17$

Now, $n((A \cap B)^c) = n(U) - n(A \cup B) = 20 - 17 = 3$.

Hence, the correct answer is (D).

40. $A \cap (A \cap B)^c = A \cap (A^c \cup B^c)$

$= (A \cap A^c) \cup (A \cup B^c) = \phi \cup (A \cup B^c) = A \cap B^c$

Hence, the correct answer is (D).

41. Let set of students who opt cricket = A

Let set of student who opt Football = B

Let set of students who opt Hockey = C

$\therefore\quad n(A) = 40, n(B) = 16, n(C) = 11$

$n(A \cap B) = 8, n(A \cap C) = 6, n(B \cap C) = 2,$

$n(C \cap F \cap H) = 1$

$\therefore\quad n(A \cup B \cup C) = 40 + 16 + 11 - 8 - 6 - 2 + 1 = 52$

$\therefore\quad (A' \cup B' \cup C') = 80 - 52 = 28.$

Hence, the correct answer is (C).

42. G.C.D. of a and b is 2 and G.C.D. of b and a is also 2.

So, it is a symmetric only.

Hence, the correct answer is (B).

43. $\sin^2 a + \cos^2 b = 1$

Reflexive: $\sin^2 a + \cos^2 a = 1 \Rightarrow aRa$

$\sin^2 a + \cos^2 b = 1, 1 - \cos^2 a + 1 - \sin^2 b = 1$

$\sin^2 b + \cos^2 a = 1 \Rightarrow bRa$

Symmetric: Let aRb, bRc

$$\sin^2 a + \cos^2 b = 1 \quad \ldots(1)$$

$$\sin^2 b + \cos^2 c = 1 \quad \ldots(2)$$

Eq. (1) + (2), we get

$\sin^2 b + \cos^2 c = 1$

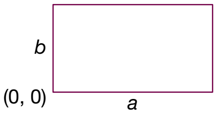
Thus, R is transitive; therefore, R is equivalence relation.

Hence, the correct answer is (D).

44. Since, R be the set of point inside the rectangle.

$\therefore\quad R = \{(x, y) : 0 < x < a$ and $0 < y < b\}$

Hence, the correct answer is (D).



45. Let H be set of persons who read Hindi and E be the set of persons who read English. Then,

$n(U) = 840, n(H) = 450, n(E) = 300, n(H \cap E) = 200$

Number of persons who read neither = $n(H' \cap F')$

$\Rightarrow\quad n(H \cup E)'$

$= n(U) - n(H \cap E)$

$= 840 - [n(H) + n(E) - n(H \cap E)]$

$= 840 - (450 + 300 - 200) = 840 - 550 = 290$

Hence, the correct answer is (B).

46. Consider that aRb, if a is congruent to b, $\forall\ a, b \in T$.

Then, $aRa \Rightarrow a \cong a$, which is true for all $a \in R$

So, R is reflexive,

Let $aRb \Rightarrow a \cong b$ $\ldots(1)$

$\Rightarrow\quad b \cong a \Rightarrow bRa$

So, R is symmetric.

Let aRb and $bRc \Rightarrow a \cong b$ and $b \cong c \Rightarrow a \cong c \Rightarrow aRc$

So, R is transitive. Therefore, R is equivalence relation.

Hence, the correct answer is (C).

47. $A - B = A \cap B^c = A \cap \overline{B}$.

Hence, the correct answer is (B).

48. Clearly, $A = \{2, 3\}$, $B = \{2, 4\}$, $C = \{4, 5\}$
$B \cap C = \{4\}$
$\therefore \quad A \times (B \cap C) = \{(2, 4), (3, 4)\}$.
Hence, the correct answer is (A).

49. $B \cap C = \{4\}$
$\therefore \quad A \cup (B \cap C) = \{1, 2, 3, 4\}$.
Hence, the correct answer is (B).

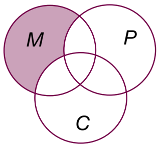
50. $P = \{(a, b) : a, b \in N, a - b = 3\} = \{((n + 3), n) : n \in N\}$
$= \{(4, 1), (5, 2), (6, 3), \ldots.\}$
Hence, the correct answer is (B).

51. $r = \{(a, b) \mid a, b \in R$ and $a - b + \sqrt{3}$ is an irrational no.$\}$
Then, r is reflexive as a $Ra = a - a + \sqrt{3} = \sqrt{3}$ which is an irrational no.
$\sqrt{3}r1 = \sqrt{3} - 1 + \sqrt{3} = 2\sqrt{3} - 1$ which is an irrational number but
$1r\sqrt{3} = 1 - \sqrt{3} + \sqrt{3} = 1$ which is not irrational number.
$\therefore \quad r$ is not symmetric. Also, r is not transitive
$\because \quad \sqrt{3}\,r1$ and $1\,r\,2\sqrt{3}$ but $\sqrt{3}\,r\,2\sqrt{3}$
Therefore, r is not an equivalence relation.
Hence, the correct answer is (B).

Integer/Numerical Answer Type Questions

1. $n(A^c \cap B^c) = n[(A \cup B)^c] = n(\text{U}) - n(A \cup B)$
$= n(\text{U}) - [n(A) + n(B) - n(A \cap B)]$
$= 700 - [200 + 300 - 100] = 300$.
Hence, the correct answer is (300).

2. $n(M \text{ alone}) = n(M) - n(M \cap B) - n(M \cap P) + n(M \cap P \cap C)$
$= 100 - 28 - 30 + 18 = 60$.

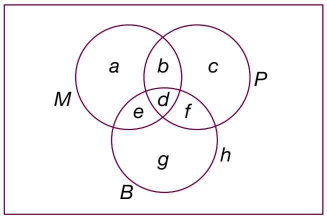


Hence, the correct answer is (60).

3. $n(A) = 4$, $n(B) = 5$, $n(A \cap B) = 3$
$n(A \times B) \cap (B \times A) = n[(A \cap B) \times (B \cap A)]$
$[\because n(A \times B) = n(A) \times n(B)]$
$= n(A \cap B) \times n(B \cap A) = 3 \times 3 = 9$.
Hence, the correct answer is (9).

4. $n(C) = 224$, $n(H) = 240$, $n(B) = 336$
$n(H \cap B) = 64$, $n(B \cap C) = 80$
$n(H \cap C) = 40$, $n(C \cap H \cap B) = 24$
$n(C^c \cap \text{H}^c \cap B^c) = n[(C \cap H \cap B)^c]$
$= n(U) - n(C \cap H \cap B)$
$= 800 - [n(C) + n(H) + n(B) - n(H \cap C) - n(H \cap B) - n(C \cap B) + n(C \cap H \cap B)]$
$= 800 - [224 + 240 + 336 - 64 - 80 - 40 + 24]$
$= 800 - 640 = 160$.
Hence, the correct answer is (160).

5. $a + b + c + d + e + f + g + h = 100$ …(1)



$h = 1$ …(2)
$a + b + d + e = 50$ …(3)
$b + c + d + f = 45$ …(4)
$d + e + f + g = 40$ …(5)
$b + e + f = 32$ …(6)
On solving, we get $d = 2$.
Hence, the correct answer is (2).

6. $R = \{(2, 2), (3, 3), (2, 3), (3, 2), (3, 4), (4, 3), (4, 4)\}$
Hence, the correct answer is (5).

7. $R = \{(1, 2), (2, 3), (1, 1), (2, 2), (2, 1), (3, 2), (1, 3), (3, 1), (3, 3)\}$
Hence, the correct answer is (7).

8. Since $A \subseteq B$, $\quad \therefore A \cup B = B$
So, $n(A \cup B) = n(B) = 6$.
Hence, the correct answer is (6).

9. Minimum value of $x = 100 - (30 + 20 + 25 + 15)$
$= 100 - 90 = 10$.
Hence, the correct answer is (10).

10. The number of non-empty subsets $= 2^n - 1$
$2^4 - 1 = 16 - 1 = 15$.
Hence, the correct answer is (15).

11. Given set is $\{(a, b) : 2a^2 + 3b^2 = 35, a, b \in Z\}$
We can see that,
$2(\pm 2)^2 + 3(\pm 3)^2 = 35$
And $2(\pm 4)^2 + 3(\pm 1)^2 = 35$
$\therefore \quad (2, 3), (2, -3), (-2, -3), (-2, 3), (4, 1), (4, -1), (-4, -1), (-4, 1)$, are 8 elements of the set.
$\therefore \quad n = 8$.
Hence, the correct answer is (8).

12. $S = \{1, 2, 3, 4\}$
Each element can be put in 3 ways either in subsets or we don't put in any subset.
So that number of unordered pairs
$= \dfrac{3 \times 3 \times 3 \times 3 - 1}{2} + 1 = 41$. [Both subsets can be empty also]
Hence, the correct answer is (41).

13. Every element has 3 options. Either set Y or set Z or none. So number of ordered pairs $= 3^5$.
Hence, the correct answer is (243).

14. $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - n(A \cap B)$
Since, maximum number of elements in $A \cap B = 3$.
$\therefore$ Minimum number of elements in $A \cup B = 9 - 3 = 6$.
Hence, the correct answer is (6).

15. $n(A) = 40\%$ of $10{,}000 = 4{,}000$;
$n(B) = 20\%$ of $10{,}000 = 2{,}000$;
$n(C) = 10\%$ of $10{,}000 = 1{,}000$
$n(A \cap B) = 5\%$ of $10{,}000 = 500$;
$n(B \cap C) = 3\%$ of $10{,}000 = 300$;
$n(C \cap A) = 4\%$ of $10{,}000 = 400$
$n(A \cap B \cap C) = 2\%$ of $10{,}000 = 200$
We want to find $n(A \cap B^c \cap C^c) = n[A \cap (B \cup C)^c]$
$= n(A) - n[A \cap (B \cup C)]$
$= n(A) - n[(A \cap B) \cup (A \cap C)]$
$= n(A) - [n(A \cap B) + n(A \cap C) - n(A \cap (B \cup C)]$
$= 4000 - [500 + 400 - 200] = 4000 - 700 = 3300$
Hence, the correct answer is (3300).

16. $\because S = \{x \mid x$ is a positive multiple of 3 less than 100$\}$
$\therefore n(\text{S}) = 33$ and $P = \{x \mid x$ is a prime number less than 20$\}$
$\therefore n(\text{P}) = 8$; $n(\text{S}) + n(\text{P}) = 33 + 8 = 41$.
Hence, the correct answer is (41).

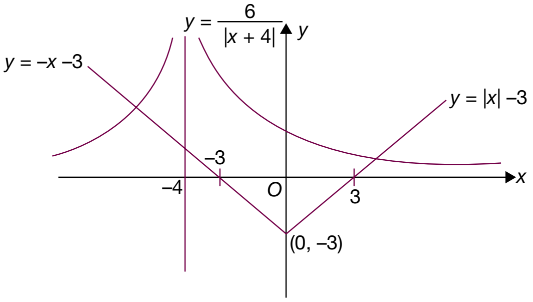
ARCHIVE: JEE MAIN

1. B and C will contain three digit numbers of the form $9k + 2$ and $9k + l$ respectively. We need to find sum of all elements in the set $B \cup C$ effectively.
Now, $S(B \cup C) = S(B) + S(C) - S(B \cap C)$
where $S(k)$ denotes sum of elements of set k.
Also, $B = \{101, 109, \ldots, 992\}$
$\therefore \quad S(B) = \frac{100}{2}(101 + 992) = 54650$
CASE 1: If $l = 2$, then $B \cap C = B$
Which is not possible as given sum is $274 \times 400 = 109600$.
CASE 2: If $l = 2$, then $B \cap C = \varphi$
$\therefore \quad S(B \cup C) = S(B) + S(C) = 400 \times 274$
$\Rightarrow \quad 54650 + \sum_{k=11}^{110} 9k + l = 109600$
$\Rightarrow \quad 9\sum_{k=11}^{110} k + \sum_{k=11}^{110} l = 54950$
$\Rightarrow \quad 9\left(\frac{100}{2}(11+110)\right) + l(100) = 54950$
$\Rightarrow \quad 54450 + 100l = 54950$
$\Rightarrow \quad l = 5$
Hence, the correct answer is (5).

2. $A = (-\infty, 1) \cup (3, \infty)$
$B = (-\infty, -2) \cup (2, \infty)$
$C = (-\infty, 2] \cup [6, \infty)$
So, $A \cap B \cap C = (-\infty, -2)\ [6, \infty)$
$z \cap (A \cap B \cap C)' = \{-2, -1, 0, -1, 2, 3, 4, 5\}$
Hence no. of its subsets $= 2^8 = 256$
Hence, the correct answer is (256).

3. $B = \{4, 7, 10, 13, 16, \ldots\}$; $C = \{2, 4, 6, 8, 10, \ldots\}$
$B - C \equiv \{7, 13, 19, \ldots 97, \ldots\}$
Now, $n^2 - n \le 100 \times 100$
$\Rightarrow \quad n(n-1) \le 100 \times 100$
$\Rightarrow \quad A = \{1, 2, \ldots, 100\}$.
So, $A \cap (B - C) = \{7, 13, 19, \ldots, 97\}$
Hence, sum $= \frac{16}{2}(7 + 97) = 832$
Hence, the correct answer is (832).

4. $x \ne -4$
$(|x| - 3)(|x + 4|) = 6$



$\Rightarrow |x| - 3 = \frac{6}{|x+4|}$
No. of solution = 2
Hence, the correct answer is (B).

5. $A = \{2, 3, 4, 5, \ldots, 30\}$
$(a, b) = (c, d) \quad \Rightarrow \quad ad = bc$
$(4, 3) = (c, d) \quad \Rightarrow \quad 4d = 3c$
$\Rightarrow \quad \frac{4}{3} = \frac{c}{d}$
$\frac{c}{d} = \frac{4}{3}$ and $c, d \in \{2, 3, \ldots, 30\}$
$(c, d) = \{(4, 3), (8, 6), (12, 9), (16, 12), (20, 15), (24, 18), (28, 21)\}$
No. of ordered pair = 7
Hence, the correct answer is (D).

6. $A \cap B \cap C$ is visible in all three venn diagram
Hence, the correct answer is (D).

7. $x^3 - 3x^2y - xy^2 + 3y^3 = 0$
$\Rightarrow \quad x(x^2 - y^2) - 3y(x^2 - y^2) = 0$
$\Rightarrow \quad (x - 3y)(x^2 - y^2) = 0$
Now, $x = y\ \forall (x, y) \in N \times N$ so reflexive
But not symmetric and transitive.
See, (3, 1) satisfies but (1, 3) does not. Also (3, 1) and (1, –1) satisfies but (3, –1) does not.
Hence, the correct answer is (A).

8. $n(A) = 25$, $n(B) = 7$, $n(A \cap B)$ = multiple of 14 between 1 and 50 = 3
$n(A \cup B) = 25 + 7 - 3 = 29$
Hence, the correct answer is (29).

9. Given, condition $2^m - 2^n = 112$
$m = 7$, $n = 4$ is a suitable solution
Now, $m \times n = 7 \times 4 = 28$
Hence, the correct answer is (28).

10. $A : x \in (-2, 2)$; $B : x \in (-\infty, -1] \cup [5, \infty)$
$A \cap B = (-2, -1]$, $A \cup B = (-\infty, 2) \cup [5, \infty) = R - [2,5)$
$B - A = (-\infty, -2] \cup [5, \infty) = R - (-2, 5)$
$A - B = (-1, -2)$
Hence, the correct answer is (D).

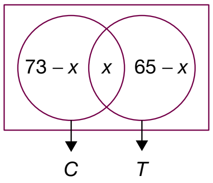
11. $R = \{(x, y) : x, y \in z, x^2 + 3y^2 \le 8\}$
For domain of R^{-1}
Collection of all integral of y's
For $x = 0$, $3y^2 \le 8 \Rightarrow y \in \{-1, 0, 1\}$ = domain of R^{-1} = Range of R
Hence, the correct answer is (B).

12. Let $a^2 + b^2 \in Q$ and $b^2 + c^2 \in Q$
E.g., $a = 2 + \sqrt{3}$ and $b = 2 - \sqrt{3}$ and $c = \left(1 + 2\sqrt{3}\right)$
$a^2 + b^2 = 14 \in Q$ and $b^2 + c^2 = 20 \in Q$
But $a^2 + c^2 = \left(2 + \sqrt{3}\right)^2 + \left(1 + 2 + \sqrt{3}\right)^2 \notin Q$
Hence, R_1 is not transitive.
For R_2: Let $a^2 = 1$, $b^2 = \sqrt{3}$ and $c^2 = 2$
$a^2 + b^2 \notin Q$ and $b^2 + c^2 \notin Q$ but $a^2 + c^2 \in Q$
Hence, R_2 is not transitive
Hence, the correct answer is (D).

13. $n(X_i) = 10$, $\overset{50}{\underset{i=1}{U}} X_i = T, \Rightarrow n(T) = 500$
Each element of T belongs to exactly 20 elements of X_i
$\Rightarrow \quad \frac{500}{20} = 25$ distinct elements
Similarly, for Y_i we have $\frac{5n}{6} = 25 \Rightarrow n = 30$
Hence, the correct answer is (D).

14. $n(B) \le n(A \cup B) \le n(U)$
$\Rightarrow \quad 76 \le 76 + 63 - x \le 100$
$\Rightarrow \quad -63 \le -x \le -39 \Rightarrow 63 \ge x \ge 39$
Hence, the correct answer is (D).

15. C person like coffee and T person like Tea.



$n(C) = 73$; $n(T) = 65$
$n(C \cup T) \le 100$
$n(C) + n(T) - n(C \cap T) \le 100$
$73 + 65 - x \le 100$
$x \ge 38$
$73 - x \ge 0 \Rightarrow x \le 73$

$65 - x \geq 0 \Rightarrow x \leq 65$ final acceptable values are $38 \leq x \leq 65$

Hence, the correct answer is (D).

16. Maths = $\{2, 4, 6, 8, \ldots, 138, 140\} \Rightarrow n(M) = 70$

Physics = $\{3, 6, 9, \ldots, 138\} \Rightarrow n(P) = 46$

Maths and Physics = $\{6, 12, 18, \ldots, 138\} \Rightarrow n(M \cap P) = 23$

Chem. = $\{5, 10, 15, 20, \ldots, 140\} \Rightarrow n(C) = 28$

Maths and Chem = $\{10, 20, \ldots, 140\} \Rightarrow n(M \cap C) = 14$

Phy and Chem = $\{15, 30, \ldots, 135\} \Rightarrow n(P \cap C) = 9$

All three subject = $\{30, 60, \ldots, 120\} = 4 \Rightarrow n(P \cap C \cap M) = 4$

Number of students without any subject opted

$= n(\cup) - n(P \cup C \cup M)$

$= 140 - 70 - 46 - 28 + 23 + 14 + 9 - 4 = 38$

Hence, the correct answer is (D).

17. Total = 2^{100}

Product is even when atleast one element of subset is even.

Thus, required number of subsets = total subsets– number of subsets all whose elements are odd

$= 2^{100} - 2^{50} = 2^{50}(2^{50} - 1)$

Hence, the correct answer is (B).

18. $A = \{x \in Z : 2^{(x+2)(x^2-5x+6)} = 1\}$

$(x + 2)(x^2 - 5x + 6) = 0 \Rightarrow (x + 2)(x - 2)(x - 3) = 0$

$\Rightarrow \quad x = -2, 2, 3$ Hence, $A = \{-2, 2, 3\}$

$B = \{x \in Z : -3 < 2x - 1 < 9\}$

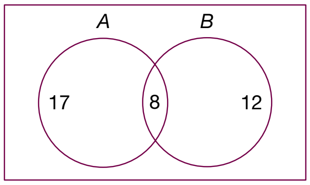
$-3 < 2x - 1 < 9 \Rightarrow -2 < 2x < 10$

$\Rightarrow \quad -1 < x < 5$ Hence, $B = \{0, 1, 2, 3, 4\}$

$A \times B$ has 15 elements so number of subset of $A \times B$ is 2^{15}

Hence, the correct answer is (D).

19.



Let population = 100

$n(A) = 25, n(B) = 20, n(A \cap B) = 8, n(A \cap \bar{B}) = 17, n(\bar{A} \cap B) = 12$

Now, % of the population who look advertisement

$$= \frac{30}{100} \times 17 + \frac{40}{100} \times 12 + \frac{50}{100} \times 8 = 13.9$$

Hence, the correct answer is (A).

20. Let $A = \{1, 2, 3, 4\}$

$B = \{3, 4, 5, 6\}$

$C = \{1, 2, 3, 4, 7, 8\}$

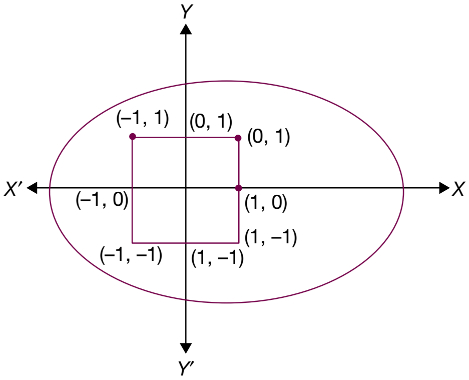
Here, $A \cap B = \{3, 4\} \subseteq C$

$A - C = \phi \subseteq B$ but $A \not\subseteq B$

So, not true (wrong) statement is 4th

Hence, the correct answer is (D).

21.



$A = \{(a, b) \in R \times R \mid |a - 5| < 1, |b - 5| < 1\}$

Let $a - 5 = x, b - 5 = y$

Set A contains all points inside $|x| < 1, |y| < 1$

$B = \{(a, b) \in R \times R : 4(a - 6)^2 + 9(b - 5)^2 < 36\}$

Set B contains all points inside or on $\dfrac{(x-1)^2}{9} + \dfrac{y^2}{4} = 1$

$\therefore \quad (\pm 1, \pm 1)$ lies inside the ellipse; hence, $A \subset B$

Hence, the correct answer is (A).

22. $n(A) = 4, n(B) = 2 \Rightarrow n(A \times B) = 8$

The number of subsets of $A \times B$ having at least

3 elements = ${}^8C_3 + {}^8C_4 + \ldots + {}^8C_8$

$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2$

$= 256 - 1 - 8 - 28 = 219$

Hence, the correct answer is (C).

23. $4^n - 3n - 1 = (1 + 3)^n - 3n - 1$

$= [{}^nC_0 + {}^nC_1 \cdot 3 + {}^nC_2 \cdot 3^2 + \ldots + {}^nC_n 3^n] - 3n - 1$

$= 9[{}^nC_2 + {}^nC_3.3 + \ldots + {}^nC_n \cdot 3^{n-2}]$

$\therefore \quad 4^n - 3n - 1$ is some multiple of 9 for all n.

$\therefore \quad X = \{x : x$ is some multiple of $9\}$

Also, $Y = \{9(n - 1) : n \in N\} = \{$All multiples of $9\}$

Clearly, $X \subset Y$.

$\therefore \quad X \cup Y = Y$

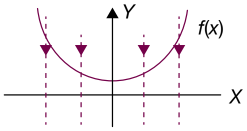
Hence, the correct answer is (B).

CHAPTER 2: FUNCTION AND ITS GRAPH

Test Your Concepts-I

1. (i) We can see in the figure that all the straight lines parallel to y-axis intersects the given curve at unique point only. Thus,

$y = x^2 + 2$ is a function.



(ii) We have $xy^2 = 16$ $y = \pm\sqrt{12 - 3x^2}$

For any values of $x \in [-2, 2]$, we get two real values of y, therefore, the given relation is not a function.

(iii) We have $y = \pm\dfrac{4}{\sqrt{x}}$

For any positive values of x, we get two real values of y, therefore, the given relation is not a function.

(iv) $2x + 3x^2y = 6y \Rightarrow y = \dfrac{2x}{6 - 3x^2}$

For any suitable values of x, we get unique real value of y, therefore, the given relation is a function.

2. $2[x] + x = 6\{x\}$

$\Rightarrow \quad 2[x] + [x] + \{x\} = 6\{x\}$

$\Rightarrow \quad 3[x] = 5\{x\} \quad \Rightarrow \quad \{x\} = \dfrac{3}{5}[x]$ …(1)

Since, $\{x\} \in [0, 1)$

$\therefore \quad 0 \le \dfrac{3}{5}[x] < 1 \quad \Rightarrow \quad 0 \le [x] < \dfrac{5}{3}$

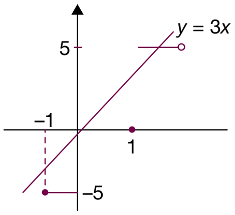
$\therefore \quad [x] = 0$ or $[x] = 1$

(i) If $[x] = 0$, then $\{x\} = 0 \Rightarrow x = 0$

(ii) If $[x] = 1 \Rightarrow x = 1 + \dfrac{3}{5} \Rightarrow x = \dfrac{8}{5}$

Hence, sum of values of $x = \dfrac{8}{5}$.

3.



$4\{x\} = x + [x] \Rightarrow 4(x - [x]) = x + [x] \quad \Rightarrow \quad 3x = 5[x]$

Graph of $y = 3x$ and $y = 5[x]$ cut at 2 points.

4. $x^2 - 12x + 35 = [x] + [-x]$

$\because \quad [x] + [-x] \begin{cases} 0; & x \in I \\ -1; & x \notin I \end{cases}$

CASE 1: When $x \in I$; $x^2 - 12x + 35 = 0$; $\Rightarrow (x - 5)(x - 7) = 0$;

$\therefore \quad x = 5, x = 7$

CASE 2: When $x \notin I$; $x^2 - 12x + 35 = -1$; $\Rightarrow x^2 - 12x + 36 = 0$;

$\Rightarrow \quad (x - 6)^2 = 0 \quad \Rightarrow \quad x = 6$

But $x \notin I$

$\therefore$ No solution from here.

Hence, 2 solutions.

5. We have $[2x^2] + x - n = 0 \Rightarrow x$ has to be an integer.

As, n is natural no. $x \in I$

$\Rightarrow \quad n = 2x^2 + x = x(2x + 1)$

$\therefore \quad n$ can be 21, 36, 55, 78 corresponding to $x = 3, 4, 5, 6$.

Hence, sum of all possible values of n is equal to 190.

6. ($\because$ $[x + I] = [x] + I$, I is an integer)

$[x]^2 + [x] - 2 < 0 \Rightarrow ([x] + 2)([x] - 1) < 0 \Rightarrow -2 < [x] < 1$

$\Rightarrow \quad x \in [-1, 1)$

As, $\{x\} = \dfrac{1}{2} \Rightarrow x = \dfrac{-1}{2}, \dfrac{1}{2}$.

7. As $\cos\dfrac{11\pi}{3} = \cos\left(4\pi - \dfrac{\pi}{3}\right) = \cos\dfrac{\pi}{3} = \dfrac{1}{2}$

and $\sin\dfrac{9\pi}{4} = \sin\left(2\pi + \dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

Now, we have

$$\log_2(3 - x) + \log_{\frac{1}{2}}\left(\frac{\sin\frac{9\pi}{4}}{5 - x}\right) = \cos\frac{11\pi}{3} - \log_{\frac{1}{2}}(x + 7)$$

$$\Rightarrow \quad \log_2\left(\frac{(3 - x)(5 - x)}{\frac{1}{\sqrt{2}}}\right) = \frac{1}{2} + \log_2(x + 7)$$

$$\Rightarrow \quad \frac{\sqrt{2}(3 - x)(5 - x)}{x + 7} = \sqrt{2}$$

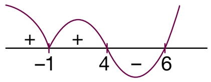
$\therefore \quad 15 - 8x + x^2 = x + 7$

$\Rightarrow \quad x^2 - 9x + 8 = 0 \quad \Rightarrow \quad (x - 8)(x + 1) = 0$

$\therefore \quad x = 1, 8$

But $x \in (-7, 3)$, so $x \ne 8 \quad \Rightarrow \quad x = 1$.

8.



$|x^2 - 3x - 4| + |x^2 - 5x - 6| = |2x + 2| = |(x^2 - 3x - 4) - (x^2 - 5x - 6)|$

As, $|x| + |y| = |x - y|$ is true, if $xy \le 0$.

$\Rightarrow \quad (x^2 - 3x - 4)(x^2 - 5x - 6) \le 0$

$\Rightarrow \quad (x - 4)(x + 1)(x - 6)(x + 1) \le 0$

Hence, $x \in \{-1\} \cup [4, 6]$.

9. $2[x] = x + \{x\}$

$2(x - \{x\}) = x + \{x\}$

$x = 3\{x\} \Rightarrow \{x\} = \dfrac{x}{3}$

$0 \le \dfrac{x}{3} \le 1 \Rightarrow 0 \le x \le 3$

$0 \le x < 1$, then $[x] = 0$; $x = [x] + \{x\} = 0 + \dfrac{x}{3} \Rightarrow x = 0$ (acceptable)

$1 \le x < 2$, then $[x] = 1$; $x = 1 + \dfrac{x}{3}$

$\Rightarrow \quad 3x = 3 + x \Rightarrow x \Rightarrow \dfrac{3}{2}$ (acceptable)

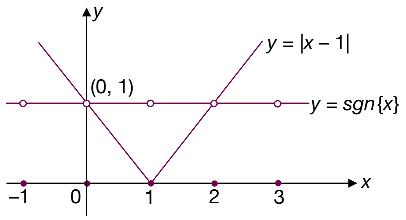
$2 \le x < 3$, then $[x] = 2$; $x = 2 + \dfrac{x}{3}$

$\Rightarrow \quad 3x = 6 + x \Rightarrow x \Rightarrow 3$ (not acceptable)

$x = 3$; $2(3) \ne 3 + 0 \Rightarrow x \ne 3$

Hence, two soln. $x = 0, \dfrac{3}{2}$

10. $\text{sgn}(\{x\}) = |1 - x|$ has solution only at $x = 1$.



11. $\sin\{x\} = \cos\{x\}$

$\tan\{x\} = 1$

$\tan(\pi/4) = 1 < \tan 1$

Graphs of $y = \tan\{x\}$ and $y = 1$ meet exactly six times in $[0, 2\pi]$

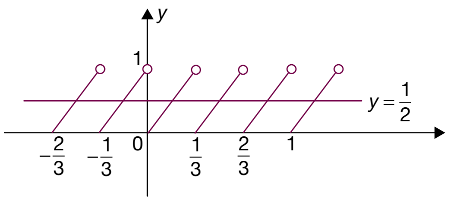
$x = \pi/4 + p$ where p can be 0, 1, 2, 3, 4, 5

12. $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \ldots + \left[x + \frac{n-1}{n}\right] = [nx]$

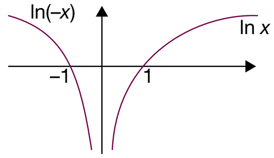
$\left[200.\frac{1}{4}\right] = [50] = 50$

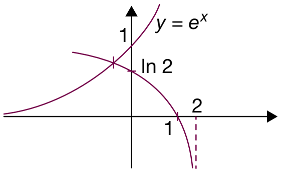
Test Your Concepts-II

1. $\{3x - 1\} = \frac{1}{2} \Rightarrow \{3x\} = \frac{1}{2}$ Hence, ∞ solutions

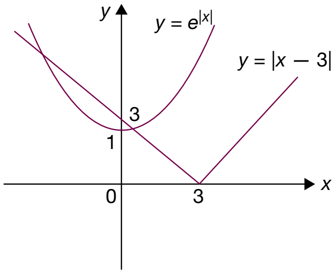


2.





3.

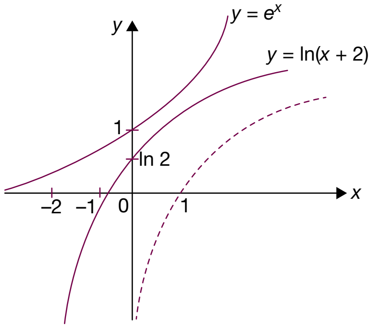


$|x - 3| = e^{|x|}$

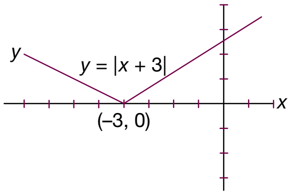
Hence, total 2 solutions exist.

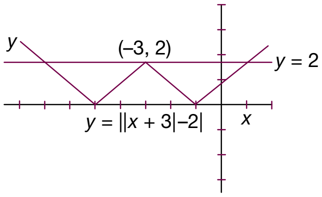
4. $\ln(x + 2) = e^x$

$\ln 2 \approx 0.693 < 1$, no point of intersection so zero solutions.



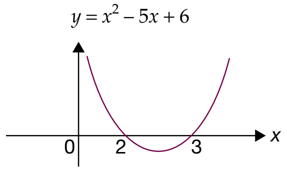
5. The following sequence of graphs gives a progression that leads directly to the answer.

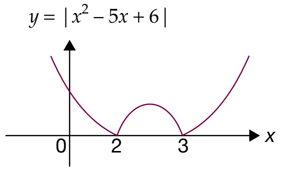


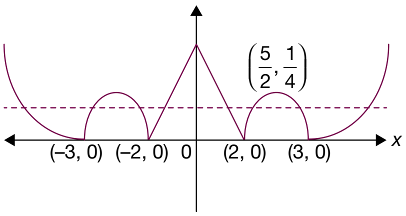


Hence, $p = 2$, no. of integral value of $p = 1$.

6. Consider $y = |x^2 - 5|x| + 6|$ and $y = k$.

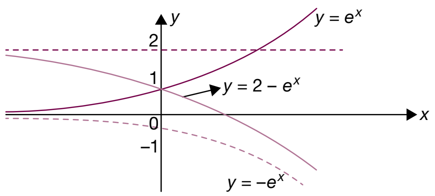






Hence, $k \in \left(0, \frac{1}{4}\right)$

7. (i) $y = 2 - e^x$

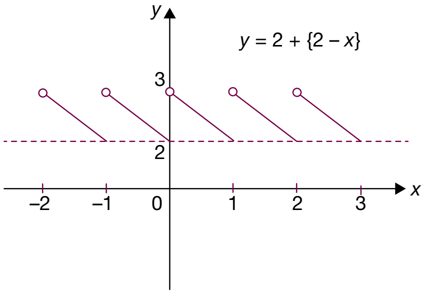


(ii) $y = 2 + \{2 - x\}$ where {.} is fractional part function

$y = 2 + \{-x\}$

$y \neq 2 - \{x\}$

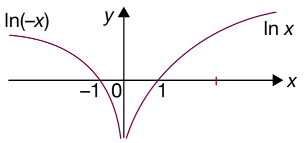
$\{x \pm m\} = \{x\}; m \in I$

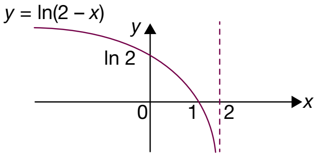


(iii) $y = \ln(2 - x)$

$2 - x > 0 \Rightarrow x < 2$

$\ln x \to \ln(-x) \to \ln(-x + 2)$

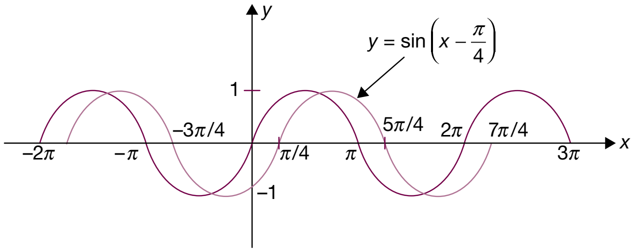




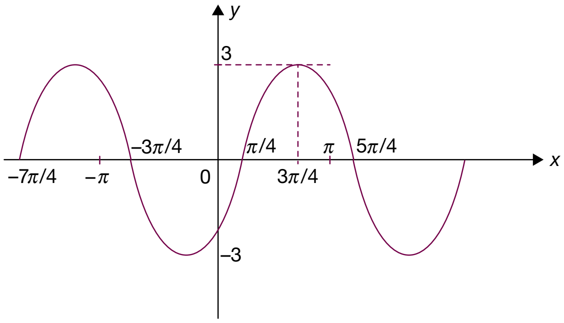
(iv) $y = 3\sin\left(x - \frac{\pi}{4}\right)$

Sequence of transformation

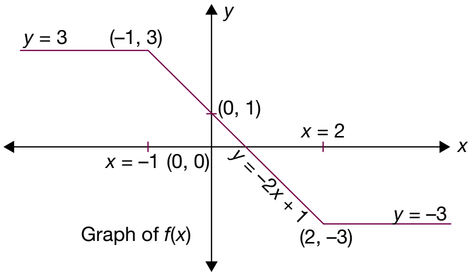
$\sin \to \sin\left(x - \frac{\pi}{4}\right) \to 3\sin\left(x - \frac{\pi}{4}\right)$



$y = 3\sin\left(x - \frac{\pi}{4}\right)$



8.



Graph of f(x)

Let $f(x) = |x-2| - |x+1| = \begin{cases} 2-x+x+1 & ; \ x<-1 \\ 2-x-x-1 & ; \ -1\le x\le 2 \\ x-2-x-1 & ; \ x>2 \end{cases}$

$\Rightarrow \quad f(x) = \begin{cases} 3 & ; \ x<-1 \\ -2x+1; & -1\le x\le 2 \\ -3 & ; \ x>2 \end{cases}$

For exactly one solution $k \in (-3, 3)$

$\therefore \quad k_{\text{integers}} = -2, -1, 0, 1, 2.$

Test Your Concepts-III

1. (i) $f(x) = \sqrt{\log_2\left(\frac{x-2}{3-x}\right)}$

$\because \quad \log_2 \frac{x-2}{3-x} \geq 0$

We have, $\left(\frac{x-2}{3-x}\right) \geq 1$

$\Rightarrow \quad \frac{x-2-3+x}{3-x} \geq 0 \Rightarrow \frac{2x-5}{x-3} \leq 0, x \neq 3 \Rightarrow x \in \left[\frac{5}{2}, 3\right)$

(ii) $f(x) = \frac{\sqrt{x+2}}{x^2-9}$

$x + 2 \geq 0 \Rightarrow x \geq -2$ and $x^2 - 9 \neq 0 \Rightarrow x \neq \pm 3$

$\therefore \quad [-2, 3) \cup (3, \infty).$

(iii) $f(x) = \sqrt{\log_2\left(\frac{1}{\log_2 x}\right)}$

$\log_2\left(\frac{1}{\log_2 x}\right) \geq 0$

$\Rightarrow \quad \frac{1}{\log_2 x} \geq 1$ Also $\log_2 x > 0, x > 1; \log_x 2 \geq 1$

$2 \geq x \Rightarrow x \leq 2$

$\therefore \quad 1 < x \leq 2$

(iv) $|\sin x - 1| \neq 0$

$\sin x \neq 1$ …(1)

$|\sin x - 1| \leq 1$

$-1 \leq \sin x - 1 \leq 1$

$0 \leq \sin x \leq 2$ …(2)

From (1) and (2),

$\sin x \in [0, 1)$

$\sin x = 0 \to x = n\pi$

$\sin x = 1 \to x = n\pi + (-1)^n \frac{\pi}{2}$

General solution is $x = n\pi + (-1)^n \alpha$ where, $\alpha \in \left[0, \frac{\pi}{2}\right)$.

2. (i) $F(x)$ is defined when $\sin(\cos x) \geq 0$

$\cos x \geq \sin^{-1} 0 \Rightarrow \cos x \geq 0$

x lies on I and IV quadrant.

$2n\pi - \frac{\pi}{2} \leq x \leq 2n\pi + \frac{\pi}{2}, n \in I$

(ii) $f(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right) \in R \Leftrightarrow -1 \leq \log_2\left(\frac{x^2}{2}\right) \leq 1$

$\Leftrightarrow \frac{1}{2} \leq \frac{x^2}{2} \leq 2 \Leftrightarrow 1 \leq x^2 \leq 4 \Leftrightarrow x \in [-2, -1] \cup [1, 2]$

(iii) $f(x) = \left[9^x + 27^{\frac{2}{3}(x-2)} - 219 - 3^{2(x-1)}\right]^{1/4}$

We must have $9^x + 27^{\frac{2}{3}(x-2)} - 219 - 3^{2(x-1)} \geq 0$

$\Rightarrow \quad (3^x)^2 + 3^{2(x-2)} - 219 - 3^{2x-2} \geq 0$

$\Rightarrow \quad (3^{2x}) + \frac{3^{2x}}{81} - 219 - \frac{3^{2x}}{9} \geq 0$

$\Rightarrow \quad \left(1 + \frac{1}{81} - \frac{1}{9}\right) 3^{2x} \geq 219 \Rightarrow \frac{73}{81} \times 3^{2x} \geq 219$

$\Rightarrow \quad 3^{2x} \geq 3 \times 81 = 3^5 \Rightarrow 2x \geq 5$

Hence, domain is $\left[\frac{5}{2}, \infty\right)$.

(iv) We must have $x^4 - 21x^2 \geq 0$ and $10 - \sqrt{x^4 - 21x^2} \geq 0$

$\Rightarrow \quad x^2(x^2 - 21) \geq 0$ …(1)

and $100 \geq x^4 - 21x^2$ …(2)

(1) gives $x = 0$ or $x \leq -\sqrt{21}$ or $x \geq \sqrt{21}$

(2) $\Rightarrow x^4 - 21x^2 - 100 \leq 0$

(v) $\log_2 x - 1 \geq 0$ and $x^3 > 0$

$\Rightarrow \quad \log_2 x \geq 1$ and $x > 0 \quad \Rightarrow \quad x \geq 2$

(vi) $\ln(\cot^{-1}x) > 0$

$\cot^{-1}x > 1$

$x < \cot 1$

$\therefore$ Domain $(-\infty, \cot 1)$

(vii) The domain excludes the roots of $x^2+3x+2=0$ i.e., $-1, -2$

Further $x+3>0 \Rightarrow x>-3$

The domain is $(-3, \infty)-\{-1, -2\}$.

(viii) $f(x) = \log_{(x^2-x+1)}(2x^2-7x+9)$

$\because$ $2x^2-7x+9>0, \forall\, x \in R$

and $x^2-x+1 \neq 1$

$\Rightarrow$ $x \neq 0$ or 1

$\Rightarrow$ $(x^2-25)(x^2+4) \le 0$

$\Rightarrow$ $x^2-25 \le 0$ (as $x^2+4>0$ always)

$\Rightarrow$ $-5 \le x \le 5$

Domain is given by $\left[-5,-\sqrt{21}\right] \cup \left[\sqrt{21},5\right]$ and $x=0$.

3. (i) $-1 \le \sin 3x \le 1$

$-4 \le 4\sin 3x \le 4$

$1 \le 5+4\sin 3x \le 9$

$$\frac{1}{9} \le \frac{1}{5+4\sin 3x} \le 1$$

$$\frac{1}{3} \le \frac{3}{5+4\sin 3x} \le 3$$

(ii) $f(x) = \dfrac{\tan\left(\pi[x^2-x]\right)}{1+\sin(\cos x)} = \{0\}$ because of $[x^2-x]$ is integer.

(iii) $y = \dfrac{x^2-6x-7}{x^2+2x-63} = \dfrac{(x+1)(x-7)}{(x+9)(x-7)}$.

$x \neq -9, 7$

$\therefore$ $y = \dfrac{x+1}{x+9} \Rightarrow x = \dfrac{1-9y}{y-1}$

$\therefore$ $y \neq 1$.

But since $x \neq 7$

$\therefore$ $y \neq \dfrac{1}{2}$

Hence, range is $R - \left\{\dfrac{1}{2}, 1\right\}$.

(iv) $y = \dfrac{x^2-3x+4}{x^2+3x+4}$

$yx^2+3xy+4y = x^2-3x+4$

$x^2(y-1)+3x(y+1)+4(y-1)=0$

Here, $D \ge 0 \Rightarrow 9(y+1)^2-4\times4(y-1)^2 \ge 0$

$(3(y+1)-4(y-1)(3(y+1)+4(y-1) \ge 0$

$(-y+7)(7y-1) \ge 0$

$(y-7)\left(y-\dfrac{1}{7}\right) \le 0$

$\dfrac{1}{7} \le y \le 7$

(v) We have, $f(x) = \dfrac{x^2+x+2}{x^2+x+1} = \dfrac{(x^2+x+1)+1}{x^2+x+1} = 1+\dfrac{1}{\left(x+\dfrac{1}{2}\right)^2+\dfrac{3}{4}}$

We can see here that as $x \to \infty$, $f(x) \to 1$ which is the min value of $f(x)$. Also $f(x)$ is max when $\left(x+\dfrac{1}{2}\right)^2+\dfrac{3}{4}$ is min which is so when $x = -\dfrac{1}{2}$ and then $\dfrac{3}{4}$.

$\therefore$ $f_{\max} = 1+\dfrac{1}{3/4} = \dfrac{7}{3}$

$\therefore$ $R_j = \left(1, \dfrac{7}{3}\right]$

(vi) $y = \dfrac{x^2-2x+4}{x^2+2x+4}$

$(y-1)x^2+(y+1)2x+4(y-1)=0;\ y \neq 1$

$D \ge 0$

$4(y^2+1+2y)-16(y^2-2y+1) \ge 0$

$3y^2-10y+3 \le 0$

$y \in \left[\dfrac{1}{3}, 3\right]$

$a = \dfrac{1}{3}$, $b=3$.

(vii) Domain of $f = \left[-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right]$

Let $\theta = \sqrt{\dfrac{\pi^2}{9}-x^2}$

Then $0 \le \theta \le \dfrac{\pi}{3}$ and $\tan\theta$ is an increasing function

Least value of $\tan\theta$ = tan 0 = 0, when $x = \pm\dfrac{\pi}{3}$ and greatest value of $\tan\theta = \tan\dfrac{\pi}{3} = \sqrt{3}$, when $x=0$

$\therefore$ Range of $f = \left[0, 3\sqrt{3}\right]$

(viii) $g(x) = \sin\left(x+\dfrac{\pi}{6}\right) + \cos\left(x-\dfrac{\pi}{6}\right)$, $x \in (-\infty, \infty)$

$$g(x) = \sin x\cos\frac{\pi}{6} + \cos x\sin\frac{\pi}{6} + \cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6}$$

$$g(x) = \frac{\left(\sqrt{3}+1\right)}{2}(\sin x + \cos x)$$

$\because$ $-\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$

$\Rightarrow$ $g(x) \in \left(-\dfrac{\left(\sqrt{3}+1\right)}{\sqrt{2}}, \dfrac{\left(\sqrt{3}+1\right)}{\sqrt{2}}\right)$

Suggested Alternate Method:

$$g(x) = \sin\left(x+\frac{\pi}{6}\right) + \cos\left(x-\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}-x\right) + \cos\left(x-\frac{\pi}{6}\right)$$

$\therefore$ $\cos A + \cos B \equiv 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

$$= 2\cos\frac{\pi}{12}\cos\left(\frac{\pi}{4}-x\right) = \frac{\sqrt{3}+1}{\sqrt{2}}\cos\left(\frac{\pi}{4}-x\right)$$

$$g(x) = \left[-\frac{(\sqrt{3}+1)}{\sqrt{2}}, \frac{(\sqrt{3}+1)}{\sqrt{2}}\right]$$

4. (i) $f(x) = e^{\sin x}$ domain $x \in R$

$-1 \le \sin x \le 1$

$\Rightarrow$ $e^{-1} \le e^{\sin x} \le e^1$

$\Rightarrow$ $\dfrac{1}{e} \le y \le e$

Required domain $x \in R$ and range $y \in \left[\dfrac{1}{e}, e\right]$

(ii) $y = e^{\cos^2 x}$ domain $x \in R$

$-1 \le \cos x \le 1 \Rightarrow 0 \le \cos^2 x \le 1$

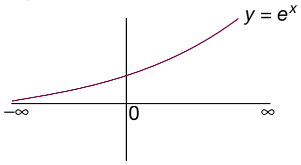
$\Rightarrow$ $e^o \le e^{\cos^2 x} \le e^1$

$\Rightarrow$ $1 \le y \le e$

Required domain $x \in R$ and range $y \in [1, e]$

(iii) $y = f(x) = \left(e^{\sin x}\right)^{\sec x}$

Domain of $f(x)$ $x \in R - \left\{(2x+1)\dfrac{\pi}{2}\right\}$ where $x \in I$



$$y = e^{\frac{\sin x}{\cos x}} = e^{\tan x}$$

$-\infty < \tan < \infty \Rightarrow e^{-\infty} < e^{\tan x} < e^{\infty}$

$\Rightarrow \quad 0 < y < \infty$

Required domain $x \in R - \left\{(2x+1)\frac{\pi}{2}\right\}$, $x \in I$ and Range $y \in (0, \infty)$

(iv) $f(x) = \dfrac{3^{\sin x}}{\left(\sqrt{3}\right)^{2\cos x}}$

$$y = \frac{3^{\sin x}}{\left(\sqrt{3}\right)^{2\cos x}} = \frac{3^{\sin x}}{3^{\cos x}} = 3^{\sin x - \cos x}$$

Domain of $f(x)$ $x \in R$

$-\sqrt{2} \le \sin x - \cos x \le \sqrt{2}$

$\Rightarrow \quad 3^{-\sqrt{2}} \le 3^{\sin x - \cos x} \le 3^{\sqrt{2}} \Rightarrow y \in \left[\frac{1}{3^{\sqrt{2}}}, 3^{\sqrt{2}}\right]$

Domain of $f(x)$ $x \in R$, range $\in \left[\frac{1}{3^{\sqrt{2}}}, 3^{\sqrt{2}}\right]$

(v) $f(x) = e^{\sin x - \sqrt{3}\cos x}$

Domain $x \in R$

$-\sqrt{1+3} \le \sin x - \sqrt{3}\cos x \le \sqrt{1+3}$

$\Rightarrow \quad -2 \le \sin x - \sqrt{3}\cos x \le 2$

$\Rightarrow \quad e^{-2} \le e^{\sin x - \sqrt{3}\cos x} \le e^{2}$

$\Rightarrow \quad y \in \left[\frac{1}{e^2}, e^2\right]$

Hence, domain of $f(x)$ $x \in R$ and range $\in \left[\frac{1}{e^2}, e^2\right]$

5. $[x] = n \in Z \Leftrightarrow \sin[x]\pi = \tan[x]\pi = 0$

So, $f(x) = \dfrac{0+0}{1+[x]^2} = 0$

6. (i) $0 < \sin x \le 1 \Rightarrow x \in (0, \pi)$ complete domain $x \in (2n\pi, 2n\pi + \pi)$ where $n \in I$

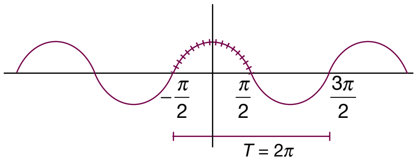
$\ln 0^{+} < \ln \sin x \le \ln 1$

$\Rightarrow \quad -\infty < y \le 0$

$y \in (-\infty, 0]$

(ii) $f(x) = \ln \cos x$

Domain $\cos x > 0$



$x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, as period of $\cos x$ is 2π

Domain $x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$ where $n \in I$

For range computation

$0 < \cos x \le 1$

$\Rightarrow \quad \ln 0^{+} < \ln \cos x \le \ln 1$

$\Rightarrow \quad -\infty < y \le 0$

Range $y \in (-\infty, 0]$

(iii) $f(x) = \ln(3 - \sin 4x)$

$3 - \sin 4x > 0$

$\Rightarrow \quad 3 > \sin 4x \quad \forall\, x \in R$ domain

$\forall\, x \in R$ domain

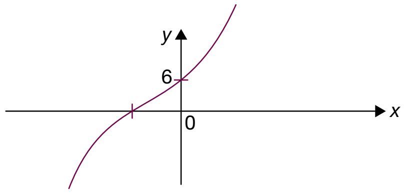
$-1 \le \sin 4x \le 1 \Rightarrow 1 \ge -\sin 4x \ge -1$

$\Rightarrow \quad 3 + 1 \ge 3 - \sin 4x \ge 3 - 1 \Rightarrow 4 \ge 3 - \sin 4x \ge 2$

$\Rightarrow \quad \ln 4 \ge \ln(3 - \sin 4x) \ge \ln 2 \Rightarrow \ln 4 \ge y \ge \ln 2$

Required domain $x \in R$ and range $y \in [\ln 2, \ln 4]$

7. $\dfrac{dy}{dx} = 3x^2 - 6x + 3 = 3(x-1)^2 \ge 0$



$f(x)$ is always increasing

for $x \in [0, 1)$

$f(0) = 0 + 6 = 6$

$f(1) = 1 - 3 + 3 + 6 = 7$

As $x \in [0, 1)$ we get $y \in [6, 7)$

Test Your Concepts-IV

1. (i) $f(x) = \cos(2\tan^{-1}x)$; $g(x) = \dfrac{1-x^2}{1+x^2}$

'f' and 'g' have same domain and range

and $f(x) = g(x) = \dfrac{1-x^2}{1+x^2}$

(ii) $f(x) = \dfrac{2x}{1+x^2}$, $g(x) = \sin(2\cot^{-1}x)$

$g(x) = \sin(2\cot^{-1}x) = 2\sin\theta\cos\theta$

(where $\theta = \cot^{-1}x$ $\left(\because \cot\theta = x; \sin\theta = \dfrac{1}{\sqrt{1+x^2}}, \cos\theta = \dfrac{1}{\sqrt{1+x^2}}\right)$

$= \dfrac{2x.1}{(1+x^2)} = f(x)$

'f' and 'g' have same domain and range. i.e., $x \in R$

(iii) $f(x) = \sin^{-1}(\sin x)$, $g(x) = \cos^{-1}(\cos x)$

$f(\pi) = 0$; $g(\pi) = \pi$

Not equal function

(iv) $f(x) = e^{\ln(\text{sgn}\cot^{-1}x)}$

sgn $(\cot^{-1}x) = 1 \quad \forall\, x \in R$ $\qquad (\because \cot^{-1}x \in (0, \pi))$

$\therefore \quad f(x) = 1$

$g(x) = e^{\ln[1+\{x\}]} = e^{\ln(1)} = 1 \quad \forall\, x \in R$

$f(x)$ and $g(x)$ are identical.

2. (i) $f(-x) = \log\left[\left(-x + \sqrt{1+x^2}\right) \times \left(\dfrac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x}\right)\right]$

$= \log\left(\dfrac{1+x^2-x^2}{\sqrt{1+x^2}+x}\right)$

$= -\log\left(x + \sqrt{1+x^2}\right) = -f(x)$

Thus, $f(x)$ is an odd function.

(ii) $f(-x) = \dfrac{a^{-x}+1}{a^{-x}-1} = \dfrac{1+a^x}{1-a^x}$

$f(-x) = -\dfrac{a^x+1}{a^x-1} = -f(x)$

Thus, $f(x)$ is an odd function.

(iii) $f(-x) = \dfrac{\cos(-x)}{\left[-\frac{x}{\pi}\right]+\frac{1}{2}} = \dfrac{\cos x}{-\left[\frac{x}{\pi}\right]-1+\frac{1}{2}}$

$\left(\text{as } x \ne n\pi \Rightarrow \frac{x}{\pi} \notin I, \text{ so as } \left[-\frac{x}{\pi}\right] = -\left[\frac{x}{\pi}\right]-1\right)$

$= -\dfrac{\cos x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} = -f(x) \Rightarrow f(x)$ is an odd function.

(iv) $f(x) = h(g(x)) = \sec^3\left\{\left(\log\left(x+\sqrt{x^2+1}\right)\right)\right\}$.

$g(x) = \log\left(x+\sqrt{x^2+1}\right)$ is an odd function.

$h(x) = \sec^3 x$ is an even function

$f(x)$ is an even function.

3. (i) $f(x) = 2/(1 - \cos 6x) + 1/(\tan 4x)$.

So the period of f = LCM (period of $\cos 6x$, period of $\tan 4x$)

$= \text{LCM}\ (\pi/3, \pi/4) = \pi$.

(ii) Since $|\sin x|$ and $\cos x$ are periodic function with period π and 2π, respectively.

Therefore, $\dfrac{|\sin x|}{\cos x}$ is periodic with period 2π.

Similarly, $\dfrac{|\cos x|}{\sin x}$ is periodic with period 2π.

So, period of $f(x)$ is LCM of $\{2\pi, 2\pi\} = 2\pi$.

(iii) $f(x) = \tan\dfrac{\pi}{2}[x]$

$$f(x+1) = \tan\frac{\pi}{2}[x+1] = -\cot\frac{\pi}{2}[x]$$

$$f(x+2) = \tan\frac{\pi}{2}[x+2] = \tan\frac{\pi}{2}(2+[x])$$

$$= \tan\left(\pi + \frac{\pi}{2}[x]\right) = \tan\frac{\pi}{2}[x] = f(x)$$

$\therefore$ 2 is period of $f(x)$, 1 is not a period of $f(x)$.

$\therefore$ 2 is the fundamental period.

(iv) $f(x) = |\cos \pi x| + |\cos 2\pi x| + \ldots + |\cos n\pi x| + e^{x-[x]}$

Period of $x - [x] = 1$

Period of $|\cos \pi x| = 1$

Period of $|\cos 2\pi x| = 1/2$

....

....

Period of $|\cos n\pi x| = 1/n$

So period of $f(x)$ will be,

LCM of all periods = LCM of $\{1, 1, 1/2, 1/3, 1/4, \ldots 1/n\} = 1$.

(v) Let $f(x) = \text{sgn}([x] + [-x]) = \begin{cases} 0; & x \in I \\ -1; & x \notin I \end{cases}$

Hence, $f(x)$ is periodic with period 1.

(vi) The period of $x - [x]$ is 1.

The period of $\sqrt{x-[x]}$ is 1.

The period of $(x - [x])^2$ is 1.

The period of $|\sin \pi x| = \dfrac{\pi}{\pi} = 1$.

The period of $|\cos \pi x| = 1$.

The period of $|\tan \pi x| = 1$.

Thus each of the above functions is a periodic function with period 1. Therefore, their LCM is 1.

Hence, the function $f(x)$ is periodic with fundamental period = 1.

(vii) $f(x) = \sin(3x - [3x])$

$f(x) = \sin\{3x\}$, where $\{.\}$ is a fractional part function.

4. Change x to $10 - x$ to obtain

$f(20 - x) = f(x)$

We have $f(20 - x) = -f(20 + x)$

$\Rightarrow$ $f(x) = -f(20 + x)$

Now change x to $20 + x$

$f(20 + x) = -f(40 + x)$

$-f(x) = -f(40 + x)$

$f(x) = f(40 + x)$, so f is periodic

Again $f(-x) = -f(20 - x) = -f(x)$

Thus, f is odd

5. $G(x) = \left(\dfrac{1}{a^x - 1} + \dfrac{1}{2}\right)F(x)$

$$G(-x) = \left(\frac{1}{a^{-x}-1} + \frac{1}{2}\right)F(-x) = -\left(\frac{a^x}{1-a^x} + \frac{1}{2}\right)F(x) = \left(\frac{a^x}{a^x-1} - \frac{1}{2}\right)F(x)$$

$$= \left(\frac{a^x - 1 + 1}{a^x - 1} - \frac{1}{2}\right)F(x) = \left(1 + \frac{1}{a^x-1}\hat{u}\frac{1}{2}\right)F(x) = \left(\frac{1}{a^x-1} + \frac{1}{2}\right)F(x) = G(x)$$

$\therefore$ $G(x)$ is an even function.

6. For $n = 2$, we have $\dfrac{\sin 2x}{\sin(x/2)} = 4(\cos x/2)\cos x$.

The period of $\cos x$ is 2π and that of $\cos(x/2)$ is 4π.

Hence, the period of $\dfrac{\sin 2x}{\sin(x/2)}$ is 4π.

Also, the period of $\dfrac{\sin 3x}{\sin(x/3)}$, $\dfrac{\sin 5x}{\sin(x/5)}$ and $\dfrac{\sin 4x}{\sin(x/4)}$ cannot be 4π.

7. $f(x) = [x] + [2x] + \ldots + [nx] - (x + 2x + \ldots + nx)$

$= [x] - x + [2x] - 2x + \ldots + [nx] - (nx)$

$= -[\{x\} + \{2x\} + \ldots + \{nx\}]$

Period of $\{rx\} = \dfrac{1}{r}$

$\therefore$ Period of $f(x) = \text{LCM}\left(1, \dfrac{1}{2}, \dfrac{1}{3}, \ldots, \dfrac{1}{n}\right) = 1$

8. $f(2 + x) - a = \{1 - [f(x) - a]^4\}^{1/4}$

$\Rightarrow$ $[f(2 + x) - a]^4 = 1 - [f(x) - a]^4$

$[f(2 + x) - a]^4 + [f(x) - a]^4 = 1$ …(1)

(1) is true for all x

Replace x by $(x + 2)$ in (1)

$[f(x + 4) - a]^4 + [f(x + 2) - a]^4 = 1$ …(2)

(1) and (2) gives, $[f(x) - a]^4 = [f(x + 4) - a]^4$

$\Rightarrow$ $f(x + 4) - a = f(x) - a$

$\Rightarrow$ $f(x + 4) = f(x)$

9. $g(I) = 1$ so many-one

$$g(-x) = \sqrt{-\{x\}} + \sqrt{1-\{-x\}} = \sqrt{1-\{x\}} + \sqrt{\{x\}} = g(x)$$

$g(x)$ is even function

10. $f(x) = \dfrac{1}{2}\cos(\sin x) + \cos(\cos x)$

Period $\dfrac{\pi}{2k} = \pi \Rightarrow K = \dfrac{1}{2}$.

11. $f(x) = \cos x + \cos ax$

Period of $\cos x = 2\pi$

Period of $\cos ax = \dfrac{2\pi}{a}$

Since $f(x)$ is periodic, LCM $\left(2\pi, \dfrac{2\pi}{a}\right)$ is feasible, so a should be a rational number.

Test Your Concepts-V

1. As $f(1) = 1 = f(2)$

$f(3) = 2 = f(4)$

So, f is not injective.

Also, $R_f = N$, so f is surjective.

Hence, f is surjective but not injective.

2. (i) f is not one–one as $f(0) = 0$ and $f(-1) = 0$. f is also not onto as for $y = 1$ there is no $x \in R$ such that $f(x) = 1$. If there is such a $x \in R$ then $e^{|x|} - e^{-x} = e^{x} + e^{-x}$. Clearly $x \neq 0$. For $x > 0$, this equation gives $-e^{-x} = e^{-x}$ which is not possible. For $x < 0$, the above equation gives $e^{x} = -e^{-x}$ which is also not possible

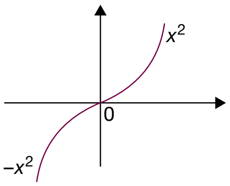
(ii) Since $f'(x) = 2 + \cos x > 0$ for all $x \in R$, so f is one – to – one. Moreover, $f(x) \to \infty$ as $x \to \infty$ and $f(x) \to -\infty$ as $x \to -\infty$, hence the range of f is R. Therefore, f is onto as well.

(iii) $f(x) = x^3 + x^2 + 100x + 5\sin x$

$\therefore f'(x) = 3x^2 + 2x + 100 + 5\cos x = 3x^2 + 2x + 94 + (6 + 5\cos x) > 0$

f is an increasing function and consequently a one-one function.

Clearly $f(-\infty) = -\infty, f(\infty) = \infty$ and $f(x)$ is continuous, therefore range $f = R =$ codomain f. Hence, f is onto.



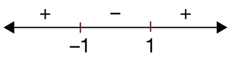
(iv) Given that $f(x) = \begin{cases} x^2 & ;\ \text{if } x > 0 \\ 0 & ;\ \text{if } x = 0 \\ -x^2 & ;\ \text{if } x < 0 \end{cases}$

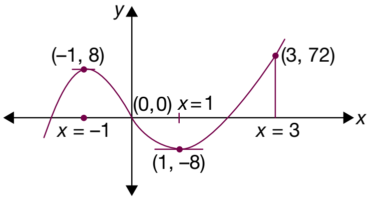
(v) $f'(x) = \dfrac{x+1}{x^2+1} \Rightarrow f(x)$ is increasing $\left[0, \sqrt{3}\right]$

3. (i) $f(x) = 4x^3 - 12x$

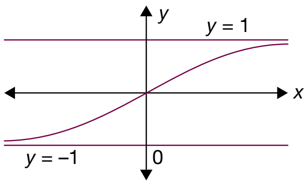
$f'(x) = 12(x^2 - 1) = 12(x + 1)(x - 1)$

$\Rightarrow$ Local maximum at $x = -1$ and local minimum at $x = 1$.





(ii) $f(x) = \begin{cases} \dfrac{x}{1-x}; & x < 0 \\ 0\ ; & x = 0 \\ \dfrac{x}{1+x}; & x > 0 \end{cases}$



$\therefore$ $f(0) = 0$. Now, $y = \dfrac{x}{1-x}, x < 0$

$\Rightarrow$ $x = \dfrac{y}{1+y}, < 0 \Rightarrow y \in (-1, 0)$

Graph of $f(x) = \dfrac{x}{1+|x|}$

Also, $y = \dfrac{x}{1+x}, x > 0 \Rightarrow x = \dfrac{y}{1-y} > 0 \Rightarrow y \in (0, 1)$

So, range of function $= (-1, 1)$.

(iii) $f(x) = 2x^3 - 6x^2 - 18x + 17$

$f'(x) = 6[x^2 - 2x - 3] = 0$

$\therefore$ $(x - 3)(x + 1) = 0$

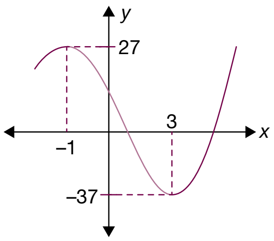
$\Rightarrow$ $x = 3$ or $x = -1$.

$\because$ $f'(x) < 0 \Rightarrow x \in (-1, 3)$

Strictly decreasing in $(-1, 3)$.

Also, $f(-1) = 27$ and $f(3) = -37$

$\Rightarrow f(x)$ is one-one and onto.



(iv) $f(x) = e^{x^2} + \cos(x)$

$\Rightarrow$ even function.

$f(-x) = e^{(-x)^2} + \cos(-x) = e^{x^2} + \cos x = f(x)$

$\because$ $e^{x^2} \geq 1$ and $\cos x \in [-1, 1]$

$\Rightarrow$ $f(x) > 0 \Rightarrow f(x)$ is into function.

(v) $f(x) = 2x - \sin x$

$f'(x) = 2 - \cos x > 0$, as $-1 \leq \cos x \leq 1$

$f(x)$ increasing one-one

co-domain $= R$

$\left.\begin{matrix} x \to -\infty & f(x) \to -\infty \\ x \to +\infty & f(x) \to \infty \end{matrix}\right]$ Range $= R$

Clearly $f(x)$ is bijective

4. $3^4 - [{}^3C_1(2^4 - 2) + {}^3C_2] = 36$

Alternate:

Distribute 4 objects to 3 different persons.

(2, 1, 1) is the only possible condition.

$= \dfrac{4!}{2!2!} \times 3! = 36$ ways

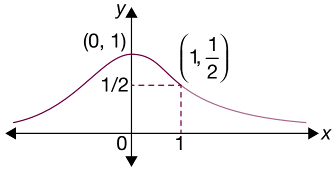
5. $\therefore$ for f to be onto range $= \left[0, \dfrac{\pi}{2}\right)$

This is possible only when $x^2 - x + a \geq 0$

$\therefore$ $D = 1^2 - 4a = 0 \Rightarrow a = \dfrac{1}{4}$

The set of values of a for which f is onto is $\left\{\dfrac{1}{4}\right\}$

6.



From above graph $f(t) \in \left(0, \dfrac{1}{2}\right]$

$\Rightarrow$ f is one-one but not onto

7. Two triangle may have equal areas

$\therefore$ f is not one-one

Since each positive real number can represent area of a triangle.

$\therefore$ f is onto

Test Your Concepts-VI

1. $0 < e^x < 1 \Rightarrow x < 0$

$0 < \ln|x| < 1 \Rightarrow 1 < |x| < e \Rightarrow x \in (-e, -1) \cup (1, e)$

$\therefore$ $x \in (-e, -1)$

2. Here, $f(x)$ is defined in [0, 1]

$\Rightarrow$ $x \in [0, 1]$ i.e., the only values of x we can substitute, lies between [0, 1].

For $f(\cot x)$ to be defined, we must have

$0 \leq \cot x \leq 1$ [as x is replaced by $\tan x$]

or $\dfrac{\pi}{4} \leq x \leq \dfrac{\pi}{2}$ [in particular]

i.e., $n\pi + \dfrac{\pi}{4} \leq x \leq n\pi + \dfrac{\pi}{2}$ [in general]

Thus, domain for $f(\cot x) \in \left[n\pi + \dfrac{\pi}{4}, n\pi + \dfrac{\pi}{2}\right]$

3. From the given domain of $f\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ we conclude that its range $(-\infty, \infty)$ i.e., whole of R.

Domain of g is $1 - x^2 \geq 0$ or $x^2 - 1 \leq 0$

or $(x + 1)(x - 1) \leq 0$ or $-1 \leq x \leq 1$ or $[-1, 1]$

and for range of g, $y = \sqrt{1 - x^2}$

Since, $x^2 \le 1$

$\therefore \quad y \in [0, 1]$

$(fog)x = f(g(x)) = f\{\sqrt{1-x^2}\} = \tan\sqrt{1-x^2}$

$(gof)x = g(f(x)) = g(\tan x) = \sqrt{1-\tan^2 x}$

4. $fof(x) = f(f(x)) = \begin{cases} 1+f(x); & 0 \le f(x) \le 2 \\ 3-f(x); & 2 < f(x) \le 3 \end{cases}$

$$= \begin{cases} 1+(1+x); & 0 \le 1+x \le 2 \Rightarrow -1 \le x \le 1 \text{ but } 0 \le x \le 2, \therefore 0 \le x \le 1 \\ 1+(3-x); & 0 \le 3-x \le 2 \Rightarrow 1 \le x \le 3 \text{ but } 2 < x \le 3, \therefore 2 < x \le 3 \\ 3-(1+x); & 2 < 1+x \le 3 \Rightarrow 1 < x \le 2 \text{ but } 0 \le x \le 2, \therefore 1 < x \le 2 \\ 3-(3-x); & 2 < 3-x \le 3 \Rightarrow 0 \le x < 1 \text{ but } 2 < x \le 3, \therefore x \in \phi \end{cases}$$

$$fof(x) = \begin{cases} 2+x; & 0 \le x \le 1 \\ 2-x; & 1 < x \le 2 \\ 4-x; & 2 < x \le 3 \end{cases}$$

5. $fof(x) = \dfrac{1}{1-f(x)} = \dfrac{1}{1-\dfrac{1}{1-x}} = \dfrac{1-x}{-x} = \dfrac{x-1}{x}$

$$fofof(x) = \frac{f-1}{f}\ \frac{\dfrac{1}{1-x}-1}{\dfrac{1}{1-x}} = x$$

$\therefore \quad g(x) = fofofofofofof(x) = fofofof(x) = f(x) = \dfrac{1}{1-x}$

Now $h(x) = \tan^{-1}\left(\dfrac{1}{1+x^2+x}\right)$

$h(0) = \tan^{-1}(1) = \dfrac{\pi}{4}$

6. $0 \le \{x\}^3 + 1 \le 1 \Rightarrow -1 \le \{x\}^3 \le 0 \Rightarrow x \in I$

7. $f(g(x)) = 2^{x^2}$

$g(f(x)) = (2^x)^2 \Rightarrow 2^{2x}$

$\Rightarrow \quad x^2 = 2x \Rightarrow x = 0 \text{ or } x = 2$

8. Given, $f(x) = \begin{cases} x; & x \in Q \\ 1-x; & x \notin Q \end{cases}$

$\Rightarrow \quad fof(x) = \begin{cases} f(x); & f(x) \in Q \\ 1-f(x); & f(x) \notin Q \end{cases}$

So, $fof(x) = \begin{cases} x; & x \in Q \\ x; & x \notin Q \end{cases}$

$\therefore \quad f(f(x)) = x, \forall x \in [0, 1]$

$\Rightarrow \quad p = 0$ and $q = 1$

Hence, $(p^2 + 2q) = 2$

9. Clearly,

$f(g(x)) = (g(x))^2 + 1 = \dfrac{1}{(x-1)^2} + 1$

and $g(f(x)) = \dfrac{1}{f(x)-1} = \dfrac{1}{x^2}$

Now, verify alternatives.

10. $fog(x) = [\,|x|\,]$

$gof(x) = |\,[x]\,|$

(i) $\operatorname{sgn}(fog(x)) + \operatorname{sgn}(gof(x)) = 0$ for $x \in (0, 1)$

(ii) $fog\left(\dfrac{7}{2}\right) = \left[\left|\dfrac{7}{2}\right|\right] = 3.$

$gof\left(\dfrac{-3}{2}\right) = \left|\left[\dfrac{-3}{2}\right]\right| = 2.$

$fog\left(\dfrac{7}{2}\right) + gof\left(\dfrac{-3}{2}\right) = 3 + 2 = 5.$

11. $k(x) = h(g(f(x))) = \dfrac{5x^2}{8-x^2}$

Domain of $k(x)$ is $R - \{0, \pm 2, \pm 2\sqrt{2}\}$

Range of $k(x)$ is $(-\infty, -5) \cup (0, \infty) - \{5\}$

or $\quad R - ([-5, 0] \cup \{5\})$

$\therefore \quad A$ is $[-5, 0] \cup \{5\}$

12. $gof(x) = 3x + 4; -1 \le x < 0$

13. We have,

$f(x) = \sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$

$= \sqrt{\sin^4 x + 4 - 4\sin^2 x} - \sqrt{\cos^4 x + 4 - 4\cos^2 x}$

$= 2 - \sin^2 x - (2 - \cos^2 x)$

$= \cos^2 x - \sin^2 x = \cos 2x$

Given, $g(\sin 2t) = \sin t + \cos t$

$\Rightarrow \quad (g(\sin 2t) = \sqrt{1+\sin 2t}$

So, $g(x) = \sqrt{1+x}, -1 \le x \le 1$

Now, $g(f(x)) = g(\cos 2x) = \sqrt{1+\cos 2x}$

$= \sqrt{2}\,|\cos|\ \forall x \in R$

$\therefore \quad$ Range of $g(f(x))$ is $[0, \sqrt{2}\,]$

$\Rightarrow \quad (a^2 + b^2) = 2$

Test Your Concepts-VII

1. $3 < x < 6$

$\Rightarrow \quad \left[\dfrac{x}{3}\right] = 1$

$\therefore \quad f(x) = x - 1$

$\therefore \quad f^{-1}(x) = x + 1$

2. $1 + \alpha x = \dfrac{x-1}{\alpha} \Rightarrow \alpha = -1$

3. We shall consider two cases.

CASE 1: When $f(x) \ge 0$, so

$x^4 f(x) - \sqrt{1-\sin 2\pi x} = f(x) - 2f(x)$

$\Rightarrow \quad f(x) = \dfrac{\sqrt{1-\sin 2\pi x}}{1+x^4} \ge 0$

CASE 2: When $f(x) < 0$

$x^4 f(x) - \sqrt{1-\sin 2\pi x} = -f(x) - 2f(x)$

$f(x) = \dfrac{\sqrt{1-\sin 2\pi x}}{3+x^4} \ge 0$ not possible.

Hence, $f(-2) = \dfrac{1}{17}$

4. $f(x) = 2x^3 - 3(k+2)x^2 + 12kx - 7, -4 \le k \le 6, k \in I$

$f'(x) = 6(x^2 - (k+2)x + 2k)$

$\therefore \quad f'(x) = 6(x-k)(x-2)$

$f'(x) \ge 0$

$\Rightarrow \quad$ For $f(x)$ to be invertible, 'k' must be '2' only.

5. $y = \ln\left(x + \sqrt{1+x^2}\right)$

$\therefore \quad \sqrt{1+x^2} + x = e^y$

$\Rightarrow \quad \sqrt{1+x^2} - x = e^{-y}$

So, $2x = e^y - e^{-y} \Rightarrow x = \dfrac{e^y - e^{-y}}{2} \Rightarrow f^{-1}(y) = \dfrac{e^y - e^{-y}}{2}$

If $y = \ln 7$

$\Rightarrow \quad f^{-1}(\ln 7) = \left(7 - \dfrac{1}{7}\right)\dfrac{1}{2} = \dfrac{48}{7 \cdot 2} = \dfrac{24}{7}.$

6. Given, $x = \log_4\left(\frac{2f(x)}{1-f(x)}\right)$

$\therefore \quad \frac{2f(x)}{1-f(x)} = 4^x \Rightarrow f(x) = \left(\frac{4^x}{2+4^x}\right)$

$\Rightarrow \quad f(1-x) + f(x) = 1$

So, $f(2010) + f(-2009) = 1$

7. $2f(x) + f(-x) = \cos x$

Put $x = \pi$ and $x = -\pi$ to get

$2f(\pi) + f(-\pi) = \cos \pi = -1$ …(1)

$2f(-\pi) + f(\pi) = \cos(-\pi) = -1$ …(2)

$\therefore$ Solving (1) and (2) we get

$f(\pi) = \frac{-1}{3}$.

8. We have $3f(x) - f\left(\frac{1}{x}\right) = \log_e x^4 = 4\log_e x$ …(1)

$\Rightarrow \quad 3f\left(\frac{1}{x}\right) - f(x) = 4\log_e\left(\frac{1}{x}\right)$ [Replacing x by $1/x$]

$\Rightarrow \quad 3f\left(\frac{1}{x}\right) - f(x) = -4\log_e x$ …(2)

Solving (1) and (2), we get

$8f(x) = 8\log_e x \Rightarrow f(x) = \log_e x \Rightarrow f(e^x) = x$.

9. Here, $f\,f(x) = x$ and $f\left(f\left(\frac{1}{x}\right)\right) = \frac{1}{x}$

Hence, $x + \frac{1}{x} \geq 2$

10. $f^{-1}(x) = \frac{x-2}{2x-1} \quad \Rightarrow \quad f(x) = f^{-1}(x)$

$\Rightarrow \quad f(f(x)) = x$

$\Rightarrow \quad f(x) + 2f^{-1}(x) + 2 = f(f(x))$ reduces to

$\Rightarrow \quad \frac{3(x-2)}{2x-1} = x - 2$

$\Rightarrow \quad \frac{(x-2)}{(2x-1)}(3 - 2x + 1) = 0$

$\Rightarrow \quad 2(x-2)(2-x) = 0 \Rightarrow x = 2$

11. $f(x) + f(x+2) = f(x+1)$ …(1)

$\Rightarrow \quad f(x+1) + f(x+3) = f(x+2)$ …(2)

Adding (1) and (2) we get

$f(x) + f(x+3) = 0$

$f(x) = -f(x+3)$

$x \to x + 3$

$\Rightarrow \quad f(x+3) = -f(x+6)$

$\Rightarrow \quad -f(x) = -f(x+6)$

$\Rightarrow \quad f(x) = f(x+6)$

$\Rightarrow$ Period of $f(x)$ is 6

Now $f(-5) = f(5) = f(5-18) = f(-13) = 4$

and $f(1) = f(-1) = f(-1-12) = f(-13) = 4$

$\Rightarrow \quad f(-5) + f(1) = 8$

12. $f(x) = \frac{2}{4^x + 2}$

$f(1-x) = \frac{2}{4^{1-x}+2} = \frac{2.4^x}{4+2.4^x} \Rightarrow f(x) + f(1-x) = 1$ …(1)

Now put $x = \frac{1}{2011} \Rightarrow f\left(\frac{1}{2011}\right) + f\left(1 - \frac{1}{2011}\right) = 1$

$f\left(\frac{1}{2011}\right) + f\left(\frac{2010}{2011}\right) = 1$

Put $x = \frac{2}{2011} \Rightarrow f\left(\frac{2}{2011}\right) + f\left(\frac{2009}{2011}\right) = 1$

Put $x = \frac{1004}{2011} \Rightarrow f\left(\frac{1004}{2011}\right) + f\left(\frac{1007}{2011}\right) = 1$

Put $x = \frac{1005}{2011} \Rightarrow f\left(\frac{1005}{2011}\right) + f\left(\frac{1006}{2011}\right) = 1$

On adding, we get sum = 1005

Single Correct Choice Type Questions

1. $x^2 + x - 2 \neq 0 \Rightarrow x \neq -2, x \neq 1$

$5 - [x-1] - [x]^2 > 0 \Rightarrow 6 - [x] - [x]^2 > 0$

$\Rightarrow \quad -3 < [x] < 2 \Rightarrow -2 \leq [x] < 2$

$\Rightarrow \quad x \in [-2, 2)$

Hence, domain $(-2, 1) \cup (1, 2)$

Hence, the correct answer is (D).

2. Consider $g(x) = 4x^3 - 8x^2 + 1$

$\Rightarrow \quad g'(x) = 12x^2 - 16x$

Now $g'(x) = 0 \Rightarrow x = 0$ or $x = \frac{4}{3} \notin [-1, 1]$

Now $g(-1) = -11, g(0) = 1$ and $g(1) = -3$

Since, $\cos x \in [-1, 1]$, the range of $f(x)$ is $[-11, 1]$.

Hence, the correct answer is (A).

3. $a, b > 0 \Rightarrow \frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$

$$\Rightarrow \quad f(x) \geq \frac{2}{\sqrt{\cos\left(\frac{\pi}{6} - x\right)\cos\left(\frac{\pi}{6} + x\right)}} \geq \frac{2}{\sqrt{\frac{1}{4} + \frac{\cos 2x}{2}}}$$

$0 \leq x \leq \frac{\pi}{3} \Rightarrow \frac{-1}{2} \leq \cos 2x \leq 1$

$\Rightarrow \quad 0 \leq \sqrt{\frac{1}{4} + \frac{\cos 2x}{2}} \leq \frac{\sqrt{3}}{2} \Rightarrow f(x) \geq \frac{4}{\sqrt{3}}$

Since 'f' is continuous range of 'f' is $\left[\frac{4}{\sqrt{3}}, \infty\right)$

Hence, the correct answer is (B).

4. For $f(x)$ to be defined

(i) $\left[\frac{3x-2}{2}\right] = -1, 0, 1$

$\Rightarrow \quad -1 \leq \frac{3x-2}{2} < 2 \Rightarrow -2 \leq 3x - 2 < 4$

$\Rightarrow \quad 0 \leq 3x < 6 \Rightarrow 0 \leq x < 2$ …(1)

(ii) $3 - x \geq 0$ and $x \neq 1, 2, 3$

$\Rightarrow \quad x < 3$ and $x \neq 1, 2$ …(2)

From (1) and (2), we have

$0 \leq x < 1$ or $1 < x < 2$

$\therefore$ Domain $= [0, 1) \cup (1, 2) = [0, 2) - \{1\}$

Hence, the correct answer is (B).

5. $x \geq 0 \Rightarrow |x| - \{x\} = x - \{x\} \geq 0$

$x < 0 \Rightarrow |x| - \{x\} = -(x + \{x\}) = [x] - 2x \geq 0$ only

if $2x \leq [x] \leq -1$ or if $x \leq -\frac{1}{2}$

Hence, the correct answer is (C).

6. We have

$y = [x^2] - [x]^2 \quad x \in [0, 2]$

$$y = \begin{cases} 0 \quad ; & 0 \leq x < 1 \\ 1 - 1 = 0; & 1 \leq x < \sqrt{2} \\ 2 - 1 = 1; & \sqrt{2} \leq x < \sqrt{3} \\ 3 - 1 = 2; & \sqrt{3} \leq x < 2 \end{cases}$$

Therefore, the range is $\{0, 1, 2\}$

Hence, the correct answer is (D).

7. $\text{sgn}\left(9-x^2\right)>0 \Rightarrow 9-x^2>0 \Rightarrow x \in (-3,3)$ ...(1)

$$[x]^3-4[x]\ge 0 \Rightarrow [x]([x]+2)([x]-2)\ge 0$$

By wavy curve, $[x]\ge 2$ (or) $[x]\in[-2,0]$

$\Rightarrow \quad x\ge 2$ (or) $[x]=-2,-1,0 \Rightarrow x\ge 2$ (or) $x\in[-2,1)$

$\Rightarrow \quad x\in[-2,1)\cup[2,\infty)$ ...(2)

For (1) and (2), $x\in[-2,1)\cup[2,3)$

Hence, the correct answer is (D).

8. For $f(x)$ to be defined, $-1\le x+[x]\le 1$. When $x\in I$, Let $x = R$, then from (1), $-1\le 2R\le 1$

$\Rightarrow \quad -\frac{1}{2}\le R\le\frac{1}{2}\Rightarrow R=0\Rightarrow x=0$ ...(1)

When $x\notin I$, let $x=R+\alpha$, where R is the integral part of x and $0<\alpha<1$

From (1), $-1\le 2R+\alpha\le 1$

$\Rightarrow \quad \frac{-1-\alpha}{2}\le R\le\frac{1-\alpha}{2}$

$\Rightarrow \quad R=0\Rightarrow 0<x<1$ ...(2)

From (1) and (2), all possible values of x are given by $0\le x<1$.

Domain of $f = [0, 1)$

Hence, the correct answer is (B).

9. For $f(x)$ to be defined:

(i) $16-x>0 \Rightarrow x<16$ (ii) $2x-1\ge 0\Rightarrow x\ge\frac{1}{2}$

(iii) $20-3x>0\Rightarrow x<\frac{20}{3}$ (iv) $4x-5\ge 0\Rightarrow x\ge\frac{5}{4}$

(v) $16-x\ge 2x-1\Rightarrow x\le\frac{17}{3}$ (vi) $20-3x\ge 4x-5\Rightarrow x\le\frac{25}{7}$

(vii) $16-x$ is an integer

∴ x should be an integer.

$\frac{5}{4}\le x\le\frac{25}{7}$ and $x\in I \Rightarrow x=2, 3$

Domain of $f = \{2, 3\}$

When $x=2$, $y={}^{14}C_3+{}^{14}C_3=728$

When $x=3$, $y={}^{13}C_5+{}^{11}C_4=1617$

Hence, the correct answer is (B).

10. $f(x)=\sum_{k=1}^{n}(1+[\sin kx])$

$= n + [\sin x] + [\sin 2x] + \ldots + [\sin nx]$ ...(1)

CASE 1: When $kx\ne\frac{\pi}{2}$ for $k = 1, 2, 3, \ldots, n$

Since $0<k\,x<\pi$ and $kx\ne\frac{\pi}{2}$

∴ $0 < \sin kx < 1$, for $k = 1, 2, \ldots, n$

∴ $[\sin kx] = 0$, for $k = 1, 2, 3, \ldots, n$

∴ From (1), $f(x) = n$

CASE 2: When exactly one of $x, 2x, 3x, \ldots, nx$ is $\frac{\pi}{2}$.

Here, not more than one of $x, 2x, 3x, \ldots, nx$ can be $\frac{\pi}{2}$. In this case one of $\sin x, \sin 2x, \ldots, \sin nx$ is 1 and others lie between 0 and 1.

From (1), $f(x) = n + 1$

Therefore, range of $f = \{n, n + 1\}$

Hence, the correct answer is (C).

11. For y to be defined:

(i) $x>0$ ...(1)

(ii) $\sin^{-1}(\log_2 x)\ge 0$

$\Rightarrow \quad \log_2 x\ge\sin 0\Rightarrow\log_2 x\ge 0$

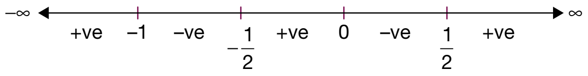
$\Rightarrow \quad x\ge 2^0\Rightarrow x\ge 1$ ...(2)

$-1\le\log_2 x\le 1\Rightarrow\frac{1}{2}\le x\le 2$ ...(3)

(iv) $\frac{2x-1}{2x^3+3x^2+x}\ge 0\Rightarrow -\infty<x<-1$

or $-\frac{1}{2}<x<0$ or $\frac{1}{2}<x<\infty$ ...(4)

Sign scheme for $\frac{2x-1}{2x^3+3x^2+x}$ is:



From (1), (2), (3) and (4), common values of x are given by $1\le x\le 2$

Hence, the correct answer is (C).

12. Given, $f(x)$ is defined when $-\frac{5}{2}\le x\le 2$

∴ $f[|x|]$ is defined when

$-\frac{5}{2}<[|x|]\le 2$

or $[x] = -2, -1, 0, 1, 2$

$[|x|] = 0, 1, 2$ $[\because |x|\ge 0]$

$\Rightarrow \quad 0\le|x|<3$

$\Rightarrow \quad |x|<3 \Rightarrow -3<x<3$

Hence, the correct answer is (D).

13. $y=\sqrt{\sin x+\cos x}+\sqrt{7x-x^2-6}$

$=\sqrt{\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)}+\sqrt{7x-x^2-6}$

For the function to be defined

(i) $\sin\left(\frac{\pi}{4}+x\right)\ge 0$

$\Rightarrow \quad 2n\pi-\frac{\pi}{4}\le x\le 2n\pi+\frac{3\pi}{4}$,

where $n = 0, \pm 1, \pm 2$ ...(1)

(ii) $7x-x^2-6\ge 0\Rightarrow 1\le x\le 6$ ...(2)

From (1) and (2),

$$x\in\left[1,\frac{3\pi}{4}\right]\cup\left[\frac{7\pi}{4},6\right]$$

Hence, the correct answer is (B).

14. $f(x)=\left[\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)\right]$

$-\sqrt{2}\le\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)\le\sqrt{2}$

∴ Range of f.

$\left[\sqrt{2}\sin\left(\frac{\pi}{4}+x\right)\right]=\{-2,-1,0,1\}$

Hence, the correct answer is (B).

15. $f(x)=\dfrac{x^3+x-2}{x^3-1} \Rightarrow f(x)=\left(\dfrac{x-1}{x-1}\right)\left(\dfrac{x^2+x+2}{x^2+x+1}\right)$

$=\dfrac{x^2+x+2}{x^2+x+1}$, when $x \neq 1$

$y=\dfrac{x^2+x+2}{x^2+x+1}$; $y=1+\dfrac{1}{x^2+x+1}$

y_{max} when $x=-\dfrac{1}{2}$; $y_{max}=1+\dfrac{4}{3}=\dfrac{7}{3}$; y_{min} when $x \to \pm\infty$

Hence, the range is $\left(1, \dfrac{7}{3}\right]$.

Hence, the correct answer is (D).

16. $-1 \le \cos x \le 1$

$-\sin 1 \le \sin(\cos x) \le \sin 1$

$\Rightarrow \quad [\sin(\cos x)] = 0, -1$

$-1 \le \sin x \le 1 \Rightarrow \cos 1 \le \cos(\sin x) \le 1$, $\qquad (\because \ \cos 0 = 1)$

$[\cos(\sin x)] = 0, 1$

$\Rightarrow \quad [\sin(\cos x)] + [\cos(\sin x)] = -1, 0, 1$.

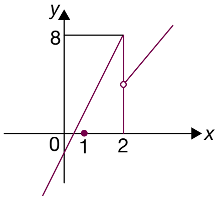
Hence, the correct answer is (D).

17. The graph of f is as follows

If $2 < f(x) < 8$ then x has two values

$\therefore \quad f: R \to R$ is not one-one

Range of f is $R \Rightarrow f: R \to R$ is onto



Hence, the correct answer is (C).

18. $f(0)=0, f(\pm 1)=0, f(\pm 2)=\pm 1, f(\pm 3)=0, f(\pm 4)=\pm 2, \ldots$

Range $= Z$ and $f(\pm 1) = 0 \Rightarrow f$ is onto but not one–one.

Hence, the correct answer is (A).

19. $0 \le \{x\} < 1 \Rightarrow \cos 1 < \cos\{x\} \le \cos 0$

$\Rightarrow \quad 1 \le \dfrac{1}{\cos\{x\}} < \dfrac{1}{\cos 1} < 2$

$\therefore \quad \left[\dfrac{1}{\cos\{x\}}\right] = 1 \Rightarrow$ Range of the function is {1}

Hence, the correct answer is (D).

20. $\tan^{-1}\sqrt{x^2-1} \ge \sec^{-1} x$ and $x^2 > 1$

$\Rightarrow \quad x > 1$

Let $x = \sec\theta$ then

$\tan^{-1}\sqrt{x^2-1} = \tan^{-1}\tan\theta = \theta = \sec^{-1} x$

So, the range of $f(x)$ is {0}.

Hence, the correct answer is (B).

21. $-1 \le \log_{2e}[x] \le 1$

$\Rightarrow \quad \dfrac{1}{2e} \le [x] \le 2e$

$\Rightarrow \quad 1 \le [x] \le 5$

$\Rightarrow \quad x = 1, 2, 3, 4, 5$

Hence, the correct answer is (B).

22. $y = \underbrace{f(-|x|)}_{\text{even function}} = f(x), x \le 0$

Even function is symmetric about y-axis.

Hence, the correct answer is (D).

23. Let $y = f(x) = \sqrt{4-x^2} + \sqrt{x^2-1}$ $\qquad$ …(1)

$\because \quad 4 - x^2 \ge 0$ and $x^2 - 1 \ge 0$

$\Rightarrow \quad x \in [-2, 2]$ and $x \in (-\infty, -1] \cup [1, \infty)$

Clearly, domain of $f(x)$ is $[-2, -1] \cup [1, 2]$

Squaring both sides of (1)

$y^2 = 4 - x^2 + x^2 - 1 + 2\sqrt{4-x^2}\ \sqrt{x^2-1}$

$y^2 = 3 + 2\sqrt{5x^2 - x^4 - 4}$

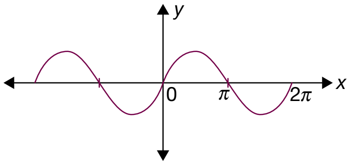
Now, $y^2 = 3 + 2\sqrt{\left(\dfrac{3}{2}\right)^2 - \left(x^2 - \dfrac{5}{2}\right)^2}$

$\therefore \quad y^2_{\text{max.}}\left(x = \pm\sqrt{\dfrac{5}{2}}\right) = 6$ and $y^2_{\text{min.}}(x = \pm 1 \text{ or } \pm 2) = 3$

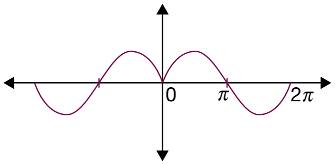
Hence, the range of $f(x)$ is $\left[\sqrt{3}, \sqrt{6}\right]$

Hence, the correct answer is (D).

24. (i) $\sin x$



(ii) $\sin|x|$



(iii) $-\sin|x|$; mirror image of $\sin|x|$ about x-axis

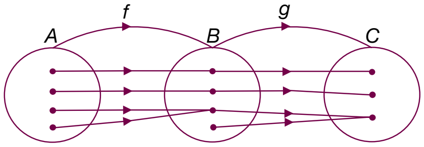
So, the given graph represents $f(x) = -\sin|x|$

Hence, the correct answer is (D).

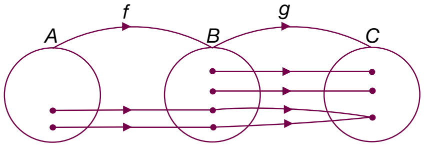
25. We have $f: A \to B$,

$g: B \to C$ and $gof: A \to C$

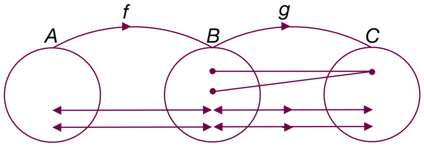
(A)



(B)

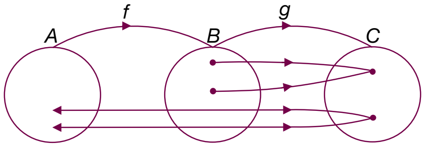


(C)



Clearly gof is one-one but g is many-one function

(D)



As $(gof)^{-1} = (f^{-1}og^{-1})$

Hence, the correct answer is (C).

26. $f(x)$ is an odd function.

$f(-1) = -f(1)$

$f(-x) = -f(x)$

Hence, the correct answer is (B).

27. $4\lambda[x]^2 = \lambda + 12$

$[x] = 1, 2, 3 \ldots$

Putting $[x] = n$

$$\lambda = \frac{12}{4n^2 - 1} = 6\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$\Rightarrow \sum_{n=1}^{\infty} \lambda = \sum_{n=1}^{\infty} 6\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)$$

$$= 6\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \ldots\right] = 6$$

Hence, the correct answer is (D).

28. $S = \{1, 2, 3, 4\}$ and $f(k) \le 2k$

So, $f(1) = 1$ or 2

$f(2), f(3)$ and $f(4)$

can be 1, 2, 3 and 4.

No. of mapping $= 2 \times 4 \times 4 \times 4 = 128$

Hence, the correct answer is (C).

29. $f(x) + f(-x) = 0$ for an odd function, $f(0) = 0$

$\therefore \quad f(f(f(-3))) = f(f(-f(3))) = f(f(-5))$

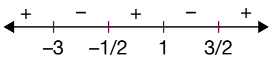
$= f(-1) = -f(1) = -2$

$3f(1) - 2f(3) - f(5) = 3(2) - 2(5) + f(-5) = 6 - 10 - 1 = -5$

$$\therefore \quad \frac{f(f(f(-3))) + f(f(0))}{3f(1) - 2f(3) - f(5)} = \frac{-2}{-5} = \frac{2}{5}$$

Hence, the correct answer is (C).

30. $f(x) = (x + 0.5)^{\log(0.5+x)\left(\frac{x^2+2x-3}{4x^2-4x-3}\right)}$



$x + 0.5 > 0$ and $x + 0.5 \ne 1$

$$\text{Also, } \frac{x^2 + 2x - 3}{4x^2 - 4x - 3} > 0 \Rightarrow \frac{(x+3)(x-1)}{(2x+1)(2x-3)} > 0$$

$$x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \cup \left(\frac{3}{2}, \infty\right)$$

Hence, the correct answer is (D).

31. $[x]^2 + [x] - 2 < 0$

$\Rightarrow \quad ([x] + 2)\,([x] - 1) < 0$

$\Rightarrow \quad -2 < [x] < 1$

$\therefore \quad [x] = -1, 0$

$\Rightarrow \quad x \in [-1, 1)$

$$\text{So, } \{x\} = \frac{1}{2} \Rightarrow x = \frac{-1}{2}, \frac{1}{2}$$

Hence, the correct answer is (B).

32. $f(x) = \dfrac{\sin 2x - \cos 2x}{4}$ $\qquad g(x) = x^2 - 4$

$$g(f(x)) = \left(\frac{\sin 2x - \cos 2x}{4}\right)^2 - 4 = \frac{1 - \sin 4x}{16} - 4$$

$$= -\frac{63}{16} - \frac{\sin 4x}{16} \quad \text{given surjective}$$

Invertible → one-one

$$4x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]; x \in \left[-\frac{\pi}{8}, \frac{\pi}{8}\right]$$

Hence, the correct answer is (C).

33. $\log(\log_{1/3}(\log_7(\sin x + \alpha))) \qquad \forall\, x \in R$

$\log_{1/3}(\log_7(\sin x + \alpha)) > 0$

$$0 < \log_7(\sin x + \alpha) < \left(\frac{1}{3}\right)^{\circ}$$

$7^{\circ} < \sin x + \alpha < 7;\ 1 < \sin x + \alpha < 7$

$-1 \le \sin x \le 1$

$$\left.\begin{aligned} 1 < 1 + \alpha < 7 \Rightarrow 0 < \alpha < 6 \\ 1 < -1 + \alpha < 7 \Rightarrow 2 < \alpha < 8 \end{aligned}\right)$$

$2 < \alpha < 6$

$\alpha = 3, 4, 5$

Hence, the correct answer is (C).

34. $f(x) = x^2 - 2x - 3;\ x \ge 1$

$x^2 - 2x - 3 = y$

$x^2 - 2x - 3 - y = 0$

$$x = \frac{2 \pm \sqrt{4 - 4(-3-4)}}{2};\ x = 1 \pm \sqrt{4+y} \quad \text{as } x \ge 1$$

$f^{-1}(x) = g(x) = 1 + \sqrt{x+4}$

$x \ge -4$

$f(x) = f^{-1}(x) = x$

$x^2 - 2x - 3 = x$

$$x^2 - 3x - 3 = 0 \Rightarrow x = \frac{3 \pm \sqrt{9+12}}{2} = \frac{3 \pm \sqrt{21}}{2} \qquad [\text{as } x \ge 1]$$

$$x = \frac{3 + \sqrt{21}}{2}$$

Hence, the correct answer is (B).

35. $f(x) = \ln\left(\dfrac{1-x}{1+x}\right)$

$$\frac{1-x}{1+x} > 0 \Rightarrow \frac{1-x}{1+x} < 0$$

$x \in (-1, 1)$

$$f(\alpha) + f(\alpha^2) = f\left(\frac{\alpha}{\alpha^2 - \alpha + 1}\right)$$

$$\ln\left(\left(\frac{1-\alpha}{1+\alpha}\right)\left(\frac{1-\alpha^2}{1+\alpha^2}\right)\right) = \ln\left(\frac{1 - \dfrac{\alpha}{\alpha^2 - \alpha + 1}}{1 + \dfrac{\alpha}{\alpha^2 - \alpha + 1}}\right)$$

$$= \ln\left(\frac{\alpha^2 - 2\alpha + 1}{\alpha^2 + 1}\right)$$

$$\ln\left(\frac{(1-\alpha)^2}{1+\alpha^2}\right) = \ln\left(\frac{\alpha^2 - 2\alpha + 1}{\alpha^2 + 1}\right) \text{ true for, all } a \in (-1, 1)$$

Hence, the correct answer is (B).

36. $\alpha = f(x) + g(y) + h(z) - [f(x)] - [g(y)] - [h(z)]$

$$= \underset{\substack{\downarrow \\ 0 \text{ to } 1}}{\{f(x)\}} + \underset{\substack{\downarrow \\ 0 \text{ to } 1}}{\{g(y)\}} + \underset{\substack{\downarrow \\ 0 \text{ to } 1}}{\{h(z)\}}$$

$\alpha \in [0, 3)$ from options $\alpha = 2$ only.

Hence, the correct answer is (B).

37. $-1 \le \sin x \le 1$

$x > 0$

$-\tan 1 \le \tan(\sin x) \le \tan 1$

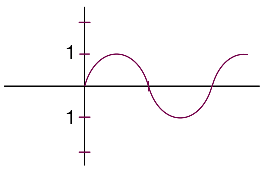
$1 \approx 57°$

$\tan 57° > 1$

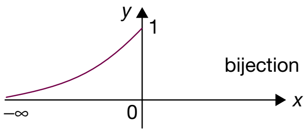
$-\tan 57° \le \tan(\sin x) \le \tan 57°$

tan(sin$x$) is best represented by (B) curve

Hence, the correct answer is (B).



38. $f(x) = \dfrac{\log x}{x}\ (x > 0)$



Equation $f(x) = K$

$\Rightarrow \quad \ln x = Kx$

$\Rightarrow \quad x = e^{Kx}$ solution of equation $(x = x_0)$

$x_0 > 0; x_0 = e^{Kx_0} = g(K)$

$K < 0$

$x_0 > 0$

Rewriting function $y = g(x) = e^{xx_0}$

$x_0 > 0$

$g : (-\infty, 0) \to (0, 1)$ It is a bijective function.

Hence, the correct answer is (A).

39. $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$

$|f(\alpha) - \alpha| \le 1; -1 \le f(\alpha) - \alpha \le 1$

$\alpha - 1 \le f(\alpha) \le \alpha + 1$

$f(1) \in \{1, 2\}$ (2 ways)

$f(2) \in \{1, 2, 3\}$ (3 ways)

$f(3) \in \{2, 3, 4\}$ (3 ways)

$f(4) \in \{3, 4\}$ (2 ways)

Total such functions $= 2 \times 3 \times 3 \times 2 = 36$

Hence, the correct answer is (B).

40. $f : N \to N$

$$f(x) = (x+1)^2 + x - \left[\sqrt{(x+1)^2 + (x+1)}\right]^2$$

$$f(1) = 4 + 1 - \left[\sqrt{6}\right]^2 = 4 + 1 - 4 = 1$$

$$f(2) = 9 + 2 - \left[\sqrt{12}\right]^2 = 9 + 2 - 9 = 2$$

$$f(3) = 16 + 3 - \left[\sqrt{20}\right]^2 = 19 - 16 = 3$$

$f(4) = 4$, clearly, we can conclude one-one and onto.

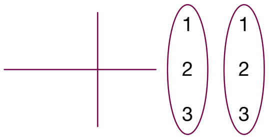
Alternate:

$\sqrt{(x+1)^2 + (x+1)} = \sqrt{(x+1)(x+2)} > (x+1)$

$(x+1) < \sqrt{(x+1)(x+2)} < (x+2)$

$f(x) = (x+1)^2 + x - (x+1)^2 = x$

$f(x) = x$



So, $f(x)$ will be one-one onto.

Hence, the correct answer is (A).

41. $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - \sin 2x}}}$

$1 - \sqrt{2 - \sqrt{3 - \sin 2x}} \ge 0; \Rightarrow 1 \ge \sqrt{2 - \sqrt{3 - \sin 2x}}$

$1 \ge 2 - \sqrt{3 - \sin 2x} \ge 0; \Rightarrow -1 \ge -\sqrt{3 - \sin 2x} \ge -2$

$\Rightarrow \quad 1 \le \sqrt{3 - \sin 2x} \le 2; \Rightarrow 1 \le 3 - \sin 2x \le 4$

$-2 \le -\sin 2x \le 1$

$2 \ge \sin 2x \ge -1$ which is always true $\forall\ x \in R$

Hence, the correct answer is (C).

42. $f(x) = \cos^2(\sin x) + \sin^2(\cos x)$

$-1 \le \sin x \le 1$ $\qquad$ $-1 \le \cos x \le 1$

$\cos 1 \le \cos(\sin x) \le 1$ $\qquad$ $-\sin 1 \le \sin(\cos x) \le \sin 1$

$$\boxed{\underset{\underset{x=\frac{\pi}{2}}{\downarrow}}{\cos^2 1} \le \cos^2(\sin x) \le \underset{\underset{x=0}{\downarrow}}{1}} \qquad \boxed{\underset{\underset{x=\frac{\pi}{2}}{\downarrow}}{0} \le \sin^2(\cos x) \le \underset{\underset{x=0}{\downarrow}}{\sin^2 1}}$$

$\cos^2 1 \le \cos^2(\sin x) + \sin^2(\cos x) \le 1 + \sin^2 1$

Hence, the correct answer is (D).

43. $f : Q \to$ Real no.

$f(x) = \sin x$

$f(x_1) = f(x_2) \Rightarrow \sin x_1 = \sin x,$

$x_1 = n\pi + (-1)^n x_2$

as $x_1, x_2 \in Q$ then acceptable solution $x_1 = x_2$ only one-one function

$f(x) = 1 = \sin x$ at $x = 2x\pi + \dfrac{\pi}{2}$ (irrational)

$\therefore \quad f(x) \ne 1$ for rational x, hence into function

Hence, the correct answer is (A).

44. $f(x) = \dfrac{x}{\ln x}$ domain $x > 0 - \{1\}$

$g(x) = \dfrac{\ln x}{x};\ x > 0$

$\dfrac{1}{f(x)} = \dfrac{\ln x}{x}$ domain $x > 0 - \{1\}$

$f(x), g(x)$ are not equal functions.

$\dfrac{1}{f(x)}, g(x)$ are also not equal functions.

$\dfrac{1}{g(x)} = \dfrac{x}{\ln x};$ domain $x > 0 - \{1\}$

$\dfrac{1}{g(x)}, f(x)$ are equal functions.

$f(x)\,g(x) = 1 \quad \forall\ x > 0 - \{1\}$

$\dfrac{1}{f(x)g(x)} = 1 \quad \forall\ x > 0 - \{1\}$

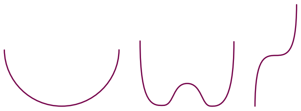
Hence, the correct answer is (A).

45. (A) $f : R \to R$

$f(x) = x^4 + 2x^3 - x^2 + 1$

$f'(x) = 4x^3 + 6x^2 - 2x$

$= 2x(x^2 + 3x - 1)$



(f is non-montonic); Range $\ne R$, not surjective not injective

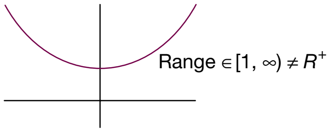
(B) $f(x) = x^3 + x + 1$

$f'(x) = 3x^2 + 1 > 0\ \forall\ x \in R$

Range $= R$

(C) $f(x) = \sqrt{1 + x^2}$ even fn.

many-one, into

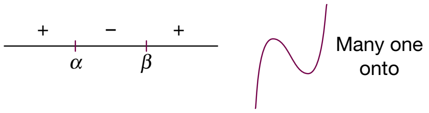


one-one onto

(D) $f(x) = x^3 + 2x^2 - x + 1$

$f'(x) = 3x^2 + 4x - 1 \qquad (\because\ D > 0)$

$f(x)$ is non-monotonic, f is many-one and onto.

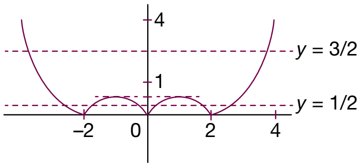


Hence, the correct answer is (D).

46. $||g(x)| - 1| = \dfrac{1}{2}$

$|g(x)| - 1 = \pm\dfrac{1}{2}$

$|g(x)| = \dfrac{3}{2}$ or $|g(x)| = \dfrac{1}{2}$



No. of solutions $= 6 + 2 = 8$

Hence, the correct answer is (D).

47. $f(x) = \tan^{-1}\sqrt{[x]+[-x]} + \sqrt{2-|-x|} + \underset{\substack{\downarrow \\ x\neq 0}}{\frac{1}{x^2}}$

$$[x]+[-x] = \begin{cases} 0 & ; \quad x \in I \\ -1 & ; \quad x \notin I \end{cases}$$

have $x \in I$

$$2 - |x| \geq 0 \Rightarrow -2 < x \leq 2$$

$f(x)$ is an even funtion.

domain $x \in \{-2, -1, 1, 2\}$

$$f(x) = \sqrt{2-|x|} + \frac{1}{x^2}; f(1) = 1 + 1 = 2$$

$$f(2) = 0 + \frac{1}{4}$$

Range $= \left\{\frac{1}{4}, 2\right\}$

Hence, the correct answer is (C).

48.

I.	$f(x)$ one-one $f(x) = x$	$g(x)$ one-one $g(x) = -x$ but $f(x) + g(x) = x - x = 0$ $=$ constant fn.(many-one)
II.	$f(x) = x$	$g(x) = -x$ $f(x)\,g(x) = -x^2$ many-one
III.	$f(x) = \sin x$ I, II, III all wrong	odd but not one-one

Hence, the correct answer is (D).

49. $f(x) = \cos^4 x + K\cos^2 2x + \sin^4 x$

$$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x + K\cos^2 2x$$

$$f(x) = 1 - \frac{1}{2}\sin^2 2x + K\cos^2 2x$$

For $f(x)$ to be constant $K = -\frac{1}{2}$

$$f(x) = 1 - \frac{1}{2}(1) = \frac{1}{2}$$

Hence, the correct answer is (B).

50. $f(x) = \begin{cases} x^2 & ; \quad x \notin Q \\ 2 - x^2 & ; \quad x \in Q \end{cases}; \; g(x) = \begin{cases} 2 - x^2 & ; \quad x \notin Q \\ x^2 & ; \quad x \in Q \end{cases}$

$$(f+g)(x) = \begin{cases} 2 & ; \quad x \notin Q \\ 2 & ; \quad x \in Q \end{cases}$$

$(f+g)(x) = 2$ for all $x \in R$ (many-one and into)

Hence, the correct answer is (D).

51. $f(x) = \frac{e^x + 1}{e^x - 1} = y$ where $e^x - 1 \neq 0; x \neq 0$

$n(d) = 1$

$(e^x - 1)y = e^x + 1$

$e^x y - y = e^x + 1 \Rightarrow e^x(y - 1) = y + 1$

$$x = \ln\left(\frac{y+1}{y-1}\right)$$

$\frac{y+1}{y-1} > 0; n(r) = 3; y \neq -1, 0, 1$ then $n(r) = 3$

$\therefore \quad n(d) + n(r) = 1 + 3 = 4$

Hence, the correct answer is (C).

52. $f(x) = \pi\left(\frac{(x+7-16)}{(x-9)\left(\sqrt{x+7}+4\right)}\right) = \frac{\pi}{\left(\sqrt{x+7}+4\right)}; \; x \neq 9$

$$0 \leq \sqrt{x+7} < \infty$$

$$\Rightarrow \quad 4 \leq \sqrt{x+7} + 4 < \infty \Rightarrow \frac{1}{4} \geq \frac{1}{\sqrt{x+7}+4} > 0$$

$$\frac{\pi}{4} \geq \frac{\pi}{\sqrt{x+7}+4} > 0 \text{ but } x \neq 9$$

$$\frac{\pi}{2} \geq 2f(x) > 0 \text{ but } 2f(x) \neq \frac{\pi}{4}$$

then $f(x) \in \left(0, \frac{\pi}{4}\right] - \left\{\frac{\pi}{8}\right\}$

$y = \sin(2f(x))$

$$y \in (0, 1] - \left\{\frac{1}{\sqrt{2}}\right\}$$

Hence, the correct answer is (C).

53. $f(x) = (x+2)e^{\ln(x+2)} = (x+2)^2; x > -2;$

For $x \neq 0, 1$

When $x > 0$

$$g(x) = \frac{e^{-\log_e x}}{x} - \frac{2x}{e^{\ln\frac{1}{2}}} = \frac{e^{\ln\frac{1}{x}}}{x} - \frac{2x}{\frac{1}{2}} = \frac{1}{x^2} - 4x$$

$$y = f(x) + g(x) = x^2 + 4 + 4x + \frac{1}{x^2} - 4x$$

$x \in (0, 1) \cup (1, \infty)$

$$= x^2 + 4 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 + 2; \; x + \frac{1}{x} > 2$$

When $x \neq 0, 1$

$$y = \left(x + \frac{1}{x}\right)^2 > 4 \Rightarrow y = \left(x + \frac{1}{x}\right)^2 + 2 > 6$$

$y \in (6, \infty)$ is the smallest interger in range $= 7$

Hence, the correct answer is (D).

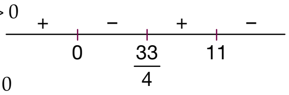
54. $f(x) = \frac{1}{\sqrt{{}^{10}C_{x-1} - 3 \cdot {}^{10}C_x}}$

$x - 1 \geq 0 \Rightarrow x \geq 1; x \in N; x \leq 10$

${}^{10}C_{x-1} - 3 \cdot {}^{10}C_x > 0$

$$\frac{10!}{(x-1)!(11-x)!} - \frac{3 \cdot 10!}{x!(10-x)!} > 0$$

$$\Rightarrow \quad \frac{10!}{(x-1)!(10-x)!}\left(\frac{1}{11-x} - \frac{3}{x}\right) > 0$$



$$\Rightarrow \quad \frac{x - 33 + 3x}{(11-x)x} > 0; \Rightarrow \frac{4x - 33}{(11-x)x} > 0$$

$$\Rightarrow \quad x \in \left(\frac{33}{4}, 11\right) \text{ final } x \in \{9, 10\}$$

Hence, the correct answer is (A).

55. $f : [0, 2] \to [2, 5]$

$f(x) = 3x^2 - 6x + 5$

$x \in [0, 2]$

$$x = \frac{-b}{2a} = \frac{6}{2(3)} = 1 \in [0, 2]$$

$$\left.\begin{array}{l} f(0) = 5 \\ f(1) = 3 - 6 + 5 = 2 \\ f(2) = 12 - 12 + 5 = 5 \end{array}\right\} \text{Range} \in [2, 5]$$

Many-one and onto

Hence, the correct answer is (B).

CHAPTER 2

56. Total fn. $= 3^4 = 81$

Group 2, 1, 1

For onto function, 4 elements will be connected to all 3 elements of set B. Exactly 2 elements will be connected to one element and remaining two elements are connected to remaining two elements.

$$\text{Onto} = \left(\frac{4!}{2!1!1!} \times \frac{1}{2!}\right) 3! = 36$$

Into $= 81 - 36 = 45$

Hence, the correct answer is (B).

57. $[2 \sin x] + [\cos x] = -3$ only if

$$[2 \sin x] = -2 \text{ and } [\cos x] = -1$$

$\therefore \quad -2 \le 2 \sin x < -1$ and $-1 \le \cos x < 0$

$\Rightarrow \quad -1 \le \sin x < -\frac{1}{2}$ and $-1 \le \cos x < 0$

$\Rightarrow \quad \frac{7\pi}{6} < x < \frac{11\pi}{6}$ and $\frac{\pi}{2} < x < \frac{3\pi}{2}$

Common values of x are $\frac{7\pi}{6} < x < \frac{3\pi}{2}$.

For these values of x,

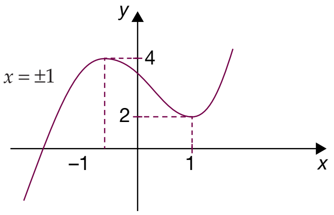
$$\sin x + \sqrt{3} \cos x = 2 \sin\left(\frac{\pi}{3} + x\right)$$

lies between –2 and –1

Range of $f(x)$ is $(-2, -1)$.

Hence, the correct answer is (B).

58. Let $g(x) = \frac{x^3 - 3x + 6}{2}$



Then $g'(x) = 0 \Rightarrow \frac{3x^2 - 3}{2} = 0 \Rightarrow x = \pm 1$

Now $g(1) = \frac{1 - 3 + 6}{2} = 2$ and

$$g(-1) = \frac{-1 + 3 + 6}{2} = 4$$

$\therefore \quad y = 2$ and $y = 4$ touch $y = g(x)$ at $x = 1$ and $x = -1$ respectively.

Now $g(x) = 2 \Rightarrow x^3 - 3x + 6 = 4 \Rightarrow x^3 - 3x + 2 = 0$

$\Rightarrow \quad (x - 1)^2 (x + 2) = 0$

Also, $g(x) = +4 \Rightarrow x^2 - 3x + 6 = 8 \Rightarrow x^2 - 3x - 2 = 0$

$(x + 1)^2 (x - 2) = 0$

$\therefore \quad g(x) = 2 \Rightarrow x = 1, 1, -2$ and $g(x) = 4 \Rightarrow x = -1, -1, 2$

Now due to $(x - \alpha)$ a void will be created in $y = g(x)$ graph. For the given condition to be true that void can be created $\forall x \in (-\infty, -2) \cup (2, \infty)$.

Hence $\alpha \in R - [-2, 2]$

Hence, the correct answer is (A).

59. $f(x) = |x - 1| = \begin{cases} 1 - x; & 0 < x < 1 \\ x - 1; & x \ge 1 \end{cases}$, $g(x) = e^x, x \ge -1$

$$fog(x) = \begin{cases} 1 - g(x); & 0 < g(x) < 1 \\ g(x) - 1; & g(x) \ge 1 \end{cases}$$

$$fog(x) = \begin{cases} 1 - e^x; & -1 \le x < 0 \\ e^x - 1; & x \ge 0 \end{cases}$$

$\therefore \quad$ Domain $= [-1, \infty)$

fog is decreasing in $[-1, 0]$ and increasing in $(0, \infty)$

When $x \to \infty, fog(x) \to \infty \quad \Rightarrow fog(0) = 0$. Range $= [0, \infty)$

Hence, the correct answer is (B).

60. $f(x + y) + f(x - y) = 2f(x) f(y)$

$$2f(0) = 2f^2(0)$$

$$f(0) \ne 0, f(0) = 1$$

$$f(y) + f(-y) = 2f(0) f(y)$$

$$f(y) + f(-y) = 2f(y)$$

$$f(-y) = f(y)$$

$f(-x) = f(x)$, even.

Hence, the correct answer is (B).

61. $f(x) + f(x + a) + f(x + 2a) + \ldots + f(x + na) = k$

Replacing $x = x + a$

$$f(x + a) + f(x + 2a) + \ldots + f(x + (n + 1)a) = k$$

Subtracting we get,

$$f(x) - f(x + (n + 1)a) = 0$$

$$f(x) = f(x + (n + 1)a$$

$\therefore \quad T = (n + 1)a$

Hence, the correct answer is (A).

62. $f(x) = f(-x)$ where $f(x) = x^2 + \sin x + xe^x$

$$F(x) = x^2 - \sin x - xe^{-x} \quad \ldots(1)$$

$g(x) = -g(-x)$, where $g(x) = \cos x + x^2 - x$

$$G(x) = -\left(\cos x + x^2 + x\right) = -\cos x - x^2 - x$$

$\therefore \quad F(x) + G(x) = -\sin x - xe^{-x} - \cos x - x$

$$= -\left(\sin x + \cos x + x + xe^{-x}\right)$$

Hence, the correct answer is (C).

63. Given, $q^2 - 4pr = 0$ and $p > 0$.

For $f(x)$ to be defined

$$px^3 + (p + q)x^2 + (q + r)x + r > 0$$

$\Rightarrow \quad px^2 (x + 1) + qx (x + 1) + r (x + 1) > 0$

$\Rightarrow \quad (x + 1)(px^2 + qx + \mathrm{r}) > 0; \Rightarrow x > -1$ and $x \ne -\frac{q}{2p}$

[Since $q^2 - 4pr = 0 \quad \therefore$ At $x = -\frac{q}{2p}, px^2 + qx + r = 0$

and at $x \ne -\frac{q}{2p}, px^2 + qx + r > 0$]

$\therefore \quad$ Domain $= R - \left[(-\infty, -1] \cup \left\{-\frac{q}{2p}\right\}\right]$

Hence, the correct answer is (B).

64. Let $g(x) = x^{12} - x^9 + x^4 - x + 1, g(x) \ge 0$

For $f(x)$ to be defined

$\Rightarrow \quad x^{12} + x^4 - x^9 - x + 1 \ge 0$

$\Rightarrow \quad x^4\left(x^8 + 1\right) - x\left(x^8 + 1\right) + 1 \ge 0$

$\Rightarrow \quad \left(x^8 + 1\right)\left(x^4 - x\right) + 1 \ge 0$

$\Rightarrow \quad \left(x^8 + 1\right)x\left(x^3 - 1\right) + 1 \ge 0 \quad \ldots(1)$

For $x \le -1$ or $-1 < x \le 0$ or $x \ge 1$, $x\left(x^3 - 1\right) \ge 0$

$g(x) > 0$

For $0 < x < 1$, we write

$$g(x) = x^{12} - \left(x^9 - x^4\right) - (x - 1) > 0$$

$$\left[\therefore 0 < x < 1 \quad \therefore x^9 < x^4 \quad \therefore x^9 - x^4 < 0 \text{ and } x - 1 < 0\right]$$

Thus for all real x, $g(x)\, x > 0$ and hence $f(x)$ is defined.

$\therefore \quad$ Domain $= (-\infty, \infty)$

Hence, the correct answer is (D).

65. For $f(x)$ to be defined,

$$2^{2x} + (64)^{(x-2)/3} - 2^{-1}\left(72 + 2^{2x}\right) \ge 0$$

$\Rightarrow \quad 4^x + 4^{x-2} - 36 - \frac{4^x}{2} \ge 0$

$\Rightarrow \quad y + \frac{y}{16} - 36 - \frac{y}{2} \ge 0$, where $y = 4^x$

$\Rightarrow \quad y \geq 64 \Rightarrow 4^x \geq 4^3 \Rightarrow x \geq 3$

Domain of $f = [3, \infty)$

Hence, the correct answer is (C).

66. $f(x) = \text{sgn}(x)$, $g(x) = x(1 - x^2)$

$$(gof)(x) = 0 \text{ and } (fog)(x) = \begin{cases} 1; & \text{if } \ x \in (-\infty, -1) \cup (0, 1) \\ 0; & \text{if } \ x = -1, 0, 1 \\ -1; & \text{if } \ x \in (-1, 0) \cup (1, \infty) \end{cases}$$

$gof(x)$ is even and odd both
$fog(x)$ is an odd function.
Hence, the correct answer is (C).

67. $f(-x) = f(x)$, $g(-x) = -g(x)$

$\therefore \quad x^2 f(x) - 2f\left(\frac{1}{x}\right) = g(x)$ and

$$x^2 f(-x) - 2f\left(\frac{-1}{x}\right) = g(-x) \Rightarrow x^2 f(x) - 2f\left(\frac{1}{x}\right) = -g(x)$$

$\Rightarrow \quad x^2 f(x) - 2f\left(\frac{1}{x}\right) = 0$

Replace x with $\frac{1}{x}$, we get $\frac{1}{x^2} f\left(\frac{1}{x}\right) - 2f(x) = 0$

$\therefore \quad -3x^2 f(x) = 0 \Rightarrow f(x) = 0 \qquad (\because x \neq 0)$

$\therefore \quad f(10) = 0$

Hence, the correct answer is (A).

68. When $x = n\pi, n \in I$, $\sin x = 0$ and $\left[\frac{x}{\pi}\right] + \frac{1}{2} \neq 0$
$f(x) = 0$
When $x = n\pi$, $f(x) = 0$ and $f(-x) = 0$
$f(-x) = f(x)$
When $x \neq n\pi, n \in I, \frac{x}{\pi} \neq$ an integer

$\therefore \quad \left[\frac{x}{\pi}\right] + \left[-\frac{x}{\pi}\right] = -1$

$\therefore \quad \left[-\frac{x}{\pi}\right] = -1 - \left[\frac{x}{\pi}\right]$

$\Rightarrow \quad \left[-\frac{x}{\pi}\right] + \frac{1}{2} = -\left[\frac{x}{\pi}\right] - \frac{1}{2} = -\left(\left[\frac{x}{\pi}\right] + \frac{1}{2}\right)$

$$\text{Now } f(-x) = \frac{\sin^{101}(-x)}{\left[-\frac{x}{\pi}\right] + \frac{1}{2}} = \frac{-\sin x}{-\left(\left[\frac{x}{\pi}\right] + \frac{1}{2}\right)} = \frac{\sin x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} = f(x)$$

Therefore, in all cases $f(-x) = f(x)$
Hence, the correct answer is (B).

69. $\sin \frac{\pi x}{(n-1)!}$ is a periodic function with period

$2\pi \div \frac{\pi}{(n-1)!} = 2(n-1)!$

Period of $\cos \frac{\pi x}{n!} = 2\pi \div \frac{\pi}{n!} = 2n!$

LCM of $2(n-1)!$ and $2(n!)$ is $2(n!)$
Period of $f(x)$ is $2(n!)$
Hence, the correct answer is (B).

70. Given: $f(T + x) = 1 + (1 - f(x))^3]^{1/3}$
$= 1 + (1 - f(x))$
$\Rightarrow \quad f(T + x) + f(x) = 2$
$\Rightarrow \quad f(2T + x) + f(T + x) = 2$
$(2) - (1) \Rightarrow f(2T + x) = f(x)$
Also T is positive and least therefore period of $f(x) = 2T$
Hence, the correct answer is (B).

71. Given, $f(g(x)) = |\cos x| = \sqrt{\cos^2 x}$ ...(1)

$g(f(x)) = \cos^2 \sqrt{x}$ ...(2)

From (1) and (2), $f(x) = \sqrt{x}$

and $g(x) = \cos^2 x$

Clearly, $f(x)$ is a non-periodic function and $g(x)$ is a periodic function.
Hence, the correct answer is (B).

72. Given $f\left(\frac{5x - 3y}{2}\right) = \frac{5f(x) - 3f(y)}{2}$

$\Rightarrow \quad f\left(\frac{5x - 3y}{5 - 3}\right) = \frac{5f(x) - 3f(y)}{5 - 3}$, which satisfies $f(x) = ax + b$

Formula for abscissa on L.H.S. and ordinate on R.H.S. Hence, $f(x)$ must be the linear function (as only) straight line satisfies such section formula).
But $f(0) = 3 \Rightarrow b = 3, f'(0) = 2 \Rightarrow a = 2$.

Thus, $f(x) = 2x + 3 \Rightarrow$ Period of $\sin(f(x)) = \sin(2x + 3)$ is $\frac{2\pi}{|2|} = \pi$
Hence, the correct answer is (B).

73. p, must be divisible by 2 and 3 both
Hence $p_1 = 6, p_2 = 2, p_3 = 3$.
Hence, the correct answer is (C).

74. Given, $f(x) = x^3 + ax^2 + 6x + \cos 3x$ ...(1)

$\therefore \quad f'(x) = 3x^2 + 2ax + 6 - 3\sin 3x$ ...(2)

Clearly $f(x)$ is continuous and $f(-\infty) = -\infty, f(\infty) = \infty$
Therefore, the range $f = (-\infty, \infty)$
$\therefore$ f is onto
$\therefore$ For f^{-1} to exist, f must be one-one

For this $f'(x) \geq 0 \; \forall x \in R$ $\quad$ [$\because$ coeff. of x^2 in $f'(x)$ is + ve]

But $f'(x) = 3x^2 + 2ax + 3 + 3(1 - \sin 3x)$

$\therefore \quad f'(x) \geq 3x^2 + 2ax + 3 \quad [\because -1 \leq \sin 3x \leq 1]$

$\therefore$ f will be necessarily increasing and hence it is also one-one if
$3x^2 + 2ax + 3 \geq 0 \; \forall \; x \in R$

$\Rightarrow \quad D \leq 0 \Rightarrow 4a^2 - 36 \leq 0 \Rightarrow a^2 - 9 \leq 0 \Rightarrow -3 \leq a \leq 3$

Hence, the correct answer is (B).

75. The function $y = f(x) = x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$ increases in the interval $\left[\frac{1}{2}, \infty\right)$ and x varying in the indicated interval we have

$y = f(x) \geq \frac{3}{4}$ i.e, $y \in \left[\frac{3}{4}, \infty\right)$

$\therefore \quad x^2 - x + 1 - y = 0$

$\Rightarrow \quad x = \frac{1}{2} + \sqrt{\left(y - \frac{3}{4}\right)} = g(y)$

$\therefore \quad y = g^{-1}(x) \Rightarrow f(x) = g^{-1}(x)$

Since the graphs of the original and inverse functions can intersect only on the straight line $y = x$.

$\therefore \quad x = f(x) \Rightarrow x = x^2 - x + 1$

$\therefore \quad x = 1$

Hence, the correct answer is (C).

76. $f(xy) = \frac{f(x)}{y} \Rightarrow f(y) = \frac{f(1)}{y}$ $\qquad$ (putting $x = 1$)

$\Rightarrow \quad f(30) = \frac{f(1)}{30}$ or $f(1) = 30 \times f(30) = 30 \times 20 = 600$

Now, $f(40) = \frac{f(1)}{40} = \frac{600}{40} = 15$

Hence, the correct answer is (A).

77. $f(2012) = f(2011 + 1) = f(2011.1) = f(2011) = 2010$.

Hence, the correct answer is (B).

78. $f(x) + f(x + 4) = f(x + 2) + f(x + 6)$ …(1)

Replacing x by $x + 2$ in equation (1), we get

$f(x + 2) + f(x + 6) = f(x + 4) + f(x + 8)$ …(2)

Adding (1) and (2)

$f(x) = f(x + 8)$

$f(5) = f(13) = 10$

$f(13) = f(21) = 10$ …

- - - - - - - - - - - -

- - - - - - - - - - - -

$f(805) = 10$

$$\therefore \sum_{r=1}^{100} f(5+8r) = f(13) + f(21) + \ldots + f(805)$$

$= 10 + 10 + \ldots$ 100 times. $= 10 \times 100 = 1000$.

Hence, the correct answer is (A).

79. Side of the triangle with vertices $(0,0)$ and $(x, g(x))$ is $a = \sqrt{x^2 + (g(x))^2}$

Area of equilateral triangle, whose side a is $\frac{\sqrt{3}}{4}a^2$

$$\therefore \frac{\sqrt{3}}{4}[x^2 + g(x)^2] = \frac{\sqrt{3}}{4} \Rightarrow x^2 + g(x)^2 = 1 \Rightarrow g(x) = \pm\sqrt{1-x^2}.$$

Thus, $g(x) = \sqrt{1-x^2}$

Hence, the correct answer is (B).

80. Put $y = a, z = b$ then $f(y) = 0, f(z) = 0$

$$\frac{f(x)}{(x-a)(x-b)} = \frac{k}{x\,ab} \Rightarrow \frac{1}{x} = \frac{k}{xab} \Rightarrow k = ab$$

Hence, the correct answer is (C).

81. $1 + 2f(2) = 6f(2) \Rightarrow f(2) = 1/4$

$$1 + \frac{1}{2} + 3f(3) = 12f(3) \Rightarrow 9f(3) = \frac{3}{2} \Rightarrow f(3) = \frac{1}{6} \ldots$$

$$\Rightarrow \quad f(n) = \frac{1}{2n}$$

$$\therefore \quad 48f(12) = 48 \times \frac{1}{24} = 2$$

Hence, the correct answer is (B).

82. $f(x) = \sin(\sin^{-1}x) = x, \forall\, x \in [-1, 1]$ Which is one-one and onto

$$f(x) = \frac{2}{\pi}\sin^{-1}(\sin x) = \frac{2}{\pi}x$$

The range of the function for $x \in [-1, 1]$ is $\left[-\frac{2}{\pi}, \frac{2}{\pi}\right]$ which is a subset of $[-1, 1]$.

Hence, the function is one-one but not onto, hence not bijective.

$$f(x) = (\text{Sgn}(x))\log(e^x) = (\text{Sgn}(x))\,x = \begin{cases} x\ ; & x > 0 \\ -x; & x < 0 \\ 0\ ; & x = 0 \end{cases}$$

This function has the range $[0, 1]$

Which is a subset of $[-1, 1]$. Hence the function is into also the function is many-to-one.

$$f(x) = x^3\,\text{sgn}(x) = \begin{cases} x^3\ ; & x > 0 \\ -x^3; & x < 0 \\ 0\ ; & x = 0 \end{cases}$$ which is many-to-one and into

Hence, the correct answer is (A).

83. $f(x) = -f(-x) = -e^{-x} + \sin x,\ x \in (0, 4)$

Hence, the correct answer is (B).

84. $f^{-1}(x) = 3 - x, g^{-1}(x) = \frac{x-2}{3}$

$g^{-1}[f^{-1}(2)] = g^{-1} = -\frac{1}{3}$

Hence, the correct answer is (B).

85. $2f(x^2) + 3f\left(\frac{1}{x^2}\right) = x^2 - 1 \Rightarrow 4f(x^2) + 6f\left(\frac{1}{x^2}\right) = 2x^2 - 2$ …(1)

$2f\left(\frac{1}{x^2}\right) + 3f(x^2) = \frac{1}{x^2} - 1 \Rightarrow 9f(x^2) + 6f\left(\frac{1}{x^2}\right) = \frac{3}{x^2} - 3$ …(2)

$(2) - (1) \Rightarrow 5f(x^2) = \frac{3}{x^2} - 2x^2 - 1 \Rightarrow f(x^2) = \frac{3 - 2x^4 - x^2}{5x^2}$

Hence, the correct answer is (C).

86. $$f(f(x)) = \frac{2\alpha f(x)}{2f(x)+1} = \frac{\frac{(2\alpha)2\alpha x}{2x+1}}{\frac{2\cdot 2\alpha x}{2x+1}+1} = \frac{4\alpha^2 x}{(4\alpha+2)x+1}$$

$$f(f(x)) = x \Rightarrow \frac{4\alpha^2 x}{(4\alpha+2)x+1} = x,$$

for all values of x $\Rightarrow 4\alpha^2 - 1 = (4\alpha+2)x$

$(2\alpha+1)\{(2\alpha-1) - 2x\} = 0$

$\Rightarrow \alpha = -\frac{1}{2}$, since $(2\alpha - 1) \neq 2x$ for all values of x

Hence, the correct answer is (B).

87. $\because k \in$ odd

$f(k) = k + 3$ (even)

CASE 1: if $\frac{k+3}{2} \in$ odd $\Rightarrow 27 = \frac{k+3}{2} + 3 \Rightarrow k = 45$ not possible

CASE 2: Now let $\frac{k+3}{2} \in$ even

$$\therefore \quad 27 = f(f(f(k))) = f\left(\frac{k+3}{2}\right) = \frac{k+3}{4}$$

$\therefore \quad k = 105$

verifying $f(f(f(105))) = f(f(108)) = f(54) = 27$.

$\therefore \quad k = 105$

Hence, the correct answer is (B).

88. Let $\log_{30}\log_{15}30 = t \Rightarrow \log_{15}30 = 30^t$

$30 = 15^{30^t}$

$1 = \log_{30} 15^{30^t}$

$1 = 30^t(\log_{30} 15)$

$30^{-t} = \log_{30} 15$

$(-t = \log_{30}\log_{30} 15)$

$\Rightarrow \quad \log_{30}(\log_{30}(15)) = -t$

$\therefore \quad f(t) + f(-t) = 12 \Rightarrow f(-t) = 5$

Hence, the correct answer is (A).

89. $[x]^2 = x + 2\{x\},\ [x]^2 = [x] + 3\{x\}$

$$\Rightarrow \quad \{x\} = \frac{[x]^2 - [x]}{3}$$

But $0 \leq \{x\} < 1 \Rightarrow 0 \leq \frac{[x]^2 - [x]}{3} < 1 \Rightarrow 0 \leq [x]^2 - [x] < 3$

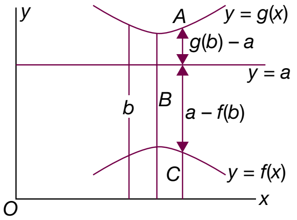
$$[x] \in \left(\frac{1-\sqrt{13}}{2}, 0\right] \cup \left[1, \frac{1+\sqrt{13}}{2}\right),$$

$$\therefore \quad [x] = -1, 0, 1, 2,\ \{x\} = \frac{2}{3}0, 0, \frac{2}{3}$$

$\therefore \quad x = -\frac{1}{3}, 0, 1, \frac{8}{3}$

Hence, the correct answer is (B).

90. $y = f(x)$ and $y = g(x)$ are mirror image of each other about line $y = a$.



$AB = g(b) - a = a - f(b)$,

$\therefore \quad g(b) + f(b) = 2a$, $h(b) = f(b) + g(b) = 2a$ (constant)

$\forall x, g(x) + f(x) =$ constant

So, $h(x)$ is constant function. Thus, it is neither one-one nor onto.

Hence, the correct answer is (D).

91. $\sin^2 x + \sin x \cos x = [x] \Rightarrow \sin 2x - \cos 2x = 2[x] - 1$

$\Rightarrow \quad -\sqrt{2} \le 2[x] - 1 \le \sqrt{2} \Rightarrow \frac{1-\sqrt{2}}{2} \le [x] \le \frac{1+\sqrt{2}}{2}$

$\Rightarrow \quad [x] = 0, 1 \Rightarrow x \in [0, 1) \cup [1, 2)$

Hence, the correct answer is (D).

92. If $f(x)$ is an even function, then

$f(x) = f(-x) \Rightarrow f(x) - f(-x) = 0 \Rightarrow \left[\frac{x^2}{a}\right] \sin x = 0$

$\Rightarrow \quad \left[\frac{x^2}{a}\right] = 0 \Rightarrow 0 \le \frac{x^2}{a} < 1 \Rightarrow a > x^2 \Rightarrow a > 400$ (since $-20 \le x \le 20$)

$\therefore \quad a \in (400, \infty)$.

Hence, the correct answer is (B).

93. For $f(x)$ to be odd, $\left[\frac{x^2}{|a|}\right]$ should not depend on value of x.

Since $x \in [-4, 4] \Rightarrow 0 \le x^2 \le 16$

$\Rightarrow \quad \left[\frac{x^2}{|a|}\right] = 0$ if $|a| > 16 \Rightarrow a \in (-\infty, -16) \cup (16, \infty)$

Hence, the correct answer is (B).

94. $0 < 1 < 2$

$\Rightarrow \quad f(0) \le f(1) \le f(2)$

$f(0) < f(1) < f(2) \Rightarrow {}^8C_3$

$f(0) < f(1) = f(2) \Rightarrow {}^8C_2$

$f(0) = f(1) < f(2) \Rightarrow {}^8C_2$

$f(0) = f(1) = f(2) = {}^8C_1$

So, number of functions $= {}^8C_3 + {}^8C_2 + {}^8C_2 + {}^8C_1 = {}^{10}C_3$

Hence, the correct answer is (C).

95. $\because$ LHS is an integer

$\because$ RHS is must be an integer for which x is multiple of 30.

$\because$ $x = 30, 60, 90, 120, \ldots, 990$

$\Rightarrow$ Number of possible values of x is 33.

Hence, the correct answer is (B).

96. $f(f(x)) = \begin{cases} f(x) = x, & \text{if } x \text{ is rational} \\ 1 - f(x) = 1 - (1 - x) = x, & \text{if } x \text{ is irrational} \end{cases}$

Hence, the correct answer is (C).

97. (A) $x^3 - x$ is not an injective function

(B) f and f^{-1} are mirror image to each other w.r.t to lien

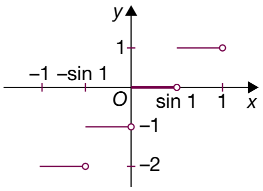
(C) $\sin |x|$ in not a periodic function

(D) $f(x) = e^x$; $g(x) = e^{-x} \Rightarrow h(x) = 1$ periodic

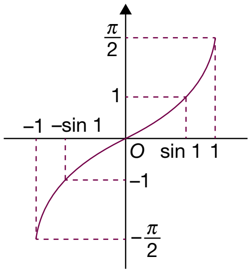
Hence, the correct answer is (B).

98. Sequence of graph transformation $\sin^{-1} x$ to $[\sin^{-1} x]$ to $[\sin^{-1}(-|x|)]$

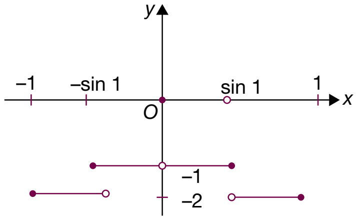
$f(x) = [\sin^{-1} x]$



$y = \sin^{-1} x$



$f(-|x|)$



Hence, the correct answer is (B).

99. $x^2 - 15x + 55 \ge 1 \cup x^2 - 15x + 55 \le -1$

$x^2 - 15x + 54 \ge \cup x^2 - 15x + 56 \le 0$

$(x - 9)(x - 6) \ge 0 \cup (x - 8)(x - 7) \le 0$

$x \in (-\infty, 6] \cup [9, \infty) \cup [7, 8]$ ...(1)

$\because \quad x^2 + 1 > 1$ and $0 < \frac{x}{8} < 1 \Rightarrow 0 < x < 8$ ...(2)

$x^2 - 6x + 5 \ne 0 \Rightarrow x \ne 5, x \ne 1$ ...(3)

From (1) and (2) and (3)

$(0, 1) \cup (1, 5) \cup (5, 6] \cup [7, 8)$

Hence, the correct answer is (A).

100. $\because \quad f^{-1}(2a - 4) = \frac{1}{2} \Rightarrow f\left(\frac{1}{2}\right) = 2a - 4$

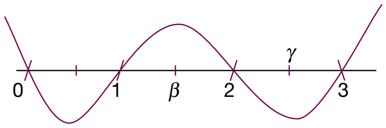
$\Rightarrow \quad \left(\frac{9}{\log_2(3-1)} - 1\right)^{\frac{1}{3}} = 2a - 4$

$2 = 2a - 4$

$a = 3$

Hence, the correct answer is (C).

101. $f(x) = x(x - 1)(x - 2)(x - 3)$



$f'(\alpha) = f'(\beta) =$ Many-one; $f(x)$ is a cubic

$\Rightarrow$ onto

Hence, the correct answer is (C).

102. $f(x) = \frac{|x^2 - x - 30|}{(x^2 - x - 30)} = \frac{|(x-6)(x+5)|}{(x-6)(x+5)}$

$$= \begin{cases} 1; & x \in (-\infty, -5) \cup (6, \infty) \\ -1; & x \in (-5, 6) \\ \text{not defined}; & x = -5, 6 \end{cases}$$

(A) option is wrong

as $y = -1$ not $y = -5$

Hence, the correct answer is (C).

103. $f(x) = \sin x + \cos x + 2\sqrt{2}$

$f(x) = \sqrt{2} \cdot \sin\left(x + \frac{\pi}{4}\right) + 2\sqrt{2}$ $\quad\because\ -1 \le \sin\left(x + \frac{\pi}{4}\right) \le 1$

$\Rightarrow\ -\sqrt{2} \le \sqrt{2}\ \sin\left(x + \frac{\pi}{4}\right) \le \sqrt{2}$

$\Rightarrow\ 2\sqrt{2} - \sqrt{2} \le f(x) \le 2\sqrt{2} + \sqrt{2}$

$\therefore\ Y = \text{Range of function} = \left[\sqrt{2}, 3\sqrt{2}\right]$

Clearly, f will be one-one also if $X = \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Hence, the correct answer is (D).

104. If $x = a$, where 'a' is an integer, then $f(a) = 2a + a + \frac{1}{2}\sin 2a$.

But $\lim\limits_{h \to 0} f(a - h) = 2a + (a - 1) + \frac{1}{2}\sin 2a$

$\therefore$ Values between $\lim\limits_{h \to 0} f(a - h)$ and $f(a)$ are never achieved.

Also, $f'(x) = 2 + \cos 2x > 0$, i.e., $f(x)$ is strictly increasing

$\Rightarrow$ f is one-one but not onto.

Hence, the correct answer is (A).

105. (A) $f(x) = \frac{e^x + e^{-x}}{2}$ is even function

(B) $f(x) = x^4 - 3x^3 + 1$ is an even degree polynomial function.

(C) $f(x) = 18x^3 - 21x^2 + 8x - 1$

$f(x)$ is an odd degree polynomial function hence, its range is R.

$f'(x) = 54x^2 - 42x + 8$

$\Rightarrow$ $D > 0$, hence, it is many-one function.

(D) $f(x) = x^3 - 4x^2 + 16x + 17$

$f(x)$ is an odd degree polynomial function hence its range is R.

$f'(x) = 3x^2 - 8x + 16$

$\Rightarrow$ $D < 0$, hence $f(x)$ is one-one as well as onto.

Hence, the correct answer is (D).

106. We have $(f - g)(x) = (f(x) - g(x))$

$$= \begin{cases} 2x + 3\ ; & x \in \text{rational} \\ 3x - \sqrt{5}; & x \in \text{irrational} \end{cases}$$

As, $f\left(\frac{-3}{2}\right) = 0 = f\left(\frac{\sqrt{5}}{3}\right)$ and so on.

$\Rightarrow$ $f(x)$ is many-one function.

Also, $-\sqrt{5}$ does not belong to the range, because it

$3x - \sqrt{5} = -\sqrt{5}$

$\therefore$ $x = 0 \notin Q$

$\Rightarrow$ $f(x)$ is into function.

Hence, the correct answer is (B).

107. Number of element in set $A = n(A) = 6$ and $n(B) = 3$

Total number of functions $= 6^3$

Number of injective function $= {}^6C_3 \,.\, 3!$

function which are not injective are $= 6^3 - {}^6C_3 3!$

$= 216 - 120 = 96$

Hence, the correct answer is (A).

108. $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + ax + b$

If $f(x)$ is one-one then $f(x)$ must be monotonic.

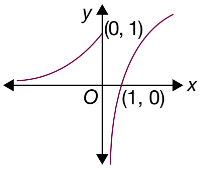
Now, $f'(x) = x^2 + x + a \ge 0\ \forall\ y \in R$

$\Rightarrow\quad D \le 0$ i.e. $1 - 4a \le 0 \Rightarrow a \ge \frac{1}{4}$

So, $a_{\min} = \frac{1}{4}$.

Hence, the correct answer is (A).

109.



Graph of $y = f(x)$

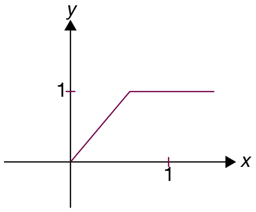
$$f(x) = \begin{cases} x - \frac{1}{x}; & x > 0 \\ e^x\ ; & x \le 0 \end{cases},$$

Clearly f is many-one.

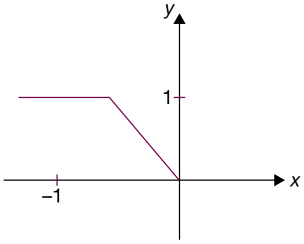
Also, $R_f = (-\infty, \infty)$ hence surjective

Hence, the correct answer is (C).

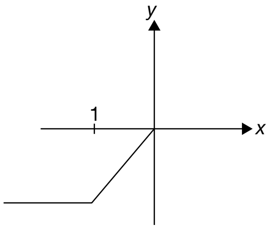
110. $f_1(x)$



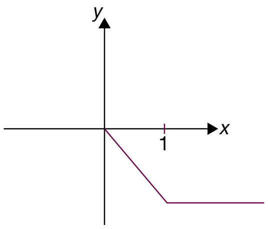
$f_2(x) = f_1(-x)$



$f_3(x) = -f_2(x)$



$f_4(x) = f_3(-x)$



Hence, the correct answer is (B).

Multiple Correct Choice Type Questions

1. $f(x) = \cos^2 x + \left(\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right)^2 - \cos x\left(\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x\right)$

$= \cos^2 x + \frac{1}{4}\cos^2 x + \frac{3}{4}\sin^2 x - \frac{\sqrt{3}}{2}\cos x \sin x - \frac{\cos^2 x}{2} + \frac{\sqrt{3}}{2}\cos x \sin x$

$= \frac{3}{4}\left(\cos^2 x + \sin^2 x\right) = \frac{3}{4}$

$\therefore$ $f(x)$ is an even function, a periodic function and $f(0) = f(1) = \frac{3}{4}$

Hence, (B), (C) and (D) are correct.

2. Given, $1900 < f(1990) < 2000$

$$\Rightarrow \quad \frac{1900}{90} < \frac{f(1990)}{90} < \frac{2000}{90}$$

$$\Rightarrow \quad \left[\frac{1900}{90}\right] \le \left[\frac{f(1990)}{90}\right] \le \left[\frac{2000}{90}\right]$$

$$\Rightarrow \quad 21 \le \left[\frac{f(1990)}{90}\right] \le 22 \;\Rightarrow \left[\frac{f(1990)}{90}\right] = 21, 22$$

Given, $x - f(x) = 19\left[\frac{x}{19}\right] - 90\left[\frac{f(x)}{90}\right]$ ...(1)

$$\therefore \quad 1990 - f(1990) = 19\left[\frac{1990}{19}\right] - 90\left[\frac{f(1990)}{90}\right]$$

$$\Rightarrow \quad 1990 - f(1990) = \begin{cases} 19 \times 104 - 90 \times 21 \\ 19 \times 104 - 90 \times 22 \end{cases}$$

$$\Rightarrow \quad 1990 - f(1990) = \begin{cases} 1976 - 1890 = 86 \\ 1976 - 1980 = -4 \end{cases}$$

$$\Rightarrow \quad f(1990) = 1904, 1994$$

Hence, (B) and (D) are correct.

3. (A) No. of onto functions $= 4^5 - 4.3^5 + 6.2^5 - 4 = 240$

(B) No. of onto functions whose range contains 3 elements

$= {}^4C_3 \;\left(3^5 - 3.2^5 + 3\right) = 4 \times 150 = 600$

(C) No. of onto functions whose range contains 2 elements

$= {}^4C_2 \;\left(2^5 - 2\right) = 60 \times 30 = 180$

(D) No. of constant function = 5

Hence, (A), (B), (C) and (D) are correct.

4. $f(x+y) + f(x-y) = 2f(x).f(y)$

Put $x = 0 \Rightarrow f(y) + f(-y) = 2f(0)f(y)$

Put $x = y = 0 \Rightarrow f(0) + f(0) = 2f(0)f(0)$

$\Rightarrow \quad f(0) = 1 \quad$ (as $f(0) \ne 0$)

$\Rightarrow \quad f(-y) = f(y)$

Hence, the function is even. Then $f(-2) = f(2) = a$

Hence, (A) and (C) are correct.

5. $(f+g)(3.5) = f(3.5) + g(3.5) = (-0.5) + 0.5 = 0$

$f(g(3)) = f(0) = 3$

$(fg)(2) = f(2)g(2) = (-1) \times (-1) = 1$

$(f-g)(4) = f(4) - g(4) = 0 - 26 = -26$

Hence, (A), (B) and (C) are correct.

6. $f(x) = x^2 - 2ax + a(a+1)$

$f(x) = (x-a)^2 + a, x \in [a, \infty)$

Let $y = (x-a)^2 + a$ clearly $y \ge a$

$\Rightarrow \quad (x-a)^2 = y - a \quad \Rightarrow \quad x = a + \sqrt{y-a}$

$\therefore \quad f^{-1}(x) = a + \sqrt{x-a}$

Now $f(x) = f^{-1}(x)$

$\Rightarrow \quad (x-a)^2 + a = a + \sqrt{x-a}$]

$(x-a)^2 = \sqrt{x-a}$

$\Rightarrow \quad (x-a)^2 = (x-a)$

$\Rightarrow \quad x = a$ or $(x-a)^3 = 1$

$\Rightarrow \quad x = a$ or $a + 1$

If $a = 5049$, then $a + 1 = 5050$

If $a + 1 = 5049$, then $a = 5048$

Hence, (B) and (D) are correct.

7. $C = 0$ and $\frac{A}{D} = -1$, then $f(x) = -x + B$ invers of itself.

$B = C = 0$ and $A = D$, then $f(x) = x$ invers of itself.

$B = C = 0$ and $\frac{A}{D} = \pm 1$, then $f(x) = \pm x$ invers of itself.

Hence, (A), (B) and (C) are correct.

8. (A) $R_f \equiv \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]; R_g = \left[0, \frac{\pi}{2}\right]$

(B) $f(x) = \cos^{-1}(1 - 2x^2) = 2\sin^{-1} x$ only for $x \in [0, 1]$

(C) $\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right) = \frac{1}{2}\cos^{-1} x \;\forall\, x \in [-1, 1]$

(D) $\tan^{-1}\left(\frac{\sqrt{1-x^2} - 1}{x}\right) = \frac{1}{2}\tan^{-1} x \;\forall\, x \in R - \{0\}$

Hence, (C) and (D) are correct.

9. $f(x+1) = \frac{f(x) - 5}{f(x) - 3}$ ...(1)

$f(x)f(x+1) = 3f(x+1) + f(x) - 5$

$\Rightarrow \quad f(x) = \frac{3f(x+1) - 5}{f(x+1) - 1}$ ...(2)

Replacing x by $(x-1)$, we get:

$$f(x-1) = \frac{3f(x) - 5}{f(x) - 1}$$

Using, (1), $f(x+2) = \frac{f(x+1) - 5}{f(x+1) - 3} = \frac{\frac{f(x)-5}{f(x)-3} - 5}{\frac{f(x)-5}{f(x)-3} - 3} = \frac{2f(x) - 5}{f(x) - 2}$ ...(3)

Using (2), $f(x-2) = \frac{3f(x-1) - 5}{f(x-1) - 1} = \frac{3\left(\frac{3f(x)-5}{f(x)-1}\right) - 5}{\frac{3f(x)-5}{f(x)-1} - 1} = \frac{2f(x) - 5}{f(x) - 2}$ ...(4)

Using (3) and (4) we have $f(x+2) = f(x-2)$

$\Rightarrow \quad f(x+4) = f(x) \Rightarrow f(x)$ is periodic with period 4

Hence, (A), (B), (C) and (D) are correct.

10. $f(x) = f(2 - x)$

$x \to x + 1$

$f(x+1) = f(1-x)$

sym about $x = 1$

$g(x) = f(x+1)$

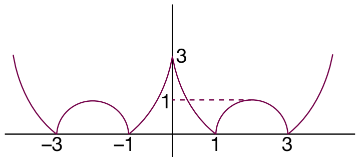
$g(-x) = f(-x+1) = f(1-x)$

$g(-x) = f(x+1)$

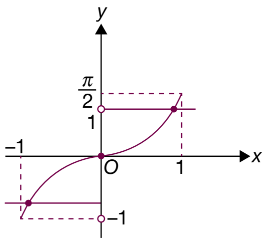
$g(-x) = g(x)$, even fn.

Hence, (B) and (D) are correct.

11. (A) $\text{sgn}(x) = \sin^{-1}x$ has three solutions



(B)



(C) $\log_{10}(-x) = 4 \Rightarrow x = -10^4 = \alpha \Rightarrow \text{sgn}(\alpha) = -1$

(D) $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x \neq 0$ and $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ can't be equal for same x.

Hence, (B) and (D) are correct.

12. $f(x) = \sqrt{a\{x\}^2 + b\{x\}} = \sqrt{(a\{x\} + b)\{x\}}$

(A) $a > 0 > b$ and $|b| > |a|$ then $\frac{b}{a} < -1$
like $(a\{x\} + b)\{x\} \geq 0$

$$a\left(\{x\} + \frac{b}{a}\right)\{x\} \geq 0;\ \{x\} + \frac{b}{a} < 0$$

$\therefore$ $\{x\} \leq 0; \Rightarrow \{x\} = 0$ then $x \in I$ (domain)

(B) $a < 0 < b$ and $|b| > |a|$ then $\frac{b}{a} < -1$

$$\underset{\underset{\ominus}{\downarrow}}{a}\ \underset{\underset{\ominus}{\downarrow}}{\left(\{x\} + \frac{b}{a}\right)}\{x\} \geq 0$$

$\{x\} \geq 0 \quad \forall x \in R$ (domain)

(C) $a > 0, b > 0$

$$\underset{\underset{\oplus}{\downarrow}}{(a\{x\} + b)}\ \underset{\underset{\oplus}{\downarrow}}{(\{x\})} \geq 0$$

$\forall x \in R$ (domain)

(D) $a < 0, b < 0$

$$\underset{\underset{\ominus}{\downarrow}}{(a\{x\} + b)}\{x\} \geq 0$$

$\{x\} \leq 0; \Rightarrow \{x\} = 0;\ \forall x \in I$ (domain)

Hence, (A), (B), (C) and (D) are correct.

13. (i) $-1 \leq |x| - 1 \leq 4 \Rightarrow x \in [-5, 5]$

(ii) Range of $f(|x| + 1)$ equals $[0, 2]$

(iii) Range of $f(-|x|)$ is $[-1, 0]$

(iv) $-1 \leq |x| \leq 4 \Rightarrow x \in [-4, 4]$

Hence, (A), (B) and (C) are correct.

14. If f is onto then range of f is R.

$\Rightarrow$ Graph of $f(x)$ is lying below x axis as well as above x-axis, therefore, if f is onto, then $g(x)$ is into.

$\therefore$ Its range is $[0, \infty)$ and hence $g(x)$ is many-one also.

Hence, (C) and (D) are correct.

15. Set of $[\sin^{-1}x] > [\cos^{-1}x]$ is $[\sin 1\ 1]$

Range of $|\sin x| + |\cos x|$ is $[1, \sqrt{2}]$

$$\text{Range of } \left[x + \frac{1}{2}\right] + \left[x - \frac{1}{2}\right] + 2[-x]$$

$$= 1 + 2\left(\left[x - \frac{1}{2}\right] + [-x]\right) \text{ is } \{-1, -3\}$$

$$\text{Range of } \left[\frac{1}{[x-3]}\right] \text{ is } \{-1, 0, 1\}$$

Hence, (A), (C) and (D) are correct.

16. (A) $f(g(x)) = (7 - x^{1/3})^3 = [7 - (7 - x^{1/3})]^3 = x$

$g(f(x) = f(g(x)$

(B) $y = f(x) = \frac{8x - 7}{5x + 8}\ x = \frac{8y - 7}{5y + 8}$

$\Rightarrow$ $y(5x - 8) = -(8x + 7)$

$$f^{-1}(x) = \frac{8x - 7}{8 - 5x} = g(x)$$

$\Rightarrow$ $f(g(x) = x$ and $g(f(x)) = x$

(C) If $x \in Q$

$g(f(x)) = g(x) = x$

If $x \notin Q$

$g(f(x)) = g(-x) = -(-x) = x$

$\therefore$ $gof(x) = x\ \forall x \in R$

Similarly $f(g(x)) = x\ \forall x \in R$ can be proved

(D) $y = \log(x - 2)$

for $x = \log(y - 2)$

$\Rightarrow$ $y = e^x + 2 = f^{-1}(x)$

$g(x) = f^{-1}(x) \Rightarrow f(g(x) = g(f(x) = x$

Hence, (A), (B), (C) and (D) are correct.

17. $f(x) = \frac{1}{\sin\{x\}}$

$\therefore$ $\{x\} \in [0, 1)$ but $\{x\} \neq 0$

$0 < \{x\} < 1 \Rightarrow 0 < \sin\{x\} < \sin 1$

$$\infty > \frac{1}{\sin\{x\}} > \frac{1}{\sin 1} = \frac{1}{\sqrt{3}/2}$$

$$f(x) = \left[\frac{1}{\sin\{x\}}\right] = 1, 2, 3, \ldots$$

Range $\rightarrow$ N, set of natural numbers

Hence, (A), (B) and (C) are correct.

18. Let $g(x) = ax^2 + bx + c$

$\because$ $[f(x)] = [g(x)]\ \forall\ x \in R$

$\therefore$ $-1 < f(x) - g(x) < 1\ \forall\ x \in R$

$\therefore$ $-1 < (1 - a)x^2 - x(1 + b) + (1 - c) < 1\ \forall\ x \in R$

$\Rightarrow$ $[x^2 - x + 1] = [x^2 - x + c] \Rightarrow [1] = [c]$

Let $x^2 - x + 1 = I + f \qquad c = 1 + f'$

$[I + f] = [I - 1 + f + c]$

$0 = [f + c] - 1$

$1 = [c + f]$

$1 = [1 + f' + f]$

$[f' + f] = 0\ \forall\ f \in (0, 1)$

$f' = 0$

$\Rightarrow$ $c = 1$

$\Rightarrow$ $f(x) = g(x)\ \forall\ x \in R$

Hence, (A) and (C) are correct.

19. $f(x) = a(x - \alpha)(x - \beta_1)$

$f(x)\, f(x + 1) = a(x - \alpha_1)(x - \beta_1)\, a(x - \alpha_1 + 1)(x - \beta_1 + 1)$

$= a^2(x - \alpha_1)(x - \beta_1)(x - \alpha_1 + 1)(x - \beta_1 + 1)$

$= a^2(x - \alpha_1)(x - \beta_1 + 1)(x - \beta_1)(x - \alpha_1 + 1)$

$= a^2[x^2 - x\beta_1 + x - \alpha_1 x + \alpha_1\beta_1 - \alpha_1][x^2 - x\alpha_1 + x - \beta_1 x + \alpha_1\beta_1 - \beta_1]$

$= a^2[x^2 + x(1 - \alpha_1 - \beta_1) + \alpha_1\beta_1 - \alpha_1][x^2 + x(1 - \alpha_1 - \beta_1) + \alpha_1\beta_1 - \beta_1]$

$g(x) = a^2(x - \alpha_1)(x - \beta_1) = a^2(x - \alpha_2)(x - \beta_2)$

$h(x) = x^2 + x(1 - \alpha_1 - \beta_1) + \alpha_1\beta_1 = (x - \alpha_3)(x - \beta_3)$

$\alpha_1 + \beta_1 = \alpha_2 + \beta_2$

$-1 + \alpha_1 + \beta_1 = \alpha_3 + \beta_3$

$\alpha_1\beta_1 = \alpha_2\beta_2$

$\alpha_1\beta_1 = \alpha_3\beta_3$

Hence, (A), (C) and (D) are correct.

20. $f(x) = \sqrt{\sin^{-1}x - \cos^{-1}x}$

$$\sin^{-1}x - \cos^{-1}x \geq 0 \Rightarrow \sin^{-1}x - \frac{\pi}{2} + \sin^{-1}x \geq 0$$

$$\Rightarrow \quad \sin^{-1}x \geq \frac{\pi}{4} \Rightarrow x \in \left[\frac{1}{\sqrt{2}}, 1\right]$$

$g(x) = \sqrt{\tan^{-1}x - \cot^{-1}x}$

$\tan^{-1}x - \cot^{-1}x \geq 0$

$$\Rightarrow \quad \tan^{-1}x - \frac{\pi}{2} + \tan^{-1}\alpha \geq 0 \Rightarrow \tan^{-1}x \geq \frac{\pi}{4} \Rightarrow x \in [1, \infty)$$

$h(x) = \sqrt{\sec^{-1}x - \text{cosec}^{-1}x}$

$\sec^{-1}x - \text{cosec}^{-1}x \geq 0$

$\Rightarrow \quad \sec^{-1}x - \frac{\pi}{2} + \sec^{-1}x \geq 0 \Rightarrow \sec^{-1}x \geq \frac{\pi}{4}$

$x \in (-\infty, -1] \cup [\sqrt{2}, \infty) =$ domain of $h(x)$

$f(x) + g(x) \quad D_f\, x \in \left[\frac{1}{\sqrt{2}}, 1\right]$ and $D_g\, x \in [1, \infty)$

domain $x \in \{1\}$

domain of $g(x) + h(x)$ is $x \in [\sqrt{2}, \infty)$

domain of $h(x) + f(x)$ is $x \in \phi$

domain of $h(x) + f(x) + g(x)$ is $x \in \phi$

Hence, (A), (B) and (D) are correct.

21. $f(x+3) = 4 + \sqrt{8f(x) - (f(x))^2} \qquad (\because f(x) \geq 4)$

$8f(x) - (f(x))^2 \geq 0$

$(f(x+3) - 4)^2 = -(f(x)^2 - 8f(x)) \Rightarrow f(\alpha)(f(\alpha) - 8) \leq 0$

$(f(x+3) - 4)^2 = 16 - (f(x) - 4)^2 \Rightarrow 0 \leq f(\alpha) \leq 8$

$x \to x + 3$

$(f(x+6) - 4)^2 = 16 - (f(x+3) - 4)^2$

$(f(x+6) - 4)^2 = 16 - (16 - (f(x) - 4)^2)$

$(f(x+6) - 4)^2 = (f(x) - 4)^2)$

$f(x+6) - 4 = f(x) - 4$

$f(x+6) = f(x) \qquad T = 6$

$f(13) - 2f(61) + f(103)$

$= f(1 + 2T) - 2f(1 + 10T) + f(1 + 17T)$

$= f(1) - 2f(1) + f(1) = 0$

Now $f(6) - 2f(10) + f(26)$

$= f(6) - 2f(4 + 6) + f(2 + 4T)$

$= f(6) - 2f(4) + f(2) \neq 0$

Hence, (B) and (C) are correct.

22. $f_n(\theta) = \sum_{k=0}^{n} \frac{1}{4^k}\sin^4(2^k\theta)$

$$f_2\left(\frac{\pi}{4}\right) = \sum_{k=0}^{2} \frac{1}{4^k}\sin^4\left(2^k\frac{\pi}{4}\right)$$

$$= \frac{1}{4^0}\sin^4\frac{\pi}{4} + \frac{1}{4^1}\sin^4\frac{\pi}{2} + \frac{1}{4^2}\sin^4(\pi) = \frac{1}{2^2} + \frac{1}{4}\cdot 1 + 0 = \frac{1}{2}$$

$$f_3\left(\frac{\pi}{8}\right) = \frac{1}{4^0}\sin^4\frac{\pi}{8} + \frac{1}{4^1}\sin^4\frac{\pi}{4} + \frac{1}{4^2}\sin^4\frac{\pi}{2} + \frac{1}{4^3}\sin^4\pi$$

$$= \left(\frac{1-\cos\frac{\pi}{4}}{2}\right)^2 + \frac{1}{4}\left(\frac{1}{\sqrt{2}}\right)^4 + \frac{1}{16} = \frac{2+1-2\sqrt{2}}{8} + \frac{1}{8}$$

$$= \frac{4-2\sqrt{2}}{8} = \frac{2-\sqrt{2}}{4}$$

$$f_4\left(\frac{3\pi}{2}\right) = \frac{1}{4^0}\sin^4\frac{3\pi}{2} + \frac{1}{4^1}\sin^4 3\pi + 0 + 0 + 0 = 1$$

$f_5(\pi) = 0$

Hence, (B), (C) and (D) are correct.

23. $f(x) = \left(\underset{\overset{\downarrow}{0}}{[\{x\}]}\tan^{-1}\left(\frac{x^2 - 3x - 1}{x^2 - 3x + 5}\right) + 3 - x^7\right)^{\frac{1}{7}}$

$f(x) = (3 - x^7)^{1/7}$

$f(f(\alpha) = (3 - (f(x))^7)^{1/7} = (3 - (3 - x^7))^{1/7} = x$

$f(f(x)) = x$

$y = (3 - x^7)^{1/7}$

$f(f(100)) = 100$

$y^7 = 3 - x^7$

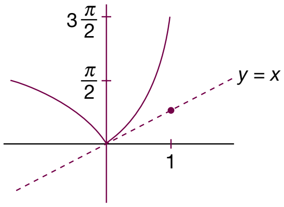
$x^7 = 3 - y^7$

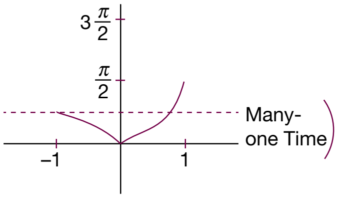
$x = (3 - y^7)^{1/7}$

$f^{-1}(x) = (3 - x^7)^{1/7} = f(x); f^{-1}(50) = f(50)$

Hence, (A) and (B) are correct.

24. $f : [-1, 1] \to \left[-\frac{3\pi}{2}, \frac{3\pi}{2}\right]$





$$f(x) = \begin{cases} 3\sin^{-1}x & x \geq 0 \\ -\sin^{-1}x & x < 0 \end{cases}; f(x) = \frac{\pi}{2}$$

Two solutions

$$\left.\begin{array}{l} \lim\limits_{x\to 0^+} \frac{f(x)}{x} = \frac{3\sin^{-1}x}{x} = 3 \\ \lim\limits_{x\to 0^-} \frac{f(x)}{x} = \frac{-\sin^{-1}x}{x} = -1 \end{array}\right\} \text{DNE}$$

$f(x) = x$

only one solution

Hence, (A), (B) and (D) are correct.

25. $f : R \to [2, 4]$

$[2, 4] \quad \underset{\overset{\downarrow}{T}}{f(x)} + \underset{\overset{\downarrow}{\frac{2\pi}{\pi}=2}}{\cos\pi x} = 1$ unique solution given for the solution

$f(x) = 2$ and $\cos \pi x = -1$

$x \to$ odd no.

As it is unique solution

So, $f(x) + \cos \pi x$ aperiodic function, $\cos \pi x$ has period 2 and $f(x)$ and T for unique solution LCM of T and $2 =$ does not exist.

$\therefore \quad T =$ irrational no.

Hence, (A), (B) and (C) are correct.

26. $f : R \to R$

$f(x) = x^{1/3} + \sin 2x \qquad$ odd

$g : R \to R \qquad g(x) = \frac{1 - x^2}{1 + x^2} \qquad$ even

$y = \frac{1 - x^2}{1 + x^2} \Rightarrow x^2y + y = 1 - x^2 \Rightarrow x^2(y + 1) = 1 - y$

$x^2 = \frac{1 - y}{y + 1} \geq 0 \Rightarrow y \in (-1, 1] \qquad g \to$ bounded

$f(x) = \underset{\overset{\downarrow}{R}}{x^{1/3}} + \underset{\overset{\downarrow}{[-1,1]}}{\sin 2x}$

Range $\equiv R$ surjective

$f \to$ unbounded $\qquad f(x) + g(x) \to$ NENO

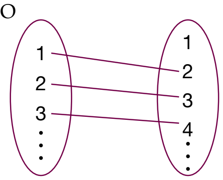
Hence, (A) and (B) are correct.

27. (A) If A is infinite set

$y = x + 1$

$f : N \to N$

one-one and into

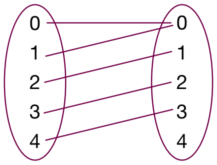


(B) $A \to A$

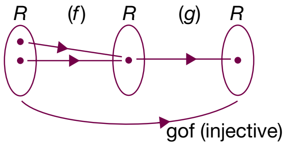
$f : W \to W$

$f(x) =$

onto and many-one

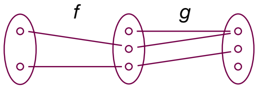


(C)



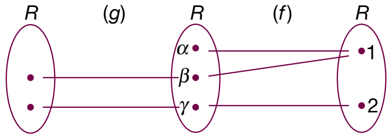
$f \rightarrow$ many-one; $gof \rightarrow$ can not be one-one 100%
gof (one-one) $f(x)$ must be one-one
$g(-x)$ may/may not be one-one

(D) $2^3 = 8$



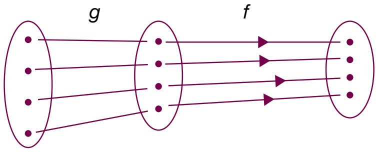
Hence, (C) and (D) are correct.

28. (A) $f : R \rightarrow R$
$g : R \rightarrow R$



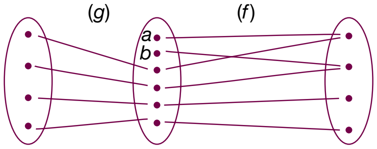
$f(g(x)) \rightarrow$ Injective
$g(x)$ 100% injective
$f(x)$ may not be injective as α has no pre-image
If $g(x)$ is many-one $f(g(x))$ can not be injective

(B) $g(x)$ is surjective
$f(g(x)) \rightarrow$ one-one
$g(x) \rightarrow$ one-one
$g(x) \rightarrow$ bijective
As $f(g(x)) \rightarrow$ one-one; $f(x)$ must be one-one



(B) wrong

(C) $g(x)$ into



$f(x) \rightarrow$ may be injective or many-one
but $fog(x) \rightarrow$ is one-one

(D) $f(g(x))$ is an injective mapping then f many or may not be injective.

Hence, (A), (C) and (D) are correct.

29. (A) $0 \le \{x\} < 1$
$[\{x\}] = 0$
$f(x) = \ln[1 + \{x\}] = \ln 1 = 0$ $\quad \forall\, x \in R$
$g(x) = \ln(1 + [\{x\}]) = \ln 1 = 0$ $\quad \forall\, x \in R$ $\quad$ equal

(B) $f(x)\ \forall\, x \notin R$ $\quad g(x)\ x \in R$ $\quad$ not equal

(C) $f(x) = 2^{+1}$ $\quad g(x) = 2(+1) = 2$
$\forall\, x \in R$ $\quad \forall\, x \in R$ $\quad$ equal

(D) $f(x) = \dfrac{\operatorname{sgn} x}{\operatorname{sgn} x}$ $\quad g(x) = \dfrac{|x|}{|x|} = 1$

$x \neq 0$ is domain for f and q equal

Hence, (A), (C) and (D) are correct.

30. (A) $f(x) = \cos(\cos^{-1} x) = x$ $\quad x \in [-1, 1]$ odd fn.

(B) $\text{FTP} = \dfrac{\pi}{2}$

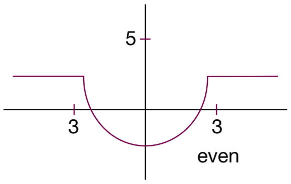
(C) $-3 \le 3 \sin x \le 3$ $\quad \cos(3 \sin x) \in [\cos 3, 1]$

(D) $f(x, y) = |x| \sqrt{\dfrac{x^2}{y^2} - 1} + x$

Homogenous of degree 1

Hence, (A), (B) and (C) are correct.

31. $f(x)$ is one even function



$$g(x) = 2\tan^{-1}(e^{-x}) - \frac{\pi}{2}$$
$$= 2\cot^{-1} e^{x} - \frac{\pi}{2}$$
$$= 2\left(\frac{\pi}{2} - \tan^{-1} e^{x}\right) - \frac{\pi}{2}$$
$$= \frac{\pi}{2} - 2\tan^{-1} e^{x}$$

$g(x) = -g(x)$; $g(x) \rightarrow$ odd
$fog(\alpha) \rightarrow$ even; $gof(\alpha) \rightarrow$ even; $gog(x) \rightarrow$ odd; $fof(x) \rightarrow$ even

Hence, (A), (B) and (C) are correct.

32. S_1 true, S_2 and S_3 are false $\Rightarrow f(x) = f(y) = 1$ contradiction

S_2 true, S_1 and S_3 are false $\Rightarrow f(z) = 2,\ f(x) \neq 1; f(y) \neq 1$ which is false

S_3 true, S_1 and S_2 are false $\Rightarrow f(x) \neq 1, f(y) = 1, f(z) \neq 2$

$\Rightarrow \quad f(y) = 1, f(z) = 3, f(x) = 2$ and $f^{-1}(1) = y$

Hence, (A), (B), (C) and (D) are correct.

33. $g(x) = \sin\left(\sin^{-1}\sqrt{\{x\}}\right) + \cos\left(\sin^{-1}\sqrt{\{x\}}\right) - 1$

$$g(x) = \sqrt{\{x\}} + \cos\left(\cos^{-1}\sqrt{1 - \{x\}}\right) - 1$$

$g(x) = \sqrt{\{x\}} + \sqrt{1 - \{x\}} - 1$ it is periodic function with period 1

If $x \in I$ then $\{x\} = 0 \Rightarrow g(x) = 0$
$\Rightarrow \quad g(-x) = g(x) \Rightarrow g$ is even
If $x \notin I$ then $\{-x\} = 1 - \{x\}$

$\Rightarrow \quad g(-x) = \sqrt{1 - \{x\}} + \sqrt{\{x\}} - 1 = g(x)$

$\Rightarrow \quad$ 'g' is even function

$$\left.\begin{array}{ll} g(x) = 0, & \text{if } x \in I \\ g(x) = g(-x), & \text{if } x \notin I \end{array}\right\} \Rightarrow \text{'}g\text{' is even function}$$

Hence, (A) and (B) are correct.

34. Replace x by 2, $2f(2) + 2f\left(\dfrac{1}{2}\right) - 2f(1) = 4$

$\Rightarrow \quad f(2) + f\left(\dfrac{1}{2}\right) = 2 + f(1)$ ...(1)

Replace x by 1, $f(1) = -1$ ...(2)

Replace x by $\dfrac{1}{2}$, $2f\left(\dfrac{1}{2}\right) + \dfrac{1}{2} f\left(\dfrac{1}{2}\right) + 2 = \dfrac{5}{2}$ ...(3)

Solve (1) and (3) $\Rightarrow f\left(\dfrac{1}{2}\right) = 0; f(2) = 1$

Hence, (A), (B) and (C) are correct.

35. Domain of $f = R$, Domain $g = R - [-1, 0)$; because in $-1 \le x < 0 \Rightarrow 1 + [x] = 0$

Domain of $f - g = R - [-1, 0)$;

Since, $e^{-x} > 0 \Rightarrow (1 + [x])y > 0$

$\Rightarrow \quad y > 0$ is $1 + [x] > 0$, $\Rightarrow y < 0$ is $1 + [x] < 0$
$\Rightarrow \quad y \in R - \{0\}$.

$$y = \frac{x}{1 + x^2} \Rightarrow y + x^2 y = x$$
$$yx^2 - x + y = 0 \Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$\Rightarrow \quad \text{Range } f \cap \text{Range } g = \left[-\frac{1}{2}, \frac{1}{2}\right] - \{0\}$$

Hence, (B) and (D) are correct.

36. $f'(x) = 3x^2 + (a+2)2x + 3a$, f is invertible if

$$f' \geq 0 \quad \forall x \in R \Rightarrow \Delta \leq 0 \Rightarrow 4(a+2)^2 - 36a \leq 0$$

$$\Rightarrow \quad a^2 - 5a + 4 \leq 0 \Rightarrow (a-1)(a-4) \leq 0 \Rightarrow 1 \leq a \leq 4$$

Hence, (A), (B) and (D) are correct.

37. $f[g(x)] = \begin{cases} 1 - g(x); & 0 \leq g(x) \leq 2 \\ g(x) - 3; & 2 < g(x) \leq 3 \end{cases}$

$$= \begin{cases} 1-(2+x) & 0 \leq 2+x \leq 2; & -1 \leq x \leq 0 \\ 2+x-3 & 2 < 2+x \leq 3; & -1 \leq x \leq 0 \\ 1-(2-x) & 0 \leq 2-x \leq 2; & 0 < x \leq 3 \\ 2-x-3 & 2 < 2-x \leq 3; & 0 < x \leq 3 \end{cases}$$

$$= \begin{cases} -1-x; & -1 \leq x \leq 0 \\ x-1 \ ; & x \in \phi \\ -1+x \ ; & 0 < x \leq 2 \\ -x-1 \ ; & x \in \phi \end{cases}$$

and $(gof)x = g\{f(x)\} = \begin{cases} 2 + f(x) & -1 \leq f(x) \leq 0 \\ 2 - f(x) & 0 < f(x) \leq 3 \end{cases}$

$$= \begin{cases} 2+1-x & -1 \leq 1-x \leq 0 & \text{and} & 0 \leq x \leq 2 \\ 2-(1-x) & 0 < 1-x \leq 3 & \text{and} & 0 \leq x \leq 2 \\ 2+x-3 & -1 \leq x-3 \leq 0 & \text{and} & 2 < x \leq 4 \\ 2-(x-3) & 0 < x-3 \leq 3 & \text{and} & 2 < x \leq 4 \end{cases} = \begin{cases} 1+x, & 0 \leq x < 1 \\ 3-x, & 1 \leq x \leq 2 \\ x-1, & 2 < x \leq 3 \\ 5-x, & 3 < x \leq 4 \end{cases}$$

Hence, (B), (C) and (D) are correct.

38. If f is odd If $f(x) = -f(-x)$

$$= \begin{cases} -x \tan x \ ; & \dfrac{-\pi}{2} \leq x < 0 \\ \dfrac{-\pi}{2}[-x]; & -2 \leq x < \dfrac{-\pi}{2} \end{cases}$$

If f is even $f(x) = f(-x)$

$$= \begin{cases} x \tan x; & \dfrac{-\pi}{2} \leq x < 0 \\ \dfrac{\pi}{2}[-x]; & -2 \leq x \leq \dfrac{-\pi}{2} \end{cases}$$

Hence, (B) and (C) are correct.

39. $1 \leq |\sin x| + |\cos x| \leq \sqrt{2}$

$$[|\sin x| + |\cos x|] = 1$$

$f(x)$ is defined if

$$\sin^2 x + 2\sin x + \frac{11}{4} \geq 2$$

$$(\sin x + 1)^2 \geq \frac{1}{4}$$

$$\sin x + 1 \geq \frac{1}{2} \text{ or } \sin x + 1 \leq -\frac{1}{2}$$

$$\sin x \geq -\frac{1}{2} \text{ or } \sin x \leq -\frac{3}{2} \left(\text{which is not true}\right)$$

Hence, (A), (B) and (C) are correct.

40. $\cos^2 x - 2\cos x + 1 = 3 - \dfrac{y+3}{2y+5}$

$$(\cos x - 1)^2 = \frac{5y+12}{2y+5}$$

$$-1 \leq \cos x \leq 1$$

$$-2 \leq \cos x - 1 \leq 0$$

Also $\quad 0 \leq (\cos x - 1)^2 \leq 4$, $\dfrac{5y+12}{2y+5} \geq 0$ and

$$0 \leq \frac{5y+12}{2y+5} \leq 4$$

$$\frac{5y+12}{2y+5} - 4 \leq 0 \quad \frac{-3y-8}{2y+5} \leq 0 \Rightarrow \frac{3y+8}{2y+5} \geq 0$$

$$\left(-\infty, -\frac{5}{2}\right) \cup \left[-\frac{12}{5}, \infty\right) \text{and} \left(-\infty, -\frac{8}{3}\right) \cup \left[-\frac{5}{2}, \infty\right)$$

$$\therefore \quad y \in \left(-\infty, -\frac{8}{3}\right] \cup \left[-\frac{12}{5}, \infty\right)$$

Hence, (A), (B) and (D) are correct.

41. (A) is correct,

(B) f, g are one-one $\Rightarrow h = gof$ is one-one.

But h need not be onto. Consider $X = \{1, 2\}$, $Y = \{a, b, c\}$ and $Z = \{p, q, r\}$; $f = \{(1, a), (2, b)\}$ and $g = \{(a, p), (b, q)\ (c, r)\}$

Then f is one-one and g is one-one and onto. But $h = \{(1, p), (2, q)\}$ is not onto.

(C) is true. If h is onto, then $y \in Y \Rightarrow g(y) \in Z \Rightarrow h(x) = g(y)$ for some $x \in X$

$\Rightarrow$ $(gof)(x) = g(y)$ for some $x \in X$

$\Rightarrow$ $f(x) = y$ for the same $x \in X$

$\therefore$ f is onto.

Hence, (A), (C) and (D) are correct.

42. $f(1) = 3$ or 5

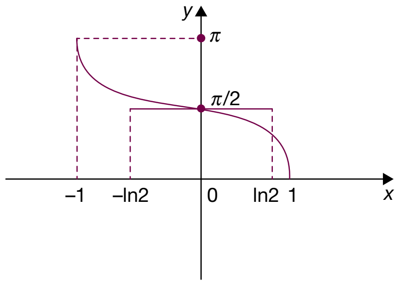
$f(3)$ can take 4, 3, 2, 1 number of values when $f(4)$ is 17, 15, 13 and 11 respectively. So number of increasing function $= 2(4 + 3 + 2 + 1) = 20$

Now for decreasing function $f(2) = 7$, $f(3) = 5$, $f(4) = 2$ and $f(1)$ can take 5 values, so the number of decreasing function is 5.

Number of one-one function is $= {}^7C_3 . 3! = 210$.

Hence, (B), (C) and (D) are correct.

43.



$$0 < e^x < 2 \text{ and } 0 < e^{-x} < 2 \Rightarrow x \in (-\ln 2, \ln 2) .$$

$$f(x) = \begin{cases} \pi; & x = 0 \\ \dfrac{\pi}{2}; & x \in (-\ln 2, 0) \cup (0, \ln 2) \end{cases}$$

Hence, (A) and (C) are correct.

44. Given $f(x) + \dfrac{3}{2} f\left(\dfrac{2x+29}{x-2}\right) = 50x + 40$ …(1)

Replacing x by $\left(\dfrac{2x+29}{x-2}\right)$ in equation (1)

$$f\left(\frac{2x+29}{x-2}\right) + \frac{3}{2} f\left(\frac{2\left(\dfrac{2x+29}{x-2}\right) + 29}{\dfrac{2x+29}{x-2} - 2}\right) = 50\left(\frac{2x+29}{x-2}\right) + 40$$

$$\Rightarrow \quad f\left(\frac{2x+29}{x-2}\right)+\frac{3}{2}f(x)=50\left(\frac{2x+29}{x-2}\right)+40 \qquad \ldots(2)$$

by solving (1) and (2) simultaneously to get

$$f(x) = 136 - 40x + \frac{1980}{x-2}.$$

Hence, (A), (C) and (D) are correct.

45. $f(15 + x) = f(15 - x)$, replace x with $15 + x$, we have

$$f(30 + x) = f(-x)$$

replace x with $15 - x \Rightarrow f(30 - x) = f(x)$

$\Rightarrow \quad f(30 + x) = -f(x)$

replace x with $30 + x$

$\Rightarrow \quad f(60+x)=-f(30+x)=f(x)$

$\therefore \quad f(x)$ is a periodic and

$f(30-x)=f(x)\Rightarrow f(30+x)=f(-x)=-f(x)$

$\therefore \quad f(x)$ is an odd function

Hence, (A) and (C) are correct.

46. $(x-y)f(x+y)-(x+y)f(x-y)=2y(x-y)(x+y)$

Let $x-y=u$ and $x-y=v \Rightarrow x=\frac{u+v}{2}, y=\frac{v-u}{2}$

$\therefore \quad uf(v)-vf(u)=2uv\left(\frac{v-u}{2}\right)$

$\frac{f(v)}{v}-\frac{f(u)}{u}=v-u \Rightarrow \frac{f(v)}{v}-v=\frac{f(u)}{u}-u=k$

$\frac{f(x)}{x}-x=k=f(x)=x^2+kx$

$f(1)=2 \Rightarrow 2=1+k\Rightarrow k=1$

$f(x)=x^2+x$

$f(0)=0$ and $f(3)=12$

Hence, (A), (B) and (C) are correct.

47. $-3 \le x \le 2$

Now $-3 \le \sin x - \cos x \le 2$

which is True $\forall x$

$\therefore \quad \sin x - \cos x$

Also lies in $\left[-\sqrt{2},\sqrt{2}\right]$

Range of $f(x)$ and $f(x + 7)$ is the same.

$\therefore \quad 2 \le f(x-2) \le 7 \Rightarrow 5 \le f(x-2) + 3 \le 10$

$$\Rightarrow \quad \frac{1}{10} \le \frac{1}{f(x-2)+3} \le \frac{1}{5}$$

$$\Rightarrow \quad \frac{1}{2} \le \frac{5}{f(x-2)+3} \le 1$$

Domain of $f(x)$ is $[-3, 2]$

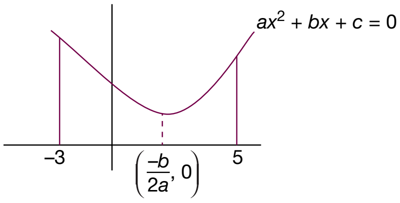
Domain of $f(x-2)$ is $-5 \le x-2 \le 0$

Range of $\frac{5}{f(x-2)+3}$ is $\left[\frac{1}{2},1\right]$

Hence, (A), (B) and (D) are correct.

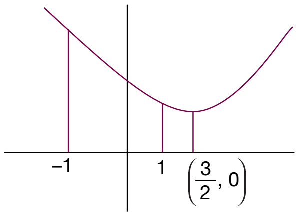
48. (A) $\quad -3<-\frac{3a}{2}<5\Rightarrow -2<-a<\frac{10}{3}\Rightarrow -\frac{10}{3}<a<2$

$\Rightarrow \quad a\in\{-3,-2,-1,0,1\}$



(B) $\quad f(x)=x^2-3x+b$

$gof(x)=\sin^{-1}\left(\frac{f(x)}{4}\right)$ is defined for $\left|\frac{f(x)}{4}\right|\le 1$, $|x|\le 1$



$\Rightarrow \quad -1\le\frac{f(x)}{4}\le 1, |x|\le 1$

$\Rightarrow \quad -4\le f(x)\le 4, |x|\le 1$

$\Rightarrow \quad -4\le x^2-3x+b\le 4, |x|\le 1$

$f\min=f(1)=b-2$

$f\max=f(-1)=b+4$

$\Rightarrow \quad b-2\ge -4$ and $b+4\le 4$

$b\ge -2$ and $b\le 0$

$\Rightarrow \quad$ Integral values of $b=-2,-1,0$

(C) $\quad f:R\to[-8,\infty)$

$f(x)=x^2+6x+b$

$f_{\min}=f(-3)=b-9$

$\Rightarrow \quad b-9=-8$

$\Rightarrow \quad b=1$

(D) $\quad h(x)=fog(x)=\left(\sin^{-1}\frac{x}{4}\right)^2+3\left(\sin^{-1}\frac{x}{4}\right)+2$

$h(t)=t^2+3t+2$ where $t=\sin^{-1}\frac{x}{4}, \; t\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$h(t)=\left(t+\frac{3}{2}\right)^2-\frac{1}{4}$

max $(h(t))$ occurs when $t=\frac{\pi}{2}$ and min $h(t)$ occurs at $t=\frac{-3}{2}$

Min $h(t)=\frac{-1}{4}$ and max of $h(t)=\frac{\pi^2}{4}+\frac{3\pi}{2}+2$

Range of $(fog)(x)=\left[-\frac{1}{4},\frac{\pi^2}{4}+\frac{3\pi}{2}+2\right]$

Hence, (A), (B) and (C) are correct.

49. Given $2f(\sin x)+f(\cos x)=x$ replace x by $\frac{\pi}{2}-x$,

$\Rightarrow \quad 2f(\cos x)+f(\sin x)=\frac{\pi}{2}-x$

$\Rightarrow \quad f(\sin x)=x-\frac{\pi}{6}\Rightarrow f(x)=\sin^{-1}x-\frac{\pi}{6}$

$\therefore \quad$ domain of $f(x)$ is [-1, 1] and range of $f(x)$ is $\left[\frac{-\pi}{2}-\frac{\pi}{6},\frac{\pi}{2}-\frac{\pi}{6}\right]$

i.e., $\left[\frac{-2\pi}{3},\frac{\pi}{3}\right]$

Hence, (B) and (D) are correct.

50. $f\left(\frac{7}{2}\right)=-0.5$, $g\left(\frac{7}{2}\right)=0.5$, $g(3)=0$, $f(0)=3$;

$f(2)=-1, g(2)=-1;\; f(4)=0;\; g(4)=26$

Hence, (A), (B), (C) and (D) are correct.

51. Given $f(x+y)-kxy=f(x)+2y^2$.

Replace y by $-x$, then $f(0)+kx^2=f(x)+2x^2$

$$\Rightarrow \quad f(x)=f(0)+kx^2-2x^2 \qquad \ldots(1)$$

Now $f(1)=f(0)+k-2=2 \Rightarrow f(0)=-k+4$

and $f(2)=f(0)+4k-8=8 \Rightarrow f(0)=-4k+16$

Which give $k=4$ and $f(0)=0$

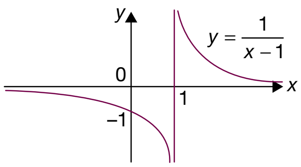
Thus, from (1) $f(x)=2x^2$

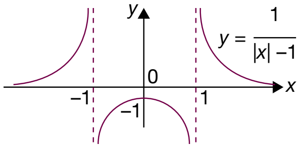
$$\therefore \quad f(x+y)f\left(\frac{1}{x+y}\right)=4=k$$

Hence, (B) and (C) are correct.

Linked Comprehension Type Questions

1.





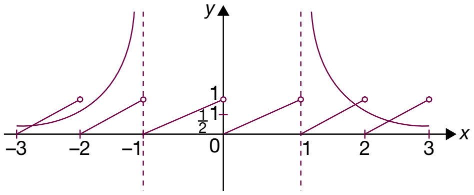
$$f(x)=g(x) \Rightarrow \left|\frac{1}{|x|-1}\right|=\{x\}$$

$x \in [-3, 3]$

1 solution $\alpha \in [2, 3)$

Hence, the correct answer is (A).

2.



$f(x)=g(x) \qquad x \in I$

$\{x\}=0$

$\left|\frac{1}{|x|-1}\right|=0$ Therefore, no solution

Hence, the correct answer is (C).

3. $f(x)=x^2-2x-1 \qquad \forall\, x \in R$

$x=\frac{-b}{2a}=1$

$y=1-2-1=-2$

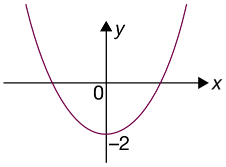
$(-\infty, 1] \to [-2, \infty)$

$a=1, b=-2$

$a+b=-1$

Hence, the correct answer is (B).

4. $f: R \to R$



$g(x)=f(x)+3x-1$

$=x^2+x-2$

$x=-\frac{1}{2}=-\frac{B}{2A}$

$g(|x|)$ least value $=-2$

Hence, the correct answer is (C).

5. $f_1(f_3(x)), f_4(f_3(x)), f_3(f_4(x))$ are constant functions. Then required probability $=\frac{3}{{}^6C_2}$

Hence, the correct answer is (D).

6. Whenever function is periodic, period is 2.

$f_1(x)+f_2(x)=\sin \pi x+\cos \pi x$ (period 2)

$f_1(x)+f_4(x)=\sin \pi x+\{x\}$ (period 2)

$f_2(x)+f_4(x)=\cos \pi x+\{x\}$ (period 2)

It is a conditional probability with given condition that sum of given two functions is a periodic function then probability is 1.

Hence, the correct answer is (D).

7. $\because \quad ff^{-1}(x)=x$

$$2^{(f^{-1}(x))^4-4(f^{-1}(x))^2}=x$$

$$\Rightarrow \quad (f^{-1}(x))^4-4(f^{-1}(x))^2-\log_2 x=0$$

$$\therefore \quad (f^{-1}(x))^2=2+\sqrt{4+\log_2 x} \qquad [\because \text{ range of } f^{-1}(x) \text{ is } [2, \infty).]$$

$$\therefore \quad f^{-1}(x)=\sqrt{2+\sqrt{4+\log_2 x}} \qquad [\because\ f^{-1}(x)>0.]$$

Hence, the correct answer is (B).

8. $g(x)=\frac{\sin x+4}{\sin x-2}$

$$\Rightarrow \quad g'(x)=\frac{-\cos x}{(\sin x-2)} \geq 0 \qquad \because x \in \left[\frac{\pi}{2}, \pi\right]$$

$\Rightarrow \quad g'(x)$ is increasing function hence one-one function.

$\therefore \quad$ range is $\left[g\left(\frac{\pi}{2}\right), g(\pi)\right]$ lie $[-5, -2]$.

Hence, the correct answer is (A).

9. Domain of $f^{-1}g^{-1}(x)=$ Range $gof(x)=[-5, -2]$

Hence, the correct answer is (A).

Combined Solution to 10–12

$f(a, b)=f(a-1, b)+a-1$

$=f(a-2, b)+(a-2)+(a-1)$

$=f(1, b)+\frac{a(a-1)}{2}$

$=f(1, b-1)-(b-1)+\frac{a(a-1)}{2}$

$=f(1, 1)-\frac{b(b-1)}{2}+\frac{a(a-1)}{2}$

$$\Rightarrow \quad \frac{a(a-1)}{2}-\frac{b(b-1)}{2}=1999$$

$$\Rightarrow \quad (a-b)(a+b-1)=2.1999$$

Here, 1999 is a prime number and $a-b<a+b-1$

of $a-b=1, a+b-1=3998 \Rightarrow a=2000, b=1999$

if $a-b=2, a+b-1=1999 \Rightarrow a=1001, b=999$.

$\therefore \quad (a, b)=(2000, 1999)$ or $(1001, 999)$

10. **Hence, the correct answer is (B).**

11. **Hence, the correct answer is (B).**

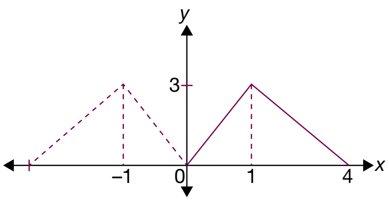
12. **Hence, the correct answer is (C).**

Combined Solution to 13–14

$f(x-2)=f(x+6)$

$\Rightarrow \quad f(x)=f(x+8)$

$$f(x)=\begin{cases} 3x, & 0 \leq x < 1 \\ 4-x, & 1 \leq x \leq 4 \end{cases}$$



(i) Required area $= 2 \times \frac{1}{2} \times \mathbf{4} \times \mathbf{3} + \mathbf{2} \times \frac{1}{2} \times \mathbf{1} \times \mathbf{3} = \mathbf{15}$

(ii) $f(-89) - f(-67) + f(46)$

$= f(-1) - f(-3) + f(-2) = 3 - 1 + 2 = 4$

13. Hence, the correct answer is (B).

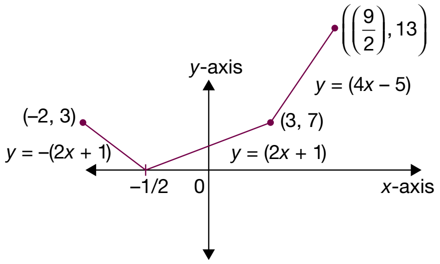
14. Hence, the correct answer is (A).

Combined Solution to 15–16

(i) $f(x) = \begin{bmatrix} -x & -3 \le x \le 0 \\ x & 0 < x \le 7 \\ 2x-7 & 7 < x \le 10 \end{bmatrix}$

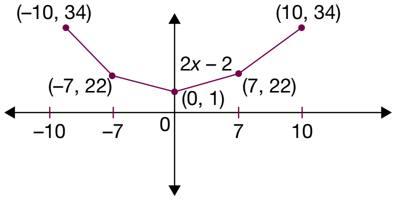
$g(x) = 2x + 1, -3 \le x \le 10$

$\therefore \quad f(g(x)) = \begin{bmatrix} -(2x+1), & -2 \le x \le \frac{-1}{2} \\ 2x+1 & \frac{-1}{2} < x \le 3 \\ 4x-5 & 3 < x \le \frac{9}{2} \end{bmatrix}$



$\therefore$ Range of $f(g(x))$ is $[0, 13]$

(ii) $f(|x|) + g(|x|) = \begin{bmatrix} -4x-6; & -10 \le x \le -7 \\ -3x+1; & -7 < x \le 0 \\ 3x+1\ ; & 0 < x \le 7 \\ 4x-6\ ; & 7 < x \le 10 \end{bmatrix}$



$\therefore$ For $f(|x|) + g(|x|) = k$ to have two distinct solution, $k \in (1, 34]$.

15. Hence, the correct answer is (C).

16. Hence, the correct answer is (D).

17. Let $k(x) = (f - g)(x)$

$= (f(x) - g(x)) = \begin{cases} 2x+3\ ; & x \in \text{rational} \\ 3x - \sqrt{5}; & x \in \text{irrational} \end{cases}$.

As, $k\left(\frac{-3}{2}\right) = 0$ and $k\left(\frac{\sqrt{5}}{3}\right) = 0$

$\Rightarrow \quad k(x)$ is not injective.

Also, $k(x)$ is not onto function. because $k(x)$ will not take irrational values like $-\sqrt{5}$, $1-\sqrt{5}$, $2-\sqrt{5}$, $3-\sqrt{5}$, $\frac{1}{2}-\sqrt{5}$ and so on.

Hence, the correct answer is (D).

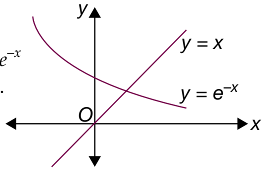
18. Put $y = (\log_2 x)^2 + 2(\log_2 x) + 3$

$\Rightarrow \quad \log_2 x = 1 \pm \sqrt{y-2}$.

So $h^{-1}(x) = 2^{-1+\sqrt{x-2}}$

Hence, the correct answer is (D).

Combined Solution to 19–21



(i) $g(x)\, h(g(x)) = 1$

$\Rightarrow \quad e^x \ln|e^x| = 1 \Rightarrow xe^x = 1 \Rightarrow x = e^{-x}$

Clearly equation has only one solution.

(ii) $f(g(x)) + f(h(x)) = f(e^x) + f(\ln|x|)$

$\therefore \quad f$ is defined in $(0, 1)$

$\therefore \quad 0 < e^x < 1$ and $0 < \ln|x| < 1$

$x \in (-\infty, 0)$ and $1 < |x| < e$

$x \in (-e, -1) \cup (1, e)$

$\therefore \quad$ Domain is $(-e, -1)$

(iii) $f : (0, 1) \to (1, 2)$

$gof\,(x) = g[f(x)] = e^{f(x)}$

Domain of $gof(x)$ is $(0, 1)$

and range of gof $(x) = e^{f(x)}$ is (e, e^2)

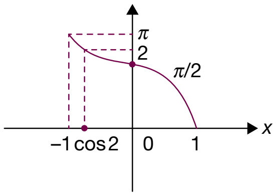
$\therefore \quad$ Domain of $(gof)^{-1} = f^{-1}og^{-1}$ is (e, e^2).

19. Hence, the correct answer is (B).

20. Hence, the correct answer is (A).

21. Hence, the correct answer is (B).

22.



$[\cos^{-1} x] \ne 1$

$\because \quad \cos^{-1} x \in [0, \pi]$

$x \in [-1, \cos 2]$

Hence, the correct answer is (B).

23. cos (sin$x$) is defined for all real x

$\log x\, \{x\} \ge 0$

CASE 1: $0 < x < 1$

$\{x\} \le 1; x \in (0, 1)$

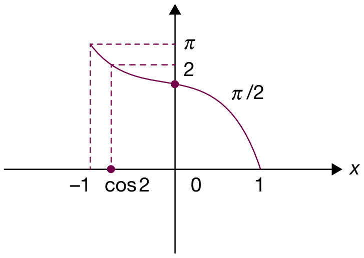
CASE 2: $x > 1$ and $\{x\} > 0$

$\{x\} \ge 1$ but $0 < \{x\} < 1$

not possible

Hence, the correct answer is (D).

24.



Least value of $x^2 + x + 1 = 3/4 \quad$ (at $x = 1/2$)

$\therefore \quad$ least value of $\sqrt{x^2+x+1} = \frac{\sqrt{3}}{2}$

Also $\sqrt{x^2+x+1} \le 1$

Thus $\frac{\sqrt{3}}{2} \le \sqrt{x^2+x+1} \le 1$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \le \sin^{-1}\sqrt{x^2+x+1} \le \sin^{-1}(1)$$

$$\Rightarrow \quad \frac{\pi}{3} \le \sin^{-1}\sqrt{x^2+x+1} \le \pi/2$$

Range of $f = \left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

Hence, the correct answer is (C).

Combined Solution to 25–27

$\because \quad f(-1) = 0 \Rightarrow a = 1$

$$f(x) = \frac{x+1}{bx^2+cx+2}$$

Having asymptote $y = 1$

$b = 0 \Rightarrow c = 1$

$$f(x) = y = \frac{x+1}{x+2}$$

$$f^{-1}(x) = \frac{2x+1}{1-x}$$

Asymptotes of $f(x) \Rightarrow x = -2$ and $y = 1$

Asymptotes of $f^{-1}(x) \Rightarrow x = 1$ and $y = -2$

$\because \quad$ Range of $f^{-1}(x) \in R - \{-2\}$

$\Rightarrow \quad f(x)$ is one-one into

Area bounded by asymptote $= 3 \times 3 = 9$

25. **Hence, the correct answer is (A).**

26. **Hence, the correct answer is (B).**

27. **Hence, the correct answer is (B).**

28. $G(x) = \dfrac{1-2\sin^2 x}{\cos 2x} = \dfrac{\cos 2x}{\cos 2x} = 1$ but $\cos 2x \ne 0$

$$\Rightarrow \quad 2x \ne (2n+1)\frac{\pi}{2} \Rightarrow x \ne (2n+1)\frac{\pi}{4}, n \in I$$

Hence, the correct answer is (D).

29. $F(x) = 1 - 2\sin^2 x + \cos 2x = 2\cos 2x$

$-1 \le \cos 2x \le 1$

$-2 \le F(x) \le 2$

$\Rightarrow \quad F$ is onto function

Hence, the correct answer is (B).

30. Periods of $f(x), g(x)$ and $F(x)$ is π each.

$\left(\because \text{ period of } \cos 2x = \dfrac{2\pi}{|2|} = \pi\right)$

Hence, the correct answer is (C).

Combined Solution to 31–32

Given that $f(x) + f\left(\dfrac{x-1}{x}\right) = x$ …(1)

For all $x \ne 0, 1$. Replacing x with $\dfrac{(x-1)}{x}$ both sides, we get that

$$f\left(\frac{x-1}{x}\right) + f\left(\frac{[(x-1)/x]-1}{(x-1)/x}\right) = \frac{x-1}{x}$$

That is $f\left(\dfrac{x-1}{x}\right) + f\left(\dfrac{1}{1-x}\right) = \dfrac{x-1}{x}$ …(2)

Again replacing x with $(x-1)/x$ in this, we get

$f\left(\dfrac{1}{1-x}\right) + f(x) = \dfrac{1}{1-x}$ …(3)

Then by taking Eq. (1) + Eq. (3) – Eq. (2), we get that

$$2f(x) = x + \frac{1}{1-x} - \frac{x-1}{x}$$

or $f(x) = \dfrac{1}{2}\left[x + \dfrac{1}{1-x} - \dfrac{x-1}{x}\right]$ …(4)

Substituting the values $x = -1$ and $1/2$ in Eq. (4).

$$f(-1) = \frac{1}{2}\left[-1 + \frac{1}{2} - 2\right]; f(-1) = \frac{1}{2}\left[-\frac{5}{2}\right]$$

$$f(-1) = -\frac{5}{4}; f\left(\frac{1}{2}\right) = \frac{1}{2}\left[\frac{1}{2} + 2 + 1\right]$$

$$f\left(\frac{1}{2}\right) = \frac{7}{4}$$

31. **Hence, the correct answer is (A).**

32. **Hence, the correct answer is (D).**

Combined Solution to 33–35

$$f(x+y) = f(x)f(a-y) + f(y)f(a-x)$$

Put $x = 0, y = 0 \Rightarrow \quad f(0) = 2f(0)f(a) \Rightarrow f(a) = \dfrac{1}{2}$

Put $y = 0, \; f(x) = f(x)f(a) + f(0)f(a-x)$

$f(x^2+1) + f(x-1) = 1$

$f(1) + f(2) + \ldots + f(100) = 50$

$$fog\left(\frac{1}{4}\right) + gof(4) = \frac{1}{2} + \frac{\frac{1}{4}+1}{\frac{1}{2}+1} = \frac{4}{3}$$

$$\Rightarrow \quad f(x) = \frac{f(x) + f(a-x)}{2}$$

$$f(x) = f(a-x)$$

Put $y = a - x$

$$f(a) = f^2(x) + f^2(a-x) \text{ but } f(x) = f(a-x)$$

$$\Rightarrow \quad f(a) = 2f^2(x) \quad \Rightarrow \quad f^2(x) = \frac{1}{4}$$

$\Rightarrow \quad f(x) = \dfrac{1}{2} \qquad \therefore\ f(x) = \dfrac{1}{2}$, a constant function $\quad \left(\therefore\ f(0) = \dfrac{1}{2}\right)$

33. **Hence, the correct answer is (C).**

34. **Hence, the correct answer is (B).**

35. **Hence, the correct answer is (C).**

36. Domain of $g(x) \qquad x \in (-2, 4]$

Range of $g(x) \qquad y \in (-2, 8]$

domain of $f(x) \qquad x \in R \qquad$ Range of $f(x)$ $[a-2, \infty)$ $(b+3, \infty)$

$g(f(x))$ is not define if

(i) $-2 + a > 4$ and (ii) $b + 3 > 4$

$a > 6$ and $b > 1$

Hence, the correct answer is (C).

37. $x \in [-1, 2] \Rightarrow -1 \le x \le 2$

$\Rightarrow -2 \le 2x \le 4 \Rightarrow -2 + a \le 2x + a \le 4 + a$

$\Rightarrow \quad -2 + a \le -2$ and $4 + a \le 4$ i.e. $a = 0$

b can take any value.

Hence, the correct answer is (D).

38. If $a = 2, b = 3$

$$f(x) = \begin{cases} 2x+2 & ; \; x \ge -1 \\ 3x^2+3 & ; \; x < -1 \end{cases}$$

Range of $f(x)$ is $[0, \infty]$, So range of $g(f(x))$ is $[4, 8]$

Hence, the correct answer is (C).

39. $f : (0, \infty) \to (0, \infty)$

$$f(x(f(y))) = x^2 y^a \, (a \in R)$$

Put $x = 1$, we get $f(f(y)) = y^a$

Put $f(y) = \dfrac{1}{x}$,

We get $f(1)=\frac{1}{(f(y))^2}\cdot y^a$ and put $y=1$ we get $(f(1))^3=1$

$\therefore \quad f(1)=1$

For $y=1$, we have $f(x(f(1))) = x^2$

$\therefore \quad f(x)=x^2$

$f(xy^2)=(xy^2)^2=x^2y^4$

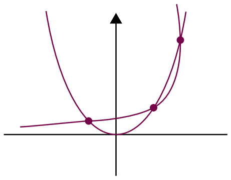
Thus $a=4$

Hence, the correct answer is (A).

40. $\sum_{r=1}^{n} f(r)\,{}^nC_r=\sum_{r=1}^{n} r^2\,{}^nC_r=\sum_{r=1}^{n}\big(r(r-1)\big)\,{}^nC_r+\sum_{r=1}^{n} r\,{}^nC_r$

$=n(n-1)2^{n-2}+n.2^{n-1}$

Hence, the correct answer is (C).



41. Since $f(x)=x^2$

$\therefore \quad 2x^2=e^x$

Hence, the correct answer is (C).

Matrix Match/Column Match Type Questions

1. **A → (r); B → (q); C → (p); D → (s)**

$f'(x)=(3x^2+2x+1)>0\ \forall\ x\in R \Rightarrow f(x)$ is strictly increasing on R.

Range of $g(x)=\frac{2x}{1+x^2}$, $x\in R$ equals $[-1, 1]$

Also domain of $h(x)$ is $[-1, 1]$ and range of $h(x)$ is $\left[\frac{-5\pi}{2},\frac{\pi}{2}\right]$

(i) Range of $f(g(x))=[0, 4]$

(ii) Range of $g(f(x))=[-1, 1]$

(iii) Maximum value of $g(x)$ occurs at $x=1$. And range of $h(x)$ is $\left[\frac{-5\pi}{2},\frac{\pi}{2}\right]$

$\therefore \quad g(h(x))|_{\max}=1$

(iv) Minimum value of $h(gf(x))=h(g(-1))=\frac{-5\pi}{2}$

$\therefore \quad |k|=5$

2. **A → (p, r, s); B → (r, s); C → (t); D → (q, t)**

(A) $f: R\to R$

$f(x+y)+f(x-y)=2f(x)\,f(y)\ \forall\ x, y\in R$

$f(0)\neq 0$

$x=y=0$

$f(0)+f(0)=2f(0)\cdot f(0)$

$2(f(0))^2-2f(0)=0;\ f(0)\neq 0;\ f(0)=1$

$x=0;\ y=x$

$f(x)+f(-x)=2f(0)\,f(x)$

$f(x)+f(-x)=2f(x)\Rightarrow f(-x)=f(x)$ even fn. many-one, into

(B) $f: R\to R$

$$f(x)=\frac{e^{|x|}-e^{-x}}{e^x+e^{-x}}=\begin{cases}\frac{e^x-e^{-x}}{e^x+e^{-x}}; & x\geq 0\\ 0\ ; & x<0\end{cases}$$

$$y=\begin{cases}\frac{e^{2x}-1}{e^{2x}+1}; & x\geq 0\\ 0\ ; & x<0\end{cases}$$ NENO

$e^{2x}y+y=e^{2x}-1$ Many-one

$e^{2x}(1-y)=y+1$ $x\geq 0\Rightarrow 2x\geq 0$

$e^{2x}=\frac{y+1}{1-y}\geq 1;$ $\frac{y+1}{1-y}-1\geq 0;\Rightarrow \frac{y+1-1+y}{1-y}\geq 0$

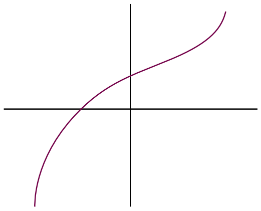
$\frac{2y}{y-1}\leq 0$ $y\in[0, 1)$ into

(C) $f(x)=1\pm x^n$ $f(3)=28=1+27=1+3^3$

$n=3$ $f(x)=1+x^3;\ f'(x)=3x^2$

Bijective, one-one onto

NENO



(D) $f: R\to R$ $f(x)=2x+\sin x$

$f'(x)=2+\cos x>0$

odd fn. Bijective onto, one-one

3. **A → (q, s); B → (p, r); C → (q, r)**

(A) $y=\frac{3x-2}{x+4}\Rightarrow xy+4y=3x-2$

$\Rightarrow \quad (y-3)x=-2-4y$

$x=\frac{4x+2}{3-y}$

$f^{-1}(x)=\frac{2+4x}{3-x}=\frac{x+\frac{1}{2}}{\frac{3}{4}-\frac{x}{4}}=\frac{x+b}{cx+d}$

$b=\frac{1}{2}, c=-\frac{1}{4}, d=\frac{3}{4}$

$b+c+d=1$

(B) $f(g(x))=g(f(x))$

$\frac{g(x)}{1+g(x)}=\frac{rf(\alpha)}{1-f(x)}$

$\Rightarrow \quad \frac{rx}{(1-x)\left(\frac{1+rx}{1-x}\right)}=\frac{rx}{(1+x)\cdot\left(1-\frac{x}{1+x}\right)}$

$\frac{rx}{1-x+rx}=\frac{rx}{1}\Rightarrow rx=rx(1-x+rx)$

$rx\,(1-x+rx-1)=0$

$rx(r-1)x=0$ $\forall\ x\in R$

$r(r-1)=0$ $r=0, 1$

(C) $f(x)=2x+1$

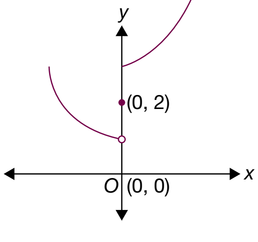
$f(x)+f(f(x))+f(f(f(x)))+f(f(f(f(x)))=116$

$2x+1+2(2x+1)+1+2(2(2x+1)+1)+1+2(2(2(2x+1)+1)+1)+1=116$

$2x+1+4x+3+2(4x+3)+1+2(2(4x+3)+1)+1=16$

$30x+26=116\Rightarrow x=3$

4. **A → (r, t); B → (q, t); C → (r, t); D → (p, s)**



(A) $f(x)=e^{\text{sgn}x}+e^{x^2}$

when $x=0, f(0)=2$

when $x>0, f(x)=e+e^{x^2}$

when $x<0, f(x)=\frac{1}{e}+e^{x^2}$

Hence, $f(x)$ is many-one and neither odd nor even.

(B) $D_f: 1-x^2>0$

$\Rightarrow \quad x^2-1<0\Rightarrow x\in(-1, 1)$

$\therefore \quad [x^4]$ when $x\in(-1, 1)$ is equal to 0.

$\therefore$ $f(x) = \dfrac{1}{\sqrt{1-x^2}}$

$\therefore$ Hence, $f(x)$ is even and many-one.

(C) $f(x) = x^4 + 1 + \dfrac{x^4+x^2+1}{x^2+x+1} = x^4 + 1 + x^2 - x + 1$

$= x^4 + x^2 - x + 2 = x(x^3 + x - 1) + 2$

Hence, $f(x)$ is many-one and neither odd nor even.

(D) $f(x) = x + 3x^3 + 5x^5 + \ldots + 101x^{101}$

Clearly, $f(x)$ is an odd function.

Now $f'(x) = 1^2 + 3^2x^2 + 5^2x^4 + \ldots + (101)^2x^{100}$.

$\Rightarrow$ $f'(x) > 0 \ \forall \ x \in R$.

Hence, $f(x)$ is one-one function.

5. A → (r); B → (p); C → (s); D → (q)

(A) When $-1 \le x < 0 \Rightarrow f(x) \in [0,1)$ and $f(x) = x+1$

$\therefore$ $f(f(x)) = (x+1)^2 - 1 = x^2 + 2x$

(B) $f\left(\dfrac{2\tan x}{1+\tan^2 x}\right) = \dfrac{(\cos 2x + 1)(\sec^2 x + 2\tan x)}{2}$

$= \dfrac{\left(1+\dfrac{1-\tan^2 x}{1+\tan^2 x}\right)(1+\tan^2 x + 2\tan x)}{2}$

$= \dfrac{1+\tan^2 x + 2\tan x}{1+\tan^2 x}$

$\therefore$ $f\left(\dfrac{2\tan x}{1+\tan^2 x}\right) = 1 + \dfrac{2\tan x}{1+\tan^2 x} \Rightarrow f(x) = 1 + x$

(C) Put, $x = 0, y = 0 \Rightarrow f(1) = (1+1)^2 = 2^2$,

Put $x = 0, y = 1 \Rightarrow f(2) = (1+2)^2 = 3^2$

By induction, $f(x) = (x + 1)^2$

(D) where $4 < x < 5 \Rightarrow \left[\dfrac{x}{4}\right] = 1$

$\Rightarrow$ $f(x) = 2x + 3 = y \Rightarrow x = \dfrac{y-3}{2}$

$\therefore$ $f^{-1}(x) = \dfrac{x-3}{2}$

6. A → (q); B → (r); C → (p); D → (s)

(A) For 1, 3, 5, 7 $\in A$ we have 4, 3, 2, 1, choices, respectively.

(B) Image of $f(7)$ should be greater than 9 $\Rightarrow$ not possible

(C) For 1, 3, 5, 7 $\in A$ we have 2, 2, 2, 2 choices, respectively.

(D) for any $i \in A$, we have 3 choices.

So, the number of functions $= 3 \times 3 \times 3 \times 3 = 81$

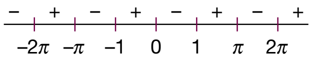
7. A→ (s); B→(r); C→ (p); D→ (q)

(A) $\left|x^2 - 1 + \sin x\right| = \left|x^2 - 1\right| + \left|\sin x\right|$

Exist only if $(x^2 - 1)$ and ($\sin x$) are of the same sign

$(\because |a+b| = |a| + |b|)$ only if ($ab \ge 0$)

$\therefore$ $(x^2 - 1) \sin x \ge 0$



$\Rightarrow$ $(x-1)(x+1)\sin x \ge 0.$

$x \in [-2\pi, -\pi] \cup [-1, 0] \cup [1, \pi] \cup \{2\pi\}$

(B) $f(x) = \dfrac{1}{\sqrt{\log_{\frac{1}{2}}\left(x^2 - 7x + 13\right)}}$

Exists if $\log_{\frac{1}{2}}\left(x^2 - 7x + 13\right) > 0$

$x^2 - 7x + 13 < 1$ and $x^2 - 7x + 13 > 0$

$\Rightarrow$ $x^2 - 7x + 13 > 0 \Rightarrow \left(x - \dfrac{7}{2}\right)^2 + \dfrac{3}{4} > 0$

which is true for all $x \in R$

again $x^2 - 7x + 13 < 1 \Rightarrow x^2 - 7x + 12 < 0$

$\Rightarrow$ $(x-3)(x-4) < 0$

$\Rightarrow$ $3 < x < 4$

Thus, D_f is (3,4)

(C) Since $10^x + 10^y = 10$

$\Rightarrow$ $10^y = 10 - 10^x$

$\Rightarrow$ $y = \log_{10}(10 - 10^x)$

Now y is defined if $10 - 10^x > 0$

$\Rightarrow$ $10^1 > 10^x$

$1 > x$ i.e. $x < 1$

$\therefore$ $D_f = (-\infty, 1)$

(D) Here, $x \in \left(0, \dfrac{\pi}{2}\right)$ $\Rightarrow$ $0 < \sin x < 1$

Again $\log_a x < b \Rightarrow x > a^b$, if $0 < a < 1$ and $x < a^b$ if $a > 1$

Thus $f(x) = \dfrac{1}{\sqrt{-\log_{\sin x} \tan x}}$ exists

If $-\log_{\sin x} \tan x > 0$

$\Rightarrow$ $\log_{\sin x} \tan x < 0$

$\Rightarrow$ $\tan x > (\sin x)^0 = 1$ $\Rightarrow$ $\tan x > 1$

$\Rightarrow$ $x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$ $\left[\because x \in \left(0, \dfrac{\pi}{2}\right)\right]$

$\therefore$ Required solution is $x \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

8. A → (r); B → (s); C → (q); D → (q)

$f(x)$ can be re-written as

$f(x) = \dfrac{1}{\pi}(\sin^{-1}x + \tan^{-1}x) + \dfrac{1}{(x+1) + \dfrac{4}{x+1}}$

Domain of $g(x)$ is [–1, 1]

$\because$ $\sin^{-1}x$ has domain [–1, 1] and $\tan^{-1}x$ has domain R.

Domain of $f(x)$ is $[-1, 1] \Rightarrow a + b = 0$

Maximum value of $g(x)$ is $g(1) = 3/4$

Maximum value of $h(x)$ occurs when $(x + 1) + \dfrac{4}{(x+1)}$ is minimum at $x = 1$.

$h(x)$ has minimum value at $x = -1$ in [–1, 1]

$\Rightarrow$ maximum value of $h(x) = \dfrac{1}{4}$

$\therefore$ Range of $f(x)$ is $\left[-\dfrac{3}{4}, 1\right]$

$\Rightarrow$ $l + m = 1 - \dfrac{3}{4} = \dfrac{1}{4}$

9. A → (q); B → (q); C → (q); D → (r)

(A) Fundamental period of $\sin^{-1}(\sin kx)$ is $\dfrac{2\pi}{|k|} = \dfrac{\pi}{2}$ i.e. $|k| = 4$ i.e. $k = \pm 4$

(B) Domain of $\tan^{-1} x$ is R, domain of $\sin^{-1} x$ is [–1, 1], domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$

$\therefore$ domain of $f(x)$ is {–1, 1}

(C) π is a period of $\sin x$. cosec x

$\therefore$ $\pi \times \dfrac{2}{\pi}$ is a period of $\sin\dfrac{\pi x}{2}$. cosec$\dfrac{\pi x}{2}$

i.e., 2 is a period of $\sin\frac{\pi x}{2}\operatorname{cosec}\frac{\pi x}{2}$

$$f(x+1)=\sin\frac{\pi}{2}(x+1)\operatorname{cosec}\frac{\pi}{2}(x+1)$$

$$=\cos\frac{\pi}{2}x\,.\sec\frac{\pi}{2}x\neq f(x)$$

(D) $f(x)=\cos^{-1}[5x]$

$[5x]$ can take the values – 1, 0, 1

$\therefore\quad \text{range} = \left\{\pi,\frac{\pi}{2},0\right\}$

$\therefore\quad a+b+c=\pi+\frac{\pi}{2}+0=\frac{3\pi}{2}$

Integer/Numerical Answer Type Questions

1. $f(x)=\dfrac{1}{\sqrt{\ln\cos^{-1}x}}$

$\ln\cos^{-1}x>0\Rightarrow\cos^{-1}x>e^{o}=1$

$\cos^{-1}x\in[0,\pi]\cap\cos^{-1}x>1$

$1<\cos^{-1}x\leq\pi\Rightarrow\cos 1>x\geq-1$

No. of integer –1, 0

Hence, the correct answer is (2).

2. $$f(f(x))=\frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1}=\frac{x+1+x-1}{x+1-x+1}=\frac{2x}{2}=x$$

$P=f(2)\,f^{2}(3)\,f^{3}(4)\,f^{4}(5)$

$$=\left(\frac{2+1}{2-1}\right)3\left(\frac{4+1}{4-1}\right)(5)=9\left(\frac{5}{3}\right)5=3^{1}\cdot5^{2}$$

No. of divisors = (1 + 1) (2 + 1) = 6

Hence, the correct answer is (6).

3. $$f(x)+f\left(\frac{x-1}{x}\right)=1+x\qquad\ldots(1)$$

Replace $x\rightarrow\dfrac{x-1}{x}$

$$f\left(\frac{x-1}{x}\right)+f\left(\frac{\frac{x-1}{x}-1}{\frac{x+1}{x}}\right)=1+\frac{x-1}{x}$$

$$f\left(\frac{x-1}{x}\right)+f\left(\frac{-1}{x-1}\right)=\frac{x+x-1}{x}$$

$$f\left(\frac{x-1}{x}\right)+f\left(\frac{1}{1-x}\right)=\frac{2x-1}{x}\qquad\ldots(2)$$

Replace $x\rightarrow\dfrac{1}{1-x}$ in (1)

$$f\left(\frac{1}{1-x}\right)+f\left(\frac{\frac{1}{1-x}-1}{\frac{1}{1-x}}\right)=1+\frac{1}{1-x}$$

$$f\left(\frac{1}{1-x}\right)+f(x)=\frac{2-x}{1-x}\qquad\ldots(3)$$

(1) + (3) – (2)

$$2f(x)=1+x+\frac{2-x}{1-x}-\frac{(2x-1)}{x}$$

$$=\frac{(1-x^{2}+2-x)x-(2x-1)(1-x)}{(1-x)x}$$

$$=\frac{x-x^{3}+2x-x^{2}-2x+2x^{2}+1-x}{(1-x)x}$$

$$2f(x)=\frac{-x^{3}+x^{2}+1}{(1-x)x}$$

$$2f(2)=\frac{-8+4+1}{(-1)2}=\frac{3}{2}$$

$4f(2)=3$

Hence, the correct answer is (3).

4. $0\leq\{x\}<1\qquad 1=57°$

$\sec\{x\}=[1,\sec 1)=[1,1{\cdot}98)$

$g(x)=2x^{2}-3x(K+1)+K(3K+1)$

$f(x)=[\sec\{x\}]=1$

$g(f(x))<0\qquad\forall\,x\in R$

$g(1)<0\Rightarrow 2-3(K+1)+K(3K+1)<0$

$2-3K-3+3K^{2}+K<0$

$3K^{2}-2K-1<0\Rightarrow(3K+1)\,(K-1)<0$

No. of integers = 1

$-\frac{1}{3}<K<1;\ K=0$

Hence, the correct answer is (1).

5. $f(x)=\cos(2010\,\{x^{3}\}\ (2011^{[x^{2}]}+2012x)\,)\ x\in R$

$x\in I;\ \{x^{3}\}=0$

$f(x)_{\max}=\cos(0)=1$

Hence, the correct answer is (1).

6. $f(1)=1;\ n>1$

$f(n)=f(f(n-1))+f(n-f(n-1))$

$f(2)=f(1)+f(1)=2$

$f(3)=f(2)+f(1)=3$

$f(4)=f(3)+f(1)=4$

$$\frac{1}{30}\sum_{r=1}^{20}f(r)=\frac{1}{30}(1+2+\ldots+20)=\frac{20.21}{2(30)}=7$$

Hence, the correct answer is (7).

7. $f:R\rightarrow R$

$f(x)=x^{3}+x+1;\ 1\leq x\leq 2$

$f(x)$ and $g(x)$ are inverse

$f'(x)=3x^{2}+1>0\qquad$ increasing

$$\left.\begin{aligned}f(1)&=3\\ f(2)&=8+2+1=11\end{aligned}\right\}$$

Range of $f(x)=[3,11)$ when $x\in[1,2]$ = domain of $g(x)$

$a=3,\ b=11$

$|a-b|=8$

Hence, the correct answer is (8).

8. Domain of $\sin^{-1}(\sin x)$ is which of R

$$-\log_{\frac{x+4}{2}}\log_{2}\left(\frac{2x-1}{3+x}\right)>0$$

i.e., $$\log_{\frac{x+4}{2}}\left(\log_{2}\left(\frac{2x-1}{3+x}\right)\right)<0$$

CASE 1: $0<\dfrac{x+4}{2}<1$ i.e. $-4<x<-2$

then $\log_{2}\dfrac{2x-1}{3+x}>1$ i.e. $\dfrac{2x-1}{3+x}>2$ i.e. $\dfrac{2x-1-6-2x}{3+x}>0$

i.e., $x+3<0$ i.e. $x<-3$

$\therefore\quad -4<x<-3\qquad\ldots(1)$

CASE 2: If $\frac{x+4}{2} > 1$ i.e. $x > -2$

then $0 < \log_2 \frac{2x-1}{3+x} < 1$ i.e. $1 < \frac{2x-1}{3+x} < 2$

i.e., $\frac{2x-1-3-x}{x+3} > 0$ and $\frac{2x-1-6-2x}{x+3} < 0$

i.e., $\frac{x-4}{x+3} > 0$ and $\frac{-7}{x+3} < 0$

i.e., $\{x < -3 \text{ or } x > 4\}$ and $x > -3$

i.e., $x > 4$ …(2)

from (1) and (2) $x \in (-4, -3) \cup (4, \infty)$

$\therefore$ $a = -4, b = -3, c = 4$ and so $a + b + 3c = 5$.

Hence, the correct answer is (5).

9. If $f(x) = \frac{ax+b}{cx-a}, x \neq \frac{a}{c}$, then $f(f(x)) = x$

$\therefore \quad f(f(x)) = x$ and $f\left(f\left(\frac{4}{x}\right)\right) = \frac{4}{x}$

$\therefore \quad f(f(x)) + f\left(f\left(\frac{4}{x}\right)\right) = x + \frac{4}{x} \geq 4$

Hence, the correct answer is (4).

10. $f(x) = \begin{cases} 1+x; & -1 \leq x < 0 \\ 1-x; & 0 \leq x \leq 1 \\ 0 \quad ; & x < -1 (\text{or}) x > 1 \end{cases}$

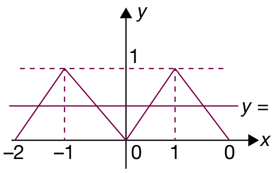
$$f(x-1) = \begin{cases} 1+(x-1); & -1 \leq x-1 < 0 \\ 1-(x-1); & 0 \leq x-1 \leq 1 \\ 0 \quad ; & x-1 < -1 (\text{or}) x-1 > 1 \end{cases}$$

$$\Rightarrow \quad f(x-1) = \begin{cases} x \quad ; & 0 \leq x < 1 \\ 2-x; & 1 \leq x < 2 \\ 0 \quad ; & x < 0 (\text{or}) x > 2 \end{cases}$$

Similarly,

$$f(x+1) = \begin{cases} x+2, & -2 \leq x < -1 \\ -x \quad , & -1 \leq x \leq 0 \\ 0 \quad , & x < -2 (\text{or}) x > 0 \end{cases}$$

$$\therefore \quad f(x-1) + f(x+1) = \begin{cases} 2-x; & 1 \leq x \leq 2 \\ x \quad ; & 0 \leq x < 1 \\ 0 \quad ; & x > 2 \\ -x \quad ; & -1 \leq x \leq 0 \\ x+2; & -2 \leq x < -1 \\ 0 \quad ; & x < -2 \end{cases}$$



Equation $f(x-1) + f(x+1) = \sin\alpha,\ 0 < \alpha < \frac{\pi}{2}$ has 4 solution.

Hence, the correct answer is (4).

11. $f(x) = ax + b \Rightarrow a[x + f(x)] + b = x + ax + b$

$\Rightarrow \quad af(x) = x \Rightarrow f(x) = \frac{x}{a}$

$\Rightarrow \quad \frac{x}{a} = ax + b \Rightarrow b = 0 \Rightarrow a^2 = 1$

$\Rightarrow \quad a = \pm 1 \Rightarrow f(x) = \pm x$

Hence, the correct answer is (2).

12. $f[f(x)] = x \Rightarrow d = -a$

$f(x) = \frac{ax+b}{cx-a}$

$f(5) = 5 \Rightarrow 5a + b = (5c - a)5 = 25c - 5a$

$f(13) = 13 \Rightarrow 13a + b = (13c - a)13 = 169c - 13a$

$-8a = -144c + 8a$

$\Rightarrow \quad a = 9c$

$\therefore \quad f(x) = \frac{9cx+b}{cx-a}$

Range $f(x)$ does not contain $\frac{9c}{c} = 9 \Rightarrow x = 9$

Hence, the correct answer is (9).

13. No. of one-one functions = ${}^5P_3 = 60$

For strictly increasing choose any 3 elements from B by ${}^5C_3$ way and fixed order by 1 way.

No. of strictly monotonic functions = 10 + 10 = 20 $\Rightarrow \frac{60}{20} = 3$

(increasing + decreasing)

Hence, the correct answer is (3).

14. The no. of different sets contains exactly 3 elements of $B = {}^5C_3 = 10$.

Then no. of onto functions from A to the set contains 3 elements = 10 $[3^4 - 3(2)^4 + 3] = 10 \times 36 = 360$.

$\Rightarrow \quad \frac{k}{60} = \frac{360}{60} = 6$

Hence, the correct answer is (6).

15. $f(2) = a\sin 2 + 2b\cos 2 + 8$

$f(-2) = -a\sin 2 - 2b\cos 2 + 8$

– – – – – – – – – – – – – – – – – – –

$f(2) + f(-2) = 16$

Hence, the correct answer is (1).

16. $f(x) = 3[2x] + 1000 \sum_{r=1}^{2008} \frac{x + r - [x+r]}{2008}$

$= 3[2x] + \frac{1000}{2008} \sum_{k=1}^{2008} \{x+k\}$ using $\{x\} = x - [x]$

$f(3) = 3[6] + \frac{1000}{2008} \sum_{k=1}^{2008} \{3+k\} = 18 + 0$ (as {integer} = 0)

$\therefore \quad f(3) = 18$

Hence, the correct answer is (18).

17. $f(2011) = (2 + 0 + 1 + 1)^2 = 16$

$f^2(2011) = (1 + 6)^2 = 49$

$f^3(2011) = (4 + 9)^2 = 169$

$f^4(2011) = (1 + 6 + 9)^2 = 256$

$f^5(2011) = (2 + 5 + 6)^2 = 169$

$\Rightarrow \quad f^{2n}(2011) = 256$

$f^{2n+1}(2011) = 169$.

$\frac{f^{2011}(2011) - f^{2010}(2011)}{f^{2013}(2011) - f^{2012}(2011)} = 1$

Hence, the correct answer is (1).

18. $[\sin x] = \frac{x}{2\pi} - \left[\frac{x}{2\pi}\right] + \frac{2x}{5\pi} - \left[\frac{2x}{5\pi}\right]$

$$[\sin x] = \left\{\frac{x}{2\pi}\right\} + \left\{\frac{2x}{5\pi}\right\}$$

Both are periodic function with period 10π.
As $x \in (30, 40) x = 10\pi$ is the only solution.
No. of solutions = 1

Hence, the correct answer is (1).

19. $24\{x\} - 5 - 16\{x\}^2 \geq 0$

$\Rightarrow \quad 16\{x\}^2 - 24\{x\} + 5 \leq 0$

$\Rightarrow \quad (4\{x\}-1)(4\{x\}-5) \leq 0 \Rightarrow \frac{1}{4} \leq \{x\} \leq \frac{5}{4}$

but $\{x\} \in [0, 1]$

$\therefore \quad \frac{1}{4} \leq \{x\} < 1$ Let $\{x\} = t$

Moreover, $24t - 5 - 16t^2$ is an increasing function $\forall t \in \left[\frac{1}{4}, 1\right)$

Now $f\left(\frac{1}{4}\right) = \sqrt{24 \times \frac{1}{4} - 5 - 16\left(\frac{1}{4}\right)^2} = 0$

$$f(1) = \sqrt{24 \times 1 - 5 - 16} = \sqrt{3}$$

$$f\left(\frac{3}{4}\right) = \sqrt{24 \times \frac{3}{4} - 5 - 16 \times \frac{9}{16}} = 2$$

Range $\in [0, 2]$

$\Rightarrow \quad a = 0,\ b = \sqrt{2} \ \Rightarrow \ b^2 - a = 4$

Hence, the correct answer is (4).

20. Let $\{x\}^{1/\sqrt{|\ln\{x\}|}} = t \Rightarrow t = e^{\ln\{x\}^{1/\sqrt{|\ln\{x\}|}}}$

$$= e^{1/\sqrt{|\ln\{x\}|}\ln\{x\}} = e^{-\sqrt{|\ln\{x\}|}}$$

$$e^{-\sqrt{|\ln\{x\}|}} - \{x\}^{1/\sqrt{|\ln\{x\}|}} = 0,\ \forall x \notin Z$$

$$[\text{sgn}(x)] = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

No solution exists

Hence, the correct answer is (0).

21. $\because$ $[\sin x] \in \{-1, 0, 1\}$
RHS is always odd and LHS is always even so no solution.

Hence, the correct answer is (0).

22. $\log_{0.5}\left(\frac{2x-8}{x-2}\right) \geq 0 \ \Rightarrow 0 < \frac{2x-8}{x-2} \leq 1$

$\Rightarrow \quad x \in (2, 6] \Rightarrow b - a = 4$

Hence, the correct answer is (4).

23. $M = {}^3C_1 \times 2! + {}^3C_2 \times 2 = 12$

$\Rightarrow \quad \frac{M}{2} = 6$.

Hence, the correct answer is (6).

24. $2\cos x \in \{-2, -1, 0, 1, 2\}$

The only acceptable values are ± 1

Its solutions are $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

Sum of roots $= \frac{\pi}{3} + \frac{2\pi}{3} = \pi \qquad \lambda = \mu = 1$

Hence, the correct answer is (2).

25. Assume $x^2 - 6x + 6 = x^2 - 4x + 4 \Rightarrow x = 1$

Put $x = 1$ in the given relation $\Rightarrow f(1) = 1$

For $x^2 - 6x + 6 = 1$ x must be 1 or 5

For $x^2 - 4x + 4 = 1$ x must be 1 or 3

Put $x = 3$ and 5 in the given relation

$$f(-3) + f(1) = 6 \Rightarrow f(-3) = 5$$

$$f(1) + f(9) = 10 \ \Rightarrow \ f(9) = 9$$

Hence, the correct answer is (9).

26. Since $f(10 - x) = f(x) = f(4 - x)$

$\Rightarrow \quad f(10 - x) = f(4 - x)$

Say, $4 - x = t$

$\Rightarrow \quad f(6 + t) = f(t) \ \Rightarrow \ f(x)$ is periodic function with period 6

for $x \in [0, 25]$

$f(x) = 101$ at $x = 0, 6, 12, 18, 24$
Total numbers = 5
since, $f(2 - x) = f(2 + x) \ \Rightarrow \ f(x)$ is symmetric about $x = 2$ line
Due to symmetry in one period length.
$f(x) = 101$ one solution at $x = 4$ other than 0 and 6.
Now $f(x) = 101$ at $x = 4, 10, 16, 22$
Total numbers = 4
Hence, atleast minimum possible number of values of $x = 9$.

Hence, the correct answer is (9).

27. $f(x) = \sin^2 x + \sin^2\left(x + \frac{\pi}{3}\right) + \cos x \cos\left(x + \frac{\pi}{3}\right)$

$$f(x) = \sin^2 x + \left[\sin\left(x + \frac{\pi}{3}\right)\right]^2 + \cos x \cos\left(x + \frac{\pi}{3}\right)$$

$$f(x) = \sin^2 x + \frac{1}{4}\left(\sin x + \sqrt{3}\cos x\right)^2 +$$

$$\frac{1}{2}\cos x\left(\cos x - \sqrt{3}\sin x\right) = \frac{5}{4}\left(\sin^2 x + \cos^2 x\right) = \frac{5}{4}$$

$$(gof)x = g[f(x)] = g(5/4) = 1$$

Hence, the correct answer is (1).

28. Domain $= R$

for range $y = \frac{e^x - e^{-x}}{e^x + e^{-x}} \Rightarrow e^x = \sqrt{\frac{y+1}{1-y}}$, $y \neq \pm 1$

$\Rightarrow \quad y \in (-1, 1)$

$B = (-1, 1) + 2 = (1, 3)$

$$\left.\begin{matrix} a = 1 \\ b = 3 \end{matrix}\right\} \Rightarrow C = \frac{1+3}{2} = 2$$

$\therefore \quad f^{-1}(2) = 0$

Hence, the correct answer is (0).

29. If $n[f(A)] = 3$, then no. of functions from A to B.

$= {}^4C_3 \times$ no. of onto functions from A to a set C

where C contains 3 elements

$$= 4 \times 150 = 600 \Rightarrow K = 600 \Rightarrow \frac{K}{600} = 6$$

Alternate:

${}^4C_3 \cdot (3^5 - {}^3C_2(2^5 - 2) - 3)$

$4(243 - 3(30) - 3) = 4(150)$

Hence, the correct answer is (6).

30. $f(-x)=-f(x)$

$$-x^3-\tan x+\left[\frac{x^2+1}{\lambda}\right]+x^3+\tan x+\left[\frac{x^2+1}{\lambda}\right]=0$$

$$\left[\frac{x^2+1}{\lambda}\right]=0 \Rightarrow 0\le\frac{x^2+1}{\lambda}<1$$

$$-2\le x\le 2\Rightarrow x^2+1\in[1,5]$$

$$1\le x^2+1\le 5\Rightarrow \lambda=6$$

Hence, the correct answer is (6).

31. Range of $[\,|\sin x|+\pi\,]$ is $\{3,4\}$

Hence, the correct answer is (7).

32. No. of functions $=4^6=2^{12}$
No. of functions such that sum

$$f(1)+f(2)+f(3)+f(4)+f(5)+f(6) \text{ is odd is } \frac{2^{12}}{2}=2^{11}$$

Hence, the correct answer is (2).

33. $\Rightarrow |x|,|y|\in[0,4], \dfrac{\pi x^2}{3}=(4n+1)\dfrac{\pi}{2}$

$$\Rightarrow \quad x^2=\frac{(4n+1)3}{2}=\frac{3}{2},\text{ as } |x|\le 4 \qquad n=0$$

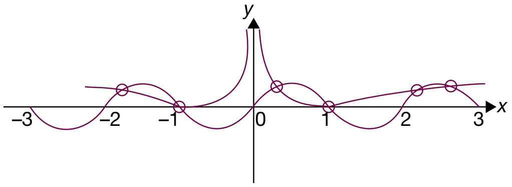
$$\Rightarrow \quad |x|=\sqrt{\frac{3}{2}},|y|=4-\sqrt{\frac{3}{2}}$$

$$n=1;\ x^2=\frac{15}{2};|y|=4-\sqrt{\frac{15}{2}}$$

$$n=2;\ x^2=\frac{27}{2};|y|=4-\sqrt{\frac{27}{2}}$$

Thus, there are 12 ordered pairs.
Hence, the correct answer is (12).

34.



No. of sol of $\sin\pi x=|\ln|x||$

$$y=\sin\pi x,\ y=|\ln|x||$$

Number of solution = 6
Hence, the correct answer is (6).

35. $f(x)+2f(1-x)=x^2+1$

Replace x by $1-x$

$$\therefore \quad f(1-x)+2f(x)=(1-x)^2+1=x^2-2x+2$$

Solving we get $3f(x)=x^2-4x+3$

$$f(3)=0$$

Hence, the correct answer is (0).

36. From the given equation, we have

$$\left(\frac{n}{2}-\left[\frac{n}{2}\right]\right)+\left(\frac{n}{3}-\left[\frac{n}{3}\right]\right)+\left(\frac{n}{5}-\left[\frac{n}{5}\right]\right)=0$$

$\Rightarrow \quad \left\{\dfrac{n}{2}\right\}+\left\{\dfrac{n}{3}\right\}+\left\{\dfrac{n}{5}\right\}=0$; where {.} is a F.P.F.

But each of the fraction part functions is positive and their sum is zero. Hence, each of the fraction part function is zero. Consequently, each of $\dfrac{n}{2},\dfrac{n}{3},\dfrac{n}{5}$ is an integer. The LCM of 2, 3, 5 is 30. Therefore, we can take $n=30k$ where k is an integer.
Hence, the number of solutions such that $1\le n\le 100$ is = 3 (viz. n = 30, 60 and 90)
Hence, the correct answer is (3).

37. Let $f\left(x+\dfrac{1}{y}\right)+f\left(x-\dfrac{1}{y}\right)$ …(1)

$$=2f(x).f\left(\frac{1}{y}\right)\forall x,y\in R$$

Given $f(0)=0$

(i) Put $x=0, y=\dfrac{1}{x}$ we get

$$f(x)+f(-x)=2f(0)f(x)$$

$\Rightarrow \quad f(x)+f(-x)=0\Rightarrow f(x)=-f(-x)$ …(2)

(ii) $\Rightarrow$ Put $x=1, y=1$

$$f(2)+f(0)=2[f(1)]^2$$

$f(2)=2[f(1)]^2$ …(3)

(iii) $\Rightarrow$ Put $x=-1, y=-2$

$f(-2)=2f(-1)f(-1)\Rightarrow -f(2)=2(f(1))^2$ …(4)

For (3) and (4) $f(2)=-f(2)\Rightarrow f(2)=0\Rightarrow f(1)=0$

$$\therefore \quad f(1)=f(2)=0$$

$$\therefore \quad f(1)+f(2)=0$$

Hence, the correct answer is (0).

38. $f(1, 1)=f(0, f(1, 0))=f(0, f(0, 1))=f(0, 2)=3$

Hence, the correct answer is (3).

39. $[x]=\dfrac{5\pm\sqrt{25+4\sin x-24}}{2.1}$

$$=\frac{5\pm\sqrt{1+4\sin x}}{2}$$

$\Rightarrow \quad -1\le\sin x\le 1 \Rightarrow -4\le 4\sin x\le 4$

$\Rightarrow \quad -3\le 1+4\sin\le 5 \Rightarrow 0\le 1+4\sin x\le 5$

$\Rightarrow \quad [x]$ is an integer $\Leftrightarrow \sin x=0$

$\Rightarrow \quad [x]=3$

$\Rightarrow \quad x=\pi$

Hence, the correct answer is (1).

40. $fog(x)=f(g(x))=f(4x(1-x))$

$\Rightarrow \quad \dfrac{1-4x(1-x)}{1+4x(1-x)}$ when $0\le 4x(1-x)\le 1$ and $0\le x\le 1$

But $4x-4x^2\ge 0 \Rightarrow 0\le x\le 1$

$$4x-4x^2\le 1\Rightarrow(2x-1)^2\ge 0\Rightarrow x\in R$$

Hence, $fog(x)=\dfrac{1-4x+4x^2}{1+4x-4x^2},\ 0\le x\le 1$

Let $y=\dfrac{4x^2-4x+1}{-(4x^2-4x)+1},\ 0\le x\le 1$

Put $0\le x\le 1 \quad t\in[-1,0]$

$$y=\frac{1+t}{1-t},\frac{dy}{dt}=\frac{1-t+1+t}{(1-t)^2}>0$$

Range of $fog(x)=[0,1]$

$\Rightarrow \quad \alpha+\beta=1$

Hence, the correct answer is (1).

41. $x + y = a, x - y = b \Rightarrow x = \frac{a+b}{2}, y = \frac{a-b}{2}$

$$f(a,b) = \left(\frac{a+b}{2}\right)\left(\frac{a-b}{2}\right) = \frac{a^2-b^2}{2}$$

$$\text{A.M} = \frac{f(x.y)+f(y,-x)}{2} = \frac{x^2-y^2+y^2-x^2}{4} = 0$$

Hence, the correct answer is (0).

42. $f(1-x) = 1 - f(x) \Rightarrow f(x) + f(1-x) = 1$

$$\Rightarrow \quad f\left(\frac{1}{11}\right) + f\left(\frac{10}{11}\right) = 1 = f\left(\frac{2}{11}\right) + f\left(\frac{9}{11}\right) = \ldots = f\left(\frac{5}{11}\right) + f\left(\frac{6}{11}\right)$$

$$\Rightarrow \quad f\left(\frac{1}{11}\right) + f\left(\frac{2}{11}\right) + \ldots + f\left(\frac{10}{11}\right) = 5$$

Hence, the correct answer is (5).

43. $f(g(x)) = \begin{cases} x^2+1 & -1 \le x \le 1 \\ 2x^2+1 & 1 < x \le \sqrt{2} \end{cases}$

$$f(g(x)) = 2 \Rightarrow x^2 + 1 = 2 \text{ and } 2x^2 + 1 = 2$$

$$x = \pm 1 \quad x = \pm\frac{1}{\sqrt{2}}\ ; x \in (1, \sqrt{2}\,]$$

No. of sol = 2

Hence, the correct answer is (2).

44. Let $x = y = 1$

$$f(x) + f(y) + f(xy) = 2 + f(x) \,.\, f(y)$$

$3f(1) = 2 + (f(1))^2 \Rightarrow f(1) = 1, 2$. But given that

$$f(1) \neq 1 \text{ so } f(1) = 2$$

Now put $y = \frac{1}{x}$

$$f(x) + f\left(\frac{1}{x}\right) + f(1) = 2 + f(x).\, f\left(\frac{1}{x}\right)$$

So $f(x) = 1 \pm x^n$

Now $f(4) = 17 \Rightarrow \pm(4)^n + 1 = 17 \Rightarrow n = 2$

$$f(x) = +(x)^2 + 1 \,. \Rightarrow f(5) = 26$$

Hence, the correct answer is (26).

45. Put $f(x) = t \Rightarrow f(t) = 4$

$\Rightarrow \quad t = -2, 0, 4.$

Hence $f(x) = -2$

$\Rightarrow \quad t = 2$ values of x.

or $\quad f(x) = 0 \Rightarrow 2$ values of x.

or $\quad f(x) = 4 \Rightarrow 3$ values of x.

Hence, the number of real solutions of the equation $f(f(x)) = 4$ are 7.

Hence, the correct answer is (7).

46. $(a-1)x^2 = x(2b+3)$

The above equation is satisfied by three distinct values of x, therefore, it is an identity.

$\therefore \quad 2 - 2a = 0 \Rightarrow a = 1$ and $2b + 3 = 0 \Rightarrow b = \frac{-3}{2}$

Now, $f(x) = 2x + 1$.

Let $g(x) = px + q \Rightarrow g'(x) = p$

$\quad f(g(x)) = 6x - 7 \Rightarrow 2(px + q) + 1 = 6x - 7$

$\Rightarrow \quad 2px + 2q + 1 = 6x - 7$

$\Rightarrow \quad 2p = 6 \Rightarrow p = 3$ and $q = -4$

$\therefore \quad g'(2012) = 3.$

Hence, the correct answer is (3).

47. $t = 2, 3, 4, 5, 6$

$$g(x) = \frac{tx^2+3x-4}{t+3x-4x^2}$$

Let $y = \frac{tx^2+3x-4}{t+3x-4x^2}$

$ty + 3xy - 4x^2y = tx^2 + 3x - 4$

$(t + 4y)x^2 + (1 - y)3x - 4 - ty = 0, \qquad \because\ x \in R, D \ge 0$

$9(1 - y)^2 + 4(t + 4y)(4 + ty) \ge 0$

$9 + 9y^2 - 18y + 16t + 4yt^2 + 64y + 16ty^2 \ge 0$

$(9 + 16t)y^2 + y(4t^2 + 46) + 9 + 16t \ge 0 \qquad$ True for all realy

$(4t^2 + 46)^2 - 4(9 + 16t)(9 + 16t) \le 0$

$(4t^2 + 46 - 18 - 32t)(4t^2 + 46 + 18 + 32t) \le 0$

$(4t^2 - 32t + 28)(4t^2 + 32t + 64) \le 0$

$(t^2 - 8t + 7)(t + 4)^2 \le 0$

$(t - 7)(t - 1)(t + 4)^2 \le 0$

$\Rightarrow \quad t \in [1, 7] \cup \{-4\}$

Hence, the correct answer is (8).

48. Total number of such mappings.

$\quad = 8 + 4 = 12 = p$

$\therefore \quad p - 7 = 5$

Hence, the correct answer is (5).

49. $\because \quad 0 \le \{x\} < 1$

$\Rightarrow \quad \cos 1 < \cos\{x\} \le \cos 0$

$\Rightarrow \quad \cos 1 < \cos\{x\} \le 1$

$$f(x) = \left[\frac{1}{\cos\{x\}}\right]$$

$\Rightarrow \quad f(x) = 1\ \forall\ x \in R \Rightarrow g(1) < 0\ \forall\ x \in R$

$\Rightarrow \quad 2 - 3(k + 1) + 3k^2 + k < 0 \Rightarrow 3k^2 - 2k - 1 < 0$

$\Rightarrow \quad (3k + 1)(k - 1) < 0 \Rightarrow k \in \left(\frac{-1}{3}, 1\right)$

Hence, the correct answer is (1).

50. $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Let $S_1 : 1, 2, 3, 4;\ S_2 : 5, 6, 7, 8, 9$

Element 10 is always included in subsets. Choose any three elements from $\{1, 2, \ldots 9\}$ but exclude the cases when all three elements selected from S_2.

Number of 4 elements subsets Y is $= {}^9C_3 - {}^5C_3 = 84 - 10 = 74$

Hence, the correct answer is (7).

51. For x-intercept $y = 0$

$\therefore \quad ||x - 2| - a| = 3$

$\Rightarrow \quad |x - 2| - a = 3$ or -3

$\Rightarrow \quad |x - 2| = a + 3$ or -3

$\quad |x - 2| = a - 3$

For 3 x-intercepts,

$\quad a + 3 = 0$ and $a - 3 = 0 \qquad \ldots(1)$

or $\quad a + 3 = 0$ and $a - 3 > 0 \qquad \ldots(2)$

From (1) $a = 3$

Hence, sum = 3.

Hence, the correct answer is (3).

52. $|x^2 - 5x + 7 - p| = 6 + |x^2 - 5x + 1 - p|$

$\Rightarrow \quad$ If $|a| + |b| = |a + b| \Rightarrow ab \ge 0$

$\quad 6(x^2 - 5x + 1 - p) \ge 0$

$\Rightarrow \quad (x^2 - 5x + 1 - p) \ge 0\ \forall\ 0 \in [-1, 3].$

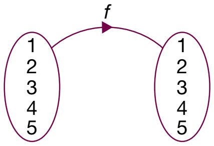
$$\Rightarrow \quad \underbrace{x^2-5x+(1-p)}_{\substack{\text{upward parabola with vertex}\\ \left(x=\frac{5}{12},\, y=-p-\frac{21}{4}\right)}} \geq 0 \ \forall\, x \in [-1, 3]$$

$$\therefore \quad -\pi - \frac{21}{4} \geq 0 \Rightarrow p + \frac{21}{4} \geq 0 \Rightarrow p \leq \frac{-21}{4}$$

$$\therefore \quad p \in \left(-\infty, \frac{-21}{4}\right]$$

$$\Rightarrow \quad \text{smallest value of } \left(\frac{-8p}{7}\right) = \frac{-8}{7} \times \frac{-21}{4} = 6.$$

Hence, the correct answer is (6).



53. Since, the range contains exactly 3 distinct values then we need to make groups to distribute 5 objects to three persons (1, 1, 3) and (1, 2, 2).

$\therefore$ Number of functions

$$= {}^5C_3\left[\frac{5!}{1!1!3!}\frac{1}{2!} + \frac{5!}{1!2!2!}\frac{1}{2!}\right] \times 3! = 1500.$$

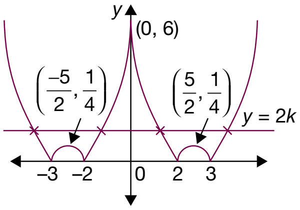
$$\text{So, } \frac{N}{300} = \frac{1500}{300} = 5.$$

Hence, the correct answer is (5).

54. If the equation $|x^2 - 5|x| + 6| = 2k$, has exactly four real distinct solutions, then

$$\frac{1}{4} < 2k < 6 \cup \{0\} \Rightarrow \frac{1}{8} < k < 3 \cup \{0\}$$

Graph of $y = |x^2 - 5|x| + 6|$



So, integral values of k are 0, 1, 2.
Hence, the number of integral values of k are 3.
Hence, the correct answer is (3).

55. $$4f\left(\frac{1}{x}+1\right) + 8f(x+1) = \log_{12}x \qquad \ldots(1)$$

$$\therefore \quad x \to \frac{1}{x},\ 8f\left(\frac{1}{x}+1\right) + 4f(x+1) = -\log_{12}x \qquad \ldots(2)$$

$$\Rightarrow \quad 2 \times (1) \Rightarrow 8f\left(\frac{1}{x}+1\right) + 16f(x+1) = 2\log_{12}x \qquad \ldots(3)$$

$(3) - (2)n \Rightarrow 12f(x+1) = 3\log_{12}x$

$\therefore \quad 4f(x+1) = \log_{12}x$

Put $x = 9,\ 4f(10) = \log_{12}9$

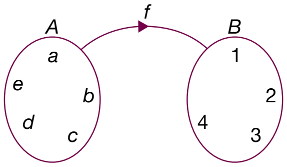
$x = 12,\ 4f(13) = \log_{12}12$

$x = 16,\ 4f(17) = \log_{12}16$

$4[f(10) + f(13) + f(17) = \log_{12}(9 \cdot 12 \cdot 16)\ \log_{12}12^3 = 3$

Hence, the correct answer is (3).

56. Since a is integer and [cot x] is also integer therefore cot x is also integer, cot x (cot x + 1) = a
$\therefore$ a is product of 2 consecutive integer.
Possible values of a are 20, 30, 42, 56, 72, 90.
Sum of all possible values of $a = 310$.
Hence, the correct answer is (310).



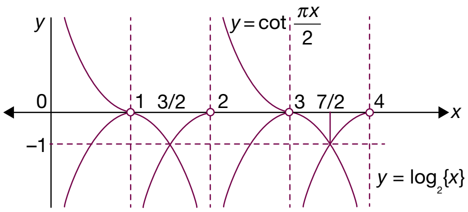
57. Exactly one elements of B can be taken ${}^4C_1$ ways. Now, 5 element of be mapped with 3 of A so that every element of
A must be the image of some element of A.

(2, 2, 1) or (3, 1, 1)

$$\left(\frac{5!}{2!2!2!} + \frac{5!}{3!2!}\right) \times 3! = 150$$

This can be done in 150 ways using grouping.
Total ways = 150 4 = 600.
Hence, the correct answer is (600).

58.



We plot the graph of $y = \cot\frac{\pi x}{2}$ and $y = \log_2\{x\}$

We can observe point of intersection are

$$\frac{3}{2} + \frac{7}{2} + \frac{11}{2} + \ldots + \frac{199}{2} = 2525$$

Hence, the correct answer is (2525).

59. Let $y = \cos x(\sin x + \sqrt{\sin^2(x)+3}\,)$

$\Rightarrow \quad (y \sec x - \sin x)^2 = \sin^2 x + 3$

$\Rightarrow \quad y^2 \sec^2 x + \sin^2 x - 2y \tan x = \sin^2 x + 3$

$\Rightarrow \quad y^2 + y^2 \tan^2 x - 2y \tan x - 3 = 0$

$\Rightarrow \quad y^2 \tan^2 x - 2y \tan x + (y^2 - 3) = 0$

As tan x is real.

$\therefore \quad D \geq 0$

$\Rightarrow \quad 4y^2 - 4y^2(y^2 - 3) \geq 0$

$\Rightarrow \quad 4 - y^2 \geq 0$

$\Rightarrow \quad y^2 - 4 \leq 0$

$\Rightarrow \quad -2 \leq y \leq 2.$

y contains –2, –1, 0, 1, 2, i.e. 5 integers.
Hence, the correct answer is (5).

60. We have $f\left(\frac{2x-3}{x-2}\right) = 5x - 2 \Rightarrow f^{-1}(5x-2) = \frac{2x-3}{x-2}$

Let $5x - 2 = 13$ then $x = 3$

$$\text{Hence } f^{-1}(13) = \frac{2(3)-3}{3-2} = 3.$$

Hence, the correct answer is (3).

61. $x([x] - 5) + 2\{x\} + 6 = 0$

$x[x] - 5x + 2x - 2[x] + 6 = 0 \Rightarrow [x]\,(x-2) - 3(x-2) = 0$

$\Rightarrow \quad ([x] - 3)\,(x - 2) = 0$

$\Rightarrow \quad x = 2 \text{ or } [x] = 3$

$\Rightarrow \quad x \in [3, 4) \cup \{2\}$

Now, $(a-3)x^2 + 2(a+3)x - 8a \leq 0$

$\Rightarrow \quad ax^2 - 3x^2 + 2ax + 6x - 8a \leq 0$

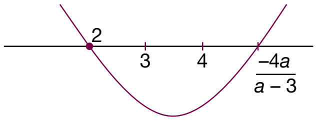
$\Rightarrow \quad ax^2 - 2ax - 3x^2 + 6x + 4ax - 8a \leq 0$

$\Rightarrow \quad ax(x-2)-3x(x-2)+4a(x-2)\le 0$

$\Rightarrow \quad (x-2)\,x(x-3)+4a)\le 0$

CASE 1: $a-3>0>3$

$$\frac{3x^2+mx+n}{x^2+1}\frac{4}{1}\frac{12n-m^2}{-12}\ge 4\Rightarrow -4a\ge 4s-12$$



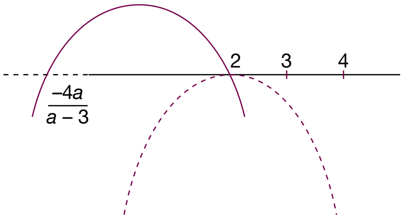
$8a\le 12\Rightarrow a\le \frac{3}{2}$

No solution.

CASE 2: $a-3<0$

$$\frac{-4a}{a-3}\le 2\Rightarrow -4a\ge 2a-6$$

$$6\ge 6a\Rightarrow a\le 1$$



$\therefore$ Greatest integral value of a is 1.

Hence, the correct answer is (1).

62. Let $y=\frac{x+a}{x^2+1}\Rightarrow y\,x^2-x+(y-a)=0$

Now, we must have $D\ge 0\ \forall\ y\in[0,1]$

Let $f(y)=4y^2-4ay-1\le 0\ \forall\ y\in[0,1]$

This is possible is $f(0)\le 0, f(1)\le 0$

$\Rightarrow \quad -1\le 0$ and $3-4a\le 0$

$\Rightarrow \quad a\ge \frac{3}{4}$

Hence, $a\in\left[\frac{3}{4},\infty\right)$. So, number of integral values of a in [0, 100] are 100.

Hence, the correct answer is (100).

63. $y=\frac{3x^2+mx+n}{x^2+1}$

$\Rightarrow \quad x^2(y-3)-mx+y-n=0$

As $x\in R$, $D\ge 0$

$\Rightarrow \quad m^2-4(y-3)\,(y-n)\ge 0$

$\Rightarrow \quad m^2-4(y^2-ny-3y+3n)\ge 0$

$4y^2+(n+3)y+12n-m^2\le 0 \quad \ldots(1)$

Given $y\in[-4,3)$

$\Rightarrow \quad (y+4)\,(y-3)\le 0$

$\Rightarrow \quad y^2+y-12\le 0 \quad \ldots(2)$

$\therefore$ Compare (1) and (2) we get $\frac{4}{1}=-\frac{4(n+3)}{1}=\frac{12n-m^2}{-12}$

$\Rightarrow \quad m=0$ and $n=-4$

Hence, the correct answer is (16).

64. If $|x|+|y|=|x+y|$, then $xy\ge 0$

Hence $|x+a-3|+|x-2a|=|2x-a-3|$

$\Rightarrow \quad (x+a-3)\,(x-2a)\ge 0$

$\Rightarrow \quad x^2-(a+3)x-2a(a-3)\ge 0\ \forall\ x\in R.$

$\Rightarrow \quad D\le 0$

$\Rightarrow \quad (a+3)^2\le -8a(a-3)\Rightarrow 9a^2-18a+9\le 0$

$\Rightarrow \quad a^2-2a+1\le 0\Rightarrow (a-1)^2\le 0\Rightarrow a=1$

Hence, the correct answer is (1).

ARCHIVE: JEE MAIN

1. CASE 1: $(x\ge 5)$

$$(x+1)^2+x-5=\frac{27}{4}\Rightarrow 4x^2+12x-43=0$$

We get $x=\frac{-3\pm 2\sqrt{13}}{2}$ not true as $x\ge 5$

CASE 2: $(x<5)$

$$(x+1)^2-x+5=\frac{27}{4}\Rightarrow 4x^2+4x-3=0$$

We get $x=\frac{1}{2},-\frac{3}{2}$ both accepted

Number of real roots = 2

Hence, the correct answer is (2).

2. $af(x)+\alpha f\left(\frac{1}{x}\right)=bx+\frac{\beta}{x}$ replace x by $1/x$

$$af(1/x)+\alpha f(x)=\frac{b}{x}+\beta x$$

$$(a+\alpha)\left(f(x)+f\left(\frac{1}{x}\right)\right)=(b+\beta)\left(x+\frac{1}{x}\right)$$

$$\frac{f(x)+f\left(\frac{1}{x}\right)}{x+\frac{1}{x}}=\frac{b+\beta}{a+\alpha}=2$$

Hence, the correct answer is (2).

3. $f(x)+f(2-x)=\frac{5^x}{5+5^x}+\frac{5^{2-x}}{5+5^{2-x}}=1$

Now $\left(f\left(\frac{1}{20}\right)+f\left(\frac{39}{20}\right)\right)+\left(f\left(\frac{2}{20}\right)+f\left(\frac{38}{20}\right)\right)+\ldots$

$\left(f\left(\frac{19}{12}\right)+f\left(\frac{21}{20}\right)\right)+f\left(\frac{20}{20}\right)$

$$=1\times 19+f(1)=19+\frac{1}{2}=\frac{39}{2}$$

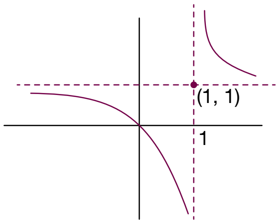
Hence, the correct answer is (C).

4. $f(g(x))=2g(x)-1=2\frac{\left(x-\frac{1}{2}\right)}{x-1}-1$

$=\frac{x}{x-1}$

$f(g(x))=1+\frac{1}{x-1}$

one-one, into



Hence, the correct answer is (B).

5. Range of $f(x)$ is even integer

$g(f(x))=f(x)\Rightarrow g(x)=x$, when x is even.

$\therefore$ So total number of functions from A to $A=10^5\times 1=10^5$

Hence, the correct answer is (D).

6. $C = \{f : A \to B \mid 2 \in f(A) \text{ and } f \text{ is not one-one}\}$

CASE 1: If $f(x) = 2\ \forall\ x \in A$, then number of function = 1

CASE 2: If $f(x) = 2$ for exactly two elements, then the total number of many-one function $= {}^3C_2\ {}^3C_1 = 9$

CASE 3: If $f(x) = 2$ for exactly one element, then the total number of many-one functions $= {}^3C_1\ {}^3C_1 = 9$

Hence, the correct answer is (19.00).

7. $f(x + y) = f(x)\, f(y)$

Put $x = y = 1$

$$f(2) = (f(1))^2 = 3^2$$

Put $x = 2, y = 1$

$$f(3) = (f(1))^3 = 3^3$$

$$\vdots$$

Similarly, $f(x) = 3^x$

$$\sum_{i=1}^{n} f(i) = 363 \Rightarrow \sum_{i=1}^{n} 3^i = 363$$

$$\Rightarrow \quad (3 + 3^2 + \ldots + 3^n) = \frac{3(3^n - 1)}{2} = 363$$

$$3^n - 1 = 242 \Rightarrow 3^n = 243 \Rightarrow n = 5$$

Hence, the correct answer is (5.00).

8. $g(x) = x^2 + x - 1$

$$g(f(x)) = 4x^2 - 10x + 5 = (2x - 2)^2 + (2 - 2x) - 1$$
$$= (2 - 2x)^2 + (2 - 2x) - 1$$

$$\Rightarrow \quad f(x) = 2 - 2x;\ f\left(\frac{5}{4}\right) = \frac{-1}{2}$$

Hence, the correct answer is (B).

9. $$f(x) = y = \frac{8^{4x} - 1}{8^{4x} + 1} = 1 - \frac{2}{8^{4x} + 1}$$

So, $8^{4x} + 1 = \dfrac{2}{1 - y} \Rightarrow 8^{4x} = \dfrac{1 + y}{1 - y}$

$$\Rightarrow \quad x = \ln\left(\frac{1 + y}{1 - y}\right) \times \frac{1}{4 \ln 8} = f^{-1}(y)$$

Hence, $f^{-1}(x) = \dfrac{1}{4} \log_8 e \ln\left(\dfrac{1 + x}{1 - x}\right)$

Hence, the correct answer is (C).

10. Let $3^x = t;\ t > 0$

$$t(t - 1) + 2 = |t - 1| + |t - 2|$$
$$\Rightarrow \quad t^2 - t + 2 = |t - 1| + |t - 2|$$

CASE 1: $t < 1$

$$t^2 - t + 2 = 1 - t + 2 - t \Rightarrow t^2 + t - 1 = 0$$

$$t = \frac{-1 \pm \sqrt{5}}{2} \text{ but } t = \frac{\sqrt{5} - 1}{2} \text{ is only acceptable}$$

CASE 2: $1 \le t \le 2$

$$t^2 - t + 2 = t - 1 + 2 - t \Rightarrow t^2 - t + 1 = 0$$

$D < 0$ no real solution

CASE 3: $t \ge 2$

$$t^2 - t + 2 = t - 1 + t - 2$$
$$t^2 - 3t + 5 = 0 \Rightarrow D < 0 \text{ no real solution}$$

So, we get only one solution.

Hence, the correct answer is (D).

11. $$f(x) = \begin{cases} \dfrac{x}{x^2 + 1}; & x \in (1, 2) \\[2mm] \dfrac{2x}{x^2 + 1}; & x \in [2, 3) \end{cases}$$

$f(x)$ is decreasing function

$$\therefore \quad f(x) \in \left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]$$

Hence, the correct answer is (B).

12. $y = \dfrac{x^2}{1 - x^2} \Rightarrow y - x^2 y = x^2 \Rightarrow x^2 = \dfrac{y}{1 - y}$

$$\Rightarrow \quad x = \sqrt{\frac{y}{1 + y}} \Rightarrow x = \frac{y}{1 + y} \ge 0$$

Range of y is $R - [-1, 0)$

For surjective function codomain = Range

$\therefore \quad A$ is $R - [-1, 0)$

Hence, the correct answer is (B).

13. $4 - x^2 \ne 0 \Rightarrow x \ne \pm 2$

for $\log_{10}(x^3 - x)$ to be defined

$$x^3 - x > 0 \Rightarrow x(x - 1)(x + 1) > 0$$
$$x \in (-1, 0) \cup (1, \infty) - \{2\}$$

Hence, the correct answer is (C).

14. $g(S) = [-2, 2];\ f(g(S)) = [0, 4] = S$

$f(S) = [0, 16] \Rightarrow f(g(S)) \ne f(S)$

$g(f(S)) = [-4, 4] \ne g(S)$ therefore, $g(f(S)) \ne S$

Hence, the correct answer is (D).

15.

CASE 1: $(2^x \ge 1)$	**CASE 2:** $(2^x < 1)$
$5 + 2^x - 1 = 2^x(2^x - 2)$	$5 + 1 - 2^x = 2^x(2^x - 2)$
Let $2^x = t$	$2^x = t$
$5 + t - 1 = t(t - 2)$	$5 + 1 - t = t(t - 2)$
$t = 4, -1$ (rejected)	$(t - 3)(t - 2) = 0 \Rightarrow t = 3, -2$
$\Rightarrow 2^x = 3, 2^x = -2$	
$2^x = 4 \Rightarrow x = 2$	not values not acceptable
only 1 solution	

Hence, the correct answer is (B).

16. $hof(x) = \dfrac{1 - x}{1 + x};\ \phi(x) = ((hof)og)(x) = \dfrac{1 - \tan x}{1 + \tan x}$

$$\phi\left(\frac{\pi}{3}\right) = \frac{1 - \tan\frac{\pi}{3}}{1 + \tan\left(\frac{\pi}{3}\right)} = \tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right) = \tan\left(\frac{11\pi}{12}\right)$$

Hence, the correct answer is (B).

17. $$\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \ldots + \left[-\frac{1}{3} - \frac{66}{100}\right] + \left[-\frac{1}{3} - \frac{67}{100}\right] + \ldots + \left[-\frac{1}{3} - \frac{99}{100}\right]$$

$$= (-1) + (-1) + (-1) + \ldots + (-1) + (-2) + (-2) + (-2) + \ldots + (-2)$$
$$= (-1)67 + (-2)33 = -133$$

Hence, the correct answer is (C).

18. CASE 1:

When $\theta \in \left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$ we have $\sin\theta \in \left(\dfrac{\sqrt{3}}{2}, 1\right)$,

$\cos\theta \in \left(-\dfrac{1}{2}, 0\right) \Rightarrow -\cos\theta \in \left(0, \dfrac{1}{2}\right)$ and $\cot\theta \in \left(\dfrac{-1}{\sqrt{3}}, 0\right)$

$[\sin\theta] = 0,\ [-\cos\theta] = 0,\ [\cot\theta] = -1$

Now equations $\left.\begin{array}{r} 0x + 0y = 0 \\ -x + y = 0 \end{array}\right]$ system will have infinitely many solutions.

CASE 2:

When $\theta \in \left(\pi, \dfrac{7\pi}{6}\right)$ we have $\sin\theta \in \left(-\dfrac{1}{2}, 0\right)$,

$\cos\theta \in \left(-1, \dfrac{-\sqrt{3}}{2}\right) \Rightarrow -\cos\theta \in \left(\dfrac{\sqrt{3}}{2}, 1\right)$

$\cot\theta \in \left(\sqrt{3}, \infty\right)$

$[\sin\theta] = -1$, $[-\cos\theta] = 0$ and $[\cot\theta] = \{1, 2, 3, \ldots\}$

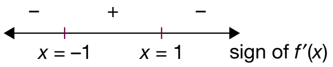
$x = 0$

$ix + y = 0$ where $i = \{1, 2, \ldots\}$

It will have unique solution is all cases $x = 0, y = 0$

Hence, the correct answer is (B).

19. We have $f ; R\left[-\frac{1}{2}, \frac{1}{2}\right]$, $f(x), \frac{x}{1+x^2}\ \forall\ x \in R$



$$\Rightarrow \quad f'(x) = \frac{(1+x^2)1 - x^2x}{(1+x^2)^2} = \frac{-(x+1)(x-1)}{(1+x^2)^2}$$

$\Rightarrow$ $f'(x)$changes sign in different intervals.

$\therefore$ Not injective

Now $y = \frac{x}{1+x^2} \Rightarrow y + yx^2 = x$

$\Rightarrow$ $yx^2 - x + y = 0$

For $y \neq 0$, $D = 1 - 4y^2 \geq 0 \Rightarrow y \in \left[\frac{-1}{2}, \frac{1}{2}\right] - \{0\}$

For $y = 0 \Rightarrow x = 0$

$\therefore$ Range is $\left[\frac{-1}{2}, \frac{1}{2}\right] \Rightarrow$ Surjective but not injective

Hence, the correct answer is (D).

20. $f(x) = ax^2 + bx + c$

$f(1) = a + b + c = 3 \Rightarrow (1) = 3$

Now $f(x + y) = f(x) + f(y) + xy$ …(1)

Put $x = y = 1$ in eqn (1)

$f(2) = f(1) + f(1) + 1 = 2f(1) + 1$

$f(2) = 7 \Rightarrow f(3) = 12$

Now, $S_n = 3 + 7 + 12 + \ldots t_n$ …(2)

$S_n = 3 + 7 + \ldots t_{n-1} + t_n$

Subtract (2) from (1)

$t_n = 3 + 4 + 5 + \ldots$ (upto n terms)

$$t_n = \frac{(n^2 + 5n)}{2}$$

$$S_n = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{5n(n+1)}{2}\right] = \frac{n(n+1)(n+8)}{6}$$

$$S_{10} = \frac{10 \times 11 \times 18}{6} = 330$$

Hence, the correct answer is (B).

21. $f(x) + 2f\left(\frac{1}{x}\right) = 3x$

Replacing x by $\frac{1}{x}$, we get $f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$

Solving (A) and (B) for $f(x)$, we get

$-3f(x) = 3x - \frac{6}{x}$ or $f(x) = \frac{2}{x} - x$

Now given $f(x) = f(-x)$

$\therefore$ $\frac{2}{x} - x = -\frac{2}{x} + x$

$\therefore$ $\frac{4}{x} = 2x$

$\therefore$ $\frac{2}{x} = x$

$\therefore$ $x = \pm\sqrt{2}$

Hence set S contains two elements.

Hence, the correct answer is (B).

22. $a^2 = 3r^2 - 2t$, where $t = \{x\} \in [0, 1)$

$\therefore$ $a^2 = 3t(t - 2/3)$

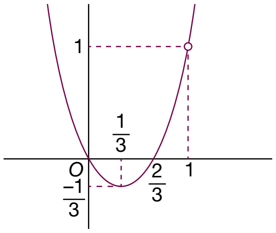
Graph of $f(t) = 3t(t - 2/3)$ is as shown in the following figure.

Clearly, from graph

$0 < a^2 < 1$

or $a \in (-1, 0) \cup (0, 1)$

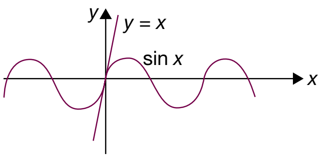
Hence, the correct answer is (A).



ARCHIVE: JEE ADVANCED

1. $f(x) = |x|\ (x - \sin x)$ is an odd and continuous function

From the graph we can conclude that



If $x \geq 0$ then $x - \sin x \geq 0$ hence $f(x) \geq 0$

If $x < 0$ then $x - \sin x < 0$ hence $f(x) < 0$

$$f(x) = \begin{cases} x(x - \sin x) & x \geq 0 \\ -x(x - \sin x) & x < 0 \end{cases} = \begin{cases} x^2 - x\sin x & x \geq 0 \\ -x^2 + x\sin x & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x - \sin x - x\cos x & x \geq 0 \\ -2x + \sin x + x\cos x & x < 0 \end{cases}$$

$$= \begin{cases} x - \sin x + x(1 - \cos x) & x \geq 0 \\ -\left[x - \sin x + x(1 - \cos x)\right] & x < 0 \end{cases}$$

We can see that $f'(x) \geq 0$ for all real x.

So, $f(x)$ is a strictly increasing function so one-one function range is R hence it is onto also

Hence, the correct answer is (C).

2. $f(1 - x) = \frac{4^{1-x}}{4^{1-x} + 2} = \frac{4}{4^x\left(\frac{4}{4x} + 2\right)} = \frac{4}{4 + 2 \cdot 4^x} = \frac{2}{2 + 4^x}$

$$f(x) + f(1 - x) = \frac{4^x + 2}{4^x + 2} = 1$$

$$f\left(\frac{1}{40}\right).+.f\left(\frac{2}{40}\right) + \ldots + f\left(\frac{19}{40}\right).+.f\left(\frac{20}{40}\right).+.f\left(\frac{21}{40}\right) + \ldots + f\left(\frac{38}{40}\right).+.f\left(\frac{39}{40}\right).-.f\left(\frac{1}{2}\right)$$

$= 1 + 1 + \ldots + 1 = 19$

Hence, the correct answer is (19).

3. $E_1 : \{x \mid x \in (-\infty, 0) \cup (1, \infty)\}$ = domain of $f(x)$

For E_2

$\sin^{-1}\left(\ln\frac{x}{x-1}\right)$ is a real number

$$-1 \leq \ln\frac{x}{x-1} \leq 1 \Rightarrow e^{-1} \leq \frac{x}{x-1} \leq e$$

$$\Rightarrow \quad x \in \left(-\infty, -\frac{1}{e-1}\right] \cup \left[\frac{1}{e-1}, \infty\right) = \text{domain of } g(x)$$

$$f(x) = y = \ln\left(\frac{x}{x-1}\right) \Rightarrow \frac{x}{x-1} = e^y \Rightarrow x = \frac{e^y}{e^y - 1}$$

$$e^y - 1 \neq 0 \Rightarrow y \neq 0$$

Range of $f(x) = R_f = R - \{0\}$

$$g(x) = y = \sin^{-1}\left(\ln\frac{x}{x-1}\right)$$

Range of $g(x) = R_g = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Hence, the correct answer is (A).

4. Set X where $n(X) = 5$, Set Y where $n(Y) = 7$

No. of one-one function from X to Y $\alpha = {}^7C_5 \cdot 5! = 2520$

No. of onto function from Y to X using inclusion and exclusion rule

$$\beta = 5^7 - {}^5C_1 4^7 + {}^5C_2 3^7 - {}^5C_3 2^7 + {}^5C_4 = 16800$$

Alternate:

No. of onto function from Y to X

= distribution of 7 different toys among 5 person such that each person receives atleast one toy

CASE 1:

No. of ways of distributing 7 different toys in groups of 1, 1, 1, 1, 3 to

5 persons $= \left(\frac{7!}{(1!)^4 3! 4!}\right) 5!$

CASE 2:

No. of ways of distributing 7 different toys in groups of 2, 2, 1, 1, 1 to

5 persons $= \left(\frac{7!}{(2!)^2 (1!)^3 3! 2!}\right) 5!$

Required total ways $\beta = \left(\frac{7!}{(1!)^4 3! 4!}\right) 5! = 16800$

Now $\frac{\beta - \alpha}{5!} = 119$

Hence, the correct answer is (119).

5. Given $g(x) = \frac{\pi}{2} \sin x \ \forall \ x \in R$ then $g(x) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\Rightarrow \quad g(g(x)) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \forall \ x \in R$$

$$\Rightarrow \quad g(g(g(x)) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \forall \ x \in R$$

Given $f(x) = \sin\left(\frac{1}{3} g\left(g(x)\right)\right)$,

Now $\frac{g(g(x))}{3} \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \forall \ x \in R$

Hence, $f(x)$ and $f(g(x)) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\sin\left(\frac{1}{3} g(g(x))\right)}{\frac{1}{3} g(g(x))} \cdot \frac{\frac{1}{3} g(g(x))}{g(x)} = \lim_{x \to 0} \frac{\frac{1}{3} g(g(x))}{g(x)}$$

$$= \lim_{x \to 0} \frac{\pi}{6} \cdot \frac{\sin\left(\frac{\pi}{2} \sin x\right)}{\frac{\pi}{2} \sin x} = \frac{\pi}{6}$$

$$g(f(x)) \neq 1$$

Range of $g(f(x)) \in \left[-\frac{\pi}{2} \sin\left(\frac{1}{2}\right), \frac{\pi}{2} \sin\left(\frac{1}{2}\right)\right]$

Hence, (A), (B) and (C) are correct.

6. Given function $f(x) = (\log(\sec x + \tan x))^3$ for $x \in (-\pi/2, \pi/2)$

$f(-x) = -f(x)$ so it is odd function

$$f'(x) = 3(\log(\sec x + \tan x))^2 \frac{1}{(\sec x + \tan x)} (\sec x \tan x + \sec^2 x)$$

$$= 3(\log(\sec x + \tan x))^2 \sec x > 0$$

$f(x)$ is an increasing function

$$f\left(-\frac{\pi}{2}^+\right) \to -\infty \quad \text{and} \quad f\left(\frac{\pi}{2}^-\right) \to \infty$$

Range of $f(x)$ is R hence it is onto function.

$$\Rightarrow \quad g(f(x)) \neq 1$$

Hence, (A), (B) and (C) are correct.

CHAPTER 3: LIMIT OF FUNCTION

Test Your Concepts-I

1. $\lim\limits_{x\to 4} \frac{[x]-4}{x^2-5x+4} = \lim\limits_{x\to 4} \frac{[x]-4}{(x-1)(x-4)}$

Put $x = 4 - h$

$$\text{LHL} = \lim_{h\to 0} \frac{[4-h]-4}{(4-h-1)(4-h-4)}$$

$$= \lim_{h\to 0} \frac{3-4}{(3-h)(-h)} = \lim_{h\to 0} \frac{1}{h(3-h)} \to \infty$$

Put $x = 4 + h$

$$\text{RHL} = \lim_{h\to 0} \frac{[4+h]-4}{(4+h-4)(4+h-1)}$$

$$= \lim_{h\to 0} \frac{4-4}{h(3+h)} = \lim_{h\to 0} \frac{0}{h(3+h)} = 0$$

Hence, required limit does not exist.

2. Put $x = 1 - h$

$$\text{LHL} = \lim_{h\to 0} \frac{\{1-h\}}{1-h} = \lim_{h\to 0} \frac{1-h}{1-h} = 1$$

Put $x = 1 + h$

$$\text{RHL} = \lim_{h\to 0} \frac{\{1+h\}}{1+h} = \lim_{h\to 0} \frac{h}{1+h} = 0$$

Hence, required limit does not exist.

3. $\text{RHL} = \lim\limits_{x\to a^+} \frac{1}{1-e^{1/(x-a)}} = \lim\limits_{h\to 0} \frac{1}{1-e^{1/(a+h-a)}}, h > 0$

$$= \lim_{h\to 0} \frac{1}{1-e^{1/h}}; \left(h \to 0 \Rightarrow \frac{1}{h} \to \infty \Rightarrow e^{1/h} \to \infty\right)$$

$$= \frac{1}{1-\infty} = 0.$$

$$\text{LHL} = \lim_{x\to a^-} \frac{1}{1-e^{1/(x-a)}} = \lim_{h\to 0} \frac{1}{1-e^{1/(a-h-a)}}, h > 0$$

$$= \lim_{h\to 0} \frac{1}{1-e^{-1/h}}; \left(h \to 0 \Rightarrow -\frac{1}{h} \to -\infty \Rightarrow e^{-1/h} \to 0\right)$$

$$= \frac{1}{1-0} = 1$$

RHL $\neq$ LHL

Hence, the limit does not exist.

4. $\lim\limits_{x\to 1^-} \frac{x^2-1}{[x-1]} = \lim\limits_{h\to 0} \frac{(1-h)^2-1}{[1-h-1]}, h > 0$

$$= \lim_{h\to 0} \frac{h^2-2h}{[-h]} = \lim_{h\to 0} \frac{h^2-2h}{-1} = 0$$

5. $\text{limit} = \lim\limits_{h\to 0}\left[\sin^{-1}(1-h)\right] = \left[\frac{\pi}{2}\right] = 1$

6. $\lim\limits_{x\to \frac{1}{2}^+} [x^2-x+2] \cdot \frac{x}{[2x]} = \lim\limits_{x\to \frac{1}{2}^+} [x^2-x+2] \frac{x}{[2x]}$

$$= \left[\frac{7}{4}\right] \cdot \frac{1/2}{1} = 1 \cdot \left(\frac{1}{2}\right) = \frac{1}{2}$$

7. We have, $f(x) = 10p + 2p[2x] - 7q + q[3x]$

Now, $f(1^+) = 10p + 4p - 7q + 3q = (14p - 4q) = f(1)$

Also, $f(1^-) = 10p + 2p - 7q + 2q = (12p - 5q)$

As, $\lim\limits_{x\to 1} f(x)$ exists.

Hence, $14p - 4q = 12p - 5q = 2p + q = 0$

8. $\lim\limits_{x\to 1} \frac{(x-1)(x-3)}{(x-1)(x-2)(x-3)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 1} \frac{1}{x-2} = -1$$

9. $\lim\limits_{x\to 3} \frac{(x-7)(x-3)(x+5)}{x(x-3)(x-8)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 3} \frac{(x-7)(x+5)}{x(x-8)} = \frac{(-4)(8)}{3(-5)} = \frac{32}{15}$$

10. $\lim\limits_{x\to 6} \frac{((x-4)-2)\left(\sqrt{x-2}+2\right)}{(x-2-4)\left(\sqrt{x-4}+\sqrt{2}\right)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 6} \frac{\sqrt{x-2}+2}{\sqrt{x-4}+\sqrt{2}} = \frac{2+2}{\sqrt{2}+\sqrt{2}} = \sqrt{2}$$

11. $\lim\limits_{x\to -2} \frac{\sqrt{x+2}\left(\sqrt{2+\sqrt{x+2}}+\sqrt{2}\right)}{\left(\sqrt{2+\sqrt{x+2}}-\sqrt{2}\right)\left(\sqrt{2+\sqrt{x+2}}+\sqrt{2}\right)}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to -2} \frac{\sqrt{x+2}\left(\sqrt{2+\sqrt{x+2}}+\sqrt{2}\right)}{\left(2+\sqrt{x+2}-2\right)} = 2\sqrt{2}$$

12. $\lim\limits_{x\to \infty} x^{3/2}\left\{\sqrt{x^3+1}-\sqrt{x^3-1}\right\}$

$$= \lim_{x\to \infty} x^{3/2}\left\{\frac{(x^3+1)-(x^3-1)}{(x^3+1)^{1/2}+(x^3-1)^{1/2}}\right\}$$

$$\left[\because a^2-b^2 = (a-b)(a+b) \quad \therefore a-b = \frac{a^2-b^2}{a+b}\right]$$

$$= \lim_{x\to \infty} x^{3/2}\left\{\frac{2}{x^{3/2}\left\{\left(1+\frac{1}{x^3}\right)^{1/2}+\left(1-\frac{1}{x^3}\right)^{1/2}\right\}}\right\}$$

$$= \lim_{x\to \infty} \frac{2}{\left(1+\frac{1}{x^3}\right)^{1/2}+\left(1-\frac{1}{3}\right)^{1/2}} = 1$$

Test Your Concepts-II

1. $\lim\limits_{x\to 0} \frac{p\sin 2x+(1-\cos 2x)}{x+\tan x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$\therefore \quad \lim_{x\to 0} \frac{2p\left(\frac{\sin 2x}{2x}\right)+\left(\frac{1-\cos 2x}{(2x)^2}\right)4x}{\left(1+\frac{\tan x}{x}\right)} = 1$$

$$\Rightarrow \quad \frac{2p+0}{1+1} = 1 \Rightarrow p = 1$$

2. $\lim\limits_{x\to 0} \frac{\tan x-\sin x}{x^3} = \lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)\frac{(1-\cos x)}{x^2}$

$$= \lim_{x\to 0} 1\left(\frac{1-\left(1-2\sin^2 {}^{x}\!/_{2}\right)}{\left({}^{x}\!/_{2}\right)^2}\right) \times \frac{1}{4}$$

$$= \lim_{x\to 0} \frac{1}{2}\left[\frac{\sin {}^{x}\!/_{2}}{{}^{x}\!/_{4}}\right]^2 = \frac{1}{2}(1)^2 = \frac{1}{2}$$

3. Put $x = \frac{\pi}{2} + h$

Required limit $= \lim\limits_{h\to 0} \dfrac{2^{-\cos(90^\circ+h)} - 1}{\left(\frac{\pi}{2}+h\right)\left(\frac{\pi}{2}+h-\frac{\pi}{2}\right)}$

$$= \lim_{h\to 0} \frac{2^{\sinh} - 1}{\left(\frac{\pi}{2}+h\right)h}$$

$$= \lim_{h\to 0} \left(\frac{2^{\sin h}-1}{\sin h}\right)\frac{1}{\pi/2}\left(\frac{\sin h}{\text{h}}\right) = \frac{\ln 2}{\pi/2} = \frac{2}{\pi}\ln 2$$

4. limit $= \lim\limits_{x\to 0} \dfrac{x(2^x-1)}{1-\cos x}$ $\left(\frac{0}{0}\text{ form}\right)$

$$= \lim_{x\to 0} \frac{\left(\frac{2^x-1}{x}\right)}{\left(\frac{1-\cos x}{x^2}\right)} = \frac{\lim\limits_{x\to 0}\left(\frac{2^x-1}{x}\right)}{\lim\limits_{x\to 0}\frac{1}{2}\left(\frac{\sin^2 x/2}{x^2/4}\right)} = 2\log 2$$

5. Put $x = \frac{1}{y}$,

Given, limit $= \lim\limits_{y\to 0^-} \dfrac{\frac{1}{y^2}\cdot\tan y - \frac{1}{y}}{1+\frac{1}{y}}$

$$= \lim_{y\to 0^-} \frac{\left(\frac{\tan y}{y} - 1\right)}{y+1} = \frac{1-1}{0+1} = 0$$

6. $\lim\limits_{x\to\frac{\pi}{2}} \left(\dfrac{\tan 2x + \sin 2x}{x - \frac{\pi}{2}}\right)$ $\left(\frac{0}{0}\right)$; Put $x = \frac{\pi}{2} + h$

$$= \lim_{h\to 0} \frac{\tan 2\left(\frac{\pi}{2}+h\right) + \sin 2\left(\frac{\pi}{2}+h\right)}{h}$$

$$= \lim_{h\to 0} \left(\frac{\tan 2h}{h} - \frac{\sin 2h}{h}\right)$$

$$= \lim_{h\to 0} \left(\frac{2\tan 2h}{2h} - \frac{2\sin 2h}{2h}\right) = 2 - 2 = 0$$

7. Let $l_1 = \lim\limits_{x\to 0} \dfrac{a^{\sin x} - 1}{1 + x - \cos x}$

$$= \lim_{x\to 0} \frac{(a^{\sin x}-1)}{\sin x}\cdot\frac{\sin x}{x}\,\frac{x}{(1+x-\cos x)}$$

$$= (\ln a)\lim_{x\to 0} \frac{1}{\frac{(1-\cos x)}{x}+1} = \ln a\left[\frac{1}{0+1}\right] = \ln a$$

Let $l_2 = \lim\limits_{x\to 1} \dfrac{x-1}{\ln(x)}$; Put $x = 1 + h$

$$= \lim_{h\to 0} \frac{h}{\ln(1+h)} = 1$$

Hence, $\ln a = 1 \Rightarrow a = e$

8. $\lim\limits_{n\to\infty} n\left(1+\tan^2\left(\frac{\pi}{4n}\right)\right)\cos^3\frac{\pi}{4n}\sin\frac{\pi}{4n}$

$$= \lim_{n\to\infty} n\left(\sec^2\frac{\pi}{4n}\right)\cos^3\frac{\pi}{4n}\sin\frac{\pi}{4n}$$

$$= \lim_{n\to\infty} \frac{n}{2}\sin\left(\frac{\pi}{2n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{2}{n}} = \lim_{n\to\infty} \frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{\pi}{2n}} \times \frac{\pi}{4} = \frac{\pi}{4}$$

9. $\lim\limits_{x\to 0} \dfrac{\sin\left(\frac{x+x^2}{a^4}\right)(e^x-1)^2}{\ln\left(1+\frac{x^3}{2}\right)} = \dfrac{2}{81}$

$$\Rightarrow \lim_{x\to 0} \frac{\sin\left(\frac{x+x^2}{a^4}\right)\left(\frac{e^x-1}{x}\right)^2}{\frac{(x+x^2)}{a^4}\,\frac{\ln\left(1+\frac{x^3}{2}\right)}{\left(\frac{x^3}{2}\right)}}\;\frac{\frac{(x+x^2)}{a^4}\times x^2}{\left(\frac{x^3}{2}\right)} = \frac{2}{81}$$

$$\Rightarrow \frac{2\times 1\times 1\times(1+0)}{a^4} = \frac{2}{81}$$

We get $a^4 = 81$
Hence, $a = \pm 3$

10. Put $x = a + h$

Given expression $= \lim\limits_{h\to 0} \tan\left(\frac{\pi}{2a}(a+h)\right)\cdot\sin\left(\frac{a+h-a}{2}\right) = \frac{2}{\pi}$

$$\Rightarrow \lim_{h\to 0} \tan\left(\frac{\pi}{2}+\frac{\pi h}{2a}\right)\sin\frac{h}{2} = \frac{2}{\pi}$$

$$\Rightarrow -\lim_{h\to 0} \cot\left(\frac{\pi h}{2a}\right)\sin\frac{h}{2} = \frac{2}{\pi}$$

$$\Rightarrow -\lim_{h\to 0} \frac{\sin\frac{h}{2}\left(\frac{\pi h}{2a}\right)}{\left(\frac{h}{2}\right)\tan\left(\frac{\pi h}{2a}\right)}\left(\frac{a}{\pi}\right) = \frac{2}{\pi}$$

$$\Rightarrow -\frac{a}{\pi}\lim_{h\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\,\frac{\left(\frac{\pi h}{2a}\right)}{\tan\left(\frac{\pi h}{2a}\right)} = \frac{2}{\pi} \Rightarrow a = -2$$

11. $\lim\limits_{x\to 0} \dfrac{(2^{\sin x}-1)\cdot\ln(1+\sin 2x)}{x(arc\tan x)}$

$$= \lim_{x\to 0} \frac{(2^{\sin x}-1)\sin x}{\sin x}\cdot\frac{\ln(1+\sin 2x)}{\sin 2x}\,\frac{\sin 2x}{2x}\cdot\frac{2}{x}\cdot\frac{1}{\tan^{-1}x/x}$$

$$= (\ln 2)\lim_{x\to 0} \frac{2\sin x}{x} = 2\ln 2 = \ln 4.$$

12. $f(x) = \dfrac{2+e^{1/x}}{1+e^{4/x}} + \dfrac{\sin x}{|x|}$

$$f(0^+) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \left(\frac{2+e^{1/h}}{1+e^{4/h}} + \frac{\sin h}{h}\right)$$

$$= \lim_{h\to 0} \left(\frac{2e^{-1/h}+1}{e^{-1/h}+e^{3/h}} + \frac{\sin h}{h}\right) = \frac{0+1}{0+\infty} + 1 = 0 + 1 = 1$$

$$f(0^-) = \lim_{h\to 0} \left(\frac{2+e^{-1/h}}{1+e^{-4/h}} + \left(\frac{\sin h}{h}\right)\right) = \frac{2+0}{1+0} - 1 = 2 - 1 = 1$$

Hence, $f(0^+) = f(0^-) = 1$

13. $\lim\limits_{x\to 0} \dfrac{5(1-\cos x) + 20\sin^{-1}x - \sin^2 x}{(10\tan^{-1}(x) - x^2 + x^4)}$

$$= \lim_{x\to 0} \frac{\left(5\frac{1-\cos x}{x^2}.x + \frac{20\sin^{-1}x}{x} - \frac{\sin^2 x}{x}\right)}{\left(\frac{10\tan^{-1}x}{x} - x + x^3\right)}$$

(Divide numerator and denominator by x)

$$= \frac{(0+20-0)}{(10-0+0)} = \frac{20}{10} = 2.$$

Test Your Concepts-III

1. $\lim\limits_{x\to 1} \dfrac{(x+x^2+x^3+\ldots+x^n)-n}{x-1}$ As, it is $\left(\dfrac{0}{0}\right)$ form.

On applying L′ Hospital′s rule, get

$$= \lim_{x\to 1} \frac{(1+2x+3x^2+\ldots+nx^{n-1})-0}{1}$$

On applying limits, we get

$$= 1+2+3+4+\ldots+n = \frac{n(n+1)}{2}$$

2. Given, $\lim\limits_{x\to 1} \dfrac{x^4-1}{x-1} = \lim\limits_{x\to k} \dfrac{x^3-k^3}{x^2-k^2}$ As, it is $\left(\dfrac{0}{0}\right)$ form.

On applying L′ Hospital′s rule, we get

$$\lim_{x\to 1} \frac{4x^3-0}{1-0} = \lim_{x\to k} \frac{3x^2-0}{2x-0} \Rightarrow 4 = \frac{3k}{2} \quad \therefore \quad k = \frac{8}{3}$$

3. $\lim\limits_{x\to\sqrt{2}} \left(\dfrac{x^4-4}{x^2+3x\sqrt{2}-8}\right)$

At $x=\sqrt{2}$, it is an indeterminate $\left(\dfrac{0}{0}\right)$ form.

On applying L′ Hospital′s rule, we get

$$= \lim_{x\to\sqrt{2}} \left(\frac{4x^3}{2x+3\sqrt{2}}\right) = \frac{4(2\sqrt{2})}{2\sqrt{2}+3\sqrt{2}} = \frac{8}{5}$$

4. $\lim\limits_{x\to 2} \dfrac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}$

At $x\to 2$, it is an indeterminate $\left(\dfrac{0}{0}\right)$ form.

On applying L′ Hospital′s rule,

$$\lim_{x\to 2} \frac{1}{2}\cdot\frac{1}{\sqrt{1+\sqrt{2+x}}}\cdot\frac{1}{2\sqrt{2+x}}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{1+2}}\cdot\frac{1}{2\sqrt{4}} = \frac{1}{8\sqrt{3}}$$

5. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x^2}-1}{x^2}$ It is an indeterminate $\left(\dfrac{0}{0}\right)$ form.

On applying L′ Hospital′s rule, we get

$$\lim_{x\to 0} \frac{\dfrac{1}{2\sqrt{1+x^2}}\cdot 2x}{2x} = \lim_{x\to 0} \frac{1}{2\sqrt{1+x^2}} = \frac{1}{2}$$

6. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{2x}$

On applying L′ Hospital′s rule, we get

$$\lim_{x\to 0} \frac{\dfrac{1}{2\sqrt{1+x}}-\dfrac{1}{2\sqrt{1-x}}(0-1)}{2} = \lim_{x\to 0} \frac{\dfrac{1}{2\sqrt{1+x}}+\dfrac{1}{2\sqrt{1-x}}}{2}$$

$$= \frac{\dfrac{1}{2}+\dfrac{1}{2}}{2} = \frac{1}{2}$$

7. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x^2}-1}{\sqrt{16+x^2}-4}$ It is an indeterminate $\left(\dfrac{0}{0}\right)$ form at $x\to 0$.

On applying, L′ Hospital′s rule, we get

$$= \lim_{x\to 0} \frac{\dfrac{1}{2\sqrt{1+x^2}}\cdot 2x}{\dfrac{1}{2\sqrt{16+x^2}}\cdot 2x} = \lim_{x\to 0} \frac{\sqrt{16+x^2}}{\sqrt{1+x^2}} = \sqrt{16} = 4$$

8. $\lim\limits_{x\to 1} \dfrac{\sqrt[3]{7+x^2}-\sqrt[3]{1-2x}}{x+x^2}$

At $x\to 1$, it is not $\left(\dfrac{0}{0}\right)$ form,

We have $= \dfrac{(8)^{1/3}-(-1)^{1/3}}{2} = \dfrac{3}{2}$

9. $\lim\limits_{x\to 2} \dfrac{\sqrt{x-2}+\sqrt{x}-\sqrt{2}}{\sqrt{x^2-4}}$

At $x\to 2$, it is an indeterminate $\left(\dfrac{0}{0}\right)$ form.

On applying L′ Hospital′s rule, we get

$$\lim_{x\to 2} \frac{\dfrac{1}{2\sqrt{x-2}}\cdot 1+\dfrac{1}{2\sqrt{x}}-0}{\dfrac{1}{2\sqrt{x-4}}\cdot 2x} = \lim_{x\to 2} \frac{\dfrac{1}{2}\left[\dfrac{1}{\sqrt{x-2}}+\dfrac{1}{\sqrt{x}}\right]}{\dfrac{x}{\sqrt{x-2}\sqrt{x-2}}}$$

$$= \lim_{x\to 2} \frac{\dfrac{1}{2}\left[\dfrac{\sqrt{x}+\sqrt{x-2}}{\sqrt{x}\sqrt{x-2}}\right]}{\dfrac{x}{\sqrt{x-2}\sqrt{x+2}}}$$

$$= \lim_{x\to 2} \frac{1}{2}\left[\frac{\sqrt{x}+\sqrt{x-2}}{x\sqrt{x}}\right]\sqrt{x+2} = \frac{1}{2}\left[\frac{\sqrt{2}}{2\sqrt{2}}\right]\sqrt{2+2}$$

$$= \frac{1}{2}\cdot\left[\frac{1}{2}\right]\cdot 2 = \frac{1}{2}$$

10. $\lim\limits_{x\to 1} \dfrac{4x^3-x^2+2x-5}{x^6+5x^3-2x-4}$ $\left[\dfrac{0}{0}\text{ form}\right]$

On applying L′ Hospital′s rule, we get

$$\lim_{x\to 1} \frac{12x^2-2x+2}{6x^5+15x^2-2} = \frac{12-2+2}{6+15-2} = \frac{12}{19}$$

Test Your Concepts-IV

1. $$\lim_{x\to 0} \frac{\sin(ax)-\sin(x)-x}{x^3} = \lim_{x\to 0} \frac{\left(ax-\dfrac{a^3x^3}{3!}\right)-\left(x-\dfrac{x^3}{3!}\right)-x}{x^3}$$

$$= \lim_{x\to 0} \frac{(a-2)x+\left(\dfrac{1}{6}-\dfrac{a^3}{6}\right)x^3}{x^3}$$

$\therefore$ For limit to exist, $a-2=0 \Rightarrow a=2$.

$$l = \frac{1}{6}-\frac{8}{6} = \frac{-7}{6}$$

2. $\lim\limits_{x\to 0} \dfrac{x\tan 2x-2x\tan x}{(1-\cos 2x)^2}$

$$= \lim_{x\to 0} \frac{x\left[2x+\dfrac{2^3x^3}{3}+2\dfrac{2^5x^5}{15}\ldots\right]-2x\left[x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}\ldots\right]}{\left(2\sin^2 x\right)^2}$$

$$= \lim_{x\to 0} \frac{x^4\left(\dfrac{8}{3}-\dfrac{2}{3}\right)+x^6\left(\dfrac{64}{15}-\dfrac{4}{15}\right)+\ldots}{4\left(\dfrac{\sin x}{x}\right)^4 x^4}$$

$$= \lim_{x\to 0} \frac{2+4x^2+\ldots}{4} = \frac{2}{4} = \frac{1}{2}$$

3. Let $l = \lim\limits_{x\to1} a^2\left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$; Put $x = 1 + h$

$$= a^2 \lim_{h\to0} \frac{h - \ln(1+h)}{\left(\dfrac{\ln(1+h)}{h}\right)h^2} = a^2 \lim_{h\to0} \frac{h - \left(h - \dfrac{h^2}{2} + \ldots\infty)\right)}{\left(\dfrac{\ln(1+h)}{h}\right)\cdot h^2}$$

$\Rightarrow \quad l = \dfrac{a^2}{2}$

$\therefore$ Required limit $= \operatorname{cosec}^{-1}\left(\dfrac{a^2}{2}\right)$ is defined, if $\left|\dfrac{a^2}{2}\right| \ge 1$

$\Rightarrow \quad |a|^2 \ge 2$

Hence, $a \in (-\infty, \sqrt{2}] \cup [\sqrt{2}, \infty)$

4. $\lim\limits_{x\to0} \dfrac{(1 + ax\ \sin x) - b\cos x}{x^4}$

$$= \lim_{x\to0}\left[\frac{\left\{1 + ax\left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}\ldots\right)\right\} - b\left(1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!}\ldots\right)}{x^4}\right]$$

$$= \lim_{x\to0}\left[\frac{(1-b) + \left(a + \dfrac{b}{2}\right)x^2 + \left(-\dfrac{a}{3!} - \dfrac{b}{4!}\right)x^4 + \ldots}{x^4}\right]$$

To have a definite limit, we must have $1 - b = 0$ and $a + \dfrac{b}{2} = 0$

$\Rightarrow \quad b = 1, a = -\dfrac{1}{2}$

Now, Limit $= \lim\limits_{x\to0}\left[\dfrac{x^4\left\{-\left(\dfrac{a}{3!} + \dfrac{b}{4!}\right) + \left(\dfrac{a}{5!} - \dfrac{b}{6!}\right)x^5 + \ldots\right\}}{x^4}\right]$

$$= -\left(\frac{a}{3!} + \frac{b}{4!}\right) = -\left[-\frac{1}{2}\times\frac{1}{6} + \frac{1}{24}\right] = \frac{1}{24}$$

5. As, $(x + \sin x - x\cos x - \tan x) = x(1 - \cos x) + \sin x\left(1 - \dfrac{1}{\cos x}\right)$

$= x(1 - \cos x) - \tan x(1 - \cos x) = (x - \tan x)\cdot(1 - \cos x)$

So, $\lim\limits_{x\to0} \dfrac{\left(\dfrac{x - \tan x}{x^3}\right)\left(\dfrac{1 - \cos x}{x^2}\right)}{x^{n-5}}$ = exist and non-zero,

Hence, $n = 5$

6. $\lim\limits_{x\to1} \sec^{-1}\left(\dfrac{k}{\ln x} - \dfrac{k}{x-1}\right) = \dfrac{\pi}{3}, \Rightarrow \lim\limits_{x\to1}\left(\dfrac{k}{\ln x} - \dfrac{k}{x-1}\right) = 2;$

Put $x = 1 + h$

$$\Rightarrow \quad \lim_{h\to0} \frac{k(h - \ln(1+h))}{h\cdot\ln(1+h)} = 2 \Rightarrow \lim_{h\to0} \frac{k\left[h - \left(h - \dfrac{h^2}{2} + \ldots\right)\right]}{h^2\,\dfrac{\ln(1+h)}{h}} = 2$$

$$\Rightarrow \quad \lim_{h\to0} \frac{k\left(\dfrac{h^2}{2} - \ldots.\right)}{h^2} = 2 \Rightarrow \frac{k}{2} = 2 \Rightarrow k = 4$$

7. Put $r = \dfrac{1}{x}$

$$\therefore \quad \lim_{x\to0} \frac{\cos(ax) - \cos(bx)\cdot\cos(cx)}{\left(\dfrac{\sin(bx)}{bx}\right)\left(\dfrac{\sin(cx)}{cx}\right)\cdot bcx^2}$$

Now, $\dfrac{1}{bc}\lim\limits_{x\to0} \dfrac{\cos(ax) - \cos(bx)\cdot\cos(cx)}{x^2} = \dfrac{1}{2}$

$$\Rightarrow \quad \frac{1}{bc}\lim_{x\to0} \frac{\left(1 - \dfrac{a^2x^2}{2} + \ldots\right) - \left(1 - \dfrac{b^2x^2}{2} + \ldots\right)\left(1 - \dfrac{c^2x^2}{2} + \ldots\right)}{x^2} = \frac{1}{2}$$

Hence, $\dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{1}{2}$

8. $\lim\limits_{x\to\infty} x\left(\dfrac{1}{e} - \left(\dfrac{x}{x+1}\right)^x\right)$

Let $x = \dfrac{1}{t}$

$$\therefore \quad \lim_{t\to0^+} \frac{1}{t}\left(\frac{1}{e} - \left(\frac{1}{1+t}\right)^{1/t}\right) = \lim_{t\to0^+} \frac{(1+t)^{1/t} - e}{e(1+t)^{1/t}}$$

$$= \lim_{t\to0^+} \frac{e^{\frac{\ln(1+t)}{t}} - e}{te^2} = \frac{1}{e}\lim_{t\to0} \frac{e^{\frac{\ln(1+t)}{t}-1} - 1}{\left(\dfrac{\ln(1+t)}{t} - 1\right)} \cdot \frac{\left(\dfrac{\ln(1+t)}{t} - 1\right)}{t}$$

$$= \frac{1}{e}\lim_{t\to0} 1.\frac{\ln(1+t) - t}{t^2} = \frac{-1}{2e}$$

Test Your Concepts-V

1. $\lim\limits_{x\to\infty}\left(\sqrt[3]{x^3 + 4x^2} - \sqrt{x^2 + \dfrac{x}{3}}\right)$

$= \lim\limits_{x\to\infty} x\left(\left(1 + \dfrac{4}{x}\right)^{1/3} - \left(1 + \dfrac{1}{3x}\right)^{1/2}\right)$; Using expansion of $(1 + x)^n$.

$$= \lim_{x\to\infty} x\left(1 + \frac{4}{3x} - 1 - \frac{1}{6x}\right)$$

$$= \lim_{x\to\infty} x\left(\frac{4}{3x} - \frac{1}{6x}\right) = \frac{4}{3} - \frac{1}{6} = \frac{7}{6} = \frac{a}{b}$$

$\therefore \quad (a + b) = 13.$

2. $\lim\limits_{x\to2}\left(\dfrac{1}{x-2} - \dfrac{2(2x-3)}{x^3 - 3x^2 + 2x}\right)$

Where $(\infty - \infty)$ indeterminate form is present

$$= \lim_{x\to2}\left(\frac{x(x-1) - 2(2x-3)}{x^3 - 3x^2 + 2x}\right)$$

$= \lim\limits_{x\to2} \dfrac{(x^2 - 5x + 6)}{(x-2)(x)(x-1)}$ $\qquad \dfrac{\infty}{\infty}$ form

$$= \lim_{x\to2} \frac{(x-2)(x-3)}{(x-2)(x)(x-1)} = \frac{-1}{2(1)} = -\frac{1}{2}$$

3. $\lim\limits_{x\to\infty} \dfrac{a + bx + cx^2 + dx^3}{e + fx^2 + gx^3} = \lim\limits_{x\to\infty} \dfrac{x^3\left(\dfrac{9}{x^3} + \dfrac{b}{x^2} + \dfrac{c}{x^{+d}}\right)}{x^3\left(\dfrac{e}{x^3} + \dfrac{f}{x} + g\right)}$

$$= \frac{0 + d}{0 + g} = \frac{d}{g}$$

4. $\lim\limits_{x\to\infty} \dfrac{3x^2 - 4x + 5}{4x^2 + x^3 - 7} = \lim\limits_{x\to\infty} \dfrac{x^2\left[3 - {}^4\!/_x + {}^5\!/_{x^2}\right]}{x^2\left[4 + x - {}^7\!/_{x^2}\right]} = \dfrac{3 - 0 + 0}{4 + \infty - 0} = 0$

5. $\lim\limits_{x\to0^+}\left(\dfrac{3\ln^2 x + 5\ln x + 6}{\ln^2 x + 1}\right)$ $\qquad \left(\dfrac{\infty}{\infty}\text{ form}\right)$

$$= \lim_{x\to0^+}\left(\frac{3 + \dfrac{5}{\ln x} + \dfrac{6}{\ln^2 x}}{1 + \dfrac{1}{\ln^2 x}}\right) = 3$$

6. $\lim\limits_{x\to\infty} \sqrt{\dfrac{x - \sin x}{x + \cos^2 x}}$; (it is in $\dfrac{\infty}{\infty}$ form)

$$= \lim_{x\to\infty} \sqrt{\frac{1 - \left(\dfrac{\sin x}{x}\right)}{1 + \left(\dfrac{\cos^2 x}{x}\right)}},$$

$$\left[\text{Using } \lim_{x\to\infty}\frac{\sin x}{x}=\frac{-1\text{ to }1}{\infty}=0 \text{ and } \lim_{x\to 0}\frac{\cos^2 x}{x}=\frac{0\text{ to }1}{\infty}=0\right]$$

Hence, required limit $=\sqrt{\dfrac{1-0}{1+0}}=1$

7. $$\lim_{x\to\infty}\frac{x^2+x+1}{3x^2+2x-5} \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to\infty}\frac{x^2(1+1/x+1/x^2)}{x^2(3+2/x-5/x^2)}$$

$$=\lim_{x\to\infty}\frac{1+1/x+1/x^2}{3+2/x-5/x^2}=\frac{1+0+0}{3+0-0}=\frac{1}{3}$$

8. $$\lim_{n\to\infty}\left(\frac{n^2+n+3}{n^2+3n+5}\right)^n \qquad (1^\infty\text{ form })$$

$$=e^{\lim\limits_{n\to\infty} n\left(\frac{n^2+n+3}{n^2+3n+5}-1\right)}=e^{\lim\limits_{n\to\infty} n\left(\frac{-2n-2}{n^2+3n+5}\right)}=e^{\lim\limits_{n\to\infty}\left(\frac{-2-2/n}{1+3/n+5/n^2}\right)}$$

$$=e^{-2}=\frac{1}{e^2}\equiv\frac{p}{e^2}, \text{ hence, } p=1$$

9. $$\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{3x-6}=\lim_{x\to-\infty}\frac{|x|\sqrt{1+\frac{1}{x^2}}}{x\left(3-\frac{6}{x}\right)}=-\frac{1}{3}$$

10. $$\lim_{x\to\pi/2}(\sin x)^{\tan x} \qquad [1^\infty\text{ form }]$$

$$=e^{\lim\limits_{x\to\pi/2}(\sin x-1)\tan x}$$

$$=e^{\lim\limits_{x\to\pi/2}\frac{(\sin x-1)\sin x}{\cos x}}=e^{\lim\limits_{x\to\pi/2}\frac{-\sin x\cos^2 x}{\cos x(1+\sin x)}}$$

$$=e^{\lim\limits_{x\to\pi/2}\frac{-\sin x\cos x}{1+\sin x}}=e^{-\frac{1\times 0}{1+1}}=e^0=1$$

11. It is 1^∞ form

$$\lim_{x\to\infty}\left(\frac{2x^2+3}{2x^2+5}\right)^{8x^2+3}=e^{\lim\limits_{x\to\infty}\left(\frac{2x^2+3}{2x^2+5}-1\right)(8x^2+3)}$$

$$=e^{\lim\limits_{x\to\infty}\left[\frac{-2\,(8x^2+3)}{2x^2+5}\right]}$$

$$=e^{\lim\limits_{x\to\infty}\frac{-2\left(8+\frac{3}{x^2}\right)}{\left(2+\frac{5}{x^2}\right)}}=e^{\frac{-2(8+0)}{2+0}}=e^{-8}$$

12. $$\lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^{\frac{8x^3}{\pi}\sin\left(\frac{\pi}{2x}\right)}, \ (1^\infty\text{ Form})$$

$$=e^{\lim\limits_{x\to\infty}\left(1+\frac{1}{x^2}-1\right)\frac{8x^3}{\pi}\sin\left(\frac{\pi}{2x}\right)}=e^{\lim\limits_{x\to\infty}\frac{8x}{\pi}\sin\left(\frac{\pi}{2x}\right)}$$

$$=e^{\lim\limits_{x\to\infty}\frac{\sin(\pi/2x)}{(\pi/2x)}4}=e^4$$

13. $$L=\lim_{x\to a}(\log_a x)^{\frac{1}{x-a}}=e^l, \ (1^\infty\text{ form})$$

Let $$l=\lim_{x\to a}\frac{\log_a x-1}{x-a}, \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$\lim_{x\to 0}\frac{\left(\frac{\ln x}{\ln a}-1\right)}{(x-a)}=\frac{1}{\ln a}\lim_{x\to a}\frac{(\ln x-\ln a)}{x-a}; \text{ Put } x=a+h$$

$$l=\frac{1}{\ln a}\lim_{h\to 0}\frac{\ln\left(1+\frac{h}{a}\right)}{a\left(\frac{h}{a}\right)}=\frac{1}{a\ln a}$$

14. $$\lim_{n\to\infty}\left(1+\frac{1}{n^2+\cos n}\right)^{n^2+n} \qquad (\text{It is } 1^\infty\text{ Form})$$

$$=\lim_{n\to\infty}e^{\left(\frac{n^2+n}{n^2+\cos n}\right)}=\lim_{n\to\infty}e^{\left(\frac{1+1/n}{1+\frac{\cos n}{n^2}}\right)}=e^{\frac{1+0}{1+0}}=e$$

Test Your Concepts-VI

1. Let $$y=\lim_{x\to 0^+}(2\cdot x^{\sin 2x}\cdot\cos 2x)$$

$$\therefore\quad \ln y=\lim_{x\to 0^+}\left(\ln 2+\frac{\ln x}{\operatorname{cosec}2x}+\ln(\cos 2x)\right)$$

Let $$t=\lim_{x\to 0^+}\frac{\ln x}{\operatorname{cosec}2x} \qquad (\text{using L-H Rule})$$

$$t=\lim_{x\to 0^+}\frac{1}{-2x\operatorname{cosec}2x\cot 2x}=-\lim_{x\to 0}\frac{\sin 2x\sin 2x}{2x\cos 2x}=0$$

Now, $\ln y=(\ln 2+0+0)$

Hence, $y=2$.

2. $\lim\limits_{x\to\frac{\pi^-}{2}}(\sec x)^{\cos x}$ it is ∞^o **form**

Put $x=\dfrac{\pi}{2}-h$

$$=\lim_{h\to 0}(\operatorname{cosec} h)^{\sin h}=e^{\lim\limits_{h\to 0}\ln(\operatorname{cosec} h)^{\sin h}}$$

$$=e^{\lim\limits_{h\to 0}\frac{\ln(\operatorname{cosec} h)}{\operatorname{cosec} h}} \qquad \left(\frac{\infty}{\infty}\text{ form, now using LH rule}\right)$$

$$=e^{\lim\limits_{h\to 0}\frac{\frac{1}{\operatorname{cosec} h}\cdot(-\operatorname{cosec} h\cot h)}{-\operatorname{cosec} h\cot h}}=e^{\lim\limits_{h\to 0}\sin h}=e^{o}=1$$

3. $$\lim_{x\to 0^+}\log_{\sin x}\sin 2x=\lim_{x\to 0^+}\left(\frac{\ln\sin 2x}{\ln\sin x}\right) \qquad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to 0^+}\left(\frac{\frac{1}{\sin 2x}(2\cos 2x)}{\frac{1}{\sin x}\cos x}\right)=\lim_{x\to 0^+}\left(\frac{2\cos 2x}{\cos x}\right)\left(\frac{\sin x}{\sin 2x}\right)$$

$$=\lim_{x\to 0^+}\left(\frac{2}{1}\right)\left(\frac{\sin x}{2\sin x\cos x}\right)=1$$

4. As $x\to 0$, $\dfrac{\sin x}{\tan x}\approx 0.99999$; $\dfrac{-\sin x}{x}$

≈ -0.99999; $\dfrac{-\tan x}{x}\approx -1.000001$

Where $\sin x<x<\tan x \qquad x\in\left(0,\dfrac{\pi}{2}\right)$

$$\lim_{x\to 0}\left[\frac{\sin x}{\tan x}\right]+\left[\frac{-\sin x}{x}\right]+\left[\frac{-\tan x}{x}\right]=0+(-1)+(-2)=-3$$

5. $\sin x<x<\tan x \ x\in\left(0,\dfrac{\pi}{2}\right)$

As $x\to 0$, $\dfrac{5\sin x}{x}\approx 4.99999$; $\dfrac{-5\tan x}{x}\approx -5.000001$

$$\lim_{x\to 0}\left[\frac{5\sin x}{x}\right]+\left[\frac{-5\tan x}{x}\right]=4-6=-2$$

6. $\sin^{-1}x>x>\tan^{-1}x$ for $0<x<1$

As $x\to 0$, $\dfrac{\sin^{-1}x}{x}\approx 1.000001$; $\dfrac{-\tan^{-1}x}{\sin^{-1}x}$

$$\approx -0.99999; \ \frac{-3\tan^{-1}x}{x} \approx -2.999999$$

$$\lim_{x\to 0}\left[\frac{\sin^{-1}x}{x}\right]+\left[\frac{-\tan^{-1}x}{\sin^{-1}x}\right]+\left[\frac{-3\tan^{-1}x}{x}\right]=1-1-3=-3$$

7. As $f(x)$ is a continuous decreasing positive function on $(1, \infty)$, so

$$0<f(8x)<f(6x)<f(4x) \Rightarrow 0<1<\frac{f(6x)}{f(8x)}<\frac{f(4x)}{f(8x)}$$

$$\Rightarrow \quad \lim_{x\to\infty} 1 \le \lim_{x\to\infty}\frac{f(6x)}{f(8x)} \le \lim_{x\to\infty}\frac{f(4x)}{f(8x)}$$

$$\Rightarrow \quad 1 \le \lim_{x\to\infty}\frac{f(6x)}{f(8x)} \le 1$$

By squeeze play theorem, $\lim_{x\to\infty}\frac{f(6x)}{f(8x)}=1$.

8. $0 \le \{x\} < 1$

$0 \le \{2x\} < 1$

…………….

$0 \le \{nx\} < 1$

$$\frac{0}{n^2} \le \frac{\{x\}+\{2x\}+.....+\{nx\}}{n^2} < \frac{n}{n^2}$$

$$\lim_{n\to\infty}\frac{0}{n^2} \le \lim_{n\to\infty}\frac{\{x\}+\{2x\}+.....\{nx\}}{n^2} \le \lim_{n\to\infty}\frac{1}{n}$$

$$\lim_{n\to\infty} g(x) = \lim_{n\to\infty}\frac{0}{n}=0 \text{ and } \lim_{n\to\infty} h(x) = \lim_{n\to\infty}\frac{1}{n}=0$$

From sandwich theorem, we can conclude

$$\lim_{n\to\infty}\frac{\{x\}+\{2x\}.......\{nx\}}{n^2}=0$$

9. First identify indeterminate form, $0+0+0+\ldots$ up to $(2n+1)$ terms ($0\times\infty$ form)

We are looking for two suitable functions for applying Sandwich Theorem

$$\frac{1}{\sqrt{n^2+2n}} \le \frac{1}{\sqrt{n^2}} \le \frac{1}{\sqrt{n^2}};$$

$$\frac{1}{\sqrt{n^2+2n}} \le \frac{1}{\sqrt{n^2+1}} \le \frac{1}{\sqrt{n^2}}$$

$$\frac{1}{\sqrt{n^2+2n}} \le \frac{1}{\sqrt{n^2+2}} \le \frac{1}{\sqrt{n^2}}$$

……………………………

$$\frac{1}{\sqrt{n^2+2n}} \le \frac{1}{\sqrt{n^2+2n}} \le \frac{1}{\sqrt{n^2}}$$

Adding all above terms

We get $\frac{2n+1}{\sqrt{n^2+2n}} \le \left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+....+\frac{1}{\sqrt{n^2+2n}}\right) \le \frac{2n+1}{\sqrt{n^2}}$

$$\Rightarrow \quad \lim_{n\to\infty}\frac{(2+1/n)}{\sqrt{1+2/n}} \le$$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+....+\frac{1}{\sqrt{n^2+2n}}\right) \le \lim_{n\to\infty}\left(2+\frac{1}{n}\right)$$

$$\Rightarrow \quad \frac{2+0}{1} \le \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+2n}}\right) \le 2+0$$

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2}}+\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+2n}}\right)=2$$

Single Correct Choice Type Questions

1. $$\lim_{x\to 1}\frac{x^{1/3}-x^{1/4}}{x^3-1}=\lim_{x\to 1}\frac{\left(x^{1/3}-1\right)-\left(x^{1/4}-1\right)}{(x-1)\left(x^2+x+1\right)}$$

$$=\lim_{x\to 1}\left\{\frac{x^{1/3}-1}{x-1}-\frac{x^{1/4}-1}{x-1}\right\}\frac{1}{x^2+x+1}=\left(\frac{1}{3}-\frac{1}{4}\right)\frac{1}{3}=\frac{1}{36}$$

Hence, the correct answer is (A).

2. $$\lim_{x\to 0}\frac{e^{ax}-e^x-x}{x^2}=\lim_{x\to 0}\frac{ae^{ax}-e^x-1}{2x}$$

As $x \to 0 \quad nr. \to 0 \Rightarrow a-2=0 \Rightarrow a=2$

$$\lim_{x\to 0}\frac{e^{2x}-e^x-x}{x^2}=\lim_{x\to 0}\frac{2e^{2x}-e^x-1}{2x}=\lim_{x\to 0}\frac{4e^{2x}-e^x}{2}=\frac{3}{2}=b$$

Hence, the correct answer is (C).

3. $$\lim_{x\to 0}\frac{f(1-\cos x)}{g(x)\sin^2 x}=\lim_{x\to 0}\frac{f\left(2\sin^2\frac{x}{2}\right)}{g(x)\left(2\sin^2\frac{x}{2}\right)^2}\times\frac{\left(\sin^2\frac{x}{2}\right)^2}{\left(\sin^2\frac{x}{2}\right)\left(\cos^2\frac{x}{2}\right)}$$

$$=\lim_{x\to 0}\frac{a}{g(x)}\times\tan^2\frac{x}{2}=b \Rightarrow \lim_{x\to 0}\frac{x^2}{g(x)}=\frac{4b}{a}$$

Thus, $\lim_{x\to 0}\frac{g(1-\cos 2x)}{x^4}=\lim_{x\to 0}\frac{g\left(2\sin^2 x\right)}{\left(2\sin^2 x\right)^2}\cdot 4\left(\frac{\sin x}{x}\right)^4$

$$=\left(\frac{a}{4b}\right)(4)\,(1)^4=\frac{a}{b}$$

Hence, the correct answer is (C).

4. $$\lim_{x\to 0}\frac{\sin(\sin x)-\sin x}{ax^5+bx^3+c}=-\frac{1}{12} \qquad \text{(given)}$$

Suppose c is not zero, then the limit on the L.H.S $=0$

$\therefore \quad c=0$

Now,

$$\lim_{x\to 0}\frac{\sin(\sin x)-\sin x}{ax^5+bx^3}=\lim_{x\to 0}\frac{2\sin\left(\frac{\sin x-x}{2}\right)\cos\left(\frac{\sin x+x}{2}\right)}{ax^5+bx^3}$$

$$=\lim_{x\to 0}\frac{2\sin\left(\frac{\sin x-x}{2}\right)}{\left(\frac{\sin x-x}{2}\right)}\cdot\frac{\sin x-x}{2}\cdot\frac{\cos\left(\frac{\sin x+x}{2}\right)}{ax^5+bx^3}$$

$$=\lim_{x\to 0}\left(\frac{\sin x-x}{ax^5+bx^3}\right)$$

Because the other terms tend to 1 as $x \to 0$.

$$=\lim_{x\to 0}\frac{\cos x-1}{5ax^4+3bx^2} \qquad \text{by L' Hospitals's Rule}$$

$$=\lim_{x\to 0}\frac{-\sin x}{20ax^3+6bx}=\lim_{x\to 0}\frac{-\sin x}{x}\cdot\frac{1}{20ax^2+6b}$$

$$=-\frac{1}{6b} \text{ for all } a\in R=-\frac{1}{12}\Rightarrow b=2$$

$\therefore \quad a\in R,\ b=2,\ c=0$

Hence, the correct answer is (D).

5. Let $l=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$$1^\infty=\lim_{x\to 0} e^{\left(\frac{\sin x}{x}-1\right)\frac{1}{x^2}}=\lim_{x\to 0} e^{\frac{\sin x-x}{x^3}}=e^{-1/6}.$$

$$\log l = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\sin x}{x}\right) \text{ by L' Hospital Rule} \Rightarrow l = e^{\frac{1}{6}}$$

Hence, the correct answer is (C).

6. Degree of numerator = degree of denominator = 2010.

$$\therefore \quad \text{Limit} = \lim_{x\to\infty}\frac{10x^{2010}+\ldots\ldots}{2x^{2010}+\ldots\ldots} = 5$$

Hence, the correct answer is (A).

7. $$\lim_{x\to 0^-}\left(\frac{1}{a}+\frac{1}{b}\right) = \lim_{x\to 0^-}\frac{a+b}{ab}$$

$$= \lim_{x\to 0^-}\sqrt{2}\frac{|\sin\theta|}{\theta} = \lim_{x\to 0^-}\sqrt{2}\frac{\sin\theta}{\theta} = -\sqrt{2}$$

Hence, the correct answer is (B).

8. Let $t = \frac{\pi}{2} - \theta$ then by substitution the problem is

$$\therefore \quad \lim_{t\to 0}\left[\frac{-t}{\tan\, t}\right] \qquad \because 0 < \frac{t}{\tan t} < 1 \Rightarrow -1 < \frac{-t}{\tan t} < 0$$

$$\left[\frac{-t}{\tan t}\right] = -1$$

Hence, the correct answer is (C).

9. Req. value $= \lim_{x\to\infty}\dfrac{x\,\sin\left(\frac{1}{x}\right)-2}{\frac{1}{x^2}+3} = \dfrac{1-2}{0+3} = \dfrac{-1}{3};\ \left(\because \lim_{n\to\infty}\dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = 1\right)$

Hence, the correct answer is (B).

10. $\pi = 3.14$, then $\left[-\pi^2\right] = -10$

$$= \lim_{x\to 0}\frac{\tan\left(\left[-\pi^2\right]x^2\right) - \tan\left(\left[-\pi^2\right]\right)x^2}{\sin^2 x}$$

Divided by x^2 we get

$$\lim_{x\to 0}\frac{\frac{-\tan 10x^2}{x^2} + \tan 10}{\frac{\sin^2 x}{x^2}} = \tan 10 - 10$$

Hence, the correct answer is (B).

11. $$\lim_{x\to 0}\left((a-n)n - \frac{\tan x}{x}\right)\frac{\sin nx}{x} = 0$$

$$\Rightarrow \quad \left((a-n)n-1\right)n = 0$$

$$\Rightarrow \quad a = n + \frac{1}{n}$$

Hence, the correct answer is (D).

12. $$\lim_{x\to\infty}\frac{\tan^{-1}\left(\frac{x+\lambda}{x+\mu}\right) - \frac{\pi}{4}}{\frac{1}{x}} = 1$$

Apply L' hospital rule and simplifying we get

$$\lim_{x\to\infty}\frac{\left(\frac{(x+\mu)^2}{(x+\lambda)^2+(x+\mu)^2}\right)\left(\frac{x+\mu-(x+\lambda)}{(x+\mu)^2}\right)}{-\frac{1}{x^2}} = 1$$

$$\lim_{x\to\infty}\frac{(\lambda-\mu)x^2}{2x^2+2x(\lambda+\mu)+\left(\mu^2+\lambda^2\right)} = 1$$

$$\Rightarrow \quad \frac{\lambda-\mu}{2} = 1 \Rightarrow \lambda - \mu = 2$$

$\therefore (\lambda, \mu)$ can be (5, 3)

Hence, the correct answer is (C).

13. $$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\cos(\sin x) - \cos x}{x^4}$$

$$= \lim_{x\to 0}\frac{-2\sin\frac{\sin x + x}{2}\sin\frac{\sin x - x}{2}}{x^4}$$

$$= -\frac{1}{2}\lim_{x\to 0}\frac{\sin\left(\frac{\sin x + x}{2}\right)}{\left(\frac{\sin x + x}{2}\right)}\frac{\sin\left(\frac{\sin x - x}{2}\right)}{\left(\frac{\sin x - x}{2}\right)} \times \frac{\sin x + x}{x} \times \frac{\sin x - x}{x^3}$$

$$= -\frac{1}{2}\lim_{u\to 0}\frac{\sin u}{u}\lim_{v\to 0}\frac{\sin v}{v}\lim_{x\to 0}\left(\frac{\sin x}{x}+1\right) \times$$

$$\frac{-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots}{x^3}\left(u = \frac{\sin x + x}{2}, v = \frac{\sin x - x}{2}\right)$$

$$= -\frac{1}{2}\times 1\times 1\times 2\times\frac{-1}{3!} = \frac{1}{6}.$$

Hence, the correct answer is (B).

14. $$\lim_{x\to 0}\frac{3^{6x}-3^{5x}-3^{4x}+3^x.3^x+3^x-1}{x^3}$$

$$= \lim_{x\to 0}\frac{3^{5x}(3^x-1)-3^{2x}(3^x-1)(3^x+1)+(3^x-1)}{x^3}$$

$$= \lim_{x\to 0}\frac{(3^x-1)(3^{5x}-3^{3x}-3^{2x}+1)}{x^3}$$

$$= \lim_{x\to 0}\frac{\left(3^x-1\right)}{x}\frac{\left(9^x-1\right)}{x}\left(\frac{27^x-1}{x}\right)$$

$= \log 3 \log 9 \log 27 = 6\ (\log 3)^3$

Hence, the correct answer is (C).

15. Req. value $= \lim_{x\to 0}\dfrac{e^{\sin x}\left(e^{x-\sin x}-1\right)}{2\ (x-\sin x)} = \dfrac{1}{2}(1) = \dfrac{1}{2}$

Hence, the correct answer is (D).

16. Req. value $= \lim_{h\to 0}\dfrac{1}{h}\left[\dfrac{1}{\sqrt[3]{8+h}} - \dfrac{1}{\sqrt[3]{8}}\right]$

$$= \lim_{h\to 0}\frac{\sqrt[3]{8}-\sqrt[3]{8+h}}{h}\times\frac{1}{\sqrt[3]{8+h}\sqrt[3]{8}} = -\frac{1}{3}8^{\frac{1}{3}-1}.\frac{1}{2.2} = \frac{-1}{48}$$

Hence, the correct answer is (A).

17. $$\lim_{x\to 2^+}\frac{\{x\}}{(x-2)}\frac{\sin(x-2)}{(x-2)} = \lim_{x\to 2+}\frac{\{x\}}{x-2} = \lim_{x\to 2+}\frac{\{x\}}{\{x\}} = 1$$

Hence, the correct answer is (C).

18. $$\lim_{x\to\infty} x\left(\tan^{-1}\frac{x+1}{x+2} - \tan^{-1}(1)\right)$$

$$= \lim_{x\to\infty} x\tan^{-1}\left[\frac{\frac{x+1}{x+2}-1}{1+\frac{x+1}{x+2}.1}\right] = \lim_{x\to\infty} x\tan^{-1}\left[\frac{-1}{2x+3}\right]$$

$$= \lim_{x\to\infty}\frac{\tan^{-1}\left(-\frac{1}{2x+3}\right)}{\left(-\frac{1}{2x+3}\right)}\left(\frac{x}{-2x+3}\right) = -\frac{1}{2}$$

Hence, the correct answer is (B).

19. $f(x)=\lim\limits_{x\to\propto}\frac{1}{3}\left(f(x+1)+\frac{5}{f(x+2)}\right),$

$$f(x)=\lim_{x\to\propto}\frac{1}{3}\left(f(\propto+1)+\frac{5}{f(\propto+2)}\right)$$

$$=\frac{1}{3}\left(f(\propto)+\frac{5}{f(\propto)}\right)$$

$$K=\frac{1}{3}\left(K+\frac{5}{K}\right),\Rightarrow 3K^2=K^2+5\Rightarrow K^2=\frac{5}{2}$$

Hence, the correct answer is (C).

20. At $x\to 0$ $\sin x<x\Rightarrow\left(\frac{\sin x}{x}\right)<1$ and $\frac{\tan x}{x}>1$

$$\lim_{x\to\pi/2}\left(\left[\frac{100\sin x}{x}\right]+\left[\frac{100\tan x}{x}\right]\right)=99+100=199$$

Hence, the correct answer is (A).

21. $$\lim_{x\to0}\frac{5\left(x-\frac{x^3}{3!}+\ldots\right)-7\left(2x-\frac{(2x)^3}{3!}+\ldots\right)+3\left[(3x)-\frac{(3x)^3}{3!}\right]+\ldots}{x^2\left(x-\frac{x^3}{3!}+\ldots\right)}$$

$$=\frac{-5}{3!}+\frac{56}{3!}-\frac{81}{3!}=-5$$

Hence, the correct answer is (A).

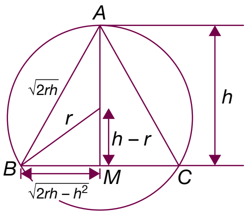
22. $$\lim_{x\to0}\frac{2\sin^2\frac{(ax^2+bx+c)}{2}}{\left(\frac{(ax^2+bx+c)}{2}\right)^2}\frac{(ax^2+bx+c)^2}{4(x-\alpha)^2}$$

$$=\lim_{x\to\alpha}\frac{2a^2\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)}{4(x-\alpha)\qquad(x-\alpha)}$$

$$\lim_{x\to\alpha}\frac{a^2(x-\alpha)^2(x-\beta)^2}{2(x-\alpha)^2};\ \frac{a^2}{2}(\alpha-\beta)^2\Rightarrow C$$

Hence, the correct answer is (C).

23. $AB=\sqrt{\cancel{h^2}+2rh-\cancel{h^2}}=\sqrt{2rh}$

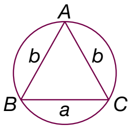


$$P=2\sqrt{2rh}+2\sqrt{2rh-h^2}\ ;$$

$$P=2\left[\sqrt{2rh}+\sqrt{2rh-h^2}\right]$$

$$\Delta=\frac{\cancel{2}\sqrt{2rh-h^2}\,.h}{\cancel{2}}=h\sqrt{2rh-h^2}$$

$$\therefore\quad\lim_{h\to0}\frac{\Delta}{p^3}=\lim_{h\to0}\frac{h\sqrt{2rh-h^2}}{8\left[\sqrt{2rh}-\sqrt{2rh-h^2}\right]^3}$$



$$=\lim_{h\to0}\frac{h^{3/2}\sqrt{2r-h}}{8h^{3/2}\left[\sqrt{2r}-\sqrt{2r-h}\right]^3}$$

$$=\frac{\sqrt{2r}}{8.8(2r)(2r)^{1/2}}=\frac{1}{128r}\Rightarrow\text{(C)}$$

Alternative:

Note that, as $h\to0$, $b=\frac{a}{2}$ or $2b=a$

Hence, $\frac{\Delta}{p^3}=\frac{ab(b)}{4r.(a+2b)^3}$ (Using $R=\frac{abc}{4\Delta}$)

$$=\frac{2b^3}{4r\ 64b^3}=\frac{1}{128\ r}\Rightarrow\text{(C)}$$

Hence, the correct answer is (C).

24. $$\prod_{n=2}^{\infty}\left(\frac{n^2-1}{n^2}\right)=\prod_{n=2}^{\infty}\left(\frac{n-1}{n}\right)\prod_{n=2}^{\infty}\left(\frac{n+1}{n}\right)$$

$$=\left(\frac{1}{2}\cdot\frac{2}{3}\frac{3}{4}\ldots\frac{n-1}{n}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\ldots\frac{n+1}{n}\right)=\frac{1}{n}\cdot\frac{n+1}{2}$$

$$=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2}=\frac{1}{2}$$

Hence, the correct answer is (D).

25. $$\lim_{x\to\frac{\pi}{2}}\frac{\sin x}{\cos^{-1}[\sin^3x]}$$

$ax\quad x\to\frac{\pi}{2}$, $[\sin^3x]\to0$ and $\sin x\to1$

$\therefore\quad l=\frac{2}{\pi}\Rightarrow$ (A)

Hence, the correct answer is (A).

26. $$\lim_{x\to0}\frac{\ln(3+x)-\ln(3-x)}{x}$$

$$=\lim_{x\to0}\frac{\ln3+\ln\left(1+\frac{x}{3}\right)-\ln3-\ln\left(1-\frac{x}{3}\right)}{x}$$

$$=\lim_{x\to0}\ln\left(1+\frac{x}{3}\right)^{1/x}-\ln\left(1-\frac{x}{3}\right)^{1/x}$$

$$=\frac{1}{x}\cdot\frac{x}{3}-\frac{1}{x}\left(-\frac{x}{3}\right)=\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$$

Hence, the correct answer is (A).

27. $f(x)=\begin{cases}-1; & \text{if } -1<x<1\\ 0; & \text{if } x=1 \text{ or } -1\\ 1; & \text{if } |x|>1\end{cases}$ and $u(x)=\text{sgn}(|x|-1)$ is same

Hence, the correct answer is (C).

28. Put $x=\frac{\pi}{4}+h$

$$\lim_{h\to0}\frac{2\sqrt{2}-\left(\frac{\cos h}{\sqrt{2}}-\frac{\sin h}{\sqrt{2}}+\frac{\sin h}{\sqrt{2}}+\frac{\cos h}{\sqrt{2}}\right)^3}{1-\sin\left(\frac{\pi}{2}+2h\right)}$$

$$=\lim_{h\to0}\frac{2\sqrt{2}-2\sqrt{2}\cos^3h}{1-\cos2h}$$

$$=\lim_{h\to0}\frac{2\sqrt{2}(1-\cos h)(1+\cos h+\cos^2h)}{2\sin^2h}$$

$$=2\sqrt{2}\times\frac{1}{2}\times3\times\frac{1}{2}$$

$$=\frac{3\sqrt{2}}{2}=\frac{3}{\sqrt{2}}$$

Hence, the correct answer is (D).

29. $\lim\limits_{x\to8^+}\frac{\sin\{x\}}{\{-x\}}=0,$

As $\{I+x\}=\{x\}$; as $\lim\limits_{x\to I^+}\{x\}\to0$ and $\lim\limits_{x\to I^-}\{x\}\to1$

$\lim\limits_{x\to8^-}\frac{\sin\{x\}}{\{-x\}}\to\infty,\ x\to8^-$ As $\sin\{x\}\to\sin(1)$ and $\{-x\}\to0$

Hence, the correct answer is (B).

30. $-1 \le \frac{4\tan^{-1}(2\pi x)}{\pi} \le 1 \Rightarrow -\frac{\pi}{4} \le \tan^{-1} 2\pi x \le \frac{\pi}{4}$

$\Rightarrow \quad -1 \le 2\pi x \le 1 \Rightarrow x \in \left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right].$

Hence, the correct answer is (C).

31. $l = \lim_{x\to 0}\left[1 + \frac{x^3 + f(x)}{x}\right]^{1/x^2}$ if $A = \frac{x^3 + f(x)}{x}$, then

$$l = \lim_{x\to 0}\left\{[1+A]^{1/A}\right\}^{A/x^2} = e^{\lim\limits_{x\to 0}\frac{A}{x^2}}$$

$$\lim_{x\to 0}\frac{A}{x^2} = \lim_{x\to 0}\frac{x^3 + f(x)}{x^3} = \lim_{x\to 0}\left(1 + \frac{f(x)}{x^3}\right)$$

Now, $1 + \lim\limits_{x\to 0}\frac{f(x)}{x^3} = 7,$ (Given)

$$\lim_{x\to 0}\frac{f(x)}{x^3} = 6$$

Hence, the correct answer is (D).

32. $\lim\limits_{x\to 0}(\sin x)^{1/x} + \lim\limits_{x\to 0}\left(\frac{1}{x}\right)^{\sin x} = 0 + e^{\lim\limits_{x\to 0}\sin x \log\left(\frac{1}{x}\right)}$

$\left[\because |\sin x| < 1 \text{ when } x \to 0\right]$

$= e^{\lim\limits_{x\to 0}\frac{-\log x}{\operatorname{cos}ec x}} = e^{\lim\limits_{x\to 0}\frac{-1/x}{-\operatorname{cosec}x \cot x}}$ $\left[\text{using L Hospital rule}\right]$

$= e^{\lim\limits_{x\to 0}\frac{\sin x}{x}.\tan x} = e^0 = 1$

Hence, the correct answer is (C).

33. $\sin x < x < \tan x \quad \forall x \in (0, \pi/2)$

$$\frac{\sin x}{x} < 1; \frac{\tan x}{x} > 1$$

$\Rightarrow \quad \left[2009\frac{\sin x}{x}\right] = 2008$ and $\left[2010\frac{\tan x}{x}\right] = 2010$

$\Rightarrow \quad \text{limit} = 2008 + 2010 = 4018$

Hence, the correct answer is (D).

34. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{y^2 + \sin x}{x^2 + \sin y^2} = \lim\limits_{x\to 0}\frac{x + \sin x}{x^2 + \sin x} = \lim\limits_{x\to 0}\frac{1 + \frac{\sin x}{x}}{x + \frac{\sin x}{x}} = 2$

Hence, the correct answer is (C).

35. $f(x) = \lim\limits_{n\to\infty}\left(\left(\frac{n}{n+1}\right)^x + \sin\frac{1}{n}\right)^n = e^{\lim\limits_{n\to\infty}\left(\left(\frac{1}{1+\frac{1}{n}}\right)^x + \sin\frac{1}{n} - 1\right)n}$

$$= e^{\lim\limits_{n\to\infty}\frac{\left(\frac{1}{\left(1+\frac{1}{n}\right)^x} + \sin\frac{1}{n} - 1\right)}{1/n}} = e^{\lim\limits_{y\to 0}\left(\frac{(1+y)^{-x} + \sin y - 1}{y}\right)},$$

Put $\frac{1}{n} = y$ as $n \to \infty \Rightarrow y \to 0$

$$= e^{\lim\limits_{y\to 0}\left(\frac{-x\left((1+y)^{-x-1} + \cos y\right)}{1}\right)}$$

$$f(x) = e^{1-x}$$

$$f(1) = e^0 = 1$$

Hence, the correct answer is (C).

36. $\lim\limits_{x\to 0}\frac{\sin 3x}{x^3} + \frac{a}{x^2} + b = \lim\limits_{x\to 0}\frac{\sin 3x + ax + bx^3}{x^3}$

$$= \lim_{x\to 0}\frac{3\frac{\sin 3x}{3x} + a + bx^2}{x^2}$$

For existence of limit $3 + a = 0 \Rightarrow a = -3$

$\therefore \quad l = \lim\limits_{x\to 0}\frac{\sin 3x - 3x + bx^3}{x^3} = 27.\frac{\sin t - t}{t^3} + b = 0$ $(3x = t)$

$$= -\frac{27}{6} + b = 0 \Rightarrow b = \frac{9}{2}$$

[OR use L′ Hospital′s rule]

Hence, the correct answer is (A).

37. $l = \lim\limits_{x\to\infty}\frac{2^{\frac{x^n}{e^x}} - 3^{\frac{x^n}{e^x}}}{x^n}$

$$= \lim_{x\to\infty}\left(\frac{2^{x^n/e^x} - 1}{x^n/e^x}\right)\frac{1}{e^x} - \left(\frac{3^{x^n/e^x} - 1}{x\,{}^{x^n}/e^x}\right)\frac{1}{e^x}$$

$$= \frac{\ln 2}{\infty} - \frac{\ln 3}{\infty} = 0$$

But $\lim\limits_{x\to\infty}\frac{x^n}{e^x} = 0 \Rightarrow l = 0$

Hence, the correct answer is (B).

38. $\lim\limits_{x\to-\infty}\left(x^{5c}\left(1 + \frac{7}{x} + \frac{2}{x^5}\right)^c - x\right)$

$$= \lim_{x\to-\infty} x\left(x^{5c-1}\left(1 + \frac{7}{x} + \frac{2}{x^5}\right)^c - 1\right)$$

This will be of the form $\infty \times 0$ only if

$$5c - 1 = 0 \Rightarrow c = \frac{1}{5}$$

Substituting $c = \frac{1}{5}$, l becomes

Hence, $l = \lim\limits_{x\to-\infty} x\left[(1+X)^{1/5} - 1\right]$ where $X = \frac{7}{x} + \frac{2}{x^5}$

$$= \lim_{x\to-\infty} x\left[1 + \frac{X}{5} + \ldots - 1\right]$$

$$= \lim_{x\to-\infty} x\left(\frac{7}{x} + \frac{2}{x^5}\right).\frac{1}{5} = \frac{7}{5}$$

Hence, $c = \frac{1}{5}$ and $l = \frac{7}{5}$

Hence, the correct answer is (A).

39. $\lim\limits_{n\to\infty}\frac{1^2 n + 2^2(n-1) + 3^2(n-2) + \ldots + n^2(n-(n-1))}{\sum n^3}$

$N^r. = n(1^2 + 2^2 + \ldots + n^2) - (1.2^2 + 2.3^2 + 3.4^2 + \ldots + (n-1)n^2)$

$$= n\sum n^2 - \sum_{r=2}^{n}(r-1).r^2$$

$$= n\sum n^2 - \sum_{r=1}^{n}(r^3 - r^2) = n\sum n^2 - \sum n^3 - \sum n^2$$

$$= (n+1)\sum n^2 - \sum n^3$$

$$\therefore \quad \ell = \lim_{n\to\infty} \frac{(n+1)\sum n^2 - \sum n^3}{\sum n^3}$$

$$\lim_{n\to\infty} \frac{4(n+1)n(n+1)(2n+1)}{6n(n+1)n(n+1)} - 1$$

$$\frac{4}{3} - 1 = \frac{1}{3} \Rightarrow A$$

Alternate:

$$l = \lim_{n\to\infty} \underbrace{\frac{1^2 n + 2^2(n-1) + \ldots + n^2 \cdot 1}{1^3 + 2^3 + 3^2 + \ldots + n^3} + 1}_{} - 1$$

$$= \lim_{n\to\infty} \frac{1^2(n+1) + 2^2(n+1) + \ldots + n^2(n+1)}{\sum n^3} - 1$$

$$l = \lim_{n\to\infty} \frac{(n+1)\sum n^2}{\sum n^3} - 1$$

$$\Rightarrow \quad \ell = \lim_{n\to\infty} \frac{(n+1)(n)(n+1)(2n+1).4}{6 \;\; n(n+1)n(n+1)} - 1 = \frac{1}{3}$$

Hence, the correct answer is (A).

40. $\lim\limits_{x\to\infty} \dfrac{\cot^{-1}\left(\dfrac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\dfrac{a^x}{\log_a x}\right)}$; $\underset{x\to\infty}{as} \left(\dfrac{\log_a x}{x^a}\right) \to 0$ and $\left(\dfrac{a^x}{\log_a x}\right) \to \infty$

(Using L-Hospital rule)

$$\therefore \quad l = \frac{\frac{\pi}{2}}{\frac{\pi}{2}} = 1$$

Hence, the correct answer is (A).

41. $\lim\limits_{x\to 2^-} f(x) \dfrac{\cos(2x-4) - 33}{2} = -16;$

$$\lim_{x\to 2^-} \frac{x^2.4|x-2|}{x-2} = -4(4) = -16$$

By sandwich theorem $\lim\limits_{x\to 2^-} f(x) = -16$

Hence, the correct answer is (B).

42. $a = (x+1)^2 + 2 \Rightarrow a = 2$

$$b = \lim_{x\to 0} \frac{\sin 2x}{\frac{2(e^{2x}-1).2x}{2x}} = \frac{1}{2}$$

Now, $\sum\limits_{r=0}^{n} a^r b^{n-r} = \sum 2^r \left(\dfrac{1}{2}\right)^{n-r} = \dfrac{1}{2^n} \sum\limits_{r=0}^{n} 2^{2r}$

$$= \frac{1}{2^n}[1 + 2^2 + 2^4 + \ldots + 2^{2n}]$$

$$= \frac{1}{2^n}\left[\frac{(2^2)^{n+1} - 1}{2^2 - 1}\right] = \frac{1}{2^n}\left[\frac{4^{n+1} - 1}{3}\right] = \frac{4^{n+1} - 1}{3.2^n}$$

Hence, the correct answer is (D).

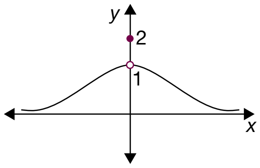
43. Put $\sin^{-1} x = \theta$

$$\lim_{\theta\to\pi/4} \frac{\sin\theta - \cos\theta}{1 - \tan\theta} = \lim_{\theta\to\pi/4} -\cos\theta = -\frac{1}{\sqrt{2}}$$

Hence, the correct answer is (B).

44. $\left[\dfrac{1}{\ln(x^2+e)}\right] = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$

$$f(x) = \begin{cases} 2 & , x = 0 \\ \dfrac{1}{\sqrt{1+x^2}} & , x \neq 0 \end{cases}$$



Hence, range of $f(x)$ is $(0, 1) \cup \{2\}$

Hence, the correct answer is (D).

45. $\lim\limits_{x\to\infty} \sqrt{x+1} - \sqrt{x} \Rightarrow \lim\limits_{x\to\infty} \dfrac{x+1-x}{\sqrt{x+1}+\sqrt{x}} = 0$

$$\Rightarrow \quad \cot^{-1}(0) = \frac{\pi}{2}$$

$$\lim_{x\to\infty} \left(\frac{2x+1}{x-1}\right)^x = \infty \Rightarrow \sec^{-1}(\infty) = \frac{\pi}{2}$$

$$\therefore \quad l = 1$$

Hence, the correct answer is (A).

46. $\lim\limits_{x\to\infty} \dfrac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})} = \lim\limits_{x\to\infty} \dfrac{\ln e^x\left(1+\dfrac{x^2}{e^x}\right)}{\ln e^{2x}\left(1+\dfrac{x^4}{e^{2x}}\right)}$

$$= \lim_{x\to\infty} \frac{x + \ln\left(1+\frac{x^2}{e^x}\right)}{2x + \ln\left(1+\frac{x^4}{e^{2x}}\right)}$$

$$= \frac{1 + \ln\left(1+\frac{x^2}{e^x}\right)^{1/x}}{2 + \ln\left(1+\frac{x^4}{e^{2x}}\right)^{1/x}}$$

Note that $\underset{x\to\infty}{as} \dfrac{x^2}{e^x} \to 0$ and

$$\underset{x\to\infty}{as} \frac{x^2}{e^{2x}} \to 0$$

$$= \lim_{x\to\infty} \frac{1 + \frac{1}{x}\left(1+\frac{x^2}{e^x}-1\right)}{2 + \frac{1}{x}\left(1+\frac{x^4}{e^{2x}}-1\right)}$$

$$= \lim_{x\to\infty} \frac{1+\frac{x}{e^x}}{2+\frac{x^3}{e^{2x}}}$$

$$\lim_{x\to-\infty} = \frac{1}{2}$$

$$\lim_{x\to-\infty} \frac{\ln(x^2+e^x)}{\ln(x^4+e^{2x})}$$

$$= \lim_{x\to-\infty} \frac{2\ln|x| + \ln\left(1+\frac{e^x}{x^2}\right)}{4\ln|x| + \ln\left(1+\frac{e^{2x}}{x^4}\right)}$$

$$= \lim_{x\to-\infty} \frac{2 + \frac{\ln\left(1+\frac{e^x}{x^2}\right)}{\ln|x|}}{4 + \frac{\ln\left(1+\frac{e^{2x}}{x^4}\right)}{\ln|x|}}$$

$$= \frac{2+0}{4+0} = \frac{1}{2} = m$$

Hence, the correct answer is (A).

47. $l = \lim\limits_{n\to\infty} \pm \cos\left(n\pi - \pi\sqrt{n^2+n}\right)$

$$\Rightarrow \quad \lim_{n\to\infty} \pm \cos\left(\pi\left(n - \sqrt{n^2+n}\right)\right)$$

$$= \pm \lim_{n\to\infty} \cos\left(\frac{(\pi(+n))}{n+\sqrt{n^2+n}}\right) = \pm \lim_{n\to\infty} \cos\left(\frac{n\pi}{n+n\sqrt{1+\frac{1}{n}}}\right)$$

$$= \pm \lim_{n\to\infty} \cos\left(\frac{\pi}{1+\sqrt{1+\frac{1}{n}}}\right) = \pm \cos\frac{\pi}{2} \to 0$$

Hence, the correct answer is (C).

48. Divide Nr. and Dr. by x^5

$$\lim_{x\to 0}\left(\frac{\frac{\sin^2 x}{\cos^2 x}\frac{(\cos x-1)^2}{x^5}-\frac{2^4\sin x}{x^5}+1}{\frac{7(\tan^{-1}x)^7}{x^5}+\frac{(\sin^{-1}x)^6}{x^5}+\frac{3\sin^5 x}{x^5}}\right) = \frac{0-0+1}{0+0+3} = \frac{1}{3}$$

Hence, the correct answer is (C).

49. $\frac{[x]^2}{x^2} = \left[\begin{array}{ll} 0, & \text{if } 0<x<1 \\ \frac{1}{x^2}, & \text{if } -1<x<0 \end{array}\right.$ $\Rightarrow l$ does not exist.

$\frac{[x^2]}{x^2} = \left[\begin{array}{ll} 0, & \text{if } 0<x<1 \\ 0, & \text{if } -1<x<0 \end{array}\right.$ $\Rightarrow m$ exists and is equal to zero.

Hence, the correct answer is (B).

50. $\lim_{x\to 0} \frac{(\tan(\{x\}-1))\ \sin\{x\}}{\{x\}(\{x\}-1)}$

$$= \lim_{x\to 0^+} f(x) = \lim_{h\to 0^+} \frac{\tan(h-1).\sinh}{h(h-1)} = \frac{\tan(-1)}{-1} = \tan 1$$

$$\lim_{x\to 0^-} \frac{\tan((1-h)-1)\ \sin(1-h)}{(1-h)\ (1-h-1)} = \frac{\sin 1}{1} = \sin 1$$

Hence, limit $x \to 0\ f(x)$ does not exist

Hence, the correct answer is (D).

51. $\lim_{x\to\infty} \frac{\frac{2}{x}+2+\frac{\sin 2x}{x}}{\left(2+\frac{\sin 2x}{x}\right)e^{\sin x}}$ (As $x \to \infty$)

$$l = \lim_{x\to\infty} \frac{2}{2.e^{\sin x}} = \text{oscillatory between } \frac{1}{e} \text{ to } \frac{1}{e^{-1}}$$

$\Rightarrow$ Limit does not exist.

Hence, the correct answer is (D).

52. $l = e^{\lim_{x\to 0} \operatorname{cosec}^2 bx(\cos ax-1)}$

Now, $-\lim_{x\to 0}\frac{1-\cos ax}{\sin^2 bx} = -\lim_{x\to 0}\frac{\sin^2 ax}{\sin^2 bx}.\frac{1}{1+\cos ax} = -\frac{a^2}{2b^2}$

$$l = e^{-\frac{a^2}{2b^2}}$$

Hence, the correct answer is (C).

53. $f(3) = 6; f'(3) = 2$

$$\lim_{x\to 3}\frac{x f(3) - 3 f(x)}{x-3}, \text{ put } x = 3+h$$

$$= \lim_{h\to 0}\frac{(3+h)\cdot f(3)-3f(3+h)}{h}$$

$$= \lim_{h\to 0}\frac{-3[f(3+h)-f(3)]}{h} + f(3)$$

$$= -3f'(3) + f(3) = -6+6 = 0$$

Hence, the correct answer is (C).

54. Multiplying $N^{r.}$ and D^r. by $(1-x)$ and proceed

$$\lim_{n\to\infty}\left[\frac{(1-x^2)(1+x^2)(1+x^4)\ldots(1+x^{2^n})}{1-x}\right]$$

$$= \lim_{n\to\infty}\frac{(1-x^{2^n})(1+x^{2^n})}{1-x}$$

$$= \lim_{n\to\infty}\frac{1-x^{2^{n+1}}}{1-x} = \frac{1}{1-x}$$

Hence, the correct answer is (D).

55. $N^r = \lim_{x\to 0} 6\frac{x}{\tan x}\cdot\frac{x}{\sin 2x} = 3$

$D^r = \lim_{x\to 0} \sec\left(\cos x + \pi\tan\left(\frac{\pi}{4\sec x}\right) - 1\right),$

$\left.\lim_{x\to 0}\sec(1+\pi-1) = -1\right] \Rightarrow \ell = -3$

Hence, the correct answer is (D).

56. $l = \lim_{x\to 1}\left(\cos^{-1}x\right)^{1-x}$ (0° form)

$\ln l = \lim_{x\to 1}(1-x)\ln(\cos^{-1}x)$ put $x = \cos\theta$

but $\lim_{\theta\to 0}(1-\cos\theta)\ln\theta = \lim_{\theta\to 0}\frac{\ln\theta}{\frac{1}{(1-\cos\theta)}} = -\lim_{\theta\to 0}\frac{(1-\cos\theta)^2}{\sin\theta\ .\ \theta}$

$$= -\lim_{\theta\to 0}\frac{(1-\cos\theta)^2}{\theta^2\ \frac{\sin\theta}{\theta}} = 0 \Rightarrow l = 1$$

Now required limit $(1+1)^2 = 4$

Hence, the correct answer is (A).

57. $\log_{\cos\frac{x}{2}}\cos x = \frac{\ln(\cos x)}{\ln\left(\cos\frac{x}{2}\right)} = \lim_{x\to 0}\frac{\frac{1-\cos x}{x^2}}{\frac{1-\cos\frac{x}{2}}{x^2}} = \frac{\frac{1}{2}}{\frac{1}{8}} = 4$

$$\Rightarrow \log^2_{\cos\frac{x}{2}}\cos x = 16 \Rightarrow l = (1+16)^2 = 289$$

Hence, the correct answer is (C).

58. Using L'hospital rule $\lim_{x\to 0}\left(\frac{a\tan^{-1}\frac{\sqrt{x}}{a} - b\tan^{-1}\frac{\sqrt{x}}{b}}{x\sqrt{x}}\right)$

$$= \lim_{x\to 0}\frac{\frac{a}{\left(1+\frac{x}{a^2}\right)}\cdot\frac{1}{a}\cdot\frac{1}{2\sqrt{x}} - \frac{b}{\left(1+\frac{x}{b^2}\right)}\cdot\frac{1}{b}\cdot\frac{1}{2\sqrt{x}}}{\frac{3}{2}\cdot\sqrt{x}}$$

$$= \lim_{x\to 0}\left[\frac{a^2}{(a^2+x)x} - \frac{b^2}{(b^2+x)}\frac{1}{x}\right]\cdot\frac{1}{3} = \frac{a^2-b^2}{3a^2b^2}$$

Hence, the correct answer is (D).

59. $f_n(x) = \tan\frac{x}{2}\left(\frac{1+\cos x}{\cos x}\right)(1+\sec 2x)(1+\sec 4x)\ldots(1+\sec 2^n x)$

$$= \tan x\left(\frac{1+\cos 2x}{\cos 2x}\right)(1+\sec 4x)\ldots(1+\sec 2^n x)$$

$$= \tan 2x\left(1+\sec 2^2 x\right)\ldots\left(1+\sec 2^n x\right)$$

$$= \tan 2^{n-1}x\left(1+\sec 2^n x\right) = \tan 2^n x$$

Now, $\lim\limits_{x\to 0}\dfrac{f_n(x)}{2x} = \lim\limits_{x\to 0}\dfrac{\tan 2^n x}{2^n x}.2^{n-1} = 2^{n-1}$

Hence, the correct answer is (C).

60. For finite value

$$-1 < \frac{x^2+a}{x^2+1} < 1 \quad\Rightarrow\quad -1 < a < 1$$

Hence, the correct answer is (B).

61. As, $|x| \geq 0$ and $-\infty < \log\{x\} < 0$

$\Rightarrow$ No solution

Here, $\alpha = 0$

$$\lim_{x\to 0}\frac{xe^{ax} - b\sin x}{x^3} = \lim_{x\to 0}\frac{x(1-b) + ax^2 + x^3\left(\dfrac{a^2}{2!}+\dfrac{b}{3!}\right)}{x^3}$$

$\Rightarrow\ b = 1$

$a = 0;\ \dfrac{a^2}{2} + \dfrac{b}{6} = \dfrac{1}{6}.$

Hence, the correct answer is (D).

62. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\dfrac{xf(x)}{x} = \lim\limits_{x\to\infty}\left(f(x) + xf'(x)\right) = a$

$\Rightarrow\ \lim\limits_{x\to\infty} xf'(x) = 0$

Now, $\lim\limits_{x\to\infty} xf'(x) = \lim\limits_{x\to\infty}\dfrac{x^2 f'(x)}{x}$ (Using L-H rule)

$$= \lim_{x\to\infty}\left(2xf'(x) + x^2 f''(x)\right) = 0$$

$\Rightarrow\ \lim\limits_{x\to\infty} x^2 f''(x) = 0.$

Hence, the correct answer is (B).

63. $$\lim_{x\to 1}\frac{\left(\dfrac{1-x}{1-x}\right)\left(\dfrac{1-x^2}{1-x}\right)\ldots\ldots\left(\dfrac{1-x^{2n}}{1-x}\right)}{\left[\left(\dfrac{1-x}{1-x}\right)\left(\dfrac{1-x^2}{1-x}\right)\ldots\ldots\left(\dfrac{1-x^n}{1-x}\right)\right]^2} = \frac{1.2.3\ldots.2n}{(1.2.3\ldots.n)^2} = {}^{2n}C_n.$$

Hence, the correct answer is (B).

64. $$\frac{\sin(x_n - x_{n+1})}{\sin x_n \sin x_{n+1}} = \frac{1}{3^{\frac{2n-1}{2}}} \Rightarrow \sum_{n=1}^{\infty}(\cot x_{n+1} - \cot x_n) = \sum_{n=1}^{\infty}\frac{1}{3^{\frac{2n-1}{2}}}$$

$$\Rightarrow\ \lim_{n\to\infty}\cot x_n - \cot x_1 = \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \frac{1}{9\sqrt{3}} + \ldots$$

$$\lim_{n\to\infty}(\cot x_n - \cot x_1) = \frac{\sqrt{3}}{2}$$

$$\Rightarrow\ \lim_{n\to\infty} x_n = \frac{\pi}{6}\ as\ \cot x_1 = \frac{\sqrt{3}}{2}.$$

Hence, the correct answer is (B).

65. $f(2x) = f(x) = f(x/2) = f(x/4) = \ldots = f\left(\dfrac{x}{2^{n-1}}\right) = f\left(\dfrac{x}{2^n}\right)$

Consider $f(x) = f\left(\dfrac{x}{2^n}\right)$

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} f\left(\frac{x}{2^n}\right) \Rightarrow f(x) = f(0)\,\forall x \in R$$

(Since $f(x)$ is given to be a continuous function)

$$\Rightarrow\ f(x) = f(2010) = \frac{\pi}{2} \Rightarrow \lim_{x\to 0}\frac{\cos\dfrac{\pi}{2} - \sqrt[3]{\cos\dfrac{\pi}{2}}}{\sin^2 x} = 0$$

Hence, the correct answer is (C).

66. $$\lim_{n\to\infty}\left(\frac{3}{7}.\frac{7}{13}.\frac{13}{21}\ldots.\frac{n^2-n+1}{n^2+n+1}\right) = \lim_{n\to\infty}\left(\frac{3}{n^2+n+1}\right) = 0$$

Hence, the correct answer is (A).

67. $$\lim_{n\to\infty}\prod_{r=2}^{n}\left(\frac{r^3+1}{r^3-1}\right) = \lim_{n\to\infty}\prod_{r=2}^{n}\left(\frac{r+1}{r-1}\right)\lim_{n\to\infty}\prod_{r=2}^{n}\left(\frac{r^2-r+1}{r^2+r+1}\right)$$

$$= \lim_{n\to\infty}\left(\frac{3}{1}\times\frac{4}{2}\times\frac{5}{3}\times\frac{6}{4}\times\ldots\ldots\frac{r}{n-2}\times\frac{n+1}{n-1}\right)$$

$$\lim_{n\to\infty}\left(\frac{3}{7}\times\frac{7}{13}\times\frac{13}{21}\times\ldots\ldots\frac{n^2-n+1}{n^2+n+1}\right)$$

$$= \lim_{n\to\infty}\frac{n(n+1)}{1.2}\frac{3}{n^2+n+1} = \frac{3}{2}$$

Hence, the correct answer is (C).

68. As $x \to 0, \dfrac{n\sin x}{x} \to n$ and $\dfrac{n\sin x}{x} < n$

Also, $\dfrac{n\tan x}{x} \to n$ and $\dfrac{n\tan x}{x} > n$

$\therefore$ Limit value $= (n-1) + n = 2n - 1$

Hence, the correct answer is (C).

69. $$\lim_{x\to\infty}\left(\sqrt[4]{(x+a)(x+b)(x+c)(x+d)} - x\right)$$

$$= \lim_{x\to\infty}\frac{[(x+a)(x+b)(x+c)(x+d) - x^4]}{\left(\sqrt[4]{(x+a)(x+b)(x+c)(x+d)} + x\right)\left(\sqrt{(x+a)(x+b)(x+c)(x+d)} + x^2\right)}$$

$$= \frac{a+b+c+d}{4} = \frac{10}{4} = \frac{5}{2}$$

Hence, the correct answer is (B).

70. $$\text{RHL} = \lim_{x\to 0^+}\left(\frac{1+x^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}} = \left(\frac{1}{e}\right)^{\infty} = 0\,,$$

$$\text{LHL} = \lim_{x\to 0^-}\left(\frac{1+(1+x)^{\frac{1}{1+x}}}{e}\right)^{\frac{1}{1+x}} = \left(\frac{2}{e}\right)^1 = \frac{2}{e}$$

Hence, the correct answer is (D).

71. Let, $L = \lim\limits_{x\to\infty} x.\ln\left(\dfrac{\alpha}{x^3} + \beta - \dfrac{\gamma}{x}\right)$; for limit to exist, $\beta = 1$

Now, $L = \lim\limits_{x\to\infty} x.\ln\left(1 + \dfrac{\alpha}{x^3} - \dfrac{\gamma}{x}\right) = \lim\limits_{x\to\infty} x\left(\dfrac{\alpha}{x^3} - \dfrac{\gamma}{x}\right)$

$$= \lim_{x\to\infty}\left(\frac{\alpha}{x^2} - \gamma\right) = -\gamma = -5 \text{ and } \alpha \in R$$

Hence, the correct answer is (D).

72. $$P = \lim_{n\to\infty}\frac{n(2+4+\ldots+2n)}{1^2+2^2+\ldots+n^2} = \lim_{n\to\infty}\frac{n.n(n+1)\times 6}{n(n+1)(2n+1)} = 3$$

$$= \lim_{x\to 0} e^{\tan x}\left[\frac{e^{\tan x - \sin x} - 1}{\tan x - \sin x}\right]\frac{(\tan x - \sin x)}{\left(\dfrac{e^{x/32} - 1}{x/32}\right)\dfrac{x^3}{32}\left(\dfrac{\sin x}{x}\right)^2}$$

$$= \lim_{x\to 0}\frac{1.1.(\tan x-\sin x)16}{1.x^3}$$

$$= \left(\frac{1}{3}+\frac{1}{6}\right)16 = 8$$

Equation of circle is

$$x^2+y^2-6x+2fy+9=0 \Rightarrow 2\sqrt{f^2-9}=8 \Rightarrow f=\pm 5$$

Radius = 5.

Hence, the correct answer is (C).

73. $e^{\lim\limits_{n\to\infty} n\left(\cos\frac{x}{\sqrt{n}}-1\right)} = e^{-\lim\limits_{n\to\infty}\frac{1-\cos\left(\frac{x}{\sqrt{n}}\right)}{\left(\frac{x}{\sqrt{n}}\right)^2}\times x^2} = e^{-\frac{x^2}{2}}$

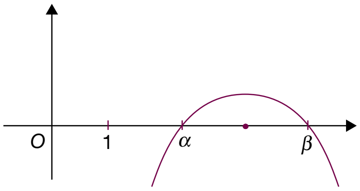
Hence, the correct answer is (D).

74. $\lim\limits_{x\to\infty}\left(e^x+\pi^x\right)^{1/x} = \left\{\pi\lim\limits_{x\to\infty}\left(\left(\frac{e}{\pi}\right)^x+1\right)^{1/x}\right\} = \{\pi\}$

$$\{\pi\} = \pi - 3$$

Hence, the correct answer is (B).

75. $\lim\limits_{x\to a}\frac{\left|px^2+qx+r\right|}{px^2+qx+r} = -1 \Rightarrow pa^2+qa+r<0$



$p < 0$, $a < \alpha$ (or) $\alpha > \beta$

Hence, the correct answer is (C).

76. $\lim\limits_{n\to\infty}\frac{3^3-8}{3^3+8}\cdot\frac{4^3-8}{4^3+8}\cdots\cdots\frac{n^3-8}{n^3+8}$

$$\lim_{n\to\infty}\left(\frac{3-2}{3+2}\cdot\frac{3^2+4+2(3)}{3^2+4-2(3)}\right)\cdot\left(\frac{4-2}{4+2}\cdot\frac{4^2+4+2(4)}{4^2+4-2(4)}\right)\cdots\left(\frac{n-2}{n+2}\cdot\frac{n^2+4+2n}{n^2+4-2n}\right)$$

$$\lim_{n\to\infty}\left(\frac{3-2}{3+2}\cdot\frac{4-2}{4+2}\cdot\frac{5-2}{5+2}\cdot\frac{n-2}{n+2}\right)\left(\frac{3^2+4+2(3)}{3^2+4-2(3)}\cdot\frac{4^2+4+2(4)}{4^2+4-2(4)}\cdots\frac{n^2+4+2n}{n^2+4-2n}\right)$$

$$= \lim_{n\to\infty}\left(\frac{1.2.3.4.5.6}{5.6.7.8}\right)\left(\frac{19.28.39.52.63}{7.12.19.28.39.52.63}\right)$$

$$= \lim_{n\to\infty}\frac{\left(n^2+5\right)\left(n^2+2n+4+4\right)(1.2.3.4)}{(n-1)n(n+1)(n+2)}\cdot\frac{1}{7.12} = \frac{2}{7}$$

Hence, the correct answer is (A).

77. $f(x) = \lim\limits_{n\to\infty}\sum\limits_{r=0}^{n}\tan\frac{x}{2^{r+1}}\sec\frac{2x}{2^{r+1}}$

$$= \lim_{n\to\infty}\frac{\sin\left(\frac{x}{2^r}-\frac{x}{2^{r+1}}\right)}{\cos\frac{x}{2^{r+1}}\cos\frac{x}{2^r}}$$

$$= \lim_{n\to\infty}\sum_{r=0}^{n}\left(\tan\frac{x}{2^r}-\tan\frac{x}{2^{r+1}}\right) = \tan x$$

$$\lim_{x\to 0}\frac{f(x)-\sin x}{x^3} = \frac{1}{2}$$

Hence, the correct answer is (D).

78. We have $a_n+\sqrt{2}b_n = \left(2+\sqrt{2}\right)^n$

$$\Rightarrow \quad a_n-\sqrt{2}b_n = \left(2-\sqrt{2}\right)^n$$

Therefore, $a_n = \frac{1}{2}\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]$

And, $b_n = \frac{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]}{2\sqrt{2}}$

Therefore,

$$\frac{a_n}{b_n} = \sqrt{2}\frac{\left[\left(2+\sqrt{2}\right)^n+\left(2-\sqrt{2}\right)^n\right]}{\left[\left(2+\sqrt{2}\right)^n-\left(2-\sqrt{2}\right)^n\right]} = \sqrt{2}\frac{\left[1+\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}{\left[1-\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n\right]}$$

Hence, $\lim\limits_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \sqrt{2}\left(\frac{1+0}{1-0}\right) \qquad \left(\because \frac{2-\sqrt{2}}{2+\sqrt{2}}<1\right) = \sqrt{2}$

Hence, the correct answer is (A).

79. Let $f(x) = \frac{(1-\sin x)\left(8x^3-\pi^3\right)\cos x}{(\pi-2x)^4}$

$$= \frac{(1-\sin x)\cos x(2x-\pi)\left(4x^2+2\pi x+\pi^2\right)}{(2x-\pi)^4}$$

$$= \frac{(1-\sin x)\cos x\left(4x^2+2\pi x+\pi^2\right)}{(2x-\pi)^3}$$

Therefore, $\lim\limits_{x\to\frac{\pi}{2}} f(x) = \lim\limits_{x\to\frac{\pi}{2}}\frac{(1-\sin x)\cos x}{(2x-\pi)^3}.\left(3\pi^2\right)$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}}\frac{(1-\sin x)\cos x}{(2x-\pi)^3}.\left(3\pi^2\right) \qquad \ldots\text{(i)}$$

Put $2x-\pi = y$ so that $y\to 0$ as $x\to\frac{\pi}{2}$. Therefore,

$$\frac{(1-\sin x)\cos x}{(2x-\pi)^3} = \frac{\left[1-\sin\left(\frac{\pi+y}{2}\right)\right]\cos\left(\frac{\pi+y}{2}\right)}{y^3}$$

$$= \frac{\left(1-\cos\frac{y}{2}\right)\left(-\sin\frac{y}{2}\right)}{y^3} = -\left(\frac{2\sin^2\frac{y}{4}}{y^2}\right)\left(\frac{\sin\frac{y}{2}}{y}\right)$$

$$= -2\left(\frac{\sin\frac{y}{4}}{y/4}\right)^2.\frac{1}{16}.\left(\frac{\sin\frac{y}{2}}{y/2}\right).\frac{1}{2}$$

$$= \frac{-1}{16}\left(\frac{\sin\frac{y}{4}}{\frac{y}{4}}\right)^2\left(\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right) \qquad \ldots\text{(ii)}$$

Therefore, from Eqs. (i) and (ii)

$$\lim_{x\to\frac{\pi}{2}} f(x) = \frac{-3\pi^2}{16}\times 1\times 1$$

Hence, the correct answer is (D).

80. Given, relation is $f\left(x.f\left(y\right)\right)=f\left(xy\right)+x$ ...(i)

Interchanging x and y in Eq. (i), we have

$$f\left(y.f\left(x\right)\right)=f\left(yx\right)+y \quad \text{...(ii)}$$

Again, replacing x with $f\left(x\right)$ in Eq. (i) we get

$$f\left(f\left(x\right).f\left(y\right)\right)=f\left(y.f\left(x\right)\right)+f\left(x\right) \quad \text{...(iii)}$$

Therefore, Eqs. (i) – (iii) imply

$$f\left(f\left(x\right).f\left(y\right)\right)=f\left(xy\right)+y+f\left(x\right) \quad \text{...(iv)}$$

Again, interchanging x and y in Eq. (iv), we have

$$f\left(f\left(y\right).f\left(x\right)\right)=f\left(yx\right)+x+f\left(y\right) \quad \text{...(v)}$$

Equations (iv) and (v) imply

$$f\left(xy\right)+y+f\left(x\right)=f\left(yx\right)+x+f\left(y\right) \quad \text{...(vi)}$$

Suppose $f\left(x\right)-x=f\left(y\right)-y=\lambda$

Substituting $f\left(x\right)=\lambda+x$ in Eq. (i), we have

$$x.f\left(y\right)+\lambda=\left(xy+\lambda\right)+x$$

$$\Rightarrow \quad x.f\left(y\right)=xy+x$$

Therefore, $x\left(y+\lambda\right)=xy+x$ $\quad\left[\because f\left(y\right)=\lambda+y\right]$

$$\Rightarrow \quad \lambda x=x \quad \Rightarrow \quad \lambda=1 \left(\because x>0\right)$$

So $\quad f\left(x\right)=x+\lambda=x+1$

Hence, $\lim\limits_{x\to 0}\dfrac{\left(f\left(x\right)\right)^{1/3}-1}{\left(f\left(x\right)\right)^{1/2}-1}=\lim\limits_{x\to 0}\dfrac{(1+x)^{1/3}-1}{(1+x)^{1/2}-1}$

$$=\lim_{x\to 0}\left(\frac{(1+x)^{1/3}-1}{1+x-1}\right).\left(\frac{1+x-1}{(1+x)^{1/2}-1}\right)$$

$$=\frac{1/3}{1/2}=\frac{2}{3}$$

Hence, the correct answer is (C).

81. We have $x_1=1, x_2=\dfrac{4+3}{3+2}=\dfrac{7}{5}$

$$x_3=\frac{4+3x_2}{3+2x_2}=\frac{4+3\left(\frac{7}{5}\right)}{3+2\left(\frac{7}{5}\right)}=\frac{41}{29}>x_2$$

We can easily verify that $x_n<x_{n+1}$ and hence $\left\{x_n\right\}$ is strictly increasing sequence of positive terms. Let $\lim\limits_{n\to\infty}x_n=l$. Therefore,

$$l=\lim_{n\to\infty}x_{n+1}$$

$$=\lim_{n\to\infty}\left(\frac{4+3x_n}{3+2x_n}\right)=\frac{4+3\lim\limits_{n\to\infty}x_n}{3+2\lim\limits_{n\to\infty}x_n}$$

$$=\frac{4+3l}{3+2l}$$

Hence, $3l+2l^2=4+3l$

or $\quad l^2=2\Rightarrow l=\sqrt{2}$ $\quad\left(\because x_n>0 \ \forall \ n\right)$.

Hence, the correct answer is (A).

82. We have $f\left(x\right)=\dfrac{x^3\left\{x^2+\sqrt{x^4+1}-2x^2\right\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$

$$=\frac{x^3\left\{\sqrt{x^4+1}-x^2\right\}}{\sqrt{x^2+\sqrt{x^4+1}}+x\sqrt{2}}$$

$$=\frac{x^3}{\left[\sqrt{x^2+\sqrt{x^4+1+}}+x\sqrt{2}\right]\left[\sqrt{x^4+1}+x^2\right]}$$

$$=\frac{1}{\left[\sqrt{1+\sqrt{1+\frac{1}{x^4}}}+\sqrt{2}\right]\left[\sqrt{1+\frac{1}{x^4}}+1\right]}$$

$$\Rightarrow \quad \lim_{x\to\infty}f(x)=\frac{1}{\left(\sqrt{1+\sqrt{1}}+\sqrt{2}\right)\left(\sqrt{1}+1\right)}=\frac{1}{2\sqrt{2}\left(2\right)}=\frac{1}{4\sqrt{2}}.$$

Hence, the correct answer is (B).

83. $a_1=1, a_2=1-\dfrac{1}{2}, a_3=1-\dfrac{1}{2}+\dfrac{1}{3}\ldots$

$$a_n=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+(-1)^{n-1}.\frac{1}{n}$$

$$\lim_{n\to\infty}a_n=\log_e 2$$

Hence, the correct answer is (D).

84. LHL $=\lim\limits_{x\to\frac{\pi^-}{2}}\left[\dfrac{[\sin x]-[\cos x]+1}{3}\right]=\left[\dfrac{0-0+1}{3}\right]=0$

RHL $=\lim\limits_{x\to\frac{\pi^+}{2}}\left[\dfrac{[\sin x]-[\cos x]+1}{3}\right]=\left[\dfrac{0-(-1)+1}{3}\right]$

$$=\left[\frac{2}{3}\right]=0$$

$\because \quad$ LHL = RHL = 0

Hence, the correct answer is (A).

85. $\lim\limits_{x\to 0}e^{\frac{1}{x^2}(\log(1+2x)-\log(1+3x)+\frac{1}{x}}$

$$=e^{\lim\limits_{x\to 0}\frac{(\log(1+2x)-\log(1+3x)+x}{x^2}}$$

$$=e^{\lim\limits_{x\to 0}\frac{\left(2x-\frac{(2x)^2}{2}+\ldots-3x+\frac{(3x)^2}{2}-\ldots+x\right)}{x^2}}$$

$$=e^{\lim\limits_{x\to 0}\frac{\left(\frac{5}{2}x^2+\ldots\right)}{x^2}}=e^{5/2}$$

Hence, the correct answer is (A).

86. $\lim\limits_{x\to\infty}x^{1/3}\left\{(x+1)^{1/3}+(x-1)^{1/3}\right\}\left\{(x+1)^{1/3}-(x-1)^{1/3}\right\}$

Rationalize $\lim\limits_{x\to\infty}\dfrac{x^{1/3}\left\{(x+1)^{1/3}+(x-1)^{1/3}\right\}\{(x+1)-(x-1)\}}{\left\{(x+1)^{2/3}+(x^2-1)^{1/3}+(x-1)^{2/3}\right\}}$

$$\lim_{x\to\infty}\frac{2.x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{1/3}\right\}}{x^{2/3}\left\{\left(1+\frac{1}{x}\right)^{2/3}+\left(1-\frac{1}{x}\right)^{1/3}+\left(1-\frac{1}{x}\right)^{2/3}\right\}}=\frac{2\times 2}{3}=\frac{4}{3}$$

Hence, the correct answer is (D).

87. Let $\dfrac{1}{n}=x, \Rightarrow x\to 0 \ \ as \ \ n\to\infty$ then required

$$\text{limit} \lim_{x\to 0}\left(\frac{a-1+b^x}{a}\right)^{\frac{1}{x}}=e^{\lim\limits_{x\to 0}\frac{b^x-1}{x.a}}$$

$$=e^{\frac{1}{a}\log e^b}=\left(b^{\frac{1}{a}}\right)$$

Hence, the correct answer is (A).

88. $(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\log(1+x)} = e^{(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}......)}$

$$\lim_{x\to 0}\frac{e^{1-\frac{x}{2}+\frac{x^2}{3}-....}-e}{x} = \lim_{x\to 0}\frac{e\left(e^{-\frac{x}{2}+\frac{x^2}{3}-....}-1\right)}{x}$$

(Using L'Hospital Rule)

$$= \lim_{x\to 0}\frac{e\left(e^{-\frac{x}{2}+\frac{x^2}{3}-....}-0\right).\left(-\frac{1}{2}+\frac{2x}{3}-.....\right)}{1} = -\frac{e}{2}$$

Hence, the correct answer is (B).

89. $\text{LHL} = \lim_{x\to 0^-} f(0-h) = \lim_{h\to 0}\frac{\sin[\cos h]}{1+[\cos h]}$

$= \frac{\sin(0)}{1+0} = 0 \quad \left(\begin{array}{l}\because\ h>0\\ \therefore\ \cos h<1\end{array}\right)$

$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{\sin[\cos h]}{1+[\cos h]}$

$= \frac{\sin(0)}{1+0} = 0 \quad \left(\begin{array}{l}\because\ h>0\\ \therefore\ \cos h<1\end{array}\right)$

$\therefore \quad \lim_{x\to 0}\frac{\sin[\cos x]}{1+[\cos x]} = 0$

Hence, the correct answer is (B).

90. $\lim_{x\to 0}\frac{\cos(\sin^2 x)-\cos(x^2)}{x^6}$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{\sin^2 x+x^2}{2}\right).\sin\left(\frac{x^2-\sin^2 x}{2}\right)}{x^6}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\dots\right)^2+x^2}{2}\right).\sin\left(\frac{x^2-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\dots\right)^2}{2}\right)}{x^6}$$

$$= \lim_{x\to 0}\frac{2\sin\left(\frac{2x^2-\frac{2x^4}{6}\dots}{2}\right)\sin\left(\frac{x^4}{6}\dots\right)}{x^2\times 6.\frac{x^4}{6}} = \lim_{x\to 0}\frac{2\sin\left(x^2-\frac{x^4}{6}.....\right)}{x^2}.\frac{1}{6} = \frac{1}{3}$$

Hence, the correct answer is (C).

91. Given, $\cos^{-1}\beta_1+\cos^{-1}\beta_2+\dots+\cos^{-1}\beta_k = k\frac{\pi}{2}$

We know that, $\cos^{-1}x \le \frac{\pi}{2}\ \forall r \ge 0$

$\therefore \quad \cos^{-1}\beta_r \le \frac{\pi}{2}\ \forall r = 1, 2, 3, \dots, k \Rightarrow \sum_{r=1}^{k}\cos^{-1}\beta_r \le \frac{k\pi}{2}$

So, the given equality holds only if

$\cos^{-1}\beta_1 = \cos^{-1}\beta_2 = ... = \cos^{-1}\beta_k = \frac{\pi}{2}$

$\Rightarrow \quad \beta_1 = \beta_2 = ... = \beta_k = 0$

Thus, $A = \sum_{r=1}^{k}(\beta_r)^r = 0$

Required limit $= \lim_{x\to 0}\frac{(1+x^2)^{1/3}-(1-2x)^{1/4}}{x+x^2}$

$$= \lim_{x\to 0}\frac{\frac{1}{3}(1+x^2)^{-2/3}(2x)-\frac{1}{4}(1-2x)^{-\frac{3}{4}}(-2)}{1+2x} = \frac{1}{2} \quad \text{(L' Hospital Rule)}$$

Hence, the correct answer is (A).

92. Let $P = \lim_{x\to[a]} = \frac{e^{\{x\}}-\{x\}-1}{\{x\}^2}$

Put $x = [a]+h,\ h>0$

Then, $P = \lim_{h\to 0}\frac{e^{\{[a]+h\}}-\{[a]+h\}-1}{\{[a]+h\}^2}$

$P = \lim_{h\to 0}\frac{e^h-h-1}{h^2}$

$\lim_{h\to 0}\frac{e^h-1}{2h} = \frac{1}{2}$ [Using L'Hospital Rule]

Next put $x = [a]-h, h>0$

Then, $P = \lim_{h\to 0}\frac{e^{\{[a]-h\}}-\{[a]-h\}-1}{\{[a]-h\}^2}$

$= \lim_{h\to 0}\frac{e^{1-h}-(1-h)-1}{(1-h)^2} = \lim_{h\to 0}\frac{e^{1-h}+h-2}{(1-h)^2} = e-2$

$\therefore$ Limit does not exist

Hence, the correct answer is (D).

93. $(f(\alpha))^2+(f'(\alpha))^2 = 0 \Rightarrow f(\alpha) = f'(\alpha) = 0$

Clearly, α is repeated root of $f(x) = 0$

$$\lim_{x\to\alpha}\frac{f(x)}{f'(x)}\left(\frac{f'(x)}{f(x)}-\left\{\frac{f'(x)}{f(x)}\right\}\right) \Rightarrow \lim_{x\to\alpha}\left(1-\frac{f(x)}{f'(x)}\left\{\frac{f'(x)}{f(x)}\right\}\right)$$

$\left[\because\ \lim_{x\to\alpha}\frac{f(x)}{f'(x)}\lim_{x\to\alpha}\frac{f'(x)}{f''(x)} = 0 \text{ and } \left\{\frac{f'(x)}{f(x)}\right\}\right)$ is bounded function)

Hence, the correct answer is (B).

94. For $a-1<x<a \Rightarrow \left[\frac{x}{a}\right] = 0$

$\ln_{x\to a^-}\left(\frac{|x|^3}{a}-0\right) = \frac{a^3}{a} = a^2$

Hence, the correct answer is (C).

95. $\sin x = t;\ x \to \frac{\pi}{2}$ than $t \to 1$

$\lim_{x\to 1}\frac{t-t^t}{1-t+\log t}$

Now, using L'Hospital rule two times.

$$\lim_{t\to 1}\frac{1-t^t(1+\ln t)}{-1+\frac{1}{t}} = \lim_{t\to 1}\frac{t-t^{t+1}(1+\ln t)}{1-t}$$

$$= \lim_{t\to 1}\frac{1-t^{t+1}\times\frac{1}{t}-(1+\ln t).t^{t+1}\left(1+\frac{1}{t}+\ln t\right)}{(-1)}$$

$$= \frac{1-1-(1+1+0)}{-1} = 2$$

Hence, the correct answer is (C).

96. $e^{\lim_{x\to 1}\left(\frac{4}{\pi}\tan^{-1}x-1\right)\frac{1}{x^2-1}} = e^{\lim_{x\to 1}\frac{4}{\pi}\frac{\left(\tan^{-1}x-\tan^{-1}1\right)}{x^2-1}} = e^{\lim_{x\to 1}\frac{4}{\pi}\frac{\tan^{-1}\left(\frac{x-1}{1+x}\right)}{\frac{(x-1)}{x+1}(x+1)^2}}$

$= e^{\frac{4}{\pi}.\frac{1}{4}} = e^{\frac{1}{\pi}}$

Hence, the correct answer is (B).

97. $$L=\lim_{x\to 0}\frac{\{(x+a)(x+b)(x+c)\}^{1/3}-(abc)^{1/3}}{x},\qquad \left(\frac{0}{0}\text{ Form}\right)$$

Let $y=\{(x+a)\,(x+b)\,(x+c)\}^{1/3}$

$$\ln y=\frac{1}{3}\,[\ln(x+a)+\ln(x+b)+\ln(x+c)]$$

Differentiation w.r. to x

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{3}\left[\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}\right]$$

$$\frac{dy}{dx}=\frac{y}{3}\left[\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}\right]$$

Now, using L' Hospital's rule

$$L=\lim_{x\to 0}\frac{\{(x+a)(x+b)(x+c)\}^{1/3}\frac{1}{3}\left[\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}\right]-0}{1}$$

$$L=(abc)^{1/3}\frac{1}{3}\left[\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right]$$

Using A.M. ≥ G.M.

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 3\left(\frac{1}{a}\cdot\frac{1}{b}\cdot\frac{1}{c}\right)^{1/3}$$

$\Rightarrow\quad L\ge 1$

Hence, the correct answer is (A).

98. $$L=\lim_{x\to\infty}\frac{f^{-1}(8x)}{x^{1/3}}-\lim_{x\to\infty}\frac{f^{-1}(x)}{x^{1/3}}$$

Put $8x=t$

$$=\lim_{t\to\infty}\frac{f^{-1}(t)}{\left(\frac{t}{8}\right)^{1/3}}-\lim_{x\to\infty}\frac{f^{-1}(x)}{x^{1/3}};=2\lim_{t\to\infty}\frac{f^{-1}(t)}{\left(\frac{t}{8}\right)^{1/3}}-\lim_{x\to\infty}\frac{f^{-1}(x)}{x^{1/3}}$$

$$=2\lim_{t\to\infty}\frac{f^{-1}(t)}{t^{1/3}}-\lim_{x\to\infty}\frac{f^{-1}(x)}{t^{1/3}}=\lim_{t\to\infty}\frac{f^{-1}(t)}{t^{1/3}}$$

$$f^{-1}(t)=x=\lim_{x\to\infty}\frac{x}{f(x)^{\frac{1}{3}}}$$

$$=\lim_{x\to\infty}\frac{x}{(8x^3+3x)^{1/3}}=\lim_{x\to\infty}\frac{1}{\left(8+\frac{3}{x^2}\right)^{1/3}}=\frac{1}{2}$$

Hence, the correct answer is (A).

99. $$\lim_{x\to\infty}\frac{\cot^{-1}\left(\frac{x+1-x}{\sqrt{x+1}+\sqrt{x}}\right)}{\sec^{-1}\left(\left(\frac{2x+1}{x-1}\right)^{x}\right)}=\lim_{x\to\infty}\frac{\cot^{-1}\left(\frac{1}{\infty}\right)}{\sec^{-1}(2^{\infty})}=\frac{\frac{\pi}{2}}{\frac{\pi}{2}}=1$$

Hence, the correct answer is (A).

100. $$\lim_{x\to 1}\frac{nx^{n+1}-(n+1)x^n+1}{(e^x-e)\sin\pi x}\qquad n=100$$

$$\lim_{x\to 1}\frac{nx^n(n+1)-(x^n-1)}{e\frac{(e^{x-1}-1)}{x-1}(x-1)\frac{\sin(\pi-\pi x)}{(\pi-\pi x)}(\pi-\pi x)}$$

$$=\lim_{x\to 1}\frac{nx^n(x-1)-(x^n-1)}{e\pi(1)(1)(x-1)(1-x)}$$

$$=\lim_{x\to 1}\frac{nx^n-(x^{n-1}+x^{n-2}+\ldots.+x+1)}{e\pi(1-x)}$$

$$=\lim_{x\to 1}\frac{n^2x^{x-1}-[(n-1)x^{n-2}+(n-2)x^{n-3}+\ldots.+2x+1]}{e\pi(0-1)}$$

$$=\frac{n^2-\left[\frac{(n-1)(n-1+1)}{2}\right]}{-e\pi}=\frac{n^2-\frac{n(n-1)}{2}}{-e\pi}$$

$$=\frac{\frac{n^2}{2}+\frac{n}{2}}{-e\pi}=\frac{n(n+1)}{-2e\pi}=\frac{100\cdot 101}{-2(e\pi)}=-\frac{5050}{e\pi}$$

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1. $$\lim_{x\to 0}\frac{x\left(1+a\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\right)\right)-b\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\right)}{x^3\left(\frac{f(x)}{x}\right)^3}=1$$

$$=\lim_{x\to 0}\left\{\left(\frac{1+a-b}{x^2}\right)+\left(\frac{-a}{2!}+\frac{b}{3!}\right)+x^2\left(\frac{a}{4!}-\frac{b}{5!}\right)+\text{higher}\ldots\text{of }x^2\right\}$$

limit exists for $1+a-b=0$ and $-\frac{a}{2}+\frac{b}{6}=1,\Rightarrow a=-\frac{5}{2},b=-\frac{3}{2}$

Hence, (A) and (D) are correct.

2. $$\lim_{x\to\infty}\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)=\lim_{x\to\infty}\sqrt{x}\frac{(x+1-x)}{\sqrt{x+1}+\sqrt{x}}=\frac{1}{\sqrt{1+0}+\sqrt{1}}$$

(A) $$\lim_{x\to 0}\frac{\log(1+x)-x}{x^2}=\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{2x}$$

$$=\lim_{x\to 0}\frac{-\frac{1}{(1+x)^2}}{2}=-\frac{1}{2}$$

(B) $$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\frac{1}{2}$$

(C) $$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}}{1}=\frac{1}{2}$$

(D) $$\lim_{x\to 0}\frac{\sqrt{x}}{\sqrt{x+\sqrt{x^2+2x}}}=\lim_{x\to 0}\frac{1}{\sqrt{1+\sqrt{1+\frac{2}{x}}}}=\frac{1}{\sqrt{2}}$$

Hence, (B) and (C) are correct.

3. Let $f(x)=\dfrac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}$

$$\text{LHL}=\lim_{x\to 0^-}\frac{e^{+\frac{1}{2x}}-1}{e^{\frac{1}{2x}}+1}=\frac{0-1}{0+1}=-1;$$

$$\text{RHL}=\lim_{x\to 0^+}\frac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}=\lim_{x\to 0^+}\frac{1-e^{-\frac{2}{x}}}{1+e^{-\frac{2}{x}}}=1$$

LHL and RHL are not equal at $x=0$

$$\lim_{x\to 0}g(x)\left[\frac{e^{\frac{1}{x}}-e^{-\frac{1}{x}}}{e^{\frac{1}{x}}+e^{-\frac{1}{x}}}\right]\text{ is exists for }x\text{ or }x^2$$

Hence, (A) and (C) are correct.

4. $$\lim_{x\to 0}\frac{a\left(1+x+\frac{x^2}{2}+\ldots\right)+b\left(1-\frac{x^2}{2}\ldots\right)+c\left(1-x+\frac{x^2}{2}\right)}{x^2},$$

$$\left(\frac{x}{e^x-1}\right)^2=4$$

$\Rightarrow \quad a+b+c=0,\ a-c=0 \Rightarrow a=c$

$$\frac{a-b+c}{2}=4 \Rightarrow a-b+c=8 \Rightarrow 2b=-8 \Rightarrow b=-4$$

$\Rightarrow \quad a=c=2$

Hence, (A), (B) and (C) are correct.

5. $$L_1=\lim_{x\to\alpha}\frac{1-\cos a(x-\alpha)(x-\beta)}{(x-\alpha)^2}=\frac{a^2(\alpha-\beta)^2}{2}$$

$$\lim_{x\to\alpha}\frac{1-\cos\{a(x-\alpha)(x-\beta)a^2(\alpha-\beta)^2}{a^2(x-\alpha)^2(x-\beta)^2}=\frac{a^2(\alpha-\beta)^2}{2}$$

$$L_2=\frac{a^2(\beta-\alpha)^2}{2},L_3=\frac{a^2}{2}$$

Hence, (A), (B), (C) and (D) are correct.

6. $$\lim_{x\to\infty}4x\left(\tan^{-1}\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right)=y^2+4y+5$$

$$\Rightarrow \quad \lim_{x\to\infty}4x\left(\frac{\tan^{-1}\frac{1}{2x+3}}{\frac{1}{2x+3}}\right)\frac{1}{2x+3}=y^2+4y+5$$

$\Rightarrow \quad y^2+4y+5=2 \Rightarrow y=-1,-3$

Hence, (B) and (D) are correct.

7. $$(\sin\alpha+\sin\beta)+\frac{\sin\alpha}{\sin\beta}=0 \Rightarrow \frac{\sin\alpha}{\sin\beta}=-(\sin\alpha+\sin\beta)$$

Now, $(\sin\alpha+\sin\beta)\left(\dfrac{\sin\alpha}{\sin\beta}\right)=-1$

$\Rightarrow \quad (\sin\alpha+\sin\beta)(\sin\alpha+\sin\beta)=1$

$\Rightarrow \quad \sin\alpha+\sin\beta=1$ (Not possible)

$\Rightarrow \quad \sin\alpha+\sin\beta=-1$

$\Rightarrow \quad \alpha=\beta=-\dfrac{\pi}{6}$

$$\lambda=\lim_{n\to\infty}\frac{1+\left(2\sin\left(-\frac{\pi}{6}\right)\right)^{2n}}{\left(2\sin\left(-\frac{\pi}{6}\right)\right)^{2n}}=\frac{1+1}{1}=2$$

Hence, (A) and (B) are correct.

8. If n is odd $\left\{\left(\sqrt{2}+1\right)^n\right\}=\left(\sqrt{2}-1\right)^n$ and $\left[\left(\sqrt{2}+1\right)^n\right]=$ an even integer

If n is even $\left\{\left(\sqrt{2}+1\right)^n\right\}=1-\left(\sqrt{2}-1\right)^n$ and $\left[\left(\sqrt{2}+1\right)^n\right]$ = an odd integer

$\Rightarrow$ Limit is 0 if n is even number

$\Rightarrow$ Limit is – 1 if n is odd number

Hence, (C) and (D) are correct.

9. $$\lim_{x\to[a]^-}f(x)=\lim_{h\to 0}\frac{e^{1-h}-(1-h)-1}{(1-h)^2}=e-2$$

$$\lim_{x\to[a]^+}f(x)=\lim_{h\to 0}\frac{e^h-h-1}{h^2}=P \quad \text{(say)} \qquad \ldots(1)$$

Replace h by $-h$ in P

$$P=\lim_{h\to 0}\frac{e^{-h}+h-1}{h^2} \qquad \ldots(2)$$

$$(1)+(2) \Rightarrow 2P=\lim_{h\to 0}\frac{(e^h-1)^2}{e^h.h^2}=1 \quad \Rightarrow \quad P=\frac{1}{2}$$

Hence, (A) and (B) are correct.

10. Given, $L=\lim\limits_{x\to 0}\dfrac{a-\sqrt{a^2-x^2}-\frac{x^2}{4}}{x^4}$, $a>0$ and L is finite.

Now, $$L=\lim_{x\to 0}\frac{\frac{x}{\sqrt{a^2-x^2}}-\frac{x}{2}}{4x^3} \qquad \text{(Using L'Hospital's rule)}$$

$$=\lim_{x\to 0}\frac{\frac{1}{\sqrt{a^2-x^2}}-\frac{1}{2}}{4x^2}$$

$\because$ L is finite, limiting value of numerator should be zero which is so when $\dfrac{1}{\sqrt{a^2}}-\dfrac{1}{2}=0$

i.e., $a=2$ $(\because a>0)$

Applying L'Hospital's rule again, we get

$$L=\lim_{x\to 0}\frac{\frac{x}{(a^2-x^2)^{3/2}}}{8x}$$

$$=\lim_{x\to 0}\frac{1}{8(a^2-x^2)^{3/2}}=\frac{1}{8\times a^3}=\frac{1}{8\times 8} \qquad (\text{Using } a=2)$$

Hence, (A) and (C) are correct.

11. (A) $$\lim_{x\to 0+}x\left(\frac{e^{|x|+[x]}-2}{|x|+[x]}\right)=\lim_{x\to 0+}x\left(\frac{e^{x+0}-2}{x+0}\right)$$

$$=\lim_{x\to 0^+}\left(e^x-2\right)=1-2=-1$$

(B) $$\lim_{x\to 0^-}x\left(\frac{e^{|x|+[x]}-2}{|x|+[x]}\right)=\lim_{x\to 0^-}x\left(\frac{e^{-x-1}-2}{-x-1}\right)=0$$

Hence, (A) and (B) are correct.

12. $$\text{LHL}=\lim_{h\to 0}\left[\frac{|\sin^{-1}h|}{0-h}\right]=\lim_{h\to 0}\left[\frac{\sin^{-1}h}{-h}\right]=-2$$

$\because$ At $x\to 0$ $\dfrac{\sin^{-1}x}{x}>1$

$$\text{RHL}=\lim_{h\to 0}\left[\frac{\left|\sin^{-1}h\right|}{h}\right]=1$$

Hence, (A) and (C) are correct.

13. $$=\lim_{x\to\infty}\frac{1+x^{-1}+\frac{1}{x^p}}{x^{q-p}+x^{q-2-p}+\frac{2}{x^p}}=\frac{1+0+0}{\infty}=0;\ p<q$$

$$\lim_{x\to\infty}\frac{x^p+x^{p-1}+1}{x^p+x^{p-2}+2}=\frac{1+\frac{1}{x}+\frac{1}{x^p}}{1+\frac{1}{x^2}+\frac{2}{x^p}}=1;\ p=q$$

$$\lim_{x\to\infty}\frac{x^{p-q}+x^{p-1-q}+\frac{1}{x^q}}{1+\frac{1}{x^2}+\frac{2}{x^q}}=\infty;\ p>q$$

Hence, (A), (B) and (C) are correct.

14. $\lim\limits_{x\to m}\dfrac{|a(x-\alpha)(x-\beta)|}{a(x-\alpha)(x-\beta)}=1$

When $a>0$, $m<1 \Rightarrow (m-\alpha)\,(m-\beta)>0$

$\Rightarrow\ a(m-\alpha)\,(x-\beta)>0 \Rightarrow \lim\limits_{x\to m}\dfrac{a(x-\alpha)(x-\beta)}{a(x-\alpha)(x-\beta)}=1$

When $a<0$, $\alpha<m<\beta \Rightarrow (m-\alpha)\,(m-\beta)<0$

$\Rightarrow\ a(m-\alpha)\,(m-\beta)>0$

$\Rightarrow\ \lim\limits_{x\to m}\dfrac{a(x-\alpha)(x-\beta)}{a(x-\alpha)(x-\beta)}=1$

Hence, (A) and (B) are correct.

15. Given, $f(x)=(e^x+2x)^{\frac{3}{x}}$

Now, $\lim\limits_{x\to 0}f(x)=\lim\limits_{x\to 0}(e^x+2x)^{\frac{3}{x}}(1^\infty)=e^L$, where

$$L=\lim_{x\to 0}(e^x+2x-1)\cdot\frac{3}{x}=3\lim_{x\to 0}\left(\frac{e^x+2x+1}{x}\right)$$

$$=3\lim_{x\to 0}\left(\frac{e^x+1}{x}+2\right)=3(1+2)=9$$

So, $\lim\limits_{x\to 0}f(x)=e^9$

Also, $\lim\limits_{x\to\infty}f(x)=\lim\limits_{x\to\infty}(e^x+2x)^{\frac{3}{x}}(\infty^0)$

$$=\lim_{x\to\infty}\left(e^x\left(1+\frac{2x}{e^x}\right)\right)^{\frac{3}{x}}=e^3\cdot\lim_{x\to\infty}\left(1+\frac{2x}{e^x}\right)^{\frac{3}{x}}=e^3(1)^0=e^3.$$

Hence, (B) and (C) are correct.

16. $\lim\limits_{x\to 0}\dfrac{af(2x)+2bf\left(\dfrac{x}{2}\right)-3f(x)}{\left(\dfrac{\sin^{-1}x^2}{x^2}\right)x^2}$

$$=\lim_{x\to 0}\frac{af(2x)+2bf\left(\frac{x}{2}\right)-3f(x)}{x^2}=3$$

For limit to exist, $a+2b=3$ …(i), using LH Rule

$$\lim_{x\to 0}\frac{2af'(2x)+bf'\left(\frac{x}{2}\right)-3f'(x)}{2x}=3$$

For limit to exist $2a+b=3$

$$\lim_{x\to 0}\frac{4af'(2x)+\frac{b}{2}f''\left(\frac{x}{2}\right)-3f''(x)}{2}=3;\ \frac{\left(4a+\frac{b}{2}-3\right)f''(0)}{2}$$

We got $a+2b=3 \Rightarrow a=3-2b$

$\left(4+\dfrac{1}{2}-3\right)f''(0)=6$ $\qquad 3b=3$

$\dfrac{3}{2}f''(0)=6$ $\qquad b=1, a=1$

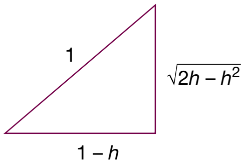
$f''(0)=4$ $\qquad a=b$

$\qquad a^2+b^2=2$

Hence, (A), (B) and (C) are correct.

17. $\lim\limits_{x\to 0^+}f(x)=\lim\limits_{h\to 0}\dfrac{\cos^{-1}(1-\{h\})(1-\{h\})}{\sqrt{2\{h\}}(1-\{h\})}$

$$=\lim_{h\to 0}\frac{\cos^{-1}(1-h)\cdot\sin^{-1}(1-h)}{\sqrt{2h}(1-h)}$$



$$=\lim_{h\to 0}\frac{\sin^{-1}\sqrt{2h-h^2}}{\sqrt{2h-h^2}}\ \frac{\sqrt{2h-h^2}}{\sqrt{2h}}\ \frac{\left(\frac{\pi}{2}\right)}{1};$$

$$=\lim_{h\to 0}(1)\sqrt{1-\frac{h}{2}}\left(\frac{\pi}{2}\right)=\frac{\pi}{2}$$

$\lim\limits_{x\to 0^-}f(x)=\lim\limits_{h\to 0}\dfrac{\cos^{-1}(1-(1-h))\cdot\sin^{-1}(1-(1-h))}{\sqrt{2(1-h)}\cdot(1-(1-h))}$

$x=-h;\ \{-h\}=1-h$

$$=\lim_{h\to 0}\frac{\cos^{-1}h}{\sqrt{2(1-h)}}\ \frac{\sin^{-1}h}{h}=\frac{\frac{\pi}{2}}{\sqrt{2}}\cdot 1=\frac{\pi}{2\sqrt{2}}$$

Hence, (A), (B) and (D) are correct.

18. $\lim\limits_{x\to\infty}\left(\left(\cos\dfrac{k\pi}{4}\right)^n-\left(\cos\dfrac{k\pi}{0}\right)^n\right)=0$

k is multiple of 24

$$\lim_{x\to\infty}((\cos 6\pi)^n-(\cos 4\pi)^n)=\lim_{x\to\infty}1^n-1^n=0$$

k is multiple of 12

$$\lim_{x\to\infty}(\cos 3\pi)^n-(\cos 2\pi)^n)=\lim_{x\to\infty}(-1)^n-1^n$$

→ n even 0

→ n even –2

k is not divisible by 4 or 6

$$\lim_{x\to\infty}\left(\cos\frac{k\pi}{4}\right)^n-\left(\cos\frac{k\pi}{6}\right)^n$$

$=(-1 \text{ to } 1)^\infty-(-1 \text{ to } 1)^\infty=0-0=0.$

Hence, (A), (C) and (D) are correct.

19. $L=\lim\limits_{x\to 1}\dfrac{\sin(6\cos^{-1}x)}{\sqrt{1-x^2}}$

$$M=\lim_{x\to 1}\frac{1-\cos(6\cos^{-1}x)}{1-x^2}$$

Let $\cos^{-1}x=\theta$

$x=\cos\theta$

$$M=\lim_{\theta\to 0}\frac{1-\cos 6\theta}{\sin^2\theta};=\lim_{\theta\to 0}\frac{1-(1-2\sin^2 3\theta)}{\sin^2\theta}$$

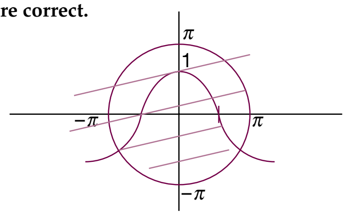
$$=\lim_{\theta\to 0}2\left(\frac{\sin 3\theta}{\sin\theta}\right)^2;=2\left(\frac{3\theta}{\theta}\right)^2=18$$

$L+M=24$

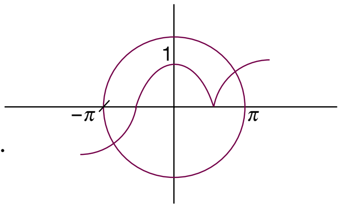
$\dfrac{M}{L}=3,\ L-M=-12,\ LM=108$

Hence, (A), (B), (C) and (D) are correct.

20. $f(x)=\begin{cases}\dfrac{12x-\pi|\cos x}{(\pi-2x)} & x\neq\dfrac{\pi}{2}\\ O & x=\dfrac{\pi}{2}\end{cases}$



$$=\begin{cases}-\cos x & x>\frac{\pi}{2}\\ 0 & x=\frac{\pi}{2}\\ \cos x & x<\frac{\pi}{2}\end{cases}$$



Hence, (B) and (C) are correct.

21. $$\lim_{x\to 0^-} g(f(x)) = \lim_{h\to 0} g\left(\frac{-h}{\tan(-h)}\right) = \lim_{h\to 0} g\left(\underset{<1}{\underset{\downarrow}{\frac{h}{\tan h}}}\right)$$

$x = 0 - h$

$$= \lim_{h\to 0} \frac{h}{\tan h} + 4 = 1 + 4 = 5$$

$$\lim_{x\to 0^+} g(f(x)) = \lim_{h\to 0} g\left(\frac{\ln(1+2h)}{h}\right)$$

$$= \lim_{h\to 0} g\left(\underset{<2}{\underset{\downarrow}{\frac{2\ln(1+2h)}{2h}}}\right) \frac{2\left(2h - \frac{(2h)^2}{2} + \ldots\right)}{2h}$$

$$= \lim_{h\to 0} 2^2\left(\frac{\ln(1+2h)}{2h}\right)^2 - 5\left(\frac{2\ln(1+2h)}{2h}\right) + 11$$

$$= 4 - 5(2) + 11 = 5$$

Hence, (A) and (C) are correct.

22. $$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} \frac{\tan^2\{h\}}{h^2 - [h]^2} = \lim_{h\to 0} \frac{\tan^2 h}{h^2 - 0}$$

$x = 0 + h$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} \sqrt{\{-h\}\cot\{-h\}}$$

$$= \lim_{h\to 0} \sqrt{(1-h)\cot(1-h)} = \sqrt{\cot^{-1}}$$

$$\cot^{-1}\left(\lim_{x\to 0^-} f(x)\right)^2 = \cot^{-1}(\cot^{-1}) = 1$$

Hence, (A) and (C) are correct.

23. $3x + x\lambda + 4y - 2y\lambda - 7 + \lambda = 0$

$(3x + 4y - 7) + \lambda(x - 2y + 1) = 0$

$3x + 4y - 7 = 0$

$x - 2y + 1 = 0 \Rightarrow 2x - 4y + 2 = 0$

$5x - 5 = 0; x = 1$

$1 - 2y + 1 = 0; y = 1$

Fixed point $(a, b) \equiv (1, 1)$

$$\lim_{x\to 0^-} \frac{[\sin x] - 2 + \{\cos x\}}{x - [x] - 1} = \lim_{h\to 0} \frac{[-\sinh] - 2 + \{\cosh\}}{-h - [-h] - 1}$$

$$x = 0 - h = \lim_{h\to 0} \frac{-1 - 2 + \cosh}{-h - (-1) - 1} \to \infty$$

Hence, (A) and (B) are correct.

24. $$L = \lim_{x\to\infty} \frac{2x^3 - (\tan^{-1} x)^3}{\frac{8}{\pi}x^3\cot^{-1}(|kx|) + k^2x^6\sin\frac{1}{x^3} - 3kx^3} = \frac{1}{2}$$

$$L = \lim_{x\to\infty} \frac{2 - \left(\frac{\tan^{-1} x}{x}\right)^3}{\frac{8}{\pi}\cot^{-1}|kx| + \frac{k^2\sin\frac{1}{x^3}}{\frac{1}{x^3}} - 3k}$$

If $k = 0$, then $L = \dfrac{2 - 0}{\frac{8}{\pi}\cdot\frac{\pi}{2}} = \dfrac{1}{2}$

If $k \neq 0$, then $L = \dfrac{2}{0 + k^2 - 3k} = \dfrac{1}{2}$

$k^2 - 3k - 4 = 0 \Rightarrow k = 4, -1.$

Hence, (A), (B) and (D) are correct.

25. (A) $\lim_{x\to 1^+} g(f(x)) = \lim_{x\to 0^-} g(x) =$ Does not exists.

(B) $\lim_{x\to \frac{\pi}{2}^-} g(f(g(x))) = \lim_{x\to 2^-} g(x) = 0$

(C) $\lim_{x\to 2^+} \dfrac{f(g(x))}{f(x) - 2} = 0$

(D) $$\lim_{x\to 0^+} \frac{g(f(x))}{(f(x) - 2)^2} = \lim_{x\to 0^+} \frac{g(2 - x)}{(2 - x - 2)^2}$$

$$= \lim_{x\to 0^+} \frac{[2 - x] - \cos(2 - x - 2)}{x^2}$$

$$= \lim_{x\to 0^+} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

Hence, (B) and (D) are correct.

26. $$f(x) = \frac{1 - \cos x\{x\}}{x^2(x^3 + ax^2 + bx + c)^2}$$

$x^3 + ax^2 + bx + c = (x - 1)(x - 2)(x - 3)$

$= x^3 - 6x^2 + 11x - 6$

$$f(x) = \frac{1 - \cos\{x\}}{x^2((x-1)(x-2)(x-3))^2}$$

Now, $l = \dfrac{1}{8}$, $m = \dfrac{1}{8}$ and $n = \dfrac{1}{72}$

$\therefore \quad l + m + n = \dfrac{19}{72}$

Hence, (B), (C) and (D) are correct.

27. $$L = \lim_{x\to 0} \frac{\sin(x^2) + 2\cos(bx) - ax^4 - 2}{e^{ax} - 1 - ax - 2x^2 - \frac{a^3x^3}{6}}$$

$$= \lim_{x\to 0} \frac{x^2 - \frac{x^6}{3!} + \ldots + 2\left(1 - \frac{b^2x^2}{2!} + \frac{b^4x^4}{4!} + \ldots\right) - ax^4 - 2}{1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \frac{a^4x^4}{4!} + \ldots - 1 - ax - 2x^2 - \frac{a^3x^3}{6}}$$

$$= \lim_{x\to 0} \frac{(1 - b^2)x^2 + \left(\frac{b^2}{12} - a\right)x^4 + \ldots}{\left(\frac{a^2}{2} - 2\right)x^2 + \frac{a^4x^4}{24} + \ldots}$$

$$\frac{1 - b^2}{\frac{a^2}{2} - 2} = \frac{25}{8} \Rightarrow 16 - 16b^2 = 25(a^2 - 4)$$

$$\Rightarrow \quad 25a^2 + 16b^2 = 116 \Rightarrow \left.\begin{matrix} a^2 = 4 \;\&\; b^2 = 1 \\ a = \pm 2, b = \pm 1 \end{matrix}\right\} \quad (\because a, b \in I)$$

Now, $L = \dfrac{\frac{b^2}{12} - a}{\frac{a^4}{24}} = \dfrac{2(b^2 - 12a)}{a^4}$

For $a = 2$, $b = \pm 1$

$\Rightarrow \quad L = \dfrac{2\cdot(1 - 24)}{16} = \dfrac{-23}{8}$

For $a = -2$, $b = \pm 1 \Rightarrow L = \dfrac{2\cdot(1 - 24)}{16} = \dfrac{25}{8}$

Hence, (B) and (D) are correct.

Linked Comprehension Type Questions

1. $$\lim_{x\to 0^+}\frac{x^2\left[-1+\frac{1+\sqrt{1+4x}}{2}\right]}{-1+\frac{1+\sqrt{1+4x^3}}{2}}=\lim_{x\to 0}\frac{x^2\left(\sqrt{1+4x}-1\right)}{\left(\sqrt{1+4x^3}-1\right)}$$

$$=\lim_{x\to 0}\frac{x^2(4x)\left(\sqrt{1+4x^3}+1\right)}{\left(4x^3\right)\left(\sqrt{1+4x}+1\right)}=\frac{(1+1)}{(1+1)}=1$$

Hence, the correct answer is (C).

2. $$\lim_{x\to 0^+}\left(\frac{-1+\frac{1+\sqrt{1+4(\tan x-\sin x)}}{2}}{-1+\frac{1+\sqrt{1+4x^3}}{2}}\right)$$

$$=\lim_{x\to 0^+}\frac{\sqrt{1+4(\tan x-\sin x)}-1}{\sqrt{1+4x^3}-1}$$

$$=\lim_{x\to 0^+}\frac{4(\tan x-\sin x)}{4x^3}\times\left(\frac{2}{2}\right)=\frac{1}{2}$$

Hence, the correct answer is (B).

3. $$\lim_{x\to\frac{\pi}{6}}-\frac{1}{2}+\sqrt{\sin x+\sqrt{\sin x+\ldots..}}=\lim_{x\to\frac{\pi}{6}}\left(-\frac{1}{2}+\frac{1+\sqrt{1+4\sin x}}{2}\right)$$

$$=\frac{1}{2}\sqrt{1+4.\frac{1}{2}}=\frac{\sqrt{3}}{2}=\cos\frac{\pi}{6}$$

Hence, the correct answer is (D).

4. $\lim_{x\to 1}\left[f(x)-\frac{1}{2}\right]$ does not exist

$\Rightarrow$ $f(1)=\frac{1}{2}$ is an integer $\Rightarrow f(1)$ is not an integer

Hence, the correct answer is (C).

5. $f(x)=\left(1-e^x\right)\frac{\sin x}{x}\quad x>0$

$\left(e^x-1\right)\frac{\sin x}{x}\quad x<0\Rightarrow f(x)<0$ for $x>0$

$f(x)<0$ for $x<0\Rightarrow\lim_{x\to 0}\left[f(x)\right]=-1$

Hence, the correct answer is (C).

6. Let $f(x)=\operatorname{cosec}\frac{\pi}{2}x$

$f\left(1^+\right)$ and $f(1)>1$

$$\Rightarrow\lim_{x\to 1}\left[\operatorname{cosec}\frac{\pi}{2}x\right]=1\Rightarrow\lim_{x\to 1}\left[\operatorname{cosec}\frac{\pi}{2}x\right]^{\frac{-1}{1-x}}=1$$

Hence, the correct answer is (B).

Combined Solution to 7–9

$$f(x)=\lim_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}\right)^n=\lim_{n\to\infty}\left(1+\left(\cos\sqrt{\frac{x}{n}}-1\right)\right)^n$$

$$=e^{\lim_{n\to\infty}\left(\cos\sqrt{\frac{x}{n}}-1\right)n}=e^{-\lim_{n\to\infty}2\sin^2\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)n}$$

$$=-2\lim_{n\to\infty}\frac{\left(\sin\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)\right)^2}{\left(\frac{1}{2}\sqrt{\frac{x}{n}}\right)^2}\frac{x}{4}=e^{-2\lim_{n\to\infty}\frac{x}{4}}=e^{-x/2}$$

$y=f(x)=e^{-x/2},x\geq 0$ range $=(0,1]$

$$g(x)=\lim_{n\to\infty}\left(1-x+x\sqrt[n]{e}\right)^n=e^{\lim_{n\to\infty}x\frac{\left(e^{1/n}-1\right)}{1/n}}=e^x\ \forall\ x\in R$$

$h(x)=\tan^{-1}\left(g^{-1}\left(f^{-1}(x)\right)\right)$

$-\frac{x}{2}=\ln y,\ x=2\ln\frac{1}{y},\ f^{-1}(x)=2\ln\frac{1}{x}\quad 0<x\leq 1$

$y=g(x)=e^x\Rightarrow x=\ln y$

$g^{-1}(x)=\ln x$

$\therefore\quad g^{-1}\left(2\ln\frac{1}{x}\right)=\ln\left(2\ln\left(\frac{1}{x}\right)\right)$ for $0<x<1.$

$\therefore\quad h(x)=\tan^{-1}\left(\ln\left(\ln\frac{1}{x^2}\right)\right)$ for $0<x<1$

7. $$\lim_{x\to 0^+}\frac{\ln f(x)}{\ln g(x)}=\lim_{x\to 0}\frac{-x/2}{x}=-\frac{1}{2}$$

Hence, the correct answer is (B).

8. Domain $h(x)$ is $(0, 1)$

Hence, the correct answer is (C).

9. $h(x)=\tan^{-1}(\ln(\ln 1/x^2))\qquad 0<x<1$

$1<\frac{1}{x^2}<\infty\Rightarrow 0<\ln\frac{1}{x^2}<\infty$

$\therefore\quad -\infty<\ln\left(\ln\left(1/x^2\right)\right)<\infty$

$\therefore$ Range of $h(x)$ is $\left(-\pi/2,\ \pi/2\right)$

Hence, the correct answer is (D).

Combined Solution to 10–12

$$\lim_{n\to\infty}f(x)=\lim_{n\to\infty}\frac{n(n-1)(x-2)\ldots(n-x+1)}{x!}\left(\frac{\lambda}{n}\right)^x\left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$=\frac{e^{-\lambda}\lambda^x}{x!}$$

$$\sum_{x=0}^{\infty}xf(x)=\sum_{x=0}^{\infty}x.e^{-\lambda}\frac{\lambda^{x-1}\lambda}{(x-1)!x}=\lambda e^{-\lambda}\sum_{x=1}^{\infty}\frac{\lambda^{x-1}}{(x-1)!}=e^{-\lambda}e^{\lambda}\lambda=\lambda$$

Similarly, $\sum_{x=0}^{\infty}x^2f(x)=\sum_{x=0}^{\infty}x(x-1)f(x)+\sum_{x=0}^{\infty}xf(x)=\lambda^2+\lambda$

$$\sum_{x=0}^{\infty}(x-\lambda)^2f(x)=\sum x^2f(x)+\lambda^2\sum f(x)-2\lambda\sum xf(x)$$

$$=\lambda^2+\lambda-\lambda^2=\lambda$$

10. **Hence, the correct answer is (A).**

11. **Hence, the correct answer is (B).**

12. **Hence, the correct answer is (A).**

13. $\lim\limits_{x\to\infty} \dfrac{x^4 f(x)}{x^8+1} = 3$

$f(x) \to$ degree 4 polynomial.

$$\left| \begin{array}{l} g(x) = f(x) - x^2 - 1 \\ g(2) = g(3) - g(-1) = g(-6) = 0 \end{array} \right.$$

Leading coff. = 3.

$$g(x) = 3(x-2)(x-3)(x+1)(x+6) = f(x) - x^2 - 1$$
$$f(x) = 3(x-2)(x-3)(x+1)(x+6) + x^2 + 1$$
$$f(x) = 3(x^2 - 5x + 6)(x^2 + 7x + 6) + x^2 + 1$$
$$f'(x) = 3(2x-5)(x^2+7x+6) + 3(x^2-5x+6)(2x+7) + 2x$$
$$f'(3) = 3(1)(9+21+6) + 0 + 6$$
$$= 3(36) + 6 = 114.$$

Hence, the correct answer is (D).

14. $\lim\limits_{x\to-6} \dfrac{f(x) - x^2 - 1}{3(x+6)}$

$$= \lim_{x\to-6} \frac{3(x-2)(x-3)(x+1)(x+6)}{3(x+6)}$$
$$= \lim_{x\to-6} (x-2)(x-3)(x+1)$$
$$= (-8)(-9)(-5) = -360 = -\frac{6!}{2}$$

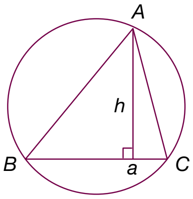
Hence, the correct answer is (D).

Combined Solution to 15–17

We have $\Delta = \dfrac{1}{2} ah = 12$

$\Rightarrow \quad ah = 24 \Rightarrow h = \dfrac{24}{a} = \dfrac{24}{2R\sin A}$

$\Rightarrow \quad h = \dfrac{24}{2\times 6\times \sin A} \Rightarrow h = 2 \operatorname{cosec} A$



So, $y = f(x) = 2$ cosec$x$

15. Clearly, least value of $f(x)$ is 2.

Hence, the correct answer is (B).

16. We have, $g(x) = f(\sin^{-1}x) = 2\operatorname{cosec}(\sin^{-1}x) = \dfrac{2}{x}$

$$g'(x) = \frac{-2}{x^2} \Rightarrow g' = \left(\frac{4}{5}\right) = \frac{-2.25}{16} = \frac{-25}{8}.$$

Hence, the correct answer is (B).

17. We have, $h(x) = \sec^{-1}\left(\dfrac{1}{2}\cdot 2\text{cosec}x\right) = \sec^{-1}(\text{cosec}x)$

$$= \left(\frac{\pi}{2}\right) - x$$

Now, $\lim\limits_{x\to\left(\frac{\pi}{2}\right)} \dfrac{e^{2h(x)} - 2e^{\left(\frac{\pi}{2}-x\right)} + \sin x}{h(x)\cos x}$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)} \frac{e^{2\left(\frac{\pi}{2}-x\right)} - 2e^{\left(\frac{\pi}{2}-x\right)} + \sin x}{\left(\frac{\pi}{2}-x\right)\cos x}$$

Put, $x = \dfrac{\pi}{2} - h$, we get

$$\lim_{h\to 0} \frac{e^{2h} - 2e^h + 1 - (1 - \cos h)}{h\cdot \sin h}$$
$$= \lim_{h\to 0} \frac{\frac{(e^h-1)}{h^2} - \frac{(1-\cos h)}{h^2}}{\frac{\sin h}{h}} = 1 - \frac{1}{2} = \frac{1}{2}$$

Hence, the correct answer is (A).

18. $a = \lim\limits_{n\to\infty}\lim\limits_{\alpha\to I^+} \dfrac{\alpha^n|\sin x| + \alpha^{-n}|\cos x|}{\alpha^n + \alpha^{-n}};$

$$a = \lim_{n\to\infty}\lim_{\alpha\to I^+} \frac{|\sin x| + \alpha^{-2n}|\cos x|}{1 + \alpha^{-2n}} = |\sin x|$$
$$b = \lim_{n\to\infty}\lim_{\alpha\to I^-} \frac{\alpha^n|\sin x| + \alpha^{-n}|\cos x|}{\alpha^n + \alpha^{-n}}$$
$$= \lim_{n\to\infty}\lim_{\alpha\to I^-} \frac{\alpha^{2n}|\sin x| + |\cos x|}{\alpha^{2n} + 1} = |\cos x|$$
$$c = \lim_{n\to\infty} \frac{\pi}{4n}\left[1 + \cos\frac{\pi}{2n} + \ldots + \cos\frac{(n-1)\pi}{2n}\right];$$
$$= \lim_{n\to\infty} \frac{\pi}{4n}\left(\frac{\sin\frac{\pi}{4}\cos\frac{(n-1)\pi}{4n}}{\sin\frac{\pi}{4n}}\right)$$
$$= \sin\frac{\pi}{4}\cos\frac{\pi}{4} = \frac{1}{2}$$

$\therefore \quad b + c - \dfrac{1}{2} = |\cos x|$

Hence, the correct answer is (A).

19. $f(x) = \max\left\{|\sin x|, |\cos x|, \dfrac{1}{2}\right\}$

$\therefore \quad$ Range of $f(x)$ is $\left[\dfrac{1}{\sqrt{2}}, 1\right]$

Hence, the correct answer is (C).

20. $y = \dfrac{h(f(x))}{g(f(x))} = \dfrac{2(4\sin^2 x - 1) + 1}{2(4\sin^2 x - 1) - 1} = \dfrac{8\sin^2 x - 1}{8\sin^2 x - 3}$

$\Rightarrow \quad 8\sin^2 x - 1 = 8(\sin^2 x)y - 3y$

$\Rightarrow \quad 3y - 1 = 8\sin^2 x(y - 1)$

$\Rightarrow \quad \sin^2 x = \dfrac{3y-1}{8(y-1)},\ 0 \le \dfrac{3y-1}{8(y-1)} \le 1$

Now, $\dfrac{3y-1}{8(y-1)} \ge 0 \Rightarrow y\ (-\infty, 1/3) \cup (1, \infty)$ …(i)

Also, $\dfrac{3y-1}{8(y-1)} \le 1 \Rightarrow \dfrac{3y - 1 - 8y + 8}{8(y-1)} \le 0 \Rightarrow \dfrac{-5y+7}{y-1}$

$\Rightarrow \quad \dfrac{y - 7/5}{y-1} \ge 0 \Rightarrow y \in (-\infty, 1) \cup \left[\dfrac{7}{5}, \infty\right)$ …(ii)

$\therefore \quad$ From (1) and (2), we get

$$y \in \left(-\infty, \frac{1}{3}\right] \cup \left[\frac{7}{5}, \infty\right)$$

Hence, the correct answer is (D).

21. $\lim\limits_{x\to\frac{1}{2}} \dfrac{f(\sin^{-1}x)}{g(x)} = \lim\limits_{x\to\frac{1}{2}} \dfrac{4\sin^2(\sin^{-1}x) - 1}{2x - 1}$

$$= \lim_{x\to\frac{1}{2}} \frac{4x^2 - 1}{2x - 1} = \lim_{x\to\frac{1}{2}} (2x + 1) = 2$$

Hence, the correct answer is (B).

22. $\lim\limits_{x\to\frac{-1}{2}} \dfrac{f(\sin^{-1}kx) - 3}{h(x)} = \lim\limits_{x\to\frac{-1}{2}} \dfrac{4\sin^2(\sin^{-1}kx) - 1 - 3}{2x - 1}$

$$= \lim_{x\to\frac{-1}{2}} \frac{4x^2x^2 - 4}{2x - 1}.$$

Here, N^r must be zero, $k^2 - 4 = 0 \Rightarrow k = \pm 2$.

$$= \lim_{x \to \frac{-1}{2}} \frac{4(4x^2-1)}{2x-1} = \lim_{x \to \frac{-1}{2}} 4(2x-1) = -8$$

So, $|k| + |l| = 2 + 8 = 10$.

Hence, the correct answer is (C).

Combined Solution to 23–24

$P(x) = (x^2 - 4)\,((x^2 - 1)\,(x^2 - 9) + 1) = (x^2 - 4)\,(x^4 - 10x^2 + 10)$

23. $$\lim_{x \to -2} \frac{\sin P(x)}{P(x)} \cdot \frac{P(x)}{(x^2-4)}$$

$$= \lim_{x \to -2} \frac{(x^2-4)(x^2-10x^2+10)}{x^2-4} = -14$$

Hence, the correct answer is (C).

24. $$\lim_{x \to \infty} \left(\frac{x^2 - \sqrt{x^4 - 10x^2 + 10}}{x \tan \frac{1}{x}} \right) = \lim_{x \to \infty} \frac{x^2 - (x^4 - 10x^2 + 10)}{x^2 - \sqrt{x^4 - 10x^2 + 10}} = 5$$

Hence, the correct answer is (B).

Matrix Match/Column Match Type Questions

1. A → (r); B → (s); C → (r); D → (p)

(A) $$\lim_{x \to \frac{\pi}{2}^-} \sqrt{\frac{\tan x - \sin\left(\tan^{-1}(\tan x)\right)}{\tan x + \cos^2(\tan x)}}$$

$$= \lim_{x \to \frac{\pi}{2}} \sqrt{\frac{\tan x - \sin x}{\tan x + \cos^2(\tan x)}}$$

$$= \lim_{x \to \frac{\pi}{2}} \sqrt{\frac{1 - \dfrac{\sin x}{\tan x}}{1 + \dfrac{\cos^2(\tan x)}{\tan x}}} = \sqrt{\frac{1-0}{1+0}} = 1$$

Also $$\lim_{x \to \frac{\pi}{2}^+} \sqrt{\frac{\tan x - \sin\left(\tan^{-1}(\tan x)\right)}{\tan x + \cos^2(\tan x)}}$$

$$= \lim_{x \to \frac{\pi}{2}} \sqrt{\frac{\tan x - \sin(x - \pi)}{\tan x + \cos^2(\tan x)}}$$

$$= \lim_{x \to \frac{\pi}{2}^+} \sqrt{\frac{1 + \dfrac{\sin x}{\tan x}}{1 + \dfrac{\cos^2(\tan x)}{\tan x}}} = 1$$

So, required limit is 1.

(B) $$\lim_{x \to 0} \frac{\log \cos x}{\log \cos \frac{x}{2}} \cdot \frac{\log \sin \frac{x}{2}}{\log \sin x}$$

$$= \lim_{x \to 0} \frac{\dfrac{\log \cos x}{\cos x - 1} \cdot \dfrac{1 - \cos x}{x^2} x^2}{\dfrac{\log \cos \frac{x}{2}}{\cos \frac{x}{2} - 1} \cdot \dfrac{1 - \cos \frac{x}{2}}{\left(\frac{x}{2}\right)^2} \left(\frac{x}{2}\right)^2} \cdot \frac{\log \sin \frac{x}{2}}{\log \sin x}$$

$$= \lim_{x \to 0} \frac{4 \log \sin \frac{x}{2}}{\log 2 + \log \sin \frac{x}{2} + \log \cos \frac{x}{2}}$$

Dividing $\log \sin \frac{x}{2}$

$$\lim_{x \to 0} \frac{4}{\dfrac{\log 2}{\log \sin \frac{x}{2}} + \dfrac{\log \sin \frac{x}{2}}{\log \sin \frac{x}{2}} + \dfrac{\log \cos \frac{x}{2}}{\log \sin \frac{x}{2}}}$$

$$= \frac{4}{0 + 1 + \frac{0}{\infty}} = 4$$

(C) $$\lim_{m \to \infty} \left(\cos \frac{x}{m} \right)^m = e^{\lim_{m \to \infty} \left(\cos \frac{x}{m} - 1 \right) m}$$

$$= e^{\lim_{m \to \infty} \left(-2 \sin^2 \frac{x}{2m} \right) \cdot m} = e^{-1 \times 0} = e^0 = 1$$

(D) $$\lim_{x \to \infty} \frac{\ln x - [x]}{[x]} = \lim_{x \to \infty} \left(\frac{\ln x}{[x]} - 1 \right) = -1$$

2. A → (s); B → (r); C → (t); D → (t)

(A) $$\lim_{x \to 0} \frac{3e^x - x^3 - 3x - 3}{\tan^2 x}$$

$$= \lim_{x \to 0} \frac{3e^x - 3x^2 - 3}{2 \tan x . \sec^2 x} \quad \text{(Using L' Hospital's rule)}$$

$$= \lim_{x \to 0} \frac{3 \dfrac{(e^x - 1)}{x} - 3x}{2 \dfrac{\tan x}{x} \cdot \sec^2 x} = \frac{3}{2}$$

(B) $$\lim_{x \to 0} \frac{2 \sin x - 2 \sin x \cos x}{x(\cos x - \cos 2x)}$$

$$= \lim_{x \to 0} \frac{2 \sin x (1 - \cos x)}{x\left(\cos - 1 + (1 - \cos 2x)\right)}$$

$$= \lim_{x \to 0} \frac{\dfrac{2 \sin x}{x} \dfrac{(1 - \cos x)}{x^2}}{\left(\dfrac{\cos x - 1}{x^2} + \dfrac{4(1 - \cos 2x)}{(2x)^2} \right)}$$

$$= \frac{2 \times \frac{1}{2} \times 1}{\left(-\frac{1}{2} + 2 \right)} = \frac{2}{3}$$

(C) $$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} = \lim_{x \to 0} \frac{(e^x - 1)^2}{1 - \cos x} = \frac{1}{\frac{1}{2}} = 2$$

(D) $$\lim_{x \to \infty} \frac{\pi - 2 \tan^{-1} x}{\ln\left(1 + \frac{1}{x}\right)}$$

$$x \to \frac{1}{y};\ y \to 0 = \lim_{y \to 0} \frac{\pi - 2 \tan^{-1}\left(\frac{1}{y}\right)}{\ln(1 + y)} = 2 \lim_{y \to 0} \frac{\frac{\pi}{2} - \cos^{-1} y}{\ln(1 + y)}$$

$$= 2 \lim_{y \to 0} \frac{\tan^{-1} y}{\ln(1 + y)} = 2 \lim_{y \to 0} \left(\frac{\dfrac{\tan^{-1} y}{y}}{\dfrac{\ln(1 + y)}{y}} \right) = 2$$

3. A → (s); B → (r); C → (q); D → (p)

(A) $$\lim_{x \to \infty} \left(\frac{x}{1 + x} \right)^x = e^{\lim_{x \to \infty} \left(\frac{x}{1+x} - 1 \right) x} = e^{\lim_{x \to \infty} \left(\frac{-x}{1+x} \right)} = e^{-1}$$

(B) $$\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^{x}=e^{\lim\limits_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}-1\right)x}$$

$$=e^{\lim\limits_{x\to\infty}\left(\frac{\sin\frac{1}{x}}{\frac{1}{x}}-\frac{1-\cos\frac{1}{x}}{\frac{1}{x}}\right)}=e^{1-0}=e$$

(C) $$\lim_{x\to0}(\cos)^{\cot^2 x}=e^{\lim\limits_{x\to0}(\cos x-1)\cot^2 x}=e^{-\lim\limits_{x\to0}\left(\frac{1-\cos x}{x^2}\right)\left(\frac{x}{\tan x}\right)^2}=e^{-1/2}$$

(D) $$\lim_{x\to0}\left(\tan\left(\frac{\pi}{4}+x\right)\right)^{1/x}=e^{\lim\limits_{x\to0}\left(\frac{1+\tan x}{1-\tan x}-1\right)\frac{1}{x}}$$

$$=e^{\lim\limits_{x\to0}\left(\frac{1+\tan x-1+\tan x}{1-\tan x}\right)\frac{1}{x}}=e^{\lim\limits_{x\to0}\frac{2\tan x}{x(1-\tan x)}}=e^2$$

4. A → (r); B → (p); C → (q); D → (s)

(A) If a is an integer $f(a)=a-1+1-a=0$

(B) If a is not an integer, suppose $n<a<n+1$ than

$$\lim_{x\to a}f(x)=n-1+(-n)=-1 \text{ and } f(a)=-1$$

$\therefore$ f is continuous at $x=a$

(C) $$\lim_{x\to0^-}f(x)=\lim_{x\to0^-}\frac{a^{x-1}-1}{x-1}=\frac{a-1}{a}$$

$$\lim_{x\to0^+}f(x)=\lim_{x\to0^+}\frac{a^{x-1}-1}{x}=\ln a$$

(D) $$\lim_{x\to\frac{\pi}{2}}f(x)=\lim_{x\to\frac{\pi}{2}}a^{\cos x}\left(\frac{a^{\cot x-\cos x}-1}{\cot x-\cos x}\right)=\ln a$$

5. A → (q); B → (p); C → (p); D → (s)

(A) $$\lim_{t\to0}\left[\min(x^2+2x-2)\frac{\sin t}{t}\right]=[-3\text{ (less than 1)}]=-3$$

(B) $$\lim_{t\to0}\left[\min(x^2+2x-2)\frac{\sin^{-1}t}{t}\right]=[-3\text{ (greater than 1)}]=-4$$

(C) $$\lim_{t\to0}\left[\max(-x^2+4x-7)\frac{t}{\tan^{-1}t}\right]=[-3\text{ (greater than 1)}]=-4$$

(D) $$\lim_{t\to0}\left[\max(-x^2+4x-7)\frac{\tan t}{t}\right]=\{-3\text{ (greater than 1)}]=1$$

6. A → (q); B → (t); C → (p); D → (q)

(A) $$\lim_{x\to0}\frac{1+x\sin x-\cos 2x}{\left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)^2\frac{x^2}{4}}$$

$$=\lim_{x\to0}\frac{1+x\left(x-\frac{x^3}{3!}+\ldots\right)-\left(1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}\ldots\right)}{x\frac{2}{4}\left(\sqrt{1+x\sin x}+\sqrt{\cos 2x}\right)}$$

$$=\lim_{x\to0}\frac{4\left[1+x^2-\frac{x^4}{3!}+\ldots-1+2x^2-\frac{16}{24}x^4+\ldots\right]}{x^2(1+1)}$$

$$=\frac{4(1+2)}{2}=\frac{12}{2}=6$$

(B) $$\lim_{x\to1}\left(\frac{\sin(x^3+x^2+x-3)}{(x^3+x^2+x-3)}\right)^2\frac{(x^3+x^2+x-3)^2}{1-(1-2\sin^2)\left(\frac{x^2-4x+3}{2}\right)}$$

$$=\lim_{x\to1}(1)^2\frac{(x^3+x^2+x-3)^2}{2\left(\frac{\sin\frac{(x-1)(x-3)}{2}}{\frac{(x-1)(x-3)}{2}}\right)^2}\frac{4}{2}$$

$$=\lim_{x\to1}\left(\frac{3x^2+2x+1}{2x-4}\right)^2\frac{4}{2}$$

$$=\left(\frac{3+2+1}{2-4}\right)^2\frac{4}{2}$$

$$=(-3)^2\frac{4}{2}=\frac{36}{2}=18$$

(C) $$\lim_{x\to0}\frac{6x^2\cot x\cdot\operatorname{cosec}2x}{\sec\left(\cos x+\pi\tan\left(\frac{\pi}{4\sec x}\right)-1\right)}$$

$$=\lim_{x\to0}\frac{\left(\frac{x}{\tan x}\right)\left(\frac{2x}{\sin 2x}\right)^3}{\sec(1+\pi(1)-1)}=\frac{(1)(1)3}{-1}=-3$$

(D) $$\lim_{h\to0}\frac{(2(a-3)h-6\sin^{-1}h)\tan^{-1}5h}{h^2}=0$$

$x=-2+h$

$$\lim_{h\to0}\left(2a-6-6\frac{\sin^{-1}h}{h}\right)\left(\frac{\tan^{-1}5h}{5h}\right)^5=0$$

$(2a-6-6(1))\,(1)\,5=0\Rightarrow a=6$

7. A → (q); B → (t); C → (p); D → (r)

(A) Put $x=\frac{1}{t}$

$$\lim_{t\to0^+}\frac{\sqrt{1+2t}-3\sqrt{1+3t}}{t^2}$$

$$=\lim_{t\to0^+}\frac{(1+2t)^{1/2}-(1+3t)^{1/3}}{t^2}=1+\frac{2t}{2}+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(2t)^2+\ldots$$

$$\frac{-\left(1+3t\frac{1}{3}+\frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2!}(3+3^2+1)\right)}{t^2}$$

$$=\lim_{t\to0^+}\frac{t^2\left[-\frac{1}{2}+1\right]+\ldots}{t^2}=\frac{1}{2}$$

(B) $$\lim_{x\to\infty}\frac{x^2(1+\sin^2 x)}{(x+\sin x)^2}=\lim_{x\to\infty}\frac{1+\sin^2 x}{\left(1+\frac{\sin x}{x}\right)^2}=\frac{1+(0\,to\,1)}{(1+0)^2}$$ DNE

(C) $$\lim_{x\to\infty}\left((4x^2-x^3)^{1/3}+ax+b\right)=\frac{1}{3}$$ put $x=\frac{1}{t}$

$$\lim_{t\to0^+}\left(\frac{(4t-1)^{1/3}}{t}+\frac{a+bt}{t}\right)=\frac{1}{3}$$

$$\lim_{t\to0^+}\frac{(4t-1)^{1/3}+a+bt}{t}=\frac{1}{3}$$

$$\Rightarrow\quad\lim_{t\to0^+}\frac{-(1-4t)^{1/3}+a+bt}{t}=\frac{1}{3}$$

$$\Rightarrow\quad\lim_{t\to0^+}\frac{-\left(1-\frac{4}{3}t+\ldots\right)+a+bt}{t}=\frac{1}{3}$$

$$\frac{(a-1)+\left(b+\frac{4}{3}\right)t+\ldots}{t}=\frac{1}{3}$$

$a=1;\ b+\frac{4}{3}=\frac{1}{3}\Rightarrow b=-1$

$a+b=0.$

(D) $\lim\limits_{x\to0}\dfrac{\ln(1+x^2+x^4)\cdot(x^2+x^4)}{(x^2+x^4)\left(\dfrac{e^x-1}{x}\right)x^2}=\lim\limits_{x\to0}\dfrac{1\cdot(1+x^2)}{1}=1$

Integer/Numerical Answer Type Questions

1. $\lim\limits_{x\to0}\dfrac{ax\left(1+x+\dfrac{x^2}{2!}+\ldots\right)-b\left(x-\dfrac{x^2}{2}+\dfrac{x^3}{3}\ldots\right)+cx\left(1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}+\ldots\right)}{x^2\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}\ldots\right)}$

$\lim\limits_{x\to0}\dfrac{(a-b+c)x+x^2\left(a+\dfrac{b}{2}-c\right)+x^3\left(\dfrac{a}{2}-\dfrac{b}{3}+\dfrac{c}{2}+\ldots\right)}{x^2\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}\ldots\ldots\right)}$

$a-b+c=0$ and $a+\frac{b}{2}-c=0$

And, $\frac{a}{2}-\frac{b}{3}+\frac{c}{2}=2$

Hence, the correct answer is (0).

2. $\lim\limits_{x\to0}|x|^{[\cos x]}$

At $x\to0$ $[\cos x]=0\Rightarrow\lim\limits_{x\to0}|x|^{[\cos x]}=|x|^{o}=1$

Hence, the correct answer is (1).

3. $f(x)=[[x]]-[x-1]$

$f(x)=[x]-[x]+1$

$f(x)=1$

$\Rightarrow\ \lim\limits_{x\to a^-}f(x)+\lim\limits_{x\to a^+}f(x)=1+1=2$

Hence, the correct answer is (2).

4. $\lim\limits_{x\to0}\dfrac{3\left\{\log\left(3+x^2\right)-\log\left(3-x^2\right)\right\}}{x^2}=\lim\limits_{x\to0}\dfrac{3\left\{\dfrac{2x}{3+x^2}+\dfrac{2x}{3-x^2}\right\}}{2x}$

(by L Hospital's rule)

$=\lim\limits_{x\to0}\dfrac{18}{9-x^4}=2$

Hence, the correct answer is (2).

5. $\lim\limits_{x\to0}\dfrac{\cos4x+a\cos2x+b}{x^4}\Rightarrow\dfrac{0}{0}$ if $1+a+b=0$

$=\lim\limits_{x\to0}\dfrac{-4\sin4x-2a\sin2x}{4x^3}\to\dfrac{0}{0}$

$=\lim\limits_{x\to0}\dfrac{-16\cos4x-4a\cos2x}{12x^2}\Rightarrow\dfrac{0}{0}$ if $-16-4a=0\Rightarrow a=-4$

And, $1+a+b=0\Rightarrow b=3$

$\therefore\quad b-a=7$

Hence, the correct answer is (7).

6. $\lim\limits_{x\to0}\ \dfrac{2f(x)-3f(2x)+f(4x)}{x^2}$ $\left(\dfrac{0}{0}\text{form}\right)$

$=\lim\limits_{x\to0}\ \dfrac{2f'(x)-6f'(2x)+4f'(4x)}{2x}$ $\left(\dfrac{0}{0}\text{form}\right)$

$=\lim\limits_{x\to0}\ \dfrac{2f''(x)-12f''(2x)+16f''(4x)}{2}$

$=\dfrac{(2-12+16)2}{2}=6$

Hence, the correct answer is (6).

7. $f'(x)=\lim\limits_{h\to0}\dfrac{f(x+h)-f(x)}{h}$

$=\lim\limits_{h\to0}\dfrac{f(x)f(h)-f(x)}{h}$

$=\lim\limits_{h\to0}\dfrac{f(x)\left[1+hg(h)H(h)-1\right]}{h}$

$=f(x)\lim\limits_{h\to0}g(h).H(h)=6f(x)$

Hence, the correct answer is (6).

8. $\lim\limits_{x\to0}\dfrac{e^{\tan x}-e^{\sin x}}{\tan x-\sin x}\cdot\dfrac{\tan x}{e^{x/32}-1}\dfrac{(1-\cos x)}{(1-\cos4x)}$

$=1.\dfrac{1}{\dfrac{1}{32}}\times\dfrac{1}{16}=2$

Hence, the correct answer is (2).

9. $\lim\limits_{x\to0}\dfrac{1-\cos^n(1-\cos x)}{(1-\cos x)^2}.\left(\dfrac{1-\cos x}{x^2}\right)^2.\dfrac{x^4}{\tan^m x}=1$

$\Rightarrow\quad m=4$ and $\frac{n}{2}.\frac{1}{4}=1\Rightarrow n=8$

Hence, the correct answer is (2).

10. $\lim\limits_{x\to0}\dfrac{3[\tan3x-\tan2x]-(\tan2x+\tan x)}{4x^2\tan x}$

$=\lim\limits_{x\to0}\dfrac{3[\tan(1+\tan3x\tan2x)-\tan3x(1-\tan x\tan2x)]}{4x^2\tan x}$

$=\lim\limits_{x\to0}\dfrac{4\tan3x\tan2x\tan x+3\tan x-\tan3x}{4x^2\tan x}$

$=\lim\limits_{x\to0}\left[\dfrac{\tan3x\tan2x}{x^2}+3\tan x-\dfrac{\dfrac{(3\tan x-\tan^3x)}{1-3\tan^2x}}{4x^2\tan x}\right]$

$=6+\lim\limits_{x\to0}\dfrac{-8\tan^3x}{4x^2\tan x}=6-2=4$

Hence, the correct answer is (4).

11. $y=(1+x)^{\frac{1}{x}}\Rightarrow\log y=\dfrac{1}{x}\log(1+x)=\dfrac{1}{x}\left[x-\dfrac{x^2}{2}+\dfrac{x^3}{3}\ldots\ldots\ldots\right]$

$=1-\dfrac{x}{2}+\dfrac{x^2}{3}\Rightarrow y=e.e^{-\frac{x}{2}+\frac{x^2}{3}}$

$=\lim\limits_{x\to0}\dfrac{x}{1-e^{-\frac{x}{2}+\frac{x^2}{3}}}=\lim\limits_{x\to0}\dfrac{x}{1-\left[1+\left(\dfrac{-x}{2}+\dfrac{x^2}{3}\ldots\ldots\right)\right]}$

$=\lim\limits_{x\to0}\dfrac{x}{\dfrac{x}{2}-\dfrac{x^2}{3}}=\lim\limits_{x\to0}\dfrac{1}{\dfrac{1}{2}-\dfrac{x}{3}}=2$

Hence, the correct answer is (2).

12. $k=\lim\limits_{x\to\pi/2}\tan\left(\dfrac{\pi}{4}-\dfrac{x}{2}\right).\dfrac{1-\cos\left(\dfrac{\pi}{2}-x\right)}{8\left(\dfrac{\pi}{2}-x\right)^3}$

$$= \lim_{x\to\pi/2} \frac{1}{8}\frac{\tan\frac{1}{2}\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)}\cdot\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{\left(\frac{\pi}{2}-x\right)^2}$$

$$= \frac{1}{8}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{32}\Rightarrow 96k = 96\times\frac{1}{32}=3$$

Hence, the correct answer is (3).

13. $\because \quad \lim_{x\to 0}\left(1+\frac{P(x)}{x^2}\right)=2$

$\Rightarrow \quad \lim_{x\to 0}\frac{P(x)}{x^2}=1$

$\Rightarrow \quad P(x) = ax^2$

$\Rightarrow \quad a = 1 \Rightarrow P(x) = x^2$

$P(3) = 3^2 = 9$

Hence, the correct answer is (9).

14. $\lim_{x\to 0}\frac{\ln(1+2x)-2\ln(1+x)}{x^2}$

$$= \lim_{x\to 0}\frac{\left(2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}-.....-2\left(x-\frac{x^2}{2}+\frac{x^3}{3}-.....\right)\right)}{x^2}$$

$$= \lim_{x\to 0}\frac{-x^2+2x^3-......}{x^2}$$

$= -1 + 0 = -1 \Rightarrow -\lambda = 1$

Hence, the correct answer is (1).

15. $\lim_{x\to 0}\frac{\cos^2 x-\cos x-e^x\cos x+e^x-\frac{x^3}{2}}{x^n}$

$$= \lim_{x\to 0}\frac{(1-\cos x)\left(e^x-\cos x)\right)-\frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to 0}\frac{\left(1-1+\frac{x^2}{2!}-\frac{x^4}{4!}+...\right)\left[1+x+\frac{x^2}{2!}+...-1+\frac{x^2}{2!}-...\right]-\frac{x^3}{2}}{x^n}$$

$$= \lim_{x\to 0}\frac{\left(\frac{x^2}{2!}-\frac{x^4}{4!}+...\right)(x+x^2+...)-\frac{x^3}{2}}{x^n}$$

For non-zero limit $n = 4$

So, $n = 4$

Hence, the correct answer is (4).

16. Let $a_n = \cot\alpha_n \Rightarrow a_{n+1} = \cot\alpha_n + \operatorname{cosec}\alpha_n$

$$= \frac{\cos(\alpha_n)}{\sin(\alpha_n)}+\frac{1}{\sin(\alpha_n)}=\frac{2\cos^2\left(\frac{\alpha_n}{2}\right)}{2\sin\left(\frac{\alpha_n}{2}\right)\cos\left(\frac{\alpha_n}{2}\right)}=\cot\left(\frac{\alpha_n}{2}\right)$$

Also $a_0 = 0 \Rightarrow a_1 = 1 \Rightarrow \cot(\alpha_1) = 1 \Rightarrow \alpha_1 = \frac{\pi}{4}$

$\Rightarrow \quad a_2 = \cot\left(\frac{\pi}{4.2}\right), a_3 = \cot\left(\frac{\pi}{4.2^2}\right), ..., a_n = \cot\left(\frac{\pi}{4.2^{n-1}}\right)$

Put $\frac{1}{2^{n-1}} = x$

Hence, $\lim_{n\to\infty}\left(\frac{a_n}{2^{n-1}}\right) = \lim_{n\to\infty}\frac{1}{\left(\frac{\tan\left(\frac{\pi x}{4}\right)}{n}\right)} = \frac{4}{\pi}.$

Hence, the correct answer is (4).

17. After rationalise, we get

$$\lim_{x\to\infty}\frac{(a-2)x^3+(3+c)x^2+(b-3)x+(2+d)}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}=4$$

Since, limit is finite, so degree of N^r must be 2

So, $a - 2 = 0 \rightarrow a = 2$

$$\lim_{n\to\infty}\frac{(3+c)x^2+(b-3)x+(2+d)}{\sqrt{x^4+ax^3+3x^2+bx+2}+\sqrt{x^4+2x^3-cx^2+3x-d}}=4$$

$$\lim_{n\to\infty}\frac{(3+c)+(b-3)/x+(2+d)/x^2}{\sqrt{1+\frac{a}{x}+\frac{3}{x^2}+\frac{6}{x^3}+\frac{2}{x^4}}+\sqrt{1+\frac{2}{x}-\frac{c}{x^2}+\frac{3}{x^3}-\frac{d}{x^4}}}=4$$

$\therefore \quad \frac{3+c}{2} = 4 \Rightarrow c = 5$

Hence, the correct answer is (7).

18. $L = \lim_{x\to 0^+}\left(\frac{\pi^2 x}{2\sqrt{2}}\right)(\ln x)^n,$ $\quad (0 \times \infty$ Form)

$L = \lim_{x\to 0^+}\frac{\pi^2}{2\sqrt{2}}\frac{(\ln x)^n}{\left(\frac{1}{x}\right)},$ $\quad \left(\frac{\infty}{\infty}\text{ Form}\right)$

Using L'Hospital's rule,

$$L = \lim_{x\to 0^+}\frac{\pi^2}{2\sqrt{2}}\frac{n(\ln x)^{n-1}}{x\left(-\frac{1}{x^2}\right)}$$

$L = -\lim_{x\to 0^+}\frac{\pi^2}{2\sqrt{2}}\frac{n(\ln x)^{n-1}}{\frac{1}{x}}$ $\quad \left(\frac{\infty}{\infty}\text{ Form}\right)$

After Differentiation n times,

$$L = (-1)^n \lim_{x\to 0^+}\frac{\pi^2}{2\sqrt{2}}n!\frac{1}{\left(\frac{1}{x}\right)}$$

$L = O$

Hence, the correct answer is (0).

19. $\frac{\sin x\tan x}{x^2} = \frac{4\tan^2\frac{x}{2}}{x^2\left(1-\tan^4\frac{x}{2}\right)} = \left(\frac{\tan\frac{x}{2}}{\frac{x}{2}}\right)\frac{1}{\left(1-\tan^4\frac{x}{2}\right)}$

Hence, when $x\frac{3}{4}\rightarrow 0$, $\frac{\tan\frac{x}{2}}{\frac{x}{2}} > 1$ and $\frac{1}{1-\tan^y\frac{x}{2}} > 1$

$\therefore \quad \frac{\sin x\tan x}{x^2} > 1$

$\therefore \quad \lim_{x\to 0}\left[\frac{x^2}{\sin x\tan x}\right] = 0$

Hence, the correct answer is (0).

20. Let $x^4 = t$

Then, $\lim\limits_{t\to 0} \dfrac{\sin t - t\cot t + t^5}{t\left(e^{2t} - 1 - 2t\right)}$

$$= \lim_{t\to 0} \frac{\left(t - \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots\right) - t\left(1 - \frac{t^2}{2!} + \ldots\right) + t^5}{t\left(1 + 2t + \frac{4t^2}{2!} + \ldots - 1 - 2t\right)}$$

$$= \lim_{t\to 0} \frac{-\frac{t^3}{3!} + \frac{t^3}{2!}}{2t^3} = -\left(\frac{1}{6} + \frac{1}{2}\right)\frac{1}{2} = \frac{1}{6}$$

[Using expansions of sin$x$, cos$x$, e^{2x}]

Hence, the correct answer is (1).

21. $n - 1 < \left[\dfrac{n\sin x}{x}\right] < n$ and $n \to 0$

$$\left[\frac{n\sin x}{x}\right] = n - 1 \text{ and } \left[\frac{n\tan x}{x}\right] = n$$

Hence, the correct answer is (7).

22. $f(3) = 3f(1) = 3, f(4) = f(2 + 1 + 1) = 2 + 1 + 1 = 4$

and so on. In general, we get $f(r) = r$ for $r \in N$

$$\therefore \quad \lim_{n\to\infty} \frac{\sum_{r=1}^{n} (4r) f(3r)}{n^3} = \lim_{n\to\infty} \frac{12n(n+1)(2n+1)}{6n^3} = 4$$

Hence, the correct answer is (4).

23. Given that,

$f(x) = \sin x, x^1 \neq n\pi, n = 0, \pm 1, \pm 2, \ldots.$

$= 2$, otherwise

$$\text{and } g(x) = \begin{cases} x^2 + 1, & x^1 \neq 0, 2 \\ 4, & x = 0 \\ 5, & x = 2 \end{cases}$$

$x = 0$

$= 5, \quad x = 2$

Then, $\lim\limits_{x\to 0} g[f(x)] = \lim\limits_{x\to 0} g(\sin x)$

$$= \lim_{x\to 0}\left(\sin^2 x + 1\right) = 1$$

Hence, the correct answer is (1).

24. $x^5 - y^5 = (x - y)\left(x^4 + x^3 y + x^2 y^2 + xy^3 + y^4\right)$

and the coefficient of x^4 in

$(x - 1)(x - 2)(x + 3)(x + 10)(x + 15) - x^5$

is $-1 - 2 + 3 + 10 + 15 = 25$

$$\therefore \quad \lim_{x\to\infty}\left(\left((x-1)(x-2)(x+3)(x+10)(x+15)\right)^{1/5} - x\right)$$

$$= \frac{25}{5} = 5$$

Hence, the correct answer is (5).

25. Using L' Hospital's Rule using $fgh' + ghf' + hfg'$

$$\lim_{x\to 0} \frac{\begin{gathered}5\cos^4 x(\sin x)\cos^3 2x\cos^3 3x \\ +3\cos^2 2x(2\sin 2x)\cos^5 x\cos^3 3x \\ +3\cos^2 3x(3\sin 3x)\cos^5 x\cos^3 2x\end{gathered}}{2x}$$

$$= \frac{5}{2} + 6 + \frac{9}{2} \times 3 = 22$$

Alternate:

Using expansion

$$\cos^5 x.\cos^3 2x.\cos^3 3x$$

$$= \left(1 - \frac{x^2}{2!} + \ldots\right)^5 . \left(1 - \frac{(2x)^2}{2!} + \ldots\right)^3 . \left(1 - \frac{9x^2}{2!} + \ldots\right)^3$$

$$= \left(1 - \frac{5x^2}{2} + \ldots\right)\left(1 - 6x^3 + \ldots\right)\left(1 - \frac{27x^2}{2} + \ldots.\right)$$

$$= \left(1 - \left(6 + \frac{5}{2}\right)x^2 + \ldots\right)\left(1 - \frac{27x^2}{2} + \ldots\right)$$

$$= 1 - \frac{27x^2}{2} - 6x^2 - \frac{5x^2}{2} + \ldots. = 1 - 22x^2 + \ldots.$$

$$\therefore \quad \lim_{x\to 0} \frac{1 - \cos^5 x.\cos^3 2x.\cos^3 3x}{x^2}$$

$$= \lim_{x\to 0} \frac{1 - \left(1 + 22x^2 + \ldots\right)}{x^2} = 22$$

Hence, the correct answer is (2).

26. Let $\left(\sqrt{3} + 1\right)^{2n} = I + f, F = \left(\sqrt{3} - 1\right)^{2n}$

$$I + f + F = \left(\sqrt{3} + 1\right)^{2n} + \left(\sqrt{3} - 1\right)^{2n}$$

$$= 2\left[{}^{2n}C_0 + {}^{2n}C_2\left(\sqrt{3}\right)^2 + {}^{2n}C_4\left(\sqrt{3}\right)^4 + \ldots.\right]$$

$I + f + F =$ even integer, $0 < f, F < 1 \Rightarrow 0 < f + F < 2$

$$\Rightarrow \quad f + F = 1 \Rightarrow \left\{\left(\sqrt{3} + 1\right)\right\}^{2n} = f = 1 - F$$

$$\lim_{n\to\infty} f = \lim_{n\to\infty} 1 - F = 1 - \lim_{n\to\infty}\left(\sqrt{3} - 1\right)^{2n} = 1 - 0 = 1$$

Hence, the correct answer is (1).

27. As $x \to [a], \{x\} \to 1 \qquad \therefore G.L = \dfrac{e^1 - 1 - 1}{1^2} = e - 2$

Hence, the correct answer is (2).

28. Given, $A = \lim\limits_{x\to -2} \dfrac{\tan \pi x}{x + 2} + \lim\limits_{\frac{1}{x}\to 0}\left(1 + \dfrac{1}{x^2}\right)^{\frac{x^2}{x}}$

$$= \lim_{x\to -2} \frac{\pi\sec^2 \pi x}{1} + \lim_{\frac{1}{x}\to 0} e^{1/x} = \pi + 1 = 3.14 + 1 = 4.14$$

$\therefore \quad A = 4.14$

$[A] = 4$

Hence, the correct answer is (4).

29. $\lim\limits_{x\to 0} \dfrac{ae^x - b\cos x + c.e^{-x}}{x\sin x} = 2 \Rightarrow a - b + c = 0 \qquad \ldots(i)$

Apply LH Rule,

$$= \lim_{x\to 0} \frac{ae^x + b\sin x - c.e^{-x}}{\sin x + x\cos x} = 2 \Rightarrow a + 0 - c = 0 \Rightarrow a = c \qquad \ldots(ii)$$

Apply LH rule,

$$\lim_{x\to 0} \frac{ae^x + b\cos x + ce^{-x}}{\cos x + \cos x - x\sin x} = 2 \Rightarrow a + b + c = 4$$

∴ $a+b+c=4$

Hence, the correct answer is (3).

30. $k=\lim\limits_{x\to 0}\dfrac{(1+x)\ln(1+x)-x}{x^2}=\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{2x}=\dfrac{1}{2}$

(On using L' Hospital's rule)

∴ $12k=6$

Hence, the correct answer is (6).

31. We have

$$\text{LHL}=\lim_{x\to\frac{\pi}{2}^-}\sqrt{\frac{\tan x-\sin\tan^{-1}(\tan x)}{\tan x+\cos^2(\tan x)}}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\sqrt{\frac{\tan x-\sin x}{\tan x+\cos^2(\tan x)}}$$

$$=\lim_{x\to\frac{\pi}{2}^-}\sqrt{\frac{1-\dfrac{\sin x}{\tan x}}{1+\dfrac{\cos^2(\tan x)}{\tan x}}}=\sqrt{\frac{1-0}{1+0}}=1$$

At $x\to\dfrac{\pi}{2}-, 0<x<\dfrac{\pi}{2}$

∴ $\tan^{-1}(\tan x)=x$

Further as, $x\to\dfrac{\pi}{2}-, \tan x\to\infty$ and $\cos^2(\tan x)$ is real number between 0 and 1.

$$\text{RHL}=\lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{\tan x-\sin\tan^{-1}(\tan x)}{\tan x+\cos^2(\tan x)}}$$

$$=\lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{\tan x+\sin x}{\tan x+\cos^2 x(\tan x)}}$$

$$=\lim_{x\to\frac{\pi}{2}^+}\sqrt{\frac{1+\dfrac{\sin x}{\tan x}}{1+\dfrac{\cos^2(\tan x)}{\tan x}}}=\sqrt{\frac{1+0}{1-0}}=1$$

(As $x\to\dfrac{\pi}{2}+, x>\dfrac{\pi}{2}\Rightarrow\tan^{-1}\tan x=\tan^{-1}\tan(x-\pi)=x-\pi$

∴ $\sin\tan^{-1}(\tan x)=\sin(x-\pi)=-\sin x$

Further as $x\to\dfrac{\pi}{2}+;\tan x\to-\infty$ and $\cos^2(\tan x)$ is a real number between 0 and 1.

LHL = RHL = 1

∴ Required limit = 1

Hence, the correct answer is (1).

32. Here, two straight line, $(\tan\alpha)x+(\sin\alpha)y=\alpha$ and $(\alpha\operatorname{cosec}\alpha)x+(\cos\alpha)y=1$ have their point of intersection as,

$$x=\frac{\alpha\cos\alpha-\sin\alpha}{\sin\alpha-\alpha}\text{ and }y=\frac{\alpha-x\tan\alpha}{\sin\alpha}$$

∴ When $\alpha\to 0$, we obtain the point P.

i.e., $\lim\limits_{\alpha\to 0}x=\lim\limits_{\alpha\to 0}\dfrac{\alpha\cos\alpha-\sin\alpha}{\sin\alpha-\alpha}\left(\dfrac{0}{0}\text{form}\right)$

$$=\lim_{\alpha\to 0}\frac{-\alpha\sin\alpha+\cos\alpha-\cos\alpha}{\cos\alpha-1}$$

(Applying L-Hospital's rule)

$$=\lim_{\alpha\to 0}\frac{-\alpha\sin\alpha}{-2\sin^2\alpha/2}=\lim_{\alpha\to 0}\frac{\alpha\left(2\sin\dfrac{\alpha}{2}\cos\dfrac{\alpha}{2}\right)}{2\sin^2\dfrac{\alpha}{2}}$$

$$\lim_{\alpha\to 0}\frac{\alpha}{\tan\alpha/2}=\lim_{\alpha\to 0}\frac{2\dfrac{\alpha}{2}}{\tan\dfrac{\alpha}{2}}=2$$

Again, $\lim\limits_{\alpha\to 0}y=\lim\limits_{\alpha\to 0}\dfrac{\alpha-x\tan\alpha}{\sin\alpha}=\lim\limits_{x\to 0}\left(\dfrac{\alpha}{\sin\alpha}-\dfrac{x}{\cos\alpha}\right)$

$$\lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha}-\lim_{\alpha\to 0}\frac{x}{\cos\alpha}=1-2=-1\qquad\left[\because\lim_{\alpha\to 0}x=2\right]$$

$\Rightarrow\ \lim\limits_{\alpha\to 0}y=-1$

Hence, in limiting position $P(2,-1)\Rightarrow h-k=2+1=3$

Hence, the correct answer is (3).

33. $L=\lim\limits_{x\to\infty}\left(x\ln x+2x\ln\sin\left(\dfrac{1}{\sqrt{x}}\right)\right)$ Put $x=\dfrac{1}{t^2}$

$$L=\lim_{t\to 0^+}\left(\frac{1}{t^2}\ln\left(\frac{1}{t^2}\right)+\frac{2}{t^2}\ln\sin t\right)$$

$$=\lim_{t\to 0^+}\frac{(-\ln t^2+2\ln\sin t)}{t^2}$$

$$=\lim_{t\to 0^+}\frac{2(\ln\sin t-\ln t)}{t^2}$$

$$=\lim_{t\to 0^+}2\frac{\left(\dfrac{\cos t}{\sin t}-\dfrac{1}{t}\right)}{2t}=\lim_{t\to 0^+}\frac{\left(\dfrac{1}{\tan t}-\dfrac{1}{t}\right)}{t}$$

$$=\lim_{t\to 0^+}\frac{(t-\tan t)}{(t\tan t)t}=\lim_{t\to 0^+}\frac{t-\left(t+\dfrac{t^3}{3}+\ldots\right)}{t^3\left(\dfrac{\tan t}{t}\right)}$$

$L=-\dfrac{1}{3}\Rightarrow-\dfrac{1}{L}=3$

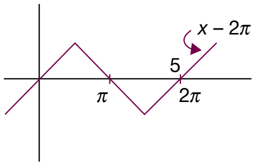
Hence, the correct answer is (3).

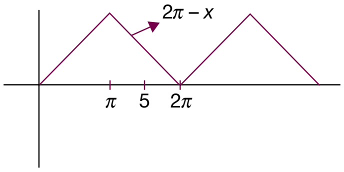
34. $\lim\limits_{x\to 2}\dfrac{\sqrt{x^2-2x+9}-\sqrt{9+2x-x^2}}{\sqrt{(x-1)^2+3}-\sqrt{5-(x-1)^2}}$

$$=\lim_{x\to 2}\frac{((x^2-2x+9)-(9+2x-x^2))}{(x-1)^2+3-(5-(x-1)^2)}\;\frac{\left(\sqrt{(x-1)^2+3}-\sqrt{5-(x-1)^2}\right)}{\left(\sqrt{x^2-2x+9}+\sqrt{9+2x-x^2}\right)}$$

$$=\lim_{x\to 2}\frac{2(x^2-2x)}{2((x-1)^2)}\,\frac{(2+2)}{(3+3)}=\lim_{x\to 2}\frac{x(x-2)}{x(x-2)}\cdot\frac{4}{6}$$

$$=\frac{2}{3}=\frac{p}{q}$$





$$\sin^{-1}(\sin(P+q)+\cos^{-1}(\cos(P+q))$$

$$=\sin^{-1}(\sin 5)+\cos^{-1}(\cos(5))$$

$$=5-2\pi+2\pi-5=0$$

Hence, the correct answer is (0).

35. $L=\lim\limits_{x\to 0}\dfrac{(1+x)^{\frac{1}{x}}+e(x-1)}{e\left(\dfrac{\sin^{-1}x}{x}\right)x}$

$$=\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}-e}{ex}+1=\lim_{x\to 0}\frac{e\left(1-\frac{x}{2}+\frac{11}{2}x^2+\ldots-1\right)}{ex}+1$$

$$=-\frac{1}{2}+1=\frac{1}{2};\ L=\frac{1}{2}$$

$$f(x)=\frac{2\tan^2 x}{\tan^2 x+(2\tan^2 x+3)^2}$$

Put $\tan^2 x=k;\ k\to(0,\infty)$

$$\frac{2k}{k+(2k+3)^2}=\frac{2k}{k+4k^2+9+12k}=\frac{2k}{4k^2+13k+9}$$

$$=\frac{1}{2k+\frac{13}{2}+\frac{9}{2k}};\ \frac{2k+\frac{9}{2k}}{2}\geq\sqrt{2k\frac{9}{2k}}$$

$$f(x)_{\max}=\frac{1}{6+\frac{13}{2}}=\frac{2}{25}=M$$

$$2k+\frac{9}{2k}\geq 6$$

$$L+\frac{1}{M}=\frac{1}{2}+\frac{25}{2}=\frac{26}{2}\qquad=13$$

Hence, the correct answer is (13).

36. $\lim\limits_{x\to 0}\sum\limits_{k=1}^{2013}\dfrac{\left\{\dfrac{x}{\tan x}+2k\right\}}{2013}=\lim\limits_{x\to 0}\left\{\dfrac{x}{\tan x}\right\}$

$$=\lim_{x\to 0}\left(\frac{x}{\tan x}-\left[\frac{x}{\tan x}\right]\right)=\lim_{x\to 0}\left(\frac{x}{\tan x}\right)=1.$$

∴ 1^{st} term of G.P. = 1

(Common ratio $=\lim\limits_{x\to 0}\dfrac{e^{x^3}(e^{\tan^3 x-x^3}-1)(\tan^3 x-x^3)}{\dfrac{2\ln(1+x^3\sin^2 x)}{x^3\sin^2 x}\cdot x^3\sin^2 x.(\tan^3 x-x^3)}$)

$$=\frac{1}{2}\lim_{x\to 0}\left(\frac{\tan x-x}{x^3}\right)\left(\frac{\tan^2 x-x^2+x\tan x}{x^2}\right)=\frac{1}{2}\cdot\frac{1}{3}\times 3=\frac{1}{2}$$

$$\therefore\quad S_\infty=\frac{1}{1-\left(\frac{1}{2}\right)}=2.$$

Hence, the correct answer is (2).

37. $-\lim\limits_{x\to 0}\dfrac{\sin x-x-x\cos x+x^2\cot x}{x^5}$

$$=-\lim_{x\to 0}\frac{(\sin x-x)+x\cot x(x-\sin x)}{x^5}$$

$$=-\lim_{x\to 0}\left(\frac{\sin x-x}{x^3}\right)\left(\frac{\tan x-x}{x^3}\right)\left(\frac{x}{\tan x}\right)$$

$$=-\left(\frac{-1}{6}\times\frac{1}{3}\right)=\frac{1}{18}=\frac{p}{q}$$

$\therefore\quad p+q=19.$

Hence, the correct answer is (19).

38. $f(x)=\lim\limits_{\theta\to 0}\dfrac{\begin{matrix}\cos(x+4\theta)+\cos x)-4(\cos(x+3\theta)\\+\cos(x+\theta)+6\cos(x+2\theta)\end{matrix}}{\theta^4}$

$$=\lim_{\theta\to 0}\frac{2\cos(x+2\theta).\cos 2\theta-8\cos(x+2\theta).\cos\theta+6\cos(x+2\theta)}{\theta^4}$$

$$=\lim_{\theta\to 0}\frac{2\cos(x+2\theta).((\cos 2\theta)-4\cos\theta+3)}{\theta^4}$$

$$=\lim_{\theta\to 0}\frac{2\cos(x+2\theta)(2\cos^2\theta-4\cos\theta+2)}{\theta^4}$$

$$=\lim_{\theta\to 0}4\cos(x+2\theta)\frac{(1-\cos\theta)^2}{\theta^4}=\cos x$$

So, maximum value of $f(x)=1$ and minimum value of $f(x)=-1$

∴ Difference between m*a*ximum and minimum value = 2

Hence, the correct answer is (2).

39. Let $f(x)=ax^2+bx+c$

As $f(0)=0\Rightarrow c=0$

$f(1)=0\Rightarrow a+b=0$

And, $f(2)=1\Rightarrow 4a+2b=0\Rightarrow 2a+b=\frac{1}{2}$

$$\therefore\quad a=\frac{1}{2},\ b=\frac{-1}{2}\Rightarrow f(x)=\frac{1}{2}x^2-\frac{1}{2}x=\frac{1}{2}x\cdot(x-1)$$

Now, $\lim\limits_{x\to\infty}\dfrac{\cos\left(\dfrac{\pi}{2}\cos^2 x\right)}{\left(\dfrac{1}{2}x(x-1)\right)^2}$ $\qquad\left(\dfrac{0}{0}\text{ Form}\right)$

$$=4\lim_{x\to 0}\frac{\cos\left(\frac{\pi}{2}(1-\sin^2 x)\right)}{x^2\cdot(x-1)^2}$$

$$=4\lim_{x\to 0}\frac{\sin\left(\frac{\pi}{2}\sin^2 x\right)\cdot\left(\frac{\pi}{2}\sin^2 x\right)}{\left(\frac{\pi}{2}\cdot\sin^2 x\right)\cdot x^2(x-1)^2}=4\cdot(1)\times\frac{\pi}{2}\times 2\pi=k\pi$$

So, $k=2$

Hence, the correct answer is (2).

40. Let $S_n=\left(\sin\dfrac{\pi}{2r}-\sin\dfrac{\pi}{2(r+2)}\right)-\left(\cos\dfrac{\pi}{2r}-\cos\dfrac{\pi}{2(r+2)}\right)$

$$\therefore\quad\Sigma S_n=\left[\left(\sin\frac{\pi}{2}+\sin\frac{\pi}{2}\right)-\left(\sin\frac{\pi}{2(n+1)}+\sin\frac{\pi}{2(n+2)}\right)\right]-\left[\left(\cos\frac{\pi}{2}+\cos\frac{\pi}{4}\right)-\left(\cos\frac{\pi}{2(n+1)}+\cos\frac{\pi}{2(n+2)}\right)\right]$$

$$\therefore\quad\lim_{n\to\infty}S_n=3$$

Hence, the correct answer is (3).

ARCHIVE: JEE MAIN

1. $f(n) = x^6 + 2x^4 + x^3 + 2x + 3$

$$\lim_{x\to1}\frac{x^n f(1) - f(x)}{x-1} = 44$$

$$\lim_{x\to1}\frac{9x^n - (x^6 + 2x^4 + x^3 + 2x + 3)}{x-1} = 44$$

$$\lim_{x\to1}\frac{9nx^{n-1} - (6x^5 + 8x^3 + 3x^3 + 2)}{1} = 44$$

$\Rightarrow \quad 9n - (19) = 44 \Rightarrow 9n = 63 \Rightarrow n = 7$

Hence, the correct answer is (7).

2. $\lim\limits_{x\to0}\dfrac{ax-(e^{4x}-1)}{ax(e^{4x}-1)}, \left(\dfrac{0}{0}\right)$

$$= \lim_{x\to0}\frac{ax-(e^{4x}-1)}{ax.4x}$$

Use $\lim\limits_{x\to0}\dfrac{e^{4x}-1}{4x} = 1$

Apply L'Hospital Rule $= \lim\limits_{x\to0}\dfrac{a-4e^{4x}}{8ax} \quad \left(\dfrac{a-4}{0}\text{form}\right)$

Limit exists only when $a - 4 = 0 \quad \Rightarrow a = 4$

$$= \lim_{x\to0}\frac{4-4e^{4x}}{32x} = \lim_{x\to0}\frac{1-e^{4x}}{8x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to0}\frac{-e^{4x}.4}{8} = -\frac{1}{2} \quad \Rightarrow \quad b = -\frac{1}{2}$$

$$a - 2b = 4 - 2\left(-\frac{1}{2}\right) = 5$$

Hence, the correct answer is (5).

3. $\lim\limits_{x\to0}\dfrac{\sin^2\left(\pi\cos^4 x\right)}{x^4}$

$$\lim_{x\to0}\frac{1-\cos\left(2\pi\cos^4 x\right)}{2x^4}$$

$$\lim_{x\to0}\frac{1-\cos\left(2\pi-2\pi\cos^4 x\right)}{\left[2\pi\left(1-\cos^4 x\right]^2\right)}4\pi^2.\frac{\sin^4 x}{2x^4}\ (1 + \cos^2 x^2)$$

$$= \frac{1}{2}.4\pi^2.\frac{1}{2}(2)^2 = 4\pi^2$$

Hence, the correct answer is (C).

4. $\alpha = \lim\limits_{x\to\frac{\pi}{4}}\dfrac{\tan^3 x - \tan x}{\cos\left(x+\dfrac{\pi}{4}\right)}; \left(\dfrac{0}{0}\text{ form}\right)$

Using L'Hopital rule $\alpha = \lim\limits_{x\to\frac{\pi}{4}}\dfrac{3\tan^2 x\sec^2 x - \sec^2 x}{-\sin\left(x+\dfrac{\pi}{4}\right)}$

$\Rightarrow \quad \alpha = -4$

$$\beta = \lim_{x\to0}(\cos x)^{\cot x} = e^{\lim\limits_{x\to0}\frac{(\cos x-1)}{\tan x}};$$

$$\beta = e^{\lim\limits_{x\to0}\frac{-(1-\cos x)}{x^2}\cdot\frac{x^2}{\left(\frac{\tan x}{x}\right)^x}}$$

$\beta = e^{\lim\limits_{x\to0}\left(\frac{-1}{2}\right)\cdot\frac{x}{1}} = e^0 \Rightarrow \beta = 1 \quad$ So, $\alpha = -4; \beta = 1$

If $ax^2 + bx - 4 = 0$ are the roots, then $16a - 4b - 4 = 0$ and $a + b - 4 = 0$

$\Rightarrow \quad a = 1$ and $b = 3$

Hence, the correct answer is (D).

5. Apply L'Hopital Rule

$$\lim_{x\to2}\left(\frac{2xf(2)-4f'(x)}{1}\right) = \frac{4(4)-4}{1} = 12$$

Hence, the correct answer is (D).

6. $\lim\limits_{x\to0}\left(\dfrac{x}{\sqrt[8]{1-\sin x}-\sqrt[8]{1+\sin x}}\right)$

$$= \lim_{x\to0}\left(\frac{x}{\sqrt[8]{1-\sin x}-\sqrt[8]{1+\sin x}}\right) = \left(\frac{\sqrt[8]{1-\sin x}+\sqrt[8]{1+\sin x}}{\sqrt[8]{1-\sin x}+\sqrt[8]{1+\sin x}}\right);$$

$$\left(\frac{\sqrt[4]{1-\sin x}+\sqrt[4]{1+\sin x}}{\sqrt[4]{1-\sin x}+\sqrt[4]{1+\sin x}}\right); = \left(\frac{\sqrt[2]{1-\sin x}+\sqrt[2]{1+\sin x}}{\sqrt[2]{1-\sin x}+\sqrt[2]{1+\sin x}}\right);$$

$$= \lim_{x\to0}\left(\frac{x}{1-\sin x-(1+\sin x)}\right)$$

$$\left(\sqrt[8]{1-\sin x}+\sqrt[8]{1+\sin x}\right)\left(\sqrt[4]{1-\sin x}+\sqrt[4]{1+\sin x}\right)\left(\sqrt[2]{1-\sin x}+\sqrt[2]{1+\sin x}\right)$$

$$= \lim_{x\to0}\frac{x}{(-2\sin x)}\left(\sqrt[8]{1-\sin x}+\sqrt[8]{1+\sin x}\right)\left(\sqrt[4]{1-\sin x}+\sqrt[4]{1+\sin x}\right)\left(\sqrt[2]{1-\sin x}+\sqrt[2]{1+\sin x}\right)$$

$$= \left(-\frac{1}{2}\right)(2)(2)(2) \qquad \left\{\because \lim_{x\to0}\frac{\sin x}{x} = 1\right\} = -4$$

Hence, the correct answer is (A).

7. Given limit is of 1^∞ form

So, $l = \exp\left(\lim\limits_{n\to\infty}\dfrac{1+\dfrac{1}{2}+\dfrac{1}{3}+......+\dfrac{1}{n}}{n}\right)$

Now, $0 \le 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n} \le 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \ldots. + \dfrac{1}{\sqrt{n}}$

So, $l = \exp(0)$ (from sandwich theorem) $= 1$

Hence, the correct answer is (D).

8. Given limit is of 1^∞ form

So, $l = \exp\left(\lim\limits_{n\to\infty}\dfrac{1+\dfrac{1}{2}+\dfrac{1}{3}+......+\dfrac{1}{n}}{n}\right)$

Now, $0 \le 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n} \le 1 + \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} + \ldots. + \dfrac{1}{\sqrt{n}}$

So, l = exp (0) (from sandwich theorem) = 1

Hence, the correct answer is (D).

9. $\lim\limits_{x\to2}\dfrac{3^x+3^{3-x}-12}{3^{-x/2}-3^{1-x}} = \lim\limits_{x\to2}\dfrac{3^{2x}-12.3^x+27}{3^{x/2}-3}$

$$= \lim_{x\to2}\frac{(3^x-9)(3^x-3)}{(3^{x/2}-3)} = \lim_{x\to2}\frac{(3^{x/2}+3)(3^{x/2}-3)(3^x-3)}{(3^{x/2}-3)} = 36$$

Hence, the correct answer is (36).

10. $\lim\limits_{x\to1}\dfrac{x+x^2+......+x^n-n}{x-1} = 820$

$\Rightarrow \quad \lim\limits_{x\to1}\left(\dfrac{x-1}{x-1}+\dfrac{x^2-1}{x-1}+.....\dfrac{x^n-1}{x-1}\right) = 820$

$\Rightarrow \quad 1 + 2 + \ldots + n = 820 \Rightarrow n(n + 1) = 2 \times 820 = 40 \times 41$

Since, $n \in N$, so $n = 40$

Hence, the correct answer is (40.00).

11. $$\lim_{x\to 0}\left\{\frac{1}{x^8}\left(1-\cos\frac{x^2}{2}-\cos\frac{x^2}{4}+\cos\frac{x^2}{2}\cos\frac{x^2}{4}\right)\right\}=2^{-k}$$

$$\Rightarrow\quad \lim_{x\to 0}\frac{\left(1-\cos\dfrac{x^2}{2}\right)}{4\left(\dfrac{x^2}{2}\right)^2}\frac{\left(1-\cos\dfrac{x^2}{4}\right)}{16\left(\dfrac{x^2}{4}\right)^2}=\frac{1}{8}\times\frac{1}{32}=2^{-k}$$

$$\Rightarrow\quad 2^{-8}=2^{-k}\Rightarrow k=8.$$

Hence, the correct answer is (8).

12. Required limit $= e^{\lim\limits_{x\to 0}\left(\frac{3x^2+2}{7x^2+2}-1\right)\frac{1}{x^2}} = e^{\lim\limits_{x\to 0}\frac{-4}{7x^2+2}} = \dfrac{1}{e^2}$

Hence, the correct answer is (1).

13. Required limit

$$=\lim_{h\to 0}\frac{(a+2(a+h))^{1/3}-(3(a+h))^{1/3}}{(3a+a+h))^{1/3}-(4(a+h))^{1/3}}$$

$$=\lim_{h\to 0}\frac{(3a)^{1/3}\left(1+\dfrac{2h}{3a}\right)^{1/3}-(3a)^{1/3}\left(1+\dfrac{h}{a}\right)^{1/3}}{(4a)^{1/3}\left(1+\dfrac{h}{4a}\right)^{1/3}-(4a)^{1/3}\left(1+\dfrac{h}{a}\right)^{1/3}}$$

$$=\lim_{h\to 0}\left(\frac{3^{1/3}}{4^{1/3}}\right)\left[\frac{\left(1+\dfrac{2h}{9a}\right)-\left(1+\dfrac{h}{3a}\right)}{\left(1+\dfrac{h}{12a}\right)-\left(1+\dfrac{h}{3a}\right)}\right]$$

$$=\left(\frac{3}{4}\right)^{1/3}\frac{\left(\dfrac{2}{9}-\dfrac{1}{3}\right)}{\left(\dfrac{1}{12}-\dfrac{1}{3}\right)}=\left(\frac{3}{4}\right)^{1/3}\left(\frac{4}{9}\right)=\frac{4^{1-\frac{1}{3}}}{3^{2-\frac{1}{3}}}=\frac{4^{2/3}}{3^{5/3}}$$

$$=\frac{(8\times 2)^{1/3}}{(27\times 9)^{1/3}}=\frac{2}{3}\left(\frac{2}{9}\right)^{1/3}$$

Hence, the correct answer is (A).

14. LHL: $\lim\limits_{x\to 0^-}\left|\dfrac{1-x-x}{\lambda-x-1}\right|=\left|\dfrac{1}{\lambda-1}\right|$

RHL: $\lim\limits_{x\to 0^+}\left|\dfrac{1-x+x}{\lambda-x+0}\right|=\left|\dfrac{1}{\lambda}\right|$

For existence of limit

$$\text{LHL}=\text{RHL}\Rightarrow\frac{1}{|\lambda-1|}=\frac{1}{|\lambda|}\Rightarrow\lambda=\frac{1}{2}\qquad\therefore L=\frac{1}{|\lambda|}=2$$

Hence, the correct answer is (B).

15. $\because\ \lim\limits_{x\to 0}\dfrac{\sqrt{1+x^2+x^4}-1}{x}\qquad\left(\dfrac{0}{0}\text{ from}\right)$

$$=\lim_{x\to 0}\frac{(1+x^2+x^4)-1}{x(\sqrt{1+x^2+x^4}+1)}=\lim_{x\to 0}\frac{x(1+x^2)}{\left(\sqrt{1+x^2+x^4}+1\right)}=0$$

So, $\lim\limits_{x\to 0}\dfrac{e^{\left(\frac{\sqrt{1+x^2+x^4}-1}{x}\right)}-1}{\dfrac{\sqrt{1+x^2+x^4}-1}{x}}=1$

Hence, the correct answer is (D).

16. $x^2-x-2=0$ has roots are 2 and –1

$$\lim_{x\to 2^+}\frac{\sqrt{1-\cos(x^2-x-2)}}{(x-2)}=\lim_{x\to 2^+}\frac{\sqrt{2\sin^2\dfrac{(x^2-x-2)}{2}}}{(x-2)}$$

$$=\lim_{x\to 2^+}\frac{\sqrt{2}\sin\left(\dfrac{(x-2)(x+1)}{2}\right)}{\dfrac{(x-2)(x+1)}{2}}\frac{(x+1)}{2}=\frac{3}{\sqrt{2}}$$

Hence, the correct answer is (A).

17. $\lim\limits_{y\to 0}\dfrac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}$ (Rationalise)

$$=\lim_{y\to 0}\frac{\sqrt{1+\sqrt{1+y^4}}-\sqrt{2}}{y^4}\times\frac{\left(\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}\right)}{\left(\sqrt{1+\sqrt{1+y^4}}+\sqrt{2}\right)}$$

$$=\lim_{y\to 0}\frac{\sqrt{1+y^4}-1}{2\sqrt{2}y^4}=\frac{1}{2\sqrt{2}}\lim_{y\to 0}\frac{\left(\sqrt{1+y^4}-1\right)\left(\sqrt{1+y^4}+1\right)}{\left(\sqrt{1+y^4}-1\right)\cdot y^4}$$

$$=\frac{1}{2\sqrt{2}}\lim_{y\to 0}\frac{1+y^4-1}{y^4\left(\sqrt{1+y^4}+1\right)}=\frac{1}{2\sqrt{2}}\times\frac{1}{2}=\frac{1}{4\sqrt{2}}$$

Hence, the correct answer is (B).

18. $\lim\limits_{x\to 0}\dfrac{\tan(\pi\sin^2 x)+(|x|-\sin(x[x]))^2}{x^2}$

$$\text{RHL}=\lim_{x\to 0^+}\frac{\tan(\pi\sin^2 x)+(x)^2}{x^2}$$

$$=\lim_{x\to 0^+}1+\frac{\tan(\pi\sin^2 x)}{x^2\pi\sin^2 x}\times\pi\sin^2 x$$

$$=1+\lim_{x\to 0^+}\pi\frac{\sin^2 x}{x^2}=1+\pi$$

$$\text{LHL}=\lim_{x\to 0^+}\frac{\tan(\pi\sin^2 x)+(x-\sin(x))^2}{x^2}$$

$$=\lim_{x\to 0^+}\frac{\tan(\pi\sin^2 x)+(x-\sin x)^2}{x^2}$$

$$=\lim_{x\to 0^+}\frac{\tan(\pi\sin^2 x)+(x^2-\sin^2 x-2x\sin x)}{x^2}$$

$$=1+\lim_{x\to 0^+}\left(\frac{\tan(\pi\sin^2 x)}{x^2}+\frac{\sin^2 x}{x^2}-\frac{2\sin x}{x}\right)$$

$=1+(\pi+1-2)=\pi$ Here, RHL $\ne$ LHL

Therefore, limit dose not exist.

Hence, the correct answer is (C).

19. $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}}$

$$=\lim_{x\to 0}\left(\frac{\sin^2 x}{\sqrt{2}-\sqrt{1+\cos x}}\right)\left(\frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}}\right)$$

$$=\lim_{x\to 0}\left(\frac{\sin^2 x}{1-\cos x}\right)(\sqrt{2}+\sqrt{1+\cos x})$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{2\sin^2\left(\dfrac{x}{2}\right)}(\sqrt{2}+\sqrt{1+\cos x})$$

$$\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin x}{x}\right)^2x^2\left(\frac{x/2}{\sin x/2}\right)^2\cdot\frac{1}{\left(\dfrac{x^2}{4}\right)}\cdot(\sqrt{2}+\sqrt{1+\cos x})$$

$$=\frac{1}{2}\times 4\times 2\sqrt{2}=4\sqrt{2}$$

Hence, the correct answer is (C).

CHAPTER 3

20. $\lim\limits_{x\to 0}\dfrac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x-x+1}}$

$= \lim\limits_{x\to 0}\dfrac{x+2\sin x}{x^2+2\sin x+1-\sin^2 x+x-1}(\sqrt{x^2+2\sin x+1}+\sqrt{\sin^2 x-x+1})$

$= \lim\limits_{x\to 0}\dfrac{2(x+2\sin x)}{x^2+2\sin x-\sin^2 x+x}2$ Applying L'H Rule

$= \lim\limits_{x\to 0}\dfrac{2(1+2\cos x)}{2x+2\cos x-2\sin x\cos x+1}=\dfrac{2(3)}{2+1}=2$

Hence, the correct answer is (D).

21. Since, $\lim\limits_{x\to 0^+} x\left(\left[\dfrac{1}{x}\right]+\left[\dfrac{2}{x}\right]+\ldots+\left[\dfrac{15}{x}\right]\right)$

$= \lim\limits_{x\to 0^+} x\left(\dfrac{1+2+3+\ldots+15}{x}\right)-\left(\left\{\dfrac{1}{x}\right\}+\left\{\dfrac{2}{x}\right\}+\ldots+\left\{\dfrac{15}{x}\right\}\right)$

$\because 0\le\left\{\dfrac{r}{x}\right\}<1 \Rightarrow 0\le+\left\{\dfrac{r}{x}\right\}<x$

$\therefore \lim\limits_{x\to 0^+}\left(\dfrac{1+2+3+9\ldots+15}{x}\right)=\dfrac{15\times 16}{2}=120$

Hence, the correct answer is (B).

22. $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x(1-\sin x)}{-8\left(x-\dfrac{\pi}{2}\right)^3}=\lim\limits_{x\to\frac{\pi}{2}}\dfrac{\cot x(1-\sin x)}{8\left(\dfrac{\pi}{2}-x\right)^3}$

Put $\dfrac{\pi}{2}-x=t \Rightarrow as\, x\to\dfrac{\pi}{2}\Rightarrow t\to 0$

$= \lim\limits_{x\to 0}\dfrac{\cot\left(\dfrac{\pi}{2}-t\right)\left(1-\sin\left(\dfrac{\pi}{2}-t\right)\right)}{8t^3}$

$= \lim\limits_{x\to 0}\dfrac{\tan t(1-\cos t)}{8t^3}=\lim\limits_{x\to 0}\dfrac{\tan t}{8t}\dfrac{1-\cos t}{t^2}=\dfrac{1}{8}.1\dfrac{1}{2}=\dfrac{1}{16}$

Hence, the correct answer is (C).

23. $p=\lim\limits_{x\to 0^+}\left(1+\tan^2\sqrt{x}\right)^{\frac{1}{2x}}$ (1^∞ form)

$= e^{\lim\limits_{x\to 0^+}(1+\tan^2\sqrt{x}-1)\frac{1}{2x}} = e^{\lim\limits_{x\to 0^+}\frac{(\tan\sqrt{x})^2}{2(\sqrt{x})^2}} = e^{\frac{1}{2}}$

$\therefore \log_e p=\log e^{\frac{1}{2}}=\dfrac{1}{2}$

Hence, the correct answer is (D).

24. $\lim\limits_{x\to 0}\dfrac{\sin(\pi\cos^2 x)}{x^2}=\lim\limits_{x\to 0}\dfrac{\sin(\pi-\pi\sin^2 x)}{x^2}$

$= \lim\limits_{x\to 0}\dfrac{\sin(\pi\sin^2 x)}{\pi\sin^2 x}\times\dfrac{\pi\sin^2 x}{x^2}=\pi$

Hence, the correct answer is (D).

25. $\lim\limits_{x\to 2}\dfrac{\tan(x-2)\{x^2+(k-2)x-2k\}}{(x-2)^2}$

$= \lim\limits_{x\to 2}\dfrac{\tan(x-2)}{(x-2)}\dfrac{\{x^2+(k-2)x-2k\}}{(x-2)}$

$= \lim\limits_{x\to 2}\dfrac{\{x^2+(k-2)x-2k\}}{(x-2)}=\lim\limits_{x\to 2}\dfrac{2x+(k-2)}{1-0}$

$= 4+k-2=5$, Thus, $k=3$

Hence, the correct answer is (A).

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. $f_n(x)=\sum\limits_{j=1}^{n}[\tan^{-1}(x+j)-\tan^{-1}(x+j-1)]$

$=\tan^{-1}(x+n)-\tan^{-1}x$

$f_n(x)=\tan^{-1}\left(\dfrac{n}{1+x(x+n)}\right)$

$f(0)$ can not be computed because zero is not in the domain, so option (A) and (B) can not be answer

(C) $\lim\limits_{x\to\infty}\tan(f_n(x))=0$

(D) $\lim\limits_{x\to\infty}\sec^2(f_n(x))=1$

Hence, the correct answer is (D).

2. $\lim\limits_{x\to\infty}\dfrac{x^2+x+1-ax^2-ax-bx-b}{x+1}$

$= \lim\limits_{x\to\infty}\dfrac{(1-a)x^2+(1-a-b)x+1-b}{x+1}=4$

For limit to exist, $1-a=0$; $1-a-b=4$

$\Rightarrow a=1; b=-4$

Hence, the correct answer is (B).

3. $\lim\limits_{a\to 0^+}(\alpha+\beta)=\lim\limits_{a\to 0^+}\dfrac{1-\sqrt{1+a}}{(1+a)^{1/3}-1}$

$= \lim\limits_{a\to 0^+}\dfrac{0-\dfrac{1}{2\sqrt{1+a}}}{\dfrac{1}{3}(1+a)^{-2/3}-0}=\lim\limits_{a\to 0^+}-\dfrac{3}{2}(1+a)^{1/6}=-\dfrac{3}{2}$

$\lim\limits_{a\to 0^+}\alpha\beta=\lim\limits_{a\to 0^+}\dfrac{(1+a)^{1/6}-1}{(1+a)^{1/3}-1}=\lim\limits_{a\to 0^+}\dfrac{1}{(1+a)^{1/6}+1}=\dfrac{1}{2}$

From options, option (B) satisfies.

Hence, the correct answer is (B).

4. $\lim\limits_{x\to 0}(1+x\ln(1+b^2))^{1/x}=e^{\lim\limits_{x\to 0}\frac{x\ln(1+b^2)}{x}}$

$= e^{\ln(1+b^2)}=1+b^2=2b\sin^2\theta\Rightarrow 2\sin^2\theta=\dfrac{1}{b}+b$

LHS $= 2\sin^2\theta\ge 2$; RHS $=\dfrac{1}{b}+b\ge 2$

Equality occurs, $\dfrac{1}{b}+b=2\Rightarrow b=1$, $\sin^2\theta=1$ For $\theta=\pm\dfrac{\pi}{2}$

Hence, the correct answer is (D).

Multiple Correct Choice Type Questions

5. $\lim\limits_{x\to 1^-}f(x)=\lim\limits_{h\to 0}\dfrac{h^2}{h}\cos\dfrac{1}{h}=0$

$\lim\limits_{x\to 1^+}f(x)=\lim\limits_{h\to 0}\dfrac{-h(2+h)}{h}\cos\dfrac{1}{h}$ does not exist

Hence, (A) and (D) are correct.

6. Given, $g(x)=\dfrac{\pi}{2}\sin x\ \forall\ x\in R$ then $g(x)\in[-\dfrac{\pi}{2},\dfrac{\pi}{2}]$

$\Rightarrow g(g(x))\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\ \forall\ x\in R$

$\Rightarrow g(g(g(x)))\in\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]\ \forall\ x\in R$

Given, $f(x) = \sin\left(\frac{1}{3}g\left(g(x)\right)\right)$,

Now, $\frac{g(g(x))}{3} \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \forall\ x \in R$

Hence, $f(x)$ and $f(g(x)) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\sin\left(\frac{1}{3}g(g(x))\right)}{\frac{1}{3}g(g(x))}\cdot\frac{\frac{1}{3}g(g(x))}{g(x)}$$

$$= \lim_{x\to 0}\frac{\frac{1}{3}g(g(x))}{g(x)}$$

$$= \lim_{x\to 0}\frac{\pi}{6}\cdot\frac{\sin\left(\frac{\pi}{2}\sin x\right)}{\frac{\pi}{2}\sin x} = \frac{\pi}{6}$$

Range of $g(f(x)) \in \left[-\frac{\pi}{2}\sin\left(\frac{1}{2}\right), \frac{\pi}{2}\sin\left(\frac{1}{2}\right)\right] \Rightarrow g(f(x)) \neq 1$

Hence, (A), (B) and (C) are correct.

Integer/Numerical Answer Type Questions

7. $$\lim_{x\to 0^+}\frac{e^{\ln(1-x)^{\frac{1}{x}}} - e^{-1}}{x^a}$$

$$= \lim_{x\to 0^+}\frac{e^{\frac{\ln(1-x)}{x}} - e^{-1}}{x^a} = \lim_{x\to 0^+}\frac{e^{\frac{-x-\frac{x^2}{2}-\frac{x^3}{3}\cdots\cdot}{x}} - e^{-1}}{x^a}$$

$$= \lim_{x\to 0^+}\frac{e^{-1-\frac{x}{2}-\frac{x^2}{3}\cdots\cdot} - e^{-1}}{x^a}$$

$$= \lim_{x\to 0^+} e^{-1}\left[\frac{e^{-\frac{x^2}{2}-\frac{x^3}{3}\cdots\cdot} - 1}{-\frac{x}{2}-\frac{x^2}{3}\cdots\cdot}\right]\frac{\left(-\frac{x}{2}-\frac{x^2}{3}\cdots\cdot\right)}{x^a}$$

$$= \lim_{x\to 0^+}\frac{1}{e}(1)\frac{\left[-\frac{x}{2}-\frac{x^2}{3}\cdots\cdot\right)}{x^a} = \text{non zero real number}$$

Thus, $a = 1$

Hence, the correct answer is (1).

8. $$\lim_{x\to\frac{\pi}{2}}\frac{4\sqrt{2}(3\sin x - 4\sin^3 x + \sin x)}{2\sin 2x\sin\frac{3x}{2} + \cos\frac{5x}{2} - \cos\frac{3x}{2} - \left(\sqrt{2}(1+\cos 2x)\right)}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{4\sqrt{2}(4\sin x(1-\sin^2 x))}{2\sin 2x\sin\frac{3x}{2} + 2\sin 2x\sin\left(\frac{-x}{2}\right) - 2\sqrt{2}\cos^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x\cos^2 x}{2\sin 2x\left[\sin\frac{3x}{2} - \sin\frac{x}{2}\right] - 2\sqrt{2}\cos^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x\cos^2 x}{2\sin 2x\left[2\cos x\sin\frac{x}{2}\right] - 2\sqrt{2}\cos^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x\cos^2 x}{8\sin x\cos^2 x\sin\frac{x}{2} - 2\sqrt{2}\cos^2 x}$$

$$= \lim_{x\to\frac{\pi}{2}}\frac{16\sqrt{2}\sin x}{8\sin x\sin\frac{x}{2} - 2\sqrt{2}}$$

$$= \frac{16\sqrt{2}(1)}{8(1)\left(\frac{1}{\sqrt{2}}\right) - 2\sqrt{2}} = \frac{16\sqrt{2}}{2\sqrt{2}} = 8$$

Hence, the correct answer is (8).

9. $$\lim_{x\to 0}\frac{x^2\sin\beta x}{\alpha x - \sin x} = \lim_{x\to 0}\frac{x^2\left(\beta x - \frac{\beta^3 x^3}{3!} + \frac{\beta^5 x^5}{5!} - \ldots\ldots.\right)}{\alpha x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots.\right)}$$

$$= \mathrm{Lim}_{x\to 0}\frac{x^3\left(\beta - \frac{\beta^3 x^2}{3!} + \ldots\ldots.\right)}{(\alpha-1)x + \frac{x^3}{3!} - \frac{x^5}{5!} + \ldots\ldots.} = 1$$

For limit to be non zero $\alpha - 1 = 0 \Rightarrow \alpha = 1$,

Then, limit $= 6\beta = 1 \Rightarrow \beta = \frac{1}{6}$

Now, required term $6(\alpha+\beta) = 6\left(1+\frac{1}{6}\right) = 7$

Hence, the correct answer is (7).

10. $$\lim_{\alpha\to 0}\frac{e^{\cos(\alpha^n)} - e}{\alpha^m} = -\frac{e}{2}$$

$$\lim_{\alpha\to 0}\frac{e\left(e^{\cos(\alpha^n)-1} - 1\right)}{\alpha^m} = \lim_{\alpha\to 0}\left(\frac{e\left(e^{\cos(\alpha^n)-1} - 1\right)}{\left(\cos\alpha^n - 1\right)}\right)\frac{\left(\cos\alpha^n - 1\right)}{\alpha^{2n}}\frac{\alpha^{2n}}{\alpha^m} = -\frac{e}{2}$$

$\lim_{\alpha\to 0}(1)\frac{\left(\cos\alpha^n - 1\right)}{\alpha^{2n}}\frac{\alpha^{2n}}{\alpha^m} = -\frac{e}{2}$ for limit to converge at given value

$2n = m$

Hence, the correct answer is (2).

11. Given, $\lim_{x\to 1}\left\{\frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1}\right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$

$$\Rightarrow \lim_{x\to 1}\left\{\frac{-a(x-1) + \sin(x-1)}{(x-1) + \sin(x-1)}\right\}^{1+\sqrt{x}} = \frac{1}{4}$$

Put $x = 1 + h$,

$$\lim_{h\to 0}\left\{\frac{-ah + \sinh}{h + \sinh}\right\}^{1+\sqrt{1+h}} = \lim_{h\to 0}\left\{\frac{-a + \frac{\sinh}{h}}{1 + \frac{\sinh}{h}}\right\}^{1+\sqrt{1+h}} = \frac{1}{4}$$

$$\Rightarrow \left(\frac{-a+1}{2}\right)^2 = \frac{1}{4}$$

or $\frac{-a+1}{2} = \frac{1}{2}$ or $-\frac{1}{2} \Rightarrow a = 0$ or 2

But at $a = 2$, base $\frac{-ah + \sinh}{h + \sinh}$ tends to negative value, so rejected.

At $a = 0$, base is positive.

Hence, $a = 0$ is accepted.

Hence, the correct answer is (0).

CHAPTER 4: METHOD OF DIFFERENTIATION

Test Your Concepts-I

1. $f(x) = (x^2 - 4x + 5)(x^3 - 2)$ $\left[\because \dfrac{d(uv)}{dx} = v\dfrac{du}{dx} + u\dfrac{du}{dx}\right]$

$= (x^2 - 4x + 5)(3x^2) + (x^3 - 2)(2x - 4)$

$= 3x^4 - 12x^3 + 15x^2 + 2x^4 - 4x^3 - 4x + 8$

Hence, $f'(x) = 5x^4 - 16x^3 + 15x^2 - 4x + 8$

2. $f(x) = (3x + 8)^{7/3} \cdot (x + 7)^{-3}$ [Using product rule]

$$f'(x) = (3x + 8)^{7/3}[-3(x + 7)^{-4}] + (x + 7)^{-3} \times \frac{7}{3}(3x + 8)^{4/3} \cdot 3$$

$= -3(3x + 8)^{7/3}(x + 7)^{-4} + 7(x + 7)^{-3} \cdot (3x + 8)^{4/3}$

$= (3x + 8)^{4/3}(x + 7)^{-4}[-3(3x + 8) + 7(x + 7)]$

$= (3x + 8)^{4/3}(x + 7)^{-4}[-9x - 24 + 7x + 49]$

Hence, $f'(x) = (25 - 2x)(3x + 8)^{4/3}(x + 7)^{-4}$

3. $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$

$$\Rightarrow \quad f'(x) = \frac{(1+\sqrt{x})\left(\dfrac{-1}{2\sqrt{x}}\right) - (1-\sqrt{x})\left(\dfrac{1}{2\sqrt{x}}\right)}{(1+\sqrt{x})^2} \quad \left[\because \left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}\right]$$

$$= \frac{1}{2\sqrt{x}} \frac{(-1-\sqrt{x}-1+\sqrt{x})}{(1+\sqrt{x})^2} = \frac{-1}{\sqrt{x}(1+\sqrt{x})^2}$$

4. $f(x) = \dfrac{(x^2+1)(x+3)}{x} = \dfrac{x^3+3x^2+x+3}{x}$

$= x^2 + 3x + 1 + \dfrac{3}{x}$

$\therefore \quad f'(x) = 2x + 3 + 0 - \dfrac{3}{x^2} = \dfrac{2x^3+3x^2-3}{x^2}$

5. $f(x) = \sqrt{a+\sqrt{a+x}}$

$$f'(x) = \frac{1}{2\sqrt{a+\sqrt{a+x}}} \times \left(0 + \frac{1}{2\sqrt{a+x}}\right) = \frac{1}{4(\sqrt{a+\sqrt{a+x}})(\sqrt{a+x})}$$

6. $f(x) = \sqrt{4+\sqrt{4+\sqrt{4+x^2}}}$

$$\therefore \quad f'(x) = \frac{1}{2\sqrt{4+\sqrt{4+\sqrt{4+x^2}}}} \times \left(0 + \frac{1}{2\sqrt{4+\sqrt{4+x^2}}}\right) \times \left(0 + \frac{1}{2\sqrt{4+x^2}}\right) \times 2x$$

$$= \frac{x}{4\sqrt{4+\sqrt{4+\sqrt{4+x^2}}}\sqrt{4+\sqrt{4+x^2}}\sqrt{4+x^2}}$$

7. $f(x) = \dfrac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}$

$$f'(x) = \frac{\left[\begin{array}{l}(\sqrt{a+x}+\sqrt{a-x})\left(\dfrac{1}{2\sqrt{a+x}} + \dfrac{1}{2\sqrt{a-x}}\right) - \\ (\sqrt{a+x}-\sqrt{a-x})\left(\dfrac{1}{2\sqrt{a+x}} - \dfrac{1}{2\sqrt{a-x}}\right)\end{array}\right]}{(\sqrt{a+x}+\sqrt{a-x})^2}$$

$$= \frac{1}{2\sqrt{a+x}\sqrt{a-x}} \frac{\left[(\sqrt{a+x}+\sqrt{a-x})^2 + (\sqrt{a+x}-\sqrt{a-x})^2\right]}{(\sqrt{a+x}+\sqrt{a-x})^2}$$

$$= \frac{2(a+x+a-x)}{2\sqrt{a+x}\sqrt{a-x}(\sqrt{a+x}+\sqrt{a-x})^2} = \frac{2a}{(\sqrt{a+x}+\sqrt{a-x})^2\sqrt{a^2-x^2}}$$

8. $f(x) = \log(x + \sqrt{x^2+a^2})$

$$\therefore \quad f'(x) = \frac{d\log(x+\sqrt{a^2-x^2})}{d(x+\sqrt{a^2-x^2})} \times \frac{d(x+\sqrt{a^2+x^2})}{dx}$$

$$= \frac{1}{x+\sqrt{a^2+x^2}} \times \left(1 + \frac{1}{2\sqrt{x^2+a^2}} \times 2x\right)$$

$$= \frac{1}{x+\sqrt{a^2+x^2}}\left(\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}}\right) = \frac{1}{\sqrt{x^2+a^2}}$$

9. $f(x) = 10^{10^x}$

$$\therefore \quad f'(x) = \frac{d10^{10^x}}{d10^x} \times \frac{d10^x}{dx} + \frac{d\log\left[\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right]}{d\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)} \times \frac{d\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}{dx}$$

$$= 10^{10^x}\log 10 \cdot 10^x \log 10 + \frac{1}{\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)} \times \sec^2\left(\frac{\pi}{4}+\frac{x}{2}\right) \times \frac{1}{2}$$

$$= 10^{10^x} \cdot 10^x(\log 10)^2 + \frac{1}{2\sin\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\cos\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}$$

$$f'(x) = 10^{10^x} \cdot 10^x(\log 10)^2 + \frac{1}{\sin\left(\dfrac{\pi}{2}+x\right)}$$

$= 10^{10^x} \cdot 10^x(\log 10)^2 + \sec x$

10. Let $g(x) = e^{ax} \cdot \cot(bx + c)$

$\therefore \quad f'(x) = e^{ax} \cdot [\cot(bx + c)]' + \cot(bx + c)(e^{ax})'$

$= e^{ax}[-\mathrm{cosec}^2(bx + c) \cdot b] + \cot(bx + c)(e^{ax} \cdot a)$

$= e^{ax}[a\cot(bx + c) - b\,\mathrm{cosec}^2(bx + c)]$

Let $h(x) = \log_{\sin x}\cos x$

$\Rightarrow \quad h(x) = \dfrac{\log\cos x}{\log\sin x}$

$$\therefore \quad h'(x) = \frac{(\log\sin x)(\log\cos x)' - (\log\cos x)(\log\sin x)'}{(\log\sin x)^2}$$

$$= \frac{(\log\sin x)(-\tan x) - \log\cos x(\cot x)}{(\log\sin x)^2}$$

$f'(x) = g'(x) + h'(x)$

$$= e^{ax}[a\cot(bx + c) - b\,\mathrm{cosec}^2(bx + c)] + \frac{(\log\sin x)(-\tan x) - \log\cos x(\cot x)}{(\log\sin x)^2}$$

11. Let $g(x) = \log_e(\sec x + \tan x)$

$$\therefore \quad g'(x) = \frac{d\log_e(\sec x+\tan x)}{d(\sec x+\tan x)} \times \frac{d(\sec x+\tan x)}{dx}$$

$$= \frac{1}{\sec x+\tan x} \times (\sec x\tan x + \sec^2 x) = \sec x$$

Let $h(x) = \log_e \sin\sqrt{x^2+1}$

$$\therefore \quad h(x) = \frac{d\log_e\sin\sqrt{x^2+1}}{d\sin\sqrt{x^2+1}}\ \frac{d\sin\sqrt{x^2+1}}{d\sqrt{x^2+1}}\ \frac{d\sqrt{x^2+1}}{d(x^2+1)}\ \frac{d(x^2+1)}{dx}$$

$$= \frac{1}{\sin\sqrt{x^2+1}}\cos\sqrt{x^2+1}\ \frac{1}{2\sqrt{x^2+1}}\,2x = \frac{x\cot\sqrt{x^2+1}}{\sqrt{x^2+1}}$$

Now, $f'(x) = g'(x) + h'(x) = \sec x + \dfrac{x\cot\sqrt{x^2+1}}{\sqrt{x^2+1}}$

Test Your Concepts-II

1. We have, $xy^3 = x$ differentiating w.r.t. x,

We get, $\left[x\left(3y^2\frac{dy}{dx}\right)+y^3\right] - \left[y(3x^2)+x^3\frac{dy}{dx}\right] = 1$

$$\Rightarrow \quad 3xy^2\frac{dy}{dx} + y^3 - 3x^2y - x^3\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx}(3xy^2 - x^3) = 1 + 3x^2y - y^3$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1+3x^2y-y^3}{3xy^2-x^3}$$

2. We have, $x^3 + y^3 = 3axy$, differentiating w.r.t. x,

We get, $3x^2 + 3y^2\frac{dy}{dx} = 3a\left(x\frac{dy}{dx}+y\right)$

$$\Rightarrow \quad \frac{dy}{dx}(3y^2 - 3ax) = 3ay - 3x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{3(ay-x^2)}{3(y^2-ax)} = \frac{ay-x^2}{y^2-ax}$$

3. $x \log y + y \log x = 5$ differentiating w.r.t. x,

We get, $\frac{x}{y}\frac{dy}{dx} + \log y + \frac{y}{x} + \log x\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{x}{y}+\log x\right) = -\left(\log y + \frac{y}{x}\right)$$

$$\therefore \quad \frac{dy}{dx} = \frac{-y}{x}\left(\frac{x\log y + y}{x + y\log x}\right)$$

4. We have, $y = \log_y x$

$$\Rightarrow \quad y = \frac{\log_e x}{\log_e y} \qquad \left[\because \log_a b = \frac{\log b}{\log a}\right]$$

$$\Rightarrow \quad y\log_e y = \log_e x$$

On differentiating w.r.t. x, we get

$$(1 + \log_e y)\frac{dy}{dx} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x(1+\log_e y)}$$

5. Let $y = x^{x^x}$

On taking log both sides, we get

$$\log y = x^x \log x$$

Again, on taking log both sides, we get

$\log(\log y) = \log x^x + \log(\log x) \Rightarrow \log(\log y) = x\log x + \log(\log x)$

On differentiating w.r.t. x, we get

$$\frac{1}{y\log y}\frac{dy}{dx} = 1 + \log x + \frac{1}{x\log x}$$

$$\frac{dy}{dx} = y\log y\left(1+\log x+\frac{1}{x\log x}\right)$$

$$= x^{x^x}\ x^x\log x\ \frac{(x\log x + x(\log x)^2 + 1)}{x\log x}$$

$$= x^{x^x}\ x^{x-1}[x\log x + x(\log x)^2 + 1]$$

6. Let $y = (\sin x)^{\cos^{-1}x}$

On taking log both sides, we get

$$\log y = \cos^{-1}x \log \sin x$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \cos^{-1}x\left(\frac{1}{\sin x}\times\cos x\right) - \frac{1}{\sqrt{1-x^2}}\log\sin x$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\cot x\cos^{-1}x - \frac{\log\sin x}{\sqrt{1-x^2}}\right)$$

$$\therefore \quad \frac{dy}{dx} = (\sin x)^{\cos^{-1}x}\left(\cot x\cos^{-1}x - \frac{\log\sin x}{\sqrt{1-x^2}}\right)$$

7. Let $y = \frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}$

On taking log both sides, we get

$$\log y = \frac{1}{2}\log x + \frac{3}{2}\log(x+4) - \frac{4}{3}\log(4x-3)$$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} + \frac{3}{2(x+4)} - \frac{4\times 4}{3(4x-3)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} + \frac{3}{2(x+4)} - \frac{16}{3(4x-3)}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left[\frac{1}{2x}+\frac{3}{2(x+4)}-\frac{16}{3(4x-3)}\right]$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\sqrt{x}(x+4)^{3/2}}{(4x-3)^{4/3}}\left[\frac{1}{2x}+\frac{3}{2x+8}-\frac{16}{12x-9}\right]$$

8. We have, $y = x^{x^2}$

On taking log both sides, we get $\log y = x^2\log_e x$

On differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{x^2}{x} + 2x\log_e x$$

$$\frac{dy}{dx} = y(x + 2x\log_e x)$$

$$\Rightarrow \quad \frac{dy}{dx} = x^{x^2}\cdot x(1 + 2\log_e x) = x^{x^2+1}(2\log_e x + 1)$$

Hence, $k = 1$

Test Your Concepts-III

1. We have, $y = x - x^2$ …(1)

$$\Rightarrow \quad \frac{dy}{dx} = 1 - 2x \qquad \text{…(2)}$$

Let $u = y^2$ and $v = x^2$

$$\frac{du}{dx} = 2y\frac{dy}{dx} \text{ and } \frac{dv}{dx} = 2x$$

$$\therefore \quad \frac{du}{dv} = \frac{2ydy/dx}{2x} = \frac{(x-x^2)(1-2x)}{x} \quad \text{[from Eqs. (1) and (2)]}$$

$$= (1-x)(1-2x) = 2x^2 - 3x + 1$$

2. We have, $x = e^{y+e^{y+\ldots\infty}}$

$$x = e^{y+x}$$

On taking log both sides, we get

$$\log x = (x+y)\log_e e$$

$$\Rightarrow \quad \log x = x + y$$

On differentiating w.r.t. x, we get

$$\frac{1}{x} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1-x}{x}$$

(i) $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots.\infty}}}$

(iii) $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots.\infty}}}$

3. (i) We have, $y = \sqrt{\sin x + \sqrt{\sin x + \sqrt{\sin x + \ldots\infty}}}$

$$y = \sqrt{\sin x + y}$$

$\Rightarrow \quad y^2 = \sin x + y \Rightarrow y^2 - y = \sin x$

On differentiating w.r.t. x, we get

$$(2y-1)\frac{dy}{dx} = \cos x$$

Hence, $\dfrac{dy}{dx} = \dfrac{\cos x}{2y-1}$

(ii) We have, $y = \sqrt{\cos x + \sqrt{\cos x + \ldots\infty}}$

$\Rightarrow \quad y = \sqrt{\cos x + y} \Rightarrow y^2 = \cos x + y \Rightarrow y^2 - y = \cos x$

On differentiating w.r.t. x, we get

$$(2y-1)\frac{dy}{dx} = -\sin x$$

Hence, $\dfrac{dy}{dx} = \dfrac{\sin x}{1-2y}$

4. $x = a\left(\cos t + \frac{1}{2}\log_e \tan^2 t/2\right) = a(\cos t + \log_e \tan t/2)$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2} \times \sec^2 t/2 \times \frac{1}{2}\right);$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{2\sin t/2\cot/2}\right)$$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\sin t}\right) = \frac{a\cos^2 t}{\sin t} \text{ and } \frac{dy}{dt} = a\cos t$$

Hence, $\dfrac{dy}{dx} = \dfrac{a\cos t}{a\dfrac{\cos^2 t}{\sin t}} = \tan t$

5. $x = \sin t\sqrt{\cos 2t}$, $\dfrac{dx}{dt} = \cos t\sqrt{\cos 2t} - \dfrac{\sin t \sin 2t}{\sqrt{\cos 2t}}$

$$\frac{dx}{dt} = \frac{\cos t\cos 2t - \sin t\sin 2t}{\sqrt{\cos 2t}} = \frac{\cos 3t}{\sqrt{\cos 2t}}$$

$y = \cos t\sqrt{\cos 2t}$; $\dfrac{dy}{dt} = -\sin t\sqrt{\cos 2t} - \dfrac{\cos t \sin 2t}{\sqrt{\cos 2t}} = -\dfrac{\sin 3t}{\sqrt{\cos 2t}}$

$$\therefore \frac{dy}{dx} = -\frac{\sin 3t}{\sqrt{\cos 2t}} \bigg/ \frac{\cos 3t}{\sqrt{\cos 2t}} = \frac{-\sin 3t}{\cos 3t} = -\tan 3t$$

6. (i) Let $u = \tan^{-1}\left(\dfrac{1+2x}{1-2x}\right)$ and $v = \sqrt{1+4x^2}$

$u = \tan^{-1}\left(\dfrac{1+\tan\theta}{1-\tan\theta}\right)$ [$\because$ Put $2x = \tan\theta$]

$u = \tan^{-1}\tan\left(\dfrac{\pi}{4}+\theta\right) \Rightarrow u = \dfrac{\pi}{4} + \theta$

$u = \dfrac{\pi}{4} + \tan^{-1} 2x \Rightarrow \dfrac{du}{dx} = \dfrac{2}{1+4x^2}$ and $\dfrac{dv}{dx} = \dfrac{8x}{2\sqrt{1+4x^2}}$

$$\therefore \frac{du}{dx} = \frac{\dfrac{2}{1+4x^2}}{\dfrac{4x}{\sqrt{1+4x^2}}} = \frac{1}{2x\sqrt{1+4x^2}}$$

7. Let $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$, $v = \tan^{-1}x$

$u = \tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$ [$\because$ Put $x = \tan\theta$]

$$u = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\tan\frac{\theta}{2}$$

$$u = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

$\Rightarrow \quad \dfrac{du}{dx} = \dfrac{1}{2(1+x^2)}$ and $\dfrac{dv}{dx} = \dfrac{1}{1+x^2}$

Hence, $\dfrac{du}{dx} = \dfrac{1}{2}$

8. Let $u = \sin^{-1}\dfrac{x}{\sqrt{1+x^2}}$

$u = \tan^{-1}x,\ x > 0$ $\left[\because \sin^{-1}\dfrac{x}{\sqrt{1+x^2}} = \tan^{-1}x\right]$

$\therefore \quad \dfrac{du}{dx} = \dfrac{1}{1+x^2}$

$v = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ $\left[\because \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = 2\tan^{-1}x\right]$

$v = 2\tan^{-1}x,\ x > 0$, we get $\dfrac{dv}{dx} = \dfrac{1}{2(1+x^2)}$

Hence, $\dfrac{du}{dv} = \dfrac{du}{dx}\bigg/\dfrac{dv}{dx} = 2$

Test Your Concepts-IV

1. We have, $g(x) = f^{-1}(x)$ let $f(x) = t$

$e^{f(x)} = \log_e x.$, we get $x = e^{e^t}$

$g(x) = e^{e^x} \Rightarrow g'(x) = e^{e^x} \cdot e^x = e^{x+e^x}$

2. We have, $g(x) = f^{-1}(x)$

$\therefore \quad fog(x) = x$

On differentiating w.r.t. x, we get

$$f'og(x) \cdot g'(x) = 1$$

Now, on putting $x = 2$, we get

$f'og(2) \cdot g'(2) = 1$ [$\because\ g(2) = a$]

$$f'(a) \cdot g'(2) = 1$$

$$\therefore \quad g'(2) = \frac{1}{f(a)} = \frac{1+a^2}{a^{10}}$$

3. Let $y = \tan^{-1}(\sqrt{1+x^2} + x)$ Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$;

$$y = \tan^{-1}(\sqrt{1+\tan^2\theta} + \tan\theta)$$

$$= \tan^{-1}(\sec\theta + \tan\theta) = \tan^{-1}\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

$$= \tan^{-1}\left[\frac{(\cos\theta/2 + \sin\theta/2)^2}{\cos^2\theta/2 - \sin^2\theta/2}\right]$$

$$= \tan^{-1}\left[\frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}\right]$$

$$= \tan^{-1}\tan(\pi/4 + \theta/2) = \pi/4 + \theta/2$$

$\Rightarrow \quad y = \pi/4 + \dfrac{1}{2}\tan^{-1}x$

On differentiating w.r.t. x, we get $\dfrac{dy}{dx} = \dfrac{1}{2(1+x^2)}$

4. Let $y = \sin^{-1}\sqrt{\dfrac{1+x^2}{2}}$, put $x^2 = \cos\theta \Rightarrow \theta = \cos^{-1}x^2$

$$y = \sin^{-1}\sqrt{\frac{1+\cos\theta}{2}} = \sin^{-1}\sqrt{\frac{2\cos^2\frac{\theta}{2}}{2}} = \sin^{-1}\cos\frac{\theta}{2}$$

$$\Rightarrow \quad y = \sin^{-1}\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\Rightarrow \quad y = \frac{\pi}{2} - \frac{1}{2}\cos^{-1}x^2$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2}\left[\frac{-1}{\sqrt{1-x^4}} \times (2x)\right] = \frac{x}{\sqrt{1-x^4}}$$

5. Let $y = \sin^{-1}(x^2\sqrt{1-x^2} + x\sqrt{1-x^4})$

$$y = \sin^{-1}x^2 + \sin^{-1}x$$

$$[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}(x\sqrt{1-y^2} + y\sqrt{1-x^2})]$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{\sqrt{1-x^4}} + \frac{1}{\sqrt{1-x^2}} = \frac{2x + \sqrt{1+x^2}}{\sqrt{1-x^4}}$$

6. We have, $y = \sin^{-1}\left(\frac{x^2-1}{x^2+1}\right) + \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$

$$y = \text{cosec}^{-1}\left(\frac{x^2+1}{x^2-1}\right) + \sec^{-1}\left(\frac{x^2+1}{x^2-1}\right)$$

$$y = \frac{\pi}{2} \qquad [\because \text{cosec}^{-1}x + \sec^{-1}x = \pi/2]$$

$$\therefore \quad \frac{dy}{dx} = 0$$

7. Given, $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin 2\alpha & \sin 2\beta & \sin 2\gamma \end{vmatrix}$, where α, β are real constants.

$f(x) = \sin 2\alpha \sin(\beta - \gamma) - \sin 2\beta\,(\sin(\gamma - \alpha)) + \sin 2\gamma\,(\sin(\beta - \alpha)) =$ Constant

Now, $f'(x) = 0$ for every $x \in R$ (Differentiate row wise)

So, $2f\left(\frac{\pi}{4}\right) - 3f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right) = 2k - 3k + k = 3k - 3k = 0$

Test Your Concepts-V

1. We have, $\log_e y = \tan^{-1}x$

$$\Rightarrow \quad y = e^{\tan^{-1}x}$$

On differentiating w.r.t. x, we get

$$y_1 = e^{\tan^{-1}x} \cdot \frac{1}{1+x^2}$$

or, $(1 + x^2)\,y_1 = y$

Again on differentiating w.r.t. x, we get

$$(1 + x^2)\,y_2 + y_1 2x = y_1$$

$$\Rightarrow \quad (1 + x^2)\,y_2 + (2x - 1)y_1 = 0$$

2. We have, $y = \tan x + \sec x$

On differentiating w.r.t. x, we get

$$y_1 = \sec^2 x + \sec x \tan x$$

or, $y_1 = \sec x(\sec x + \tan x)$

$$= \frac{1+\sin x}{\cos^2 x} = \frac{1+\sin x}{1-\sin^2 x}$$

$$\therefore \quad y_1 = \frac{1}{1-\sin x}$$

Again on differentiating w.r.t. x, we get

$$y_2 = -\frac{1}{(1-\sin x)^2} \times (-\cos x)$$

$$\therefore \quad y_2 = \frac{\cos x}{(1-\sin x)^2}$$

3. We have, $x = a\cos^3\theta$ and $y = a\sin^3\theta$

Now, $\frac{dx}{d\theta} = 3a\cos^2\theta\,(-\sin\theta) = -3a\cos^2\theta \times \sin\theta$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{3a\sin^2\theta\cdot\cos\theta}{-3a\cos^2\theta\cdot\sin\theta} = -\tan\theta; \quad \frac{d^2y}{dx^2} = -\sec^2\theta\cdot\frac{d\theta}{dx}$$

$$= -\sec^2\theta\cdot\frac{1}{-3a\cos^2\theta\cdot\sin\theta}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{3a}\sec^4\theta\cdot\text{cosec}\,\theta$$

4. We have, $y = e^x \cdot \tan^{-1}x$,

$$y_1 = e^x \cdot \frac{1}{1+x^2} + \tan^{-1}x \cdot e^x$$

$$(1 + x^2)\,y_1 = e^x + y(1 + x^2) \qquad \ldots(1)$$

$$(1 + x^2)\,y_2 + y_1 2x = e^x + y\cdot 2x + (1 + x^2)\,y_1$$

$$\Rightarrow \quad (1 + x^2)\,y_2 + y_1(2x - 1 - x^2) = 2xy + e^x \text{ using (1)}$$

$$\Rightarrow \quad (1 + x^2)\,y_2 + y_1(2x - 1 - x^2) = 2xy + (1 + x^2)\,y_1 - y(1 + x^2)$$

$$\Rightarrow \quad (1 + x^2)\,y_2 + y_1(2x - 2 - 2x^2) + y(1 + x^2 - 2x) = 0$$

$$\therefore \quad (1 + x^2)\,y_2 - 2(1 - x + x^2)\,y_1 + (1 - x)^2 y = 0$$

5. Let $y = f(e^x) \Rightarrow \frac{dy}{dx} = f'(e^x)\cdot e^x$

$$\therefore \quad \frac{d^2y}{dx^2} = e^x[f'(e^x)]' + [f'(e^x)](e^x)'$$

$$= e^x f''(e^x)\cdot e^x + f'(e^x)\cdot e^x = f''(e^x)\cdot e^{2x} + f'(e^x)\cdot e^x$$

Single Correct Choice Type Questions

1. We have,

$$y = \frac{1+x^2+x^4}{1+x+x^2} = \frac{(1-x+x^2)(1+x+x^2)}{(1+x+x^2)}$$

$$y = 1 - x + x^2$$

$$\frac{dy}{dx} = 2x - 1 = ax + b$$

We get, $a = 2$, $b = -1$

Hence, the correct answer is (C).

2. We have,

$$y = \frac{1}{1+x^{n-m}+x^{p-m}} + \frac{1}{1+x^{m-n}+x^{p-n}} + \frac{1}{1+x^{m-p}+x^{n-p}}$$

$$= \frac{x^m}{x^m+x^n+x^p} + \frac{x^n}{x^n+x^m+x^p} + \frac{x^p}{x^p+x^m+x^n}$$

$$\Rightarrow \quad y = \frac{x^m+x^n+x^p}{x^m+x^n+x^p} = 1$$

$$\Rightarrow \quad y = \frac{dy}{dx} = 0 \;\therefore \left(\frac{dy}{dx}\right)_{x=e^{m^{n^p}}} = 0$$

Hence, the correct answer is (D).

3. We have, $f(x) = \log_x(\log x)$

$$\Rightarrow \quad f(x) = \frac{\log(\log x)}{\log x} \text{ or } f'(x) = \frac{\log x\dfrac{1}{x\log x} - \dfrac{\log\log x}{x}}{(\log x)^2}$$

$$\therefore \quad f'(e) = \frac{\dfrac{\log e}{e\log e} - \dfrac{\log(\log e)}{e}}{(\log e)^2} = \frac{1}{e}$$

Hence, the correct answer is (A).

4. Given, $f(x)$ is a polynomial function of second degree.

Let $f(x) = \alpha x^2 + \beta x + \gamma$

We have, $f(1) = f(-1)$ $\quad\therefore\ \alpha + \beta + \gamma = \alpha - \beta + \gamma$

$\Rightarrow \quad \beta = 0$ $\quad\therefore\ f(x) = \alpha x^2 + \gamma$

Now, $f'(1) = 2\alpha x$

$f'(a) = 2\alpha a,\ f'(b) = 2\ \alpha b$ and $f'(c) = 2\ \alpha c$

Here, a, b, c are in an AP.

$\therefore\ f'(a), f'(b)$ and $f'(c)$ are also in an AP.

Hence, the correct answer is (A).

5. $$y = \frac{x}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\tan^{-1}\left(\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\left(\frac{1}{1+\left(\frac{\sin x}{a+\sqrt{a^2-1}+\cos x}\right)}\right)$$
$$\frac{\left(a+\sqrt{a^2-1}+\cos x\right)\cos x - \sin x(-\sin x)}{\left(a+\sqrt{a^2-1}+\cos x\right)^2}$$

At $x = \dfrac{\pi}{2}$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\left(\frac{1}{1+\frac{1}{\left(a+\sqrt{a^2-1}\right)^2}}\right)\frac{1}{\left(a+\sqrt{a^2-1}\right)^2}$$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\left(\frac{1}{\left(a+\sqrt{a^2-1}\right)^2+1}\right)$$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}}\frac{1}{\left(a^2+a^2+1+2a\sqrt{a^2-1}\right)}$$

$$= \frac{1}{\sqrt{a^2-1}}\left(1-\frac{2}{2a\left(a+\sqrt{a^2-1}\right)\left(a-\sqrt{a^2-1}\right)}\right)$$

$$= \frac{1}{\sqrt{a^2-1}}\left(1-\frac{2\left(a-\sqrt{a^2-1}\right)}{2a\left(a^2-a^2+1\right)}\right)$$

$$= \frac{1}{\sqrt{a^2-1}}\left(1-\frac{1}{a}\right)\left(a-\sqrt{a^2-1}\right)$$

$$= \frac{1}{\sqrt{a^2-1}}\left(1-1+\frac{\sqrt{a^2-1}}{a}\right) = \frac{1}{a}$$

Hence, the correct answer is (A).

6. $$\frac{dy}{dz} = \left.\frac{e^x\ln^2 x + e^x\left(2\ln x\cdot\frac{1}{x}\right)}{\frac{1}{x}}\right|_{x=e} = \frac{e^e\cdot 1 + e^e 2\cdot\frac{1}{e}}{\frac{1}{e}} = e^e\left(e+2\right)$$

Hence, the correct answer is (C).

7. $f(x) = \left(2015+x\right)^n$

$f'(x) = n(2015+x)^{n-1}$

$f''(x) = n(n-1)(2015+x)^{n-2}$

$f'''(x) = n(n-1)(n-2)(2015+x)^{n-3}$

$\vdots$

$f^{n-1}(x) = n(n+1).....2\cdot(2015+x)^1$

$$f(0) + \frac{f'(0)}{\underline{|1}} + \frac{f''(0)}{\underline{|2}} + \ldots + \frac{f^{(n-1)}(0)}{\underline{|n-1}}$$

$$= (2015)^n + \frac{n}{1}(2015)^{n-1} + nC_2(2015)^{n-2} + \ldots + {}^nC_{n-1}(2015)^1$$

$$= (2015+1)^n - 1 = (2016)^n - 1$$

Hence, the correct answer is (D).

8. $f''(x) = e^x\cos e^x \Big|\ \lim\limits_{x-1-\infty} f'(x) = 0$

$$f'(x) = \int e^x\cos e^x dx$$
$$= \int \cos t\ dt = \sin e^x + c$$

$\lim\limits_{x-\infty} f'(x) = 0 = \sin 0 + c \Rightarrow c = 0$

$f'(x) = \sin e^x$

$y = f\left(\dfrac{x-1}{x+1}\right)$

$$\frac{dy}{dx} = f'\left(\frac{x-1}{x+1}\right)\frac{d}{d_x}\left(1-\frac{2}{x+1}\right) = \sin e^{\left(\frac{x-1}{x+1}\right)}\cdot\frac{2}{(x+1)^2} = \frac{\sin 1}{2}$$

Hence, the correct answer is (A).

9. $f\left(x\right) = \log x$

$f'\left(x\right) = \dfrac{1}{x};\ f'\left(\log x\right) = \dfrac{1}{\log x}$

Hence, the correct answer is (D).

10. $\dfrac{du}{dx} = f'\left(\tan x\right).\sec^2 x\,;$ $\qquad \dfrac{dv}{dx} = g'\left(\sec x\right).\sec x\tan x$

$\dfrac{du}{dv} = \dfrac{\tan^{-1}\left(\tan x\right).\sec^2 x}{\cos ec^{-1}\left(\sec x\right).\sec x.\tan x}\,;$ $\qquad \dfrac{du}{dv} = \dfrac{x.\sec^2 x}{\left(\frac{\pi}{2}-x\right)\sec x\tan x}$

$\Rightarrow du/dv = \sqrt{2}\ at\ x = \pi/4$

Hence, the correct answer is (A).

11. $$y = \frac{(1-x)(1+x)\left(1+x^2\right)\left(1+x^{2^2}\right)\ldots\left(1+x^{2^n}\right)}{(1-x)}$$

$$y = \frac{1-x^{2^{n+1}}}{1-x} \quad \frac{dy}{dx} = \frac{(1-x)\left(-2^{n+1}.x^{2^{n+1}-1}\right)+(1-x^{2^{n+1}})}{(1-x)^2}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{0+1}{(1-0)^2} = 1$$

Hence, the correct answer is (A).

12. $$y = \cos^{-1}\left[x\sqrt{x} - \sqrt{1-\left(\sqrt{x}\right)^2}\sqrt{1-x^2}\right]$$

Let $x = \cos A\ \ \sqrt{x} = \cos B\,;\ y = \cos^{-1}\left[\cos\left(A+B\right)\right] = A+B$

$y = \cos^{-1}x + \cos^{-1}\sqrt{x}\,;\ \dfrac{dy}{dx} = \dfrac{-1}{\sqrt{1-x^2}} - \dfrac{1}{2\sqrt{x-x^2}}$

Hence, the correct answer is (B).

13. $y^{-1} = 1 + x + x^2 + x^3$ $\qquad$ as $x = 0 \Rightarrow y = 1$

$-1.y^{-1-1}\dfrac{dy}{dx} = 1 + 2x + 3x^2\,;$ $\qquad \dfrac{dy}{dx} = -y^2\left(1+2x+3x^2\right)$

At $x = 0, y = 1 \Rightarrow \dfrac{dy}{dx} = -1$

$$\frac{d^2y}{dx^2} = -\left[y^2\left(2+6x\right)+2y\,y^1\left(1+2x+3x^2\right)\right]$$

At (0, 1) $\frac{d^2y}{dx^2} = -[1(2) + 2(-1)(1 + 0 + 0)]$

$= -(2-2) = 0$

Hence, the correct answer is (A).

14. $5f(x)+3f(1/x)=x+2 \quad \ldots(1)$ $\quad$ [Put $\frac{1}{x}$ for x]

$3f(x)+5f(1/x)=\frac{1}{x}+2 \quad \ldots(2)$

Solving (1) and (2) $16f(x)=5x-\frac{3}{x}+4$

$y=\frac{1}{16}\left[5x^2+4x-3\right]$

$\frac{dy}{dx}=\frac{10x+4}{16}$ at $x=1 \Rightarrow \frac{dy}{dx}=\frac{7}{8}$

Hence, the correct answer is (B).

15. $y'(x)=f'\left(\frac{2x-1}{x^2+1}\right).\frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$

$=f'\left(\frac{2x-1}{x^2+1}\right).\frac{\left(x^2+1\right)2-(2x-1)2x}{\left(x^2+1\right)^2}$

$=2\sin\left(\frac{2x-1}{x^2+1}\right)^2.\frac{1+x-x^2}{\left(x^2+1\right)^2}$ $\quad$ $[\therefore y'(0)=2\sin 1]$

Hence, the correct answer is (C).

16. $y_1=\frac{\alpha(\gamma x+\delta)-(\alpha x+\beta)\gamma}{(\gamma x+\delta)^2}$

$y_1=\frac{\alpha-\gamma}{(\gamma x+\delta)^2}$

$y_2=(\alpha-\gamma)\left[\frac{-2}{(\gamma x+\delta)^3}(r)\right]$ and $y_3=(\alpha-\gamma)\left[\frac{+6}{(\gamma x+\delta)^4}\left(r^2\right)\right]$

$[\therefore y_3^2=\frac{2}{3}y_1y_2]$

Hence, the correct answer is (C).

17. $f(x)=-\cos x$ when $x=\frac{3\pi}{4}$

$f'(x)=\sin x \Rightarrow f'\left(\frac{3\pi}{4}\right)=\frac{1}{\sqrt{2}}$

Hence, the correct answer is (C).

18. $y=\frac{1-\cos x+\sin x}{1+\cos x+\sin x}$

$=\frac{2\sin^2(x/2)+2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)+2\sin(x/2)\cos(x/2)}$

$y=\tan(x/2); \quad y_1=\frac{1}{2}\sec^2\frac{x}{2}$

$y_2=\frac{1}{2}\cdot 2\sec\frac{x}{2}\cdot\sec\frac{x}{2}\ \tan\frac{x}{2}\cdot\left(\frac{1}{2}\right)$

$y_2=\frac{1}{2}\tan\frac{x}{2}\sec^2\frac{x}{2}$

Hence, the correct answer is (D).

19. $\frac{d}{dx}\left[\frac{\sin x+\cos x}{\sqrt{(\sin x+\cos x)^2}}\right]$

$(\because\ |\sin x + \cos x| = (\sin x + \cos x);\ x \in \left(0, \frac{\pi}{4}\right))$

$\frac{d}{dx}\left[\frac{\sin x+\cos x}{|\sin x+\cos x|}\right]=\frac{d}{dx}(1)=0$

Hence, the correct answer is (B).

20. $f'_{n+1}(x)=e^{f_n(x)}\ f'_n(x)$

$f'_{n+1}(x)=e^{f_n(x)}e^{f_{n-1}(x)}f'_{n-1}(x)$

$f'_{n+1}(x)=e^{f_n(x)}e^{f_{n-1}(x)}e^{f_{n-2}(x)}f'_{n-2}(x)$ and so on ….

$f'_{n+1}(x)=f'_1(x)e^{f_1(x)}e^{f_2(x)}\ \ldots\ldots\ e^{f_n(x)}=f_1(x)f_2(x)\ldots\ldots f_{n+1}(x)$

Putting $n=9$, we get, $f'_{10}(x)=\prod_{r=1}^{10}f_r(x)$

Hence, the correct answer is (D).

21. $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy/dt}{dx/dt}\right)=\frac{\frac{dx}{dt}\frac{d^2y}{dt^2}\cdot\frac{dx}{dt}-\frac{dy}{dt}\frac{d^2x}{dt^2}\cdot\frac{dt}{dx}}{\left(\frac{dx}{dt}\right)^2}$

$\frac{d^2y}{dx^2}=\frac{\left(\frac{dx}{dt}\right)\frac{d^2y}{dt^2}-\left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)^3}$

Hence, the correct answer is (A).

22. Put $x=a\sin\theta$

Then, $f(x)=\frac{d}{dx}\left[\frac{a^2}{4}\sin 2\theta+\frac{a^2}{2}\theta\right]$

$=\frac{a^2}{2}[1+\cos 2\theta]\frac{d\theta}{dx}=\frac{a^2}{2}\times\frac{2\cos\theta}{a}$

$=a\cos\theta=\sqrt{a^2-x^2}$

$\therefore y=\left(f(x)\right)^2+x^2=a^2-x^2+x^2=a^2$

$\therefore$ It is a constant function

Hence, the correct answer is (A).

23. $F'(x)=f(x)\phi'(x)+f'(x)\phi(x)$

$F''(x)=f(x)\ \phi''(x)+\phi'(x)f'(x)+f'(x)\ \phi'(x)+\phi(x)f''(x)$

$F''(x)=f(x)\phi''(x)+2f'(x)\phi'(x)+f''(x)\ \phi(x)$

$\frac{F''(x)}{F(x)}=\frac{\phi''(x)}{\phi(x)}+\frac{2C}{f(x)\phi(x)}+\frac{f''(x)}{f(x)}$

Hence, the correct answer is (A).

24. $\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\left(\frac{1}{\frac{dx}{dy}}\right)$

$=\frac{d}{dy}\left(\frac{1}{\frac{dx}{dy}}\right)\frac{dy}{dx}=\frac{-\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^2}\cdot\frac{dy}{dx}=-\frac{-\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$

$\Rightarrow \frac{d^2y}{dx^2}\cdot\left(\frac{dx}{dy}\right)^3+\frac{d^2x}{dy^2}=0$

Hence, the correct answer is (D).

25. $y_1 = \dfrac{2\sin^{-1}x}{\sqrt{1-x^2}} - \dfrac{2\cos^{-1}x}{\sqrt{1-x^2}} \Rightarrow \sqrt{1-x^2}\ y_1 = 2(\sin^{-1}x - \cos^{-1}x)$

$$\Rightarrow \quad \sqrt{1-x^2}\ y_2 - \frac{2x}{2\sqrt{1-x^2}}\ y_1 = 2\left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right)$$

$$(1-x^2)y_2 - xy_1 = 4$$

Hence, the correct answer is (B).

26. $y_1 = \sin(\log x) + x\cos(\log x)\left(\dfrac{1}{x}\right) + \log x + 1$

$$y_2 = \frac{\cos(\log x)}{x} - \frac{\sin(\log x)}{x} + \frac{1}{x}$$

$$x^2y_2 - x\cos\log x + x\sin\log x = x$$

$$x^2y_2 - x(\sin\log x + \cos\log x + \log x + 1) + 2(x\sin(\log x) + x\log x) = x\log x$$

$$x^2y_2 - xy_1 + 2y = x\log x$$

Hence, the correct answer is (B).

27. $t = \cos^{-1}x \Rightarrow \dfrac{dt}{dx} = \dfrac{-1}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = -\frac{1}{\sqrt{1-x^2}}\cdot\frac{dy}{dt} \Rightarrow \sqrt{1-x^2}\ \frac{dy}{dx} = -\frac{dy}{dt}$$

$$\Rightarrow \quad \sqrt{1-x^2}\ \frac{d^2y}{dx^2} + \frac{dy}{dx}\ \frac{1\cdot(-2x)}{2\sqrt{1-x^2}} = -\frac{d^2y}{dt^2}\cdot\frac{dt}{dx}$$

$$\Rightarrow \quad (1-x^2)\ \frac{d^2y}{dx^2} - x\frac{dy}{dx} = \frac{d^2y}{dt^2}$$

Hence, the correct answer is (A).

28. $(fg)' = 1 \Rightarrow fg' + f'g = 0$

$$\Rightarrow \quad fg'' + gf'' + 2f'g' = 0$$

$$\Rightarrow \quad fg''' + gf''' = -3[f'g'' + f''g']$$

$$\Rightarrow \quad \frac{f'''}{f'} - \frac{g'''}{g'} = 3\left[\frac{f''}{f} - \frac{g''}{g}\right] \qquad \text{(Using } fg' = -f'g\text{)}$$

Hence, the correct answer is (B).

29. $y = \left(x + \sqrt{1+x^2}\right)^m$

$$\Rightarrow \quad y_1 = m\left(x + \sqrt{1+x^2}\right)^{m-1}\cdot\left\{1 + \frac{2x}{2\sqrt{1+x^2}}\right\}$$

$$\Rightarrow \quad \sqrt{1+x^2}\ y_1 = m\left(x + \sqrt{1+x^2}\right)^m \Rightarrow \sqrt{1+x^2}\ y_1 = my$$

$$\sqrt{1+x^2}\ y_2 + \frac{y_1.2x}{2\sqrt{1+x^2}} = my_1$$

$$\Rightarrow \quad (1+x^2)\,y_2 + xy_1 = my_1\sqrt{1+x^2} \Rightarrow (1+x^2)y_2 + xy_1 = m^2y$$

Hence, the correct answer is (A).

30. $\dfrac{dy}{dx} = \dfrac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \dfrac{8\sec^8\theta\tan\theta + 8\cos^7\theta\sin\theta}{\sec\theta\tan\theta + \sin\theta} = \dfrac{8\tan\theta(\sec^8\theta + \cos^8\theta)}{\tan\theta(\sec\theta + \cos\theta)}$

$$\left(\frac{dy}{dx}\right)^2 = \frac{64(\sec^8\theta + \cos^8\theta)^2}{(\sec\theta + \cos\theta)^2}$$

$$= \frac{64[(\sec^8\theta - \cos^8\theta)^2 + 4]}{[(\sec\theta - \cos\theta)^2 + 4]} = \frac{64(y^2+4)}{(x^2+4)}$$

$$\left(\frac{x^2+4}{y^2+4}\right)\left(\frac{dy}{dx}\right)^2 = 64$$

Hence, the correct answer is (C).

31. $f''(x)$ is an odd function & increasing function.

$\therefore \quad \phi(\alpha, \beta) = 0$

$\Rightarrow \quad (\sin^4\alpha - 1)^2 + (\cos^4\beta - 1)^2 + 2(\sin^2\alpha - \cos^2\beta)^2 = 0$

We get, $\sin^2\alpha = \cos^2\beta = 1,\ \sin^2\beta = 0$

$\Rightarrow \quad \sin^2\alpha + \sin^2\beta = 1$

Hence, the correct answer is (B).

32. $\dfrac{dy}{dx} = \cos(\sin x)\cos x$

$$\frac{d^2y}{dx^2} = -\cos(\sin x)\sin x + \cos x\,[-\sin(\sin x)]\cos x$$

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{dy}{dx}\tan x = -\cos^2x\sin(\sin x)$$

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{dy}{dx}\tan x + \cos^2x\sin(\sin x) = 0$$

$\therefore \quad f(x) = \cos^2x\sin(\sin x)$

Hence, the correct answer is (D).

33. $\dfrac{y}{x} = \log x - \log(a + bx)$

On differentiating w.r.t. x, we get

$$\frac{x\dfrac{dy}{dx} - y}{x^2} = \frac{1}{x} - \frac{1}{a+bx}b = \frac{a}{x(a+bx)}$$

$$\therefore \quad x\frac{dy}{dx} - y = \frac{ax}{a+bx} \qquad \ldots(1)$$

On differentiating again w.r.t x, we get

$$\frac{xd^2y}{dx^2} + \frac{dy}{dx} - \frac{dy}{dx} = \frac{(a+bx)a - ax.b}{(a+bx)^2}$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} = \frac{a^2}{(a+bx)^2}$$

$$\therefore \quad x^3\frac{d^2y}{dx^2} = \frac{a^2x^2}{(a+bx)^2} = \left(\frac{xdy}{dx} - y\right)^2$$

Hence, the correct answer is (B).

34. $y = \tan^{-1}\left(\dfrac{1-2\log_e x}{1+2\log_e x}\right) + \tan^{-1}\left(\dfrac{3+2\log_e x}{1-6\log_e x}\right)$

$$= \tan^{-1}(1) - \tan^{-1}(2\log_e x) + \tan^{-1}(3) + \tan^{-1}(2\log_e x)$$

$$= \tan^{-1}(1) + \tan^{-1}(3) \qquad \left(\therefore \frac{dy}{dx} = 0\right)$$

Hence, the correct answer is (A).

35. $f'(x) = \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0}\dfrac{f(x) + f(h) - f(x)}{h}$

$$= \lim_{h\to 0}\frac{(2h^2 + 3h)g(h)}{h} = \lim_{h\to 0}(2h+3)\,g(h)$$

$$= 3g(0) = 3.9 = 27$$

Hence, the correct answer is (C).

36. $\log y = (\tan x)^{\tan x}\log(\tan x)$ …(1)

Taking log again, we get from (1)

$\log(\log y) = \tan x\log(\tan x) + \log(\log(\tan x))$

On differentiating w.r.t. x, we get

$$\frac{1}{\log y}\cdot\frac{1}{y}\ \frac{dy}{dx} = \sec^2x\log(\tan x) + \tan x\ \frac{1}{\tan x}\ \sec^2x + \frac{1}{\log(\tan x)}\ \frac{1}{\tan x}\ \sec^2x$$

$$\therefore \quad \frac{dy}{dx} = y \log y \sec^2 x\left(\log(\tan x)+1+\frac{1}{\tan x \log(\tan x)}\right)$$

$$= y(\tan x)^{\tan x} \log \tan x.\sec^2 x$$

$$\left[\left(\log(\tan x)+1\right)+\frac{1}{\tan x \log(\tan x)}\right]$$

$$= y(\tan x)^{\tan x} \sec^2 x[\log(\tan x)\ (\log \tan x + 1) + \cot x]$$

When at $x = \frac{\pi}{4}, y = 1$, we get $\frac{dy}{dx} = 1.1.2(0 + 2) = 2$

Hence, the correct answer is (B).

37. $\frac{dy}{dx} = 2x + 2t \ . \ \frac{dt}{dx}$

$$t = \frac{x}{1+x^2}, \Rightarrow \frac{dt}{dx} = \frac{1-x^2}{(1+x^2)^2}.\ \text{Put } x = 2$$

$$\Rightarrow \quad \frac{dy}{dx} = 2 \times 2 + 2 \times \left(\frac{2}{5}\right)\left(-\frac{3}{25}\right) = \frac{488}{125}$$

Hence, the correct answer is (C).

38. $x = a \sin 2\theta(2\cos^2\theta) = 2a \sin 2\theta \cos^2\theta$

$$\frac{dx}{d\theta} = 2a[2\sin 2\theta \cos\theta \cdot (-\sin\theta) + 2\cos^2\theta \cos 2\theta]$$

$$y = b \cos 2\theta(2\sin^2\theta) = 2b \cos 2\theta \sin^2\theta$$

$$\frac{dy}{dx} = \frac{2b(-2\sin 2\theta \sin^2\theta + \cos 2\theta \sin 2\theta)}{2a(2\cos 2\theta \cos^2\theta + \sin 2\theta(-\sin 2\theta))}$$

$$\frac{dy}{d\theta} = 2b[-\sin^2\theta \sin 2\theta \cdot (2) + 2\cos 2\theta \cdot \sin\theta \cos\theta]$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{\theta=\pi/3} = \frac{b}{a}.\sqrt{3}$$

Hence, the correct answer is (C).

39. $\frac{dx}{dt} = -\frac{3}{t^4} - \frac{2}{t^3} = -\left(\frac{3+2t}{t^4}\right)$

$$\frac{dy}{dt} = -\frac{3}{t^3} - \frac{2}{t^2} = -\left(\frac{3+2t}{t^3}\right); \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{(3+2t)}{t^3}}{-\frac{(3+2t)}{t^4}} = t$$

Since $f(x)\left(\frac{dy}{dx}\right)^3 = 1 + \frac{dy}{dx} \Rightarrow f(x)t^3 = 1 + t \Rightarrow t(x) = \frac{1+t}{t^3} = x$

Hence, the correct answer is (A).

40. Multiplying numerator & denominator by $(1 - x)$

$$y = \frac{(1-x)(1+x)(1+x^2)(1+x^4)}{1-x} \quad \Rightarrow y = \frac{1-x^8}{1-x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(1-x)(-8x^7)-(1-x^8)(-1)}{(1-x)^2} = \frac{0+1}{1}\left(\frac{dy}{dx}\right)_{x=0} = 1$$

Hence, the correct answer is (A).

41. $x = \frac{\cos y}{\cos(a+y)} \Rightarrow \frac{dx}{dy} = \frac{\sin a}{\cos^2(a+y)}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos^2 y}{x^2 \sin a} = \frac{\sin a}{1+x^2-2x\cos a}$$

$\Rightarrow \quad k = \sin a$

Hence, the correct answer is (A).

42. $f'(0) = \lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0} \frac{\frac{hg(h)}{|h|}-0}{h} = \lim_{h\to 0} \frac{g(h)}{|h|} = 0 = \lim_{h\to 0} \frac{g'(h)}{1} = 0$$

Hence, the correct answer is (D).

43. $f(x) = x + \tan x$

$$f(f^{-1}(y)) = f^{-1}(y) + \tan(f^{-1}(y))$$

$$y = g(y) + \tan(g(y))$$

$$x = g(x) + \tan(g(x))$$

On differentiating w.r.t. x, we get

$$1 = g'(x) + \sec^2(g(x)).g'(x)$$

$$g'(x) = \frac{1}{\tan^2 g(x)+2};\ g'(x) = \frac{1}{2+[g(x)-x]^2}$$

Hence, the correct answer is (C).

44. $\frac{1}{2}\ln(x^2 + y^2) = \ln a + \tan^{-1} {}^{y}\!/\!{}_{x}\,;\ y(0) = ae^{\pi/2}$

$$\Rightarrow \quad \frac{1}{2}\ \frac{(2x+2yy')}{(x^2+y^2)} = 0 + \frac{1}{1+{}^{y^2}\!/\!{}_{x^2}} \cdot \frac{x\,{}^{dy}\!/\!{}_{dx} - y\cdot 1}{x^2}$$

$$\Rightarrow \quad \frac{x+yy'}{x^2+y^2} = \frac{x\,{}^{dy}\!/\!{}_{dx} - y}{x^2+y^2} \Rightarrow x + yy' = xy' - y$$

$$\Rightarrow \quad y'(x-y) = x + y \Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y}$$

$$\frac{d^2y}{dx^2} = \frac{(x-y)(1+y')-(x+y)(1-y')}{(x-y)^2}$$

$$= \frac{2xy'-2y}{(x-y)^2} = \frac{2\left(x\left(\frac{x+y}{x-y}\right)-y\right)}{(x-y)^2}$$

$$= \frac{2(x^2+xy-xy+y^2)}{(x-y)^2} = \frac{2(x^2+y^2)}{(x-y)^3}$$

On differentiating with respect to x two times, we get

$$\frac{d^2y}{dx^2} = \frac{2(x^2+y^2)}{(x-y)^3}\ \text{Put } x = 0$$

$$= \frac{2y(0)^2}{(-y(0))^3} = -\frac{2}{y(0)} = -\frac{2}{ae^{\pi/2}} = -\frac{2}{a}e^{-\pi/2}$$

Hence, the correct answer is (C).

45. $y = e^{-x}\cos x$

$$y_1 = -e^{-x}\sin x - e^{-x}\cos x = -e^{-x}\sin x - y$$

$$y_2 = -e^{-x}\cos x + e^{-x}\sin x - y_1$$

$$\Rightarrow \quad y_2 = -y - y_1 + e^{-x}\sin x$$

$$\Rightarrow \quad y_2 = -2\left(y + y_1\right) \Rightarrow y_3 = -2\left(y_1 + y_2\right)$$

$$\Rightarrow \quad y_3 = -2\left(e^{-x}\sin x - y\right)$$

$$\Rightarrow \quad y_4 = 2y_1 - 2y - 2y - 2y_1$$

or, $\quad y_4 + 4y = 0 \quad \Rightarrow k = 4$

Hence, the correct answer is (A).

46. $y = e^{2x}$

$$\therefore \frac{dy}{dx} = 2e^{2x} \text{ and } \frac{d^2y}{dx^2} = 4e^{2x}$$

$$\frac{dx}{dy} = \frac{1}{2e^{2x}} = \frac{1}{2y}$$

$$\therefore \frac{d^2x}{dy^2} = -\frac{1}{2y^2} = -\frac{1}{2}e^{-4x}$$

$$\therefore \quad \frac{d^2y}{dx^2}\cdot\frac{dx^2}{dy^2} = 4.\ e^{2x}.\left(\frac{-e^{-2x}}{2e^{2x}}\right) = -2e^{-2x}$$

Hence, the correct answer is (D).

47. $y^2 = P(x) \Rightarrow 2y\dfrac{dy}{dx} = P'(x)$

$$\text{or,} \quad 2\left(\frac{dy}{dx}\right)^2 + 2y\frac{d^2y}{dx^2} = P''(x)$$

$$\text{or,} \quad 2y\frac{d^2y}{dx^2} = P''(x) - 2\left(\frac{dy}{dx}\right)^2 = P'' - \frac{P'^2}{2y^2}$$

$$\because \quad 2y^3\frac{d^2y}{dx^2} = y^2P'' - \frac{1}{2}P'^2 = PP'' - \frac{1}{2}P'^2$$

$$\therefore \quad 2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right) = P'P'' + PP''' - P'P'' = PP'''$$

Hence, the correct answer is (C).

48. $F'(x) = \left[f\left(\frac{x}{2}\right).\ f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right)g'\left(\frac{x}{2}\right)\right]$

Here, $g(x) = f'(x)\ \&\ g'(x) = f''(x) = -f(x)$

So, $F'(x) = f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right)g\left(\frac{x}{2}\right) = 0$

$\Rightarrow$ $F(x)$ is constant function

So, $F(10) = 5$

Hence, the correct answer is (A).

49. $f'(x) = \begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$

$$f''(x) = \begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix};\ f'''(x) = \begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix} = 0$$

Hence, the correct answer is (D).

50. $x^y = e^{x-y}$ i.e. $y\,l\,n\,x = x - y$ i.e. $y = \dfrac{x}{1 + l\,n\,x}$

$$\therefore \quad \frac{dy}{dx} = \frac{l\,n\,x}{(1 + l\,n\,x)^2}$$

Hence, the correct answer is (C).

51. $y = \tan^{-1}\left(\dfrac{2^{x+1} - 2^x}{1 + 2^x.2^{x+1}}\right) = \tan^{-1}2^{(x+1)} - \tan^{-1}2^x$

$$\Rightarrow \quad y' = \frac{2^{x+1}\ln 2}{1 + \left(2^{x+1}\right)^2} - \frac{2^x\ln 2}{1 + \left(2^x\right)^2}$$

$$\Rightarrow \quad y'(0) = -\frac{1}{10}\ln 2$$

Hence, the correct answer is (D).

52. $g'\left(x^2\right) = x^3$

$$\Rightarrow \quad g'(t) = t\cdot\sqrt{t} = t^{3/2}, \text{ where } x^2 = t$$

$$\Rightarrow \quad g(t) = \frac{t^{5/2}}{5/2} + c$$

$$\therefore g(1) = \frac{(1)^{5/2}}{5/2} + c$$

$$\therefore \quad c = \frac{3}{5}$$

$$\therefore g(t) = \frac{t^{5/2}}{5/2} + \frac{3}{5}$$

$$g(4) = \frac{(4)^{5/2}}{5/2} + \frac{3}{5} = \frac{67}{5}$$

Hence, the correct answer is (C).

53. $y = f(x) \Leftrightarrow x = f^{-1}(y) \Leftrightarrow x = g(y)$ since $g = f^{-1}$

$$\frac{dx}{dy} = g'(y) = \frac{1}{f'(x)} = \sqrt{1 + x^3}$$

$$\frac{d^2x}{dy^2} = g''(y) = \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\sqrt{1 + x^3}$$

$$= \frac{d}{dx}\left(\sqrt{1 + x^3}\right)\times\frac{dx}{dy} = \frac{3x^2}{2\sqrt{1 + x^3}}\times g'(y) = \frac{3x^2}{2}$$

$$g''(y) = \frac{3}{2}g^2(y)$$

Hence, the correct answer is (C).

54. $f'(0) = \lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0}\dfrac{f(x)}{x}$

$$= \lim_{x\to 0}\frac{g(x)}{x^2} = \lim_{x\to 0}\frac{g'(x)}{2x} = \lim_{x\to 0}\frac{g''(x)}{2} = \frac{g''(0)}{2} = \frac{17}{2}$$

Hence, the correct answer is (D).

55. $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$

$f'(x) = 30x^9 - 56x^7 + 30x^5 - 63x^2 + 6x$

$f'(1) = 30 - 56 + 30 - 63 + 6 = 4 - 57 = -53$

$$265\left(\lim_{x\to 0}\frac{h^4 + 3h^2}{(f(1-h) - f(1))\sin 5h}\right)$$

$$= 265\left(\lim_{h\to 0}\frac{h^4 + 3h^2}{\left(\frac{f(1-h) - f(1)}{-h}\right)\left(\frac{\sin 5h}{5h}\right)(-h)(5h)}\right)$$

$$= 265\left(\lim_{h\to 0}\frac{h^4 + 3h^2}{f'(1)\cdot(1)(-5h^2)}\right)$$

$$= 265\left(\lim_{h\to 0}\frac{h^2 + 3}{(-5f'(1))}\right) = \frac{-265}{5}\ \frac{3}{f'(1)} = \frac{(-53)3}{(-53)} = 3$$

Hence, the correct answer is (C).

56. $\lim\limits_{x\to 0}\dfrac{d}{dx}\left(\dfrac{e^{e^{x^2}} - e}{x}\right) = \lim\limits_{x\to 0}\dfrac{x\left(e^{e^{x^2}}\cdot e^{x^2}\cdot 2x\right) - \left(e^{e^{x^2}} - e\right)1}{x^2}$

$$= \lim_{x\to 0} 2e^{e^{x^2}}\cdot e^{x^2} - \left(\frac{e^{e^{x^2}} - e}{x^2}\right) \quad \text{[using LH Rule]}$$

$$= \lim_{x\to 0} 2(e)(1) - \frac{e^{e^{x^2}}\cdot e^{x^2}\cdot 2x}{2x} = 2e - \frac{(e)(1)}{1} = 2e - e = e$$

Hence, the correct answer is (B).

57. $y(x) = e^{2\sin^{-1}x} \Rightarrow y'(x) = e^{2\sin^{-1}x}\dfrac{2}{\sqrt{1 - x^2}}$

$$y''(x) = e^{2\sin^{-1}x}\frac{4}{(1 - x^2)} + e^{2\sin^{-1}x}\cdot 2\left(\frac{-1}{2}\frac{(-2x)}{(1 - x^2)^{3/2}}\right)$$

$$(1-x^2)\,y''(x)=4y(x)+\frac{2x}{\sqrt{1-x^2}}e^{2\sin^{-1}x}$$
$$=4y(x)+xy'(x)$$

We get, $\lambda=4$

Hence, the correct answer is (D).

58. $f(x)=(2x-\pi)^3+2x-\cos x$

$$f'(x)=3(2x-\pi)^2(2)+2+\sin x$$
$$f'(\pi)=6\pi^2+2+0$$
$$f(x)=\pi=(2x-\pi)^3+2x-\cos x=\pi, we\ get\ \ x=\frac{\pi}{2}$$
$$f^{-1}(x)=g,\text{then } gof(x)=x$$
$$g'(f(x))f'(x)=1$$
$$g'(f(x))=\frac{1}{f'(x)} \qquad [\text{Put } x=\pi/2]$$
$$g'(\pi)=\frac{1}{f'\left(\frac{\pi}{2}\right)}=\frac{1}{\left(3(2x-\pi)^2.2+2+\sin x\right)}$$
$$=\frac{1}{2+1}=\frac{1}{3}=\frac{p}{q}\qquad \Rightarrow p+q=4$$

Hence, the correct answer is (D).

59. $\ln(\ln x-\ln y)=e^{x^2y}(1-\ln x)$

$$\text{At } x=e\Rightarrow \ln(1-\ln y)=e^{e^2y}(1-1)=0$$
$$1-\ln y=1\Rightarrow y=y(e)=1$$

On differentiating w.r.t. x, we get

$$\frac{1}{(\ln x-\ln y)}\left(\frac{1}{x}-\frac{1}{y}\frac{dy}{dx}\right)=e^{x^2y}\left(2xy+x^2\frac{dy}{dx}\right)\left(1=\ln x+e^{x^2y}\left(0-\frac{1}{x}\right)\right)$$

At $x=e$

$$\frac{1}{(1-0)}\left(\frac{1}{e}-\frac{1}{1}y'(e)\right)=e^{e^2}\left(2e+e^2y'(e)(1-1)\right)-\frac{1}{e}e^{e^2}$$
$$\frac{1}{e}-y'(e)=-\frac{1}{e}e^{e^2}$$
$$y'(e)=\frac{1+e^{e^2}}{e}$$

Hence, the correct answer is (A).

60. $y=\left(1+\frac{1}{x}\right)^x\Rightarrow y(2)=\left(\frac{3}{2}\right)^2=\frac{9}{4}$

$$\ln y=x\ln\left(1+\frac{1}{x}\right)$$
$$\frac{1}{y}\frac{dy}{dx}=\ln\left(1+\frac{1}{x}\right)+x\frac{1}{\left(1+\frac{1}{x}\right)}\left(\frac{-1}{x^2}\right)$$
$$\frac{y'(2)}{\frac{9}{4}}=\ln\frac{3}{2}+\frac{2}{3/2}\left(\frac{-1}{4}\right)$$
$$y'(2)=\frac{9}{4}\left(\ln\frac{3}{2}-\frac{1}{3}\right)$$

Hence, the correct answer is (A).

61. $f(x)=\tan^{-1}(\sin x+\cos x)$

$$f'(0)=\frac{1}{1+(\sin x+\cos x)^2}(\cos x-\sin x)\ \ x=0$$
$$=\frac{1-0}{1+(0+1)^2}=\frac{1}{2}$$

Hence, the correct answer is (A).

62. $y=6\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots\right)$

$$\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\ldots+x-1\right)-3x^3-x^4-x^5$$
$$=6\left(\frac{x^3}{2}+\frac{x^4}{6}+\frac{x^5}{24}+\frac{x^6}{120}+\frac{x^5}{6}+\frac{x^6}{18}+\frac{x^7}{15}+\ldots\right)-3x^3-x^4-x^5$$
$$y=\frac{x^5}{4}+\ldots\ldots.$$
$$\frac{d^5y}{dx^5}=\frac{5!}{4}+\ldots\ldots=5$$

Hence, the correct answer is (C).

63. $f_n(x)=e^{f_{n-1}(x)}$ all $n\in N$

$$\frac{d}{dx}\left(fn(x)\right)=\frac{d}{dx}\left(e^{f_{n-1}(x)}\right)=e^{f_{n-1}(x)}\frac{d}{dx}\left(f_{n-1}(x)\right)$$
$$=f_n(x)\cdot\frac{d}{dx}\left(e^{f_{n-2}(x)}\right)=f_n(x)\cdot f_{(n-1)}(x)\frac{d}{dx}\left(f_{n-2}(x)\right)$$
$$=f_n(x)\,f_{(n-1)}(x)\cdot f_{(n-2)}(x)\ldots f_1(x)\frac{d}{dx}f_0(x)$$
$$=f_n(x)f_{n-1}(x)f_{(n-2)}(x)\ldots f_1(x)\frac{d}{dx}(x)$$
$$=f_n(x)f_{n-1}(x)f_{(n-2)}(x)\ldots f_1(x)$$

Hence, the correct answer is (A).

64. $\frac{u(x)}{v(x)}=7,\frac{u'(x)}{v'(x)}=p,$

$$u(x)=7v(x)$$
$$u'(x)=7v'(x),p=7$$
$$\frac{d}{dx}\left(\frac{u'(x)}{v'(x)}\right)=\frac{d}{dx}(p)=q=0$$

Now, $\frac{p+q}{p-q}=\frac{7+0}{7-0}=1$

Hence, the correct answer is (A).

65. $y=\frac{3}{2t^2}+\frac{2}{t}\Rightarrow\frac{dy}{dt}=\frac{-3}{t^3}-\frac{2}{t^2}=-\frac{(3+2t)}{t^3}$

$$x=\frac{1+t}{t^3}=\frac{1}{t^3}+\frac{1}{t^2}$$
$$\frac{dx}{dt}=\frac{-3}{t^4}\frac{-2}{t^3}=\frac{-(3+2t)}{t^4}$$
$$\frac{dy}{dx}=t$$
$$\left(\frac{dy}{dx}\right)-x\left(\frac{dy}{dx}\right)^3=t-\left(\frac{1+t}{t^3}\right)(t)^3=t-(1+t)=-1$$

Hence, the correct answer is (C).

66. $f'(x)=\frac{x^{10}}{1+x^2}$ $\qquad g(2)=a$

$$f(g(x))=x\Rightarrow f'(g(x))g'(x)=1$$

$$f'(g(2))\,g'(2)=1 \Rightarrow f'(a)g'(2)=1$$

$$\frac{a^{10}}{1+a^2}\cdot g'(2)=1 \Rightarrow g'(2)=\frac{1+a^2}{a^{10}}$$

Hence, the correct answer is (B).

67. $f''(x)=-f(x) \qquad f'(x)=g(x)$

$$h'(x)=(f(x))^2+(g(x))^2$$
$$h''(x)=2f(x)f'(x)+2g(x)g'(x)$$
$$=2f(x)f'(x)+2f'(x)(-f(x))=0$$

$h'(x)=\text{Constant}=k$

$h(x)=kx+C \qquad ;\, k=2$

$h(0)=2=C$

$h(1)=4=k+C$

$h(x) = 2x + 2$

Hence, the correct answer is (C).

68. $f(x)=e^{ax}+e^{bx} \Rightarrow f'(x)=ae^{ax}+be^{bx}$

$f''(x)=a^2e^{ax}+b^2e^{bx}$

$$f''(x)-2f'(x)-15f(x)=0$$

$$\left(a^2-2a-15\right)e^{ax}+\left(b^2-2b-15\right)e^{bx}=0 \qquad \forall x\in R$$

$(a-5)(a+3)=0 \Rightarrow a=5,-3$

$(b-5)(b+3)=0 \Rightarrow b=5,-3 \quad ;$

$a\neq b$

$ab=-15$

Hence, the correct answer is (C).

69. $x=t^3+t+5; \quad y=\sin t$

$$\frac{dx}{dt}=3t^2+1; \quad \frac{dy}{dt}=\cos t$$

$$\frac{dy}{dx}=\frac{\cos t}{3t^2+1}; \quad \frac{dy^2}{dx^2}=\frac{\left(3t^2+1\right)(-\sin t)-6t(\cos t)}{\left(3t^2+1\right)^2}\,\frac{dt}{dx}$$

$$=-\frac{\left(\left(3t^2+1\right)\sin t+6+\cos t\right)}{\left(3t^2+1\right)^2}\,\frac{1}{\left(3t^2+1\right)}$$

Hence, the correct answer is (A).

70. $y^2e^{xy}=9e^{-3}x^2$

$$e^{xy}2y\frac{dy}{dx}+y^2e^{xy}\left(1\cdot y+x\frac{dy}{dx}\right)=18e^{-3}x$$

At $x=-1$, $y=3$, we get

$$e^{-3}6y^1+9e^{-3}\left(3-1y^1\right)=-18e^{-3}$$

$\Rightarrow \quad e^{-3}6y^1-9e^{-3}y^1+27e^{-3}=-18e^{-3}$

$\Rightarrow \quad -3e^{-3}y^1=-45e^{-3} \Rightarrow y^1=15$

Hence, the correct answer is (D).

71. $f(x)=f'(x)f''(x)$

$$(a_nx^n+....)=\left(a_nnx^{n-1}+...\right)\left(a_nn(n-1)x^{n-2}+...\right)$$

LHS is of degree n = RHS is of degree $(n-1)+(n-2)$

$n=n-1+n-2 \Rightarrow n=3$, it is a cubic polynomial

Now, comparing the leading term $a_n=na_n\cdot n(n-1)a_n$

$$a_n=\frac{1}{n^2(n-1)}=\frac{1}{9(2)}=\frac{1}{18}$$

Hence, the correct answer is (D).

72. $x^2+y^2=R^2 \;\; (R>0)$

$$2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\left(\frac{x}{y}\right)_{(x_1y_1)}$$

$$1+\left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=0 \Rightarrow yy''=-\left(1+(y')^2\right)$$

$$y''=-\left(\frac{1+(y')^2}{y}\right)$$

$$k=\frac{y''}{\sqrt{(1+(y')^2)^3}}=\frac{-(1+(y')^2)}{y\left(1+(y')^2\right)^{3/2}}=\frac{-1}{y\sqrt{1+(y')^2}}=\frac{-1}{y\sqrt{1+\frac{x^2}{y^2}}}$$

$$=\frac{-1}{\sqrt{y^2+x^2}}=\frac{-1}{R}$$

Hence, the correct answer is (B).

73. $f'(x)-2f'(2x)\big|_{x=1} \;=5$

$f'(1)-2f'(2)=5$

$f'(2)-2f(4)=7$

Using above two relations, we get $f'(1)-4f'(4)=19$

$y=f(x)-f(4x)$

$$\frac{dy}{dx}=f'(x)-4f'(4x)\Big|_{x=1}=f'(1)-4f'(4)=19$$

Hence, the correct answer is (A).

74. $f(x)=\cos^{-1}\left(\frac{2}{\sqrt{13}}\cos x-\frac{3}{\sqrt{13}}\sin x\right)+\sin^{-1}\left(\frac{2}{\sqrt{13}}\cos x+\frac{3}{\sqrt{13}}\sin x\right)$

$$=\cos^{-1}\left(\cos(x+\alpha)\right)+\sin^{-1}\left(\sin(x+\beta)\right)$$

Where $\alpha=\cos^{-1}\frac{2}{\sqrt{13}}, \beta=\cos^{-1}\frac{3}{\sqrt{13}}, \alpha+\beta=\pi/2$

$\Rightarrow \quad f(x)=x+\alpha+x+\beta=2x+\pi/2$

And $g(x)=\sqrt{1+x^2}$

$$\Rightarrow \quad \frac{df(x)}{dg(x)}=\frac{2}{\frac{1}{2\sqrt{1+x^2}}2x}\Bigg|_{x=3/4}=\frac{2\sqrt{1+9/16}}{3/4}=\frac{8}{3}\cdot\frac{5}{4}=\frac{10}{3}$$

Hence, the correct answer is (C).

75. $f(x)=2-e^{-x}\cdot g(x)=f^{-1}(x)$

$y=2-e^{-x} \Rightarrow e^{-x}=2-y \quad ;$

$\Rightarrow \quad -x=\ln(2-y) \Rightarrow x=-\ln(2-y)$

$$y = -\ln(2-x)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-1}{(2-x)}(0-1)\bigg|_{x=1} = 1$$

$$\frac{d^2y}{dx^2} = \frac{-1}{(2-x)^2}(-1)\bigg|_{x=1} = 1$$

Now, $g'(1) + g''(1) = 2$

Hence, the correct answer is (D).

76. Opening determinant along third column

$$f(x) = (\sin(3x-2x))\cos x - 2\cos x\left(\sin(3x-x)\right) + 3\cos 3x\left(\sin(2x-x)\right)$$

$$= \sin x \cos x - 2\cos 2x \sin 2x + 3\cos 3x \sin x$$

$$= \frac{\sin 2x}{2} - \sin 4x + \frac{3}{2}\left(\sin 4x + \sin(-2x)\right)$$

$$f(x) = \frac{\sin 4x}{2} - \sin 2x$$

$$f'(x) = \frac{4\cos 4x}{2} - 2\cos 2x\Big|_{x=\pi/2} = 2\,(1) - (-1)2 = 4$$

Hence, the correct answer is (C).

77. $y = \ln(1+\cos x)^2$

$$\frac{dy}{dx} = \frac{1}{(1+\cos x)^2}\cdot 2(1+\cos x)(-\sin x)$$

$$\frac{dy}{dx} = \frac{-2\sin x}{(1+\cos x)}$$

$$\frac{d^2y}{dx^2} = \frac{-2\left[(1+\cos x)\cos x - \sin x(-\sin x)\right]}{(1+\cos x)^2}$$

$$= -2\frac{\left(\cos x + \cos^2 x + \sin^2 x\right)}{(1+\cos x)^2} = \frac{-2}{(1+\cos x)}$$

$$\frac{d^2y}{dx^2} + \frac{2}{e^{y/2}} = \frac{-2}{(1+\cos x)} + \frac{2}{e^{1/2}\ln(1+\cos x)^2}$$

$$= \frac{-2}{(1+\cos x)} + \frac{2}{1+\cos x} = 0$$

Hence, the correct answer is (A).

78. $$\lim_{h\to 0}\left(\frac{g(2-x^2)}{g(2)}\right)^{4/x^2} = e^{\lim\limits_{x\to 0}\frac{g(2-x^2)-g(2)}{g(2)}\cdot\frac{4}{x^2}}$$

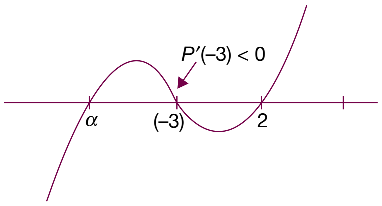
$$= e^{\lim\limits_{x\to 0}\frac{g(2-x^2)-g(2)}{-x^2}\left(\frac{-4}{g(2)}\right)} = e^{\frac{-4}{g(2)}g'(2)} = e^{\frac{-4(5)}{-40}} = e^{-1/2} = \frac{1}{\sqrt{e}}$$

Hence, the correct answer is (C).

79. $m > 1 \quad m = 4$ least

$P(x) = x^3 + ax^2 + bx + c$ where $a, b, c \in R$

$= (x+3)(x-2)(x-\alpha)$



As, given cubic has leading coefficient positive and

$P'(-3) < 0$ then α is smallest among all three roots

Product of roots = $-c = (3)(-2)(\alpha) = -6\alpha \Rightarrow c = 6\alpha$

As, we can see that α is smallest then –3

So, $c = 6\alpha < -18 \Rightarrow c < -18$

Hence, the correct answer is (A).

Multiple Correct Choice Type Questions

1. $f(x) = 2\tan^{-1}x + \pi - 2\tan^{-1}x$, When $x > 1$

$= \pi$ (When $x > 1$) $\qquad \Rightarrow f'(x) = 0$

$f(x) = 2\tan^{-1}x + 2\tan^{-1}x$ When $|x| \le 1$ $= 4\tan^{-1}x$

$$f'(x) = \frac{4}{1+x^2} \Rightarrow f'(1/2) = \frac{16}{5}$$

Hence, (A), (B) and (C) are correct.

2. $$y^2 = x + y \Rightarrow 2y\frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y-1}$$

$$y = \frac{x}{y} + 1 \Rightarrow \frac{dy}{dx} = \frac{y}{2x+y}$$

$$y^2 - y - x = 0 \Rightarrow y = \frac{1 \pm \sqrt{1+4x}}{2} \qquad \because y > 0,\ y = \frac{1+\sqrt{1+4x}}{2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2}\cdot\frac{1.(4)}{2\sqrt{1+4x}} = \frac{1}{\sqrt{1+4x}}$$

Hence, (A), (C) and (D) are correct.

3. $$\frac{dx}{dt} = 1 + 3t^2,\ \frac{dy}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{2t}{1+3t^2} = \frac{1}{2} \Rightarrow 3t^2 - 4t + 1 = 0 \quad t = 1, 1/3$$

Hence, (A) and (D) are correct.

4. $g(-x) = e^x f(-x) + e^{-x} f(x)$

$\Rightarrow \quad g(-x) = g(x)$

Clearly, $g(x)$ in even, hence, $f(x)$ is odd as $f(0) = 0$

$g'(0) = 0;\ f'(0) = g(0) = f(0) = 0$

Hence, (A), (B) and (C) are correct.

5. $$f(x) = -\frac{x^3}{3} + x^2\sin 1.5a - x\sin a.\sin 2a - 5\sin^{-1}\left(a^2 - 8a + 17\right)$$

$f(x)$ is defined when $-1 \le a^2 - 8a + 17 \le 1$

$$-1 \le (a-4)^2 + 1 \le 1 \quad \Rightarrow \quad a = 4$$

$$\therefore \quad f(x) = -\frac{x^3}{3} + x^2\sin 6 - x\sin 4\sin 8 - \frac{5\pi}{2}$$

$$\therefore \quad f'(x) = -x^2 + 2x\sin 6 - \sin 4\sin 8$$

$$f'(\sin 8) = -\sin^2 8 + 2\sin 8\sin 6 - \sin 4\sin 8$$

$$= \sin 8\left[2\sin 6 - (\sin 8 + \sin 4)\right]$$

$$= \sin 8\,[2\sin 6 - 2\sin 6\cos 2]$$

$$= 2\sin 6\sin 8\,(1 - \cos 2)$$

$\sin 6 < 0$, $\sin 8 > 0$, $1 - \cos 2 > 0$

$\therefore \quad f'(\sin 8) < 0$

Hence, (A) and (D) are correct.

6. $$f(0) = \frac{2}{g(0)},\ f'(0) = 2g'(0) = 4g(0)$$

$$g''(0) = 5f''(0) = 6f(0) = 3$$

$$f(0) = \frac{1}{2};\ g(0) = 4;\ g'(0) = 8,\ f'(0) = 16$$

$$g''(0) = 3, \qquad f''(0) = \frac{3}{5}$$

(A) $$h'(x)=\left.\frac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}\right|_{x=0}$$

$$=h'(0)=\frac{g(0)f'(0)-f(0)g'(0)}{g^2(0)}$$

$$h'(0)=\frac{4(16)-\frac{1}{2}(8)}{16}\quad=4-\frac{1}{4}=\frac{15}{4}$$

(B) $$k(x)=f(x)g(x)\sin x$$

$$k'(0)=\left(f(0)g'(0)+f'(0)g(0)\right)\sin 0+f(x)g(x)\cos x$$

$$=\left(\frac{1}{2}8+16(4)\right)0+f(0)g(0)\cdot 1\quad=0+\frac{1}{2}\cdot 4=2$$

(C) $$\lim_{x\to 0}\frac{g'(x)}{f'(x)}=\frac{g'(0)}{f'(0)}=\frac{8}{16}=\frac{1}{2}$$

Hence, (A), (B) and (C) are correct.

7. $f'gh=2f^2g^2h^2+1$ …(1)

$$g'fh=f^2g^2h^2+4$$

$$h'fg=3f^2g^2h^2+1$$

Add all $\Rightarrow\ f'gh+g'fh+h'fg=6f^2g^2h^2+6$

Now, put $fgh=t$

$$\Rightarrow\quad (f'gh+g'fh+h'fg)=\frac{dt}{dx}$$

$$\Rightarrow\quad dt=6(t^2+1).dx\quad\Rightarrow\quad \tan^{-1}(t)=6x+c$$

$$\Rightarrow\quad \tan^{-1}(f.g.h)=6x+c$$

For $x=0,\Rightarrow\ \tan^{-1}=c\quad\Rightarrow\quad\frac{\pi}{4}$

$$\therefore\quad fgh=\tan\left(6x+\frac{\pi}{4}\right)\qquad\text{…(2)}$$

Eq. (1) $\Rightarrow$ multiply both sides by $f(x)$

$$\Rightarrow\quad f'(fgh)=2f(fgh)^2+f$$

$$\Rightarrow\quad \frac{dy}{dx}\tan(6x+\frac{\pi}{4})\ =\ y\left[2\tan^{-1}\left(6x+\frac{\pi}{4}\right)+1\right]$$

$$\Rightarrow\quad \frac{dy}{6y}=\frac{2\tan^2\left(6+\frac{\pi}{4}\right)+1}{\tan\left(6+\frac{\pi}{4}\right)}$$

For $\tan\left(6+\frac{\pi}{4}\right)=z$

$$=\frac{2z^2+1}{(z)}\times\frac{dz}{(z^2+1)}=\frac{z^2+1+z^2}{(z)(z^2+1)}.dz$$

$$=\int\frac{dy}{6y}=\int\frac{z.dz}{z^2+1}+\int\frac{dz}{z}$$

$$\Rightarrow\quad \ln y^{1/6}=\frac{1}{2}\ln(z^2+1)+\ln z+\ln c$$

$$\Rightarrow\quad y^{1/6}=z\sqrt{z^2+1}\times c\quad\Rightarrow\quad y^{1/6}=\tan\left(6x+\frac{\pi}{4}\right).\sec\left(6x+\frac{\pi}{4}\right).c$$

For $x=0,y=1\quad 1=\tan\left(\frac{\pi}{4}\right).\sec\left(\frac{\pi}{4}\right).c\quad\Rightarrow\quad c=\frac{1}{\sqrt{2}}$

$$\therefore\quad f(x)=2^{-1/12}\times\left[\frac{\sin\left(6x+\frac{\pi}{4}\right)}{\cos^2\left(6x+\frac{\pi}{4}\right)}\right]^{1/6}\qquad\text{(A)}$$

Also, $y^{1/6}=2^{-1/2}.\dfrac{\sin\left(6x+\frac{\pi}{4}\right)}{\cos^2\left(6x+\frac{\pi}{4}\right)}$

$$=\frac{2^{-1/2}}{2^{1/2}}\times\frac{2\times(\sin 6x+\cos 6x)}{(\cos 6x-\sin 6x)^2}$$

$$=\frac{(1+\sin 12x)^{1/2}}{(1-\sin 12x)}=\frac{\left(1+\frac{\tan 12x}{\sec 12x}\right)^{1/2}}{\left(1-\frac{\tan 12x}{\sec 12x}\right)}$$

$$y^{1/6}=(\sec 12x)^{1/2}.(\tan 12x+\sec 12x)^{1/2}.(\tan 12x+\sec 12x)$$

$$\Rightarrow\quad f(x)=(\sec 12x)^{1/12}.(\tan 12x+\sec 12x)^{1/4}\qquad\text{(C)}$$

Hence, (A) and (C) are correct.

8. $f(x)=x^2+xg'(1)+g''(2)$; $\qquad f'(x)=2x+g'(1)$

$f''(x)=2$; $\qquad f'(1)=2+g'(1)$

$f'(1)=2+2+f'(2)\qquad$ (A)

$g(x)=x^2+xf'(2)+f''(3)$; $\qquad g'(x)=2x+f'(2)$

$g'(1)=2+f'(2)$; $\qquad g''(2)=2$

$g''(x)=2$

$g'(2)=4+f'(2)=8+g'(1)\qquad$ (B)

$f'(2)=4+g'(1)$

$f''(3)=2$

$f''(3)+g''(2)=4\qquad$ (C)

Hence, (A), (B) and (C) are correct.

9. $f(x)=ax^4+bx^3+cx^2+dx+e$

$f'(x)=4ax^3+3bx^2+2cx+d\qquad f'(0)=1=d$

$f''(x)=12ax^2+6bx+2c$; $\qquad f'''(0)=6=6b\Rightarrow b=1$

$f(x)$ odd function; therefore, $a=c=e=0$

$$f(x)=x^3+x$$

$f(f(x))=(x^3+x)^3+(x^3+x)$ degree 9.

$$f'(x)=3x^2+1>0$$

$f(x)$ is one-one function

$$\sum_{i=1}^{n}f(i)=\sum_{i=1}^{n}(i^3+i)=\left(\frac{n(n+1)}{2}\right)^2+\frac{(n+1)n}{2}$$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+1\right]=\frac{n(n+1)}{4}(n^2+n+2)$$

Hence, (A), (B) and (C) are correct.

10. $f(x+y)+f(x-y)=2f(x)\qquad\forall x,y\in R;\qquad f(0)=0$

$x=0\quad f(y)+f(-y)=2f(0)=0$

$f(-y)=-f(y)$

$f(-x)=-f(x)\qquad$ [$f(x)$ odd]

$f(x+y)+f(x-y)=2f(x)$

$y=x\Rightarrow\ f(2x)=2f(x)$

Since, $f(x)$ odd

$f(x)=kx$

$f'(x)=k\ \Rightarrow\ f'(1)=f'(2)=f'(3)=f'(4)=k$

Hence, (A), (C) and (D) are correct.

11. $\dfrac{d(x^{\ln x})}{d(\ln x)} = \dfrac{d(e^{\ln x^{\ln x}})}{\left(\dfrac{1}{x}\right)dx} = x\dfrac{d}{dx}\left(e^{(\ln x)^2}\right)$

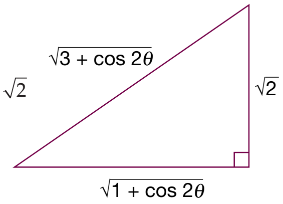
$= xe^{(\ln x)^2}\cdot\left(2\ln x\cdot\dfrac{1}{x}\right)$

$= 2\ln x\cdot e^{(\ln x)^2}$ (A)

$= 2\ln x\cdot(e^{\ln x^{\ln x}}) = 2\ln x\cdot x^{\ln x}$ (B)

Hence, (A) and (B) are correct.

12. $f(\theta) = \tan\left(\sin^{-1}\sqrt{\dfrac{2}{3+\cos 2\theta}}\right)$



$f(\theta) = \sqrt{\dfrac{2}{1+\cos 2\theta}} = \sec\theta \Rightarrow f\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$f'(\theta) = \sec\theta\tan\theta \Rightarrow f'\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$\therefore \left.\dfrac{d(f(\theta))}{d(\cos(\theta))}\right|_{\theta=\frac{\pi}{4}} = \left(\dfrac{-1}{\cos^2\theta}\right)_{\theta=\frac{\pi}{4}} = -2.$

Hence, (A), (B) and (D) are correct.

13. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f(x).g'(x) - g(x)f'(x)}{(g(x))^2}$

$= -\dfrac{(f'(x).g(x) - f(x)g'(x))}{(g(x))^2}$

$\Rightarrow \dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = 0$

$\Rightarrow \dfrac{f(x)}{g(x)} = C,$ where C is constant

$\therefore$ $F(n) = C\ \forall\ n\ \in$ N.

$\therefore$ $F(n) = F(n+1) = F(n+2)$ and $F(1), F(2), F(3)\ldots.$

are in G.P. with common ratio 1.

Hence, (A), (B), (C) and (D) are correct.

14. $f(x-y) + f(x+y) = 2f(x)f(y)\quad \forall x, y \in R$

$y = 0$ $f(x) + f(x) = 2f(x)f(0)$

$2f(x) = 2f(x)f(0)$ $f(0) = 1$

$x = 0$ $f(-y) + f(y) = 2f(0)f(y)$

$f(-y) = f(y)$ $f(x) \rightarrow$ even function

$f'(x) \rightarrow$ odd function

Hence, (B) and (D) are correct.

15. $14 = 2^x + 4^x + 8^x \Rightarrow x = 1$

$f'(x) = 2^x\ln 2 + 4^x\ln 4 + 8^x\ln 8$

$f'(1) = 2\ln 2 + 8\ln 2 + 24\ln 2$

$f'(1) = 34\ln 2$

As $(g(f(x))) = x \Rightarrow g'(f(x))\,f'(x) = 1$ we have $g'(14) = \dfrac{1}{34(\ln 2)}$,

Hence, (A) and (B) are correct.

16. Let $u = \sec^{-1}\left(\dfrac{1}{2x^2-1}\right) = 2\cos^{-1}x$

For $0 \le x \le 1$ (except at $x = \dfrac{1}{\sqrt{2}}$)

$\therefore \dfrac{du}{dx} = -\dfrac{2}{\sqrt{1-x^2}}$ and $v = \sqrt{1-x^2} \Rightarrow \dfrac{dv}{dx} = -\dfrac{x}{\sqrt{1-x^2}}$,

at $x = \pm 1, \pm\dfrac{1}{\sqrt{2}}$

$f'(x)$ does not exist.

For $-1 < x < 1$

$\therefore \dfrac{du}{dv} = \dfrac{\dfrac{du}{dx}}{\dfrac{dv}{dx}} = \dfrac{2}{x}$

Hence, (A) and (C) are correct.

17. $y = \sin\left(\dfrac{x}{y}\right) \Rightarrow \dfrac{dy}{dx} = \cos\left(\dfrac{x}{y}\right)\dfrac{y - xdy/dx}{y^2}$

$$\frac{dy}{dx} = \frac{\cos\left(\dfrac{x}{y}\right).\dfrac{1}{y}}{1-\cos\left(\dfrac{x}{y}\right)\left(-\dfrac{x}{y^2}\right)}$$

$$= \frac{y\cos\left(\dfrac{x}{y}\right)}{y^2 + x\cos\left(\dfrac{x}{y}\right)} = \frac{\sin\left(\dfrac{x}{y}\right)\cos\left(\dfrac{x}{y}\right)}{\sin^2\left(\dfrac{x}{y}\right) + x\cos\left(\dfrac{x}{y}\right)}$$

$$= \frac{\sin\left(\dfrac{2x}{y}\right)}{1-\cos\left(\dfrac{2x}{y}\right) + 2x\cos\left(\dfrac{x}{y}\right)}$$

Hence, (A), (C) and (D) are correct.

18. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{e^{\theta}(\cos\theta - \sin\theta)}{e^{-\theta}(\cos\theta - \sin\theta)} = e^{2\theta} > 0\,\forall\,\theta \in R$

$\Rightarrow f(x)$ is an increasing function and $\dfrac{dy}{dx} \ne -2$ and

$x^2\dfrac{dy}{dx} + y^2\dfrac{dx}{dy} = e^{-2\theta}\sin^2\theta.e^{2\theta} + e^{2\theta}\cos^2\theta.e^{-2\theta} = 1$

$y^{-2}\dfrac{dy}{dx} - \dfrac{x^2}{y^2}\left(\dfrac{dy}{dx}\right) = \dfrac{e^{2\theta}}{e^{2\theta}\cos^2\theta} - \dfrac{e^{-2\theta}\sin^2\theta}{\cos^2\theta}e^{2\theta}$

$= \sec^2\theta - \tan^2\theta = 1$

Hence, (A), (B) and (C) are correct.

19. $f'(x) = e^x,\ f''(x) = e^x \ldots f''' = e^x$

Sum $= ne^x = nf(x) = nf'(x)$

Hence, (A) and (B) are correct.

20. $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\theta(2\sin\theta + \theta\cos\theta)}{2\sin\theta + \theta\cos\theta} = \theta = \dfrac{\theta^2\sin\theta}{\theta\sin\theta} = \dfrac{y}{x+\cos\theta}$ for (–1, 0)

$\Rightarrow \theta = 0 = \dfrac{dy}{dx}$, tangent is parallel to x axis.

$f'(x) = \theta$, $f(x)$ is non monotonic and manyone

Hence, (B) and (D) are correct.

21. Given,

$f: (0, \infty) \rightarrow (-\infty, \infty)$ be defined as $f(x) = e^x + \ell n\,x$

Also, $f'(x) = e^x + \dfrac{1}{x} > 0\ \forall\ x \in (0, \infty)$

Also, $\lim\limits_{x\to+\infty} f(x) = \infty$ and $\lim\limits_{x\to 0^+} f(x) = -\infty$

$\Rightarrow$ $f(x)$ is a bijective mapping.

As, $f(1) = e \Rightarrow g(e) = 1$

$f(g(x) = x$

$$f'(g(x))\,g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}$$

$$g''(x) = \frac{-1}{(f'(g(x)))^2}\,f''(g(x))\,g'(x) = \frac{-f''(g(x))}{f'(g(x))^3}$$

Also, $g'(e) = \dfrac{1}{f'(1)}$

$$f'(x) = e^x + \frac{1}{x} \Rightarrow f'(1) = (e+1)$$

$$f''(x) = e^x - \frac{1}{x^2} \Rightarrow f'(1) = (e-1)$$

So, $g'(e) = \dfrac{1}{e+1}$

Also, $g''(e) = \dfrac{-f''(1)}{(f(1))^3}\left(\text{As, } g''(x) = \dfrac{-f''(x)}{(f'(x))^3}\right) = \dfrac{-(e-1)}{(e+1)^3} = \dfrac{1-e}{(e+1)^3}$

Now, verify alternatives.

Hence, (A) and (D) are correct.

22. Given that,

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \frac{x^{98}}{98} + \ldots. + \frac{x^2}{2} + x + 1.$$

$$\therefore f'(x) = x^{99} + x^{98} + \ldots + x + 1 = \left(\frac{x^{100}-1}{x-1}\right)$$

So, $f'(0) = 1$ and $f'(i) = 0$, where $i^2 = -1$.

Hence, (A) and (D) are correct.

23. $f(x) = \dfrac{x^4 - x^2 + 1}{x^2 + \sqrt{3}x + 1} = \dfrac{(x^4 + 2x^2 + 1) - 3x^2}{x^2 + \sqrt{3}x + 1}$

$$f(x) = \frac{(x^2 + 1 - \sqrt{3}x)(x^2 + 1 + \sqrt{3}x)}{(x^2 + \sqrt{3}x + 1)}$$

$$f'(x) = 2x - \sqrt{3}$$

$\therefore \quad a = 1, b = -\sqrt{3} \Rightarrow a^2 + b^2 = 4$

Hence, (A) and (B) are correct.

24. Now, $g'(2\pi) = \dfrac{1}{f'\left(\dfrac{3\pi}{2}\right)} = \dfrac{3}{7}$

$$f'(x) = 2 \times 3(2x - 3\pi)^2 + \frac{4}{3} - \sin x$$

$$\therefore \quad f'\left(\frac{3\pi}{2}\right) = \frac{4}{3} + 1 = \frac{7}{3}$$

Also, $g''(2\pi) = \dfrac{-f''\left(\dfrac{3\pi}{2}\right)}{\left(f'\left(\dfrac{3\pi}{2}\right)\right)^3} = 0$

Hence, (A) and (C) are correct.

25. $f(x) = e^{x + \sin x}$

$f'(x) = e^{x + \sin x}.\,(1 + \cos x)$

$f''(x) = e^{x + \sin x}.\,(-\sin x) + e^{x + \sin x}(1 + \cos x)^2$

$\therefore f'(0) = 2$ and $f''(0) = 4$.

Now, $g'(1) = \dfrac{1}{f'(0)} = \dfrac{1}{2}$.

Also, $g''(1) = \dfrac{-f''(0)}{(f'(0))^3} = \dfrac{-4}{(2)^3} = \dfrac{-4}{8} = \dfrac{-1}{2}$.

Hence, (B) and (C) are correct.

26. Put $x = \tan\theta$

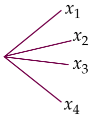
$$\Rightarrow f(x) = \sin^{-1}\left|\sin\frac{\theta}{2}\right| = \begin{cases} \sin^{-1}\left(\sin\dfrac{\theta}{2}\right); & 0 \le \theta < \dfrac{\pi}{2} \\ -\sin^{-1}\left(\sin\dfrac{\theta}{2}\right); & \dfrac{-\pi}{2} < \theta < 0 \end{cases}$$

$$f(x) = \begin{cases} \dfrac{1}{2}\tan^{-1}x; & x \ge 0 \\ -\dfrac{1}{2}\tan^{-1}x; & x < 0 \end{cases} \qquad f'(x) = \begin{cases} \dfrac{1}{2(1+x^2)}; & x > 0 \\ -\dfrac{1}{2(1+x^2)}; & x < 0 \end{cases}$$

$\therefore f'(1) = \dfrac{1}{4}; f'(-1) = \dfrac{-1}{4}$ and $f'(x)$ is an odd function.

Hence, (A), (B) and (C) are correct.

27. $P(x) = x^4 - ax^3 + bx^2 - cx + 480$



Given, $|P(4)| + |P'(4)| = 0 \therefore P(4) = 0 = P'(4)$

Hence, 4 is a root repeated twice in P(x) say $x_1 = 4; x_2 = 4$

Now, $x_1x_2x_3x_4 = 480 = 2^5 \times 3^1 \times 5^1$

$x_1 = 2^2; x_2 = 2^2$

we have $x_3x_4 = 30$

Now, possibility for x_3 and x_4 is 5 and 6 or 10 and 3 or 15 and 2.

Hence, 3 such polynomials are possible, i.e.,

$P(x) = (x-4)^2(x^2 + Bx + 30)$ B can be –11, –13, –17

A : $a = x_1 + x_2 + x_3 + x_4$

$a_{\min} = 4 + 4 + 6 + 5 = 19$

B : $a_{\max} = 4 + 4 + 15 + 2 = 25$

P(2) = 4(34 + 2B)

$P(2)_{\min} = 0$

Hence, (A), (B) and (D) are correct.

28. [Note : [y] denotes greatest integer function less than or equal to y.]

$$\frac{dy}{dx} = \frac{1}{dxldy}; \frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{1}{dx/dy}\right).\frac{dy}{dx} = -\frac{1}{(dx/dy)^3}\frac{d^2x}{dy^2}$$

Hence, $x\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 - \dfrac{dy}{dx} = 0$

Becomes, $-x.\ \dfrac{1}{(dx/dy)^3}\dfrac{d^2x}{dy^2} + \dfrac{1}{(dxldy)^3} - \dfrac{1}{(dx/dy)} = 0$

$$x\frac{d^2x}{dy^2} - 1 + \left(\frac{dx}{dy}\right)^2 = 0 \Rightarrow x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 = 1$$

$\therefore k = 1$

Hence, (B), (C) and (D) are correct.

29. $f(x) = e^{x^3 + x^2 + x + 1} - e^{-x^3 + x^2 - x - 3} + 2x + 5 \quad f(-1) = 3$

$$f'(x) = e^{x^3 + x^2 + x + 1}(3x^2 + 2x + 1) - e^{-x^3 + x^2 - x - 3} \times (-3x^2 + 2x - 1) + 2$$

$\therefore f(-1) = 3 - 2 + 1 - (-3 - 2 - 1) + 2 = 10$

Now, verify the options.

Hence, (A) and (B) are correct.

30. (A) If $x \in (0, 1)$; $f(x) = \dfrac{\pi}{2} - 2\tan^{-1}x \Rightarrow f'(x) = \dfrac{-2}{1+x^2}$

(B) If $0 < x < 1$; $g(x) = 2\tan^{-1}x \Rightarrow g'(x) =$ obviously $= \dfrac{2}{1+x^2}\left(\dfrac{x}{|x|}\right)$

$\therefore$ If $x \in (-1, 1) \Rightarrow h'(x) = \dfrac{2}{1+x^2}$

(C) If $x \in (-1, 1)$; $h(x) = \dfrac{\pi}{2} - 2\tan^{-1} x \quad \Rightarrow h'(x) = \dfrac{-2}{1+x^2}$

(D) If $-1 < x < 1$; $k(x) = 2\tan^{-1} x \Rightarrow k'(x)$ obviously will be $\dfrac{2}{1+x^2}$

Hence, (B) and (D) are correct.

Linked Comprehension Type Questions

1. $$D^* f(x) = \lim_{h\to 0} \frac{f^2(x+h) - f^2(x)}{h}$$
$$= \lim_{h\to 0} \frac{f(x+h)+f(x)}{h} \cdot \frac{f(x+h)-f(x)}{h}$$

$D^*f(x) = 2f(x)\,f'(x)$

$D^*(fg)(x) = 2fg(x)\,(fg(x))' = f^2(x).2g(x)g'(x) + g^2(x).2f(x)f'(x)$

$$= \tan^2 x.2\frac{\ln x}{x} + (\ln x)^2.2\tan x \sec^2 x \qquad \therefore D^*(fg)(1) = 0$$

Hence, the correct answer is (B).

2. $D^*(fg)(x) = x^{10}.2e^{2x} + e^{2x}.10x^9$

$\Rightarrow \quad D^*(fg)(1) = 2e^2 + 10e^2 = 12e^2$

Hence, the correct answer is (D).

3. $(fog)(x) = \sin(\cos x) \Rightarrow D^*((fog)(x).= 2(fog)(x)\,(fog)'(x)$

$$= 2\sin(\cos x).\cos(\cos x).(-\sin x) = -\sin x \sin(2\cos x)$$

Hence, the correct answer is (A).

4. We have, $\dfrac{f(b)-f(a)}{F(b)-F(a)} = \dfrac{e^b - e^a}{e^{-b} - e^{-a}} = \dfrac{f'(c)}{F'(c)} = \dfrac{e^c}{-e^{-c}}$

$$\Rightarrow \quad \frac{\left(e^b - e^a\right)e^a e^b}{e^a - e^b} = -e^{2c}$$

$\Rightarrow \quad e^{2c} = e^{a+b} \Rightarrow 2c = a + b$

Hence, the correct answer is (A).

5. We have, $\dfrac{\sin b - \sin a}{\cos b - \cos a} = \dfrac{\cos c}{-\sin c}$

$$\Rightarrow \quad \frac{\sin\dfrac{b-a}{2}\cos\dfrac{a+b}{2}}{\sin\dfrac{a-b}{2}\sin\dfrac{a+b}{2}} = -\cot c \Rightarrow \cot\frac{a+b}{2} = \cot c$$

$\Rightarrow \quad c = \dfrac{a+b}{2}$

Hence, the correct answer is (A).

6. Let $F(x) = x^2$

Hence, $\dfrac{f(b)-f(a)}{b^2-a^2} = \dfrac{f'(c)}{2c} \qquad (\because\ (a^2 - b^2) f'(c) = k[f(a) - f(b)])$

$\Rightarrow \quad k = 2c$

Hence, the correct answer is (B).

7. $f(x)g(x) = x^3 g(x)$

$3x^2.\,g'(x) = 3x^2 g(x) + x^3 g'(x)$;

$3g'(x) = 3g(x) + xg'(x)$

$(3-x)g'(x) = 3g(x)$;

$$\Rightarrow \quad \int \frac{g'(x)}{g(x)}\,dx = \int \frac{3}{3-x}\,dx + \ln c$$

$\Rightarrow \quad \ln g(x) = -3\ln|3-x| + \ln c$

$\therefore \quad g(x) = \dfrac{c}{|3-x|^3}$

$\Rightarrow \quad g(0) = \dfrac{c}{27} = \dfrac{1}{3}$

$\therefore \quad c = 9$

$\therefore \quad g(x) = \dfrac{9}{|3-x|^3}$

Hence, the correct answer is (C).

8. $f(x-3).g(x) = (x-3)^3.\,g(x) = 9$

$\left(\because g(x) = \dfrac{9}{|3-x|^3}\right)$

$\therefore$ Derivative of $f(x-3).\,g(x)$ is 0.

Hence, the correct answer is (A).

9. $$\lim_{x\to 0} \frac{f(x).g(x)}{x(1+g(x))} = \lim_{x\to 0} \frac{x^3 \dfrac{9}{|x-3|^3}}{x\left(1 + \dfrac{9}{|x-3|^3}\right)} = 0$$

Hence, the correct answer is (A).

10. $f(x) = 3x - \sin\dfrac{\pi}{2}x$

Let $K = f^{-1}$; $\qquad K(f(x)) = x$

$K'(f(x))f'(x) = 1$; $\qquad K'(f(x)) = \dfrac{1}{f'(x)}$

$g(x) = x^3 + 2x - \sin\dfrac{\pi}{2}x$

Let $J = g^{-1}$; $J(g(x)) = x$

$J'(g(x)) = \dfrac{1}{g'(x)}$

Now, $\left.\dfrac{d}{dx} f^{-1}(g^{-1}(x))\right|_{x=12} = \dfrac{d}{dx}K(J(x)) \quad = K'(J(12))J'(12)$

$J'(12) = J'(g(x)) = \dfrac{1}{g'(x)}$

$g(x) = 12 = x^3 + 2x - \sin\dfrac{\pi}{2}x$ we get $x = 2$

$$J'(12) = \frac{1}{g'(2)} = \frac{1}{\left.3x^2 + 2 - \dfrac{\pi}{2}\cos\dfrac{\pi}{2}x\right|_{x=2}} = \frac{1}{14 + \dfrac{\pi}{2}} = \frac{2}{28+\pi}$$

If $g(2) = 12$; $J(12) = 2$; $\qquad K'(J(12)) = K'(2)$

$K'(f(x)) = \dfrac{1}{f'(x)}$; $\qquad K'(2) = \dfrac{1}{f'(x)}$

$f(x) = 2 \quad \Rightarrow 3x - \sin\dfrac{\pi}{2}x = 2$;we get $x = 1$

$$K'(2) = \frac{1}{\left.3 - \dfrac{\pi}{2}\cos\dfrac{\pi}{2}x\right|_{x=1}} = \frac{1}{3-0} = \frac{1}{3}$$

Now, required $= K'(J(12))J'(12) = \dfrac{1}{3}\left(\dfrac{2}{28+\pi}\right)$

Hence, the correct answer is (D).

11. $f'(x) = g'(x) \Rightarrow \quad 3 - \cancel{\dfrac{\pi}{2}\cos\left(\dfrac{\pi}{2}x\right)} = 3x^2 + 2 - \cancel{\dfrac{\pi}{2}\cos\left(\dfrac{\pi}{2}x\right)}$

$x = \pm\dfrac{1}{\sqrt{3}}$; no real root

Hence, the correct answer is (C).

12. $x = f(t) = a^{\ln b^t} = a^{t\ln b} = (a^{\ln b})^t$

$$y = g(t) = a^{-\ln b^t} = \frac{1}{f(t)}; \quad y = \frac{1}{x} \quad g(t) = \frac{1}{f(t)}$$

$$\frac{dy}{dx} = \frac{-1}{x^2} = \frac{-1}{(f(t))^2} = -(g(t))^2 = -\frac{g(t)}{f(t)}$$

Hence, the correct answer is (D).

13. $y = \frac{1}{x}; \frac{dy}{dx} = \frac{-1}{x^2}$

$x = y$ [Curve $y = \frac{1}{x}$]

$x = \pm 1$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}\bigg|_{x=\pm 1} = 2$$

Hence, the correct answer is (D).

14. $y = \frac{1}{x} \Rightarrow g(t) = \frac{1}{f(t)};$ $\quad x = f(t) = a^{t\ln b}$

$f(t) = (a^{\ln b})^t = K^t;$ $\quad f'(t) = K^t \ln K$

$f''(t) = K^t(\ln k)^2;$

$$\frac{f(t)f''(-t)}{f'(t)f'(-t)} + \frac{f(-t)f''(t)}{f'(-t)f'(t)} = \frac{K^t \cdot K^{-t}(\ln K)^2}{(K^t \ln K)(K^{-t}\ln K)} + \frac{K^{-t}\cdot K^t(\ln K)}{(K^{-t}\ln K)(K^t \ln K)}$$

$= 1 + 1 = 2$

Hence, the correct answer is (B).

15. $r^2 = a^2\cos^2\theta + b^2\sin^2\theta = a^2 + (b^2 - a^2)\sin^2\theta$ …(1)

On differentiating w.r.t. θ, we get

$$r\frac{dr}{d\theta} = \left(b^2 - a^2\right)\sin\theta\cos\theta \quad \text{…(2)}$$

Again Differentiating we get,

$$r\frac{d^2r}{d\theta^2} + \left(\frac{dr}{d\theta}\right)^2 = \left(b^2 - a^2\right)\cos 2\theta$$

$$= (b^2 - a^2)\left(1 - 2\sin^2\theta\right)$$

$$= \left(b^2 - a^2\right)\left(1 - 2\left(\frac{r^2 - a^2}{b^2 - a^2}\right)\right)$$

$$= a^2 + b^2 - 2r^2 \quad \text{…(3)}$$

Hence, the correct answer is (B).

16. Using (2) and (3),

$$r\frac{d^2r}{d\theta^2} + \frac{\left(b^2 - a^2\right)^2 \sin^2\theta\cos^2\theta}{r^2} = a^2 + b^2 - 2r^2$$

$$\Rightarrow \quad r^3\frac{d^2r}{d\theta^2} + \left(b^2 - a^2\right)^2 \times \left(\frac{r^2 - a^2}{b^2 - a^2}\right)\left(\frac{b^2 - r^2}{b^2 - a^2}\right) = \left(a^2 + b^2 - 2r^2\right)r^2$$

(Using (1))

$$\Rightarrow \quad r^3\frac{d^2r}{d\theta^2} + \left(a^2 + b^2\right)r^2 - a^2b^2 - r^4 = \left(a^2 + b^2\right)r^2 - 2r^4$$

$$\Rightarrow \quad r^3\frac{d^2r}{d\theta^2} + r^4 = a^2b^2.$$

Hence, the correct answer is (A).

17. We have, $f(x) = \tan^{-1}|x - 1| = \begin{cases} \tan^{-1}(1 - x); & x < 1 \\ \tan^{-1}(x - 1); & x \ge 1 \end{cases}$

Clearly, $f(x)$ is non-derivable at $x = 1$ as $f'(1^-) = -1$ and $f'(1^+) = 1$

Also, $g(x) = \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right)$

$$= \pi - \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) = \begin{cases} \pi + 2\tan^{-1}x; & x < 0 \\ \pi - 2\tan^{-1}x; & x \ge 0 \end{cases}$$

As $f(x)$ is non-derivable at $x = 1$, so derivative of $f(x)$ with respect to $g(x)$, when $x = 1$ is non-existent.

Hence, the correct answer is (D).

18. $g(x) = \pi + 2\tan^{-1}x, x < 0$

$$\therefore g'(x) = 0 + \frac{2}{1 + x^2}$$

$$\Rightarrow g''(x) = \frac{-2\cdot(2x)}{(1 + x^2)^2}$$

$$\therefore g''(x = -1) = \frac{4}{4} = 1$$

Hence, the correct answer is (C).

19. $f(x) = g(x)$

$x^{\ln x} = e^2 x$

$\Rightarrow$ $(\ln x)^2 = 2 + \ln x$

$\Rightarrow$ $(\ln x)^2 - \ln x - 2 = 0$

$\Rightarrow$ $(\ln x - 2)(\ln x + 1) = 0$ $\quad \Rightarrow \ln x = -1, 2 \quad \Rightarrow x = \frac{1}{e}, e^2$

$\therefore$ $\alpha = \frac{1}{e}$ and $\beta = e^2$

$$\alpha\beta = \frac{1}{e}\cdot e^2 = e$$

Hence, the correct answer is (C).

20. $l = \lim_{x\to\beta} \frac{f(x) - c\cdot\beta}{g(x) - \beta^2} = \lim_{x\to e^2} \frac{f(x) - c\cdot e^2}{g(x) - e^4}$

$\because$ $g(x) - e^4 \to 0$ as $x \to e^2$

$\therefore$ $f(x) - ce^4 \to 0$ as $x \to e^2$

$\Rightarrow$ $e^4 - ce^2 = 0 \Rightarrow c = e^2.$

$$l = \lim_{x\to e^2} \frac{f(x) - e^4}{g(x) - e^4} \Rightarrow \lim_{x\to e^2} \frac{x^{\ell n x} - e^4}{e^2 x - e^4} \Rightarrow \lim_{x\to e^2} \frac{x^{(\ln x)^2} - e^4}{e^2(x - e^2)}$$

$$\Rightarrow \quad \lim_{x\to e^2} \frac{e^2(\ln x + 2)(\ln x - 2)}{x - e^2}$$

$$\Rightarrow \quad \lim_{x\to e^2} e^2\cdot 4\cdot\left(\frac{\ln x - 2}{x - e^2}\right) \Rightarrow 4e^2 \lim_{x\to e^2} \frac{\frac{1}{x}}{1} \Rightarrow 4e^2\cdot\frac{1}{e^2} = 4.$$

Hence, the correct answer is (B).

21. $h(x) = \frac{f(x)}{g(x)} = \frac{x^{\ln x}}{e^2\cdot x} = \frac{x^{(\ln x)^2}}{e^2\cdot x}$

$$\therefore \quad h'(x) = \frac{1}{e^2}\left(\frac{x\cdot e^{(\ln x)^2}\cdot 2\ln x\cdot\frac{1}{x} - e^{(\ln x)^2}\cdot 1}{x^2}\right)$$

$$= \frac{1}{e^2}\, e^{(\ln x)^2}\left(\frac{2\ln x - 1}{x^2}\right)$$

$$h' = \left(\frac{1}{e}\right) = \frac{1}{e^2}\,\frac{e(-2 - 1)}{\frac{1}{e^2}} = -3e.$$

Hence, the correct answer is (D).

22. Let degree of $f(x)$ be n.

$\therefore n^3 = n.$

Hence, $n = 1$

$\therefore f(x)$ must be linear.

Let $f(x) = 2x + b \Rightarrow f(f(x))$

$= 2(2x + b) + b = 4x + 3b$

$f(f(f(x))) + (1 - p)\, f(x) = 3\ \forall\ x \in R$

$\therefore 2(4x + 3b) + b + (1 - p)(2x + (B) = 3$

$(10 - 2p)x + (8b - bp) = 3$

$10 - 2p = 0 \Rightarrow p = 5,\ 8b - bp = 3$

$\Rightarrow \quad p = 1 \therefore f(x) = 2x + 1$

$$\frac{d}{dx}(f(f(x))) = f'(f(x))\, f'(x)$$

Hence, the correct answer is (B).

23. $f(x) = ax + 4,\ a < 0$

$f(f(x)) = a(ax + 4) + 4 = a^2x + 4a + 4$

$f(f(f(x))) = a(a^2x + 4a + 4) + 4 = a^3x + 4a^2 + 4a + 4$

Now, $a^3x + 4a^2 + 4a + 4 + (1 - p)\,(ax + 4) = 3\ \forall\ x \in R$

$\Rightarrow \quad a^3 + (1 - p)a = 0 \Rightarrow a^2 = p - 1\ (a \neq 0)$

$\Rightarrow \quad 4a^2 + 4a + 4 + 4\,(1 - p) = 3$

$\therefore \quad a = \dfrac{-1}{4} \therefore f(x) = \dfrac{-1}{4}x + 4$

$\Rightarrow \quad x = 16 - 4y = f^{-1}(y)$

$\therefore \quad f^{-1}(4) = 0$

Hence, the correct answer is (C).

Matrix Match/Column Match Type Questions

1. **A $\rightarrow$ (s); B $\rightarrow$ (q); C $\rightarrow$ (q); D $\rightarrow$ (p)**

(A) $y = \tan^{-1}\left(x + \sqrt{1+x^2}\right)$

$$y' = \frac{1}{1+\left(x+\sqrt{1+x^2}\right)^2}\left[1+\frac{x}{\sqrt{1+x^2}}\right]$$

$$y'(0) = \frac{1}{2}$$

(B) $y = \dfrac{1 + x\tan^{-1}x}{\sqrt{1+x^2}}$

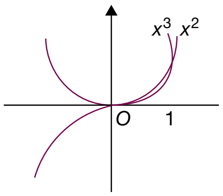
$$y' = \frac{\left(\sqrt{1+x^2}\right)\left[\dfrac{x}{1+x^2}+\tan^{-1}x\right] - \dfrac{\left(1+x\tan^{-1}x\right)x}{\sqrt{1+x^2}}}{(1+x^2)}$$

$$y'(0) = 0$$

(C) $f(x) = x^3$, when $x \in (-\infty, 1]$

$f'(x) = 3x^2$

$f'(0) = 0$



(D) $y = \dfrac{2(x+1)-1}{x+1} - \dfrac{1}{2}\left[\dfrac{2x+1+1}{2x+1}\right]$

$$= 2 - \frac{1}{x+1} - \frac{1}{2}\left(1+\frac{1}{2x+1}\right)$$

$$y' = \frac{1}{(x+1)^2} + \frac{1}{2}\left(\frac{2}{(2x+1)^2}\right) = \frac{1}{(x+1)^2} + \frac{1}{(2x+1)^2}$$

$$y'(0) = 2$$

2. **A $\rightarrow$ (r); B $\rightarrow$ (s); C $\rightarrow$ (s); D $\rightarrow$ (q)**

(A) $f(x) = \cos^{-1}\left(\dfrac{2x}{1+x^2}\right) = \dfrac{\pi}{2} - \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$\therefore f(x) = \dfrac{\pi}{2} - 2\tan^{-1}x,\ -1 \le x \le 1$

So, $f'(x)$ at$_{x=0} = \left(\dfrac{-2}{1+x^2}\right) = -2.$

(B) $g(x) = \cos^{-1}(2x^2 - 1)$

$g'(x) = \dfrac{-1}{\sqrt{1-(2x^2-1)^2}}4x$ $\qquad \therefore g'(0) =$ non-existent.

(C) $h(x) = \sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$

$$= \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} \dfrac{\pi}{2} + 2\tan^{-1}x, & -\infty < x < 0 \\ \dfrac{\pi}{2} - 2\tan^{-1}x & x \ge 0 \end{cases}$$

As, $h'(0^-) = 2$ and $h'(0^+) = -2$. So, $h'(0) =$ non-existent.

(D) $k(x) = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) = 3\tan^{-1}x,\ \dfrac{-1}{\sqrt{3}} < x < \dfrac{1}{\sqrt{3}}$

$\therefore \quad k'(x) = \dfrac{3}{1+x^2}$. So, $k'(0) = 3$

3. **A $\rightarrow$ (q); B $\rightarrow$ (r); C $\rightarrow$ (p); D $\rightarrow$ (p)**

(A) $y_1(x) = f'\big(f(f(f(x)))\big)\, f'\big(f(f(x))\big).\, f'\big(f(x)\big).f'(x)$

$$\Rightarrow y_1(0) = f'\big(f(f(f(0)))\big)\ f'\big(f(f(0))\big).\ f'\big(f(0)\big).f'(0)$$

$$= f'\big(f(f(0))\big)\ f'\big(f(0)\big).\ f'(0).\ f'(0)$$

$$= f'\big(f(0)\big)\ f'(0).\ f'(0).\ f'(0)$$

$$= f'(0)\, f'(0)\, f'(0)\, f'(0) = 16$$

(B) $y = f(x) = l\text{n} \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$

$$\therefore \quad \frac{dy}{dx} = \frac{\sec^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}{\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)}.\frac{1}{2}$$

$$= \frac{1}{2}\,\frac{1}{\sin\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\cos\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)} = \frac{1}{\sin\left(\dfrac{\pi}{2}+x\right)} = \sec x$$

(C) Let $\cos\alpha = \dfrac{5}{13}$ and $\sin\alpha = \dfrac{12}{13}$

So, $y = \cos^{-1}(\cos\alpha.\cos x - \sin\alpha.\sin x)$

$\Rightarrow y = \cos^{-1}(\cos(x+\alpha)) = x + \alpha$

$\because \ x + \alpha$ is in the first quadrant.

$\Rightarrow \dfrac{dy}{dx} = 1$

(D) $xe^{xy} - y = \sin^2 x$

On differentiating w.r.t x, we get

$$e^{xy} + x\,e^{xy}\left(x\frac{dy}{dx}+y\right) - \frac{dy}{dx} = 2\sin x\cos x$$

$$\therefore \frac{dy}{dx} = \frac{2\sin x\cos x - e^{xy} - xye^{xy}}{x^2e^{xy} - 1}$$

When $x=0, y=0$

$$\therefore \frac{dy}{dx}=\frac{0-1-0}{0-1}=1$$

4. A → (Q); B → (R); C → (P); D → (Q)

(A) $fog = I$

$$\Rightarrow fog(x)=x, \forall x$$

$$\Rightarrow f'(g(x))g'(x)=1, \forall x$$

$$\Rightarrow f'(g(x))=\frac{1}{g'(x)}$$

$$\Rightarrow f'(g(a))=\frac{1}{2} \Rightarrow f'(b)=\frac{1}{2}$$

(B) We know that,

$$\cos x.\cos 2x.\cos 2^2x \ldots \cos 2^{n-1}x=\frac{\sin 2^n x}{2^n \sin x}$$

$$\therefore f(x)=\frac{\sin 32x}{32 \sin x}$$

$$\Rightarrow f'(x)=\frac{1}{32}\left[\frac{\sin x \ \cos(32x).32-(\sin 32x)\cos x}{\sin^2 x}\right]$$

$$\therefore f'\left(\frac{\pi}{4}\right)=\sqrt{2}$$

(C) $y=f(x^2) \Rightarrow \frac{dy}{dx}=f'(x^2)2x$

$$=2x\sqrt{2(x^2)^2-1}$$

At $x=1 \Rightarrow \frac{dy}{dx}=2$

(D) Put $x=\cos\theta \Rightarrow \theta=\cos^{-1}x$

$$\Rightarrow y=\sin^2 \cot^{-1}\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right) = \sin^2\left(\frac{\pi}{2}-\frac{\theta}{2}\right)$$

$$=\cos^2\left(\frac{\theta}{2}\right)=\frac{1+\cos\theta}{2} \ = \frac{1+x}{2}$$

$$\therefore \frac{dy}{dx}=\frac{1}{2}$$

5. A → (t); B → (r); C → (t); D → (p)

(A) $y=f(x^3)$

$\therefore \quad \frac{dy}{dx}=f'(x^3).\,3x^2$

$\therefore \quad \left.\frac{dy}{dx}\right|_{x=1}=f'(1).\,3=9 \qquad (\because f'(1)=\sqrt{9}=3)$

(B) $f(xy)=f(x)+f(y)$

$f(1) = f(1) + f(1) \qquad \therefore \ f(1)=0$

Also, $f(1)=f(e)+f\left(\frac{1}{e}\right)$

$\therefore \quad f(e)+f\left(\frac{1}{e}\right)=0$

(C) $f''(x)=-f(x),\ f'(x)=g(x)$

$\therefore g'(x)=f''(x)=-f(x)$

$h(x)=(f(x))^2+(g(x))^2$

$\therefore \quad h'(x)=2f(x).\ f'(x)+2g(x).g'(x)$

$= 2 f(x).\ g(x) + 2 gx\,(-f(x)) = 0$

$\therefore h(x)=c, x\in R$

$\therefore \quad h(10) = h\,(5) = 9.$

(D) $y=\tan^{-1}(\cot x)+\cot^{-1}(\tan x), \frac{\pi}{2}<x<\pi$

$$\frac{dy}{dx}=\frac{-\operatorname{cosec}^2 x}{1+\cot^2 x}+\frac{-1}{1+\tan^2 x}.\ \sec^2 x \ =-1-1=-2$$

6. A → (t); B → (q); C → (r); D → (s)

(A) $y=\cos^{-1}(\cos x)$

$$y'=\frac{-1}{\sqrt{1-\cos^2 x}}.\,(-\sin x)=\frac{\sin x}{|\sin x|}$$

$\therefore \quad y'$ at $x = 5$ is -1

(B) $y = f(x) = \ln\ |\tan x|$

For $x < 0$; $f(x) = \ln\ |-\tan x|$

$\therefore \quad f'(x)=(1/\tan x)(\sec^2 x)$

Hence, $f'\left(-\frac{\pi}{4}\right)=-2$

(C) $$\frac{d}{dx}\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}.\frac{d}{dx}\left(\frac{1+x}{1-x}\right)$$

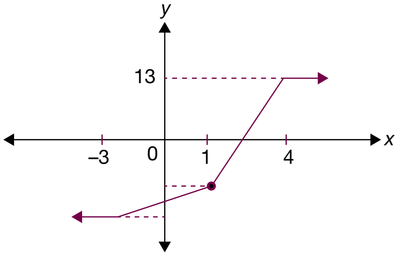
$$=\frac{(1-x)^2}{2(1+x^2)}.\frac{2}{(1-x)^2}=\frac{1}{1+x^2} \text{ at } x=-1$$

Hence, $\frac{d}{dx}\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{1}{2}$

(D) $$\frac{d}{dx}\frac{\ln|x|}{x}=\frac{x.\frac{1}{x}-\ln|x|}{x^2}=\frac{1-\ln|x|}{x^2} \text{ at } x=-1$$

Hence, $\frac{d}{dx}\frac{\ln|x|}{x}=1$

7. A → (p); B → (p); C → (q); D → (r)



Clearly, $m = -13$, $M = 13$

(B) $x^2 + y^2 - 2xy = x^2 + y^2 \qquad \Rightarrow xy = 0$

$xy' + y \times 1 = 0 \qquad \Rightarrow y' = \frac{-y}{x}$

$\therefore \quad y'|_{(x,\,0)} = 0$ and $y'|_{(0,\,y)} = \infty$

(C) $\because f'''(4) = 0 \Rightarrow (x - 4)$ must be a factor of $(x^3 + bx^2 + cx + d)$

and two other factors are $(x - 1)$ and $(x - 2)$

$\therefore\ x^3 + bx^2 + cx + d = (x - 1)\,(x - 2)\,(x - 4)$

$1 + b + c + d = 0$

$\Rightarrow |b + c + d| = 1$

(D) $f(x) = \sin\left(\cos^{-1}\dfrac{x}{2}\right) + \sin\left(\tan^{-1}\dfrac{\sqrt{4-x^2}}{x}\right)$

$$= \begin{cases} \sqrt{4-x^2}; & 0 < x \le 2 \\ 0; & -2 \le x < 0 \end{cases}$$

∴ Range of f is : [0, 2]

Integer/Numerical Answer Type Questions

1. $\ln(x + y) = 2xy$

At $x = 0$, $y = 1$,

$\therefore \quad \dfrac{1+y'}{x+y} = 2\,(xy' + y)$

Put $x = 0$, $y = 1 \Rightarrow 1 + y' = 2(0 + 1) = 2$

$\Rightarrow \quad y' = 1$

Now, $\dfrac{(x+y)y'' - (1+y')^2}{(x+y)^2} = 2(xy'' + 2y')$

Put $x = 0$, $y = 1$, $y' = 1$, we get

$\dfrac{y''-4}{1} = 2\,(0 + 2) = 4 \Rightarrow y''\,(0) = 8$

Hence, the correct answer is (8).

2. Let $y = f^2(x) - g^2(x)$

Now, $\dfrac{dy}{dx} = 2\,(f(x)f'(x) - g(x)\,g'(x))$

$= 2\,(f(x)g(x) - g(x)\,f(x)) = 0$

$\Rightarrow \quad f^2(x) - g^2(x) = k$ (constant)

So, $f^2(3) - g^2(3) = k \Rightarrow (5)^2 - (4)^2 = k \Rightarrow k = 9, \forall\ x \in R$

Hence, $f^2(\pi) - g^2(\pi) = 9$

Hence, the correct answer is (9).

3. L is $\left(\dfrac{0}{0}\right)$ from, By L.H. Rule,

$$L = \lim_{h\to 0} \frac{f'\left(2h+2+h^2\right)(2+2h)}{f'\left(h-h^2+1\right)(1-2h)} = \frac{2f'(2)}{f'(1)} = \frac{2\times 6}{4} = 3$$

Hence, the correct answer is (3).

4. $y = \cos^{-1}(\cos x) = 4\pi - x$

$\left.\dfrac{dy}{dx}\right|_{x=\frac{13\pi}{4}} = -1$

Hence, the correct answer is (0).

5. $\left(x^2+y^2\right)^2 = \left(t - \dfrac{1}{t}\right)^2 = t^2 + \dfrac{1}{t^2} - 2$

$\Rightarrow \quad x^4 + y^4 + 2x^2y^2 = x^4 + y^4 - 2 \ \Rightarrow \ x^2y^2 = -1$

On differentiating with respect to x, we get

$\Rightarrow \quad x^2.2y\dfrac{dy}{dx} + y^2.2x = 0 \ \Rightarrow x^3y\dfrac{dy}{dx} = 1$

Hence, the correct answer is (1).

6. $\log(x+y) = 2xy$; When $x = 0; y = 1$

$\dfrac{1}{x+y}\left[1+\dfrac{dy}{dx}\right] = 2\left[x\dfrac{dy}{dx}+y\right]$

At (0,1) we get $\dfrac{dy}{dx} = 1$

Hence, the correct answer is (1).

7. Taking log on both sides,

$n\log\,(x+y) = \log x + \log y$

On differentiating w.r.t. x $\dfrac{n}{x+y}\left[1+\dfrac{dy}{dx}\right] = \dfrac{1}{x} + \dfrac{1}{y}\dfrac{dy}{dx}$

$\dfrac{dy}{dx} = \dfrac{y}{x}\left[\dfrac{x+y-nx}{ny-x-y}\right] \Rightarrow n = 2$ satisfies

Hence, the correct answer is (2).

8. $y' = f'(x) - 2f'(2x)$

$y'(1) = f'(1) - 2f'(2) = 5$ …(1)

And $y'(2) = f'(2) - 2f'(4) = 7$ …(2)

Now, let $y = f(x) - f(4x)$

$y' = f'(x) - 4f'(4x)$

$y'(1) = f'(1) - 4f'(4)$ …(3)

On substituting the value of $f'(2) = 7 + 2f'(4)$ in (1)

$f'(1) - 2[7 + 2f'(4)] = 5$

$f'(1) - 4f'(4) = 19$.

Hence, the correct answer is (9).

9. $y = \dfrac{2x^2}{(x-2)(x-3)(x-4)} + \dfrac{3x+x^2-3x}{(x-3)(x-4)} = \dfrac{x^3}{(x-2)(x-3)(x-4)}$

$\log y = 3\log x - \log(x-2) - \log(x-3) - \log(x-4)$

$\dfrac{\frac{dy}{dx}}{y} = \dfrac{1}{x} - \dfrac{1}{x-2} + \dfrac{1}{x} - \dfrac{1}{x-3} + \dfrac{1}{x} - \dfrac{1}{x-4}$

$= \dfrac{-2}{x(x-2)} - \dfrac{3}{x(x-3)} - \dfrac{4}{x(x-4)}$

$\dfrac{x}{y}\dfrac{dy}{dx} = \dfrac{2}{2-x} + \dfrac{3}{3-x} + \dfrac{4}{4-x} \Rightarrow a + b + c = 9$

Hence, the correct answer is (9).

10. Clearly, $g'[f(0)] = g'[1] = \dfrac{1}{f'(0)}$ (Here, $f(0) = 1$)

$f'(x) = 20x^4 + 9x^2 + 4x + \dfrac{e^{x/7}}{7}$

$f'(0) = \dfrac{1}{7} \because gof(x) = x$ as ‘g’ is inverse of f.

$\Rightarrow g'[f(x)].f'(x) = 1 \qquad \therefore g'(1) = 7$

Hence, the correct answer is (7).

11. $f(x) = x^3 + Ax^2 + Bx + C$ where $A = f'(1)$

$f'(x) = 3x^2 + 2Ax + B \qquad B = f''(2)$

$f''(x) = 6x + 2A; f'''(x) = 6 \qquad C = -f'''(3)$

$B = 12 + 2A$ …(1)

$C = -6$; $A = 3 + 2A + B$ …(2)

$A + B = -3$; $2A - B = -12$

$3A = -15$, A = –5, B = 2 $\qquad \therefore f(x) = x^3 - 5x^2 + 2x - 6$

$f(0) = -6$

$f(3) = 27 - 5\times 9 + 2\times 3 - 6 = -18$

$\therefore f(0) + f(3) = -12$

Hence, $|f(0) + f(3) + 16| = 8$

Hence, the correct answer is (8).

12. $f''(x)=g''(x) \Rightarrow f'(x)=g'(x)+c_1$

$\Rightarrow \ f(x)=g(x)+c_1x+c_2$

$f'(1)=g'(1)+c_1$ and $f(2)=g(2)+2c_1+c_2$

$c_1=f'(1)-g'(1)=2-4=-2$

$c_2=f(2)-g(2)-2c_1=-2$

$\therefore f(4)-g(4)=4c_1+c_2 \ =-10$

Hence, the correct answer is (5).

13. $y=\tan^{-1}\dfrac{4x}{1+5x^2}+\tan^{-1}\dfrac{2+3x}{3-2x}=\tan^{-1}\dfrac{5x-x}{1+5x.x}+\tan^{-1}\dfrac{\frac{2}{3}+x}{1-\frac{2}{3}.x}$

$=\tan^{-1}5x-\tan^{-1}x+\tan^{-1}\dfrac{2}{3}+\tan^{-1}x=\tan^{-1}5x+\tan^{-1}\dfrac{2}{3}$

$\therefore \dfrac{dy}{dx}=\dfrac{5}{1+25x^2}$

Hence, the correct answer is (5).

14. $y=e^{\sin^{-1}x}+e^{-\cos^{-1}x}$

$y=e^{\sin^{-1}x}+e^{\frac{-\pi}{2}+\sin^{-1}x}=e^{\sin^{-1}x}\left(1+e^{-\frac{\pi}{2}}\right);$

$\ln y=\sin^{-1}x+\ln\left(1+e^{\frac{-\pi}{2}}\right)$

$\dfrac{1}{y}\dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-x^2}}+0\,;\ (y')^2=\dfrac{y^2}{(1-x^2)}\Rightarrow(1-x^2)(y')^2=y^2$

$\Rightarrow -2x(y')^2+(1-x^2)2y'y''=2yy' \ \Rightarrow -xy'+(1-x^2)y''=y$

$-xy'=y+(x^2-1)y''$

$\left|\dfrac{(x^2-1)y''+y}{xy'}\right|=|-1| \ =1$

Hence, the correct answer is (1).

15. $(f(2x+2y)-f(2x-2y))\sin x\cos y=\cos x\sin y(f(2x+2y)+f(2x-2y))$

$f(2x+2y)(\sin x\cos y-\cos x\sin y) \ =(\cos x\sin y+\sin x\cos y)f(2x-2y)$

$f(2x+2y)\sin(x-y)=\sin(x+y)f(2x-2y)$

$x=0 \ \ f(2y)\sin(-y)=\sin y f(-2y)$

$f(2y)=-f(-2y)\ ;\ \dfrac{f(2x+2y)}{f(2x-2y)}=\dfrac{\sin(x+y)}{\sin(x-y)}$

$f(x)=k\sin\dfrac{x}{2} \ \Rightarrow \ f'(x)=\dfrac{k}{2}\cos\dfrac{x}{2}$

$f'(0)=\dfrac{1}{2}=\dfrac{k}{2} \ \Rightarrow k=1$

$f''(x)=\dfrac{-k}{4}\sin\dfrac{x}{2}$

$4f''(x)+f(x)=-k\sin\dfrac{x}{2}+k\sin\dfrac{x}{2}=0$

Hence, the correct answer is (0).

16. $f(x)+f(y)=f\left(\dfrac{x+y}{1-xy}\right) \qquad xy\neq 1$

$f(x)=k\tan^{-1}x \qquad y,x<1$

$\lim\limits_{x\to 0}\dfrac{f(x)}{x}=2=k$

$\dfrac{15}{\pi}\dfrac{f\left(\sqrt{3}\right)}{f'(-2)}=\dfrac{15}{\pi}\left(\dfrac{(2,\tan^{-1}\sqrt{3})}{\frac{k}{1+4}}\right) \ =\dfrac{(15)}{\pi}\dfrac{\pi}{3}\cdot 5=25$

Hence, the correct answer is (25).

17. $f(xy)=2f(x)+f(y)+x+2y+1$ …(1)

Let $x=y=1 \ \Rightarrow f(1)=-2$

On differentiating (1), w.r.t. x

$yf'(xy)=2f'(x)+1$

Now let $x=1$ and $y=x$

$\Rightarrow \ xf'(x)=2f'(1)+1=3$

$\Rightarrow \ f(x)=3\ln x+c$ and $c=-2$ $(\therefore f(1)=-2)$

$\therefore \ f(x)=3\ln x-2$

Hence, the correct answer is (7).

18. $y'=\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}-\dfrac{e^{-\sqrt{x}}}{2\sqrt{x}}$

$\Rightarrow \ 2\sqrt{x}y'=e^{\sqrt{x}}-e^{-\sqrt{x}}$

$\Rightarrow \ \dfrac{y'}{\sqrt{x}}+2\sqrt{x}y''=\dfrac{e^{\sqrt{x}}}{2\sqrt{x}}+\dfrac{e^{\ \sqrt{x}}}{2\sqrt{x}}$

$\Rightarrow \ 2y'+4xy''=y\Rightarrow 4xy+2y'-y=0$

A + B + C = 5

Hence, the correct answer is (5).

19. We have,

$f(x)=\lambda x^2+\mu x+12$

$f'(x)=2\lambda x+\mu$

$f'(4)=8\lambda+\mu=15$ …(1)

And $f'(2)=4\lambda+\mu=11$ …(2)

On solving Eqs. (1) and (2), we get

$\lambda=1$ and $\mu=7 \ \ \therefore \ \lambda+\mu=1+7=8$

Hence, the correct answer is (8).

20. We have,

$f(x)=(ax+b)\sin x+(cx+d)\cos x$

$\Rightarrow \ f'(x)=(ax+b)\cos x+a\sin x+c\cos x-(cx+d)\sin x$

$=ax\cos x+b\cos x+a\sin x+c\cos x-c\,x\sin x-d\sin x$

$=ax\cos x+(b+c)\cos x+(a-d)\sin x-c\,x\sin x$

But we have, $f'(x)=x\cos x$

$\therefore \ b+c=0,\ a-d=0,\ c=0$ and $a=1$

$\Rightarrow \ a+b+c+d=(1+0+0+1)=2$

Hence, the correct answer is (2).

21. We have, $y=f(x^3)$ and $f'(x)=\sqrt{3x^2+6}$

Now, $\dfrac{dy}{dx}=f'(x^3)\cdot 3x^2=\sqrt{3x^6+6}\cdot 3x^2$ $\left[\because f'(x)=\sqrt{3x^2+6}\right]$

$\therefore \ \left(\dfrac{dy}{dx}\right)_{x=1}=\sqrt{3+6}\times 3=3\times 3=9$

Hence, the correct answer is (9).

22. We have,

$f(x)=\tan^{-1}\dfrac{1}{x^2+x+1}+\tan^{-1}\dfrac{1}{x^2+3x+3}+\tan^{-1}\dfrac{1}{x^2+5x+7}+\ldots\infty$

$\Rightarrow \ f(x)=\tan^{-1}\left[\dfrac{(x+1)-x}{1+(x+1)(x)}\right]+\tan^{-1}\left[\dfrac{(x+2)-(x+1)}{1+(x+2)(x+1)}\right]+$

$\tan^{-1}\left[\dfrac{(x+3)-(x+2)}{1+(x+3)(x+2)}\right]+\ldots$

$\Rightarrow \ f(x)=(\tan^{-1}(x+1)-\tan^{-1}x)+(\tan^{-1}(x+2)-\tan^{-1}(x+1))+(\tan^{-1}(x+3)-\tan^{-1}(x+2))+\ldots+(\tan^{-1}(n+1)-\tan^{-1}n)$

$$\Rightarrow \quad f(x) = \frac{\pi}{2} - \tan^{-1}x \qquad \left[\because \tan^{-1}\infty = \frac{\pi}{2}\right]$$

$$\Rightarrow \quad f'(x) = \frac{-1}{1+x^2} \Rightarrow f'(0) = -1 \Rightarrow |f'(0)| = |-1| = 1$$

Hence, the correct answer is (1).

23. We have, $g(x) = f^{-1}(x)$ and $f(x) = x^2 - 4x - 3$

$f'(x) = 2x - 4$

Now, $gof(x) = x$,

On differentiating we get $g'(f(x)) \cdot f'(x) = 1$

$$g'(f(x)) = \frac{1}{2x-4}$$

Put $x = 6$, $g'(f(6)) = g'(9) = \frac{1}{8}$

$\therefore$ $8g'(x)$ at $f(x) = 9$ is 1.

Hence, the correct answer is (1).

24. $f'(x) = (k\,x + e^{x)}\,h'(x) + h(x)\cdot(k + e^{x)}$

$f'(0) = h'(0) + h(0)\,(k + 1)$

$18 = -2 + 5(k + 1)$

$\therefore$ $k = 3$

Hence, the correct answer is (3).

25. Let $y = f(x) - f(2x)$

$y'(1) = f'(1) - f'(2) \times 2 = 5 \Rightarrow f'(1) - 2f'(2) = 5$ …(1)

$y'(2) = f'(2) - 2f'(4) = 7 \Rightarrow f'(2) - 2f'(4) = 7$ …(2)

Given, $g(x) = f(x) - f(4x) - 16x$

$\Rightarrow g'(x) = f'(x) - 4f'(4x) - 16$

So, $g'(1) = f'(1) - 4f'(4) - 16$

From (1) and (2), we get

$\therefore f'(1) - 4f'(4) = 19$

$\therefore g'(1) = 19 - 16 = 3$

Hence, the correct answer is (3).

26. $f(x) = x + \cos(x) + 2$

$f'(x) = 1 - \sin x$ and $f'(0) = 1$

$f''(x) = -\cos x$ and $f''(0) = -1$

Now, $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \Rightarrow g'(y) = \frac{1}{f'(x)}$

When $y = 3$ then $x = 0$,

$$\therefore \quad g'(3) = \frac{1}{f'(0)} = 1 \qquad \text{…(1)}$$

$$\Rightarrow \quad g''(y) = \frac{-f''(x)}{(f'(x))^2} \text{ or } g''(3) = \frac{-f''(0)}{(f'(0))^3} = \frac{1}{1}$$

$g''(3) = 1$ …(2)

Hence, $g'(3) + g''(3) = 1 + 1 = 2$

Hence, the correct answer is (2).

27. $P(x) - P'(x) = (x + 1)^2$

Let degree of $P(x) = n$

Degree of $P'(x) = n - 1$

$$\therefore \quad \left.\begin{matrix}\text{LHS has a degree } n\\ \text{RHS has a degree } 2\end{matrix}\right] \quad \Rightarrow n = 2$$

Let $P(x) = ax^2 + bx + c$

$\therefore \quad ax^2 + bx + c - (2ax + b) \equiv x^2 + 2x + 1$

$\therefore \quad a = 1;\ b = 4$ and $c = 5$

$\therefore \quad P(x) = x^2 + 4x + 5$

$P(4) = 37$

Hence, the correct answer is (37).

28. Given, $f(x) = \dfrac{(10\cos x + 5\cos 3x + \cos 5x)}{(\cos 6x + 6\cos 4x + 15\cos 2x + 10)}$

Consider, Dr. $= \cos 6x + 6\cos 4x + 15\cos 2x + 10$

$= (\cos 6x + \cos 4x) + 5(\cos 4x + \cos 2x) + 10(1 + \cos 2x)$

$= (2\cos 5x \cdot \cos x) + 5(2\cos 3x \cdot \cos x) + 10(2\cos^2 x)$

$= 2\cos x\,(10\cos x + 5\cos 3x + \cos 5x)$

$$\therefore \quad f(x) = \frac{1}{2\cos x} = \left(\frac{1}{2}\sec x\right) \Rightarrow f(0) = \frac{1}{2}.$$

$$f'(x) = \frac{1}{2}\sec x \cdot \tan x \Rightarrow f'(0) = 0$$

$$f''(x) = \frac{1}{2}\sec x \cdot \sec^2 x + \tan^2 x \cdot \sec x) \Rightarrow f''(0) = \frac{1}{2}$$

Hence, $f(0) + f'(0) + f''(0) = \frac{1}{2} + 0 + \frac{1}{2} = 1$

Hence, the correct answer is (1).

29. $g(x) = f\left(\dfrac{x}{f(f(x))}\right)$

$$\Rightarrow \quad g'(x) = f'\left(\frac{x}{f(f(x))}\right) \cdot \left(\frac{f(f(x)) - xf'(f(x)) \cdot f'(x)}{(f(f(x)))^2}\right)$$

$$\therefore \quad g'(x) = f'\left(\frac{1}{1}\right) \cdot \left(\frac{f(1) - 1f'(1) \cdot f'(1)}{(f(1))^2}\right)$$

So, $g'(1) = f'(1)\left(\dfrac{1-1}{f(1)^2}\right) = 0.$

Hence, the correct answer is (0).

30. We have

$(a - 1)^2 + (b + 2)^2 + (c - 1)^2 = 0$

$\Rightarrow \quad a = 1,\ b = -2,\ c = 1$

$\therefore \quad f(x) = (x - 1)^2$ and $g(x) = x^2 - 3x + 2$

So, $(f(1) + g(1)) = (0)^2 + (1 - 3 + 2) = 0$

Hence, the correct answer is (0).

31. $p(x) = a(x - 1)(x - 2)(x - 3)(x - b)$

$p'(x) = a[1.(x - 2)(x - 3)(x - b) + (x - 1).1.(x - 3)(x - b) + (x - 1)(x - 2).1.(x - b) + (x - 1)(x - 2)(x - 3)]$

$p'(4) = a[(4 - b)(2.1 + 3.1 + 3.2) + 3.2.1] = 0$

$\Rightarrow \quad (4 - b)(11) + 6 = 0 \Rightarrow 44 - 11b + 6 = 0 \Rightarrow 11b = 50$

$b = \frac{50}{11} = \frac{p}{q}$ So, $p + q = 61$.

Hence, the correct answer is (61).

32. We have,

$$x = \sin\left(t + \frac{7\pi}{12}\right) + \sin\left(t - \frac{\pi}{12}\right) + \sin\left(t + \frac{3\pi}{12}\right)$$

$$= 2\sin\left(t + \frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \sin\left(t + \frac{\pi}{4}\right) = 2\sin\left(t + \frac{\pi}{4}\right)$$

$$y = \cos\left(t + \frac{7\pi}{12}\right) + \cos\left(t - \frac{\pi}{12}\right) + \cos\left(t + \frac{3\pi}{12}\right)$$

$$= 2\cos\left(t + \frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(t + \frac{\pi}{4}\right) = 2\cos\left(t + \frac{\pi}{4}\right)$$

Now, $\frac{x}{y} = \tan\left(t+\frac{\pi}{4}\right) = \frac{1+\tan t}{1-\tan t}$

And $\frac{x}{y} = \frac{1}{\tan\left(t+\frac{\pi}{4}\right)} = \frac{1-\tan t}{1+\tan t}$

∴ $\left(\frac{x}{y}-\frac{y}{x}\right) = \left(\frac{1+\tan t}{1-\tan t}\right) - \left(\frac{1-\tan t}{1+\tan t}\right) = \frac{4\tan t}{1-\tan^2 t} = 2\tan 2t$

Hence, $\frac{d}{dt}\left(\frac{x}{y}-\frac{y}{x}\right) = \frac{d}{dt}(2\tan 2t) = 4\sec^2 2t\Big]_{t=\frac{\pi}{8}}$

$= 4\sec^2\frac{\pi}{4} = 8$

Hence, the correct answer is (8).

33. $F'(x_0) = f'(x_0)\,g(x_0)\,h(x_0) + f(x_0)\,g'(x_0)\,h(x_0) + f(x_0)\,g(x_0)\,h'(x_0)$

⇒ $5f(x_0)\,g(x_0)\,h(x_0) = 2f(x_0)\,g(x_0)\,h(x_0) - 5f(x_0)\,g(x_0)\,h(x_0) + kf(x_0)\,g(x_0)\,h(x_0)$

⇒ $5 = (2 - 5 + k)$

∴ $k = 8$

Hence, the correct answer is (8).

34. $\frac{1-e^{f(x)}}{1+e^{f(x)}} = x \quad \Rightarrow f(x) = \ln\left(\frac{1-x}{1+x}\right)$. Domain of $f(x)$ is $(-1, 1)$

∴ $f'(x) = -\frac{1}{1-x} - \frac{1}{1+x} = \frac{2}{x^2-1} < 0$

⇒ $f(x)$ is decreasing $f(x) = \ln 3 \Rightarrow x = \frac{-1}{2}$

$g'(y) = \frac{1}{f'(x)} \qquad \Rightarrow g'(\ln 2) = \frac{1}{f'\left(\frac{-1}{2}\right)} = \frac{1}{\frac{-8}{3}} = \frac{-8}{3}$

Also, $g''(y) = -\frac{f''(x)}{(f'(x))^3} \qquad \Rightarrow g''(\ln 2) = -\frac{f''\left(\frac{-1}{2}\right)}{\left(f'\left(\frac{-1}{2}\right)\right)^3}$

$f''(x) = -\frac{4x}{(x^2-1)^2}$

$f''(x)\left(\frac{-1}{2}\right) = \frac{-4\left(\frac{-1}{2}\right)}{\frac{9}{16}} = \frac{32}{9}$

∴ $g''(\ln 2) = -\frac{\left(\frac{32}{9}\right)}{\left(\frac{-8}{3}\right)^3} = \frac{32\times 3}{2^9} = \frac{3}{16}$

∴ $g''(\ln 2) = g'(\ln 2) = \frac{3}{16} - \left(\frac{-3}{8}\right) = \frac{9}{16}$

∴ $p + q = 25$

Hence, the correct answer is (25).

35. $z = \cos^5 x \Rightarrow \frac{dz}{dx} = 5\cos^4 x \cdot (-\sin x)$

$y = \sin x \quad \Rightarrow \frac{dy}{dx} = \cos x$

∴ $\frac{dz}{dy} = \frac{\frac{dz}{dx}}{\frac{dy}{dx}} = -5\cos^3 x \cdot \sin x$

So, $\frac{d^2z}{dy^2} = -5\cdot\frac{d}{dx}(\cos^3 x\cdot\sin x)\frac{dx}{dy} = -5\cos x(\cos^2 x - 3\sin^2 x)$

∴ $\frac{d^2z}{dy^2} = -5\cos 3x$

So, $\frac{d^2z}{dy^2}\Bigg]_{x=\frac{2\pi}{9}} = 5\left(\frac{-1}{2}\right) = \frac{5}{2}$

Hence, the correct answer is (5/2).

36. $G(x) = \frac{x^2}{g(x)}$

⇒ $G'(x) = \frac{g(x)\cdot 2x - x^2 g'(x)}{g^2(x)}$

⇒ $G'(4) = \frac{8g(4) - 16g'(4)}{g^2(4)}$ …(1)

Now, $f(x) = xe^{\tan^{-1}(x+2)} + 6e^{(x+2)}$

At $x = -2$,

$f(-2) = -2 + 6 = 4 \qquad \Rightarrow g(4) = -2$

∴ $g'(4) = \frac{1}{f'(-2)}$

Now, $f'(x) = \frac{x.e^{\tan^{-1}(x+2)}}{1+(x+2)^2} + e^{\tan^{-1}(x+2)} + 6e^{(x+2)}$

∴ $f'(-2) = -2 + 1 + 6 = 5$

∴ $g'(4) = \frac{1}{5}$

Now, $G'(4) = \frac{8\times(-2) - 16\times\frac{1}{5}}{4}$

$= -4 - \frac{4}{5} = \frac{-24}{5}$

∴ $|5G'(4)| = 24$

Hence, the correct answer is (24).

37. $y = \begin{cases} 3\cos^{-1}x; & x\in\left[\frac{1}{2},1\right] \\ 2\pi - 3\cos^{-1}x; & x\in\left(\frac{-1}{2},\frac{1}{2}\right) \\ 3\cos^{-1}x - 2\pi; & x\in\left[-1,-\frac{1}{2}\right] \end{cases}$

∴ $\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}}; & x\in\left(\frac{-1}{2},\frac{1}{2}\right) \\ \frac{-3}{\sqrt{1-x^2}}; & x\in\left(-1,-\frac{1}{2}\right)U\left(\frac{1}{2},1\right) \end{cases}$

∴ $\left(\frac{dy}{dx}\right)_{x=0} = 3$

$\left(\frac{dy}{dx}\right)_{x=4/5} = -5$

Hence, $(2p - 3q) = 6 + 15 = 21$

Hence, the correct answer is (21).

38. $\frac{dy}{dx} = \frac{-3}{2}\sin 2t$ and $\frac{d^2y}{dx^2} = \frac{3\cos 2t}{\sin t}$.

Now, substitute these value and compute required term.

Hence, the correct answer is (3).

39. $y^{2/3} - 2xy^{1/3} + 1 = 0$

∴ $y^{1/3} = (x \pm \sqrt{x^2-1}\,) \qquad \Rightarrow \ln y = 3\ln(x \pm \sqrt{x^2-1}\,)$

$$\frac{y_1}{y} = \frac{\pm 3}{\sqrt{x^2-1}}$$

$\Rightarrow \quad (x^2-1)\ y_1^2 = 9y^2$

$\Rightarrow \quad 2x\ y_1^2 = (x^2-1)\ 2y_1y_2 = 18yy_1$ (on differentiating)

$xy_1 + (x^2-1)y_2 = 9y$ (As y_1 is not equal to 0, because y is not constant)
Dividing by y, we get

$\therefore \quad x\ \dfrac{y_1}{y} + (x^2-1)\ \dfrac{y_2}{y} = 9$

Hence, the correct answer is (9).

40. $e^y = \dfrac{1+\sqrt{\dfrac{1-\alpha}{1+\alpha}}}{1-\sqrt{\dfrac{1-\alpha}{1+\alpha}}} = \tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$

$\therefore \quad e^y \times y' = \dfrac{1}{2}\ \sec^2\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}} = 2$$

Hence, the correct answer is (2).

ARCHIVE: JEE MAIN

1. $\ln(x+y) = 4xy \quad$ (At $x = 0$, $y = 1$)

$x + y = e^{4xy} \quad \Rightarrow 1 + \dfrac{dy}{dx} = e^{4xy}\left(4x\dfrac{dy}{dx}+4y\right)$

At $x = 0$, $\dfrac{dy}{dx} = 3$

$$\frac{d^2y}{dx^2} = e^{4xy}\left(4x\frac{dy}{dx}+4y\right)^2 + e^{4xy}\left(4x\frac{d^2y}{dx^2}+4y\right)$$

At $x = 0$, $\dfrac{d^2y}{dx^2} = e^0(4)^2 + e^0(24)$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} = 40$

Hence, the correct answer is (40).

2. $f(x) = \sin\left(\cos^{-1}\left(\dfrac{1-2^{2x}}{1+2^{2x}}\right)\right)$ at $x = 1$; $2^{2x} = 4$

For $\sin\left(\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)\right)$;

Let $\tan^{-1} x = \theta$; $\theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right) \qquad \therefore \sin(\cos^{-1}\cos 2\theta) = \sin 2\theta$

$$\left[\begin{array}{l} \text{If} \quad x > 1 \Rightarrow \dfrac{\pi}{2} > \theta > \dfrac{\pi}{4} \\ \therefore \qquad \pi > 2\theta > \dfrac{\pi}{2} \end{array}\right] = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$$

$= \dfrac{2x}{1+x^2}$

Hence, $f(x) = \dfrac{2\cdot 2^x}{1+2^{2x}}$

$\therefore \quad f'(x) = \dfrac{(1+2^{2x})(2.2^x\ln 2) - 2^{2x}\cdot 2\cdot\ln 2\cdot 2\cdot 2^x}{(1+2^{2x})}$

$\therefore \quad f'(1) = \dfrac{20\ln 2 - 32\ln 2}{25} = -\dfrac{12}{25}\ln 2$

So, $a = 25$, $b = 12 \quad \Rightarrow |a^2 - b^2| = 25^2 - 12^2$

$= 625 - 144 = 481$

Hence, the correct answer is (481).

3. $\ln f(x+1) = \ln(xf(x))$

$\ln f(x+1) = \ln x + \ln f(x) \quad \Rightarrow g(x+1) = \ln x + g(x)$

$\Rightarrow \quad g(x+1) - g(x) = \ln x$

$g'(x+1) - g'(x) = \dfrac{1}{x} \quad \Rightarrow g''(x+1) - g''(x) = -\dfrac{1}{x^2}$

Put $x = 1, 2, 3, 4$

$g''(2) - g''(1) = -\dfrac{1}{1^2}$ …(1)

$g''(3) - g''(2) = -\dfrac{1}{2^2}$ …(2)

$g''(4) - g''(3) = -\dfrac{1}{3^2}$ …(3)

$g''(5) - g''(4) = -\dfrac{1}{4^2}$ …(4)

Add all the equation, we get

$g''(5) - g''(4) = -\dfrac{1}{1^2} - \dfrac{1}{2^2} - \dfrac{1}{3^2} - \dfrac{1}{4^2}$

$|g''(5) - g''(1)| = \dfrac{205}{144}$

4. $f(x) = \cos\left(2\tan^{-1}\sin\left(\cos^{-1}\sqrt{\dfrac{1-x}{x}}\right)\right)$

$\because \quad \cot^{-1}\sqrt{\dfrac{1-x}{x}} = \sin^{-1}\sqrt{x}$

or $\quad f(x) = \cos(2\tan^{-1}\sqrt{x}) = \cos\tan^{-1}\left(\dfrac{2\sqrt{x}}{1-x}\right)$

$f(x) = \dfrac{1-x}{1+x}$

Now $f'(x) = \dfrac{-2}{(1+x^2)}$ or $f'(x)(1-x)^2 = -2\left(\dfrac{1-x}{1+x}\right)^2$

or $\quad (1-x)^2 f'(x) + 2(f(x))^2 = 0.$

5. $y(x) = \cot^{-1}\left[\dfrac{\cos\frac{x}{2}+\sin\frac{x}{2}+\sin\frac{x}{2}-\cos\frac{x}{2}}{\cos\frac{x}{2}+\sin\frac{x}{2}-\sin\frac{x}{2}+\cos\frac{x}{2}}\right]$

$y(x) = \cot^{-1}\left(\tan\dfrac{x}{2}\right) = \dfrac{x}{2} - \dfrac{x}{2}$

$y'(x) = \dfrac{-1}{2}$

6. $\dfrac{dy}{dx} = \dfrac{2^x 2y - 2^x}{2^y}$;

$2^y\dfrac{dy}{dx} = 2^x(2^y - 1)$

$$\int \frac{2^y}{2^y - 1}\, dy = \int 2^x\, dx;$$

$$\frac{\ln(2^y - 1)}{\ln 2} = \frac{2^x}{\ln 2} + C \Rightarrow \log_2(2^y - 1) = 2^x \log_2 e + C$$

$\because\ y(0) = 1 \quad \Rightarrow 0 = \log_2 e + C$

$C = -\log_2 e \quad \Rightarrow \log_2(2^y - 1) = (2^x - 1)\log_2 e$

Put $x = 1$, $\log_2(2^y - 1) = \log_2 e$

$2^y = e + 1$

$y = \log_2(e + 1)$

7. Put $\cos\alpha = \frac{3}{5}.\sin\alpha = \frac{4}{5}, 0 < \alpha < \frac{\pi}{2}$

Now, $\frac{3}{5}\cos kx - \frac{4}{5}\sin kx = \cos\alpha\,.\,\cos kx - \sin\alpha\,.\,\sin kx$

$= \cos(\alpha + kx)$

We have, $\cos^{-1}(\cos(\alpha + kx)) = (\alpha + kx)$

$$y = \sum_{k=1}^{6} k\cos^{-1}\left\{\frac{3}{5}\cos kx - \frac{4}{5}\sin kx\right\} = \sum_{k=1}^{6}(\alpha + kx)k$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{at\, x=0} = \sum_{k=x}^{6} k^2 = \frac{6\times 7\times 13}{6} = 91$$

Hence, the correct answer is (91).

8. $y(\alpha) = \sqrt{2\left(\frac{\tan\alpha + \cot\alpha}{1 + \tan^2\alpha}\right) + \frac{1}{\sin^2\alpha}}, \alpha \in \left(\frac{3\pi}{4}, \pi\right)$

$$y(\alpha) = \frac{|\sin\alpha + \cos\alpha|}{|\sin\alpha|} = \frac{-(\sin\alpha + \cos\alpha)}{\sin\alpha} = -1 - \cot\alpha$$

$y'(\alpha) = \text{cosec}^2\alpha \Rightarrow y'\left(\frac{5\pi}{6}\right) = 4$

Hence, the correct answer is (A).

9. $x^k + y^k = a^k\ (a, k > 0) \Rightarrow kx^{k-1} + ky^{k-1}\frac{dy}{dx} = 0$

$$\frac{dy}{dx} + \left(\frac{x}{y}\right)^{k-1} = 0 \Rightarrow k - 1 = -\frac{1}{3} \Rightarrow k = 2/3$$

Hence, the correct answer is (C).

10. Put $x = \sin\theta$, $y = \sin\alpha$

$\sin\alpha\,.\,\cos\theta + \cos\alpha\,.\,\sin\theta = k \Rightarrow \sin(\alpha + \theta) = k$

$\Rightarrow \quad a + \theta = \sin^{-1}k \Rightarrow \sin^{-1}x + \sin^{-1}y = \sin^{-1}k$

$$\Rightarrow \frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - y^2}}\frac{dy}{dx} = 0$$

At $x = \frac{1}{2}, y = \frac{-1}{4}$ we get $\frac{dy}{dx} = \frac{-\sqrt{5}}{2}$

Hence, the correct answer is (B).

11. Let $\tan^{-1}x = \theta$, $\theta \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$f(x) = (\sin\theta + \cos\theta)^2 - 1 = \sin 2\theta = \frac{2x}{1 + x^2}$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}(\sin^{-1}(f(x)) =$$

$$\frac{1}{2}\frac{f'(x)}{\sqrt{1 - f^2(x)}} = \frac{1}{2}\frac{1}{\sqrt{1 - \left(\frac{2x}{1 + x^2}\right)^2}}\frac{(1 + x^2)2 - 4x^2}{(1 + x^2)^2} = \frac{1 - x^2}{|x^2 - 1|(1 + x^2)}$$

For $|x| > 1$, we have $\frac{dy}{dx} = \frac{-1}{(1 + x^2)}$ we get

$y(x) = -\tan^{-1}x + c$

$y(\sqrt{3}) = \frac{\pi}{6}$, then $\frac{\pi}{6} = -\frac{\pi}{3} + c$ we get, $c = \pi/2$

$$y\left(-\sqrt{3}\right) = \frac{\pi}{2} - \tan^{-1}(-\sqrt{3}) = \frac{5\pi}{6}$$

Hence, the correct answer is (A).

12. $y^2 + \ln(\cos^2 x) = y \qquad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

for $x = 0$ $\qquad$ we have $y = 0$ or 1

On differentiating wrt x, we get

$\Rightarrow \quad 2yy' - 2\tan x = y'$

At $(0, 0)$ $y' = 0$ and at $(0, 1)$ $y' = 0$

On differentiating wrt x, we get

$2yy'' + 2(y')^2 - 2\sec^2 x = y''$

At $(0, 0)$ $y'' = -2$ and at $(0, 1)$ $y'' = 2$

$\therefore \quad |y''(0)| = 2$

Hence, the correct answer is (A).

13. Let $f = \tan^{-1}\left(\frac{\sqrt{1 + x^2} - 1}{x}\right)$ Put $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$$f = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1 - \cos\theta}{\sin\theta}\right) = \tan^{-1}(\tan\frac{\theta}{2}) = \frac{\theta}{2}$$

$$f = \frac{\tan^{-1}x}{2} \Rightarrow \frac{df}{dx} = \frac{1}{2(1 + x^2)} \qquad \ldots(1)$$

Let $g = \tan^{-1}\left(\frac{2x\sqrt{1 - x^2}}{1 - 2x^2}\right)$ Put $x = \sin\theta \Rightarrow \theta = \sin^{-1}x$

$$g = \tan^{-1}\left(\frac{2\sin\theta\cos\theta}{1 - 2\sin^2\theta}\right) = \tan^{-1}(\tan 2\theta) = 2\theta$$

$$g = 2\sin^{-1}x \Rightarrow \frac{dg}{dx} = \frac{2}{\sqrt{1 - x^2}} \qquad \ldots(2)$$

$\frac{df}{dg} = \frac{1}{2(1 + x^2)}\frac{\sqrt{1 - x^2}}{2}$; at $x = \frac{1}{2}$ we have $\left(\frac{df}{dg}\right)_{x=\frac{1}{2}} = \frac{\sqrt{3}}{10}$

Hence, the correct answer is (B).

14. $f'(x) = f(x) \Rightarrow \ln(f(x)) = x + c$

$f(x) = e^{x + c} \Rightarrow f(x) = ke^x$

$f(x) = ke \Rightarrow 2 = k.e \Rightarrow k = 2/e$

$f(x) = ke^x$

$f'(x) = f'(f(x)).f'(x)$

$\Rightarrow \quad h'(1) = f'(f(1)).f'(1) = f'(2).\left(\frac{2}{e}\times e\right) = \frac{2}{e}.e^2.2 = 4e$

Hence, the correct answer is (A).

15. $2y = \left(\cot^{-1}\left(\frac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2$

$$= \left(\cot^{-1}\left(\frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x}\right)\right)^2$$

$$= \left(\frac{\pi}{2} - \tan^{-1}\left(\tan\left(\frac{\pi}{3} + x\right)\right)\right)^2$$

CASE 1: $0 < x < \frac{\pi}{6} \Rightarrow \frac{\pi}{3} < x + \frac{\pi}{3} < \frac{\pi}{2}$;

$$2y = \left(\frac{\pi}{2} - \frac{\pi}{3} - x\right)^2$$

On differentiating, we get

$$\Rightarrow \quad 2 \cdot \frac{dy}{dx} = 2\left(\frac{\pi}{6} - x\right)(-1) \Rightarrow \frac{dy}{dx} = x - \frac{\pi}{6}$$

CASE 2: $\frac{\pi}{6} < x < \frac{\pi}{2} \Rightarrow \frac{\pi}{2} < x + \frac{\pi}{3} < \frac{5\pi}{6}$

$$2y = \left(\frac{\pi}{2} - \left(\frac{\pi}{3} + x - \pi\right)\right)^2 = \left(\frac{7\pi}{6} - x\right)^2$$

On differentiating, we get

$$\Rightarrow \quad \frac{dy}{dx} = x - \frac{7\pi}{6}$$

Hence, the correct answer is (A).

16. $f(1) = 1\ f'(1) = 3$

$f(f(f(x))) + (f(x))^2$ at $x = 1$

$$\Rightarrow \quad f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x) + 2f(x)\,f'(x)$$

$$\Rightarrow \quad f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x) + 2f(x)\,f'(1)$$

$$\Rightarrow \quad f'(f(f(1))) \cdot f'(1) \cdot f'(1) + 2f'(1)$$

$$\Rightarrow \quad f'(f(1) \cdot f'(1) \cdot f'(1) + 2f'(1) = 3 \times 3 \times 3 + (2 \times 3) = 27 + 6 = 33$$

Hence, the correct answer is (D).

17. $f(g(x)) = -x \Rightarrow (f(g(x))' = -1$

$f(g(\alpha)) = -\alpha = b \quad \Rightarrow \quad (f(g(x)))' = -1 = a$

$\therefore \quad b = -\alpha,\ a = -1$

Now, check from options

Hence, the correct answer is (D).

18. Given, $y = \tan^{-1}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right) = \tan^{-1}\left(\frac{\tan x - 1}{\tan x + 1}\right)$

$$\Rightarrow \quad y = -\tan^{-1}\left[\tan\left(\frac{\pi}{4} - x\right)\right]$$

$$\because \quad 0 < x < \frac{\pi}{2} \Rightarrow -\frac{\pi}{2} < -x < 0$$

$$\Rightarrow \quad -\frac{\pi}{4} < \frac{\pi}{4} - x < \frac{\pi}{4}$$

$$\Rightarrow \quad y = -\left(\frac{\pi}{4} - x\right) = x - \frac{\pi}{4} \qquad \left\{\because \tan^{-1}\tan x = x\ \forall x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)\right\}$$

$$\frac{dy}{d(x/2)} = \frac{1}{(1/2)} = 2$$

Hence, the correct answer is (C).

19. Let $F(x) = \tan^{-1}\left(\frac{6x\sqrt{x}}{1 - 9x^3}\right)$ where $x \in \left(0, \frac{1}{4}\right)$.

$$= \tan^{-1}\left(\frac{2.(3x^{3/2})}{1 - (3x^{3/2})^2}\right) = 2\tan^{-1}(3x^{3/2})$$

As, $3x^{3/2} \in \left(0, \frac{3}{8}\right)$

$$\left[\because 0 < x < \frac{1}{4} \Rightarrow 0 < x^{3/2} < \frac{1}{8} \Rightarrow 0 < 3x^{3/2} < \frac{3}{8}\right]$$

So $\frac{dF(x)}{dx} = 2 \times \frac{1}{1 + 9^3} \times 3 \times \frac{3}{2} \times x^{1/2} = \frac{9}{1 + 9x^3}\sqrt{x}$

On comparing, $g(x) = \frac{9}{1 + 9x^3}$

Hence, the correct answer is (B).

20. Since, g is inverse of f, then $f(g(x)) = x$

$$\Rightarrow \quad f'(g(x))\,g'(x) = 1$$

$$\Rightarrow \quad g'(x) = 1 + (g(x))^5 \qquad \left(\because f'(x) = \frac{1}{1 + x^5}\right)$$

Hence, the correct answer is (D).

21. $y = \sec(\tan^{-1}x)$

$$\Rightarrow \quad \frac{dy}{dx} = \sec(\tan^{-1}x)\,\tan(\tan^{-1}x) \cdot \frac{1}{1 + x^2}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)_{x=1} = \sec\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)\frac{1}{1 + 1} = \frac{1}{\sqrt{2}}$$

Hence, the correct answer is (A).

ARCHIVE: JEE ADVANCED

1. $f(x) = |x|$

$\lim\limits_{h \to 0} \frac{|h| - 0}{\sqrt{|h|}} = \lim\limits_{h \to 0} \sqrt{|h|} = 0$ $f(x) = |x|$ has property 1

$f(x) = x^{2/3}$

$\lim\limits_{h \to 0} \frac{h^{2/3} - 0}{\sqrt{|h|}} = \lim\limits_{h \to 0} |h|^{1/6} = 0.$ $f(x) = x^{2/3}$ has property 1

$f(x) = x\ |x|$

$\lim\limits_{h \to 0} \frac{h|h| - 0}{h^2} = \lim\limits_{h \to 0} \frac{|h|}{h}$ LHL is -1, RHL $= 1$ does not exist.,

$f(x) = x\ |x|$ does not have property 2

$f(x) = \sin x$

$\lim\limits_{h \to 0} \frac{\sin h - 0}{h^2}$ does not exist.,

$f(x) = \sin x$ does not have property 2

Hence, (A) and (B) are correct.

2. We know that, $g(f(x)) = x$

$g'(f(x))f'(x) = 1$

Now, $f(x) = 2$ then $x^3 + 3x + 2 = 2$ we get $x = 0$

$g'(f(0))\,f'(0) = 1$

$$g'(2) = \frac{1}{f'(0)} = \frac{1}{3}$$

Let consider $h(g(g(f(x)) = f(x)$

$h(g(x)) = f(x)$ (by definition) and $h(g(3)) = f(3) = 27 + 9 + 2 = 38$

$h(g(f(x)) = f(f(x))$

$h(x) = f(f(x))$ and $h(0) = f(f(0)) = f(2) = 8 + 6 + 2 = 16$

$h'(x) = f'(f(x))f'(x)$

$h'(1) = f'(6)f'(1) = 111 \times 6 = 666$

And $h(0) = f(f(0)) = f(2) = 8 + 6 + 2 = 16$

And $h(g(3)) = f(3) = 27 + 9 + 2 = 38$

Hence, (B) and (C) are correct.

3. Let $\tan\phi = \dfrac{\sin\theta}{\sqrt{\cos 2\theta}}$

$\Rightarrow \quad \sec^2\phi = 1 + \tan^2\phi$

$$= 1 + \frac{\sin^2\theta}{\cos 2\theta} = \frac{\cos 2\theta + \sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta}{\cos 2\theta}$$

$\Rightarrow \quad \sec\phi = \dfrac{\cos\theta}{\sqrt{\cos 2\theta}}$ or $\cos\phi = \dfrac{\sqrt{\cos 2\theta}}{\cos\theta}$

Thus, $\sin\phi = \dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\left(\dfrac{\sqrt{\cos 2\theta}}{\cos\theta}\right) = \tan\theta$

$\therefore \quad f(\theta) = \sin[\tan^{-1}(\tan\phi)] = \sin\phi = \tan\theta$

Thus, $\dfrac{d}{d(\tan\theta)}(f(\theta)) = 1$

Hence, the correct answer is (1).

4. $g(x) = f^{-1}(x) \qquad \Rightarrow f(g(x)) = x$

By the chain rule, we get

$f'(g(x))\, g'(x) = 1$

Let $g(1) = k$, then $f(g(1)) = 1$

$\Rightarrow \quad f(k) = 1 \Rightarrow k = 0$

Putting $x = 1$ in (1), we get

$f'(0)\, g'(1) = 1$

But $f'(x) = 3x^2 + (1/2)e^{x/2} \Rightarrow f'(0) = 1/2$

$\therefore \quad \dfrac{1}{2}g'(1) = 1 \qquad \Rightarrow g'(1) = 2$

Hence, the correct answer is (2).

5. $g(x + 1) - g(x) = \log f(x + 1) - \log f(x) = \log\dfrac{f(x+1)}{f(x)} = \log x$

$\Rightarrow \quad g'(x + 1) - g'x = \dfrac{1}{x}$

$g''(x + 1) - g''(x) = \dfrac{1}{x^2}$

Putting $x = k - 1/2$,

We get $g''\left(k + \dfrac{1}{2}\right) - g''\left(k - \dfrac{1}{2}\right) = -\dfrac{4}{(2k-1)^2}$

Now, $\displaystyle\sum_{k=1}^{N}\left[g''\left(k + \frac{1}{2}\right) - g''\left(k - \frac{1}{2}\right)\right] = \sum_{k=1}^{N}\frac{-4}{(2k-1)^2}$

$$g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left[1 + \frac{1}{9} + \frac{1}{25} + \ldots. + \frac{1}{(2N-1)^2}\right]$$

Hence, the correct answer is (A).

6. $f(x) = g(x) \sin x$

$\Rightarrow \quad f'(x) = g'(x) \sin x + g(x) \cos x$

$\Rightarrow \quad f'(0) = g'(0) \sin x + g(0) \cos 0 = g(0)$

Also, $f''(x) = g''(x) \sin x + 2g'(x) \cos x - g(x) \sin x$

$f''(0) = g''(0) \sin 0 + 2g'(0) \cos 0 - g(0) \sin 0$

$= 0 + 0 - 0 = 0$

Next, $\displaystyle\lim_{x\to 0}[g(x) \cot x - g(0) \operatorname{cosec} x]$

$$= \lim_{x\to 0}\left[\frac{g(x)\cos x - g(0)}{\sin x}\right]\left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x\to 0}\left[\frac{g'(x)\cos x - g(0)\sin x}{\cos x}\right]$$

$$= \left[\frac{g'(0) - g(0)\sin x}{\cos 0}\right] = g'(0) = f''(0)$$

Thus, statement-1 as well as statement-2 is true but statement-2 is not a correct explanation for statement-1.

Hence, the correct answer is (B).

7. We have, $\dfrac{d^2x}{dy^2} = \dfrac{d}{dy}\left(\dfrac{dx}{dy}\right)$

$$= \frac{d}{dy}\left(\frac{1}{\frac{dy}{dx}}\right) = \frac{d}{dx}\left(\frac{1}{\frac{dy}{dx}}\right)\cdot\frac{dx}{dy}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\cdot\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$= -\frac{1}{\left(\frac{dy}{dx}\right)^3}\cdot\frac{d^2y}{dx^2} = -\left(\frac{dy}{dx}\right)^{-3}\left(\frac{d^2y}{dx^2}\right)$$

Hence, the correct answer is (D).

8. $f(x) = x^3 + 3x + 2, \qquad g(f(x)) = x$

And $h(g(g(x))) = x, \qquad x \in R$

$f'(x) = 3x^2 + 3 \qquad \Rightarrow f'(0) = 3 \qquad \because f(0) = 2$

$g'(f(x))\, f'(x) = 1$

Put $x = 0 \qquad g'(f(0))\, f'(0) = 1$

$g'(2)\, 3 = 1 \qquad \Rightarrow g'(2) = \dfrac{1}{3}$

$\because\ h(g(g(x))) = x$

$h(g(g(f(x)))) = f(x); \qquad h(g(x)) = f(x)$

Put $x = 3 \qquad \Rightarrow h(g(3)) = f(3)$

$\Rightarrow \quad h(g(3)) = 3^3 + 3 \times 3 + 2 = 38$

$\Rightarrow \quad h(g(f(x)) = f(f(x));$

$h(x) = f(f(x)); \qquad h(0) = f(f(x)) = f(2) = 8 + 6 + 2 = 16$

$h'(x) = f'(f(x))\, f'(x)$

$h'(1) = f'(f(1))\, f'(1) = f'(6))\, f'(1) = (3 \times 36 + 3)\ (3 + 3)$

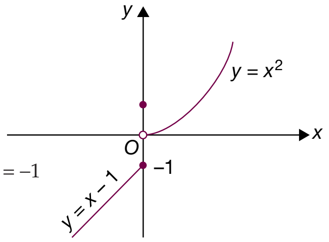
$= 111 \times 6$

$= 666$

Hence, (B) and (C) are correct.

CHAPTER 5: CONTINUITY AND DIFFERENTIABILITY OF FUNCTION

Test Your Concepts-I



1. Graph for given function is,
Since, the graph of $f(x)$ breaks at point $x = 0$
Hence, $f(x)$ is discontinuous at $x = 0$.

From graph, LHL $= \lim\limits_{x\to 0^-} f(x) = -1$

RHL $= \lim\limits_{x\to 0^+} f(x) = 0$

LHL and RHL are not equal, non-removable finite type discontinuity. Hence, jump of discontinuity = | LHL – RHL | = 1

2. $f(x)$ is not defined at $x = 0$ but $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x\sin\frac{1}{x} = 0$

To remove discontinuity at $x = 0$, value of $f(0)$ must be zero. Hence, we define $f(x)$ as

$$f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

3. $f(x)$ is continuous at $x = \frac{\pi}{4}$

$$\Rightarrow \quad k = \lim_{x\to\frac{\pi}{4}} f(x) = \exp\left(\lim_{x\to\frac{\pi}{4}}\left(\frac{2-\tan x\sqrt{\cot x}-1}{\sin x-\cos x}\right)\right)$$

$$= \exp\left(\lim_{x\to\frac{\pi}{4}}\frac{\left(1-\frac{\sqrt{\sin x}}{\sqrt{\cos x}}\right)}{(\sin x-\cos x)}\right)$$

$$= \exp\left(\lim_{x\to\frac{\pi}{4}}\frac{\sqrt{\cos x}-\sqrt{\sin x}}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)\left(\sqrt{\sin x}-\sqrt{\cos x}\right)\sqrt{\cos x}}\right)$$

$$= \exp\left(\lim_{x\to\frac{\pi}{4}}\frac{-1}{(\sqrt{\sin x}+\sqrt{\cos x})\sqrt{\cos x}}\right) = e^{\frac{-1}{\sqrt{2}}}$$

Hence, $k = e^{\frac{-1}{\sqrt{2}}}$

4. Since, $f(x)$ is continuous at $x = 0$.

$$\Rightarrow \quad f(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$$

Therefore, $k = \lim\limits_{x\to 0^+} f(x)$; Put $x = t^2$

$$k = \lim_{t\to 0^+}\frac{e^{t^4}-\frac{2}{\pi}\sin^{-1}\sqrt{1-t^2}}{\frac{\ln(1+t)}{t}\cdot t}$$

$$= \lim_{t\to 0^+}\frac{e^{t^4}-\frac{2}{\pi}\sin^{-1}\sqrt{1-t^2}}{t}, \left(\frac{0}{0}\right) \text{form}$$

(Using L'Hospital's rule)

$$k = \lim_{t\to 0^+}\left(\frac{e^{t^4}(4t^3)-\frac{2}{\pi}\frac{1}{\sqrt{1-(1-t^2)}}\frac{(-2t)}{2\sqrt{1-t^2}}}{1}\right)$$

$$= \lim_{t\to 0^+}\left(e^{t^4}\cdot 4t^3+\frac{2}{\pi}\cdot\frac{1}{\sqrt{1-t^2}}\right)$$

Hence, $k = \frac{2}{\pi}$

5. Given, $f(x)$ is continuous at $x = 0$.

$$\text{Hence, } f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(\frac{x\sin x+2\cos 2x}{2}\right)^{\frac{1}{x^2}}$$

$$= e^{L}\ (1^{\infty}\text{ Form})$$

$$\text{Where} \quad L = \lim_{x\to 0}\frac{1}{x^2}\left(\frac{x\sin x+2\cos 2x}{2}-1\right)$$

$$= \lim_{x\to 0}\left(\frac{x\sin x+2(\cos 2x-1)}{2x^2}\right)$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{2x}-\frac{4(1-\cos 2x)}{(2x)^2}\right) = \frac{1}{2}-4\times\left(\frac{1}{2}\right) = \frac{-3}{2}.$$

Hence, $f(0) = e^{\frac{-k}{2}} = e^{\frac{-3}{2}} \Rightarrow k = 3$.

6. We have, $f(x) = \begin{cases} kx+1, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$

$f(x)$ is continuous at $x = \pi$.

$$\therefore \quad \lim_{x\to\pi^-} f(x) = \lim_{x\to\pi^+} f(x) = f(\pi)$$

$$\Rightarrow \quad \lim_{x\to\pi^-} kx+1 = \lim_{x\to\pi^+} \cos x \quad \Rightarrow k\pi+1 = \cos\pi$$

$$\Rightarrow \quad k\pi+1 = -1$$

Hence, $k = -\frac{2}{\pi}$

7. We have, $f(x) = \begin{cases} \frac{\sin x}{x}+\cos x, & x \neq 0 \\ 2, & x = 0 \end{cases}$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{\sin x}{x}+\cos x\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin h}{h}+\cos h\right) = 1+1 = 2$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{\sin x}{x}+\cos x\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin h}{h}+\cos h\right) = 1+1 = 2$$

$$\therefore \quad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 2$$

Hence, $f(x)$ is continuous at $x = 2$.

8. We have, $f(x) = \begin{cases} \frac{(\sin x)}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

$$\Rightarrow \quad f(x) = \begin{cases} \frac{-\sin x}{x}, & \text{if } x < 0 \\ 1, & \text{if } x = 0 \\ \frac{\sin x}{x}, & \text{if } x > 0 \end{cases}$$

For continuous at $x = 0$, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^+} f(x) = f(0)$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{-\sin x}{x} = -1 \text{ and}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

$$\therefore \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

Hence, $f(x)$ is discontinuous at $x = 0$.

9. We have, $f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$ is continuous at $x = 1$.

$$\therefore \lim_{x\to 1} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3ax + b = 3a + b$$

$\Rightarrow 3a + b = f(1) = 11$ [$\because f(1) = 11$]

$\Rightarrow 3a + b = 11$ …(1)

And, $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} 5ax - 2b = 5a - 2b$

$\Rightarrow 5a + 5b = f(1) = 11 \Rightarrow 5a - 2b = 11$ …(2)

On solving Eqs. (1) and (2), we get $a = 3$ and $b = 2$

10. We have, $f(x) = \begin{cases} x\sin\dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

For continuous at $x = 0$.

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x\sin\frac{1}{x}$$

$= \lim\limits_{h\to 0} h\sin\dfrac{1}{h} = 0$ [$\because 1 \leq \sin\dfrac{1}{h} \leq 1$]

And, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} x\sin\dfrac{1}{x} = \lim\limits_{h\to 0} h\sin\dfrac{1}{h} = 0$

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0$$

Hence, $f(x)$ is continuous at $x = 0$.

11. We have, $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x}, & \text{if } x < 0 \\ c, & \text{if } x = 0 \\ \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}}, & \text{if } x > 0 \end{cases}$

$f(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(a+1)x + \sin x}{x}$$

$$= \lim_{x\to 0^-} \frac{\sin(a+1)x}{x} + \lim_{x\to 0^-} \frac{\sin x}{x}$$

$$= \lim_{x\to 0^-} \frac{(a+1)\sin(a+1)x}{(a+1)x} + \lim_{x\to 0^-} \frac{\sin x}{x}$$

$$= a + 1 + 1 = a + 2$$

$\Rightarrow \lim\limits_{x\to 0^-} f(0) = f(0) = c \qquad \therefore a + 2 = c$ …(1)

And, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{\sqrt{x + bx^2} - \sqrt{x}}{bx^{3/2}}$

$$= \lim_{x\to 0^+} \frac{x + bx^2 - x}{bx^{3/2}(\sqrt{x + bx^2} + \sqrt{x})}$$

$$= \lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{x}\left(\sqrt{1 + \frac{b}{x}} + 1\right)} = \frac{1}{2}$$

$\therefore \lim\limits_{x\to 0^+} f(x) = f(0) \Rightarrow \dfrac{1}{2} = c$ …(2)

From Eqs. (1) and (2), $a = -\dfrac{3}{2}$

Hence, $a = -\dfrac{3}{2}$, $b \in R$ and $c = \dfrac{1}{2}$

12. We have, $f(x) = \begin{cases} \dfrac{1 - \cos 4x}{8x^2}, & x \neq 0 \\ k, & x = 0 \end{cases}$

$f(x)$ is continuous at $x = 0$. $\therefore \lim\limits_{x\to 0} f(x) = f(0)$

$\Rightarrow \lim\limits_{x\to 0} \dfrac{1 - \cos 4x}{8x^2} = k$ [$\because f(0) = k$]

$$\Rightarrow \lim_{x\to 0} \frac{2\sin^2 2x}{8x^2} = k$$

$$\Rightarrow \lim_{x\to 0} \left(\frac{\sin 2x}{2x}\right)^2 = k$$

Hence, $k = 1$

Test Your Concepts-II

1. f is continuous where

$x^2 + 6x + 2 = x^2 + 5x - 4. \quad \Rightarrow x = -6$

2. $f(x) = [3x] - \{2x\}$, where $x \in \left[0, \dfrac{3}{2}\right)$

$\because$ $[x]$ and $\{x\}$ functions are discontinuous when x is an integer.

$f(x) = [3x] - 2x + [2x] = [3x] + [2x] - 2x$

For $[3x]$ we get $x = \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{4}{3}, 1$ and for $[2x]$ we get $x = \dfrac{1}{2}, 1$ For point of discontinuity.
At one point sum of discontinuous and continuous function results as discontinuous at point.

Hence, $f(x)$ is discontinuous at $x = \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{1}{2}, \dfrac{4}{3}$.

At $x = 1$, $[3x]$ and $[2x]$ are discontinuous so we should check continuity of $f(x)$.

$f(1) = 3 + 2 - 2 = 3$

$f(1^-) = 2 + 1 - 2 = 1$

Hence, number of points of discontinuity of $f(x) = 5$, $x = \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{1}{2}, \dfrac{4}{3}, 1$.

3. $\because$ $f(x)$ is continuous for all $x \in R$.

$f(0^-) = f(0) = f(0^+)$

$$f(0^-) = \lim_{x\to 0^-} \frac{\sin(p+1)x + \sin x}{x}$$

$$= \lim_{x\to 0^-} (p+1)\frac{\sin(p+1)x}{(p+1)x} + \lim_{x\to 0^-} \frac{\sin x}{x}$$

$= p + 1 + 1 = p + 2$

$f(0) = q$

$$f(0^+) = \lim_{x\to 0^+} \frac{\sqrt{1+x} - 1}{x} = \lim_{x\to 0^+} \frac{x}{x\left(\sqrt{1+x} + 1\right)} = \frac{1}{2}$$

$\Rightarrow q = \dfrac{1}{2}$; $p + 2 = q \Rightarrow p = \dfrac{-3}{2}$.

Hence, $2p + 8q = -3 + 4 = 1$

4. $\because$ $f(x)$ is continuous for all $x \in R$

$\therefore$ $f(x)$ is also continuous at $x = 2$.

$f(2^-) = (2^+) = f(2) \Rightarrow 4c + 4 = 8 - 2c$

$\Rightarrow$ $6c = 4$

Hence, $c = \dfrac{2}{3}$.

5. $f(x)$ will be continuous when $\cos^2 x = -\cos^2 x$

$\Rightarrow$ $\cos^2 x = 0$ $\Rightarrow \cos x = 0$

Hence, $x = (2k+1)\dfrac{\pi}{2}$, $k \in$ I.

6. $f(x) = [2x^3 - 5] = [2x^3] - 5$

Let $2x^3 = t$

$\therefore$ $g(t) = [t]$ where $t \in [2, 16)$

$\because$ $[x]$ is discontinuous when x is an integer

g is continuous at $t = 2$ and discontinuous $t = 3, 4, \ldots, 15$

Hence, no. of point of discontinuity = 13 points.

7. $f(x) = [x] (\sin kx)^p$

Let $(\sin kx)^p$ is continuous and differentiable function $\forall\ x \in R,\ k \in R$ and $p > 0$.

$[x]$ is discontinuous at $x \in I$

For $k = n\pi,\ n \in I$

$$f(x) = [x](\sin(n\pi x))^p$$

$\lim\limits_{x \to a} f(x) = 0,\ a \in I$ and $f(a) = 0$

Hence, $f(x)$ becomes continuous for all $x \in R$

8. $f(x) = \begin{cases} x+2, & x < 0 \\ -x^2 - 2, & 0 \le x < 1 \\ x, & x \ge 1 \end{cases}$

$$|f(x)| = \begin{cases} -x-2, & x < -2 \\ x+2, & -2 \le x < 0 \\ x^2 + 2, & 0 \le x < 1 \\ x, & x \ge 1 \end{cases}$$

Discontinuous at $x = 1$

Hence, number of points of discontinuity = 1.

9. $f(x)$ is always discontinuous, $g(x)$ is continuous at $x = 0$ only.

Hence, we have $p + q = 0 + 1 = 1$.

10. $f(r) = K \in Q'$

$f(x) = \sqrt{5}$ constant function

$$\sum_{r=1}^{20} f(r) = 20\sqrt{5}$$

11. $f(x) = \begin{cases} -x, & \text{if } -1 \le x < 0 \\ 0, & \text{if } 0 \le x < 1 \\ x, & \text{if } 1 \le x < 2 \\ 4, & \text{if } x = 2 \end{cases}$

Hence, $f(x)$ is discontinuous at $x = 1, 2$.

Test Your Concepts-III

1. First we obtain LHD $= f'(0^-) = \lim\limits_{h \to 0} \left[\dfrac{f(-h) - f(0)}{-h}\right]$

$$= \lim_{h \to 0} \left[\left(-\frac{1}{h}\right)\left\{\frac{1 - \cos h}{h \sin h} - \frac{1}{2}\right\}\right]$$

$$= \lim_{h \to 0} \left[\frac{h \sin h + 2(1 - \cos h)}{2h^2 \sin h}\right]; \left(\text{in } \frac{0}{0} \text{ form}\right)$$

$$= \lim_{h \to 0} \left[\frac{h\left(h - \frac{h^3}{3!} + \frac{h^5}{5!} - \ldots\right) - 2\left(\frac{h^2}{2!} - \frac{h^4}{4!} + \frac{h^6}{6!} - \ldots\right)}{2h^2\left(h - \frac{h^3}{3!} + \ldots\right)}\right]$$

$$= \lim_{h \to 0} \left[\frac{h^4\left\{\left(\frac{1}{12} - \frac{1}{3!}\right) + \left(\frac{1}{5!} - \frac{2}{6!}\right)h^2\right\}}{2h^3\left(1 - \frac{h^2}{3!} + \ldots\right)}\right]$$

$$= \lim_{h \to 0} \left[\frac{h\left\{\left(\frac{1}{12} - \frac{1}{3!}\right) + \left(\frac{1}{5!} - \frac{2}{6!}\right)h^2 + \ldots\right\}}{2\left(1 - \frac{h^2}{3!} + \ldots\right)}\right] = 0$$

RHD $= f'(0^+) = \lim\limits_{h \to 0} \left(\dfrac{f(0+h) - f(0)}{h}\right)$

$$= \lim_{h \to 0} \left\{\frac{1}{h}\left(\frac{1 - \cos h}{h \sin h} - \frac{1}{2}\right)\right\} = 0,$$

We get LHD = RHD

Hence, $f(x)$ is differentiable at $x = 0$.

2. At $x = 3$, function $f(x)$ is continuous at $x = 3$ as $f(3^+) = f(3^-) = f(3) = 5$

For function to be differentiable

$f'(3+h) = f'(3-h)$

RHD $= f'(3+h) = \lim\limits_{h \to 0} \dfrac{f(3+h) - f(3)}{h}$

$$= \lim_{h \to 0} \frac{(3+h+2) - 5}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

LHD $= f'(3-h) = \lim\limits_{h \to 0} \dfrac{f(3-h) - f(3)}{-h}$

$$= \lim_{h \to 0} \frac{8 - (3-h) - 5}{-h} = \lim_{h \to 0} \frac{h}{-h} = -1$$

$\therefore$ $f'(3-h) \neq f'(3-h)$

Hence, function is not differentiable.

3. First will check continuity at $x = 1$

$f(1^-) = \lim\limits_{h \to 0} f(1-h) = \lim\limits_{h \to 0} |1-h-1|([1-h] - (1-h))$

$= \lim\limits_{h \to 0} h(0 - (1-h)) = 0$

$f(1^+) = \lim\limits_{h \to 0} f(1+h) = \lim\limits_{h \to 0} |1+h-1|([1+h] - (1+h))$

$= \lim\limits_{h \to 0} h(1 - (1+h)) = 0$

$f(1) = 0$

Thus, continuous at $x = 1$.

Check the differentiability at $x = 1$

$f'(1^+) = \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{h}$

$$= \lim_{h \to 0} \frac{|1+h-1|([1+h] - (1+h)) - 0}{h}$$

$$= \lim_{h \to 0} \frac{h(1 - 1 - h)}{h} = \lim_{h \to 0} (-h) = 0$$

$f'(1^-) = \lim\limits_{h \to 0} \dfrac{f(1-h) - f(1)}{-h}$

CHAPTER 5

$$= \lim_{h \to 0} \frac{|1-h-1|([1-h]-(1-h))-0}{-h}$$

$$= \lim_{h \to 0} \frac{h(0-1+h)}{-h} = 1$$

$$f'(1^+) \neq f'(1^-)$$

Hence, $f(x)$ is not differentiable at $x = 1$.

4. Since, the function is continuous at $x = 1$.

We have, $\lim_{h\to 0} f(1+h) = \lim_{h\to 0} f(1-h) = f(1)$

Now, $\lim_{h\to 0} f(1-h) = \lim_{h\to 0} [(1-h)^2 + b] = 1 + b$

$\lim_{h\to 0} f(1+h) = \lim_{h\to 0} [a(1+h)+3] = a + 3$ and $f(1) = a + 3$

Hence, $a + 3 = 1 + b$, or $b = a + 2$

Since, the function is differentiable at $x = 1$, we have $f'(1^+) = f'(1^-)$.

We obtain, $f'(1^+) = \lim_{h\to 0} \frac{1}{h}[f(1+h) - f(1)]$

$$= \lim_{h\to 0} \frac{1}{h}[\{a(1+h)+3\} - (a+3)] = a$$

$$f'(1^-) = \lim_{h\to 0} \frac{1}{(-h)}[f(1-h) - f(1)]$$

$$= \lim_{h\to 0} \frac{1}{(-h)}[\{(1-h)^2 + b\} - (a+3)]$$

$$= \lim_{h\to 0} \frac{1}{(-h)}[\{1 - 2h + h^2 + b\} - (1+b)]$$

$$= \lim_{h\to 0} \frac{1}{(-h)}[h^2 - 2h] = 2$$

Hence, $a = 2$ and $b = a + 2 = 4$

5. $l = f'(0^+) = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$

$$= \lim_{h\to 0} \frac{h\tan^{-1} h + \sec^{-1}\frac{1}{h} - \frac{\pi}{2}}{h}$$

$$= \lim_{h\to 0} \left(\tan^{-1} h + \frac{\cos^{-1} h - \frac{\pi}{2}}{h}\right)$$

$$= \lim_{h\to 0} \left(\tan^{-1} h - \frac{\sin^{-1} h}{h}\right) = 0 - 1 = -1$$

$$m = f'(0^-) = \lim_{h\to 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h\to 0} \frac{h\tan^{-1} h + \sec^{-1}\left(\frac{-1}{h}\right) - \frac{\pi}{2}}{-h}$$

$$= \lim_{h\to 0} \left(-\tan^{-1} h + \frac{\frac{\pi}{2} - \cos^{-1} h}{-h}\right)$$

$$= \lim_{h\to 0} \left(-\tan^{-1} h - \frac{\sin^{-1} h}{h}\right) = 0 - 1 = -1$$

$\therefore \quad l = f'(0^+) = -1$ and $m = f'(0^-) = -1$

Hence, $l^2 + m^2 = 2$

6. $f(x) = \left[\begin{array}{ll} \frac{\sin x^2}{x}, & x \neq 0 \\ 0, at & x = 0 \end{array}\right.,$

$$f'(0^+) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\sinh^2}{h^2} - 0 = 1$$

$$f'(0^-) = \lim_{h\to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h\to 0} \frac{\sinh^2}{h^2} - 0 = 1$$

Hence, $f'(0^+) + f'(0^-) = 2$

7. $b = g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h}$

$$b = \lim_{h\to 0} \frac{h^2 + h\tan h - h\tan 2h}{h(ah + \tan h - \tan 3h)} = \lim_{h\to 0} \frac{h + \tan h - \tan 2h}{ah + \tan h - \tan 3h}$$

$$= \lim_{x\to 0} \frac{h + \left(h + \frac{h^3}{3} + \frac{2}{15}h^5 + \ldots\infty\right) - \left(2h + \frac{8h^3}{3} + \frac{2}{15}\cdot 32h^5 + \ldots\infty\right)}{ah + \left(h + \frac{h^3}{3} + \frac{2}{15}h^5 + \ldots\infty\right) - \left(3h + \frac{27h^3}{3} + \frac{2}{15}\cdot 243h^5 + \ldots\infty\right)}$$

$$= \lim_{x\to 0} \frac{h^3\left(-\frac{7}{3}\right) - \frac{62}{15}h^5 + \ldots\infty}{(a-2)h + \left(\frac{1}{3} - 9\right)h^3 + \frac{2}{15}(-242)h^5 + \ldots\infty}$$

$\therefore \quad b$ can be finite if $a - 2 = 0 \quad \Rightarrow a = 2$

$$\therefore \quad b = \frac{-\frac{7}{3}}{\frac{1}{3} - 9} = \frac{7}{26} \quad \Rightarrow \frac{b}{a} = \frac{7}{52}$$

8. For continuity:

$\text{LHL} = f(3^-) = \lim_{x\to 3^-} \sqrt{10 - x^2} = \sqrt{10 - 3^2} = 1$

$\text{RHL} = f(3^+) = \lim_{x\to 3^+} (2 - e^{x-3}) = 2 - e^0 = 1$

$f(3) = 2 - e = 1$

As we can see that limit exists and its equal to value of function hence, $f(x)$ is continuous at $x = 3$ for differentiability:

$$f'(3^+) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{(2 - e^h) - 1}{h} = -\lim_{h\to 0} \left(\frac{e^h - 1}{h}\right) = -1$$

$$f'(3^-) = \lim_{h\to 0} \frac{f(3-h) - f(3)}{-h} = \lim_{h\to 0} \frac{\sqrt{10 - (3-h)^2} - 1}{-h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{1 + 6h - h^2} - 1\right)\left(\sqrt{1 + 6h - h^2} + 1\right)}{-h\left(\sqrt{1 + 6h - h^2} + 1\right)}$$

$$= \lim_{h\to 0} \frac{h(6-h)}{-h\left(\sqrt{1 + 6h - h^2} + 1\right)} = -3$$

LHD ≠ RHD at $x = 3$

Hence, $f(x)$ is not differentiable at $x = 3$.

9. If function is differentiable in R, then it is also be differentiable in $x_0 = a$.

$$\text{RHD} = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(a+h+1)h - 0}{h} = a + 1$$

$$\text{LHD} = \lim_{h\to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h\to 0} \frac{(a-h+1)h - 0}{-h} = -a - 1$$

For differentiable

LHD = RHD

Hence, $a + 1 = -(a + 1) \Rightarrow a = -1$

10. $\because$ $g(x)$ is continuous at $x = 0$

$g(0^-) = g(0^+) = g(0)$

$\Rightarrow \quad 2(0) + \tan^{-1}.0 + a = 0 + 0 + 0 \Rightarrow a = 0$

$g(x)$ is also differentiable at $x = 0$

$$g'(x) = \begin{cases} 2 + \dfrac{1}{1+x^2}, & x < 0 \\ 3x^2 + 2x + b, & x > 0 \end{cases}$$

$\because \quad g'(0^-) = g'(0^+) \quad \Rightarrow 2 + \dfrac{1}{1+0^2} = 3(0)^2 + 2(0) + b$

$\Rightarrow \quad 1 + 2 = b \quad \Rightarrow b = 3$

Hence, $a^2 + b^2 = 9$

11. $g(x) = |2 - x|$

$$g(x) = \begin{cases} x - 2, & x \geq 2 \\ 2 - x, & x < 2 \end{cases}$$

$\therefore \quad g'(2^+) = 1 = p$

Now, $\lim\limits_{x \to 0} f(x) = p \Rightarrow \lim\limits_{x \to 0} \dfrac{x^3}{\sqrt{a + x}(bx - \sin x)} = 1$

$\Rightarrow \quad \dfrac{1}{\sqrt{a}} \lim\limits_{x \to 0} \left(\dfrac{x^3}{bx - \sin x} \right) = 1$

$\therefore \quad$ for existence of limit $b = 1$

Now, $\dfrac{1}{\sqrt{a}} \lim\limits_{x \to 0} \dfrac{x^3}{(x - \sin x)} = 1$

$\Rightarrow \quad \dfrac{1}{\sqrt{a}} \lim\limits_{x \to 0} \dfrac{x^3}{x - \left(x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \ldots \right)} = 1$

$\Rightarrow \quad \dfrac{1}{\sqrt{a}} \lim\limits_{x \to 0} \dfrac{x^3}{\dfrac{x^3}{3!} - \dfrac{x^5}{5!} \ldots\ldots} = 1 \Rightarrow \dfrac{1}{\sqrt{a}} \times 6 = 1 \qquad \therefore a = 36$

Hence, $a + 2b = 38$

12. $f(x) = \lim\limits_{t \to x} \left(\dfrac{\sec t - \sec x}{t - x} \right)$ put $t = x + h$

$$= \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h} = \frac{d}{dx}(\sec x) = \sec x \tan x$$

$\therefore \quad f'(x) = \sec x \cdot \sec^2 x + \tan^2 x \cdot \sec x$

$$f'\left(\frac{\pi}{3}\right) = 8 + 2 \cdot 3 = 14; \; f'\left(-\frac{\pi}{3}\right) = 8 + 6 = 14$$

$$\frac{f'\left(-\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)}{4} = \frac{28}{4} = 7$$

13. $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ x^3, & -1 \leq x \leq 0 \end{cases} \quad f(0^+) = 0 \quad f(0^-) = 0 = f(0)$

$$f'(0^-) = 0 \; ; \; f'(0^+) = 1$$

14. $\lim\limits_{x \to 0+} e^{-\frac{1}{x}} = 0, \; \lim\limits_{x \to 0-} e^{\frac{1}{x}} = 0;$

$$f(0^-) = \lim_{x \to 0^-} \left(\frac{e^{\frac{2}{x}} - 1}{e^{\frac{2}{x}} + 1} \right) = \lim_{x \to 0^-} \left(\frac{0 - 1}{0 + 1} \right) = -1$$

$$f(0^+) = \lim_{x \to 0^+} \left(\frac{1 - e^{-\frac{2}{x}}}{1 + e^{-\frac{2}{x}}} \right) = 1$$

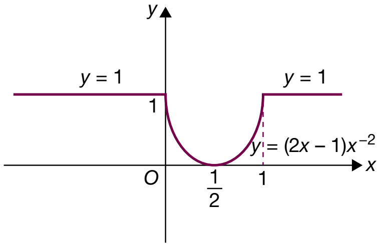
Hence, $\lim\limits_{x \to 0} f(x)$ does not exist, neither continuous nor differentiable at $x = 0$.

Test Your Concepts-IV

1. $|x| - |x - 1| = \begin{cases} -x + x - 1 & x < 0 \\ x + x - 1 & 0 \leq x < 1 \\ x - (x - 1) & x \geq 1 \end{cases} = \begin{cases} -1 & x < 0 \\ 2x - 1 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$

$\therefore \quad f(x) = \{|\, x \,| - |\, x - 1 \,|\}2$

$$f(x) = \begin{cases} 1 & x < 0 \\ (2x - 1)^2 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$$



From graph it is clear that $f(x)$ is continuous for all $x \in R$ and at point $x = 0, 1$ there are sharp edges at these points, hence, differentiable for $x \in R - \{0, 1\}$.

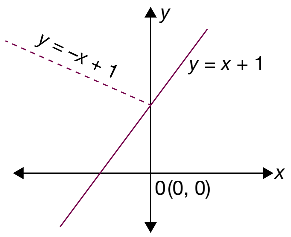
2. Since, $\tan \pi [x] = 0$ for all real x

$\therefore \quad f(x)$ is actually $f(x) = \tan x$

Therefore, this is discontinuous and non-differentiable at $x = \left(\dfrac{2n+1}{2} \right) \pi, n \in I$.

3. $f(x) = \min. (x + 1, |x| + 1) = x + 1 \; \forall \; x \in R$

$\because \quad f(x) = x + 1$ so $f(x)$ is differential $\forall \; x \in R$



4. $\because \; f(x)$ is both continuous and differentiable at $x = -1$.

$f(-1^-) = f(-1^+) \quad \Rightarrow -p + q = p - 1 + 2q \quad \Rightarrow 2p + q = 1.$

$$f'(x) = \begin{cases} p & x < -1 \\ 2px + 1 & x > -1 \end{cases}$$

Also, $f'(-1^-) = f'(-1^+) \quad \Rightarrow p = -2p + 1 \quad \Rightarrow p = 1/3, q = 1/3,$

$$(p, q) = \left(\tfrac{1}{3}, \tfrac{1}{3} \right)$$

5. $f(x) = \begin{cases} 2x + 1, & x \leq 1 \\ x^2 + 2, & 1 < x \leq 2 \\ 4x - 2, & x > 2 \end{cases}$

We can say that $f(x)$ is continuous everywhere.

$$f'(x) = \begin{cases} 2, & x < 1 \\ 2x, & 1 < x < 2 \\ 4, & x > 2 \end{cases}$$

$f'(1^-) = 2 = f'(1^+)$ and $f'(2^-) = 4 = f'(2^+)$

Hence, $f(x)$ is differentiable $\forall \; x \in R$.

6. In both the polynomial functions, we should check continuity at $x = 1$,

$$g(1^-) = g(1) = g(1^+) = 1/2$$

Now, we can say that $g(x)$ is continuous $\forall\ x \in R$.
Let discuss differentiability at $x = 1$

$$\text{Now, } g'(1^-) = \lim_{h\to 0} \frac{g(1-h) - g(1)}{-h}$$

$$= \lim_{h\to 0} \frac{2(1-h)^2 - 3(1-h) + 3/2 - 1/2}{-h}$$

$$= \lim_{h\to 0} \frac{2h^2 - 4h + 3h}{-h} = 1$$

$$\text{And, } g'(1^+) = \lim_{h\to 0} \frac{g(1+h) - g(1)}{h} = \lim_{h\to 0} \frac{\frac{(1+h)^2}{2} - \frac{1}{2}}{h}$$

$$= \lim_{h\to 0} \frac{1 + h^2 + 2h - 1}{2h} = \lim_{h\to 0} \left(1 + \frac{h}{2}\right) = 1$$

Hence, $g'(1) = 1$, $g(x)$ is continuous and differentiable $\forall\ x \in R$.
So, $L = 0$, $M = 0$
Hence, $L + M = 0 + 0 = 0$.

7. $f(x) = \begin{cases} 3 - x, & x < -3 \\ 6, & -3 \le x \le 3 \\ 3 + x, & x > 3 \end{cases}$

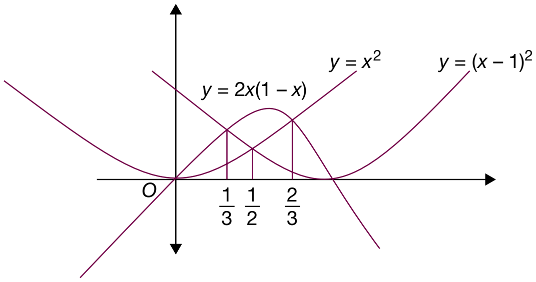
Since, these expressions are linear function in x or a constant.
It is clearly differentiable at all points except at the border points at -3 and 3.

At $x = -3$, LHD $= -1$, RHD $= 0$
At $x = 3$, LHD $= 0$, RHD $= 1$

Hence, at $x = -3$ and $x = 3$ it is not differentiable.

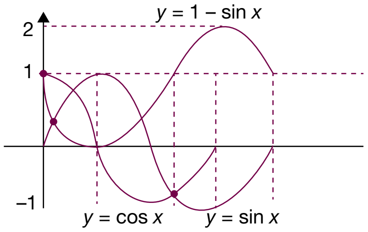
8. $2\pi[\pi^2 x]$ is integral multiple of π, therefore $f(x) = 0\ \forall\ x$
$\Rightarrow$ $f(x)$ is constant function
Hence, $f(x)$ is continuous and differentiable any number of points.

9. $f(x) = \begin{cases} (x-1)^2, & \text{when } 0 \le x \le \frac{1}{3} \\ 2x(1-x), & \text{when } \frac{1}{3} \le x \le \frac{2}{3} \\ x^2, & \text{when } \frac{2}{3} \le x \le 1 \end{cases}$



From the figure, it is clear that the function f is not differentiable at $x = \frac{1}{3}$ and $\frac{2}{3}$.

10.



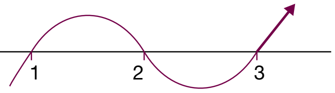
By figure it is clear

$$f(x) = \begin{cases} \sin x, & 0 \le x \le \pi/6 \\ 1 - \sin x, & \pi/6 \le x \le \pi/2 \\ \cos x, & \pi/2 \le x \le 5\pi/4 \\ \sin x, & 5\pi/4 \le x \le 2\pi \end{cases}$$

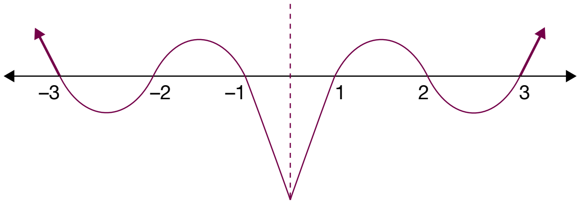
Hence, $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{4}$ are the points where $f(x)$ is not differentiable.

11. Here, $f(x) = (x - 1)(x - 2)(x - 3)$, $x \in R$.

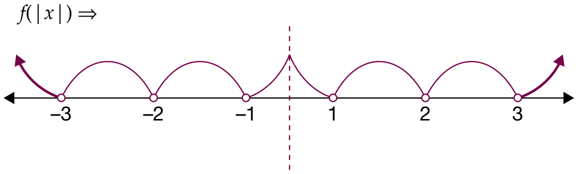
$f(x) \Rightarrow$



$f(|x|) \Rightarrow$



$f(|x|) \Rightarrow$



Hence, 7 points of non-differentiability.

12. Given, $f(x) = \sqrt[3]{x^2|x|} - |x| - 1$

$$= \begin{cases} \sqrt[3]{-x^2} + x - 1 = -x + x - 1 = -1, & x < 0 \\ x - x - 1 = -1, & x \ge 0 \end{cases}$$

Hence, $f(x) = -1\ \forall\ x \in R$
So, $f(x)$ is continuous and differentiable $\forall\ x \in R$.
Hence, number of points of non-derivability of $f(x)$ is 0.

Test Your Concepts-V

1. $\lim_{h\to 0} \frac{f((\pi+h)^3) - f(\pi^3)}{h} = \pi$

$$\Rightarrow \lim_{h\to 0} \frac{f((\pi+h)^3) - f(\pi^3)}{(3h^2\pi + 3\pi^2 h + h^3)} \cdot \lim_{h\to 0} \frac{(3h^2\pi + 3\pi^2 h + h^3)}{h} = \pi$$

Hence, $f'(\pi^3) \cdot 3\pi^2 = \pi \Rightarrow f'(\pi^3) = \frac{1}{3\pi}$

2. We have, $f\left(\frac{x+2y}{3}\right) = \frac{f(x) + 2f(y)}{3}\ \forall x, y \in R \rightarrow (1)$ Replacing x by $3x$ and putting $y = 0$ in (1), we get $f(x) = \frac{f(3x) + 2f(0)}{3}$.
$\Rightarrow f(3x) = 3f(x) - 2f(0) \rightarrow (2)$.

$$\text{Now, } f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f\left(\frac{3x + 2.\frac{3h}{2}}{3}\right) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{f(3x)+2.f\left(\dfrac{3h}{2}\right)}{3} - f(x)}{h} \quad \text{(from (1))}$$

$$= \lim_{h \to 0} \frac{f(3x)+2f\left(\dfrac{3h}{2}\right)-3f(x)}{3h}$$

$$= \lim_{h \to 0} \frac{2f\left(\dfrac{3h}{2}\right)-2f(0)}{3h} \quad \text{(from (2))}$$

$$= \lim_{h \to 0} \frac{f\left(\dfrac{3h}{2}\right)-f(0)}{\dfrac{3h}{2}} = f'(0) = 1 \quad \text{(given)}$$

$\Rightarrow f'(x) = 1 \Rightarrow f(x) = x + c$

Hence, $f(x)$ is a linear function in x, continuous $\forall x \in R$ and differentiable $\forall x \in R$.

3. $L = \lim_{x \to 0} \frac{1}{x} f\left(\frac{x}{n}\right) = \lim_{x \to 0} \frac{f(x/n) - f(0)}{x - 0}$

$$= \lim_{x \to 0} \frac{f(x/n) - f(0)}{x/n - 0} . \frac{1}{n}$$

$$\therefore \quad \lim_{x \to 0} \frac{1}{x} \sum_{n=1}^{\infty} (-1)^n f\left(\frac{x}{n}\right) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = -\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = -\ln 2$$

4. $f(x) = ax + b$

$x = y = 0 \Rightarrow f(0) = \dfrac{2 + 2f(0)}{3}$

$3f(0) = 2 + 2f(0)$; $f(0) = ax + b \quad b = 2$

$f'(\alpha) = a = 2$; $f(x) = 2x + 2$

$f\left(\frac{1}{2}\right) = 3$. $f(3) = 8$, $f(5) = 12$

5. Given, $f(u) = \dfrac{3}{2u^2 + 5u - 3}$ and $u = \dfrac{1}{x+2}$

Since, $u = \dfrac{1}{x+2}$ is not defined for $x = -2$.

$\therefore$ u is discontinuous at $x = -2$.

Also, $f(u) = \dfrac{3}{2u^2 + 5u - 3} = \dfrac{3}{(2u-1)(u+3)}$

Clearly, $f(u)$ is not defined at $u = -3, \dfrac{1}{2}$

When $u = -3$, then $\dfrac{1}{x+2} = -3$

$\Rightarrow \; -3x - 6 = 1 \quad \Rightarrow x = \dfrac{-7}{3}$

When $u = \dfrac{1}{2}$, then $\dfrac{1}{x+2} = \dfrac{1}{2} \qquad \Rightarrow x = 0$

Hence, $f(u)$ is discontinuous at $x = 0, -2, \dfrac{-7}{3}$.

6. Given, $f(x^2 + 1) = \dfrac{2}{f(2^+) - 1}$, $f(x) > 0$ …(1)

Put $t = x^2 + 1$, then $x = \sqrt{t-1}$

$\therefore \quad f(t) = \dfrac{2}{f(2^{\sqrt{t-1}}) - 1}$

$\Rightarrow \quad f(t) = \dfrac{2}{f(2^{\sqrt{x-1}}) - 1}$ …(2)

From Eq. (1), $f(1) = \dfrac{2}{f(1) - 1}$ $\qquad \Rightarrow f(1)^2 - f(1) = 2$

$\Rightarrow \; f(1)^2 - f(1) - 2 = 0$ $\qquad \Rightarrow f(1) = -2, -1$

$\Rightarrow \; f(1) - 2$ $\qquad [\because f(x) > 0$ for all $x \in R]$

Now, $\lim_{x \to 1} f(x) = \lim_{x \to 1} \dfrac{2}{f(2^{\sqrt{t-1}}) - 1}$ [using Eq. (2)]

$= \dfrac{2}{f(1) - 1} = \dfrac{2}{2 - 1}$ $\qquad [\because f(1) = 2]$

$= 2$

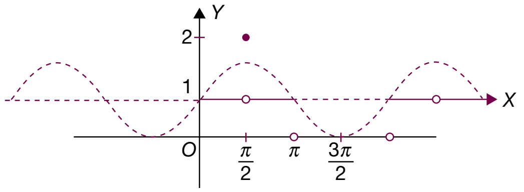
7. We have, $\lim_{h \to 0} \dfrac{f(2h + 2 + h^2) - f(2)}{f(h - h^2 + 1) - f(1)}$

It is $\dfrac{0}{0}$ form, hence, apply L'Hospital's rule,

$$\lim_{h \to 0} \frac{f'(2h + 2 + h^2) - (2 + 2h)}{f'(h - h^2 + 1)(1 - 2h)} = \frac{f'(2) \cdot 2}{f'(1)} = \frac{2 \cdot 6}{4} = \frac{12}{4} = 3$$

8. $h(x) = gof(x) = [f(x) + 1] = [\sin x + 1]$

$\therefore \quad h(x) = 1 + [\sin x] \quad$ or $\quad h(x) = \begin{cases} 1, & \text{if } 0 \le x < \pi/2 \\ 1, & \text{if } \pi/2 < x < \pi \end{cases}$



Clearly, $h(x)$ discontinuous at $x = \dfrac{\pi}{2}$.

Hence, $h'\left(\dfrac{\pi}{2}\right)$ does not exist.

9. We have, $f\left(\dfrac{x}{y}\right) = \dfrac{f(x)}{f(y)}$

On differentiating w.r.t. x, treating y constant, we get

$$f'\left(\frac{x}{y}\right) \cdot \frac{1}{y} = \frac{f'(x)}{f(y)} \qquad \text{…(1)}$$

Again, differentiating w.r.t. y, treating x constant, we get

$$f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) = f(x) \cdot \left(-\frac{1}{[f(y)]^2}\right) f'(y) \qquad \text{…(2)}$$

On dividing Eq. (1), we get

$$-\frac{y}{x} = -\frac{f(y) \cdot f'(x)}{f(x) \cdot f'(y)} \qquad \text{…(3)}$$

$$\Rightarrow \quad \frac{yf'(y)}{f(y)} = \frac{xf'(x)}{f(x)} = k, \quad \frac{f'(x)}{f(x)} = \frac{k}{x}$$

$$\because \quad f'(x) = \frac{kf(x)}{x} \qquad \text{…(4)}$$

Now, putting $x = y = 1$ in the given relation, we get $f(1) = 1$
Putting $x = 1$ in Eq. (4), we get

$$f'(1) = \frac{k(1)}{1}; \quad 2 = k$$

Hence, $f'(x) = \dfrac{2f(x)}{x}$

10. We have, $f(x+y) = 2f(x) \cdot f(y)$

On differentiating w.r.t. x, treating y constant, we get

$$f'(x+y) = 2f'(x) \cdot f(y) \qquad \ldots(1)$$

Again, differentiating w.r.t. y treating x constant, we get

$$f'(x+y) = 2f(x) \cdot f'(y) \qquad \ldots(2)$$

On dividing Eq. (1) by Eq. (2), we get

$$= \frac{f'(x) \cdot f(y)}{f(x) f'(y)} \Rightarrow \frac{f'(x)}{f'(x)} = \frac{f'(y)}{f(y)} = k$$

$$f'(x) = kf(x) \qquad \ldots(3)$$

Put $x = 0$ in Eq. (3), we get

$$f'(0) = kf(0) = \frac{k}{2} = 3 \quad \Rightarrow k = 6 \qquad \because\ f'(x) = 6f(x)$$

Hence, $f'(4) = 6f(4) = 6 \times 2 = 12$ [given, $f(4) = 2$]

11. We have, $f\left(\frac{x+y}{2}\right) = \frac{f(x) + f(y)}{2}$ …(1)

On differentiating Eq.(1) w.r.t. x, treating y constant, we get

$$f'\left(\frac{x+y}{2}\right) \cdot \frac{1}{2} = \frac{1}{2} f'(x) \qquad \ldots(2)$$

Again, differentiating Eq. (1) w.r.t. y treating x constant, we get

$$f'\left(\frac{x+y}{2}\right) \cdot \frac{1}{2} = \frac{1}{2} f'(y) \qquad \ldots(3)$$

From Eqs. (2) and (3), we get

$$1 = \frac{f'(x)}{f'(y)} \quad \Rightarrow f'(x) = f'(y) = k$$

$\Rightarrow$ $f'(x) = k$, which is constant and $f'(0) = -1$

Hence, $f'(2) = -1$

Single Correct Choice Type Questions

1. $$f(x) = \begin{cases} \dfrac{[x^2]-1}{x^2-1}, & \text{for } x^2 \neq 1 \\ 0, & \text{for } x^2 = 1 \end{cases} = \begin{cases} \dfrac{-1}{x^2-1}, & \text{for} \quad 0 < x^2 < 1 \\ 0, & \text{for} \quad x^2 = 1 \\ 0, & \text{for} \quad 1 < x^2 < 2 \end{cases}$$

$\therefore$ RHL at $x = 1$ is 0

Also, LHL at $x = 1$ is ∞

Hence, the correct answer is (B).

2. $f(0) = 2$

$$\lim_{x \to 0} \frac{\cos x(a \sin x - b) + b}{x^2}$$

$$= \lim_{x \to 0} \frac{\left(1 - \frac{x^2}{2!} + \ldots\right)\left(a\left(x - \frac{x^3}{3!} + \ldots\right) - b\right) + b}{x^2}$$

$$= \lim_{x \to 0} \frac{a\left(x - \frac{x^3}{3!} + \ldots\right) - b - 2!\,x^2\left(\left(x - \frac{x^3}{3!} + \ldots\right) - b\right) + b}{x^2}$$

$$= \lim_{x \to 0} \frac{a\left(x - \frac{x^3}{3!} + \ldots\right) - \frac{ax^3}{2!} + \frac{x^5 a}{2!3!} + \ldots + \frac{bx^2}{2!}}{x^2}$$

For limit to exist, $a = 0$;

Limit $= \frac{b}{2!} = 2; \quad b = 4$

Hence, the correct answer is (A).

3. $g(x) = \cos x \frac{1}{\sqrt{2}} + \sin x \frac{1}{\sqrt{2}} - (\sin x \cdot \frac{1}{\sqrt{2}} - \cos x \cdot \frac{1}{\sqrt{2}})$

$g(x) = \sqrt{2} \cos x$ always continuous

$f(x)$ discount at $x = 0$; $f(g(x))$ discount when $g(x) = 0$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Hence, the correct answer is (A).

4. $$\lim_{x \to 0} \frac{e^{2x} - 1 - 2x}{x(e^{2x} - 1)} = f(0) = \lambda$$

$$\lim_{x \to 0} \frac{1 + (2x) + \frac{(2x)^2}{2!} + \ldots. - 1 - 2x}{x\left(\frac{e^{2x} - 1}{2x}\right)2x} = \lambda$$

$$\lim_{x \to 0} \frac{2x^2 + \ldots}{2x^2} = \lambda$$

We get, $\lambda = 1$

Hence, the correct answer is (B).

5. Point of discontinuities when $-1 < x < 2$

$-1 < x^3 < 8$ at all integers except zero (0), function is discontinuous.

Hence, 7 points of discontinuity.

And, 7 point of non differentiability.

Hence, the correct answer is (A).

6. Given, $f(x) = sgnx = \begin{cases} -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$

And $g(x) = x(1 - x^2)$ always continuous

Now, $fog(x) = -1$ if $x(1 - x^2) < 0$ or $x \in (-1, 0) \cup (1, \infty)$

$(fog)(x) = 0$ if $x(1 - x^2) = 0$, for $x \in \{-1, 0, 1\}$

$(fog)(x) = 1$ if $x(1 - x^2) > 0$ for $x \in (-\infty, -1) \cup (0, 1)$

$\therefore$ $fog(x)$ is discontinuous at $x = -1, 0, 1$

Hence, the correct answer is (C).

7. $$f(x) = \lim_{n \to -\infty} x(x+1) \cdot (x^2 - 1)^2 (\cos \pi x)^n$$

$$= \lim_{n \to -\infty} x(x+1)^3 (x-1)^2 \left(\cos^2 \pi x\right)^n$$

At $x = 1$

LHD RHD

$$\lim_{h \to 10} \frac{f(1-h) - f(1)}{-h} \qquad = \lim_{h \to 10} f\frac{(1+h) - f(1)}{h}$$

$$\lim_{n \to -\infty} \lim_{h \to 10} \frac{(1-h)(2-h)^3(-h)^2(\cos^2 \pi h)^n}{-h}$$

$$= \lim_{n \to -\infty} \lim_{h \to 10} \frac{(1+h)(2+h)^3 h^2 (\cos^2 \pi h)^n - 0}{h}$$

$0 = 0$

At $x = -1$

$$\lim_{h \to 10} \frac{f(-1-h) - f(+1)}{-h} = \lim_{h \to 10} \frac{f(-1+h) - f(-1)}{h}$$

$$\lim_{h \to 10} \frac{(-1-h)(-h)^3(-2-h)^2(\cos^2 \pi h)^n}{-h}$$

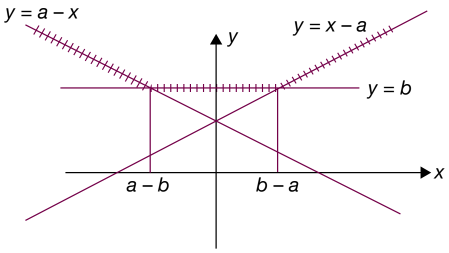
$$= \lim_{h \to 10} \frac{(-1+h)(h)^3(-2+h)^2 \left(\cos^2 \pi h\right)^n}{h}$$

$0 = 0$ diff at $x = -1$

$f'(x)$ is continuous only at $x = 1, -1$. Discontinuous at all integer except $x = \pm 1$.

Hence, the correct answer is (C).

8. $$f(x)=\begin{cases} a-x, & \text{if} \quad x<a-b \\ b, & \text{if} \quad a-b\le x\le b-a \\ a+x, & \text{if} \quad x>b-a \end{cases}$$



Hence, f is not differentiable at $x=a-b, b-a$

Hence, the correct answer is (A).

9. Here, cos (| x |) = cos x

$$f(x)=-\left(x^2-1\right)\left(x^2-3x+2\right)+\cos x, 1\le x\le 2$$

$$f(x)=\left(x^2-1\right)\left(x^2-3x+2\right)+\cos x,\ x\le 1 \text{ or } x\ge 2$$

Clearly, $f(1) = \cos 1$, $\lim\limits_{x\to 1} f(x)=\cos 1$

$f(2) = \cos 2$, $\lim\limits_{x\to 2} f(x)=\cos 2$

Hence, $f(x)$ is continuous at $x = 1, 2$

Now,

$$f'(x)=-2x\left(x^2-3x+2\right)-\left(x^2-1\right)(2x-3)-\sin x,\ 1\le x<2$$

$$=2x\left(x^2-3x+2\right)+\left(x^2-1\right)(2x-3)-\sin x, x<1 \text{ or } x>2$$

$$f'(1-0)=-\sin 1, f'(1+0)=-\sin 1$$

$$f'(2-0)=-3-\sin 2\,,$$

$$f'(2+0)=3-\sin 2$$

Hence, $f(x)$ is not differentiable at $x = 2$.

Hence, the correct answer is (D).

10. $f(x)=x(x), g(x)=\sin x$

$$h(x)=gof(x)=\sin(x\,|\,x\,|)$$

$$h(x)=\begin{cases} \sin x^2 & x\ge 0 \\ -\sin x^2 & x<0 \end{cases} \qquad \text{Continuous and differentiable}$$

$$h'(x)=\begin{cases} 2x\cos x^2 & x\ge 0 \\ -2x\cos x^2 & x<0 \end{cases}$$

$$h'(0^+)=h'(0^-)=0$$

$h'(x)$ is continuous

$$h''(x)=\begin{cases} 2\cos x^2+2x(-\sin x^2)2x & x\ge 0 \\ -2\cos x^2-2x(-\sin x^2)2x & x<0 \end{cases}$$

$$h''(0^+)=2\ \&\ h''(0^-)=-2$$

Thus, $h'(x)$ not diff but continuous.

Hence, the correct answer is (C).

11. Given, $f(0) = a$, $f'(0)=ab$, $f''(0)=ab^2$, $f'''(0)=ab^3$ and so on.

$\therefore \quad f(x)=ae^{bx}$

$\therefore \quad \lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty} ae^{bx}=0 \ [\because b>0]$

Hence, the correct answer is (C).

12. $$f(x)=\begin{cases} p\sin x+qe^x+rx^3 & 0\le x<\pi \\ -p\sin x+qe^{-x}-rx^3 & -\pi<x<0 \end{cases}$$

$$f'(0^+)=p\cos x+qe^x+3rx^2\Big|_{x=0^+}$$
$$=p+q$$
$$f'(0^-)=-p\cos x-qe^{-x}-3rx^2\Big|_{x=0^-}$$
$$=-p-q$$

For differentiable at $x = 0$

$f'(0^+)=f'(0^-)\Rightarrow p+q=-p-q$

$\Rightarrow p+q=0$

$p=q=0$ is valid for it, $r\in R$

Hence, the correct answer is (B).

13. $$f'(0)=\lim_{h\to 0}\frac{\dfrac{\tan 7h-\tan 4h-\tan 3h}{h^2}-0}{h}$$

$$=\lim_{h\to 0}\frac{\tan 7h-\tan 4h-\tan 3h}{h^3}$$

$$=\frac{\left(7h+\dfrac{(7h)^3}{3}+\right)-\left(4h+\dfrac{(4h)^3}{3}+\right)-\left(3h+\dfrac{(3h)^3}{3}+\right)}{h^3}$$

$$=\frac{7^3-4^3-3^3}{3}=\frac{343-91}{3}=84$$

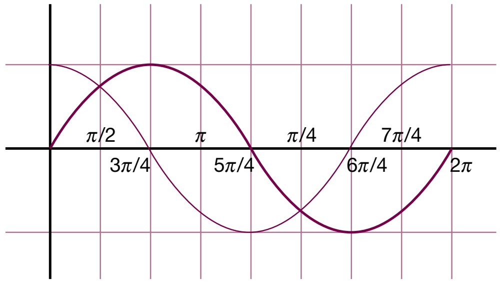
Hence, the correct answer is (C).

14. ∃ infinitely many continuous functions of the form $f(x)=mx$. When m is irrational, and when slope is irrational the line obviously will not pass through any of the points in the co-ordinate plane with integral co-ordinates. We know a straight line is always continuous.

Hence, the correct answer is (C).

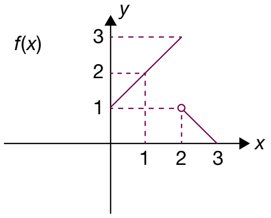
15. Consider the graph of $f(x)$ = max(sin x, cos x), which is non-differentiable at $x = \pi/4$, hence, statement (A) is false. From the graph $y = f(x)$ is differentiable at $x = \pi/2$, hence, Statement (B) is false. Statement (C) is false.

Statement (D) is false as consider $g(x)$ = max(x, x^2) at x = 0, for which $x = x^2$ at $x = 0$, but $f(x)$ is differentiable at $x = 0$.



Hence, the correct answer is (C).

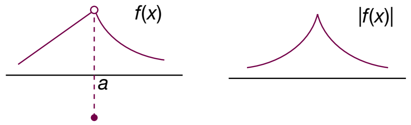
16. $$g(x)=\begin{cases} 1+f(x), & 0\le x\le 1, & 1\le f(x)\le 2 \\ 3-f(x), & 1<x\le 2, & 2<f(x)\le 3 \\ 1+f(x), & 2<x\le 3, & 0\le f(x)<1 \end{cases}$$



$$g(x)=\begin{cases}2+x, & 0\le x\le 1\\ 2-x, & 1<x\le 2\\ 4-x, & 2<x\le 3\end{cases}$$

Hence, the correct answer is (C).

17.



$$\lim_{x\to a} f(x)=-f(a) \qquad \Rightarrow \qquad \lim_{x\to a} f(x)+f(a)=0.$$

Hence, the correct answer is (B).

18. Check the function at $x=0$

$$f(0)=0, f(0^+)=\lim_{h\to 0}\frac{h}{5}+\left[\frac{h}{2}\right]=0 \text{ continuous at } x=0$$

at $x=100$

$$f(100)=\{20\}+[50]=0+50=50$$

$$f(100^-)=\left\{20-\frac{h}{5}\right\}+\left[50-\frac{h}{2}\right]=1-\frac{h}{5}+49=50 \text{ continuous at } x=100$$

$$f(x)=\left\{\frac{x}{5}\right\}+\left[\frac{x}{2}\right]$$

$$\frac{x}{5}\in I\Rightarrow x=5, 10, 15, \ldots, 95$$

$$\frac{x}{2}\in I\Rightarrow x=2, 4, \ldots, 98, 100$$

Common $x\in$ $10, 20, \ldots., 90$

$$f(10)=\left\{\frac{10}{5}\right\}+\left[\frac{10}{2}\right]=0+5$$

$$f(10^-)=\lim_{h\to 0}\left\{\frac{10-h}{5}\right\}+\left[\frac{10-h}{2}\right]$$

$$=\lim_{h\to 0}\left\{2-\frac{h}{5}\right\}+\left[5-\frac{h}{2}\right]=\lim_{h\to 0}1-\frac{h}{5}+4=5$$

$$f(10^+)=\lim_{h\to 0}\left\{\frac{10+h}{5}\right\}+\left[\frac{10+h}{2}\right]$$

$$=\lim_{h\to 0}\left\{2+\frac{h}{5}\right\}+\left[5+\frac{h}{2}\right]=\lim_{h\to 0}\frac{h}{5}+5=5$$

All common point, it is continuous.
Function is not continuous at. $x=5, 15, 25, \ldots., 95 \to 10$ points
$x=2, 4, 6, 8, 12, \ldots., 98 \to 40$ points
50 points of discontinuity.

Hence, the correct answer is (A).

19. RHL $(x=0)=\alpha+0=\alpha$

$$\frac{\sin x-x}{x^3}=\frac{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\ldots-x}{x^3}=\frac{-1}{3!}+\frac{x^2}{5!}-\ldots\ldots$$

$$\lim_{x\to 0}\frac{\sin x-x}{x^3}=\frac{-1}{6}; \text{ LHL}=\beta-1$$

Now, $\alpha=2$ and $\beta=3\Rightarrow\beta-\alpha=1$

Hence, the correct answer is (A).

20. $f(x)=\dfrac{\sqrt{x^2+px+1}}{x^2-p}$

$x\neq\pm\sqrt{p}$ for two values $p>0$

$x^2+Px+1\ge 0 \qquad \forall\, x\in R.$

$D\le 0$

$P^2-4\le 0$ $P\in[-2, 2]$ but $\qquad P>0$

$P\in(0, 2]$

$P=1, 2$

Hence, the correct answer is (B).

21. $f(x)=[2x]^2-\{2x\}^2$

$=([2x]+\{2x\})([2x]-\{2x\})$

$f(x)=2x([2x]-\{2x\})=2x(2x-\{2x\}-\{2x\})$

$f(x)=2x(2x-2\{2x\})$

$f(x)=2x(2x-2\{2x\})$

$x\in(-2, 2);\quad 2x\in(-4, 4)$

Discontinuous at $2x=-3, -2, -1, 1, 2, 3$

At $x=0$ $f(x)$ is continuous.

Hence, the correct answer is (A).

22. $f:(-2, 2)\to R$

$$f(x)=\left[2^{-x^2}\left[2^{x^2}\right]\right]=\left[\frac{\left[2^{x^2}\right]}{2^{x^2}}\right] \qquad \begin{array}{c}-2<x<2\\ 0\le x^2<4\\ 1\le 2^{x^2}<16\end{array}$$

$$f(0)=\left[\frac{[2^0]}{2^0}\right]=\left[\frac{1}{1}\right]=1$$

$$f(0^-)=f(0^+)=\lim_{h\to 0}\left[\frac{[2^{h^2}]}{2^{h^2}}\right]$$

$$=\lim_{h\to 0}\left[\frac{1}{\underset{>1}{\underset{\downarrow}{2^{h^2}}}}\right]=0, \text{ discontinuous at } x=0$$

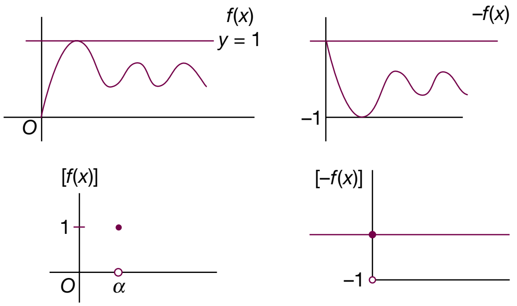
For $x\in(0, 2)$, we get $1<2^{x^2}<16$ $\quad 2, 3, \ldots, 15\to 14$ point

For $x\in(-2, 0)\to 14$ point

At $x=0$ also discontinuous, then total 29 points of discontinuity.

Hence, the correct answer is (B).

23.



One point discount $x=\alpha$ $\qquad$ One point discount $x=0$.

$f(x)=[f(x)] \qquad f(x)=[-f(x)]$

$x=0, \alpha$ at $x=0$ only

Two solution

Hence, the correct answer is (D).

24. $$f(x)=\left[\frac{x^2+1}{x^2+[\,|x|\,]+1}\right]=\begin{cases}0 & x\le -1, x\ge 1\\ 1 & -1<x<1\end{cases}$$

$$f(1)=\left[\frac{2}{1+1+1}\right]=0, f(1^-)=\left[\frac{(1-h)^2+1}{(1-h)^2+0+1}\right]=1$$

$$f(1^+)=\left[\frac{(1+h)^2+1}{(1+h)^2+0+1}\right]=0$$

Discontinuous at $x=1, -1$; two points

Hence, the correct answer is (B).

25. $$f(a^+) = \lim_{h\to 0} \sin\left(\frac{a-a-h}{2}\right) \tan\left[\frac{\pi}{2a}(a+h)\right]$$

$$= \lim_{h\to 0} \left(-\sin\frac{h}{2}\right) \tan\left[\frac{\pi}{2}+\frac{\pi}{2a}h\right] = (0)\tan 1 = 0$$

$$f(a^-) = \lim_{h\to 0} \frac{\left[\cos\frac{\pi}{2a}(a-h)\right]}{a-a+h} = \lim_{h\to 0} \frac{\left[\cos\left(\frac{\pi}{2}-\frac{\pi}{2a}h\right)\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[\sin\frac{\pi}{2a}h\right]}{h} = \frac{\text{Exact zero}}{\text{approaching zero}} = 0$$

$f(0^+) = f(a^-) = 0$

Removable discontinuity at $x = a$.

Hence, the correct answer is (B).

26. $$\lim_{x\to 0} \frac{e^{2x}-(1+4x)^{1/2}}{\left(\frac{\ln(1-x^2)}{-x^2}\right)(-x^2)}$$

$$= \lim_{x\to 0} \frac{\left(1+2x+\frac{(2x)^2}{2!}+\ldots\right)-\left(1+2x+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}(4x)^2+\ldots\ldots\right)}{1(-x^2)}$$

$$= \lim_{x\to 0} \frac{(2x^2+\ldots)+2x^2\ldots}{-x^2} = -4$$

Hence, the correct answer is (B).

27. $$f(0) = \lim_{x\to 0} \frac{e^x x\cos x - x\ln(1+x) - x}{x^2}$$

$$= \lim_{x\to 0} \frac{e^x\cos x - \ln(1+x) - 1}{x}$$

$$= \lim_{x\to 0} \frac{e^x\cos x - e^x\sin x - \frac{1}{1+x} - 0}{1} = \frac{1-0-1}{1} = 0$$

Hence, the correct answer is (A).

28. $$\text{RHL} = \lim_{x\to 0^+} \frac{\ln\left(e^{x^2}+2\sqrt{\{x\}}\right)}{\tan\frac{\sqrt{x}}{\sqrt{x}}\cdot\sqrt{x}}, \text{ put } x = 0+h$$

$$= \lim_{h\to 0} \frac{\ln(e^{h^2}+2\sqrt{h})}{\sqrt{h}}$$

$$= \lim_{h\to 0} \frac{\ln\left(1+\left(e^{h^2}+2\sqrt{h}-1\right)\right)\left(e^{h^2}+2\sqrt{h}-1\right)}{\left(e^{h^2}+2\sqrt{h}-1\right)\sqrt{h}}$$

$$= \lim_{h\to 0} 1\ \frac{e^{h^2}+2\sqrt{h}-1}{\sqrt{h}}$$

$$= \lim_{h\to 0} \left(\frac{e^{h^2}-1}{h^2}\right)\frac{h^2}{\sqrt{h}} + 2 = 1\cdot 0 + 2 = 2$$

$$\text{LHL} = \lim_{x\to 0^-} x[x]^2\log_{(1+x)} 2 \text{ put } x = 0-h$$

$$= \lim_{h\to 0} (-h)\,[-h]^2\log_{1-h}2 = \lim_{h\to 0} \frac{(-h)(-1)^2\ln 2}{\ln(1-h)}$$

$$= \lim_{h\to 0} \left(\frac{-h}{\ln(1-h)}\right)\ln 2 = 1(\ln 2) = \ln 2$$

Irremovable discont. at $x = 0$

Hence, the correct answer is (D).

29. $$f(x) = \frac{e^{\tan x}-e^x+\ln(\sec x+\tan x)-x}{\tan x - x}$$

$$= \lim_{x\to 0} \frac{e^x(e^{\tan x - x}-1)}{\tan x - x} + \frac{\ln(\sec x+\tan x)-x}{\tan x - x}$$

$$= e^0\,(1) + \lim_{x\to 0} \frac{\frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}-1}{\sec^2 x - 1};$$

$$= 1 + \lim_{x\to 0} \frac{\sec x - 1}{\sec^2 x - 1}$$

$$= 1 + \lim_{x\to 0} \frac{1}{\sec x + 1} = 1 + \frac{1}{1+1} = \frac{3}{2}$$

Hence, the correct answer is (C).

30. $$f(x) = \frac{1}{x+2^{\frac{1}{x-2}}} \quad x \neq 2$$

$$f(2^+) = f(2+h) = \lim_{h\to 0} \frac{1}{2+h+2^{1/h}} = \frac{1}{2+0+\infty} = 0$$

$$f(2^-) = f(2-h) = \lim_{h\to 0} \frac{1}{2-h+2^{-\frac{1}{h}}} = \frac{1}{2-0+0} = \frac{1}{2}$$

Non-removable discontinuity.

Hence, the correct answer is (C).

31. Given, that $f(x)$ is differentiable so, it must be continuous at $x = 1$.

$f(1+0) = f(1-0) \Rightarrow a + b = 1$

$$f'(x) = \begin{cases} 2x^2e^{2(x-1)} + e^{2(x-1)}\cdot 2x, & 0 < x < 1 \\ -2a\sin(2x-2) + 2bx, & 1 < x < 2 \end{cases}$$

$f'(1-0) = f'(1+0) \Rightarrow 4 = 2b$

$\Rightarrow\ b = 2,\ a = -1$

Hence, the correct answer is (A).

32. The given function is $f(x) = \begin{cases} \tan^{-1} x & \text{if } |x| \le 1 \\ \frac{1}{2}(|x|-1) & \text{if } |x| > 1 \end{cases}$

$$\Rightarrow\ f(x) = \begin{cases} \frac{1}{2}(-x-1) & \text{if} \quad x < -1 \\ \tan^{-1} x & \text{if} \quad -1 \le x \le 1 \\ \frac{1}{2}(x-1) & \text{if} \quad x > 1 \end{cases}$$

Clearly, L.H.L at $(x = -1) = \lim_{h\to 0} f(-1-h)$

RHL at $(x = -1) = \lim_{h\to 0} f(-1+h) = \lim_{h\to 0} \tan^{-1}(-1+h) = -\pi/4$

$\therefore$ LHL $\neq$ RHL at $x = -1$

$\therefore$ $f(x)$ is discontinuous at $x = -1$

Also, we can prove in the same way, that $f(x)$ is discontinuous at $x = 1$.

$\therefore$ $f(x)$ can not be found for $x = \pm 1$ or domain of $f'(x) = R - \{-1, 1\}$

Hence, the correct answer is (D).

33. $\{[x+\{x\}]\} = 0$ as $[x+\{x\}] \in I$

$$\text{for } x \neq 0,\ \frac{[\{|x|\}]e^{x^2}\{[x+\{x\}]\}}{\left(e^{1/x^2}-1\right)\text{sgn}(\sin x)} = 0$$

Hence, $f(x) = 0$ for all real x, it is continuous at $x = 0$

Hence, the correct answer is (A).

34. Let $\sin x = t$

$$\lim_{t\to1}\frac{t^2}{1-t^2}\left[\sqrt{2t^2+3t+4}-\sqrt{t^2+6t+2}\right] \text{ by rationalization}$$

$$=\lim_{t\to1}\frac{t^2}{1-t^2}\left(\frac{2t^2+3t+4-t^2-6t-2}{\sqrt{2t^2+3t+4}+\sqrt{t^2+6t+2}}\right)$$

$$=\lim_{t\to1}\left(\frac{t^2}{1-t^2}\right)\left(\frac{t^2-3t+2}{\sqrt{2t^2+3t+4}+\sqrt{t^2+6t+2}}\right)$$

$$=\lim_{t\to1}\frac{t^2}{(1+t)}\left(\frac{-(t-2)}{\sqrt{2t^2+3t+4}+\sqrt{t^2+6t+2}}\right)$$

$$=-\frac{1}{2}\left(\frac{-1}{3+3}\right)=\frac{1}{12}$$

Hence, the correct answer is (B).

35. $f(x) = a_1 \sin x + a_2 \sin 2x + \ldots\ldots + a_8 \sin 8x$

$$|a_1+2a_2+\ldots\ldots+8a_8| = |f'(0)| = \lim_{x\to0}\left|\frac{f(x)-0}{x}\right|$$

$$|p| = \lim_{x\to0}\left|\frac{f(x)}{\sin x}\right| \le \lim_{x\to0}\left|\frac{\sin x}{x}\right| = 1$$

$$|p| = \lim_{x\to0}\left|\frac{f(x)}{\sin x}\right| \le 1$$

$$|p| \le 1$$

Hence, the correct answer is (A).

36. $f(0^-)=0-(-1)^2=-1$ and $f(0)=0$. Hence, f is not continuous at $x=0$

$f(1^-)=0-0=0$, $f(1^+)=1-1=0$, $f(1)=0$

Thus, f is continuous at $x=1$

Hence, the correct answer is (C).

37. $f(n\pi^+)=\sec^{-1}1=0$ and $f(n\pi^-)=\sec^{-1}1=0$ and $f(n\pi)=0$

$\therefore f$ is continuous at $x=n\pi$

$f((2n-1)\frac{\pi}{2}+)=\sec^{-1}1=0$ but $f((2n-1)\frac{\pi}{2})=\sec^{-1}2=\frac{\pi}{3}$

$\therefore f$ is discontinuous at $x=(2n-1)\frac{\pi}{2}$ for all $n\in I$

Hence, the correct answer is (B).

38. Here, $1<x<3$ and in this interval x^2 is an increasing function.

$\therefore \quad 1<x^2<9$

$$[x^2]=1, 1\le x<\sqrt{2},\ [x^2]=2, \sqrt{2}\le x<\sqrt{3}$$

$$[x^2]=3, \sqrt{3}\le x<2,\ [x^2]=4, 2\le x<\sqrt{5}$$

$$[x^2]=5, \sqrt{5}\le x<\sqrt{6},\ [x^2]=6, \sqrt{6}\le x<\sqrt{7}$$

$$[x^2]=7, \sqrt{7}\le x<\sqrt{8},\ [x^2]=8, \sqrt{8}\le x<3$$

Clearly, $[x^2]$ and also $a^{[x^2]}$ is discontinuous and not differentiable at only 7 points $x=\sqrt{2},\sqrt{3},\sqrt{4},\sqrt{5},\sqrt{6},\sqrt{7},\sqrt{8}$.

Hence, the correct answer is (D).

39. $$f(x)=\lim_{n\to\infty}\frac{2^x-x^n\sin x}{1+x^n}=\begin{cases}2^x, & 0\le x<1\\ \frac{2^x-\sin x}{2}, & x=1\\ -\sin x & x>1\end{cases}$$

Now, $f(1)=\frac{2-\sin 1}{2}$; $\lim_{x\to1^-}f(x)=\lim_{x\to1^-}2^x=2$

Hence, $f(x)$ is discontinuous at $x=1$

Hence, the correct answer is (C).

40. $f(x)=\left[\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)\right]$

But $[x]$ is discontinuous only at integral points.

Also, $-\sqrt{2}\le\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)\le\sqrt{2}$

Integral values of $\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)$ when $0<x<2\pi$ are

$$\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)=-1, \text{ at } x=\pi,\frac{3\pi}{2}$$

$$\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)=0, \text{ at } x=\frac{3\pi}{4},\frac{7\pi}{4}$$

$$\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)=1, \text{ at } x=\frac{\pi}{2}$$

$\therefore$ In $(0,2\pi)$, $f(x)$ is discontinuous at $x=\frac{\pi}{2},\frac{3\pi}{4},\pi,\frac{3\pi}{2},\frac{7\pi}{4}$.

Hence, the correct answer is (C).

41. For $0<x<\frac{\pi}{2}$ *or* $\frac{\pi}{2}<x<\pi$

We have $0<\sin x<1$ $\quad\therefore$ for $n>1$, $\sin x>\sin^4 x$

$$\therefore \quad f(x)=\left[\frac{3(\sin x-\sin^n x)}{\sin x-\sin^n x}\right]=3, x\ne\frac{\pi}{2} \quad =3, x=\frac{\pi}{2}$$

Thus, in $(0,\pi)$, $f(x)=3$.

Hence, $f(x)$ is continuous and differentiable at $x=\frac{\pi}{2}$.

Hence, the correct answer is (B).

42. $[x]$ is not differentiable at integral points.

Also, $[n+p\sin x]=n+[p\sin x]$

$\therefore$ $[p\sin x]$ is not differentiable, where

$p\sin x$ is an integer. But p is prime and $0<\sin x\le1\,[\because 0<x<\pi]$

$\therefore$ $p\sin x$ is an integer only when

$\sin x=\frac{r}{p}$, where $0<r\le p$ and $r\in N$

For $r=p$, $\sin x=1 \Rightarrow x=\frac{\pi}{2}$ in $(0,\pi)$

For $0<r<p$, $\sin x=\frac{r}{p}$

$\therefore \quad x=\sin^{-1}\frac{r}{p}$ or $\pi-\sin^{-1}\frac{r}{p}$

Number of such values of $x=p-1+p-1=2p-2$

$\therefore$ Total number of points where $f(x)$ is not differentiable $=1+2p-2=2p-1$.

Hence, the correct answer is (C).

43. $$l=\lim_{x\to0}\left\{\frac{f(x)-f(0)}{x-0}+\frac{f\left(\frac{x}{2}\right)-f(0)}{x-0}+---+\frac{f\left(\frac{x}{k}\right)-f(0)}{x-0}\right\}$$

$$=\left(1+\frac{1}{2}+\frac{1}{3}+---+\frac{1}{k}\right)f'(0).$$

Hence, the correct answer is (B).

44. Replace x by $-x$

$$f(2x^2-1)=-2x^3f(-x)$$

$$2x^3f(x)=-2x^3f(-x)$$

$$\Rightarrow \quad f(-x)=-f(x)$$

$f(x)$ or odd function $f^{(n)}(0)=0$. when n is even

Hence, the correct answer is (C).

45. $t_{r+1}=\dfrac{x}{(rx+1)\{(r+1)x+1\}}$

$$=\frac{(r+1)x+1-(rx+1)}{(rx+1)[(r+1)x+1]}=\frac{1}{(rx+1)}-\frac{1}{(r+1)x+1}$$

$$\therefore\ S_n=\sum_{r=0}^{n-1}t_{r+1}=\sum_{r=0}^{n-1}\left(\frac{1}{rx+1}-\frac{1}{(r+1)x+1}\right)$$

$$=1-\frac{1}{1+x}+\frac{1}{1+x}-\frac{1}{2x+1}+\ldots+\frac{1}{(n-1)x+1}-\frac{1}{nx+1}$$

$$S_n=\frac{nx}{nx+1}=1,\ x\neq 0$$

$$S_n=0,\ x=0$$

$$\therefore \lim_{n\to\infty} S_n=\lim_{n\to\infty}\left(1-\frac{1}{nx+1}\right)$$

Thus, $f(x)$ is neither continuous nor differentiable at $x=0$.
Clearly, $f(x)$ is not a periodic function.
Hence, the correct answer is (C).

46. $\sin\pi[x+1]=0$. Also $[x+1]=[x]+1$

$$\therefore f(x)=\frac{[x]}{1+[x]}\sin\frac{\pi}{[x]+1}$$

At $x=n,\ n\in I,\ f(x)=\dfrac{n}{1+n}\sin\dfrac{\pi}{n+1}$

For $n<x<n+1,\ n\in I,\ f(x)=\dfrac{n}{1+n}\sin\dfrac{\pi}{n+1}$

For $n-1<x<n$, $[x]=n-1$ $\qquad\therefore f(x)=\dfrac{n-1}{n}\sin\dfrac{\pi}{n}$

Hence, $\displaystyle\lim_{x\to n=0} f(x)=\frac{n-1}{n}\sin\frac{\pi}{4}$,

$$f(n)=\frac{n}{1+n}\sin\frac{\pi}{n+1}$$

$\therefore$ $f(x)$ is discontinuous at all $n\in I$
Hence, the correct answer is (D).

47. If $-1<x<1$ then $f(x)=x+4$

If $1<x<3$ then $f(x)=3x+c$

But f is continuous at $x=1$

$\therefore f(1)=1+4=3+c\Rightarrow c=2$ and $f(1)=5$ $\qquad\therefore f(2)=8$

Hence, the correct answer is (C).

48. $f(0)=\displaystyle\lim_{h\to0}\frac{1-\cos(1-\cos h)}{h^4}\times\frac{1+\cos(1-\cos h)}{1+\cos(1-\cos h)}$

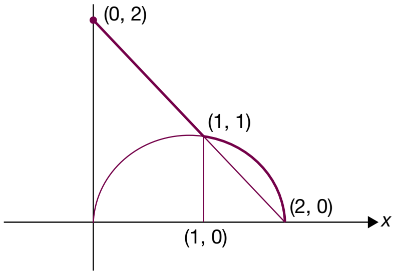
$$=\lim_{h\to0}\frac{\sin^2(1-\cos h)}{h^4.(1+\cos(1-\cos h)}\cdot\frac{(1-\cos h)^2}{(1-\cos h)^2}$$

$$=\lim_{h\to0}\left[\frac{\sin(1-\cos h)}{(1-\cos h)}\right]^2.\lim_{h\to0}\left(\frac{1-\cos h}{h^2}\right)^2.\lim_{h\to0}\frac{1}{1+\cos(1-\cos h)}$$

$$=(1)^2\times\frac{1}{4}\times\frac{1}{2}=\frac{1}{8}\ .$$

Hence, the correct answer is (A).

49.



At $x=1, 2$ not differentiable.
Hence, the correct answer is (D).

50. $\displaystyle\lim_{x\to0^+}\frac{e^{e/x}-e^{-e/x}}{e^{1/x}+e^{-1/x}}=\lim_{x\to0^+}\frac{e^{\frac{e-1}{x}}\left(1-e^{-2e/x}\right)}{\left(1+e^{-2/x}\right)}=+\infty$

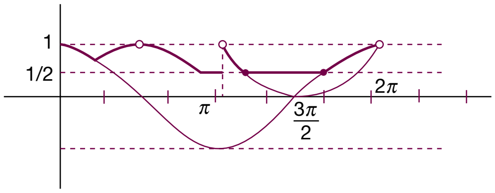
$$\lim_{x\to0^-}\frac{e^{e/x}-e^{-e/x}}{e^{1/x}+e^{-1/x}}=\lim_{x\to0^-}\frac{e^{-e/x}\left(e^{2e/x}-1\right)}{e^{-\frac{1}{x}}\left(e^{+2/x}+1\right)}$$

$$=\lim_{x\to0^-}e^{-\left(\frac{e-1}{x}\right)}\left(\frac{e^{2e/x}-1}{e^{2/x}+1}\right)=-\infty$$

Limit doesn't exist. So, $f(x)$ is discontinuous.
Hence, the correct answer is (B).

51. See figure

There are 4 points $x=\dfrac{\pi}{4},\dfrac{3\pi}{4},\dfrac{5\pi}{4},\dfrac{7\pi}{4}$



Hence, the correct answer is (D).

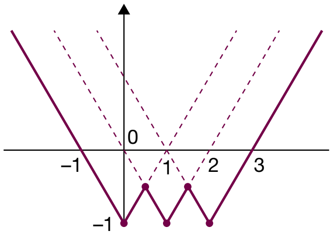
52. $\displaystyle\lim_{x\to0}f(x)=0$ {$\because \displaystyle\lim_{x\to0}x^2=0$ and $\{e^{1/x}\}$ is a bounded function}; $k=0$

$$\lim_{x\to0}\frac{f(0+x)-f(0)}{x}=\lim_{x\to0}x\{e^{1/x}\}=0$$

$\therefore\quad f'(0)=0$

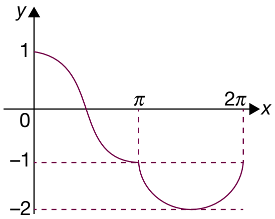
$f(x)$ is not continuous at $x=\log_2 e,\ \log_3 e,\ldots$ etc.
Hence, the correct answer is (A).

53. From the graph,



It is clear that function is non-differentiable at 0, ½, 1,3/2, 2.
Hence, the correct answer is (B).

54. $g(x)=\begin{cases}\cos x, & 0\le x\le\pi\\ \sin x-1, & x>\pi\end{cases}$



Adjacent figure represents the graph of $g(x)$.
Clearly, $g(x)$ is continuous but non-differentiable at $x=\pi$.
Hence, the correct answer is (B).

55. $f(x) = (x \bmod 2)^2 + (x \bmod 4)$

$$= \begin{cases} x^2 + x & 0 < x < 2 \\ (x-2)^2 + x & 2 \le x < 4 \\ (x-4)^2 + x - 4 & 4 \le x < 6 \\ (x-6)^2 + x - 4 & 6 \le x < 8 \\ (x-8)^2 + x - 8 & 8 \le x < 9 \end{cases}$$

$\therefore$ $x = 2, 4, 6, 8$ are points of discontinuity.

Hence, the correct answer is (C).

56. $f(x+y) = f(x) \, . \, f(y)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x)\,f(h) - f(x)}{h} = f(x) \lim_{h\to 0} \frac{f(h)-1}{h}$$

Now, $f(h) = 1 + hp(h) + h^2\, q(h)$

$$\Rightarrow f'(x) = f(x) \lim_{h\to 0} \frac{hp(h) + h^2 q(h)}{h}$$

$$= f(x) \cdot \lim_{h\to 0} p(h)$$

$$f'(x) = a\, f(x)$$

Hence, the correct answer is (A).

57. $f(x) = \begin{cases} [x] + \sqrt{\{x\}} & x < 1 \\ \dfrac{1}{[x] + \{x\}^2} & x \ge 1 \end{cases}$, Consider the function $f(x)$ in the interval (0, 2).

$$f(x) = \begin{cases} \sqrt{x} & 0 < x < 1 \\ \dfrac{1}{1 + (x-1)^2} & 1 \le x < 2 \end{cases}$$

$$f(1) = 1 \quad \lim_{x\to 1} f(x) = 1$$

Continuous at $x = 1$

$$\lim_{h\to 0_+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0_+} \frac{\frac{1}{1+h^2} - 1}{h} = \lim_{h\to 0_+} \frac{-h}{1+h^2} = 0$$

$$\lim_{h\to 0_+} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0_+} \frac{\sqrt{1-h} - 1}{-h}$$

$$= \lim_{h\to 0_+} \frac{1}{\sqrt{1-h} + 1} = \frac{1}{2}$$

$\therefore$ $f(x)$ is not differentiable at $x = 1$

Hence, the correct answer is (A).

58. $$\lim_{h\to 0} \frac{f(2+2h+h^2) - f(2)}{(2h+h^2)} \cdot \lim_{h\to 0} \frac{(h^2+h)}{f(1+h^2+h) - f(1)} \cdot \lim_{h\to 0} \frac{2h+h^2}{h^2+h}$$

$$= \frac{f'(2)}{f'(1)} . 2 = \frac{6 \times 2}{4} = 3$$ [Or use L'Hospital's rule]

Hence, the correct answer is (A).

59. $f(0) = g(0) = 0$

$$f'(0) = \lim_{h\to 0} \frac{g(0+h)\cos(1/h) - 0}{h}$$

$$= \lim_{h\to 0} \frac{g(h)\cos(1/h)}{h} = \lim_{h\to 0} g'(0)\cos(1/h) = 0$$

$$g(x) = -g'(-x) \Rightarrow g'(0) = 0$$

Hence, the correct answer is (B).

60. By theorem, if g and h are continuous functions on the open interval (a, b), then g/h is also continuous at all x in the open interval (a, b) where $h(x)$ is not equal to zero.

Hence, the correct answer is (D).

61. $$\lim_{x\to 0} \frac{x - e^x + \cos 2x}{x^2}$$

$$= \lim_{x\to 0} \frac{x - 1 - x - \frac{x^2}{2!} - \frac{x^3}{3!} - .... + 1 - (1 - \cos 2x)}{x^2}$$

$$= -\frac{1}{2} - 2 = -\frac{5}{2}; \text{ hence, for continuity } f(0) = -\frac{5}{2}$$

$$\therefore \quad [f(0)] = -3; \ \{f(0)\} = \left\{-\frac{5}{2}\right\} = \frac{1}{2};$$

Hence, $[f(0)]\,\{f(0)\} = -\dfrac{3}{2} = -1.5$

Hence, the correct answer is (D).

62. $g(x+y) = e^y g(x) + e^x g(y) \,\&\, g(5) = 32, g'(0) = 2$

$x = y = 0$

$g(0) = g(0) + g(0)$; $g'(x) = \lim\limits_{h\to 0} \dfrac{g(x+h) - g(x)}{h}$

$g(0) = 0$

$$g'(x) = \lim_{h\to 0} \frac{e^h g(x) + e^x g(h) - g(x)}{h}$$

$$g'(x) = g(x)\left(\lim_{h\to 0} \frac{e^h - 1}{h}\right) + e^x \lim_{h\to 0} \frac{g(h)}{h}$$

$$g'(x) = g(x) + e^x g'(0) = g(x) + e^x \cdot 2$$

$$g'(5) = g(5) + 2e^5 \Rightarrow g'(5) - 2e^5 = g(5) = 32$$

Hence, the correct answer is (C).

63. RHL Let $2^x = t$; $t \to 4$

$$\text{RHL} = \lim_{t\to 4} \frac{t + \frac{8}{t} - 6}{\sqrt{1/t - 2/t}} = \lim_{t\to 4} \frac{(t-4)(t-2)}{\left(\sqrt{t} - 2\right)} = 8$$

$$\text{LHL} = \lim_{x\to 2^-} \frac{(x^2-4)}{\left(x - \sqrt{3x-2}\right)} = \lim_{x\to 2} \frac{(x-2)(x+2)\left(x + \sqrt{3x-2}\right)}{(x^2 - 3x + 2)}$$

$$= \lim_{x\to 2^-} \frac{(x+2)\left(x + \sqrt{3x-2}\right)}{(x-1)} = 4 \times 4 = 16$$

$$f(2^+) = 8; f(2^-) = 16$$

Hence, the correct answer is (C).

64. $\because$ $g'(x) = f''(x) = -f(x)$

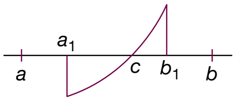
$$h''(x) = 2f(x)\,f'(x) + 2g(x)\,g'(x) = 2f(x)\,f'(x) - 2f'(x)f(x) = 0$$

$$h'(x) = C; \quad h(x) = Cx + C_1; \quad h(x) = 2x + 2$$

Hence, the correct answer is (C).

65. I and II are false.

The function $f(x) = 1/x$, $0 < x < 1$, is a counter example.



Statement III is true.

Apply the intermediate value theorem to f on the closed interval $[a_1, b_1]$.

Hence, the correct answer is (D).

66. $f(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases} \Rightarrow f$ is discontinuous for all $x \in I$

Hence, the correct answer is (C).

67. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} \Rightarrow \cot 2\theta = \dfrac{1-\tan^2\theta}{2\tan\theta}$

$\Rightarrow \cot 2\theta = \dfrac{\cot\theta - \tan\theta}{2}$

$\tan\theta = \cot\theta - 2\cot 2\theta$

$$f(x) = \lim_{n\to\infty} \sum_{n=1}^{n} \frac{1}{2^n}\left(\cot\frac{x}{2^n} - 2\cot\frac{x}{2^{n-1}}\right)$$

$$= \sum_{n=1}^{n} \left(\frac{1}{2^n}\cot\frac{x}{2^n} - \frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}\right)$$

$$= \left(\frac{1}{2}\cot\frac{x}{2^1} - \frac{1}{2^0}\cot\frac{x}{2^0}\right) + \left(\frac{1}{2^2}\cot\frac{x}{2^2} - \frac{1}{2}\cot\frac{x}{2^1}\right) + \ldots\ldots\ldots + \left(\frac{1}{2^n}\cot\frac{x}{2^n} - \frac{1}{2^{n-1}}\cot\frac{n}{2^{n-1}}\right)$$

$$f(x) = \lim_{n\to\infty} \frac{1}{2^n}\cot\frac{x}{2^n} - \cot x\,;$$

$$f(x) = \lim_{n\to\infty} \frac{1}{x}\left(\frac{x/2^n}{\tan x/2^n}\right) - \cot x$$

$$f\left(\pi/2^+\right) = f\left(\pi/2^-\right) = \frac{1}{\pi/2}(1) - 0 = \frac{2}{\pi}\,;$$

$$f\left(\pi/2\right) = 2/\pi$$

Continuous at $x = \dfrac{\pi}{2}$

Hence, the correct answer is (C).

68. $m_{PQ} = \dfrac{r-2}{s-1} = \dfrac{s^2+2s-3}{s-1}$

$r = s^2 + 2s - 1$

$f(s) = s^2 + 2s - 1 \Rightarrow f'(s) = 2s + 2$

$f'(1) = 4$

Hence, the correct answer is (C).

69. $h(x) = (x-a)^2 f(x)$ at $x = a$ continuous and differentiable

$$h(a^+) = \lim_{h\to 0} h^2 f(a^+) = 0$$

$$h(a^-) = \lim_{h\to 0} h^2 f(a^-) = 0,\, h(a) = 0$$

$$h'(a^+) = \lim_{h\to 0} \frac{h^2 f(a+h) - 0}{h} = 0$$

$$h'(a^-) = \lim_{h\to 0} \frac{h^2 f(a-h) - 0}{-h} = 0$$

$f(a) = 0$

$g(x) = (x-a)f(x)$ is continuous but not differentiable at $x = a$

$$g(a^+) = \lim_{h\to 0} hf(a+h) = 0$$

$$g(a^-) = \lim_{h\to 0} -hf(a-h) = 0,\, g(a) = 0$$

$$g'(a^+) = \lim_{h\to 0} \frac{hf(a+h) - 0}{h} = \lim_{h\to 0} f(a+h) = f(a^+)$$

$$g'(a^-) = \lim_{h\to 0} \frac{-hf(a-h)}{-h} = \lim_{h\to 0} f(a-h) = f(a^-)$$

As, $g(x)$ is not differentiable then $f(a^+) \neq f(a^-)$ which means $f(x)$ is not continuous at $x = a$.

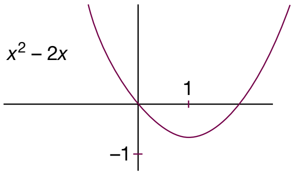
Hence, the correct answer is (D).

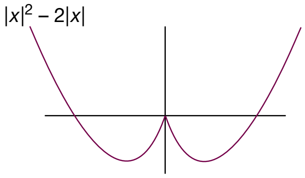
70. $g(x) = [x^2]\left(\left\{\cos^2 4x\right\} + \left[\cos^2 4x\right]\right) + \left\{x^2\right\}\left(\left[\cos^2 4x\right]\right) + \left\{\cos^2 4x\right\} + x^2\sin^2 4x = \cos^2 4x\left(\left[x^2\right] + \left\{x^2\right\}\right) + x^2\sin^2 4x$

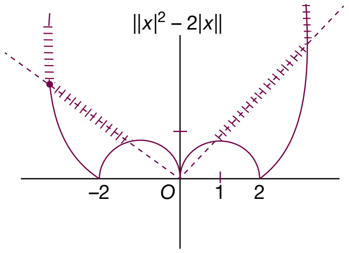
$g(x) = x^2$ always differentiable for all x

Hence, the correct answer is (D).

71. $\left|x^2 - 2|x|\right| = \left||x|^2 - 2|x|\right|$







5 points of non-differentiability $x = -3, -1, 0, 1, 3$

Hence, the correct answer is (D).

72. $f(-x) = -f(x)$ it is an odd function

Atleast three roots of $f(x) = 0$.

Hence, the correct answer is (D).

73. $f(1) = 0$

For $x > 1$, $f(x) = \lim\limits_{n\to\infty} \dfrac{ax(x-1)\left(\cot\frac{\pi x}{4}\right)^n + (px^2+2)}{\left(\cot\frac{\pi x}{4}\right)^n + 1} = px^2 + 2$

$$\left(\because \cot\frac{\pi^+}{4} < 1\right)$$

For $x < 1$, $f(x) = \lim\limits_{n\to\infty} \dfrac{ax(x-1) + \dfrac{(px^2+2)}{\left(\cot\frac{\pi x}{4}\right)^n}}{1 + \dfrac{1}{\left(\cot\frac{\pi x}{4}\right)^n}}$

$$\left(\because \cot\frac{\pi^+}{4} > 1\right) = ax(x-1)$$

$$f(x) = \begin{cases} px^2 + 2 \text{ for } x \in (1,2) \\ 0 \quad x = 1 \\ ax(x - 1) \text{ for } x \in (0,1) \end{cases}$$

Continuous at $x = 1$, we have $p + 2 = 0$, we get $p = -2$

Differentiable at $x = 1$, $2p = 2a - a$

We get, $a = -4$

Now, $a^2 + p^2 = 20$

Hence, the correct answer is (B).

74. $f'(0) = \lim\limits_{h\to 0} \dfrac{e^{-1/h^2}}{h} = \lim\limits_{h\to 0} \dfrac{1/h}{e^{1/h^2}}$

$$= \frac{-1/h^2}{-e^{1/h^2}\cdot\dfrac{2}{h^3}} = \lim_{h\to 0} \frac{2h^3}{h^2 e^{1/h^2}} = 0$$

Hence, f is differentiable at $x = 0$. Also $\lim\limits_{x\to\pm\infty} e^{-\frac{1}{x^2}} \to 1$

Alternatively:

Check concavity by finding $\dfrac{d^2y}{dx^2}$ and eliminate D.

Hence, the correct answer is (C).

75. $f(x) = \big||\sin(|x|-1)|-2\big| = 2-|\sin(|x|-1)|$

Around $x = 2$, $\sin(|x|-1) = \sin(x-1)$

And $\sin(x-1) > 0 \Rightarrow |\sin(x-1)| = \sin(x-1)$

we know $\sin(x-1) < 2$

Hence, $|\sin(x-1)-2| = 2-\sin(x-1)$

$f(x) = 2 - \sin(x-1)$ for $x \in (2-\delta, 2+\delta)$

$f'(2) = -\cos 1$

Hence, the correct answer is (D).

76. As 'x' approaches to zero $\sin x^2 \simeq x^2$

Limit $= \lim\limits_{x\to 0} \dfrac{2f'(x)-6f'(2x)+4f'(4x)}{2x}$ (Using L. H. rule)

$$\lim_{x\to 0} \frac{2f''(x)-12f''(2x)+16f''(4x)}{2} = 3f''(0) = 12$$

Hence, the correct answer is (D).

77. $f(x) = \begin{cases} 1+x^2, & x > 0 \\ 0, & x = 0 \\ -(1+x^2), & x < 0 \end{cases}$

$$f^{-1}(x) = \begin{cases} \sqrt{x-1}, & x \in [1,\infty] \\ 0, & x = 0 \\ -\sqrt{-x-1}, & x \in (-\infty,-1] \end{cases}$$

Clearly, $f^{-1}(x)$ is continuous on $[1,\infty)$ and on $(-\infty,-1]$, and as $x = 0$ is isolated point of its domain the function is continuous at that point as well.

Hence, the correct answer is (C).

78. $g(x) = \tan^{-1}|x| - \cot^{-1}|x| = 2\tan^{-1}|x| - \pi/2$

$h(x) = \left|2\tan^{-1}\left|\dfrac{[x]}{[x+1]}\{x\}\right| - \dfrac{\pi}{2}\right|$, Domain: $R - [-1, 0)$,

$h(0) = \dfrac{\pi}{2} = h(0^+)$

Hence, the correct answer is (A).

79. $f(x) = \begin{cases} x-2 \text{ if } x \ge 1 \\ -x \text{ if } x < 1 \end{cases}$, $g(x) = \begin{cases} 1-x \text{ if } x \ge -1 \\ x+3 \text{ if } x < -1 \end{cases}$

$$gof(x) = \begin{cases} 1-f(x) \text{ if } f(x) \ge -1 \\ f(x)+3 \text{ if } f(x) < -1 \end{cases}$$

$$= \begin{cases} 1-(x-2) \text{ if } x-2 \ge -1 \text{ and } x \ge 1 \\ 1-(-x) \text{ if } -x \ge 1 \text{ and } x < 1 \\ x-2+3 \text{ if } x-2 < -1 \text{ and } x \ge 1 \\ -x+3 \text{ if } -x < -1 \text{ and } x < 1 \end{cases}$$

$$\Rightarrow g(f(x)) = \begin{cases} 3-x \text{ if } x \ge 1 \\ 1+x \text{ if } x < 1 \end{cases}$$

Hence, the correct answer is (D).

80. $f(x) = \begin{cases} \sin\left(\sin^{-1}\sqrt{1-x^2}\right) + \cos\left(\cos^{-1}\sqrt{1-x^2}\right), & x \le 0 \\ \sin\left(\sin^{-1}\sqrt{1-x^2}\right) - \cos\left(\cos^{-1}\sqrt{1-x^2}\right), & x > 0 \end{cases}$

$$f(x) = \begin{cases} 2\sqrt{1-x^2}, & x \le 0 \\ 0 \quad , & x > 0 \end{cases}$$

$f(x)$ is neither differentiable not continuous at $x = 0$

Hence, the correct answer is (D).

81. $t > 0 \Rightarrow x = 3t \;\&\; y = (t-1)^2 \Rightarrow y = \left(\dfrac{x}{3}-1\right)^2, x > 0$

$t < 0 \Rightarrow x = t \;\&\; y = (t+1)^2 \Rightarrow y = (x+1)^2, x < 0$

$t = 0 \Rightarrow x = 0, y = 1$, point $(0,1)$ lies on it

$$y = \begin{cases} \left(\dfrac{x}{3}-1\right)^2, \text{if } x > 0 \\ 1, \quad \text{if } x = 0 \\ (x+1)^2, \text{if } x < 0 \end{cases}$$

Clearly, continuous every where but not differentiable at $x = 0$

Hence, the correct answer is (B).

82. Since, $\lim\limits_{x\to 3} f(x) = 1$

$$\therefore \lim_{x\to 3} \frac{\ln(1+3(f(x)-1))}{-2(f(x)-1)} = -\frac{1}{2}\lim_{x\to 3} \frac{\ln(1+3(f(x)-1))}{3(f(x)-1)} \cdot (3)$$

$$= -\frac{3}{2}$$

Hence, the correct answer is (C).

83. $f(x) = \begin{cases} [x]+\sqrt{\{x\}} & x < 1 \\ \dfrac{1}{[x]+\{x\}^2} & x \ge 1 \end{cases}$, Consider the function $f(x)$ in the interval (0, 2)

$$f(x) = \begin{cases} \sqrt{x} & 0 < x < 1 \\ \dfrac{1}{1+(x-1)^2} & 1 \le x < 2 \end{cases}$$

$f(1) = 1 \quad \lim\limits_{x\to 1} f(x) = 1$ continuous at $x = 1$

$$\lim_{h\to 0_+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0_+} \frac{\dfrac{1}{1+h^2}-1}{h} = \lim_{h\to 0_+} \frac{-h}{1+h^2} = 0$$

$$\lim_{h\to 0_+} \frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0_+} \frac{\sqrt{1-h}-1}{-h}$$

$$= \lim_{h\to 0_+} \frac{1}{\sqrt{1-h}+1} = \frac{1}{2}$$

$\therefore$ $f(x)$ is not differentiable at $x = 1$

Hence, the correct answer is (A).

84. $f(x) = \text{sgn}(\sin^2 x - \sin x - 1)$ is discontinuous.

When $\sin 2x - \sin x - 1 = 0$

$\sin x = \dfrac{1 \pm \sqrt{5}}{2}$; $\sin x = \dfrac{1-\sqrt{5}}{2}$ is possible

$$\sin x = \frac{1+\sqrt{5}}{2} \text{ is acceptable}$$

For exactly four points of discontinuous, n can take value 4 or 5.
Hence, the correct answer is (A).

85. $|x-1|, |x-1|^2$ etc are all continuous everywhere and the algebraic sum of continuous function is also continuous. $|x-1|, |x-1|^3$, etc., are not differentiable at $x=1$ whereas $|x-1|^2, |x-1|^4$, etc., are all differentiable at $x=1$.
Hence, the correct answer is (C).

86. $f\left(5^-\right) = \lim\limits_{x\to 5^-} \sin\left(\frac{\pi}{2}(x-[x])\right) = \lim\limits_{x\to 5^-} \sin\left(\frac{\pi}{2}(x-4)\right) = 1$

$$f\left(5^+\right) = \lim_{x\to 5^+} \frac{ab^2\left|x^2-11x+24\right|}{x-3}$$

$$= \lim_{x\to 5^+} \frac{-ab^2(x-3)(x-8)}{x-3} = 3ab^2$$

$$\Rightarrow 5(b-1) = 1 = 3ab^2 \quad \Rightarrow b = \frac{6}{5},\ a = \frac{25}{108}.$$

Hence, the correct answer is (A).

87. $f(x) = \lim\limits_{n\to\infty} \frac{\left(x^2+ax+1\right)+x^{2n}\left(2x^2+x+b\right)}{1+x^{2n}}$

$$f(x) = \begin{cases} 2x^2+x+b & \text{for} \quad x<-1 \\ \dfrac{-a+b+3}{2} & \text{for} \quad x=-1 \\ x^2+ax+1 & \text{for} \quad -1<x<1 \\ \dfrac{a+b+5}{2} & \text{for} \quad x=1 \\ 2x^2+x+b & \text{for} \quad x>1 \end{cases}$$

$f(x)$ is continuous at $x=-1$ and at $x=1$

$\Rightarrow\ a+b=1$ and $a-b=1$

$\Rightarrow\ a=1, b=0$

Hence, the correct answer is (A).

88. At $x=0$

$$f'(0) = \lim_{x\to 0} \frac{x^p \sin\left(\frac{1}{x}\right)+x|x|}{x-0} = \lim_{x\to 0} x^{p-1} \sin\left(\frac{1}{x}\right)+|x|$$

If $P=1 \Rightarrow\ f'(0) = \lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right)+|x| = \sin(\infty)$ undefined

$\therefore$ At $P=1, f(x)$ is not differentiable.

Hence, the correct answer is (C).

89. $f(x) = x-[x] \Rightarrow f(x) = \{x\}$

$0 \le \{x\} < 1 \Rightarrow$ as $n\to\infty \Rightarrow \{f(x)\}^{2n} \to 0$

$\therefore g(x) = -1$ constant function.

Hence, the correct answer is (A).

90. $f(e) = [\log_e e]+\sqrt{\{\log_e e\}} = [1]+\sqrt{\{1\}} = 1+0 = 1$

$$f(e^+) = [\log_e e^+]+\sqrt{\{\log_e e^+\}} = \lim_{h\to 0}[1+h]+\sqrt{\{1+h\}} = 1+0 = 1$$

$$f(e^-) = [\log_e e^-]+\sqrt{\{\log_e e^-\}} = \lim_{h\to 0}[1-h]+\sqrt{\{1-h\}} = 0+1 = 1$$

Hence, $f(x)$ is continuous at $x=e$

Now, $=[\log_e e^-]+\sqrt{\{\log_e e^-\}}$

$$= \lim_{h\to 0}[1-h]+\sqrt{\{1-h\}} = 0+1 = 1$$

$$f'(e^-) = \lim_{h\to 0} \frac{[\log_e(e+h)]+\sqrt{\{\log_e(e+h)\}}-1}{h}$$

$$f'(e^-) = \lim_{h\to 0} \frac{1+\sqrt{\log_e(e+h)-1}-1}{h}$$

$$f'(e^-) = \lim_{h\to 0} \sqrt{\frac{\log_e(1+h/e)}{h^2}} \to \infty$$

Hence, $f(x)$ is non-differentiable at $x=e$
Hence, the correct answer is (A).

91. $y = f(x)-f(2x) \Rightarrow y' = f'(x)-2f'(2x)$

$\Rightarrow y'(1) = f'(1)-2f'(2) = 5$, and
$y'(2) = f'(2)-2f'(4) = 7$

Now, let $y = f(x)-f(4x)$

$\Rightarrow y' = f'(x)-4f'(4x)$
$\Rightarrow y'(1) = f'(1)-4f'(4)$

Substituting the value of $f'(2) = 7+2f'(4)$ …(1)

$f'(1)-2(7+2f'(4)) = 5$
$f'(1)-4f'(4) = 19$

Hence, the correct answer is (A).

92. $f(x) = \begin{cases} 1-4x^2, & 0 \le x < \frac{1}{2} \\ 4x^2-1, & \frac{1}{2} \le x < 1 \\ -1, & 1 \le x < 2 \end{cases}$

$$f'\left(\frac{1^-}{2}\right) = (-4)\, 2 \times \left(\frac{1}{2}\right) = -4;$$

$$f'\left(\frac{1^+}{2}\right) = 4 \times \left(2 \times \frac{1}{2}\right) = 4$$

LHD at $x = \frac{1}{2}$ is –4, RHD at $x = \frac{1}{2}$ is 4

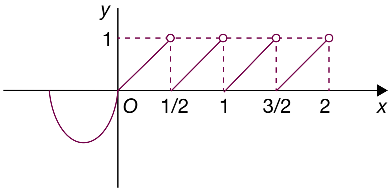
$f(x)$ is not differentiable at $x = \frac{1}{2}$

$$f'\left(\frac{1^-}{2}\right) = 4(2\times 1) = 8$$

$f'(1^+) = 0$
LHD at $x=1$ is not defined $\Rightarrow$ not differentiable at $x=1$
Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1.



Discontinuous at $x = \frac{1}{2}, 1, \frac{3}{2}, 2$

Continuous and non–differentiable at $x=0$
Hence, (B) and (D) are correct.

2. $\max\left(x, \frac{1}{x}\right) = \begin{cases} x, & x \in [1,\infty) \\ \frac{1}{x}, & x \in (0,1) \\ x, & x \in [-1,0) \\ \frac{1}{x}, & x \in (-\infty,-1) \end{cases}$

And $\min\left(x, \frac{1}{x}\right) = \begin{cases} \frac{1}{x}, & x \in [1, \infty) \\ x, & x \in (0, 1) \\ \frac{1}{x}, & x \in [-1, 0) \\ x, & x \in (-\infty, -1) \end{cases}$

Now, we get,

$$g(x) = \begin{cases} x^2, & x \in [1, \infty) \\ \frac{1}{x^2}, & x \in (0, 1) \\ x^2, & x \in [-1, 0] \\ \frac{1}{x^2}, & x \in (-\infty, -1) \\ 1, & x = 0 \end{cases} = \begin{cases} x^2, & x \in [-1, 0] \cup [1, \infty) \\ \frac{1}{x^2}, & x \in (-\infty, -1) \cup (0, 1) \\ 1, & x = 0 \end{cases}$$

$$\lim_{x \to 0^+} g(x) \to \infty\ ;\ \lim_{x \to 0^-} g(x) = 0$$

$g(x)$ is continuous for all x except at $x = 0$

$g(x)$ is differentiable for all x except at $x = 0, -1, 1$

Hence, (B), (C) and (D) are correct.

3. $f(1) = \lim_{x \to 1} \frac{\sin 3\pi x + A \sin 5\pi x + B \sin \pi x}{(x-1)^5}$ put $x = 1 + h$

$$= \lim_{h \to 0} \frac{-\sin 3\pi x - A \sin 5\pi h - B \sin \pi h}{h^5}$$

$$= \lim_{h \to 0} - \frac{\left[(3\pi h) - \frac{(3\pi h)^3}{3!} + \frac{(3\pi h)^5}{5!}\right] + A\left[\frac{5\pi h}{1} - \frac{(5\pi h)^3}{3!} + \frac{(5\pi h)^5}{5!} + \ldots\right] + B\left(\pi h - \frac{(\pi h)^3}{3!} + \frac{(\pi h)^5}{5!} \ldots\right)\Bigg]}{h^5}$$

$$= \lim_{h \to 0} - \frac{\left[\pi h(3 + 5A + B) - \frac{\pi^3 h^3}{3!}(27 + 5^3 A + B) - \frac{\pi^5 h^5}{5!}(3^5 + 5^5 A + B)\right]}{h^5}$$

For limit to exist $3 + 5A + B = 0 \,\&\, 27 + 125A + B = 0$, we get

$$A = -\frac{1}{5}, B = -2$$

Required limit $= -\frac{\pi^5}{5!}(243 + 5^5 A + B) = -\frac{\pi^5}{5!}(243 - 625 - 2)$

$$= -\frac{\pi^5}{120}(-384) = \frac{16\pi^5}{5}$$

Hence, (A) and (C) are correct.

4. (A) $f = [x]$ $\quad g = \{x\}$

$f + g = x$ continuous but f, g are discontinuous at x = integer (Incorrect)

(B) $\lim_{x \to 0} x \cdot \frac{1}{x} = 1$

$\lim_{x \to 0} x = 0$ and $\lim_{x \to 0} \frac{1}{x} \to$ DNE (Incorrect)

(C) discontinuity at $x = a$, may be removable type discontinuity (Incorrect)

(D) $f(x) = \{x\}$ where $\{.\}$ is fractional part function $x \in [0, 2]$

Range [0,1)

Maximum not defined.

Hence, (A), (B), (C) and (D) are correct.

5. (A) $f(x) = \{x\}$ $\quad x \in [0, 2]$

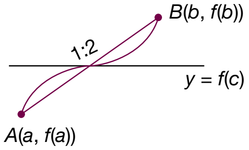
$f(x) \in [0, 1)$ bounded function but discontinuous.

$(x, \phi(x))$

(B) $f(x)$ cont. $f(a) \neq f(b)$

$f(c) = \frac{2f(a) + f(b)}{3}$

so (exist $c \in (a, b)$)



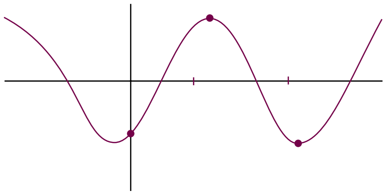
(C) $f(x) = 2x + 3\cos x$ cont.

Range $\in (-\infty, \infty)$

Now, by IMT

$f(c) = 10$ for some $c \in R$

(D) $f(x) = 5x^8 + ax^5 + bx^2 - cx - 10$



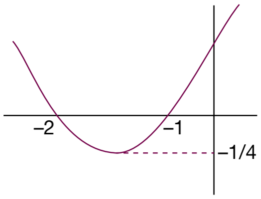
Atleast four real root.

Hence, (B), (C) and (D) are correct.

6. $f(0^-) = f(0^+) = f(0) \Rightarrow a = 2$

$f(0^-) = f'(0^+) \Rightarrow b = 3$

Graph of $f(x)$



Hence, (A) (B), (C) and (D) are correct.

7. Since, $0 \leq 2\sin x < 1$ for $0 \leq x < \frac{\pi}{6}$, $f(x) = \begin{cases} 0 & 0 \leq x \leq \frac{\pi}{6} \\ 1 & x = \frac{\pi}{6} \end{cases}$ Clearly,

$f(x)$ has removable discontinuity at $x = \frac{\pi}{6}$ and $f(x)$ is continuous in $\left(0, \frac{\pi}{6}\right)$.

Hence, (C) and (D) are correct.

8. $f(x) = \sqrt{x^2} + [x]^2 = |x| + [x]^2$ $\qquad x \in [-2, 2]$

$$f(x) = |x| + [x]^2 = \begin{cases} -x + (-2)^2 & -2 \leq x < -1 \\ -x + (-1)^2 & -1 \leq x < 0 \\ x + (0)^2 & 0 \leq x < 1 \\ x + (1)^2 & 1 \leq x < 2 \\ 2 + 2^2 & x = 2 \end{cases}$$

$$= \begin{cases} 4 - x & -2 \leq x < -1 \\ 1 - x & -1 \leq x < 0 \\ x & 0 \leq x < 1 \\ x + 1 & 1 \leq x < 2 \\ 6 & x = 2 \end{cases}$$

Discontinuous at $x = -1, 0, 1, 2$

Range $y \in (5, 6] \cup (1, 2] \cup [0, 1) \cup [2, 3)$

$y \in [0, 1) \cup (1, 3) \cup (5, 6]$

Integers 0, 2, 6

$f(-2) = f(2) = 6$ Many-one, non-periodic, neither even nor odd function.

Hence, (A), (B) and (D) are correct.

9. $$f(x) = \lim_{n\to\infty} \begin{cases} \cos^{2n} x & x<0 \\ \sqrt[n]{1+x^n} & 0 \le x \le 1 \\ \dfrac{1}{1+x^n} & x>1 \end{cases}$$

$$f(0) = 1, f(0^+) = \lim_{n\to\infty}\ \lim_{h\to 0} (1+h^n)^{1/n}$$

$$= \lim_{n\to\infty}\ \lim_{h\to 0} \left(1 + \frac{1}{n}\cdot h^n + ...\right)$$

$$= 1 + 0 = 1$$

$$f(0^-) = \lim_{x\to 0^-}\ \lim_{n\to\infty} (\cos(-h))^{2n} \text{ put } x = 0 - h$$

$$= \lim_{n\to\infty}\ \lim_{h\to 0} \underset{<1}{\underset{\downarrow}{(\cosh)}}{}^{2n} = 0, \text{ discontinuous at } x = 0$$

$$f(1) = (1+1^n)^{1/n} = \lim_{n\to\infty}(2)^{1/n} = 2'' = 1$$

$$f(1^-) = \lim_{n\to\infty}\lim_{h\to 0}(1+(1-h)^n)^{1/n} = (1)^{1/n} = 1$$

$$f(1^+) = \lim_{n\to\infty}\lim_{h\to 0}\frac{1}{1+(1+h)^n} = \frac{1}{1+\infty} = 0$$

Discontinuous at $x = 1$

Hence, (A), (B) and (C) are correct.

10. f continuous in $x \in [a, b]$ $\qquad f(b^-) = f(b)$

g continuous in $x \in [b, c]$ $\qquad g(b) = g(b^+)$

$$h(x) = \begin{cases} f(x) & x \in [a,b) \\ g(x) & x \in (b,c] \end{cases} \qquad f(b) = g(b) = f(b^-) = g(b^+)$$

$h(b^-) = f(b^-) = g(b^+)$; $\qquad h(b^+) = g(b^+) = f(b^-)$

$h(x)$ is not defined at $x = b$

So, removable type of discount at $x = b$.

Hence, (A) and (C) are correct.

11. $$f(x) = \begin{cases} 3-2x & x<1 \\ 1 & 1\le x<2 \\ 2x-3 & 2\le x\le 3 \end{cases} \qquad g(x) = \begin{cases} 3-2x & x<1 \\ 1 & 1\le x\le 3 \\ x-2 & x>3 \end{cases}$$

$g(x)$ is not differentiable at $x = 1$ and 3.

Hence, (A) and (C) are correct.

12. $f(e^{-e^{100}}) = 4.$

$$f(1) = a = f(1^+) = c + d \Rightarrow a = c + d$$

$$f(-1) = a = f(-1^+) = ce^4 + d \Rightarrow a = e^4c + d$$

$$a - d = c = e^4c.$$

$$a = d \text{ and } c = 0$$

At $x = 1$ RHD = LHD

$$\lim_{h\to 0}\frac{ce^{h^2} + d - a}{h} = \lim_{h\to 0}\frac{a + b\sqrt{2h-h^2} - a}{-h}$$

$$0 = \lim_{h\to 0} -b\sqrt{\frac{2}{h} - h} \qquad b = 0$$

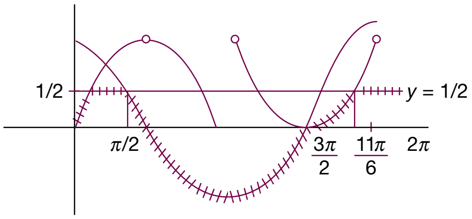
Now, $$f(x) = \begin{cases} a & |x| \le 1 \\ d & |x| > 1 \end{cases}$$

$$f(e^{-e^{100}}) = 4 = a = d$$

$$b = c = 0$$

Hence, (B) and (C) are correct.

13. $$f(x) = \min\left(\cos x, \frac{1}{2}, \{\sin x\}\right) \qquad 0 < x < 2\pi$$



not differentiable at $x = \dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$

$$\sum_{i=1}^{n} x_i = \frac{\pi}{6} + \frac{\pi}{3} + \frac{3\pi}{2} + \frac{11\pi}{6} = \frac{23\pi}{6} = \frac{\lambda\pi}{6}$$

$\lambda = 23$

Hence, (A), (B) and (C) are correct.

14. $f(x+y) - f(x) = f(y) + (e^x - 1)(e^y - 1) \qquad \forall x, y \in R$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \qquad f'(0) = 2$$

$$= \lim_{h\to 0}\frac{f(h) + (e^x - 1)(e^h - 1)}{h}$$

Put $x = 0; y = 0$ we get $f(0) = 0$

$$= \lim_{h\to 0}\frac{f(h)}{h} + (e^x - 1)\lim_{h\to 0}\frac{e^h - 1}{h}$$

$$f'(x) = f'(0) + e^x - 1; \ f(x) = xf'(0) + e^x - x + c$$

$$f(0) = 0 = 0 + 1 - 0 + c \Rightarrow c = -1;$$

$$f(x) = e^x + x - 1$$

$$\lim_{x\to 0}\frac{f(f(x))}{f(x) - x} = \lim_{x\to 0}\frac{e^{f(x)} + f(x) - 1}{\left(\dfrac{e^x - 1}{x}\right)x} = \lim_{x\to 0}\frac{e^{f(x)} + f(x) - 1}{f(x)}\cdot\frac{f(x)}{x}$$

$$= \lim_{x\to 0}\frac{1 + f(x) + \dfrac{(f(x))^2}{2!} + \cdots + f(x) - 1}{f(x)}\cdot\frac{(e^x + x - 1)}{x}$$

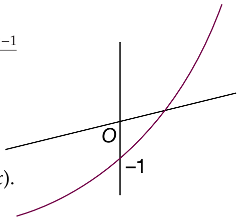
$$= \lim_{x\to 0}\left(2 + \frac{(f(x))}{2!} + \cdots\right)\left(\frac{e^x + 1}{1}\right) = 2(2) = 4$$

$$\lim_{x\to 0}(f(x) + \cos x)^{\frac{1}{e^x - 1}} = \lim_{x\to 0}(e^x + x - 1 + \cos x)^{\frac{1}{e^x - 1}}$$

$$= e^{\lim\limits_{x\to 0}\frac{(e^x + x - 1 + \cos x - 1)}{e^x - 1}} = e^{\lim\limits_{x\to 0}\frac{e^x - 1}{x} + \frac{x}{x} + \frac{\cos x - 1}{x}}$$

$$= e^{1+1+0} = e^2$$

$$f(x) = e^x + x - 1 = 0$$



$f'(x) = e^x + 1 > 0$ SMI only one of zero $f(x)$.

Range of $f(x)$ R

Hence, (A), (B) and (D) are correct.

15. $\displaystyle\lim_{x\to 0}\frac{g(x)(\cos x - 1)}{\sin x}$ apply L'Hospital's rule

$$\lim_{x\to 0}\frac{g(x)(-\sin x) + (\cos x - 1)g'(x)}{\cos x} = \frac{0 + (1-1)\cdot 0}{1} = 0 = g'(0)$$

$f'(x) = g(x)\cos x + \sin x\, g'(x)$

$f''(x) = -g(x)\sin x + \cos x\, g'(x) + \sin x\, g''(x) + g'(x)\cos x$

$f''(0) = 0 + 0 + 0 + 0 = 0$

Hence, (B), (C) and (D) are correct.

16. $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} \dfrac{\tan^2 x}{\left(x^2-[x]^2\right)} = \lim\limits_{x\to 0^+} \dfrac{\tan^2 x}{x^2} = 1$

As $x \to 0; [x]=0)$

Also, $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \sqrt{(x-[x])\cot(x-[x])} = \sqrt{\cot 1}$

$$\cot^{-1}\left(\lim_{x\to 0^-} f(x)\right)^2 = \cot^{-1}(\cot 1) = 1$$

Hence, (B) and (C) are correct.

17. (A) $f(1)=5 \quad f(1^+)=3+2=5$ continuous at $x=1$.

$f'(1^+)=3 \;\; f'(1^-)=4$ not differentiable at $x=1$.

(B) $f(x)=\begin{cases} x^3 & x\ge 0 \\ -x^3 & x<0 \end{cases}$ continuous

$f'(0^+)=f'(0^-)=0$

$f'(x)=\begin{cases} 3x^2 & x\ge 0 \\ -3x^2 & x<0 \end{cases}$

$f''(0^+)=0=f''(0^-)$ True

(C) $\lim\limits_{x\to 2} f(4x^2-11)=f(5)=2$

(D) $\left.\begin{array}{l} \lim\limits_{x\to a} f(x)+\lim\limits_{x\to a} g(x)=2 \\ \lim\limits_{x\to a} f(x)-\lim\limits_{x\to a} g(x)=1 \end{array}\right\} \lim\limits_{x\to a} f(x)=3/2 \,\&\, g(x)=1/2$

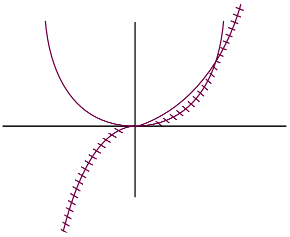
$\lim\limits_{x\to a} f(x).g(x) = 3/4$ must exist.

Hence, (B) and (C) are correct.

18. $f(x)=\min\{x^3,x^2\} = \begin{cases} x^3 & x\le 1 \\ x^2 & x>1 \end{cases}$

$f'(x)=\begin{cases} 3x^2 & x\le 1 \\ 2x & x>1 \end{cases}$

It continuous but not differentiable at $x=1$ for continuous.



$g(x)=[x]^2+\{x\}$;

$\left.\begin{array}{l} g(1)=1+0 \\ g(1^-)=0+\{1-h\}=1 \\ g(1^+)=1+0 \end{array}\right\}$ continuous

at $x=1$

Let take $x=k$ which is an integer.

$g(k)=k^2+0=k^2$

$g(k^-)=\lim\limits_{h\to 0}[k-h]^2+\{k-h\} = (k-1)^2+1$

$g(k^+)=\lim\limits_{h\to 0}[k+h]^2+\{k+h\} = \lim\limits_{h\to 0} k^2+h=k^2$

For $g(x)$ to be continuous at $x=k$

$g(k)=g(k^+)=g(k^-)$

$\Rightarrow k^2=(k-1)^2+1$ we get $k=1$ only. It means $g(x)$ is continuous at $x=1$ only.

RHD = $g'(1^+)=\lim\limits_{h\to 0} \dfrac{1^2+h-1^2}{h}=1$;

LHD = $g'(1^-)=\lim\limits_{h\to 0} \dfrac{(1-1)^2+1-h-1^2}{-h}=\lim\limits_{h\to 0} \dfrac{-h}{-h}=1$

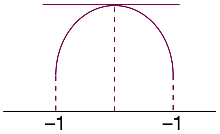
$g(x)$ is discontinuous and non differentiable at $x\in I-\{1\}$.

$f(x)$ is always continuous but differentiable except $x=1$.

Hence, (A) and (C) are correct.

19. (A) $\lim\limits_{x\to 1} \dfrac{f(x^3)-f(1)}{x-1} = \lim\limits_{x\to 1} \dfrac{f(x^3)-f(1)}{x^3-1}\cdot\dfrac{x^3-1}{x-1}$

$= \lim\limits_{x\to 1} f'(1)(x^2+x+1) \;= 3f'(1)$



(B) $\lim\limits_{x\to 0} \dfrac{f(ax)-f(bx)}{cx}$ (Using LH rule)

$= \lim\limits_{x\to 0} \dfrac{af'(ax)-bf'(bx)}{c} = f'(0)\dfrac{a-b}{c}$

(C) $\lim\limits_{x\to x_0} \dfrac{xf(x_0)-x_0 f(x)}{x-x_0} = \lim\limits_{x\to x_0} \dfrac{f(x_0)-x_0 f'(x)}{1}$ (Using LH rule)

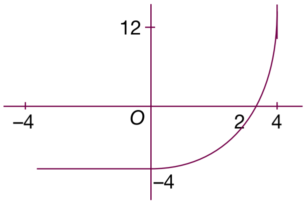
$= f(x_0)-x_0 f'(x_0)$

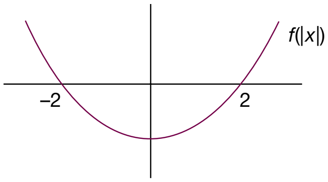
(D) $f(x)\le f(0)$ $f(0)$ max $f(x)$ differentiable.

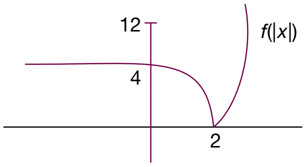
$f'(0)=0$

Hence, (A), (B), (C) and (D) are correct.

20. $f(x)=\begin{cases} -4 & -4\le x<0 \\ x^2-4 & 0\le x\le 4 \end{cases}$







$g(x)=f(|x|)+|f(x)|$

As, $f(|x|)$ and $|f(x)|$ both are continuous so $g(x)$ is also continuous.

As, $f(|x|)$ is always differentiable but $|f(x)|$ is not differentiable at $x=2$.

So, $g(x)$ is not differentiable at $x=2$.

Hence, (A), (C) and (D) are correct.

21. $f(x)=\begin{cases} \cos\dfrac{1}{x}(\ln^2(1+x)) & x>0 \\ 0 & x\le 0 \end{cases}$

$$f(0^+)=\lim_{h\to 0}\cos\frac{1}{h}\left(\frac{\ln(1+h)}{h}\right)^2 h^2 \;=\lim_{h\to 0} h^2\cos\frac{1}{h}=0$$

$f(0)=f(0^-)=0$ hence, $f(x)$ is continuous at $x=0$

RHD = $f'(0^+)=\lim\limits_{h\to 0}\dfrac{\cos\frac{1}{h}\ln^2(1+\mathrm{h})-0}{h}$

$$=\lim_{x\to\infty} h\cos\frac{1}{h}\left(\frac{\ln(1+h)}{h}\right)^2 = 0(-1to1)=0$$

$f'(0^-)=\lim\limits_{h\to 0}\dfrac{0-0}{-h}=0$

$f(x)$ is differentiable at $x=0$.

$$f'(x)=\begin{cases} -\sin\dfrac{1}{x}\left(\dfrac{1}{x^2}\right)\ln^2(1+x)+\cos\dfrac{1}{x}2\ln(1+x)\cdot\dfrac{1}{1+x} & x>0 \\ 0 & x<0 \end{cases}$$

$$f'(x)=\begin{cases}-\left(\dfrac{\ln(1+x)}{x}\right)^2\sin\dfrac{1}{x}+\cos\dfrac{1}{x}2\ln(1+x)\dfrac{1}{1+x} & x>0\\ 0 & x<0\end{cases}$$

$f'(0^-)=0; f'(0^+)$ is not having finite value

$f'(0)$ is discontinuous at $x=0$ not differentiable.

Hence, (A), (B) and (D) are correct.

22. $f(0^+)=0; f(0^-)=0$

$$\lim_{x\to0^+}\left(3-\left[\cot^{-1}\left(\frac{2x^3-3}{x^2}\right)\right]\right)=\lim_{x\to0^+}(3-[\pi])=0$$

$$\lim_{x\to0^-}\{x^2\}\cos e^{1/x}=\lim_{x\to0^-}x^2\cos(e^{1/x})$$

$$=\lim_{x\to0^-}\frac{\sin(e^{1/x})e^{1/x}}{-x^2}=0$$

Hence, (B) and (D) are correct.

23. $\text{RHL}=\lim\limits_{x\to0}f(x)=\lim\limits_{h\to0}\left\{1+\left(\dfrac{ch+dh^3}{h^2}\right)\right\}^{1/h}=e^d$ to exist the limit $c=0$.

$\text{LHL}=\lim\limits_{h\to0}f(0-h)$

$$=\lim_{h\to0}\frac{a(1-h\sin h)+b\cos h+5}{h^2}$$

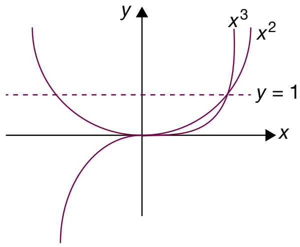
$$=\lim_{h\to0}\frac{a+b+\cosh+5}{h^2}-\frac{ah\sin h}{h^2}$$

Limit is possible if $a+b+5=0$ and $-a-\dfrac{b}{2}=3$.

On solving, $a=-1$, $d=\log e\,3$, $c=0$, $b=-4$

Hence, (A), (B) and (C) are correct.

24.

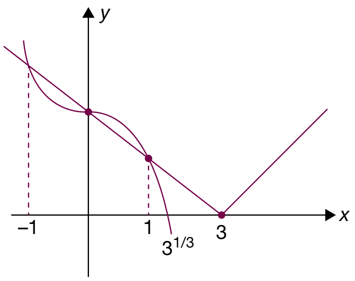


$$f(x)=\begin{cases}x^3, \text{when } x\le1\\ 1 \quad \text{when } x>1\end{cases}$$

$\therefore f(x)$ is continuous at $x=1$and not differentiable at $x=1$.

Hence, (A) and (C) are correct.

25. Draw graph and find points of non-differentiability i.e. – 1, 0, 1, 3.



Hence, (A) and (C) are correct.

26. $0\le\left[x^2\right]\le x^2$

$\Rightarrow 0\le x^{-2}\left[x^2\right]\le1$

$\Rightarrow\left[x^{-2}\left[x^2\right]\right]=0, 1$.

$f(x)$ is discontinuous at $x^2=n, n\in N\Rightarrow x=\sqrt{n}$ and non-differentiable.

Hence, (B), (C) and (D) are correct.

27. $f(x)=\displaystyle\int_0^1 5+|1-y|\,dy+\int_1^x 5+|1-y|\,dy;\qquad x>2$

$$=\int_0^1(5+1-y)\,dy+\int_1^x(5+y-1)\,dy=\frac{x^2}{2}+4x+1;$$

$$\therefore f(x)=\begin{cases}\dfrac{x^2}{2}+4x+1, & x>2\\ 5x+1, & x\le2\end{cases}$$

Hence, (A) and (C) are correct.

28. $f(x)=\lim\limits_{n\to\infty}\dfrac{2x^{2n}+x+2}{x^{2x}-x^2+3}; f(1)=\dfrac{2+1+2}{1-1+3}=\dfrac{5}{3}$

$$f(1^-)=\lim_{h\to0}\lim_{n\to\infty}\frac{2(1-h)^{2n}+1-h+2}{(1-h)^{2n}-(1-h)^2+3}=\frac{3}{2}$$

Discontinuous at $x=1$

$$\lim_{x\to2}\lim_{n\to\infty}\frac{x^{2n}\left(2+\dfrac{(x+2)}{x^{2n}}\right)}{x^{2n}\left(1+\dfrac{3-x^2)}{x^{2n}}\right)}=\frac{2+0}{1+0}=2$$

$$\lim_{x\to-2}\lim_{n\to\infty}f(x)=2$$

Hence, (B), (C) and (D) are correct.

29. $f(x)=[2^n x]+\{2^n x\}$

$\downarrow \qquad\qquad \downarrow$

discont discont

$\dfrac{1}{2^n},\dfrac{2}{2^n},\dfrac{3}{2^n},\ldots,\dfrac{2^n-1}{2^n}$ $\qquad$ $\dfrac{1}{2^m},\dfrac{2}{2^m},\dfrac{3}{2^m},\ldots\ldots,\dfrac{2^m-1}{2^m}$

$p=2^n-1$ points $\qquad$ $q=2^m-1$ points

If $n=m$ then fn is cont $n\ne m$.

P will be common in q is $(p<q)$ vice and verse and at common point cont so discontinuous at no. of points $=2^m-1-2^n+1$

$2^m-2^n=24\ (m=5, n=3)$

$=28\ (m=5, n=2)$

$=496\ (m=8, n=4)$

Hence, (A), (B) and (D) are correct.

30. $f(x)=\begin{cases}(x+1)e^{-\frac{2}{x}} & x>0\\ 0 & x=0\\ (x+1) & x<0\end{cases}$

$$f(0^+)=(h+1)\ e^{-\left(\frac{1}{h}+\frac{1}{h}\right)}=0$$

$f(0^-)=(-h+1)=1$ discontinuous at $x=0$

$x>0\quad f(x)=(x+1)e^{-2/x}$

$$f'(x)=e^{-2/x}+(x+1)e^{-2/x}\left(\frac{2}{x^2}\right)$$

$$= e^{-2/x}\left(1+\frac{2(x+1)}{x^2}\right)$$

$$= \frac{e^{-2/x}}{x^2}((x+1)^2+1) > 0 \text{ increasing}$$

$$\left.\begin{aligned} f(x)_{\max} &= f(2) = 3e^{-2/2} = \frac{3}{e} \\ f(x)_{\min} &= f(-2) = -1 \end{aligned}\right)$$

Range $[f(-2), f(2)]$

Hence, (B), (C) and (D) are correct.

31. Clearly, for $x = y = 0;\ f(0) = 1$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = 2.f(x)$$

Integrating $f(x) = e^{2x}$ from this all the remaining follows.

$\Rightarrow$ e^{2x} is every where continuous

$\Rightarrow$ $f\left(\frac{1}{2}\right) = e \to$ an irrational number.

Hence, (A), (B), (C) and (D) are correct.

32. $f(0^+) = \lim\limits_{x\to 0^+}\dfrac{e^x-1+ax}{x^2} = \dfrac{1+x+\frac{x^2}{2!}+\ldots-1+ax}{x^2}$

$$= \lim_{x\to 0^+}\frac{(a+1)x+\frac{x^2}{2!}+\ldots}{x^2} = \frac{1}{2}\text{; for limit existence } a+1=0$$

$\Rightarrow$ $a = -1$

$$f(0^-) = \lim_{x\to 0^-}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)\frac{1}{2} = \frac{1}{2}$$

$f(0) = b = \frac{1}{2}$; $b = \frac{1}{2}$ for continuous

Hence, (A), (B), (C) and (D) are correct.

33. Take $p = (x, \phi(x)); Q = (y, \phi(y))$ be any two points the curve $y = \phi(x)$.

Let 'R' divides the line segment $\overline{PQ}$ in the ratio 2:1 then

$$R = \left(\frac{x+2y}{3}, \frac{\phi(x)+2\phi(y)}{3}\right)$$

Clearly, TM > RM

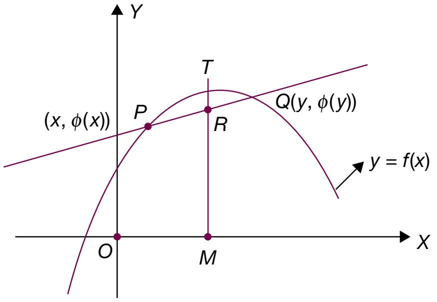
$$\Rightarrow \quad \phi\left(\frac{x+2y}{3}\right) > \frac{\phi(x)+2\phi(y)}{3}$$

Equality holds iff $\phi(x)$ is a linear function.

$\therefore$ $\phi(x) = ax+b$

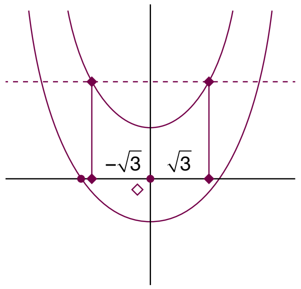
$\because$ $\phi'(0) = 1 \Rightarrow a = 1$ $\because \phi(0) = 2$

$b = 2$ $\because \phi(x) = x+2$



Hence, (A), (B) and (C) are correct.

34. Draw graph, clearly at $x = \pm\sqrt{3},\ f(x)$ is not differentiable.



Hence, (A), (B) and (C) are correct.

35. $f(1000)\,f(f(1000)) = 1$

$\Rightarrow f(1000)\,f(999) = 1 \quad \Rightarrow 999\,f(999) = 1$

$\therefore f(999) = \dfrac{1}{999}$

The numbers 999 and $\frac{1}{999}$ are in the range of f. Hence, by intermediate value property (IVP) of continuous function, function takes all values between 999 and $\frac{1}{999}$, then there exists $\alpha \in \left(\frac{1}{999}, 999\right)$ such that $f(\alpha) = 500$.

Then, $f(\alpha)f(f(\alpha)) = 1 \Rightarrow f(500) = \dfrac{1}{500}$

Similarly, $199 \in \left(\frac{1}{199}, 999\right)$, thus $f(199) = \dfrac{1}{199}$.

But there is nothing to show that 1999 lies in the range of f
Thus, (D) is not correct and so 'C' also
Hence, (A) and (B) are correct.

36. $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$

$$\frac{1}{a} = a \quad \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$$

$$\lim_{x\to\sqrt{2}^-} f(x) = \lim_{x\to\sqrt{2}^+} f(x)\ a = \frac{2b^2-4b}{2}$$

$$a = b^2 - 2b$$

(i) When $a = 1 \quad b^2-2b-1 = 0 \quad \Rightarrow b = 1 \pm \sqrt{2}$

(ii) When $a = -1 \quad b^2-2b+1 = 0 \quad \Rightarrow b = 1$

We get $(a, b) = (-1, 1), \left(1, 1+\sqrt{2}\right), \left(1, 1-\sqrt{2}\right)$

Hence, (A) and (C) are correct.

37. $f(x)$ is obviously differentiable at $x = 0$ and for $x_n = \dfrac{2}{2n+1}$ where $n = 0, 2, 4, \ldots.$

We get $f'\left(x_n^+\right) = \pi$ and $f'\left(x_n^-\right) = -\pi$ and for $x_n = \dfrac{2}{2n+1}$ where $n = 1, 3, 5, \ldots$

We get $f'\left(x_n^+\right) = \pi$ and $f'\left(x_n^-\right) = -\pi$ and

$\because$ $f(x)$ is even function $f(x)$ is not differentiable at $x_n = \dfrac{2}{2n+1}$; $n \in$ Z.

Hence, (B), (C) and (D) are correct.

38. $f(x+y) = f(x) + f(y) + xy\,(x+y)$

$$f(0) = 0 \quad \therefore \quad \lim_{h\to 0}\frac{f(h)}{h} = -1$$

$$\therefore \quad \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x)+f(h)+xh(x+h)-f(x)}{h}$$

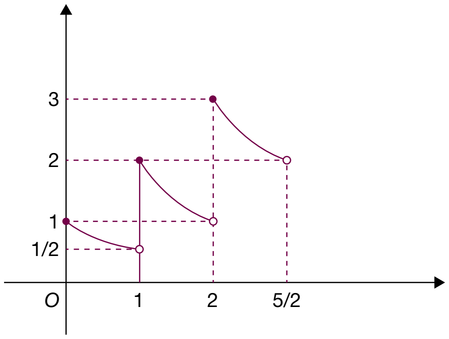
$$= \lim_{h\to 0}\frac{f(h)}{h} + \lim_{h\to 0} x(x+h) = -1 + x^2$$

$\therefore \quad f'(x) = -1 + x^2$ $\qquad \therefore \ f(x) = \frac{x^3}{3} - x + c$

$\therefore$ $f(x)$ is a polynomial function, $f(x)$ is twice differentiable for all $x \in R$ and $f'(3) = 3^2 - 1 = 8$.

Hence, (A), (C) and (D) are correct.

39. $f(x) = \begin{cases} \frac{1}{x+1} & , \quad 0 \le x < 1 \\ \frac{2}{x} & , \quad 1 \le x < 2 \\ \frac{3}{x-1} & , \quad 2 \le x < \frac{5}{2} \end{cases}$



Clearly, $f(x)$ is discontinuous and bijective function.

$$\lim_{x\to 1^-} f(x) = \frac{1}{2}; \quad \lim_{x\to 1^+} f(x) = 2$$

$$\min\left(\lim_{x\to 1} f(x), \lim_{x\to 1^-} f(x)\right) = \frac{1}{2} \ne f(1)$$

Except(1, 2,) = 2 = $f(1)$

Hence, (A), (B) and (D) are correct.

40. $gof(x) = \begin{cases} 0 & f(x) \in I \\ (f(x))^2 & f(x) \notin I \end{cases}$

$gof(x) = 0, \forall x \in R$ Continuous and differentiable

Constant function $gof(1) = 0$

$$fog(x) = [g(x)] = \begin{cases} 0 & x \in I \\ [x^2] & x \notin I \end{cases}$$

At $x = 0$, $fog(x)$ is continuous and differentiable

Discontinuous at $x \in I - \{0\}$

Discontinuous at $x = \sqrt{2}, \sqrt{3}....$ also

Hence, (A), (B) and (D) are correct.

41. Discontinuous when $a + \sin \pi x = 1$

If $a = 1$ then $\sin \pi x = 0$ for $x = 1, 2, 3, 4, 5$

If $a = 3$ then $\sin \pi x = -2$ not possible.

If $a = \frac{1}{2}$ the $\sin \pi x = \frac{1}{2}$ for six values of x

If $a = 0$ then $\sin \pi x = 1$ for $x = \frac{1}{2}, \frac{5}{2}, \frac{9}{2}$

Hence, (A), (B), (C) and (D) are correct.

Linked Comprehension Type Questions

1. $\lim_{x\to 1^-} f(x) = a + b$

$\lim_{x\to 1^+} f(x) = 4b$

For continuity $a + b = 4b$ i.e. $a = 3b$ ...(1)

$$f'(1^+) = \lim_{h\to 0^+}\frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0^+}\frac{2b(1+h) + 2b - (a+b)}{h} = \lim_{h\to 0^+}\frac{3b - a + 2bh}{h} = 2b$$

$$f'(1^-) = \lim_{h\to 0^+}\frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0^+}\frac{a(1-h)^2 + b - a - b}{-h} = \lim_{h\to 0^+}\frac{a(-2h + h^2)}{-h} = 2a$$

$2a \ne 2b, \ a \ne b$

Hence, the correct answer is (C).

2. $\lim_{x\to 2^-} g(x) = \lim_{x\to 2^-} cx^2 + d = 4c + d$

$\lim_{x\to 2^+} g(x) = \lim_{x\to 2^+} (dx + 3 - c) = 2d + 3 - c$

$g(2) = 4c + d$

$\therefore \ 4c + d = 2d + 3 - c$

$\therefore \ d = 5c - 3$

Hence, the correct answer is (A).

3. $\lim_{x\to 3^-} f(x) = 8b, \ \lim_{x\to 3^+} f(x) = 3(a-1) + 2a - 3 = 5a - 6$

Since, $f(x)$ is continuous at $x = 3$

$\therefore \ 8b = 5a - 6$

$$f'(3^-) = \lim_{h\to 0^+}\frac{f(3+h) - f(3)}{h} = \lim_{h\to 0^+}\frac{(a-1)(3+h) + 2a - 3 - 8b}{h}$$

Since, f is differentiable at $x = 3$

$\therefore \ \lim_{h\to 0^+} (a-1)(3+h) + 2a - 3 - 8b = 0$ i.e. $5a - 8b - 6 = 0$

$\therefore \ f'(3^+) = a - 1$

Thus, $a - 1 = 2b$ ...(2)

From (1) and (2), we get $a = 2, \ b = \frac{1}{2}$

Hence, the correct answer is (D).

4. Put $x = y = 1 \Rightarrow f(1) = 1, 2$

But f is one-one $\Rightarrow f(1) = 2$

Replace y by $1/x$

$f(x) f(1/x) = f(x) + f(1/x)$, since $f(1) = 2$

We get $f(x) = 1 + x^2$

Hence, the correct answer is (B).

5. $f(x) = x^2 + 1 \Rightarrow f(x) \ge 1$

$$0 < \frac{1}{f(x)} \le 1$$

$$\left[\frac{1}{f(x)}\right] = \begin{cases} 1, & x = 0 \\ 0, & \text{other wise} \end{cases}$$

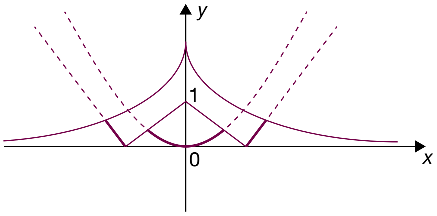
Hence, $g(x)$ has only one point of discontinuity.

Hence, the correct answer is (A).

6. $h(x)=\min\left\{\dfrac{2}{1+x^2},x^2,\left||1-|x|\right|\right\}$,

As all functions are even It is sufficient to look for $x\geq 0$

The required number will be double those of $x\in[0,\infty)$.



Hence, the correct answer is (D).

7. $f\left(\dfrac{x+y}{3}\right)=\dfrac{2+f(x)+f(y)}{3}$ $\quad f'(2)=2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f\left(\frac{3x+3h}{3}\right)-f\left(\frac{3x+0}{3}\right)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{2+f(3x)+f(3\mathrm{h})}{3}-\left(\frac{2+f(3x+f(0))}{3}\right)}{h}$$

$$=\lim_{h\to 0}\frac{f(3h)-f(0)}{3h}=f'(0)$$

$f'(x)=f'(0)=2$; $\int dy=\int 2dx$

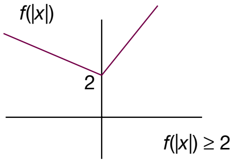
$\Rightarrow$ $y=2x+c$

$f(x)=2x+c$; $f(0)=2=0+\mathrm{c}$

$c=2;\quad x=y=0$

$f(0)=\dfrac{2+f(0)+f(0)}{3}$; $f(0)=2$

$f(x)=2x+2$



Range of $f(|x|)=2|x|+2$, range is $[2,\infty)$

$g(x)=|f(|x|)-3|$ is not- differentiable at 3 points.

Hence, the correct answer is (C).

8. **Hence, the correct answer is (C).**

9. $x^2+(f(|x|))^2=9$

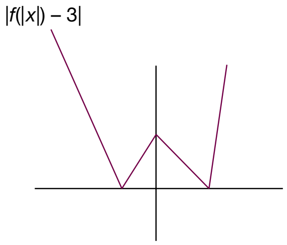
$x^2+(2|x|+2)^2=9$

$5x^2+4+8|x|=9$

$5x^2+8|x|-5=0$

$|x|=\dfrac{-8\pm\sqrt{164}}{10}$

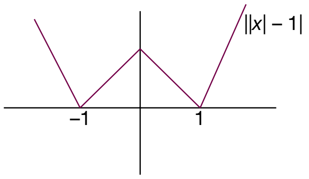
$|x|=\dfrac{\sqrt{164}-8}{10}$ only is acceptable solution.

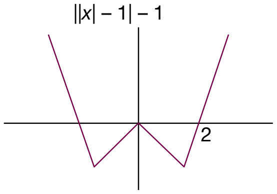


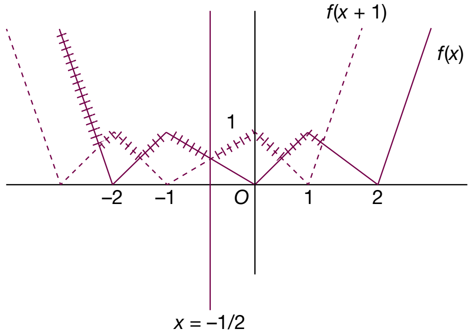
Hence, it has 2 solutions.

Hence, the correct answer is (B).

10.







$g(x)$ sym. about $x=-½$

Hence, the correct answer is (B).

11. $g(x)$ has 9 points of non-differentiability.

Hence, the correct answer is (C).

12. $f(x)=\dfrac{2}{3}\ln^{3/2}(\sin x+\cos x)$ for $x\in\left(0,\dfrac{\pi}{2}\right)$

$$\lim_{x\to\frac{\pi}{2}}\frac{f(x)}{\left(\frac{\pi}{2}-x\right)^{3/2}}=\lim_{x\to\frac{\pi}{2}}\frac{2}{3}\left(\frac{\ln(\sin x+\cos x)}{\frac{\pi}{2}-x}\right)^{3/2}$$

Where $x\in\left(0,\dfrac{\pi}{2}\right)$ $x=\dfrac{\pi}{2}-h$ only LHL is defined

$$=\lim_{h\to 0}\frac{2}{3}\left(\frac{\ln(\cos h+\sin h)}{h}\right)^{3/2}$$

$$=\lim_{h\to 0}\frac{2}{3}\left(\frac{\frac{1}{(\cos h+\sin h)}(-\sin h+\cos h)}{1}\right)^{3/2}$$

$$=\frac{2}{3}\left(\frac{1}{(1+0)}(0+1)\right)^{3/2}=\frac{2}{3}$$

Hence, the correct answer is (A).

13. $g(x)=\begin{cases}\ln(\sin x+\cos x) & x\in\left(0,\dfrac{\pi}{2}\right)\\ \dfrac{e^{\ln\left(\sin\left(x-\frac{\pi}{2}\right)+\cos\left(x-\frac{\pi}{2}\right)\right)}}{\sqrt{2}} & x\in\left(\dfrac{\pi}{2},\pi\right)\end{cases}$

$$=\begin{cases}\ln(\sin x+\cos x) & x\in\left(0,\frac{\pi}{2}\right)\\ \frac{-\cos x+\sin x}{\sqrt{2}} & x\in\left(\frac{\pi}{2},\pi\right)\end{cases}$$

$$\text{LHL}=\lim_{x\to\frac{\pi}{2}}\ln(\sin x+\cos x)=\ln(1+0)=0$$

$$\text{RHL}=\lim_{x\to\frac{\pi^+}{2}}\frac{0+1}{\sqrt{2}}=\frac{1}{\sqrt{2}}$$

Non-removable type of discontinuity at $x=\pi/2$

Jump of discontinuity $=\dfrac{1}{\sqrt{2}}$

Hence, the correct answer is (D).

14. $f_1(x)=\begin{cases}x^2 \text{ for } x\in[-1,1]\\ x^2 \text{ for } |x|>1\end{cases}$; $f_2(x)=\begin{cases}|x| \text{ for } x\in[-1,1]\\ |x| \text{ for } |x|>1\end{cases}$

$f(x)=x^2-2|x|$

$$f(x)=\begin{cases} x^2-2x & x>0 \\ x^2+2x & x<0 \end{cases}$$

$f'(0)$ does not exist $f'(x)$ vanishes at $x=1$ and $x=-1$

Hence, the correct answer is (C).

15. $g(x) = \left|f(x)\right| = \left|x^2-2|x|\right|$

Not differentiable at $x=-2, 0, 2$

$\therefore \lambda = 2$

Hence, the correct answer is (C).

16. $f_1(x) = \lim\limits_{n\to\infty} \lim\limits_{a\to1^+} \dfrac{|\sin x|+|\cos x|a^{-2n}}{1+a^{-2n}} = |\sin x|$

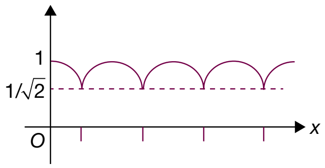
$$f_2(x) = \lim_{n\to\infty} \lim_{a\to1^-} \frac{a^{2n}|\sin x|+|\cos x|}{a^{2n}+1} = |\cos x|$$

$$f_3(x) = \lim_{x\to0} \frac{\frac{1}{2}g'\left(\frac{x}{2}\right)-g'(x^2)\cdot 2x}{g'(x)} = \frac{1}{2}$$

$$f_1(x) = |\sin x|,\ f_2(x) = |\cos x|, f_3(x) = \frac{1}{2}$$

Hence, the correct answer is (A).

17. $f(x) = \max\{f_1(x), f_2(x), f_3(x)\}$



Non-differentiable $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$

Hence, the correct answer is (B).

18. $\because \quad [f_3(x)] = \left[\dfrac{1}{2}\right] = 0; [f_1(x)] \neq [f_2(x)]$

$\Rightarrow \ [|\sin x|] \neq [|\cos x|]$

No solution $x = 0, \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}, 2\pi$

Hence, the correct answer is (C).

Combined solution to 19–21

$$f(x)=\begin{cases} [x], & -2\le x\le -\frac{1}{2} \\ 2x^2-1, & -\frac{1}{2}<x\le 2 \end{cases} = \begin{cases} -2, & -2\le x<-1 \\ -1, & -1\le x\le -\frac{1}{2} \\ 2x^2-1, & -\frac{1}{2}<x\le 2 \end{cases}$$

$$|f(x)|=\begin{cases} 2, & -2\le x<-1 \\ 1, & -1\le x\le -\frac{1}{2} \\ \left|2x^2-1\right|, & -\frac{1}{2}<x\le 2 \end{cases} = \begin{cases} 2, & -2\le x<-1 \\ 1, & -1\le x\le -\frac{1}{2} \\ 1-2x^2, & -\frac{1}{2}<x\le \frac{1}{\sqrt{2}} \\ 2x^2-1, & \frac{1}{\sqrt{2}}<x\le 2 \end{cases}$$

$$f(|x|)=\begin{cases} -2, & -2\le |x|<-1 \\ -1, & -1\le |x|\le -\frac{1}{2} \\ 2|x|^2-1, & -\frac{1}{2}<|x|\le 2 \end{cases} = 2x^2-1, -2\le x\le 2;$$

$$g(x) = \begin{cases} 2x^2+1, & -2\le x<-1 \\ 2x^2, & -1\le x\le -\frac{1}{2} \\ 0, & -\frac{1}{2}<x<\frac{1}{\sqrt{2}} \\ 4x^2-2, & \frac{1}{\sqrt{2}}\le x\le 2 \end{cases}$$

$$g\left(-1^-\right) = \lim_{x\to-1}\left(2x^2+1\right) = 3, g\left(-1^+\right) = \lim_{x\to-1} 2x^2 = 2$$

$$g\left(-\frac{1^-}{2}\right) = \lim_{x\to-\frac{1}{2}} 2x^2 = \frac{1}{2}, g\left(-\frac{1^+}{2}\right) = \lim_{x\to-\frac{1}{2}} 0 = 0$$

$$g\left(-\frac{1^-}{\sqrt{2}}\right) = \lim_{x\to-\frac{1}{\sqrt{2}}} 0 = 0, g\left(-\frac{1^+}{\sqrt{2}}\right) = \lim_{x\to-\frac{1}{\sqrt{2}}}\left(4x^2-2\right) = 0$$

Hence, $g(x)$ is discontinuous at $x = -1, \dfrac{1}{2}$

$g(x)$ is continuous at $x = -1, -\dfrac{1}{\sqrt{2}}$

Now, $g'\left(\dfrac{1^-}{\sqrt{2}}\right) = 0, g'\left(\dfrac{1^+}{\sqrt{2}}\right) = 8\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{8}{\sqrt{2}}$

Hence, $g(x)$ is non - differentiable at $x = \dfrac{1}{\sqrt{2}}$

19. Hence, the correct answer is (A).

20. Hence, the correct answer is (D).

21. Hence, the correct answer is (B).

22. Let $0<x<1$ then $f(x) = \log(2+x)$ $\left(\because \lim\limits_{n\to\infty} x^{2^n} = 0\right)$

Hence, the correct answer is (C).

23. Let $x=1, f(x) = \dfrac{1}{2}(\log 3+\sin 1)$

Hence, the correct answer is (B).

24. If $x>1$, then

$$\lim_{n\to\infty} \frac{\log(2+x)+x^{2^n}\sin x}{1+x^{2^n}} = \lim_{n\to\infty} \frac{\left[\frac{\log(2+x)}{x^{2^n}}+\sin x\right]}{\frac{1}{x^{2^n}}+1}$$

$$= \sin x.\left(\because \lim_{n\to\infty}\frac{1}{x^{2^n}} = 0\right)$$

$$\therefore \quad f(x) = \begin{cases} \log(2+x) & \text{if } 0<x<1 \\ \frac{1}{2}(\log 3+\sin 1) & \text{if } x=1 \\ \sin x & \text{if } x>1 \end{cases}$$

All the 3-results obviously follows.

Hence, the correct answer is (A).

25. At $x=0$,

$$\text{LHL} = b\sin^{-1}\frac{c}{2}$$

$$\text{RHL} = \lim_{x\to0^+} \frac{e^{\frac{ax}{2}}-1}{x}$$

$$= \lim_{x\to0^+}\left(\frac{e^{\frac{ax}{2}}-1}{\frac{ax}{2}}\right)\left(\frac{a}{2}\right) = \frac{a}{2}$$

For $f(x)$ to be continuous at $x = 0$ and $f(0) = \frac{1}{2}$

LHL $\big|_{x=0}$ = RHL$\big|_{x=0} = f(0)$

$$\frac{a}{2} = \frac{1}{2} = b\sin^{-1}\left(\frac{c}{2}\right); a = 1$$

Hence, the correct answer is (A).

26. $f(x)$ need to be continuous first $\Rightarrow$ $a = 1$

Now, let $f(x)$ be differentiable also at $x = 0$, then

$$\text{LHD}\big|_{x=0} = \frac{1}{2}\frac{b}{\sqrt{1-\left(\frac{x+c}{2}\right)^2}}\Bigg|_{x=0} = \frac{b}{2\sqrt{1-\frac{c^2}{4}}} = \frac{b}{\sqrt{4-c^2}} \qquad \ldots(1)$$

$$\text{RHD}\big|_{x=0} = \lim_{h\to 0+}\frac{\frac{e^{\frac{h}{2}}-1}{n}-\frac{1}{2}}{h} = \lim_{h\to 0}\frac{e^{w2}-1-\frac{h}{2}}{h^2}$$

$$= \lim_{h\to 0}\frac{e^{\frac{h}{2}}\frac{1}{2}-0-\frac{1}{2}}{2h}$$

$$= \lim_{h\to 0}\frac{1}{8}\left(\frac{e^{\frac{h}{2}}-1}{\frac{h}{2}}\right) = \frac{1}{8} \qquad \ldots(2)$$

$$\text{LHD} = \text{RHD} \Rightarrow \frac{b}{\sqrt{4-c^2}} = \frac{1}{8} \Rightarrow 64b^2 = 4-c^2.$$

Hence, the correct answer is (B).

27. If $f(x)$ is differentiable at $x = 0$, then

$$\frac{1}{2} = b\sin^{-1}\left(\frac{c}{2}\right) \Rightarrow 2\sin\left(\frac{1}{2b}\right) = c$$

Hence, the correct answer is (B).

Matrix Match/Column Match Type Questions

1. **A $\rightarrow$ r; *B* $\rightarrow$ p,q; *C* $\rightarrow$ q; D $\rightarrow$ s**

(A) $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$\lim_{h\to 0}\frac{f(x)[f(h)-f(0)]}{h} = f(x).f'(0) \Rightarrow \frac{f'(x)}{f(x)} = 2$$

(B) $f(x) = \begin{cases} \frac{x^2-1}{4}, & x \in (-1, 1) \\ \tan^{-1}x, & x \in (-\infty, -1] \cup [1, \infty) \end{cases}$

$f(x)$ is not continuous at $x = -1$ and $x = 1$

Hence, not differentiable at these points.

(C) $f(x)$ is differentiable at $x = 1$ if $1 + a = 2 + b$ ...(1)

$g(x)$ is differentiable at $x = 1$ if $3 + b = 1$ ...(2)

From (1) and (2) $b = -2$, $a = -1$

(D) $$\frac{dy}{dx} = \frac{\sec^2(x+y)}{1-\sec^2(x+y)} = \frac{1+y^2}{1-(1+y^2)} \Rightarrow \frac{dy}{dx} = -\frac{1}{y^2}-1$$

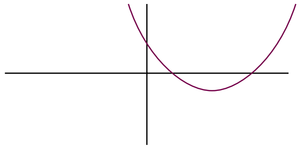
$$\Rightarrow \frac{d^2y}{dx^2} = \frac{2}{y^3}\left(\frac{-1}{y^2}-1\right) = -\frac{2}{y^5}-\frac{2}{y^3}$$

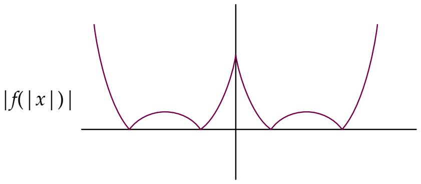
$$\Rightarrow \frac{d^3y}{dx^3} = \left(\frac{10}{y^6}+\frac{6}{y^4}\right)\left(-\frac{1}{y^2}-1\right) = -\left(\frac{6y^4+16y^2+10}{y^8}\right)$$

$$\lambda = 10 \Rightarrow \left[\frac{\lambda}{3}\right] = \left[\frac{10}{3}\right] = [3.33] = 3$$

2. **A $\rightarrow$ t; B $\rightarrow$ r; C $\rightarrow$ p; D $\rightarrow$ q**

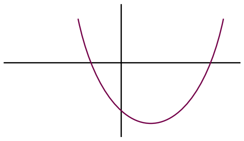
(A) If $D > 0$, $b < 0$ and $c > 0$, we have vertex x coordinate $\frac{-b}{2a} > 0$

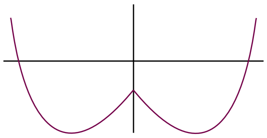


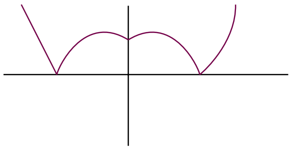


At 5 points not differentiable.

(B) If $D > 0$, $b < 0$ and $c < 0$, we have vertex x coordinate $\frac{-b}{2a} > 0$

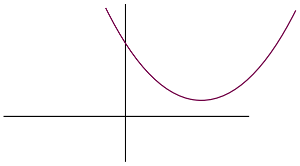


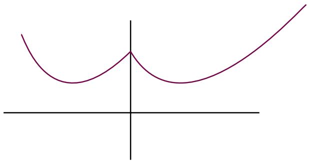




At 3 points not differentiable.

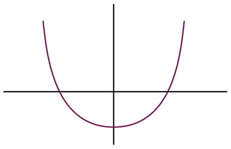
(C) If $D < 0$, $b \neq 0$ and $c > 0$

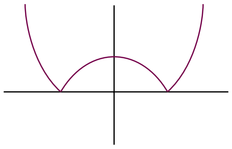




At 1 points not differentiable.

(D) If $D > 0$, $b = 0$ and $c < 0$





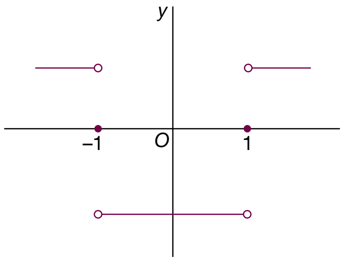
At 2 points not-differentiable.

3. **A $\rightarrow$ q; *B* $\rightarrow$ p; *C* $\rightarrow$ s; D $\rightarrow$ p**

(A) $f(x) = \begin{cases} 0 & , \quad 1 < x \le 2 \\ 1-x & , \quad 0 \le x < 1 \\ -\sin\pi x & , \quad -1 \le x < 0 \end{cases}$ continuous at $x = 1$ but not differentiable.

(B) $$f'(0^-) = \lim_{h\to 0}\frac{h^2e^{-1/h}-0}{-h} = \lim_{h\to 0^-}\left(-he^{-1/h}\right) = 0$$

(C) $$g(x) = \frac{1}{1+\frac{1}{x}(2+2x)} = \frac{x}{3x+2}$$



Thus, the points where $g(x)$ is not differentiable are $x = 0, -1, -\frac{2}{3}$

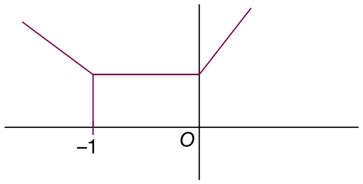
(D) Vertical tangents exist at $x = 1$

And $x = -1$ else where horizontal tangents exist.

$\therefore$ Number of points where tangent does not exist is 0.

4. A → r; B → t; C → s; D → r

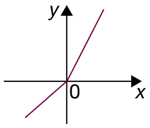
(A) (i) 2 point not differentiable.



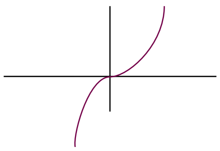
2 point not diff

(ii) $g(x) = \begin{cases} 3x & x \ge 0 \\ x & x < 0 \end{cases}$

one point not differentiable.

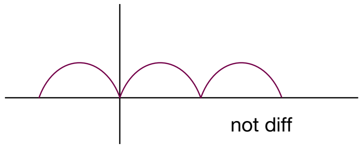


(iii) $h(x) = \begin{cases} x^2 & x \ge 0 \\ -x^2 & x < 0 \end{cases}$



always diff

(iv) Not differentiable at $x = n\pi$, infnite points.



not diff

So, we get three non-differentiable functions.

(B) $\lim\limits_{x\to 0} \dfrac{(3a+4b-2)x^2 + 10x^3 + (a-b-1)x + 2a+b-5}{x^2}$

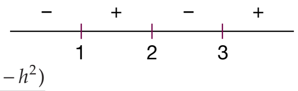
$a - b - 1 = 0 \qquad \Rightarrow a = b + 1$

$2a + b - 5 = 0 \qquad \Rightarrow 2b + 2 + b = 5$

$3b = 3\,;\, b = 1 \qquad a = 2$

$a^2 + b^2 = 4 + 1 = 5$

(C) $\lim\limits_{h\to 0} \dfrac{\ln f(2+h^2) - \ln f(2-h^2)}{h^2}$



$$= \lim_{h\to 0} \frac{\dfrac{f'(2+h^2)2h}{f(2+h^2)} - \dfrac{(-2h)f'(2-h^2)}{f(2-h^2)}}{2h}$$

$$= \frac{f'(2)}{f(2)} + \frac{f'(2)}{f(2)} = \frac{4}{2} + \frac{4}{2} = \frac{8}{2} = 4.$$

(D) $f(x) = (x-1)(x-2)\,|(x-1)(x-2)(x-3)| + \left|\sin\left(x + \frac{\pi}{4}\right)\right|$

Polynomial sub-function is not differentiable at $x = 3$.

Trigonometric function is not differentiable at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$.

$f(x)$ is not differentiable at 3 Points in given interval.

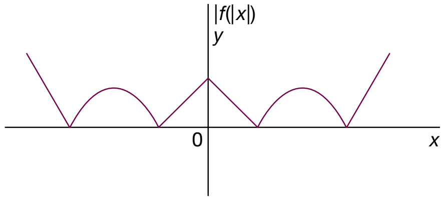
5. A → s; B → q; C → r; D → p

(A) RHL $= f\left(\frac{\pi}{2}^+\right) = \lim\limits_{h\to 0} \dfrac{e^{\tan\left(\frac{\pi}{2}+h\right)} - 1}{e^{\tan\left(\frac{\pi}{2}+h\right)} + 1} = \lim\limits_{h\to 0} \dfrac{e^{-\cot h} - 1}{e^{-\cot h} + 1} = -1$

LHL $= f\left(\frac{\pi}{2}^-\right) = \lim\limits_{h\to 0} \dfrac{e^{\tan\left(\frac{\pi}{2}-h\right)} - 1}{e^{\tan\left(\frac{\pi}{2}-h\right)} + 1} = \lim\limits_{h\to 0} \dfrac{e^{\cot h} - 1}{e^{\cot h} + 1} = 1$

Jump of discontinuity = 2

(B)



non-differentiable at 5 points

(C) $f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{4} \\ b, & x = 0 \\ \sin x, & -\frac{\pi}{4} < x < 0 \end{cases} \quad f(0)$

$= f(0 + h) = f(0 - h) \Rightarrow b = 0$

$f(x) = \begin{cases} x, & 0 \le x < \frac{\pi}{4} \\ \sin x, & -\frac{\pi}{4} < x < 0 \end{cases} \quad f'(0^+) = f'(0^-) = 1$

(D) $\ell = \lim\limits_{x\to 0} \dfrac{1 - \cos 9x}{(e^x - 1)} = \dfrac{81}{2}$

$\left[\dfrac{\ell}{9}\right] = [4.5] = 4$

6. A → q; B → q; C → p; D → p

(A) $f(x) = \begin{cases} -1, x < 0 \\ 0, x = 0 \\ 1, x > 0 \end{cases}$ discontinuous at $x = 0$

(B) $f(x) = \begin{cases} 0 & , \quad x = 0 \\ \frac{1}{x^2} & , \quad 0 < |x| < \frac{1}{2} \\ 4 & , \quad \frac{1}{2} \le |x| \le 2 \\ x^2 & , \quad |x| > 2 \end{cases}$ discontinuous at $x = 0$, not differentiable

at $x = 0, 1/2, -1/2, 2, -2$

(C) $f(x) = \begin{cases} \ell n|x| & if \quad |x| > e \\ 1 & if \quad |x| \le e \end{cases}$ always continuous, not differentiable

at $x = e, -e$

CHAPTER 5

(D) $f(x) = \begin{cases} |\cos x| & \text{if } x \in \bigcup\limits_{n\in I}\left[n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{4}\right] \\ |\sin x| & \text{otherwise} \end{cases}$

Always continuous, not differentiable at infinite points.

7. **A → r, s; B → p, q; C → p, q; D → p, r**

(A) The given function is clearly continuous at all points except possibly at $x = \pm 1$.

As, $f(x)$ is an even function, so we need to check its continuity only at $x = 1$.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x\to 1^-}\left(ax^2 + b\right) = \lim_{x\to 1^+}\frac{1}{|x|} \Rightarrow a + b = 1 \quad \text{(r)}$$

Clearly, $f(x)$ is differentiable for all x, except possibly at $x = \pm 1$. As $f(x)$ is even function, so we need to check its differentiability at $x = 1$ only.

$$\lim_{x\to 1^-}\frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^+}\frac{f(x) - f(1)}{x - 1}$$

$$\Rightarrow \lim_{x\to 1}\frac{ax^2 + b - 1}{x - 1} = \lim_{x\to 1^+}\frac{\frac{1}{|x|} - 1}{x - 1}$$

$$\Rightarrow \lim_{x\to 1}\frac{ax^2 - a}{x - 1} = \lim_{x\to 1}\frac{(1 - x)}{x(x - 1)} \Rightarrow 2a = -1 \Rightarrow a = -\frac{1}{2}$$

(B) $f(x) = \operatorname{sgn}\left(x^2 - ax + 1\right)$ is discontinuous then $x^2 - ax + 1 = 0$ must have only one real root. Hence, $a = \pm 2$.

(C) $f(x) = \left[2 + 3|n|\sin x\right], n \in N$ has exactly 11 points of discontinuity in $x \in (0, \pi)$ for $n = 2, -2$.

∵ $[x]$ is discontinuous at every integer

(D) If $f(x) = ||\,|x| - 2| + a|$ has exactly three points of non-differentiability.

$f(x)$ is non-differentiable at $x = 0, |x| - 2 = 0$ or $x = 0, \pm 2$.
Hence, the value of a must be positive, as negative value of a allows $||x| - 2| + a = 0$ to have real roots, which gives two more points of non - differentiability.

8. **A → p; B → q; C → r; D → s**

(A) $\frac{K}{2} = \lim_{x\to 0}\frac{\sin 3x}{x} \Rightarrow \frac{K}{2} = 3 \Rightarrow K = 6$

(B) $f(5) = \lim_{x\to 5}\frac{(x - 5)^2}{(x - 5)(x - 2)} = \lim_{x\to 5}\frac{(x - 5)}{x - 2} = \frac{0}{3} = 0$

(C) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

$$\lim_{x\to 0^+} e^{ax+b} = \lim_{x\to 0}\,(a^2\cos^2 x + b^2\sin^2 x) \quad \Rightarrow e^b = a^2$$

$\Rightarrow\ b = 2\ln|a|$

(D) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

$$\Rightarrow\ \lim_{x\to 0^+}\cos[x] = \lim_{x\to 0^-}|x| + a \Rightarrow \cos(0) = a$$

$\Rightarrow\ a = 1$

9. **[A → p; B → q, r; C → q, s; D → r]**

(A) $\lim_{x\to 0^-}\frac{a + 3\cos x}{x^2} = \lim_{x\to 0^+} b\tan\left(\frac{\pi}{x + 3}\right) = f(0)$

∵ $a + 3 = 0$ and $b\sqrt{3} = f(0)$

$$\Rightarrow\ a = -3 \ \Rightarrow \lim_{x\to 0^-}\frac{-3 + 3\cos x}{x^2} = b\sqrt{3}$$

$$\Rightarrow\ -\lim_{x\to 0^-}\frac{3(1 - \cos x)}{x^2} = b\sqrt{3} \Rightarrow -\frac{3}{2} = b\sqrt{3} \Rightarrow b = -\frac{\sqrt{3}}{2}$$

$[a - 2b] = [-3 + \sqrt{3}\,] = -2$

(B) $f(x) = \begin{cases} -2\sin x, & -\pi \le x \le -\frac{\pi}{2} \\ a\sin x + b, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x, & \frac{\pi}{2} \le x \le \pi \end{cases}$

$$\lim_{x\to -\frac{\pi}{2}^-} f(x) = \lim_{x\to -\frac{\pi}{2}^+} f(x)$$

$$\Rightarrow\ \lim_{x\to -\frac{\pi}{2}^-}(-2\sin x) = \lim_{x\to -\frac{\pi}{2}^+}(a\sin x + b)$$

$\Rightarrow 2 = -a + b$ …(1)

$$\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}^+} f(x)$$

$\Rightarrow a + b = 0$ …(2)

From (1) and (2) $a = -1, b = 1$

$\Rightarrow |a - b| = 2$

$|a + 2b| = 1$

(C) $\lim_{x\to -\frac{\pi}{2}^-}\left(\frac{3}{2}\right)^{\frac{\cos x}{\cot 2x}} = \lim_{x\to -\frac{\pi}{2}^+} = b + 3$

$$\Rightarrow\ \lim_{x\to -\frac{\pi}{2}^-}\left(\frac{3}{2}\right)^{\frac{-3\sin 3x}{-2\operatorname{cosec}^2 2x}} = b + 3 \ \Rightarrow \left(\frac{3}{2}\right)^{\frac{3}{-2}\times\frac{1}{\infty}} = b + 3$$

$\Rightarrow b + 3 = 1 \Rightarrow b = 2$ and $\lim_{x\to \frac{\pi}{2}^-} f(x) = \lim_{x\to \frac{\pi}{2}^+} f(x) \quad a = 2\ln 5$

$\Rightarrow b + 3 = \lim_{x\to \frac{\pi}{2}^+}(1 + \cos x)\left(\frac{a|\tan x|}{b}\right)$ for existence of $\lim_{x\to \frac{\pi}{2}^+} f(x)$

$\Rightarrow |a + 2b| = 4$ and $|a - b| = 2$

(D) $\lim_{x\to 0} f(x) = 1$

$$\Rightarrow\ \lim_{x\to 0}\frac{a\sin x + b}{x} = 1 \quad \Rightarrow b = 0$$

$$\Rightarrow\ \lim_{x\to 0}\frac{a\sin x}{x} = 1 \quad \Rightarrow a = 1 \Rightarrow |a + 2b| = 1$$

Integer/Numerical Answer Type Questions

1. $f(x) = [x] + \left[x + \frac{1}{4}\right] + \left[x + \frac{2}{4}\right] + \left[x + \frac{3}{4}\right] = [4x]$

∴ $f(x) = [4x]$ which will become discontinuous at $x = \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$

Hence, the correct answer is (4).

2. $1 \le x^3 \le 8$

Discontinuous at $x^3 = 2, 3, 4, 5, 6, 7, 8$

$f(1^+) = f(1) = -2$ continuous at $x = 1$

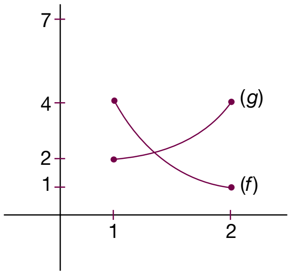
$f(2) = (-2)^8$ and $f(2-h) = \lim_{h\to 0}(-2)^{[8-h]} = (-2)^7$

Non differentiable for $x^3 = 2, 3, 4, 5, 6, 7, 8$

Total points of non- differentiability are = 7

Hence, the correct answer is (7).

3. $f : [1, 2] \to [1, 4],\ g : [1, 2] \to [2, 7]$



f and g both are bijective function (one-one and onto) biject.

$f(1) = 4, f(2) = 1$. Decreasing

$g(2) = 7, g(1) = 2$ Increasing

No. of solutions of the equation $f(x) = g(x)\ \forall\ x \in [1, 2]$.

One solution

Hence, the correct answer is (1).

4. $f(x) = \begin{cases} x^a \sin\left(\dfrac{1}{x^b}\right) & x > 0 \\ 0 & x = 0 \end{cases}$ $a > 1$ for $f(x)$ is differentiable

$$f'(x) = \begin{cases} ax^{a-1}\sin\left(\dfrac{1}{x^b}\right) + x^a\cos\left(\dfrac{1}{x^b}\right)\dfrac{-b}{x^{b+1}} & x > 0 \\ 0 & x = 0 \end{cases}$$

$$f'(x) = \begin{cases} ax^{a-1}\sin\left(\dfrac{1}{x^b}\right) - bx^{a-b-1}\cos\left(\dfrac{1}{x^b}\right) & x > 0 \\ 0 & x = 0 \end{cases}$$

For $f'(x)$ to be differentiable

$a - 1 > 1$ and $a - b - 1 > 1$

or $a > 2$ and $a - b > 2$

$$f''(x) = \begin{cases} a(a-1)x^{a-2}\sin\dfrac{1}{x^b} - ax^{a-1}\cos\dfrac{1}{x^b}\left(\dfrac{-b}{x^{b+1}}\right) \\ -b(a-b-1)x^{a-b-2}\cos\dfrac{1}{x^b} - bx^{a-b-1}\left(-\sin\left(\dfrac{1}{x^b}\right)\left(\dfrac{-b}{x^{b+1}}\right)\right) \end{cases} \quad x > 0$$

$$f''(x) = a(a-1)x^{a-2}\sin\frac{1}{x^b} + abx^{a-b-2}\cos\frac{1}{x^b} - b(a-b-1)\left(\cos\frac{1}{x^b}\right)x^{a-b-2} + b^2\left(\sin\frac{1}{x^b}\right)x^{a-2b-2} \text{ For } x > 0$$

For continuous

$a - 2 > 0$ and $a - b - 2 > 0$ and $a - 2b - 2 > 0$

$a - 2b - 2 > 0$

$a - 2b > 2$

Integral value

$a - 2b \ge 3$ $[0, 50]$

$a - 2b$ can take values 3, 4, 5, … 50, which are 48 values.

Hence, the correct answer is (48).

5. $f(x) = |x| + \left|\dfrac{x}{3} - 1\right| + \left||x| - \left|\dfrac{x}{3} - 1\right|\right| \ \forall\ x \in R$

$\downarrow$ $\downarrow$ $\downarrow$

cont cont cont

No point of discontinuity.

Hence, the correct answer is (0).

6. $f(x) = \begin{cases} x^2 - 5x + \alpha & x \le 0 \\ |7 - 3x| + \sin[x] & 0 < x < 1 \\ \beta x + 2 & 1 \le x < \infty \end{cases}$

$f(0) = \alpha = f(0^-), f(0^+) = 7 + \sin(0) = 7$

$\alpha = 7$

$f(1^-) = 4 + \sin 0 = 4$

$f(1) = f(1^+) = \beta + 2$

$\beta = 2$

$\alpha - \beta = 5$

Hence, the correct answer is (5).

7. $a = \lim\limits_{x\to\frac{\pi}{6}} \dfrac{\sin\left(x - \frac{\pi}{6}\right)}{\sqrt{3} - 2\cos x} = \lim\limits_{x\to\frac{\pi}{6}} \dfrac{\cos\left(x - \frac{\pi}{6}\right)}{0 + 2\sin x}$ using LH Rule

$$= \frac{1}{2\cdot\frac{1}{2}} = 1$$

$$r = \lim_{x\to 0}\left(\frac{\sin x^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right)\frac{x^{\frac{1}{3}}}{\left(\frac{\tan^{-1}\sqrt{x}}{\sqrt{x}}\right)^2(\sqrt{x})^2}\left(\frac{\ln(1+3x)}{3x}\right)\frac{3x}{(e^{5x^{1/3}} - 1)}$$

$$= \lim_{x\to 0}\frac{1}{(1)^2}\frac{x^{1/3}}{x}\frac{(1)3x}{(e^{5x^{1/3}} - 1)}$$

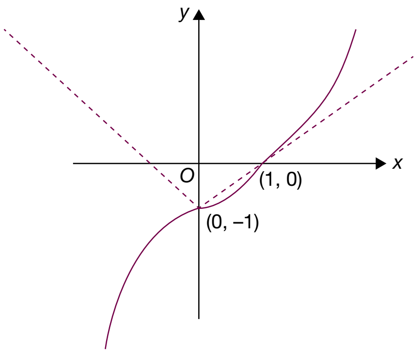
$$= \lim_{x\to 0}\frac{3}{5}\frac{5x^{1/3}}{(e^{5x^{1/3}} - 1)} = \frac{3}{5}(1) = \frac{3}{5}$$

$$S = \frac{a}{1-r} = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}$$

$2S = 5$

Hence, the correct answer is (5).

8. $h(x) = x^3 - 1, x \in R$.



Clearly, $h(x)$ is differentiable for all $x \in R$.

Hence, the correct answer is (0).

9. $g(x)$ must assume only one integer value i.e. $g(x) = 0\ \forall x \in R$

$$\Rightarrow \frac{2}{c} < \frac{f(x)}{c} < \frac{\sqrt{21}}{c}$$

$$\Rightarrow \quad \frac{\sqrt{21}}{c} < 1$$

$C = 5, 6, 7, \ldots$; least value $= 5$

Hence, the correct answer is (5).

10. $$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{(4^x - 1)^3}{(x/p)(x^2/4)}$$

$$\text{As, } x\to 0 \begin{cases} \sin(x/p) \simeq x/p \\ \ln(1+\frac{x^2}{4}) \simeq x^2/4 \end{cases}$$

$$\text{Limit } = \lim_{x\to 0}\left(\frac{4^x - 1}{x}\right)^3 4p = 4p \ (\ln 4)^3$$

$$4p(\ln 4)^3 = 12(\ln 4)^3 \Rightarrow p = 3$$

Hence, the correct answer is (3).

11. $$0<x<2\,,\, 0<\frac{x}{2}<1\,,\, 0<\frac{\pi x}{2}<\pi \;:\; \left(\cos\frac{\pi x}{2}\right) \to -1^{+}$$

$$\left(\cos\frac{\pi x}{2}\right)^{2n} \to 0 \text{ as } n\to\infty$$

But at $x = 2$, $\cos\frac{\pi x}{2} = -1 \Rightarrow \lim_{x\to\infty}\left(\cos\frac{\pi x}{2}\right)^{2n} = 0$ at $x = 2$ $f(2) = 1$

Hence, $f(x)$ is discontinuous at $x = 2$.

Hence, the correct answer is (2).

12. $$\text{RHL} = \lim_{x\to -1+} \frac{\pi - \cos^{-1}(x)}{\sqrt{x+1}} \cdot \frac{1}{\sqrt{\pi} + \sqrt{\cos^{-1} x}}$$

$$= \lim_{x\to -1+} \frac{\cos^{-1}(-x)}{\sqrt{x+1}} \cdot \frac{1}{\sqrt{\pi} + \sqrt{\cos^{-1} x}} \qquad \text{(Using LH Rule)}$$

$$= \sqrt{2} \cdot \frac{1}{\sqrt{\pi} + \sqrt{\pi}} = \frac{1}{\sqrt{2\pi}} \qquad \therefore \lambda = 2$$

Hence, the correct answer is (2).

13. Partial Differentiation w.r.t. x

$$y\, f'(xy) = \frac{f'(x)}{y} - \frac{f(y)}{x^2} \quad \text{put } x = 1$$

$$y\, f'(y) = \frac{1}{y} - f(y) \Rightarrow f'(y) + \frac{f(y)}{y} = \frac{1}{y^2}$$

$$f(x) = \frac{\ln x}{x} \Rightarrow M = \frac{1}{e}$$

$$[M + e] = \left[\frac{1}{e} + e\right] = 3$$

Hence, the correct answer is (3).

14. $$f(x) = \begin{cases} \frac{1}{|x|} & |x| \ge 1 \\ ax^2 + b & |x| < 1 \end{cases}; \quad f'(x) = \begin{cases} \frac{-1}{x^2} & x > 1 \\ \frac{1}{x^2} & x < -1 \\ 2ax & |x| < 1 \end{cases}$$

$1 = a + b$

$$f'(1) = 2a = \frac{-1}{1} \quad \Rightarrow a = \frac{-1}{2}, b = 3/2$$

$$f'(-1) = -2a = \frac{1}{1}$$

$$\left|\frac{b}{a}\right| = \left|\frac{3}{2\left(\frac{-1}{2}\right)}\right| = 3$$

Hence, the correct answer is (3).

15. $$e^{-|x|} = \begin{cases} e^{-x} & x \ge 0 \\ e^{x} & x < 0 \end{cases} \quad \text{LHD} = e^{+x}\Big|_{x=0} = 1$$

LHD of $e^{-|x|}$ at $x = 0 = 1$

$$\lim_{x\to 0} \frac{1 - \cos\left(1 - \cos\frac{x}{2}\right)}{2^m x^n} = \lim_{x\to 0} \frac{1 - \cos\left(1 - \left(1 - 2\sin^2\frac{x}{4}\right)\right)}{2^m x^n} = 1$$

$$\Rightarrow \lim_{x\to 0} \frac{1 - \cos\left(2\sin^2\frac{x}{4}\right)}{2^m x^n} = \lim_{x\to 0} \frac{1 - \left(1 - 2\sin^2\left(\sin^2\frac{x}{4}\right)\right)}{2^m x^n} = 1$$

$$\Rightarrow \lim_{x\to 0} 2\left(\frac{\sin\left(\sin^2\frac{x}{4}\right)}{\sin^2\frac{x}{4}}\right)^2 \left(\frac{\sin\frac{x}{4}}{\frac{x}{4}}\right)^4 \frac{x^4}{2^8 x^m} \cdot \frac{1}{2^n} = 1$$

$$\Rightarrow \lim_{x\to 0} (1)^2 (1)^4 \left(\frac{x^4}{x^n}\right) \frac{1}{2^{m+7}} = 1$$

We get $n = 4$, $m = -7$

$$\frac{n - 10m}{37} = \frac{4 + 70}{37} = 2$$

Hence, the correct answer is (2).

16. $2f(x+y) + f(x-y) = 3f(x) + 3f(y) \quad \forall x, y \in R \quad f'(0) = 0$

$y = 0 \qquad f(0) = 0$

$x = y \qquad 2f(2x) + f(0) = 3f(x) + 3f(x)$

$2f(2x) = 6f(x)\,;\; f(2x) = 3f(x)$

$x = 0 \qquad 2f(y) + f(-y) = 3f(0) + 3f(y)$

$f(-y) = f(y)$ even function.

$x = x \qquad y = h$

$2f(x+h) - 2f(x) = f(x) - f(x-h) + 3f(h)$

$$\lim_{h\to 0} 2\left(\frac{f(x+h) - f(x)}{h}\right) = \lim_{h\to 0} \frac{f(x) - f(x-h)}{h} + \frac{3f(h)}{h}$$

$2f'(x) = f'(x) + 3f'(0)\,; \qquad f'(x) = 0$

$f(x) = $ constant $\qquad f(0) = 0$

$f(x) = 0$

$f'(5) + f(5) = 0$

Hence, the correct answer is (0).

17. $$f(x) = \begin{cases} x^3 & x < 1 \\ ax^2 + bx + c & x \ge 1 \end{cases};$$

From continuity of $f(x)$ at $x = 1$, $1 = a + b + c$

$$f'(x) = \begin{cases} 3x^2 & x < 1 \\ 2ax + b & x \ge 1 \end{cases}; f'\,(1) \text{ exists then } 3 = 2a + b$$

$$f''(x) = \begin{cases} 6x & x < 1 \\ 2a & x \ge 1 \end{cases}; f''\,(1) \text{ exists then } 6 = 2a$$

$a = 3$; $b = -3$; $c = 1$

$a^3 + b^3 + c^3 = 27 - 27 + 1$

Hence, the correct answer is (1).

18. $f(0^-) = f(0) = f(0^+)$

$$\Rightarrow \quad a\cot^{-1}\left(\frac{b}{4}\right) = 2 = \lim_{x\to 0^+}\left(\frac{\ln(1-cx)}{-cx}\right)(-c)$$

$$\Rightarrow \quad a\cot^{-1}\frac{b}{4} = 2 = -c \quad \Rightarrow \frac{b}{4} = \cot\frac{2}{a} \text{ and } c = -2$$

LHD = RHD at $x = 0$

$$\lim_{h\to 0}\frac{a\cot^{-1}\left(\frac{b-h}{4}\right)-2}{-h} = \lim_{h\to 0}\frac{\frac{\ln(1-ch)}{h}-2}{h}$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{\frac{-a}{1+\left(\frac{b-h}{4}\right)^2}\left(\frac{-1}{4}\right)-0}{-1} = \lim_{h\to 0}\frac{l\text{n}(1+2h)-2h}{h^2}$$

$$= \lim_{h\to 0}\frac{\cancel{2h}-\frac{(2h)^2}{2}-\cdots\cancel{-2h}}{h^2}$$

$$\Rightarrow \quad \frac{-a}{4\left(1+\frac{b^2}{16}\right)} = -2 \;\Rightarrow \frac{a}{4} = 2+\frac{2b^2}{16}$$

$$\Rightarrow \quad b^2-2a = -16$$

Now, $b^2-2a+c^6 = -16+(-2)^6 \;= -16+64 = 48$

Hence, the correct answer is (48).

19. $f(0^-) = f(0) = f(0^+)$

$a^2+1 = 0+2 \;\Rightarrow a^2 = 1 \;\Rightarrow a = \pm 1$

$f(3^-) = f(3^+) \;\Rightarrow\; 5 = c-\frac{b^2}{3}$ …(1)

$f'(3^-) = f'(3^+) \;\Rightarrow\; 1 = 0+\left.\frac{b^2}{x^2}\right|_{x=3^+} \quad \Rightarrow b^2 = 9$

From (1) we get $a = \pm 1; b = \pm 3; c = 8$

Number of ordered triplets $(a, b, c) = 4$

Hence, the correct answer is (4).

20. Putting $\tan x = t$, we get $f(t)$ is a polynomial function of the form, $f(t) = \pm t^n + 1$.

When $t = 2$, $f(2) = 9 \Rightarrow n = 3$ $\quad \therefore f(t) = t^3+1$

$f'(2) = 12$ $\quad \therefore \frac{f'(2)}{6} = 2$

Hence, the correct answer is (2).

21. $f(x) = \lim_{n\to\infty}\frac{1-\left(\sin^2\pi x\right)^n}{1+\left(\sin^2\pi x\right)^n} = \begin{cases}1 & \text{if } x \neq \frac{(2n+1)}{2}\\ 0 & \text{if } x = \frac{(2n+1)}{2}\end{cases}$

$g(x) = e^{\frac{2x-1}{2}}$ it can not be zero , it can be zero.

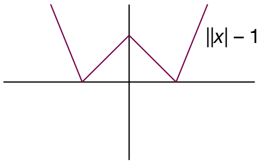
$e^{\frac{2x-1}{2}} = 1 \Rightarrow$ we get $x = 1/2$ but when $x = 1/2$ we get $f(x) = 0$

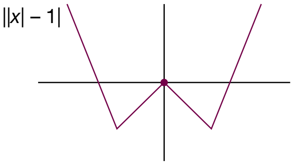
Hence, no solution of given equation.

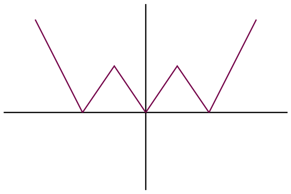
Hence, the correct answer is (0).

22. $f(x) = ||x|-1|$

$g(x) = f(f(x)) = ||f(x)|-1| \;= |||x|-1|-1|$



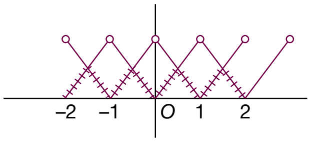


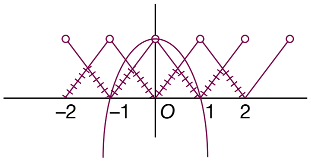


5 point not differentiable.

Hence, the correct answer is (5).

23. $f(x) = \min\{\{x\},\{-x\}\}$





$f(x) = 1-x^2$

$f(x) = 1-x^2$

2 solution $x = 1, -1$

Hence, the correct answer is (2).

24. $\frac{1}{17} \le f(x) \le \frac{1}{9} \Rightarrow \frac{k}{17} \le kf(x) \le \frac{k}{9} < 1$

For it to be continuous, then $k < 9$

$k_{\max} = 8$

Hence, the correct answer is (8).

25. Using 1st principles for $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f(x)+2h^2+k2x-f(x)}{h} = kx$$

$f(x) = \frac{kx^2}{2} + C$

$2 = \frac{k}{2} + C$ …(1)

$8 = 2k + C$ …(2)

$\frac{3k}{2} = 6;\quad k = 4$

$f'(x) = 4x$

Hence, the correct answer is (4).

26. $\because\; f(x) = \left|x^2-3x+2\right| + \cos|x|$

$= |(x-1)||(x-2)| + \cos|x|$

$$f(x) = \begin{cases}x^2-3x+2+\cos x, \; \text{x<1}\\ -x^2-3x-2+\cos x, 1 \le x < 2\\ x^2-3x+2+\cos x, x > 2\end{cases}$$

$$\therefore f'(x) = \begin{cases}2x-3-\sin x, x < 1\\ -2x+3-\sin x, 1 \le x < 2\\ 2x-3-\sin x, x > 2\end{cases}$$

It is clear $f(x)$ is not differentiable at $x = 1$ and $x = 2$.

$\therefore\; f'\left(1^-\right) = -1-\sin 1$ and $f'\left(1^+\right) = 1-\sin 1$.

Hence, the correct answer is (2).

27. $f(x) = [\tan x\,[\cot x]]$

If $\cot x \notin I,\ [\cot x] < \cot x \Rightarrow 0 < \tan x[\cot x] < 1$

$\therefore\ f(x) = 0$ If $\cot x \in I,\ f(x) = 1$

Points of discontinuity occurs if $\cot x \in I$.

$$\frac{\pi}{12} \le x < \frac{\pi}{2} \Rightarrow \cot 15^\circ \ge \cot x > \cot \frac{\pi}{2}^{-}$$

$0 < \cot x \le 2 + \sqrt{3}$ i.e. $\cot x = 1, 2, 3$

Hence, the correct answer is (3).

28. The points of discontinuities will be same as points where the function is not defined i.e., $\sin 1/x = 0$

$$\Rightarrow \quad \frac{1}{x} = n\pi \Rightarrow x = \frac{1}{n\pi}$$

$$\Rightarrow \quad \frac{1}{2013\pi} < \frac{1}{n\pi} < \frac{1}{2007\pi} \Rightarrow \frac{1}{2013} < \frac{1}{n} < \frac{1}{2007}$$

$\Rightarrow\ 2007 < n < 2013$

Number of integers between 2007 and 2013 are 5. $\therefore\ K = 5$

Hence, the correct answer is (5).

29. $[x^2]$ is discontinuous at $\sqrt{2}, \sqrt{3}$

$[x^3]$ is discontinuous at $\sqrt[3]{2}, \sqrt[3]{3}, \sqrt[3]{4}, \sqrt[3]{5}, \sqrt[3]{6}, \sqrt[3]{7}$

$\Rightarrow [x^2] + [x^3]$ is discontinuous at 8 points.

Hence, the correct answer is (8).

30. $$f(x) = \left[\frac{\sin^2 x + (1 - \sin^2 x)}{\cos^2 x + \sin^4 x}\right] = \left[\frac{\sin^2 x + 1 + \sin^4 x - 2\sin^2 x}{\cos^2 x + \sin^4 x}\right]$$

$f(x) = [1] = 1$ which is constant function

Number of discontinuous $= 0$

Hence, the correct answer is (0).

31. Clearly, $\left[\dfrac{(x-2)^3}{a}\right] = 0$, for $f(x)$ to be continuous.

$4 \le x \le 6 \Rightarrow (x-2)^3 \in [8, 64]$

$\Rightarrow a > 64 \Rightarrow a = 72, 81, 90$ are numbers which are two digit numbers whose sum of digits are 9.

No of values $= 3$

Hence, the correct answer is (3).

32. $[\sin^{-1} x]$ is discontinuous at

$x = -\sin 1, 0, \sin 1$

$[x]$ is discount at $x = -1, 0, 1$

$$\left.\begin{array}{l} f(0^-) = (-1) - (-1) = 0 \\ f(0^+) = 0 - 0 = 0 \end{array}\right\} \text{at 0 continuous } f(0) = 0$$

$\therefore$ Discontinuous at $x = -\sin 1, \sin 1, -1, 1$

Number of points $= 4$

Hence, the correct answer is (3).

33. $x^2 - (a-1)x + 4 > 0\ \forall x \in R$

$\Rightarrow \Delta < 0$

$\Rightarrow (a-1)^2 - 4^2 < 0 \ \Rightarrow (a-5)(a+3) < 0$

$\Rightarrow (a-5)(a+3) < 0$

$\Rightarrow a = -2, -1, 0, 1, 2, 3, 4$

Number of values $= 7$

Hence, the correct answer is (7).

34. Given, $f(x) = ([2x] + \{2x\})([2x] - \{2x\})$.

$f(x) = 2x(2x - 2\{2x\}) = 4x^2 - 4x\{2x\}$

Hence, 6 points of discontinuity where $2x$ is an integer $x = -3/2, -1, -1/2, 1/2, 1, 3/2$.

are there as the function is continuous at $x = 0$.

Hence, the correct answer is (6).

35. Hence, the continuity must be checked at $x = 1, 2, e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}, 3.5$

Further, $f(1) = 0$ and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \operatorname{sgn}(x-2) \times [\log_e x] = 0$$

Hence, $f(x)$ is continuous at $x = 1$;

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \operatorname{sgn}(x-2) \times [\log_e x] = (-1) \times 0 = 0$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \operatorname{sgn}(x-2) \times [\log_e x] = (+1) \times 0 = 0$$

Hence, $f(x)$ is continuous at $x = 2$;

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \operatorname{sgn}(x-2) \times [\log_e x] = 1$$

$$\Rightarrow \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \{x^2\} = 0$$

Hence, $f(x)$ is discontinuous at $x = 3$.

Also $\{x^2\}$ is discontinuous at $x = \sqrt{10}, \sqrt{11}, \sqrt{12}$

Therefore, $\lim\limits_{x \to 3.5^-} f(x) = \lim\limits_{x \to 3^-} \{x^2\} = 0.25 = f(3.5)$

Hence, $f(x)$ is discontinuous at $x = e, 3, \sqrt{10}, \sqrt{11}, \sqrt{12}$

Hence, the correct answer is (5).

36. $$\text{LHL} = \lim_{x \to \left(\frac{\pi^-}{2}\right)} \left(\frac{1 - \sin^3 x}{3\cos^2 x}\right) = \lim_{h \to 0} \frac{1 - \sin^3\left(\frac{\pi}{2} - h\right)}{3\cos^2\left(\frac{\pi}{2} - h\right)}$$

$$= \lim_{h \to 0} \frac{1 - \cos^3 h}{3\sin^2 h} = \lim_{h \to 0} \frac{(1 - \cos h)(1 + \cos^2 h + \cos h) \cdot h^2}{h^2 3\sin^2 h}$$

$$= \frac{1}{2} \times \frac{1}{3} \times (3) = \frac{1}{2}$$

$$a = \frac{1}{2};$$

$$\text{RHL} = \lim_{x \to \frac{\pi^+}{2}} \frac{b(1 - \sin x)}{(\pi - 2x)^2} = \lim_{h \to 0} \frac{b(1 - \cos h)}{4h^2} = \frac{b}{8}$$

$$\Rightarrow \quad \frac{b}{8} = \frac{1}{2} \Rightarrow b = 4$$

Hence, the correct answer is (8).

37. $$f'(x) = \begin{cases} \dfrac{1}{1 + x^2} \ if\ |x| \le 1 \\ \dfrac{|x|}{2x} \ if\ |x| > 1 \end{cases}$$

It is not differentiable at $x = \{-1\}$

Hence, the correct answer is (7).

38. $f(x) = x\sin\left([x]^2 - 1\right)\left([x]^2 - 4\right)$

$$\lim_{x \to 0} f(x) = 0$$

$\therefore f(x)$ is continuous at $x = 0$

$$\lim_{x\to 1^-} f(x) = -1\sin((-1)(-4)) = -\sin 4$$

$$\lim_{x\to 1^+} f(x) = 1\sin\theta = 0$$

$\therefore$ Discontinuous at $x = 1$

$$\lim_{x\to 2^+} f(x) = 0 = \lim_{x\to 2^-} f(x) = 0$$

$\therefore$ Continuous at $x = 2$

Similarly, it is continuous at $x = -1$ but not at -2

$f(x)$ is discontinuous at $-4, -3, -2, 1, 3, 4$.

Hence, the correct answer is (6).

39. $|1-x|$ is differentiable at all points except $x = 1$.

$[x]$ is not differentiable at $x = 0, 1, 2, 3$ but differentiable at $x = -1$ at with right derivative of $f(x)$ equal to $-1, -1 \le x < 0$.

Hence, the correct answer is (4).

40. $\frac{\pi}{2} = \frac{3.14}{2} = 1.57$

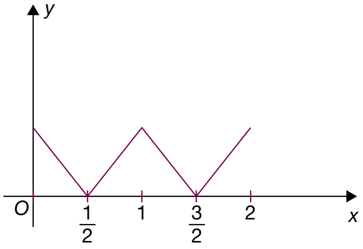
$\frac{3\pi}{2} = \frac{9.42}{2} = 4.71$

From figure it is not diff

at $x = n/2,\ n \in Z$

$x = 2, 2.5, 3, 3.5, 4, 4.5$

No. of points = 6



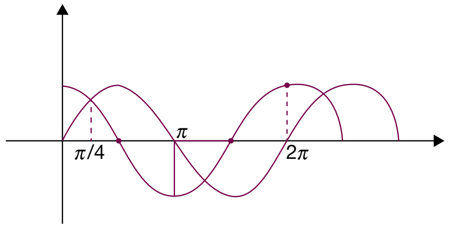
Hence, the correct answer is (6).

41. $f(x) = 2x + 3x - 6 - 15 + 20x;\ \frac{3}{4} < x < 2$

$f(x) = 25x - 21$ *for* $\frac{3}{4} < x < 2$; $f'(x) = 25$

Hence, the correct answer is (5).

42.



$$f(x) = \begin{cases} \cos x & \text{if } 0 < x < \frac{\pi}{4} \\ \sin x & \text{if } \frac{\pi}{4} \le x < \pi \\ 0 & \text{if } \pi \le x < \frac{3\pi}{2} \\ \cos x & \text{if } \frac{3\pi}{2} \le x < 2\pi \end{cases}$$

f is not differentiable at $x = \frac{\pi}{4}, \pi, \frac{3\pi}{2}$

Hence, the correct answer is (3).

43. $\because$ $f(x)$ is continuous and get only irrational values.

$\Rightarrow f(x)$ is constant let $f(x) = C \quad \Rightarrow \lambda(10C) = 40C \quad \Rightarrow \lambda = 4$

Hence, the correct answer is (4).

44. $f(|x|) = \begin{cases} \sin|x|, & |x| < 0 \\ \cos(x) - \big||x|-1\big|, & |x| \ge 0 \end{cases}$

$\Rightarrow f(|x|) = \cos(x) - \big||x|-1\big|, x \in R$

[As, $|x| < 0$ is not possible and $|x| \ge 0$ is true $\forall x \in R$]

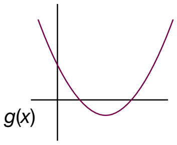
$\Rightarrow f(|x|) = \cos(x) - \big||x|-1\big|, x \in R$

Which is non-differentiable at $x = 0$ and when $|x| - 1 = 0$ or $x = \pm 1$.

Hence, $f(|x|)$ has exactly three points of non - differentiability.

Hence, the correct answer is (3).

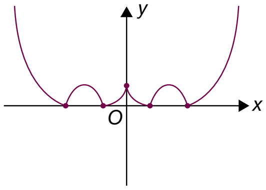
45. given $f(x) = \left|x^2 + (\alpha - 1)|x| - \alpha\right|$



Take $g(x) = x^2 + (\alpha - 1)x - \alpha = (x + \alpha)(x - 1)$

$\Rightarrow f(x) = (|x| - 1)(|x| + \alpha)$

From graph it is clear that $f(x)$ is not differentiable at '5' points.



Hence, the correct answer is (5).

46. $1 = f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x(1 + a(1 - \frac{x^2}{\underline{|2}} + ...) - b(x - \frac{x^3}{\underline{|3}} + ...}{x^3}$

$$= \lim_{x\to 0} \frac{x(1 + a - b) + x^3(\frac{-a}{2} + \frac{b}{6}) + x^5(\lambda) + ...}{x^3}$$

$\Rightarrow 1 + a - b = 0$ and $\frac{-a}{2} + \frac{b}{6} = 1 \Rightarrow a = \frac{-5}{2}, b = \frac{-3}{2}$ and $2a - 8b = 7$

Hence, the correct answer is (7).

47. In the vicinity of $x = 0$, we have

$$x^2 \sum_{r=0}^{\left[\frac{1}{|x|}\right]} r \quad = x^2\left(1 + 2 + 3 + ...\left[\frac{1}{|x|}\right]\right)$$

Use sandwich theorem

$$P = \left(1 + 2 + 3 + ...\left[\frac{1}{|x|}\right]\right) = \frac{x^2\left(1 + \left[\frac{1}{|x|}\right]\right)}{2}\left[\frac{1}{|x|}\right]$$

So, $\frac{1}{2}(1 - |x|) < P \le \frac{1}{2}(1 + |x|)$

Then, the limit is $\frac{1}{2}$

Hence, the correct answer is (1).

48. Rewrite the equation as

$$1 + f(x + y) = (1 + f(x))(1 + f(y))$$

Put $g(x) = 1 + f(x)$ to get

$$g(x + y) = g(x)\, g(y)$$

We get $g(x) = a^{kx} = 1 + f(x) \Rightarrow f(x) = a^{kx} - 1$

$$f'(x) = a^{kx} \ln a^k - 0$$

$$f'(0) = 1 = \ln a^k \textit{ gives } a^k = e$$

$\therefore f(x) = e^x - 1$,

$$\left[\frac{f(4)}{f(2)}\right]=\left[\frac{e^4-1}{e^2-1}\right]=[e^2+1]=8$$

Hence, the correct answer is (8).

49. $$f(0)=\lim_{x\to 0}\frac{\ln(1+x)^{1+x}-x}{x^2}=\lim_{x\to 0}\frac{(1+x)\ln(1+x)-x}{x^2}$$

$$=\lim_{x\to 0}\frac{1+\ln(1+x)-1}{2x}=\frac{1}{2}\ \therefore 6f(0)=3$$

Hence, the correct answer is (3).

50. $$f(x)=\begin{cases} ax^3+b & ,\ 0\le x\le 1 \\ 2\cos\pi x+\tan^{-1}x & ,\ 1<x\le 2 \end{cases}$$

$$f'(x)=\begin{cases} 3ax^2 & ,\ 0<x<1 \\ -2\pi\sin\pi x+\dfrac{1}{1+x^2} & ,\ 1<x<2 \end{cases}$$

As the function is differentiable in [0, 2]

$\Rightarrow$ Function is differentiable at $x=1$

$\therefore\quad f'(1^-)=f'(1^+)\ \Rightarrow\ 3a=\frac{1}{2}\ \Rightarrow a=\frac{1}{6}$

Function will also be continuous at $x=1$

$\therefore\quad \lim_{x\to 1^-} f(x)=\lim_{x\to 1^-} f(x)$

$\Rightarrow\quad a+b=-2+\frac{\pi}{4}$

$\therefore\quad b=-2-\frac{1}{6}+\frac{\pi}{4}=\frac{\pi}{4}-\frac{13}{6}\Rightarrow k_1=6\ \&\ k_2=12$

$\Rightarrow\quad k_1^2+k_2^2=180$

Hence, the correct answer is (180).

51. $$f'(x)=\lim_{x\to 0^+}\frac{|x|^p\sin\frac{1}{x}+x|\tan x|^q-0}{x}$$

$$=\lim_{x\to 0^+}\left(x^{p-1}\sin\frac{1}{x}+|\tan x|^q\right)=0$$

If $p-1>0$ and $q>0$ ...(1)

$$\lim_{x\to 0^-}\left((-1)^p x^{p-1}\sin\frac{1}{x}+|\tan x|^q\right)=0$$

If $p-1>0$ and $q>0$ ...(2)

$p+q>1$

Least value of $[p\,q]=1$.

Hence, the correct answer is (1).

52. We have, $h(x)=\lim_{n\to\infty}\dfrac{x^{n+1}t(x)+x^2}{x^n+g(x)}$, $x>0$

$$\therefore\quad h(x)=\begin{cases} \dfrac{x^2}{g(x)}, & 0<x<1 \\ f(x), & x>1 \\ \dfrac{f(1)+1}{1+g(1)}, & x=1 \end{cases}$$

Since, $h(x)$ is continuous for $x>1$. $\therefore h(1)=h(1^+)=h(1^-)$

$\Rightarrow\quad \frac{1}{g(1)}=f(1)\quad\Rightarrow f(1)\cdot g(1)=1$

Hence, the correct answer is (1).

ARCHIVE: JEE MAIN

1. $f(x)=[x]\,|x^2-1|+\sin\dfrac{\pi}{[x]+3}-[x+1]$

$$f(x)=\begin{cases} 3-2x^2, & -2<x<-1 \\ x^2, & -1\le x<0 \\ \dfrac{\sqrt{3}}{2}+1 & 0\le x<1 \\ x^2+1+\dfrac{1}{\sqrt{2}}, & 1\le x<2 \end{cases}$$

discontinuous at $x=0, 1$

Hence, the correct answer is (2).

2. $f(x)=|2x+1|-3|x+2|+|x^2+x-2|$

$=|2x+1|-3|x+2|+|x+2|\,|x-1|$

$=|2x+1|+|x+2|(|x-1|-3)$

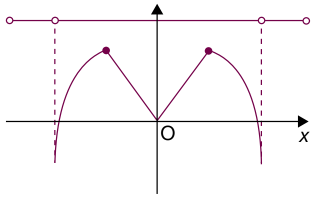
Critical points are $x=\frac{-1}{2}, -2, -1$

but $x=-2$ is making a zero.

twice in product so, points of non-differentiability are $x=\frac{-1}{2}$ and $x=-1$

$\therefore$ Number of points of non-differentiability = 2

Hence, the correct answer is (2).

3. 

$$f(x)=\begin{cases} \min\{|x|,2-x^2\}, & -2\le x\le 2 \\ [|x|], & 2<|x|\le 3 \end{cases}$$

$\Rightarrow\ x\in[-3,-2)\cup(2,3]$

Number of points of non-differentiability in (–3, 3) = 5

Hence, the correct answer is (5).

4. $$g[f(x)]=\left[\begin{array}{ll} f(x)+1 & f(x)<0 \\ (f(x)-1)^2+b & f(x)\ge 0 \end{array}\right.$$

$$g[f(x)]=\left[\begin{array}{ll} x+a+1 & x+a<0 \text{ and } x<0 \\ |x-1|+1 & |x-1|<0 \text{ and } x\ge 0 \\ (x+a-1)^2+b & x+a\ge 0 \text{ and } x\ge 0 \\ (|x-1|-1)^2+b & |x-11|\ge 0 \text{ and } x\ge 0 \end{array}\right.$$

$$g[f(x)]=\left[\begin{array}{ll} x+a+1 & x\in(-\infty,-a) \text{ and } x\in(-\infty,0) \\ |x-1|+1 & x\in\phi \\ (x+a-1)^2+b & x\in[-a,\infty) \text{ and } x\in(-\infty,0) \\ (|x-1|-1)^2+b & x\in R \text{ and } x\in[0,\infty) \end{array}\right. ;$$

$$g[f(x)]=\left[\begin{array}{ll} x+a+1 & x\in(-\infty,-a) \\ (x+a-1)^2+b & x\in[-a,0) \\ (|x-1|-1)^2+b & x\in[0,\infty) \end{array}\right.$$

$g(f(x))$ is continuous

at $x=-a$ and at $x=0$

$1=b+1$ and $(a-1)^2+b=b$

$b=0$ and $a=1\Rightarrow a+b=1$

Hence, the correct answer is (1).

5. $f(x) = x^3 - 6x^2 + 9x - 3$

$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$

$f(1) = 1$

$f(3) = -3$

$$g(x) = \begin{bmatrix} f(x) & 0 \le x \le 1 \\ 1 & 1 \le x \le 3 \\ 4-1 & 3 < x \le 4 \end{bmatrix}$$

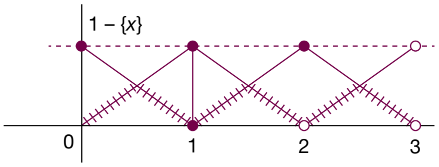
$g(x)$ is continuous

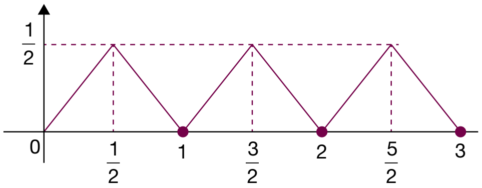
$$g'(x) = \begin{bmatrix} 3(x-1)(x-3) & 0 \le x \le 1 \\ 0 & 1 \le x \le 3 \\ -1 & 3 < x \le 4 \end{bmatrix}$$

$g(x)$ is non-differentiable at $x = 3$

Hence, the correct answer is (1).

6. $1 - \{x\} = 1 - x;\ 0 \le x < 1$





Non-differentiable at

$$x = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$$

$\Rightarrow\ p = 0, Q = 5$

Hence, the correct answer is (5).

7. $f(x)$ is continuous on R

$\Rightarrow\ f(1^-) = f(1) = f(1^+)$

$|a + 1 + b| = \lim\limits_{x \to 1} \sin(\pi x)$

$|a + 1 + b| = 0 \Rightarrow a + b = -1$ …(1)

$\Rightarrow$ Also $f(-1^-) = f(-1) = f(-1^+)$

$$\lim_{x \to -1} 2\sin\left(\frac{-\pi x}{2}\right) = |a - 1 + b|$$

$|a - 1 + b| = 2$

Either $a - 1 + b = 2$ or $a - 1 + b = -2$

$a + b = 3$ …(2)

or

$a + b = -1$ …(3)

from (1) and (2) $\Rightarrow a + b = 3 = -1$ (reject)

from (1) and (3) $\Rightarrow a + b = -1$

Hence, the correct answer is (B).

8. $$f(g(x)) = \begin{cases} g(x) + 2, & g(x) < 0 \\ (g(x))^2, & g(x) \ge 0 \end{cases}$$

$$= \begin{cases} x^3 + 2, & x < 0 \\ x^6, & x \in [0, 1] \\ (3x-2)^2, & x \in [1, \infty] \end{cases};\ (fog(x))' = \begin{cases} 3x^3, & x < 0 \\ 6x^5, & x \in (0, 1) \\ 2(3x-2) \times 3, & x \in (1, \infty) \end{cases}$$

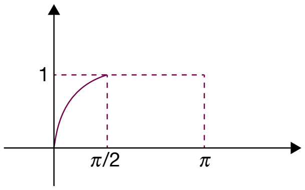
At ‘O’

L.H.L. $\ne$ R.H.L. (Discontinuous) $\Rightarrow$ not differentiable at $x = 0$

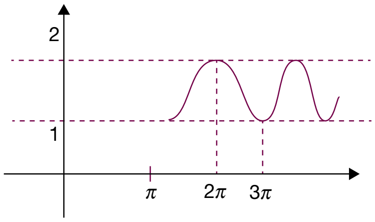
At ‘I’; L.H.D. = 6 = R.H.D. $\Rightarrow fog(x)$ is differentiable for $x \in R - \{0\}$

Hence, the correct answer is (B).

9. Graph of $\max\{\sin t : 0 \le t \le x\}$ in $x \in [0, \pi]$

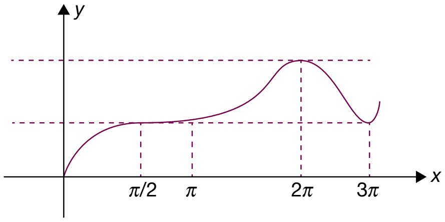


and graph of $2 + \cos x$ for $x \in [\pi, \infty)$



So, graph of

$$f(x) = \begin{cases} \max\{\sin t : 0 \le t \le x, & 0 \le x \le \pi \\ 2 + \cos x & x > \pi \end{cases}$$



$f(x)$ is differentiable everywhere in $(0, \infty)$

Hence, the correct answer is (C).

10. If $f(x)$ is continuous at $x = 0$, RHL = LHL = $f(0)$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\cos^2 x - \sin^2 x - 1}{\sqrt{x^2+1} - 1} \cdot \frac{\sqrt{x^2+1}+1}{\sqrt{x^2+1}+1} \text{ (Rationalisation)}$$

$$\lim_{x \to 0^+} -\frac{2\sin^2 x}{x^2} \cdot \left(\sqrt{x^2+1}+1\right) = -4;\ \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} \frac{1}{x} \ln\left(\frac{1+\frac{x}{a}}{1-\frac{x}{b}}\right)$$

$$\lim_{x \to 0^-} \frac{\ln\left(1+\frac{x}{a}\right)}{\left(\frac{x}{a}\right).a} + \frac{\ln\left(1-\frac{x}{b}\right)}{\left(-\frac{x}{b}\right).b} = \frac{1}{a} + \frac{1}{b} \text{ So, } \frac{1}{a} + \frac{1}{b} = -4 = k$$

$$\Rightarrow\ \frac{1}{a} + \frac{1}{b} + \frac{4}{k} = -4 - 1 = -5$$

Hence, the correct answer is (A).

11. $$k = \lim_{x \to 0}\left(\frac{\ln(1+3x)}{x} - \frac{\ln(1-2x)}{x}\right)$$

$k = 3 + 2 = 5$

Hence, the correct answer is (5).

12. $x \in (-10, 10) \Rightarrow \frac{x}{2} \in (-5, 5)$ 9 integers

$f(0) = f(0^+) = f(0^-) = 0$ continuous at $x = 0$

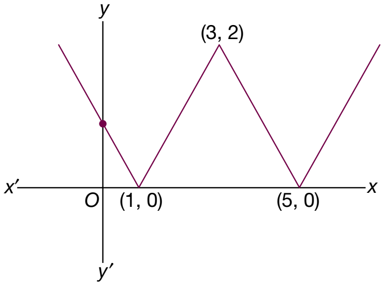
Function will be discontinuous when $\frac{x}{2} = \pm 4, \pm 3, \pm 2, \pm 1$

8 points of discontinuity

Hence, the correct answer is (8).

13. $f(x) = |2 - |x-3||$

f is not differentiable at $x = 1, 3, 5$



$$\sum_{x\in S} f(f(x)) = f(f(1)) + f(f(3)) + f(f(5))$$

$= f(0) + f(2) + f(0) = 1 + 1 + 1 = 3$

Hence, the correct answer is (3).

14. Since, $\lim\limits_{x\to 0} \dfrac{f(x)}{x}$ exist $\Rightarrow f(0) = 0$

Now, $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim\limits_{h\to 0} \dfrac{f(h) + xh^2 + x^2h}{h}$ (take $y = h$)

$= \lim\limits_{h\to 0} \dfrac{f(h)}{h} + \lim\limits_{h\to 0} (xh) + x^2$

$\Rightarrow f'(x) = 1 + 0 + x^2 \Rightarrow f'(3) = 10$

Hence, the correct answer is (10).

15.
$$f'(x) = \begin{cases} -x^3\cos\left(\dfrac{1}{x}\right) + 5x^4\sin\left(\dfrac{1}{x}\right) + 10x &, \quad x < 0 \\ 0 &, \quad x = 0 \\ x^3\sin\left(\dfrac{1}{x}\right) + 5x^4\cos\left(\dfrac{1}{x}\right) + 2\lambda x &, \quad x > 0 \end{cases}$$

LHD of $f'(x)$ at $x = 0$ is 10 and RHD of $f'(x)$ at $x = 0$ is 2λ

If $f''(0)$ exists then $2\lambda = 10 \Rightarrow \lambda = 5$

Hence, the correct answer is (5).

16.
$$f(x) = \begin{cases} \dfrac{\pi}{4} + \tan^{-1} x &, \quad x \in [-1, 1] \\ -\dfrac{(x+1)}{2} &, \quad x \in (-\infty, -1) \\ \dfrac{x-1}{2} &, \quad x \in (1, \infty) \end{cases}$$

For continuity at $x = -1$

LHL $= \dfrac{\pi}{4} - \dfrac{\pi}{4} = 0$ and RHL $= 0$ so continuous at $x = -1$

$f'(-1^+) = 1/2$

$f'(-1^-) = -1/2$ non –differentiable at $x = -1$

for continuity at $x = 1$

RHL = 0 and LHL = $\dfrac{\pi}{4} + \dfrac{\pi}{4} = \dfrac{\pi}{2}$ so discontinuous at $x = 1$, not differentiable.

Hence, the correct answer is (A).

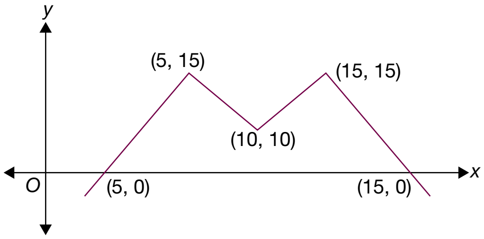
17. $f(x) = 15 - |x - 10|$

$g(x) = f[f(x)] = 15 - |f(x) - 10|$

$= 15 - |15 - |x - 10| - 10|$

$= 15 - |5 - |x - 10||$

$\therefore$ $g(x)$ is not differentiable at $x = 5, 10, 15$



Hence, the correct answer is (B).

18. $g'(c^+) = \lim\limits_{h\to 0} \dfrac{|f(c+h)| - |f(c)|}{h} = \lim\limits_{h\to 0} \dfrac{|f(c+h)|}{h}$ $\qquad (\because f(c) = 0)$

$= \lim\limits_{h\to 0} \left|\dfrac{f(c+h) - f(c)}{h}\right| . \dfrac{|h|}{h} = \lim\limits_{h\to 0} |f'(c)| \dfrac{|h|}{h} = |f'(c)|$

$g'(c^-) = \lim\limits_{h\to 0} \dfrac{|f(c-h)| - |f(c)|}{-h} = \lim\limits_{h\to 0} \dfrac{|f(c-h)|}{-h}$ $\qquad (\because f(c) = 0)$

$= \lim\limits_{h\to 0} \left|\dfrac{f(c-h) - f(c)}{-h}\right| . \dfrac{|-h|}{-h} = \lim\limits_{h\to 0} -|f'(c)| = -|f'(c)|$

i.e. $g(x)$ is differentiable at $x = c$ if $|f'(c)| = -|f'(c)|$

Hence, $f'(c) = 0$

Hence, the correct answer is (A).

19. For f to be continuous at $x = 1$

$f(1^-) = f(1^+) \Rightarrow 5 = a + b$ …(1)

For f to be continuous at $x = 3$

$f(3^-) = f(3^+) \Rightarrow a + 3b = b + 15 \Rightarrow a + 2b = 15$ …(2)

From (1) and (2)

$5 + b = 15 \Rightarrow b = 10$ and $a = -5$

For f to be continuous at $x = 5$

$f(5^-) = f(5^+) \Rightarrow b + 25 = 30 \Rightarrow b = 5$

For $f(x)$ to be continuous we are not getting same values of a and b hence, $f(x)$ is not continuous for any values of a and b.

Hence, the correct answer is (D).

20.
$$f(x) = \begin{cases} |x| + [x] & -1 \le x < 1 \\ x + |x| & 1 \le x < 2 \\ x + [x] & 2 \le x \le 3 \end{cases} = \begin{cases} -x-1 & x \in [-1, 0) \\ x & x \in [0, 1) \\ 2x & x \in [1, 2) \\ x+2 & x \in [2, 3) \\ 6 & x = 3 \end{cases}$$

$f(x)$ is discontinuous at $x = 0, 1, 3$

Hence, the correct answer is (B).

21. $\lim\limits_{x\to\frac{\pi}{4}} \left(\dfrac{\sqrt{2}\cos x - 1}{\cot x - 1}\right) = f\left(\dfrac{\pi}{4}\right) \Rightarrow \lim\limits_{x\to\frac{\pi}{4}} \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1} = k$

Using L'Hospital's Rule

$\Rightarrow \lim\limits_{x\to\frac{\pi}{4}} \dfrac{\sqrt{2}(-\sin x)}{-\text{cosec}^2 x} = \sqrt{2}\left(\dfrac{1}{\sqrt{2}}\right)^3 = k \Rightarrow k = \dfrac{1}{2}$

Hence, the correct answer is (C).

22. Continuous at $x = 5$ $\qquad \therefore$ LHL = RHL = $f(5)$

$\Rightarrow b|\pi - 5| + 3 = a|\pi - 5| + 1$

$\Rightarrow -b(\pi - 5) + 3 = a(\pi - 5) + 1$

$\Rightarrow (a - b)(\pi - 5) = -2$

$\Rightarrow a - b = \dfrac{-2}{\pi - 5} = \dfrac{2}{5 - \pi}$

Hence, the correct answer is (D).

23. $f(x) = [x] - \left[\dfrac{x}{4}\right]$

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} \left\{[x] - \left[\frac{x}{4}\right]\right\} = 4 - 1 = 3$$

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \left\{[x] - \left[\frac{x}{4}\right]\right\} = 3 - 0 = 3$$

$f(4) = 3$ $\quad \therefore$ continuous at $x = 4$

Hence, the correct answer is (D).

24. RHL $= \displaystyle\lim_{x\to 0^+} \frac{\sqrt{x+x^2} - \sqrt{x}}{x^{3/2}} = \lim_{x\to 0^+} \frac{\sqrt{1+x} - 1}{x} = \frac{1}{2}$

$$\text{LHL} = \lim_{x\to 0^-} \frac{\sin(p+1)x + \sin x}{x} = p + 2$$

for function to be continuous

$$\text{LHL} = \text{RHL} = f(0) \Rightarrow (p, q) = \left(-\frac{3}{2}, \frac{1}{2}\right)$$

Hence, the correct answer is (C).

25. $f(x) = |x - \pi|\ (x^{|x|} - 1)\sin|\ x\ |$

$$\text{At } x = \pi: \text{RHD} = \lim_{h\to 0} \frac{|\pi + h - \pi|(e^{|x+h|} - 1)\sin|\pi + h| - 0}{h}$$

$$\text{LHD} = \lim_{h\to 0} \frac{|\pi - h - \pi|(e^{|x+h|} - 1)\sin|\pi - h| - 0}{-h} = 0$$

$\because$ RHD = LHD

Therefore, function is differentiable at $x = \pi$

$$\text{At } x = 0: \text{ R.H.D} = \lim_{h\to 0} \frac{|h - \pi|(e^{|h|} - 1)\sin|h| - 0}{h} = 0$$

$$\text{L.H.D} = \lim_{h\to 0} \frac{|-h - \pi|(e^{|-h|} - 1)\sin|-h| - 0}{-h} = 0$$

$\therefore$ RHD = LHD

Therefore, function is differentiable at $x = 0$

Since, the function $f(x)$ is differentiable at all the points including π and 0.

i.e., $f(x)$ is every where differentiable

Therefore, there is no element in the sec S.

$\Rightarrow$ $S = f$ (an empty set)

Hence, the correct answer is (4).

26. $f(x) = |\log_e 2 - \sin x|$

$\therefore$ $f(f(x)) = |\log_e 2 - \sin|\log_e 2 - \sin x||$

In the neighbourhood of $x = 0$

$g(x) = \log_e 2 - \sin(\log_e 2 - \sin x)$

Hence, $g(x)$ is differentiable at $x = 0$ as it is sum and composite of differentiable function.

$g'(x) = \cos(\log_e 2 - \sin x).\cos x$

$\therefore$ $g'(0) = \cos(\log_e 2)$

Hence, the correct answer is (C).

27. $\displaystyle\lim_{x\to 0} f\left(\frac{1 - \cos 3x}{x^2}\right) = f\left(\lim_{x\to 0} \frac{1 - \cos 3x}{x^2}\right)$

$$= f\left(\lim_{x\to 0} \frac{3\sin 3x}{2x}\right) = f\left(\frac{9}{2}\right) = \frac{2}{9}$$

Hence, the correct answer is (C).

ARCHIVE: JEE ADVANCED

1. $f(x) = x^2(x - 1) + (x - 1)\sin x = (x - 1)(x^2 + \sin x)$

$f\,g(x) = (x - 1)(x^2 + \sin x)\,g(x)$

$f\,g(1^+) = f\,g(1^-) = f\,g(1) = 0$ $f\,g(x)$ always continuous

$$\text{RHD} = \lim_{h\to 0} \frac{fg(1+h) - fg(1)}{h}$$

$$= \lim_{h\to 0} \frac{h((1+h)^2 + \sin(1+h))}{h}\, g(1+h)$$

$$= \lim_{h\to 0} (1 + \sin 1)\, g(1+h)$$

$$\text{LHD} = \lim_{h\to 0} \frac{fg(1-h) - fg(1)}{-h}$$

$$= \lim_{h\to 0} \frac{-h((1-h)^2 + \sin(1-h))g(1-h)}{-h}$$

$$= \lim_{h\to 0} (1 + \sin 1)g(1 - h)$$

Clearly, we can say that if g is continuous at $x = 1$, then $f\,g$ is differentiable at $x = 1$.

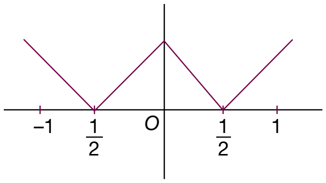
If g is differentiable at $x = 1$, then $f\,g$ is differentiable at $x = 1$

But we can not comment on converse of above two

Hence, (A) and (C) are correct.

2. $f(g(x)) = |2\{x\} - 1| + |2\{x\} + 1|$ for $x \in (-1, 1)$ we have $0 \le \{x\} < 1$

$f(g(x)) = |2\{x\} - 1| + 2\{x\} + 1$



$|2\{x\} - 1|$

$|2\{x\} - 1|$ is always continuous for $x \in (-1, 1)$ and not differentiable at $x = \pm\dfrac{1}{2}$

$2\{x\} + 1$ is discontinuous at $x = 0$ and not differentiable at $x = 0$

$f(g(x)) = |2\{x\} - 1| + 2\{x\} + 1$ is discontinuous at $x = 0$ only and not differentiable at $x = \pm\dfrac{1}{2}, 0$

$c + d = 1 + 3 = 4$

Hence, the correct answer is (4).

3. $f(x + y) = f(x) + f(y) + f(x)f(y)$

replacing y by h

$f(x + h) - f(x) = f(h)(1 + f(x))$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(h)(1 + f(x))}{h}$$

$$= \lim_{h\to 0} \frac{hg(h)(1 + f(x))}{h}$$

$$f'(x) = (1 + f(x)) \lim_{h\to 0} g(h) = 1 + f(x)$$

$$\int \frac{f'(x)}{f(x)+1}\,dx = \int dx \Rightarrow \ln(f(x) + 1) = x + c$$

$\Rightarrow$ $f(x) + 1 = e^{x+c}$ $(f(0) = 0)$

$f(x) = e^x - 1\ \forall\ x \in R$ it is always differentiable $f'(x) = e^x$

$\Rightarrow$ $f'(0) = 1$ and $f'(1) = e$

$f(x) = xg(x);\ \displaystyle\lim_{x\to 0} g(x) = 1$

$$g(x)=\frac{f(x)}{x}\text{; let } g(0)=1$$

$$g(x)=\begin{cases}1 & x=0\\ \dfrac{e^x-1}{x} & x\neq 0\end{cases}$$

$$\text{RHD}=g'(0^+)=\lim_{h\to 0}\frac{\dfrac{e^h-1}{h}-1}{h}$$

$$=\lim_{h\to 0}\frac{e^h-1-h}{h^2}=\lim_{h\to 0}\frac{1+h+\dfrac{h^2}{2!}+....-1-h}{h^2}=\frac{1}{2}$$

$$\text{LHD}=g'(0^-)=\lim_{h\to 0}\frac{\dfrac{e^{-h}-1}{-h}-1}{-h}$$

$$=\lim_{h\to 0}\frac{e^{-h}-1+h}{h^2}=\lim_{h\to 0}\frac{1-h+\dfrac{h^2}{2!}+....-1+h}{h^2}=\frac{1}{2}$$

At $x=0$ it is differentiable hence, $g(x)$ is differentiable at every real x.

Hence, (A), (B) and (D) are correct.

4. Given, relation is $f(x+y)=f(x)f'(y)+f'(x)f(y)$ and $f(0)=1$

Put $x=y=0$, $f(0)=2f'(0)\Rightarrow f'(0)=1/2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(x)f'(h)+f'(x)f(h)-f(x)}{h}$$

For limit to exist, numerator $\dfrac{f(x)}{2}-f(x)+f'(x)=0$

$$\Rightarrow f'(x)=\frac{f(x)}{2}\Rightarrow\int\frac{f'(x)}{f(x)}dx=\int\frac{dx}{2}$$

$$\Rightarrow \ln(f(x))=x/2+C$$

$$f(0)=1\Rightarrow C=0$$

$$\Rightarrow f(x)=e^{x/2}$$

$$\ln(f(4))=\ln(e^2)=2.$$

Hence, the correct answer is (2).

5. (i) $\text{RHL}=\lim\limits_{h\to 0^+}\sin(\sqrt{1-e^{-h^2}})=0=\text{LHL}=f_1(0)$

$f_1(x)$ is continuous at $x=0$

$$\text{RHD}=\lim_{h\to 0}\frac{\sin\left(\sqrt{1-e^{-h^2}}\right)}{h}$$

$$=\lim_{h\to 0}\left(\frac{\sin\left(\sqrt{1-e^{-h^2}}\right)}{\sqrt{1-e^{-h^2}}}\right)\frac{\sqrt{1-e^{-h^2}}}{h}=1$$

$$\text{LHD}=\lim_{h\to 0}\frac{\sin(\sqrt{1-e^{-h^2}})}{-h}=-1\neq\text{RHD}$$

$f_1(x)$ is not differentiable at $x=0$

(ii) $\text{LHL}=\lim\limits_{h\to 0}\dfrac{-\sin(-h)}{\tan^{-1}(-h)}=\lim\limits_{h\to 0}\dfrac{\sin h}{-\tan^{-1}h}=-1$

$\text{RHL}=\lim\limits_{h\to 0}\dfrac{\sin h}{\tan^{-1}h}=1\Rightarrow$ Not continuous , not differentiable at $x=0$

(iii) $\text{LHL}=\lim\limits_{h\to 0}[\sin(\ln(-h+2))]=0\quad(\because 0<\sin(\ln 2)<1)$

$\text{RHL}=\lim\limits_{h\to 0}[\sin(\ln(2+h))]=0=f_3(0)$ continuous at $x=0$

$$\text{LHD}=\lim_{h\to 0}\frac{[\sin(\ln(-h+2))]-0}{-h}=0$$

$\text{RHD}=\lim\limits_{h\to 0}\dfrac{[\sin(\ln(h+2))]-0}{h}=0$ differentiable at $x=0$

$f_3'(x)$ is 0 in neighbourhood of $x=0$ so $f_3'(x)$ is continuous at $x=0$

(iv) $f_4(x)=x^2\sin\left(\dfrac{1}{x}\right)$ is continuous and differentiable at $x=0$

$$f_4'(x)=2x\sin\left(\frac{1}{x}\right)+x^2\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$

$$=2x\sin\left(\frac{1}{x}\right)-\cos\left(\frac{1}{x}\right)$$

$f_4'(x)$ is not continuous at $x=0$

Correct Match $(P)\to(2)$, $(Q)\to(1)$, $(R)\to(4)$, $(S)\to(3)$

Hence, the correct answer is (D).

6. $$f(x)=x\cos\pi(x+[x])=\begin{cases}x\cos\pi(x-2) & -2\le x<-1\\ x\cos\pi(x-1) & -1\le x<0\\ x\cos\pi(x) & 0\le x<1\\ x\cos\pi(x+1) & 1\le x<2\\ x\cos\pi(x+2) & 2\le x<3\end{cases}$$

Continuous at $x=0$ but discontinuous at $x=-1, 1, 2$

Hence, (A), (C) and (D) are correct.

7. $f(x)=a\cos|x^3-x|+b|x|\sin|x^3+x|=a\cos|x^3-x|+b|x|\sin(|x|(x^2+1))$ is an even function

$f(x)=a\cos(x^3-x)+b\,x\sin(x(x^2+1))$

$f(x)$ is always differentiable for all real x

Hence, (A) and (B) are correct.

8. $$f(x)=[x^2-3]=[x^2]-3=\left[\begin{array}{ll}-3 & -\frac{1}{2}\le x<1\\ -2 & 1\le x<\sqrt{2}\\ -1 & \sqrt{2}\le x<\sqrt{3}\\ 0 & \sqrt{3}\le x<2\\ 1 & x=2\end{array}\right.$$

$f(x)$ is discontinuous at 4 points $x=1, \sqrt{2}, \sqrt{3}, 2$ in $[-1/2, 2]$

$|x|+|4x-7|$ is always continuous and not differentiable at $x=0, 7/4$

$g(x)=(|x|+|4x-7|)f(x)$ for $x\in(-1/2, \sqrt{3})$

$g(x)=0$ for $x\in[\sqrt{3}, 2)$

g is not differentiable at 4 points $x=0, 1, \sqrt{2}, \sqrt{3}$,

Hence, (B) and (C) are correct.

9. Let check differentiability of $f(x)$ at $x=0$

$$\text{LHD}=f'(0^-)=\lim_{\delta\to 0}\left(\frac{f(0-\delta)-f(0)}{-\delta}\right)$$

$$=\lim_{\delta\to 0}\frac{0+g(-\delta)}{\delta}=g'(0)=0$$

$$\text{RHD}=f'(0^+)=\lim_{\delta\to 0}\left(\frac{f(0+\delta)-f(0)}{\delta}\right)$$

$$=\lim_{\delta\to 0}\frac{g(\delta)}{\delta}=g'(0)=0$$

$\Rightarrow f(x)$ is differentiable at $x=0$.

Let check differentiability of $h(x)$ at $x=0$

$h'(0^+)=1$, $h(x)$ is an even function

$h'(0^-)=-1$ hence, non diff. at $x=0$

Let check differentiability of $f(h(x))$ at $x = 0$

$$f(h(x)) = g\left(e^{|x|}\right)\ \forall\ x \in R$$

LHD $f'(h(0^-)) = \lim\limits_{\delta\to 0}\left(\dfrac{f(h(0)) - f(h(0-\delta))}{\delta}\right)$

$$= \lim_{\delta\to 0}\frac{g(1) - g(e^{\delta})}{\delta} = -g'(1)$$

RHD $f'(h(0^+)) = \lim\limits_{\delta\to 0}\left(\dfrac{f(h(0+\delta)) - f(h(0))}{\delta}\right)$

$$= \lim_{\delta\to 0}\frac{g(e^{\delta}) - g(1)}{\delta} = g'(1)$$

Since, $g'(1) \neq 0 \Rightarrow f(h(x))$ is non diff. at $x = 0$

Differentiability of $h(f(x))$ at $x = 0$

$$h(f(x)) = \begin{cases} e^{|f(x)|}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

LHD $h'(f(0-\delta)) = \lim\limits_{\delta\to 0}\left(\dfrac{h(f(0)) - h(f(0-\delta))}{\delta}\right)$

$$= \lim_{\delta\to 0}\frac{1 - e^{|g(-\delta)|}}{|g(-\delta)|}\cdot\frac{|g(-\delta)|}{\delta} = g'(0) = 0$$

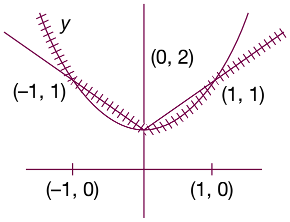
RHD $h'(f(0+\delta)) = \lim\limits_{\delta\to 0}\left(\dfrac{h(f(0+\delta)) - h(f(0))}{\delta}\right)$

$$= \lim_{\delta\to 0}\frac{e^{|g(\delta)|} - 1}{|g(\delta)|}\cdot\frac{|g(\delta)|}{\delta} = g'(0) = 0.$$

Hence, (A) and (D) are correct.

10. $f(x) = |x| + 1 = \begin{cases} x+1 & x \geq 0 \\ -x+1 & x < 0 \end{cases}$ and $g(x) = x^2 + 1$

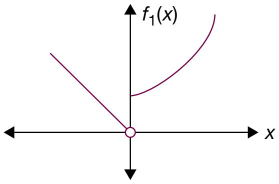
$$h(x) = \begin{cases} x^2+1 & x \leq -1 \\ -x+1 & -1 < x < 0 \\ x^2+1 & 0 \leq x < 1 \\ x+1 & x \geq 1 \end{cases}$$



Number of Non-differential points 3. $x = -1, 0, 1$

Hence, the correct answer is (3).

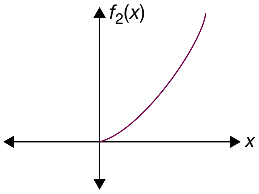
11. $f_1(x)$ is many-one, into



Discontinuous and not differentiable at $x = 0$

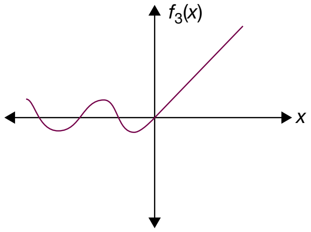
$f_2(x)$ is one-one, into

continuous and differentiable in domain



$f_3(x)$ is many-one, into

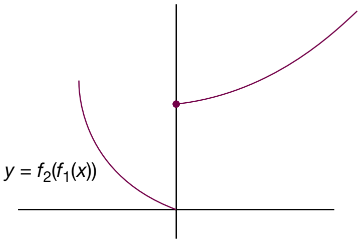
Continuous and differentiable in domain.



$$f_2(f_1(x)) = (f_1(x))^2 = \begin{cases} x^2 & x < 0 \\ e^{2x} & x \geq 0 \end{cases}$$

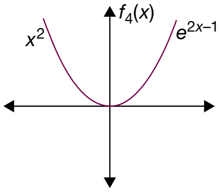
It is many-one, into

Discontinuous and not differentiable at $x = 0$.



$$f_4(x) = \begin{cases} x^2 & x < 0 \\ e^{2x} - 1 & x \geq 0 \end{cases}$$

$f_4(x)$ is many-one onto, continuous and non-derivable at $x = 0$



P → 1, Q → 3, R → 2, S → 4

Hence, the correct answer is (D).

12. Let $f(x)$ and $g(x)$ have their maximum values at point $x = a, b$ respectively in $x \in [0,1]$, Assume $a < b$

and $f(a) = g(b) = \lambda$

Consider a function $h(x) = f(x) - g(x)$

$h(a) = \lambda - g(a) > 0$ and $h(b) = f(b) - \lambda < 0$

$f(x)$ and $g(x)$ are continuous functions in $x \in [0,1]$ then $h(x)$ is also a continous function

By intermediate value theorem $\Rightarrow h(c) = 0 \Rightarrow f(c) = g(c)$ for some $c \in [0,1]$ ...(1)

$(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c) \Rightarrow (f(c) - g(c))\,(f(c) + g(c) + 3) = 0$

So, there exist a 'c': $f(c) - g(c) = 0$ Hence, (A) is correct.

Similarly, $(f(c))^2 = (g(c))^2 \Rightarrow (f(c) - g(c))\,(f(c) + g(c)) = 0$

So, there exist a 'c': $f(c) - g(c) = 0$ (D) is correct.

B and C are wrong as by counter

If $f(c) = g(c) = k \neq 0$, then

(B) $k^2 + k = k^2 + 3k$ is not possible.

(C) $k^2 + 3k = k^2 + k$ is not possible.

Hence, (A) and (D) are correct.

CHAPTER 6: APPLICATION OF DERIVATIVES

Test Your Concepts-I

1. $x = 1 : y = \dfrac{1-1}{1+1} = 0$

Point (1, 0)

$$m = \frac{dy}{dx} = \left.\frac{(x^2+1)(3x^2-1)-(x^3-x)(2x)}{(x^2+1)^2}\right|_{x=1}$$

$$= \frac{2(3-1)-(1-1)(2)}{4} = 1$$

Slope of tangent $m_T = 1$

Equation of tangent $y - 0 = 1(x - 1) \Rightarrow x - y - 1 = 0$

Slope of normal $m_N = -1$

Equation of normal $y - 0 = -1(x - 1) \Rightarrow x + y - 1 = 0$

2. $x = 0 : y = be^{-0} = b$

Point (0, b)

$$\frac{dy}{dx} = be^{-\frac{x}{a}}\left.\left(-\frac{1}{a}\right)\right|_{x=0}$$

Slope of tangent $= m_T = \dfrac{-b}{a}e^{-0} = -\dfrac{b}{a}$

Equation of tangent $y - b = -\dfrac{b}{a}(x - 0) \Rightarrow ay - ab = -bx$

$$\Rightarrow\ bx + ay = ab \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

3. The equation of the curve is $x^2 + 3y + y^2 = 5$.
Differentiating w.r.t. x, we get

$$2x + 3\frac{dy}{dx} + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y+3} \Rightarrow \left(\frac{dy}{dx}\right)_{(1,\,1)} = -\frac{2}{5}$$

$\therefore$ Slope of the tangent at (1, 1) $= \left(\dfrac{dy}{dx}\right)_{(1,1)} = \dfrac{-2}{5}$.

And, Slope of the normal at (1, 1) $= \dfrac{-1}{\left(\dfrac{dy}{dx}\right)_{(1,1)}} = \dfrac{-1}{\frac{-2}{5}} = \dfrac{5}{2}$.

4. Let the required point be $P(x_1, y_1)$. Then,

$y_1 = x_1{}^3 - 11x_1 + 5$ …(1)

[$\because$ (x_1, y_1) lies on $y = x^3 - 11x + 5$]

Now, $y = x^3 - 11x + 5$

$$\Rightarrow\ \frac{dy}{dx} = 3x^2 - 11 \Rightarrow \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)} = 3x_1^2 - 11$$

Since, the line $y = x - 11$ is tangent at the point (x_1, y_1). Therefore, Slope of the tangent at (x_1, y_1) = slope of the line $y = x - 11$.

$$\Rightarrow\ \left(\frac{dy}{dx}\right)_{(x_1,\,y_1)} = \text{Slope of the line } x - y - 11 = 0$$

$$\Rightarrow\ 3x_1{}^2 - 11 = \frac{-1}{-1} \Rightarrow 3x_1{}^2 = 12 \Rightarrow x_1 = \pm 2$$

When, $x_1 = 2$, $y_1 = 2^3 - 22 + 5 = -9$ [Using (1)]

When, $x_1 = -2$, $y = (-2)^3 - 11(-2) + 5 = 19$ [Using (1)]

So, two points are (2, –9) and (–2, 19). Of these two point (–2, 19) does not lie on $y = x - 11$. Therefore, the required point is (2, –9).

5.
$$\lim_{x\to e^a}\frac{\frac{1}{1+7f(x)}7f'(x) - \cos(f(x).f'(x)}{3f'(x)}$$

(Using L'Hospital's Rule)

$$= \frac{\frac{7}{1+0} - \cos 0}{3} = \frac{7-1}{3} = 2$$

6. $\dfrac{dy}{dx} = \dfrac{3\alpha x^2}{2y}$; Slope of normal at (2, 3) $= \dfrac{-1}{2\alpha} = \dfrac{-1}{4} \Rightarrow \alpha = 2$

And $9 = 8\alpha - \beta \ \Rightarrow \beta = 7$

7. $x = t^2 + 3t - 8$ and $y = 2t^2 - 5$, $M(2, -1)$

$2 = t^2 + 3t - 8 \Rightarrow t^2 + 3t - 10 = 0$

$\Rightarrow$ $(t + 5)(t - 2) = 0 \Rightarrow t = -5, 2$ …(1)

$-1 = 2t^2 - 2t - 5 \Rightarrow 2t^2 - 2t - 4 = 0$

$\Rightarrow$ $t^2 - t - 2 = 0 \Rightarrow (t - 2)(t + 1) = 0 \Rightarrow t = 2, -1$ …(2)

From (1) and (2) $t = 2$

$$\frac{dx}{dt} = 2t + 3 \Rightarrow \left.\frac{dx}{dt}\right|_{t=2} = 7$$

$$\frac{dy}{dt} = 4t - 2;\ \left.\frac{dy}{dt}\right|_{t=2} = 6$$

$$\left.\frac{dy}{dx}\right|_{M(2,-1)} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6}{7}$$

8. Given equation of the curve is $x + y - \ln(x + y) = 2x + 5$

$$\Rightarrow\ 1 + \frac{dy}{dx} - \frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) = 2$$

$$\Rightarrow\ \frac{dy}{dx} = \left(\frac{x+y+1}{x+y-1}\right)_{(\alpha,\beta)} \qquad \because \frac{dx}{dy} = 0$$

$\therefore$ $\alpha + \beta = 1$

9. $y = x\log x$

$$\frac{dy}{dx} = 1 + \log x$$

Slope of normal $= 1 - \dfrac{1}{1+\log x} = 1 \Rightarrow x = e^{-2}$ and $y = -2e^{-2}$

Equation of normal $y + 2e^{-2} = 1(x - e^{-2})$

Hence, $x - y = 3e^{-2}$

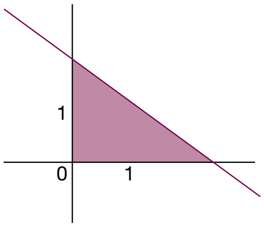
10. $y = \log, x \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{x}$

$\therefore$ $\left(\dfrac{dy}{dx}\right)_{1,0} = 1.$

The equation of the normal at (1, 0) is

$$y - 0 = -\frac{1}{1}(x - 1) \text{ or } x + y = 1$$

$\therefore$ Area $= \dfrac{1}{2}.1.1$



11. Slope of tangent $= \dfrac{-1}{\tan\frac{3\pi}{4}} \Rightarrow f'(3) = 1$

12. $y = 1 \Rightarrow x = \pm 1$

Points are (1, 1), (–1, 1)

$$\Rightarrow \frac{dy}{dx} = \frac{16x}{(3+x^2)}, \left(\frac{dy}{dx}\right)_{(1,1)} = 1, \left(\frac{dy}{dx}\right)_{(-1,1)} = -1$$

Eq. of tangent at (1, 1) is $y - 1 = (x - 1) \Rightarrow x - y = 0$

Eq. of tangent at (–1, 1) $y - 1 = -1\ (x + 1) \Rightarrow x + y = 0$

Hence, both tangents pass through origin.

13. Slope of line $(4 - t)x + ty + (a^3 - 1) = 0$ is

$$\frac{-(4-t)}{t} \text{ (or) } \frac{t-4}{t}$$

$\because\ xy = 1$

$$\therefore\ \frac{dy}{dx} = \frac{-y}{x} = \frac{-1}{x^2}$$

$\therefore$ Slope of normal $= x^2 = \frac{t-4}{t}$

$\therefore\ x^2 > 0$ or $\frac{t-4}{t} > 0$

$t \in (-\infty, 0) \cup (4, \infty)$

14. (t^2, t^3) is a point on the curve.

$$\frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 3t^2;\ \frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3}{2}t$$

$A = (1, 1)$ and $B = (9, 27)$

$$\text{Slope of } AB = \frac{27-1}{9-1} = \frac{26}{8} = \frac{13}{4}$$

$$\frac{3}{2}t = \frac{13}{4} \Rightarrow t = \frac{13}{6}$$

Hence, tangent at $\left(\left(\frac{13}{6}\right)^2, \left(\frac{13}{6}\right)^3\right)$ is parallel to AB.

Test Your Concepts-II

1. Assume point $A(x_1, y_1)$ on curve $x^2 = 4y$ so $x_1^{\ 2} = 4y_1$ …(1)

$$x^2 = 4y \Rightarrow 2x = 4\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2}\bigg|_{(x_1, y_1)} = \frac{x_1}{2}$$

Slope of tangent $m_T = \frac{x_1}{2}$;

Slope of normal $m_N = -\frac{2}{x_1}$

Equation of normal, $y - y_1 = \frac{-2}{x_1}(x - x_1)$

Point (1, 2) lies on it,

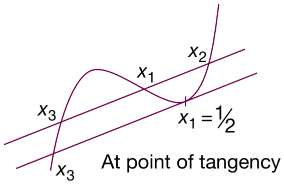
$$2 - y_1 = \frac{-2}{x_1}(1 - x_1) = -\frac{2}{x_1} + 2 \Rightarrow y_1 = \frac{2}{x_1} \quad \text{…(2)}$$

From (1) $y_1 = \frac{x_1^{\ 2}}{4}$

$$y_1 = \frac{2}{x_1} = \frac{x_1^{\ 2}}{4}$$

$\Rightarrow\ x_1^{\ 3} = 8 \Rightarrow x_1 = 2$

$x_1^{\ 2} = 4y_1 \Rightarrow 4 = 4y_1 \Rightarrow y_1 = 1$



Slope of normal, $m_N = -\frac{2}{x_1} = -\frac{2}{2} = -1$

Equation of normal, $y - 1 = -1\ (x - 2)$

$\Rightarrow\ y - 1 + x - 2 = 0 \Rightarrow x + y = 3$

2. $2y\frac{dy}{dx} = -2x(2 - x) + (2 - x)^2$

At (1, 1)

$$\frac{dy}{dx} = \frac{-2(2-1)+(2-1)^2}{2} = \frac{-2+1}{2} = \frac{-1}{2}$$

$y - 1 = -\frac{1}{2}(x - 1);\ 2y - 2 = -x + 1$

$x + 2y = 3 \Rightarrow y = \frac{3-x}{2}$

$\left(\frac{3-x}{2}\right)^2 = x(2-x)^2$

$(3 - x)^2 = 4x(2 - x)^2$

$9 + x^2 - 6x = 4x(4 + x^2 - 4x)$

$9 + x^2 - 6x = 16x + 4x^3 - 16x^2$

$4x^3 - 17x^2 + 22x - 9 = 0$

$(x - 1)^2\left(x - \frac{9}{4}\right) = 0 \Rightarrow \left(\frac{9}{4}, \frac{3}{8}\right)$

3. $\frac{dy}{d\theta} = +a\sin\theta;\ \frac{dx}{d\theta} = a(1 + \cos\theta)$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \tan\frac{\theta}{2};$$

$$m_1 = \tan\frac{\frac{\pi}{3}}{2} = \frac{1}{\sqrt{3}},\ m_2 = -\frac{1}{\tan\frac{\pi}{3}} = \frac{-1}{\sqrt{3}}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \sqrt{3} \Rightarrow \theta = 60^\circ$$

4. Let point (x_1, y_1) lies on curve $x^{2/3} + y^{2/3} = a^{2/3}$ then

$x_1^{\ 2/3} + y_1^{\ 2/3} = a^{2/3}$ …(1)

Equation of tangent, $xx_1^{\ -1/3} + yy_1^{\ -1/3} = a^{2/3}$

$A \equiv (a^{2/3}x^{1/3}, 0),\ B \equiv (0, a^{2/3}y_1^{\ 1/3})$

$$h = \frac{a^{2/3}x_1^{\ 1/3}}{2};\ k = \frac{a^{2/3}y_1^{\ 1/3}}{2}$$

$$\Rightarrow \frac{4h^2}{a^2} + \frac{4k^2}{a^2} = 1 \Rightarrow x^2 + y^2 = \left(\frac{a^2}{4}\right) \text{ circle.}$$

5. $x = 8\cos^3\theta,\ y = 8\sin^3\theta,\ \frac{dy}{dx} = -\frac{\sin\theta}{\cos\theta}$

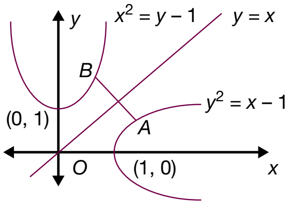
Tangent at θ, $y - 8\sin^2\theta = -\frac{\sin\theta}{\cos\theta}(x - 8\cos^3\theta)$

$x\sin\theta + y\cos\theta = 8\sin\theta\cos\theta$

$OA = 8\cos\theta,\ OB = 8\sin\theta$

$AB = \sqrt{OA^2 + OB^2} = 8$

6. Both curves are symmetrical about the line $y = x$. If line AB is the line of shortest distance then at A and B slopes of curves should be equal to one.



For $y^2 = x - 1 \Rightarrow \frac{dy}{dx} - \frac{1}{2y} = 1$

$\Rightarrow\ y = \frac{1}{2},\ x = \frac{5}{4}$

$\Rightarrow\ B = \left(\frac{1}{2}, \frac{5}{4}\right),\ A = \left(\frac{5}{4}, \frac{1}{2}\right)$

Hence, minimum distance AB

$$= \sqrt{\left(\frac{5}{4} - \frac{1}{2}\right)^2 + \left(\frac{5}{4} - \frac{1}{2}\right)^2} = \frac{3\sqrt{2}}{4} \text{ units.}$$

7. For the equation of the curve, we have

$$2y\frac{dy}{dx} = 2p, \text{ or } \frac{dy}{dx} = \frac{p}{y}.$$

At $P(a, b)$, we get, $\frac{dy}{dx} = \frac{p}{b}$ and $\frac{dx}{dy} = \frac{b}{p}$

We also have $b^2 = 2ap$. Hence, we have at $P(a, b)$

$$\text{Length of tangent} = \left| y\sqrt{1+\left(\frac{dx}{dy}\right)^2} \right|$$

$$= \left| b\sqrt{1+\frac{b^2}{p^2}} \right| = \left|\frac{b}{p}\right| \sqrt{b^2+p^2}$$

$$\text{Length of normal} = \left| y\sqrt{\left(\frac{dy}{dx}\right)^2+1} \right| = \left| b\sqrt{\frac{p^2}{b^2}+1} \right| = \sqrt{b^2+p^2}$$

$$\text{Length of the sub-tangent} = \left| y\frac{dx}{dy} \right| = \left| \frac{b^2}{p} \right| = \frac{b^2}{p}$$

$$\text{Length of the sub-normal} = \left| y\frac{dy}{dx} \right| = \left| b\frac{p}{b} \right| = p$$

8. Range of 2 sin$x$ is [–2, 2] and minimum value of $5x^2+2x+3$ is $\frac{-(4-4.5.3)}{20} = \frac{14}{5} > 2$

No. of point of intersections = 0.

9. Curve $y^2 = (x + a)^3$

$2yy' = 3(x + a)^2$

$$m = y' = \frac{3(x+a)^2}{2y} = \frac{3y^{4/3}}{2y} = \frac{3y^{1/3}}{2}$$

$$\text{Length of sub normal} = |y_1 m| = \frac{3{y_1}^{4/3}}{2}$$

$$\text{Length of sub tangent} = \left|\frac{y_1}{m}\right| = \frac{2{y_1}^{2/3}}{3}$$

$$p(\text{S.N}) = q(\text{S.T})^2 \Rightarrow p\,\frac{3{y_1}^{4/3}}{2} = q\,\frac{4{y_1}^{4/3}}{9}$$

$$\Rightarrow \frac{p}{q} = \frac{8}{27}$$

10. Get m and compute

$$\text{Length of tangent} = \left|\frac{y_1\sqrt{1+m^2}}{m}\right|$$

$$\text{Length of sub-tangent} = \left|\frac{y_1}{m}\right|$$

11. Clearly, (1, 0) is the point of intersection of the given curves.

Now, $f'(x) = \frac{2^x}{x} + 2^x(\log_e 2)(\log_e x)$

∴ Slope of tangent to the curve $f(x)$ at (1, 0), $m_1 = 2$.

$$g'(x) = \frac{d}{dx}(e^{2x \log x} - 1)$$

$$= x^{2x}\left(2x \times \frac{1}{x} + 2\log_e x\right)$$

∴ Slope of tangent to the curve $g(x)$ at (1, 0), $m_2 = 2$.

Since, $m_1 = m_2 = 2$.

∴ Two curves touch each other, so the angle between them is 0.

Hence, $\cos\theta = \cos 0 = 1$.

12. Let point of contact is (h, k) is on curve $y = \cos x$

$\Rightarrow\ k = \cos h$ …(1)

Equation of tangent at (h, k) $y - k = -\sin h(x - h)$,

It passes through origin $\Rightarrow -k = h.\sin h$ …(2)

$$\cos^2 h + \sin^2 h = k^2 + \frac{k^2}{h^2}$$

$\Rightarrow\ 1 = y^2 + \frac{y^2}{x^2}$ is the locus of point of contact

Hence, point of contact lies on curve $x^2 - y^2 = x^2y^2$

13. Let the point of intersection is (x, y)

$$x^3 - 3xy^2 + 2 = 0 \Rightarrow \frac{dy}{dx} = \frac{x^2-y^2}{2xy}$$

$$3x^2y - y^3 = 2 \Rightarrow \frac{dy}{dx} = \frac{2xy}{y^2-x^2},\ m_1.m_2 = -1 \Rightarrow \theta = 90^\circ$$

14. Let $x = a\sin^2\theta$; $y = 2a\sin\theta$

$$\frac{dy}{dx} = \frac{2a\cos\theta}{2a\sin\theta\cos\theta} = \frac{1}{\sin\theta}$$

Equation of tangent $\frac{y-2a\sin\theta}{x-a\sin^2\theta} = \frac{1}{\sin\theta}$

$$\Rightarrow \frac{x}{-a\sin^2\theta} + \frac{y}{a\sin\theta} = 1$$

Sum of intercepts $= a\left(\sin^2\theta + \sin\theta\right)$

Which is maximum when $\sin\theta = 1$

(Sum of intercepts)$_{\max} = 2a$

15. $y = \frac{1+3x^2}{3+x^2}$

$$y = 1 = \frac{1+3x^2}{3+x^2} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Hence, points (+1, 1) and (–1, 1) would be points of tangent

$$\frac{dy}{dx} = \frac{6x(3+x^2)-(1+3x^2)(2x)}{(3+x^2)^2}$$

$$\frac{dy}{dx} = \frac{18x+6x^3-2x-6x^3}{(3+x^2)^2}$$

$$\frac{dy}{dx} = \frac{16x}{(3+x^2)^2};\ \left(\frac{dy}{dx}\right)\Bigg|_{(1,3)} = \frac{16}{16} = 1$$

Equation of tangent $y - 1 = (x - 1) \Rightarrow y = x$

$$\left.\frac{dy}{dx}\right|_{(-1,1)} = -\frac{16}{16} = -1;$$

Equation of tangent at (–1, 1) is $y - 1 = -1\,(x + 1) \Rightarrow y = -x$

Hence, solving $y = x$ and $y = -x$, we will point of intersection (0, 0).

16. (i) $x^2 - y^2 = 5$ and $\frac{x^2}{18} + \frac{y^2}{8} = 1$

We have to first find point of intersection above two curves.

$$x^2 = y^2 + 5 \text{ and } \frac{x^2}{18} + \frac{y^2}{8} = 1$$

$$\Rightarrow \frac{y^2+5}{18} + \frac{y^2}{8} = 1 \Rightarrow y = \pm 2$$

When $y = 2$, $x^2 = y^2 + 5 = 9 \Rightarrow x = \pm 3$

When $y = -2$, $x^2 = y^2 + 5 = 9 \Rightarrow x = \pm 3$

Hence, point of intersection would be as follows: (3, 2), (–3, 2) (3, –2), (–3, –2)

First curve, $x^2 - y^2 = 5$

$$\Rightarrow 2x - 2y \times \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y} \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

At point (3, 2), we have $\left.\frac{dy}{dx}\right|_{(3,2)} = \left.\frac{y}{x}\right|_{(3,2)} = \frac{3}{2} = m_1$

Second curve $\frac{x^2}{18} + \frac{y^2}{8} = 1$

On differentiation, we get $\frac{2x}{18} + \frac{2y}{8}\frac{dy}{dx} = 0$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{4}{9}\frac{x}{y}$$

At point (3, 2), we have $\left(\frac{dy}{dx}\right)\Big|_{(3,2)} = -\frac{4}{9}\frac{3}{2} = -\frac{2}{3} = m_2$

Now, we have $m_1 m_2 = -1$

$\theta = \pi/2$

(ii) $y^2 = \frac{2x}{\pi}$ and $y = \sin x$

$$y^2 = \frac{2x}{\pi}, y = \sin x$$

Intersection point $\left(\frac{\pi}{2}, 1\right)$, $2y \cdot \frac{dy}{dx} = \frac{2}{\pi}$

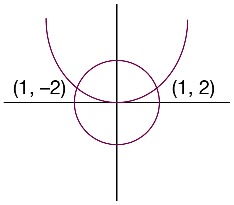
$$2 \times 1 \times \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{2},1\right)} = \frac{\pi}{2} \Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{2},1\right)} = \frac{1}{\pi};$$

$$m_2 = \left(\frac{dy}{dx}\right)_{\left(\frac{\pi}{2},1\right)} = (\cos x)_{\pi/2} = 0$$

Angle between them $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{1}{\pi}\right|$

$\Rightarrow \quad \theta = \cot^{-1}(\pi)$

(iv) $y^2 = 4x; x^2 + y^2 = 5; y^2 = 4x$



$$2y \cdot \frac{dy}{dx} = 4; \left(\frac{dy}{dx}\right) = \left(\frac{2}{y}\right)_{(1,2)}$$

$$m_1 = \frac{2}{2} = 1; x^2 + y^2 = 5$$

$$x^2 + 4x - 5 = 0; x^2 + 5x - x - 5 = 0$$

$$(x + 5)(x - 1) = 0$$

$x = 1$, $x = -5$ not possible

$$2x + 2y, \frac{dy}{dx} = 5; 2 \times 1 + 2 \times 4\frac{dy}{dx} = 5 \Rightarrow \frac{dy}{dx} = \frac{3}{8}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1.m_2}\right| = \left|\frac{1 - \frac{3}{8}}{1 + \frac{3}{8}}\right| = \left|\frac{\frac{5}{8}}{\frac{11}{8}}\right|$$

$$\tan\theta = \frac{5}{11} \Rightarrow \theta = \tan^{-1}\left(\pm\frac{5}{11}\right)$$

Test Your Concepts-III

1. $x^2 + y^2 = 25$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$(3, 4) \qquad \left.\begin{matrix} x = 3 \\ y = 4 \end{matrix}\right] \frac{dx}{dt} = 2 \text{ cm/sec}$$

$$2(3)(2) + 2(4)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{3}{2} \text{ cm/sec}$$

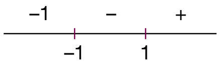
2. $y^3 = 27x \Rightarrow y = 3x^{1/3}$

$$\frac{dy}{dx} = 3 \cdot \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{x^{\frac{2}{3}}}$$

$$\frac{dy}{dx} > \frac{dx}{dt} \Rightarrow \frac{dy}{dx} > 1$$

$$\frac{1}{x^{\frac{2}{3}}} > 1 \Rightarrow 1 > x^{\frac{2}{3}}$$

$$\Rightarrow \left(x^{\frac{1}{3}} - 1\right)\left(x^{\frac{1}{3}} + 1\right) < 0$$



$$x^{\frac{1}{3}} \in (-1, 1) \Rightarrow x \in (-1, 1)$$

3. $u = \sqrt{x^2 + 16}$

$$\frac{du}{dx} = \frac{2x}{2\sqrt{x^2+16}} = \frac{x}{\sqrt{x^2+16}},$$

$$V = \frac{x}{x-1} \Rightarrow \frac{dv}{dx} = \frac{-1}{(x-1)^2}$$

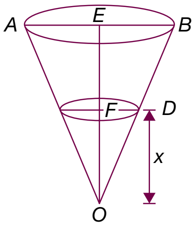
$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{-12}{5}$$

4. $\frac{dr}{dt} = 0.1$ cm/sec; $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = \frac{4}{3}\frac{dr}{dt}; \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}\Big|_{r=5cm}$$

$$= 4\pi(5)^2 \times 0.1 = 10\pi \text{ cm}^3/\text{sec}$$

5. Let the depth of water be x cm inside the cone at any instant t. Then the volume v of the water inside the cone is given by $v = \frac{1}{3}\pi r^2 x$



From similar triangles OFD and OEB, we have

$$\frac{FD}{OF} = \frac{EB}{OE}$$

or $\frac{r}{x} = \frac{6}{12}$ or $r = \frac{1}{2}x$,

$\therefore \quad v(x) = \frac{1}{3}\pi\left(\frac{1}{2}x\right)^2 x = \frac{1}{12}\pi x^3$; given $\frac{dv}{dt} = 5.5$ c.c/sec.

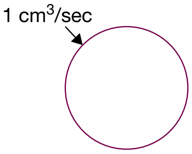
Diff. v about. t, we have

$$\frac{dv}{dt} = \frac{1}{12}\pi\left(3x^2\frac{dx}{dt}\right) \text{ or } \frac{dx}{dt} = \frac{4}{\pi x^2}\frac{dv}{dt}$$

$\therefore \quad$ Rate at which the water level is rising

$$= \frac{dx}{dt} = \frac{4}{\pi(3.5)^2} \times 5.5 = 0.571 \text{ cm/sec}$$

6. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^3 \frac{dr}{dt} = 1$



$$\Rightarrow r^2\frac{dr}{dt} = \frac{1}{4\pi}$$

$$S = 4\pi r^2$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi\left(\frac{r^2 dr}{dt}\right)\frac{1}{r}$$

$$= 8\pi\left(\frac{1}{4\pi}\right)\left(\frac{2\pi}{3}\right)^{\frac{1}{3}} = 2\left(\frac{2\pi}{3}\right)^{\frac{1}{3}}$$

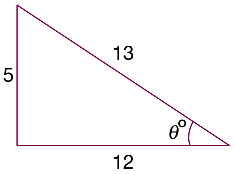
$$\int_0^r r^2\,dr = \int_0^t \frac{1}{4\pi}\,dt$$

$$\frac{r^3}{3} = \frac{t}{4\pi}$$

$$t = 2 \text{ sec}$$

$$r = \left(\frac{3}{2\pi}\right)^{\frac{1}{3}}$$

7. $x = 13\cos\theta$



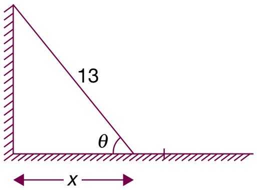
$$\frac{dx}{dt} = -13\sin\theta\frac{d\theta}{dt}$$

$$S = -13\left(\frac{5}{13}\right)\frac{d\theta}{dt}$$

$$\frac{d\theta}{dt} = -1 \text{ rad/sec}$$

$$\theta = 0°$$

$$\sin\theta_0 = \frac{5}{13}$$



8. Let x is edge of cube.

$$V = 125 \Rightarrow x^3 = 125$$

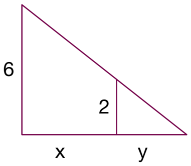
$$\Rightarrow \quad x = 5 \text{ cm}$$

$$\frac{dv}{dt} = 4\,;\ 3x^2\,\frac{dx}{dt} = 4$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{4}{3x^2}$$

$$\frac{ds}{dt} = 12x\frac{dx}{dt} = 12x.\frac{4}{3x^2} = \frac{16}{x} = \frac{16}{5}$$

9. $\frac{dx}{dt} = 54$ mt/min



$$\frac{x+y}{6} = \frac{y}{2}$$

$$\Rightarrow \quad y = \frac{1}{2}x \Rightarrow \frac{dy}{dx} = 27 \text{ m/min.}$$

10. Let radius and height of cylinder = R

Given that radius and height of cylinder is equal to radius of sphere = R.

Volume of cylinder $V_1 = \pi r^2 h$

$$V_1 = \pi R^2(R);\ V_1 = \pi R^3$$

$$\frac{dV_1}{dt} = 3\pi R^2\,\frac{dR}{dt} \qquad \ldots(1)$$

Volume of sphere $V_2 = \frac{4}{3}\pi R^2$

$$\frac{dV_2}{dt} = \frac{4}{3}\pi^3 R^2\,\frac{dR}{dt}$$

$$\frac{dV_2}{dt} = 4\pi R^2\,\frac{dR}{dt}$$

$$\frac{\frac{dV_1}{dt}}{\frac{dV_2}{dt}} = \frac{3\pi R^2\frac{dR}{dt}}{4\pi R^2\frac{dR}{dt}} = \frac{3}{4}$$

Hence, ratio volumes of sphere and cylinder is 4 : 3

11. Let V be the volume and r the radius of the ballon at any time t.

$$\therefore \quad V = \frac{4}{3}\pi r^3$$

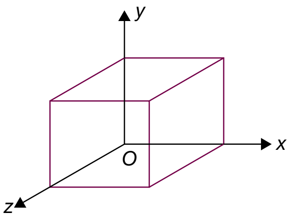
$$\therefore \quad \frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\right)\frac{dr}{dt} \Rightarrow 10 = \frac{4\pi}{3}\cdot 3\cdot(15)^2\,\frac{dr}{dt}$$

Hence, $\frac{dr}{dt} = \frac{1}{90\pi}$ inch/min

12. Volume of cube,

$V = xyz$; Put $x = y = z$; $V = x^3$

Differentiate above equation w.r.t. time, we get



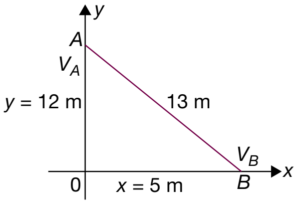
$$\frac{dv}{dt} = 3x^2 \times \frac{dx}{dt}$$

Given, $\frac{dx}{dt} = \frac{3 \text{ cm}}{\text{sec}}$, $x = 10$ cm

$$\frac{dv}{dt} = 3 \times 100 \text{ cm}^2 \times \frac{3 \text{ cm}}{\text{sec}}$$

Hence, $\frac{dv}{dt} = 900 \text{ cm}^3/\text{sec}$.

13. $x^2 + y^2 = (13)^2 = 169$



Differentiating above curve w.r.t. time, we get

$$2x\,\frac{dx}{dt} + 2y\,\frac{dx}{dt} = 0$$

$$y\,\frac{dx}{dt} = -x\,\frac{dx}{dt}\,;\ y\,.\,V_A = -x\,V_B$$

$$V_A = -\frac{x}{y} \times V_B;\ V_A = -\frac{5}{12} \times 2 \text{ m/sec.}$$

Hence, $V_A = \frac{-5}{6}$ m/sec.

14. $16x^2 + 9y^2 = 400$

Differentiate above curve w.r.t. x, we get

$$16 \times 2x + 9 \times 2y\,\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-16 \times 2x}{9 \times 2y} = \frac{-16}{9} \times \frac{x}{y}$$

As given condition $\frac{16}{9}\ \frac{x}{y} = \frac{9y}{16x} \Rightarrow (16x)^2 = (9y)^2$

Now, solving $16x^2 + 9y^2 = 400$

$$\frac{81y^2}{16} + 9y^2 = 400 \Rightarrow 81y^2 + 16 \times 9y^2 = 400 \times 16$$

$$\Rightarrow \quad 9y^2\,(9 + 16) = 400 \times 16 \Rightarrow y^2 = \frac{400 \times 16}{9 \times 25}$$

$$\Rightarrow \quad y = \frac{20 \times 4}{3 \times 5} = \frac{16}{3}$$

$$16x^2 + 9 \times \left(\frac{16}{3}\right)^2 = 400;\ 16x^2 = 400 - 256 = 144$$

$$x^2 = 9 \Rightarrow x = \pm 3$$

Hence, required point $\left(3, \frac{16}{3}\right)$

Test Your Concepts-IV

1. The given function is a polynomial in [0, 1] and [1, 2]. Also $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 1$.

Hence, $f(x)$ is continuous in [0, 2].

Now, $f'(1^-) = \lim_{h\to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0} \frac{1-h-1}{-h} = 1$

And $f'(1^+) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \frac{2-(1+h)-1}{h} = -1$.

Therefore, the function is not differentiable at $x = 1$, that is in the open interval (0, 2). Thus, Rolle's theorem is not applicable.

2. By LMVT, applied to $f(x) = \log_e x$ on $[1, e]$, $\exists$ an $x_0 \in (1, e)$

$$\Rightarrow \quad f'(x_0) = \frac{f(e) - f(1)}{e - 1}$$

$$\Rightarrow \quad x_0 = e - 1$$

3. Given $a, b \in R; a < b$

Let $f(x) = x^3$

From Lagrange's Mean Value Theorem, we get

$$F'(c) = \frac{f(b)-f(a)}{b-a} = \frac{b^3-a^3}{b-a} = (a^2+ab+b^2)$$

Also, $f(x) = x^3$

Differentiating w.r.t. x, we get $f'(x) = 3x^2$

$$f'(c) = 3c^2 = a^2 + b^2 + ab$$

Hence, c must be between a and b

i.e. $a < c < b$; proved.

4. $f'(c) = \frac{f(2)-f(0)}{2-0} = \frac{8-5}{2-0} = \frac{3}{2}$

$$g'(c) = \frac{g(2)-g(0)}{2-0} = \frac{1-0}{2-0} = \frac{1}{2}$$

$$f'(c) = 3g'(c)$$

5. Let $g(x) = \frac{f(x)}{x}$

$g(x)$ is continuous in $[a, b]$ and

Differentiable in (a, b)

$g(a) = g(b)$ hence, Rolle's theorem can be applicable

So, $g'(x_0) = 0$ for $x_0 \in (a, b)$; $\frac{x_0 f'(x_0) - f(x_0)}{x^2} = 0$

$\Rightarrow \quad x_0 f'(x_0) - f(x_0) = 0$

Hence, proved

6. Let $g(x) = f(x) - x$

$$g(a) = g(b) = 0$$

$$g'(c_1) = \frac{g\left(\frac{a+b}{2}\right) - g(a)}{\frac{a+b}{2} - a} \quad \forall\, c_1 \in \left(a, \frac{a+b}{2}\right)$$

$$g'(c_1) = \frac{g\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}};$$

$$g'(c_2) = \frac{g(b) - g\left(\frac{a+b}{2}\right)}{b - \left(\frac{a+b}{2}\right)}, \forall\, c_2 \in \left(\frac{a+b}{2}, b\right)$$

$$g'(c_2) = -\frac{g\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}}; \ \frac{g'(c_1)}{g'(c_2)} = -1 \Rightarrow g'(c_1) = -g'(c_2)$$

Hence, $g'(c_1) + g'(c_2) = 0 \Rightarrow f'(c_1) + f'(c_2) = 2$

7. $g(x) = f(x) - x$

$$g'(x) = f'(x) - 1 \Rightarrow g'(x) < 0 \text{ or } = 0$$

If $g'(x) < 0$ then $g(x)$ is strictly decreasing and continuous.

Also, $g(a) = g(-a)$, then $g(x)$ cannot strictly decreasing.

If $g'(x) = 0$, as $g(x)$ is continuous, $g(x) = 0$

So, $f(x) = x \Rightarrow f(0) = 0$

8. $f(a) = f(b) = f(c) = f(d) = 0$

$f(x) = 0 \quad \Rightarrow \quad$ '4' times

Using Rolles theorem, $f'(x) = 0$ for atleast '3' times

Again using Rolles theorem, we can say that $f''(x) = 0$ (minimum 2 times)

9. Let $g(x) = f^3(x)$

$\Rightarrow \quad g'(x) = 3f^2(x) \cdot f'(x)$

$\because \quad f:[2,7] \to [0,\infty] \Rightarrow g:[2,7] \to [0,\infty]$

Using Lagrange's Mean value theorem on $g(x)$ we get

$$g'(c) = \frac{g(7)-g(2)}{5}; c \in [2,7]$$

$\Rightarrow \quad 3f^2(c)f'(c) = (f(7) - f(2)) \ \frac{(f(7))^2 + (f(2))^2 + f(2)f(7)}{5}$

10. Using LMVT,

$$f'(c) = \frac{f(5)-f(2)}{5-2} \Rightarrow f'(c) = \frac{\frac{1}{2}-\frac{1}{5}}{3} \Rightarrow f'(c) = \frac{1}{10}$$

Single Correct Choice Type Questions

1. $f(1) = \int_0^1 \frac{dt}{1+t^2} = \tan^{-1} t\Big|_0^1 = \frac{\pi}{4}$, Point we get $\left(1, \frac{\pi}{4}\right)$

$$f'(x) = \frac{1}{1+x^2}\Bigg|_{x=1} = m_T = \frac{1}{2}, \text{ slope of normal } m_N = -2$$

Equation of normal, $y - \frac{\pi}{4} = -2(x-1) \Rightarrow 2x + y = 2 + \frac{\pi}{4}$

$$\Rightarrow \quad \frac{x}{1+\frac{\pi}{8}} + \frac{y}{2+\frac{\pi}{4}} = 1$$

$X = 1 + \frac{\pi}{8}$; $Y = 2 + \frac{\pi}{4}$ then clearly $Y = 2X$

Hence, the correct answer is (B).

2. Point (1, 1) lies on it given curve $f(x) = e^{a+bx^2}$ so we have

$1 = e^{a+b}$ …(1)

Slope of tangent $= f'(x) = e^{a+bx^2} \cdot 2bx\,|_{x=1} = 2$

$\Rightarrow \quad 2 = 2be^{a+b}$ using (1)

$m = 2b = 2 \Rightarrow b = 1$ and $1 = e^{a+1}$

$a + 1 = \ln 1 = 0 \Rightarrow a = -1$

$\therefore \quad a^2 + b^2 = 2$

Hence, the correct answer is (B).

3. Point of intersection of given curves $y = \frac{1}{1+x^2} = \frac{1}{1+x}$

$\Rightarrow \quad x^2 = x$

$x = 0, y = 1 \qquad P(0, 1)$

$x = 1, y = \frac{1}{2} \qquad Q\left(1, \frac{1}{2}\right)$

$$\frac{dy}{dx} = \frac{-1}{(1+x^2)^2} 2x\Bigg|_{P(0,\,1)} = 0,$$

We get tangent at P, $y - 1 = 0$

$$\frac{dy}{dx} = \frac{-2x}{(1+x^2)^2} 2x\Bigg|_{Q(1,1/2)} = \frac{-1}{2},$$

We get tangent at Q, $y - \frac{1}{2} = -\frac{1}{2}(x-1)$ or $x + 2y = 2$

Hence, the correct answer is (C).

4. $y = (5 - x^{2/3})^{\frac{3}{2}}$

Put $x = 1 \Rightarrow y = 8 \Rightarrow M(1, 8)$

CHAPTER 6

$$\frac{dy}{dx} = \frac{3}{2}\left(5 - x^{\frac{2}{3}}\right)^{\frac{1}{2}}\left(-\frac{2}{3}x^{-\frac{1}{3}}\right)$$

$$\frac{dy}{dx} = -\left(5 - x^{\frac{2}{3}}\right)^{\frac{1}{2}} x^{-\frac{1}{3}}$$

$$\left.\frac{dy}{dx}\right]_{M(1,8)} = -\left(5 - 1^{\frac{2}{3}}\right)^{\frac{1}{2}} (1)^{-\frac{1}{3}}$$

$$\left.\frac{dy}{dx}\right]_{M(1,8)} = -2$$

When $x = 1, y = 8$

∴ Tangent is $(y - 8) = -2(x - 1)$

⇒ $2x + y = 10 \Rightarrow \frac{x}{5} + \frac{y}{10} = 1$

∴ Length of intercept $= \sqrt{100 + 25} = \sqrt{125}$

⇒ $N = 125$

Hence, the correct answer is (D).

5. Given curve is $x + y = x^y$

Put $y = 0$, we get $x = 1$.

Now, differentiate $x + y = x^y$, take log on both sides, we get $\ln(x + y) = y \ln x$

$$\Rightarrow \frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) = y \cdot \frac{1}{x} + (\ln x)\frac{dy}{dx}$$

Put $x = 1, y = 0$, we get

$$\left.\frac{dy}{dx}\right]_{(1,0)} = -1 = \text{Slope of tangent}$$

⇒ Equation of normal, is $\left(\frac{y-0}{x-1}\right) = 1 \Rightarrow y = x - 1.$

Hence, the correct answer is (C).

6. $f(x) = \lim\limits_{h \to 0} \frac{e^{\sin^{-1}(x+h)} - e^{\sin^{-1} x}}{h}$

$$f(x) = \frac{d}{dx}\left(e^{\sin^{-1} x}\right) = e^{\sin^{-1} x} \frac{1}{\sqrt{1 - x^2}}$$

At $x = 0, y = 1$

$$f'(x) = -\frac{1}{2} e^{\sin^{-1} x} \cdot \frac{(-2x)}{(1 - x^2)^{3/2}} + \frac{1}{\sqrt{1 - x^2}} \cdot e^{\sin^{-1} x} \frac{1}{\sqrt{1 - x^2}}$$

$$f'(0) = 1$$

∴ Equation of tangent at (0, 1) is $(y - 1) = 1. (x - 0)$

⇒ $y = x + 1.$

Hence, the correct answer is (C).

7. $2x + y = 6$

$$y = \frac{a^2}{x - 1}$$

$$\frac{dy}{dx} = \frac{-a^2}{(x - 1)^2} = -2$$

(Slope of line joining (–1, 8) and (0, 6))

⇒ $x - 1 = \pm\frac{a}{\sqrt{2}}$

∴ The possible points are $\left(1 \pm \frac{a}{\sqrt{2}}, \pm a\sqrt{2}\right)$

Now, equation of tangent is,

$$y - (\pm a\sqrt{2}) = -2\left(x - \left(1 \pm \frac{a}{\sqrt{2}}\right)\right)$$

i.e., $y - (\pm a\sqrt{2}) = -2x + 2 \pm \sqrt{2}a$

As it passes through (0, 6), we get

$$6 - (\pm a\sqrt{2}) = 2 \pm \sqrt{2}a$$

i.e., $4 = \pm 2\sqrt{2}a \Rightarrow a^2 = 2$

Hence, the correct answer is (A).

8. Given, $g(x) = \begin{vmatrix} \cos x & x & 1 \\ 2\sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$

$$= \cos x(x^2 - 2x^2) - x(2\sin x - 2x\tan x) + 1(2x\sin x - x^2 \cdot \tan x)$$

$$= -x^2 \cdot \cos x + 2x^2 \cdot \tan x - x^2 \cdot \tan x$$

$$= -x^2 \cdot \cos x + x^2 \cdot \tan x$$

∴ $g(x) = x^2 (\tan x - \cos x)$

Also, $g(0) = 0$ and $g'(x) = x^2(\sec^2 x + \sin x) + 2x \cdot (\tan x - \cos x)$

∴ $g'(0) = 0$

So, equation of normal drawn to $g(x)$ at $x = 0$, is $x = 0$

$$= g'(0) = 0$$

Hence, the correct answer is (A).

9. Values of parameter 'a'

Given curve is: $xy = 1 \Rightarrow y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = \frac{-1}{x^2}$

∴ Slope of normal $= x^2 > 0$ (As $x \neq 0$)

∴ Slope of given line $= \frac{\log_2(1 + 5a - a^2)}{5} > 0$

⇒ $\log_2(1 + 5a - a^2) > 0 \Rightarrow 1 + 5a - a^2 > 1$

⇒ $a^2 - 5a < 0$

Hence, $a \in (0, 5)$

Hence, the correct answer is (C).

10. $\frac{dy}{dx} = e^{x_1} = \frac{1}{4}$

∴ $x = -\ln 4$

⇒ $y_1 = e^{-\ln 4} = \frac{1}{e^{\ln 4}} = \frac{1}{4}$

Equation of line is $\left(y - \frac{1}{4}\right) = \frac{1}{4}(x + \ln 4)$

⇒ $4y - 1 = x + \ln 4$

Put $y = 0$, we get

$$x = -(1 + \ln 4) = -\ln(4e) = \ln\left(\frac{1}{4e}\right).$$

Hence, the correct answer is (B).

11. $x = t^2$ and $y = t^3 - 3t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

∴ Slope $= \frac{3t^2 - 3}{2t}$ at $t = 2$

Slope $= \frac{9}{4}$.

Equation of tangent at (4, 2) is

$$(y - 2) = \frac{9}{4}(x - 4) \Rightarrow y = \frac{9}{4}x - 7.$$

Hence, the correct answer is (D).

12. $S = 4\pi r^2;\ \dfrac{dS}{dt} = 8$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{\pi r} \qquad \left(\because \frac{ds}{dt} = 8\pi r\frac{dr}{dt}\right)$$

Now, $v = \dfrac{4}{3}\pi r^3 \Rightarrow \dfrac{dv}{dt} = 4\pi r^2;\ \dfrac{dr}{dt} = 4r \Rightarrow \dfrac{dv}{dt} \propto r$

Hence, the correct answer is (C).

13. For $x^2 = 9a(9 - y)$;

Differentiation with respect to x

$$2x = 9a(-1)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = -\frac{2x}{9a}$$

Let $m_1 = -\dfrac{2x}{9a}$

For $x^2 = a(y + 1);\ \dfrac{dy}{dx} = \dfrac{2x}{a} = m_2$

$\therefore\ m_1m_2 = -1$ gives $4x^2 = 9a^2$ …(1)

Also, by eliminating 'y' between the equation of curves

$x^2 = 9a$

Adding $\dfrac{x^2}{9a} = 9 - y$ and $\dfrac{x^2}{a} = y + 1 \Rightarrow \dfrac{x^2 + 9x^2}{9a} = 10$

From (1) and (2), we get $a = 4$ …(2)

Hence, the correct answer is (D).

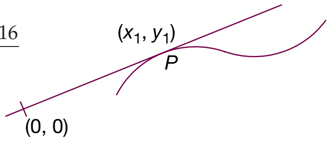
14. We have $y = x^3 + x + 16$

$$\left.\frac{dy}{dx}\right]_P = 3x^2 + 1 = 3x_1^2 + 1$$

$$\therefore\ 3x_1^2 + 1 = \frac{y_1}{x_1} = \frac{x_1^3 + x_1 + 16}{x_1}$$

$$\Rightarrow 3x_1{}^3 + x_1 = x_1{}^3 + x_1 + 16$$

$$\Rightarrow 2x_1{}^3 = 16 \Rightarrow x_1 = 2$$



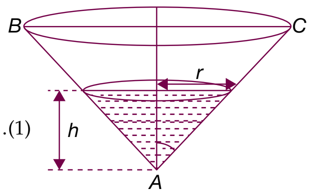
Hence, slope of line $l = 3(2)^2 + 1 = 13$

Hence, the correct answer is (C).

15. $\dfrac{dV}{dt} = 2 \Rightarrow \dfrac{d}{dt}\left(\dfrac{1}{3}\pi r^3\right) = 2$

[Here, $r = h$, as $\theta = 45^\circ$]

$$\Rightarrow \pi r^2\frac{dr}{dt} = 2 \Rightarrow \frac{dr}{dt} = \frac{2}{\pi r^2} \quad \text{…(1)}$$



Now, perimeter $= 2\pi r = p$ (let)

$$\Rightarrow \frac{d}{dt}(2\pi r) = 2\pi\left(\frac{2}{\pi r^2}\right) = \frac{4}{r^2} \quad \text{…(2)}$$

(Using equation (1))

When $h = 2$ meters $\Rightarrow r = 2$ meters

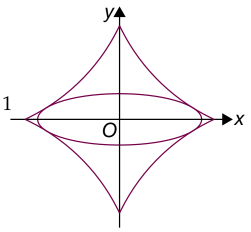
Hence, $\dfrac{dp}{dt} = \dfrac{4}{4} = 1$ m/sec.

Hence, the correct answer is (A).

16. $x^{2/3} + y^{2/3} = c^{2/3};\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\therefore\ \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} = 0;\ \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y_1^{1/3}}{x_1^{1/3}}$$



$$\therefore\ (y - y_1) = \frac{y_1^{1/3}}{x_1^{1/3}}(x - x_1)$$

$$yx_1^{1/3} - y_1x_1^{1/3} = -y_1^{1/3}x + x_1y_1{}^{1/3}$$

$$y_1^{1/3} + x_1^{1/3}y = x_1^{1/3}y_1^{1/3}(x_1^{2/3} + y_1^{2/3})$$

$$\Rightarrow \text{Similarly} \left.\begin{aligned} x_1^{1/3} &= \frac{x}{c^{1/3}} \\ y_1^{2/3} &= \frac{a}{c^{1/3}} \end{aligned}\right] \Rightarrow x_1^{2/3} + y_1^{2/3} = \frac{a + b}{c^{2/3}}$$

$$\Rightarrow c^{2/3} = \frac{a + b}{c^{1/3}}$$

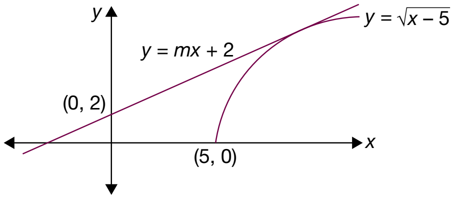
$$\Rightarrow a + b = c$$

Hence, the correct answer is (A).

17. $\sqrt{x - 5} = mx + 2$

$\Rightarrow x - 5 = (mx + 2)^2 = m^2x^2 + 4mx + 4$

$\Rightarrow m^2x^2 + (4m - 1)x + 9 = 0$



Condition of tangency

Put $D = 0$,

$\Rightarrow (4m - 1)^2 - (6m)^2 = 0$

$\Rightarrow (10m - 1)(-2m - 1) = 0$

$\Rightarrow m = \dfrac{-1}{2}, \dfrac{1}{10}$

Line always passes through (0, 2) hence for two distinct points m must be less than $\dfrac{1}{10}$ and greater than zero.

$\Rightarrow m \in \left(0, \dfrac{1}{10}\right)$

Hence, the correct answer is (B).

18. Let the point of intersection of curves $y^2 = 2ax (a > 0)$ and $xy = 4\sqrt{2}$ be $P(x_1, y_1)$.

Now, $y^2 = 2ax \Rightarrow \left.\dfrac{dy}{dx}\right]_{P(x_1, y_1)} = \dfrac{a}{y_1} = m_1$ …(1)

Also, $xy = 4\sqrt{2}$

$$x\frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \left.\frac{dy}{dx}\right]_{P(x_1, y_1)} = \frac{-y_1}{x_1} = m_2$$

As, $m_1 \times m_2 = -1 \Rightarrow x_1 = a$

As, $x_1y_1 = 4\sqrt{2} \Rightarrow y_1 = \dfrac{4\sqrt{2}}{a}$

So, $P\left(x_1 = a, y_1 = \dfrac{4\sqrt{2}}{a}\right)$

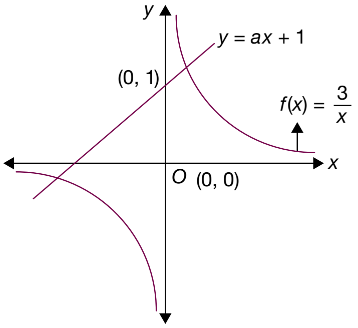
Also, point P must satisfy

$$y_1{}^2 = 2ax_1$$

$$\frac{32}{a^2} = 2a^2 \Rightarrow a^4 = 16 \Rightarrow a = 2$$

Hence, the correct answer is (C).

19.



CASE 1: If slope of line = $a > 0$, then two intersection point,

So, $a \in (0, \infty)$ …(1)

CASE 2: If $a < 0$, let point P is point of contact of both curve.

$$\left.\frac{dy}{dx}\right|_B = \frac{-3}{a^2} = \frac{\frac{3}{a} - 1}{a - 0} \Rightarrow \frac{-3}{a^2} = \frac{3-a}{a^2} \Rightarrow a = 6$$

$\therefore$ Slope at $P = \frac{-1}{12}$ (For tangency)

So, $a \in \left(\frac{-1}{12}, 0\right)$ …(2)

Hence, (1) ∪ (2) $\Rightarrow \left(\frac{-1}{12}, 0\right) \cup (0, \infty)$

Hence, the correct answer is (C).

20. $\because$ $f(x) = x^2 - 3x - 1$ is continuous and in [0, 4] and differentiable in (0, 4)

$$\Rightarrow \frac{f(b) - f(a)}{b - a} = f'(C)$$

$$\frac{f(4) - f(0)}{4 - 0} = 2c - 3$$

$$2C - 3 = \frac{(4^2 - 3 \times 4 - 1) - (0 - 0 - 1)}{4 - 0}$$

$$2C - 3 = 1 \Rightarrow C = 2$$

Hence, the correct answer is (C).

21. Using LMVT in [0, 2]

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) \text{ where } c \in (0, 2)$$

$$\Rightarrow \frac{f(2) + 3}{2} \le 5$$

$$f(2) \le 7$$

Hence, the correct answer is (A).

22. (A) $f(x) = |x|$, is not derivable at $x = 0$

(B) $f(x) = \tan x$ is discontinuous at $x = \frac{\pi}{2}$

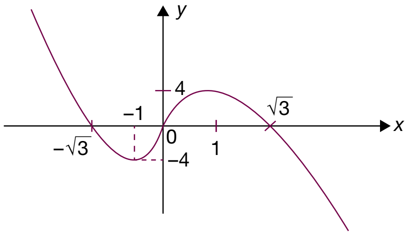
(C) $f(x) = 1 + (x - 2)^{2/3}$ is non-derivable at $x = 2$.

(D) Only function which satisfies Rolle's theorem, is $f(x) = x(x - 2)^2$, $0 \le x \le 2$.

$\because$ $f(x)$ is continuous in [0, 2] and differentiable in (0, 2). Also, $f(0) = f(2)$

Hence, the correct answer is (D).

23.



Let $f(x) = 6x - 2x^3 = 2x(3 - x^2)$

$f'(x) = 6 - 6x^2 = 0$, we get $x = \pm 1$

$f(1) = 4$; $f(-1) = -4$

Equation $a = 6x - 2x^3$ has 3 real and district roots

$-4 < a < +4$

$a \in \{-3, -2, -1, 0, 1, 2, 3\}$, we are getting, 7 integral values of a

Hence, the correct answer is (C).

24. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$$\Rightarrow \frac{dr}{dt} = \left(\frac{35}{4\pi r^2}\right) \qquad \text{…(1)}$$

Also, $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt} = (8\pi r)\left(\frac{35}{4\pi r^2}\right) = \frac{70}{r}$

$$\therefore \left.\frac{dS}{dt}\right]_{r=7} = \frac{70}{7} = 10$$

Hence, the correct answer is (D).

25. $\frac{dy}{dx}, \frac{4}{3}$

$$\frac{x^2}{c} + \frac{y^2}{4} = 1 \text{ and } y^3 = 16x$$

Let (x_1, y_1) be their point of intersection.

So, $m_1 \times m_2 = -1$

$$\frac{2x}{c} + \frac{2y}{4}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{c} \cdot \frac{4}{2y} = -\frac{4x}{cy}$$

$$y^3 = 16x$$

$$3y^2 \frac{dy}{dx} = 16 \Rightarrow \frac{dy}{dx} = \frac{16}{3y^2}$$

$$-\frac{4x}{cy} \cdot \frac{16}{3y^2} = -1$$

$$64x = 3cy^3, \text{ at } (x_1, y_1)$$

$$64x_1 = 3cy_1^3$$

$$64x_1 = 3c(16x_1) \Rightarrow c = \frac{4}{3}$$

Hence, the correct answer is (B).

26. $\frac{dy}{dx} = \frac{2x}{x^2 + 4} - \frac{x}{1 + \frac{x^2}{4}} - \tan^{-1}\frac{x}{2}$

At $x = 2$; $y = \ln 8 - \frac{\pi}{2}$

$$\left.\frac{dy}{dx}\right]_{x=2} = \frac{4}{8} - \frac{1}{2}\left(\frac{2}{1+1}\right) - \frac{\pi}{4} = -\frac{\pi}{4}$$

$\therefore$ Equation is $y - \left(\ln 8 - \frac{\pi}{2}\right) = \frac{-\pi}{4}(x - 2)$

$$\Rightarrow \frac{\pi x}{4} + y = \ln 8.$$

Hence, the correct answer is (A).

27. $f(x) = y = x^2 - 6x + \lambda$

$y = -x^2 + 6x - 8$

$\therefore$ $f(3) \ge 1 \Rightarrow \lambda \ge 8$

Tangent at (2, 0) at parabola (1)

$$\frac{dy}{dx} = -2x + 6$$

$$\left.\frac{dy}{dx}\right]_{(2,0)} = 2$$

$\therefore$ Equation of tangent at (2, 0) is $(y - 0) = 2(x - 2)$

$\Rightarrow$ $y = 2x - 4$

Now, this will be secant to parabola (2)

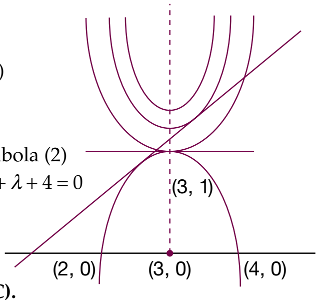
If $2x - 4 = x^2 - 6x + \lambda \Rightarrow x^2 - 8x + \lambda + 4 = 0$

$D > 0$

$\Rightarrow$ $\lambda < 12$

$\Rightarrow$ $\lambda = 10$ or 11.

Hence, the correct answer is (C).



28. Curve $\ln(x^2 + y^2) = C\tan^{-1}\left(\frac{y}{x}\right)$ on differentiation

$$\Rightarrow \frac{2x + 2yy'}{x^2 + y^2} = \frac{c}{1 + \frac{y^2}{x^2}} \cdot \frac{xy' - y \cdot 1}{x^2} \Rightarrow 2x + 2yy' = c(xy' - y)$$

$\Rightarrow$ $y'(cx - 2y) = cy + 2x$

$$\Rightarrow y' = m_2 = \left.\frac{cy_1 + 2x_1}{cx_1 - 2y_1}\right|_{(x_1, y_1)} \quad \text{let point } P(x_1, y_1)$$

Slope of OP $m_1 = \frac{y_1}{x_1}$; required angle is 45°, we get

$$\tan 45° = 1 = \left|\frac{\frac{cy_1 + 2x_1}{cx_1 - 2y_1} - \frac{y_1}{x_1}}{1 + \frac{y_1}{x_1}\left(\frac{cy_1 + 2x_1}{cx_1 - 2y_1}\right)}\right|$$

$$1 = \left|\frac{cx_1y_1 + 2x_1^2 - cx_1y_1 + 2y_1^2}{cx_1^2 - 2x_1y_1 + cy_1^2 + 2x_1y_1}\right|$$

$$\Rightarrow 1 = \left|\frac{2x_1^2 + 2y_1^2}{cx_1^2 + cy_1^2}\right|, \text{ final value } c = 2$$

Hence, the correct answer is (B).

29. $f'(x) = 2x - a\,|_{x=2}$

$m_T = 4 - a$

Point (2, 4)

$$m_N = \frac{1}{a - 4}$$

Normal $y - 4 = \frac{1}{a - 4}(x - 2)$

Normal cuts x axis at A $(18 - 4a, 0)$ in II$^{\text{nd}}$ quadrant, $18 - 4a < 0$

$\Rightarrow$ $18 < 4a \Rightarrow a > \frac{9}{2}$

Normal cuts x axis at $B\left(0, 4 - \frac{2}{a - 4}\right)$ in II$^{\text{nd}}$ quadrant,

$$4 - \frac{2}{a - 4} > 0 \Rightarrow \frac{2a - 8 - 1}{a - 4} > 0 \Rightarrow \frac{2a - 9}{a - 4} > 0; a > 4$$

$$\text{Area} = \frac{1}{2}(4a - 18)\left(4 - \frac{2}{a - 4}\right) = 2 \Rightarrow (2a - 9)\left(2 - \frac{1}{a - 4}\right) = 1$$

$\Rightarrow$ $(4a - 17)(a - 5) = 0$

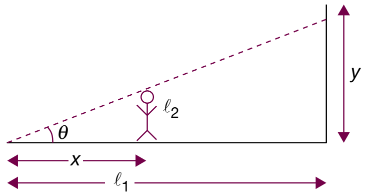
$a = \frac{17}{4}$ or $|$ $a = 5$ but $a > \frac{9}{2}$

Then, final $a = 5$

Hence, the correct answer is (D).

30. $\tan\theta = \frac{\ell_2}{x} = \frac{y}{\ell_1} \Rightarrow xy = \ell_1\ell_2$ …(1)

Given, $\frac{dx}{dt} = \frac{\ell_1}{10}$ m/sec



When man in is in midway $x = \frac{\ell_1}{2}$; from (1) $\frac{\ell_2}{\ell_1/2} = \frac{y}{\ell_1}$

$\Rightarrow$ $y = 2\ell_2$

$xy = \ell_1\ell_2$

$$\frac{dx}{dt}y + \frac{xdy}{dt} = 0 \Rightarrow \frac{\ell_1}{10}(2\ell_2) + \frac{\ell_1}{2}\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = -\frac{\ell_1\ell_2}{5} \cdot \frac{2}{\ell_1} = -\frac{2\ell_2}{5} \text{ m/sec}$$

Hence, the correct answer is (B).

31. $|f'(x)| \le \frac{1}{2}$ for $x \in (0, 2)$

From LMVT in $[0, c]$, we have $\left|\frac{f(x) - f(0)}{c - 0}\right| \le \frac{1}{2}$

$|f(c)| \le \frac{c}{2} \le 1$ as $c \le 2$

$|f(c)| \le 1$

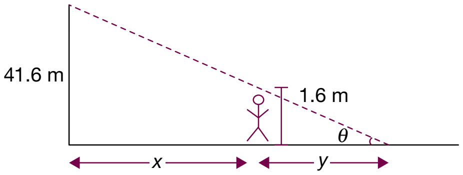
Hence, the correct answer is (B).

32. $\tan\theta = \frac{41 \cdot 6}{x + y} = \frac{1 \cdot 6}{y}$

$\Rightarrow$ $\frac{26}{x + y} = \frac{1}{y} \Rightarrow x = 25y$ …(1)

$$\frac{dx}{dt} = 25\frac{dy}{dt} = 2 \Rightarrow \frac{dy}{dt} = \frac{2}{25} \text{ given } \frac{dx}{dt} = 2 \text{ m/sec}$$

$$\tan\theta = \frac{1 \cdot 6}{y} \Rightarrow \sec^2\theta\frac{d\theta}{dt} = -\frac{106}{y^2}\frac{dy}{dt}$$



When $x = 30$, as $x = 25y \Rightarrow y = \frac{6}{5}$ and $\tan\theta = \frac{1 \cdot 6}{\frac{6}{5}} = \frac{4}{3}$

$$\left(1 + \frac{16}{9}\right)\frac{d\theta}{dt} = -\frac{1 \cdot 6}{18}$$

$$\Rightarrow \frac{d\theta}{dt} = -\frac{1}{25}(0.8) = -\frac{4}{125} \text{ radius/sec}$$

Hence, the correct answer is (A).

33. Given, $3(a + 2c) = 4(b + 3d)$

$\Rightarrow$ $3a + 6c = 4b + 12d$

$$\Rightarrow \frac{a}{4} + \frac{c}{2} = \frac{b}{3} + d \Rightarrow \frac{a}{4} - \frac{b}{3} + \frac{c}{2} - d = 0 \quad \text{…(1)}$$

Let $f(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx$ for $x \in [-1, 0]$ it is continuous and differentiable

$f(0) = 0$ and $f(-1) = \frac{a}{4} - \frac{b}{3} + \frac{c}{2} - d = 0$

From Rolle's theorem, we have $f'(x) = ax^3 + bx^2 + cx + d = 0$ for atleast one x in $(-1, 0)$;
Hence, the correct answer is (B).

34. $y = \frac{x^2}{4} - 2$

$$\frac{dy}{dx} = \frac{2x}{4} = \left.\frac{x}{2}\right|_{(x_1, y_1)}$$

$m_N = -\frac{2}{x_1}$ let point $P(x_1, y_1)$

Normal $y - y_1 = -\frac{2}{x_1}(x - x_1)$

Point $(10, -1)$ lies on it $-1 - \frac{x_1^2}{4} + 2 = -\frac{2}{x_1}(10 - x_1)$

$$\Rightarrow 1 - \frac{x_1^2}{4} = -\frac{20}{x_1} + 2 \Rightarrow \frac{x_1^2}{4} - \frac{20}{x_1} + 1 = 0$$

$$\Rightarrow x_1^3 - 80 + 4x_1 = 0 \Rightarrow x_1^3 + 4x_1 - 80 = 0$$

$$x_1 = 4;\ y_1 = \frac{x_1^2}{4} - 2 = 2$$

Hence, required point $P(4, 2)$.
Hence, the correct answer is (D).

35. $f(x) = \int_2^x (2t - 5)\, \mathrm{d}t = \left. t^2 - 5t \right|_2^x = 0$

$f(x) = x^2 - 5x - 4 + 10 = 0$

$f(x) = x^2 - 5x + 6 = 0$ or $x = 2, 3$

$f'(x) = 2x - 5$

$x = 2$ $m_1 = -1$ and $x = 3$ $m_2 = 1$

$m_1 m_2 = -1$

Required angle $\theta = 90^{\circ}$

Hence, the correct answer is (D).

36. $x^2 y = c^3 \Rightarrow y = \frac{c^3}{x^2}$

$$P\left(t, \frac{c^3}{t^2}\right);\ \frac{dy}{dx} = -\left.\frac{2c^3}{x^3}\right|_t = -\frac{2c^3}{t^3}$$

Tangent $y - \frac{c^3}{t^2} = -\frac{2c^3}{t^3}(x - t) \Rightarrow \frac{2c^3}{t^3}x + y = \frac{3c^3}{t^2}$

$$\Rightarrow \frac{x}{\frac{3t}{2}} + \frac{y}{\frac{3c^3}{t^2}} = 1;\ a = \frac{3t}{2};\ b = \frac{3C^3}{t^2}$$

$$\Rightarrow a^2 b = \left(\frac{9t^2}{4}\right)\left(\frac{9c^2}{t^2}\right) = \frac{27}{4}c^3$$

Hence, the correct answer is (C).

37. $y = x^3 \quad (x \in R)$

$A(t, t^3)$

$$\frac{dy}{dx} = 3x^2 = 3t^2$$

Tangent $y - t^3 = 3t^2(x - t)$

Cure $y = x^3$

$x^3 - t^3 = 3t^2 x - 3t^3$

$x^3 - 3t^2 x + 2t^3 = 0$

$x_1 + x_2 + x_3 = 0$

Point of tangency $= x_2 = x_1 = t$

$2t + x_3 = 0 \Rightarrow x_3 = -2t$

$m_B = km_A \Rightarrow 3(-2t)^2 = k(3t^2)$

$k = 4$

Hence, the correct answer is (A).

38. $f : [-1, 2] \to R$

$0 \le f'(t) \le 1$

$t \in [-1, 0]$

$0 \le f(0) - f(-1) \le 1$ …(1)

$-1 \le f'(t) \le 0$

$t \in [0, 2]$

$$-1 \le \frac{f(2) - f(0)}{2 - 0} \le 0$$

$-2 \le f(2) - f(0) \le 0$ …(2)

Add (1) and (2)

$-2 \le f(2) - f(-1) \le 1$

Hence, the correct answer is (A).

39. $y = x^{3/2}$

Let point $\left(x_1, x_1^{3/2}\right)$ on curve, let required distance is P.

$p^2 = x_1^2 + x_1^3$

$$\frac{2pdp}{dt} = 2x_1 \frac{dx_1}{dt} + 3x_1^2 \frac{dx_1}{dt}$$

$$2\sqrt{x_1^2 + x_1^3}\,(11) = \frac{dx_1}{dt} x_1(2 + 3x_1)$$

At $x_1 = 3$; $2\sqrt{36}\,(11) = \frac{dx_1}{dt}(3)(11)$

$$\Rightarrow \frac{dx_1}{dt} = 4$$

Hence, the correct answer is (A).

40. $f(x) = 3\tan x + x^3 - 2$

$f'(x) = 3\sec^2 x + 3x^2 > 0$

$f(x)$ is an increasing function

$f(0) = -2$

$$f\left(\frac{\pi}{4}\right) = 3 + \left(\frac{\pi}{4}\right)^3 - 2 > 0$$

$f(x) = 0$ has exactly one solution

Hence, the correct answer is (B).

41. $p(a, a^n)$

Curve $y = x^n$

$$\frac{dy}{dx} = \left. nx^{n-1}\right|_{x=a} = m_T = na^{n-1}$$

$$m_N = -\frac{1}{na^{n-1}}$$

Normal $y - a^n = -\frac{1}{na^{n-1}}(x - a)$

y-axis, $x = 0$

$$b = a^n + \frac{a}{na^{n-1}} = a^n + \frac{1}{n}a^{2-n}$$

$$\lim_{a\to 0} b = \lim_{a\to 0} a^n + \frac{1}{n}a^{2-n} = \frac{1}{2} \qquad (n \in N)$$

$$= \lim_{a\to 0} 0 + \frac{1}{n}a^{2-n} = \frac{1}{2} \text{ we get } n = 2$$

Hence, the correct answer is (C).

42. $y = x^3 + x^2 - x$

$$\frac{dy}{dx} = 3x^2 + 2x - 1 = 0 \Rightarrow (3x - 1)(x + 1) = 0$$

$x = \frac{1}{3}$ and $x = -1$

At $x = \frac{1}{3}$; $y = \frac{1}{27} + \frac{1}{9} - \frac{1}{3} = -\frac{5}{27}$

At $x = -1$; $y = 1$

Distance between parallel horizontal tangents

$$= 1 + \frac{5}{27} = \frac{32}{27}$$

Hence, the correct answer is (D).

43. $f(x) = x^2 \sin\frac{1}{x} + x^3 \cos\frac{1}{2x}$

Continuous and differentiable in $\left[\frac{1}{3\pi}, \frac{1}{\pi}\right]$

$$\left.\begin{aligned} f\left(\frac{1}{\pi}\right) &= 0 \\ f\left(\frac{1}{3\pi}\right) &= 0 \end{aligned}\right\}$$

Hence, the correct answer is (B).

44. $\because$ $f(t) = t^2 \Rightarrow f'(t) = 2t$ and $g(t) = t^3 \Rightarrow g'(t) = 3t^2$,

We have $\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(c)}{g'(c)}$

$\Rightarrow \frac{1-0}{1-0} = \frac{2c}{3c^2} \Rightarrow \frac{3c}{2} = 1$ (As $0 < c < 1$)

$\Rightarrow c = \frac{2}{3}$

Hence, the correct answer is (A).

45. $f(x) = 4 - \left(\frac{1}{2} - x\right)^{2/3}$ where $x \in [0, 1]$

Not differentiable at $x = \frac{1}{2}$

$g(x)$ discount at $x = 0$; as $g(0^+) = 0$ and $g(0) = 1$

$h(x) = \{x\}$

$h(0) = h(1) = 0$; $h(1^-) = 1$ $h(x)$ is discontinuous at $x = 1$

$k(x) = 5^{\log_2(x+3)}$ continuous and differential

LMVT not applicable for f, g, h

Hence, the correct answer is (A).

46. $f(x) = \begin{cases} \frac{|x-1|}{x^2+1} & x > -1 \\ x^2 & x \le -1 \end{cases} = \begin{cases} x^2 & x \le -1 \\ \frac{1-x}{x^2+1} & -1 < x < 1 \\ \frac{x-1}{x^2+1} & x \ge 1 \end{cases}$

$f(x)$ is continuous

$$f'(x) = \begin{cases} 2x & x < -1 \\ N.D. & x = -1 \\ \frac{-1}{x^2+1} - \frac{(1-x)2x}{(x^2+1)^2} & -1 < x < 1 \\ N.D. & x = 1 \\ \frac{1}{x^2+1} - \frac{(x-1)2x}{(x^2+1)^2} & x > 1 \end{cases}$$

$f'(x)$ does not exist at $x = \pm 1$ (critical points)

For $-1 < x < 1$

$$f'(x) = 0 \Rightarrow \frac{-x^2 - 1 - (1-x)2x}{(x^2+1)^2} = 0$$

$\Rightarrow x^2 + 1 - 2x(x - 1) = 0$

$\Rightarrow x^2 - 2x - 1 = 0$

We get $x = 1 + \sqrt{2}$ (not acceptable)

$x = 1 - \sqrt{2}$ (acceptable)

For $x > 1$; $f'(x) = 0 \Rightarrow \frac{x^2 + 1 - 2x^2 + 2x}{(x^2+1)^2} = -x^2 + 2x + 1 = 0$

$\Rightarrow x^2 - 2x - 1 = 0$

$\Rightarrow (x - 1)^2 - (\sqrt{2})^2 = 0$

$x = 1 + \sqrt{2}$ (acceptable); $x = 1 - \sqrt{2}$ (not acceptable)

Total 4 critical point

$x = \pm 1, 1 + \sqrt{2}, 1 - \sqrt{2}$

Hence, the correct answer is (A).

47. $y = \frac{1}{2\sin^2 x + 3\cos^2 x} = \frac{1}{1 - \cos 2x + \frac{3}{2}(1 + \cos 2x)}$

$y = \frac{2}{5 + \cos 2x}$

$\frac{dy}{dx} = \frac{-2}{(5 + \cos 2x)^2}(-\sin 2x) = 0$

$\sin 2x = 0$

$2x = \pi, 2\pi, 3\pi$; $2x = n\pi$

$2x = n\pi(n \text{ even})$

$y = \frac{2}{5+1} = \frac{2}{6} = \frac{1}{3}$

$2x = n\pi(\text{odd})$

$y = \frac{2}{5-1} = \frac{2}{4} = \frac{1}{2}$

Hence, the correct answer is (D).

48. $f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ (x-2)^2 & 1 < x \le 3 \\ 1 & 3 < x \le 4 \end{cases}$

$f'(x) = \begin{cases} 2 & 0 \le x < 1 \\ 2(x-2) & 1 < x < 3 \\ 0 & 3 < x \le 4 \end{cases}$

Discontinuous at $x = 1$

Not diff at $x = 1, 3$

$f'(x) = 0$

For $x = \{2\} \cup (3, 4]$ infinite point

Rolles theorem can not apply.

But $f'(x) = 0$

So, Rolles theorem is not necessary for $f'(x)$ to be zero.

Hence, the correct answer is (A).

49. Let $F(x) = x^{2013} - x^{2012} - 8x^2 + 8x = (x^{2012} - 8x)(x - 1)$

$F(x) = x(x^{2011} - 8)(x - 1)$ where $x \in [0, 8^{1/2011}]$

$F(0) = F(8^{1/2011}) = F(1) = 0$

From Rolles theorem $F'(x) = P(x) = 0$ has atleast 2 real roots

Hence, the correct answer is (D).

50. Curve $3x^2 - 4y^2 = 72$

On differentiation $6x - 8yy' = 0$

Point (x_0, y_0) lies on given curve $3x_0^{\,2} - 4y_0^{\,2} = 72$ …(1)

Slope of tangent at P, $y' = \frac{3x_0}{4y_0}$

For nearest point on curve from line $3x + 2y + 1 = 0$, tangent should be parallel to given line

$\frac{3x_0}{4y_0} = -\frac{3}{2} \Rightarrow x_0 = -2y_0$

From (1), we have $12y_0^{\,2} - 4y_0^{\,2} = 72 \Rightarrow y_0^{\,2} = 9$

We get $y_0 = \pm 3$ and $x_0 = \mp 6$ so Points are $(-6, 3)$ $(6, -3)$ $(-6, 3)$ lies near to given line, nearest point is $x_0 = -6$, $y_0 = 3$

$x_0 + y_0 = -3$

Hence, the correct answer is (B).

51. $\frac{dT}{dt} = -k(T-30) \Rightarrow T-30 = Ae^{-kt}\,(t=0, T=100 \Rightarrow A=70)$

$\therefore\quad T-30 = 70e^{-kt}$

$t=5, T=80 \Rightarrow 5 = 7e^{-5k}$

$T=60 \Rightarrow 30 = 70e^{-kt}$

$5 = 7e^{-5k} \Rightarrow \ln\frac{5}{7} = -5k$

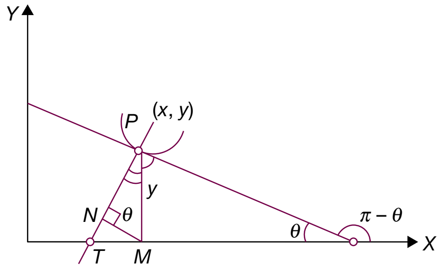
$3 = 7e^{-kt} \Rightarrow \ln\frac{3}{7} = -kt$

$$t = \frac{\ln\left(\frac{3}{7}\right)5}{\ln\frac{5}{7}}$$

Eliminating e^k, $t = 5\log_{\frac{7}{5}}\left(\frac{7}{3}\right)$

Hence, the correct answer is (D).

52.



Ordinate = PM

Let $P \equiv (x, y)$

Projection of ordinate on normal = PN

$\therefore\quad PN = PM\cos\theta = a$

$$= \frac{y}{\sqrt{1+\tan^2\theta}} = a \Rightarrow y = a\sqrt{1+\left(\frac{dy}{dx}\right)}^{2} \Rightarrow \int\frac{a\,dy}{\sqrt{y^2-a^2}} = \int dx$$

$\Rightarrow\quad a\log(y+\sqrt{y^2-a^2}) = x+c$

Hence, the correct answer is (A).

53. Since, the curve $y = ax^3+bx^2+cx+5$ touches x-axis at $P(-2,0)$ then x-axis is the tangent at (–2, 0). The curve meets y-axis in (0, 5). We have

$\frac{dy}{dx} = 3ax^2+2bx+c$

$\therefore\quad \left.\frac{dy}{dx}\right|_{(0,5)} = 0+0+C = 3 \qquad (\text{given}) \qquad \ldots(1)$

$\therefore\quad c = 3$

And $\left.\frac{dy}{dx}\right|_{(-2,0)} = 0 \Rightarrow 12a-4b+c = 0 \qquad \ldots(2)$

And $(-2,0)$ lies on the curve then

$0 = -8a+4b-2c+5 \Rightarrow 0 = -8a+4b-1 \qquad (\because c=3)$

$\Rightarrow\quad 8a-4b+1 = 0 \qquad \ldots(3)$

From (2) and (3) we get $a = -\frac{1}{2}, b = -\frac{3}{4}$

Hence, $a = -\frac{1}{2}, b = -\frac{3}{4}$ and $c = 3$.

Hence, the correct answer is (D).

54. Given, $f(x) = x^3 - ax^2 + 2x$, $x \in [0, 2]$

Now, $f'(x) = 3x^2 - 2ax + 2$

$f(2) = (2)^3 - 4a + 4 = 12 - 4a$ and $f(0) = 0$

$\therefore$ Using LMVT, we have

$$f'\left(\frac{1}{2}\right) = \frac{f(2)-f(0)}{2-0} \Rightarrow \frac{3}{4} - a + 2 = \frac{(12-4a)-0}{2}$$

$\Rightarrow\quad \frac{11}{4} - a = 6 - 2a \Rightarrow a = 6 - \frac{11}{4} \Rightarrow a = \frac{13}{4}$

Hence, the correct answer is (C).

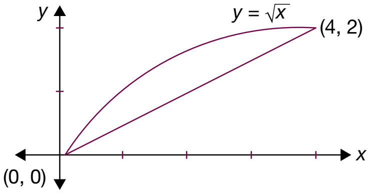
55. $f(x)$ satisfy the Rolle's theorem in [1, 3]

$\Rightarrow\quad f(1) = f(3)$

$$\int_1^3 f'(x)dx = f(x)\Big|_1^3 = f(3) - f(1) = 0$$

Hence, the correct answer is (A).

56.



$y = \sqrt{x}\,;\ y' = \frac{1}{2\sqrt{x}}$

By the Mean Value Theorem we have

$f'(C) = \frac{f(b)-f(a)}{b-a}$

$\Rightarrow\quad f'(C) = \frac{f(4)-f(0)}{4-0}$

$\Rightarrow\quad \frac{1}{2\sqrt{C}} = \frac{2-0}{4-0}$

$\frac{1}{2\sqrt{c}} = \frac{2}{4} \Rightarrow c = 1$

The point is (1, 1).

Hence, the correct answer is (B).

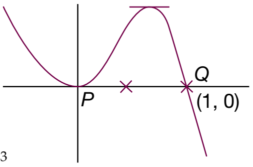
57. $y = x^2 - x^3$

$\frac{dy}{dx} = 2x - 3x^2$

$\Rightarrow\quad \frac{dy}{dx} = 0$ at $x = 0,\ x = \frac{2}{3}$

$M = \left(\frac{1}{2}, 0\right)\quad y = 1 - 9x + 28x^2 - 28x^3$

$0 = 1 - \frac{9}{2} + \frac{28}{4} - \frac{28}{8}$ satisfies



Hence, the correct answer is (A).

58. At the nearest point tangent line must be parallel to $y = 2x - 1$

$y = x^4 + 3x^2 + 2x$

$\frac{dy}{dx} = 4x^3 + 6x + 2 \Rightarrow 4x^3 + 6x + 2 = 2$

$\Rightarrow\quad 2x(x^2+3) = 0 \Rightarrow x = 0$ and $y = 0$

Distance = $\left|\frac{0-0+1}{\sqrt{1^2+2^2}}\right| = \frac{1}{\sqrt{5}}$

Hence, the correct answer is (D).

59. $y = \frac{1}{x} \Rightarrow \frac{dy}{dx} = -\frac{1}{x^2}$

$-\frac{1}{x^2} = +\frac{a}{3-a}\,;\ x^2 = \frac{a-3}{a}$

$\frac{a-3}{a} > 0 \Rightarrow a \in (-\infty, 0) \cup (3, \infty)$

Hence, the correct answer is (A).

60. Using LMVT in [1, 4] for $y = f(x)$ there must exist atleast one $c \in (1, 4)$ such that

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{8}{3} \text{ but } f'(x) > 3 \ \forall \ x \in R$$

Hence, no such function exists.

Hence, the correct answer is (A).

61. $y^3 - x^2y + 5y - 2x = 0$

$$3y^2 \frac{dy}{dx} - x^2 \frac{dy}{dx} - y(2x) + 5\frac{dy}{dx} - 2 = 0$$

Slope at (0, 0) $\Rightarrow \dfrac{dy}{dx} = \dfrac{2}{5}$ …(1)

$$x^4 - x^3y^2 + 5x + 2y = 0$$

$$4x^3 - x^3\, 2y \frac{dy}{dx} - y^2\, 3x^2 + 5 + 2\frac{dy}{dx} = 0$$

Slope at (0, 0) $\dfrac{dy}{dx} = -\dfrac{5}{2}$ …(2)

From (1) and (2) product of slopes = –1

$\Rightarrow \ \theta = \dfrac{\pi}{2}$

Hence, the correct answer is (D).

62. By Rolle's theorem

$f(1) = f(3)$

$\Rightarrow \ a + b + 11 - 6 = 27a + 9b + 33 - 6$

$\Rightarrow \ 13a + 4b + 11 = 0$

Now, $f'(x) = 3ax^2 + 2bx + 11$

So, $f'\left(2 + \dfrac{1}{\sqrt{3}}\right) = 0$

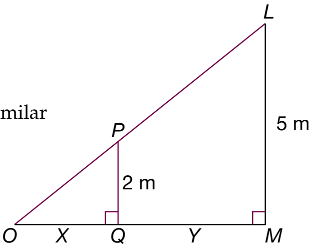
$\Rightarrow \ 3a\left(2 + \dfrac{1}{\sqrt{3}}\right)^2 + 2b\left(2 + \dfrac{1}{\sqrt{3}}\right) + 11 = 0$ [Using equation (1)]

$\Rightarrow \ 6a + b = 0$ …(2)

$\therefore$ On solving (1) and (2), we get $a = 1, b = –6$.

Hence, the correct answer is (D).

63. Let L be the lamp and PQ be the man and $OQ = x$ metre be his shadow and let $MQ = y$ metre.



$\therefore \ \dfrac{dy}{dx}$ = speed of the man

$= 3$ m/s (given)

$\because \ \Delta OPQ$ and $\because \Delta OLM$ are similar

$\therefore \ \dfrac{OM}{OQ} = \dfrac{LM}{PQ}$

$\Rightarrow \ \dfrac{x + y}{x} = \dfrac{5}{2} \Rightarrow y = \dfrac{3}{2}x$

$\therefore \ \dfrac{dy}{dt} = \dfrac{3}{2}\dfrac{dx}{dt}$

$\Rightarrow \ 3 = \dfrac{3}{2}\dfrac{dx}{dt} \Rightarrow \dfrac{dx}{dt} = 2$ m/s

Hence, the correct answer is (B).

64. $\dfrac{dy}{dx} = -\dfrac{x^{-1/3}}{y^{-1/3}}$ Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$

We have, $\dfrac{dy}{dx} = \dfrac{x^{1/3}}{y^{1/3}} \Rightarrow \displaystyle\int y^{1/3}dy = \int x^{1/3}dx \Rightarrow x^{4/3} - y^{4/3} = c$

Hence, the correct answer is (C).

65. $y = x + \sin x \Rightarrow$ If $\dfrac{dy}{dx} = 1 + \cos x = 0,$ then $\cos x = -1$

$\Rightarrow \ x = \pm\pi, 3\pi \ldots$

Also $y = \pm\pi, \pm 3\pi$......

But for the given constrain on x and y, no such y exists.

Hence, no such tangent exists.

Hence, the correct answer is (A).

66. Slope of $3x + 2y = 1$ is $\dfrac{-3}{2}$

$\Rightarrow \ f^1\left(\dfrac{1}{2}\right) = \dfrac{-3}{2}$

$$p = \lim_{x \to 0} \frac{x(x-1)}{f\left(\frac{e^{2x}}{2}\right) - f\left(\frac{e^{-2x}}{2}\right)} \left(\frac{0}{0}\right)$$

$$= \frac{-1}{f^1\left(\frac{1}{2}\right) + f^1\left(\frac{1}{2}\right)} = \frac{1}{3}$$

$$\therefore \ \sum_{r=1}^{\infty} p^r = \frac{1}{3} + \frac{1}{3^2} + \ldots \infty = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

Hence, the correct answer is (A).

67. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ …(1)

$$\frac{2x}{a^2} + \frac{2y}{b^2}y' = 0 \Rightarrow y' = -\frac{xb^2}{a^2y} = m_1$$

$$\frac{x^2}{\ell^2} - \frac{y^2}{m^2} = 1; \ \frac{2x}{\ell^2} - \frac{2yy'}{m^2} = 0$$

$$y' = \frac{xm^2}{\ell^2 y} = m_2$$

$\because \ m_1\, m_2 = 1 \Rightarrow y^2 = \dfrac{x^2b^2m^2}{a^2\ell^2}$

Put y^2 in equation (2) $\Rightarrow x^2 = \dfrac{\ell^2a^2}{a^2 - b^2}$

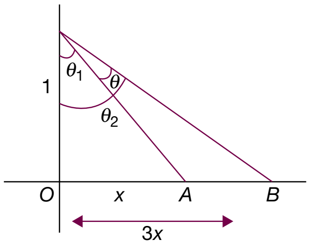
Now, putting value of x^2 and y^2 in equation (1)

$$\frac{\ell^2a^2}{a^2(a^2 - b^2)} + \frac{\ell^2a^2b^2m^2}{(a^2 - b^2)a^2\ell^2b^2} = 1$$

$\Rightarrow \ \ell^2 + m^2 = a^2 - b^2$

Hence, the correct answer is (C).

68. $\tan\theta = \tan(\theta_2 - \theta_1)$



$\Rightarrow \ \tan\theta = \dfrac{3x - x}{1 + 3x.x} = \dfrac{2x}{1 + 3x^2}$

Let $y = \dfrac{2x}{1 + 3x^2} \ \dfrac{dy}{dx} = \dfrac{2(1 - 3x^2)}{(1 + 3x^2)^2}$

$\dfrac{dy}{dx} = 0 \Rightarrow x = \dfrac{1}{\sqrt{3}}$ and

$\dfrac{d^2y}{dx^2} = \dfrac{-24x}{(1 + 3x^2)} < 0$

For $x = 1/\sqrt{3}$

$\Rightarrow \ \theta = \pi/6$

Hence, the correct answer is (B).

69. Let $g'(1) = a, g''(a) = b$ then $f(x) = x^2 + ax + b$ then $f(1) = 1 + a + b$

$g(x) = (1 + a + b)x^2 + x(2x + a) + b$

$g'(x) = 2x(3 + a + b) + a$

$g'(1) = a \Rightarrow a + b + 3 = 0, g''(2) = b \Rightarrow 2a + b = -6$

$\Rightarrow \ a = -3$ and $b = 0$

$f(x) = x^2 - x$

$f(3) = 0$

And $g(x) = -3x + 2$

$g(3) = -7$

Hence, the correct answer is (B).

70. Equation of tangent is

$$\frac{2x}{4} + \frac{2y}{3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{3x}{4y};\ \frac{dy}{dx} = \frac{-3(2\cos\theta)}{4\sqrt{3}\sin\theta}$$

$$\frac{dy}{dx} = \frac{-\sqrt{3}\cos\theta}{\sin\theta}$$

$$y - \sqrt{3}\sin\theta = \frac{-\sqrt{3}}{2}\ .\ \cot\theta\,(x - 2\cos\theta)$$

$\Rightarrow$ x intercept $(x_0) = \dfrac{2}{\cos\theta} \Rightarrow \cos\theta = \dfrac{2}{x_0}$,

y intercept $(y_0) = \dfrac{\sqrt{3}}{\sin\theta} \Rightarrow \sin\theta = \dfrac{\sqrt{3}}{y_0}$, if mid point is (h, k)

$$h = \frac{2}{x_0},\ k = \frac{y_0}{2},\ \cos\theta = \frac{1}{h},\ \sin\theta = \frac{\sqrt{3}}{2k} \Rightarrow \frac{1}{h^2} + \frac{3}{4k^2} = 1$$

Hence, the correct answer is (B).

71. $\dfrac{dy}{dt} = 4t + 3,\ \dfrac{dx}{dt} = 3t^2 - 8t - 3$

Tangents are horizontal if $\dfrac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dt} = 0,\ \frac{dx}{dt} \neq 0,\ 4t + 3 = 0 \Rightarrow t = \frac{-3}{4}$$

Tangents are vertical if

$$\frac{dx}{dy} = 0\ \frac{dx}{dt} = 0,\ \frac{dy}{dt} \neq 0$$

$$3t^2 - 8t - 3 = 0 \Rightarrow t = 3,\ \frac{-1}{3}$$

Hence, the correct answer is (B).

72. Apply LMVT with $a = 2010$, $b = 2012$

$$f(x) = 2010x^2 + 2011x - 2011$$

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

$$\Rightarrow f'(c) = \frac{[2010\times(2012)^2 + 2011\times2012 - 2010\times(2010)^2 - 2011(2010)]}{2012 - 2010}$$

$$\Rightarrow f'(c) = \frac{2010\times2\times4022 + 2011\times2}{2}$$

$$2\times2010\,c + 2011 = 2011(4021)$$

$$c = \frac{2011\times4020}{2\times2010}$$

$$c = 2011$$

$$y = f(2011) = 2010\cdot20112 + 2011\cdot2010 = 2010\cdot2011\cdot2022$$

Hence, the correct answer is (B).

73. Let $P_1 = \left(t_1, t_1^3\right),\ P_2 = \left(t_2, t_2^3\right),\ P_3 = \left(t_3, t_3^3\right)\ldots$

Solving tangent equation at P_1 with the curve again we get $t_2 = -2t_1$. Repeating the process we have

$$t_3 = 4t_1$$

$$\frac{dy}{dx} = 3x^2;\ \left.\frac{dy}{dx}\right|_{(t_1, t_1^3)} = 3t_1^2$$

Equation of tangent

$$y - t_1^3 = 3t_1^2\,(x - t_1)$$

Intersection with $p_2(t_2,\ t_2^3)$

$$t_2^3 - t_1^3 = 3t_1^2\,(t_2 - t_1)$$

$$t_4 = -8t_1\ldots$$

$$\therefore\ \frac{\Delta P_1P_2P_3}{\Delta P_2P_3P_4} = \begin{vmatrix} t_1 & t_1^3 & 1 \\ t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \end{vmatrix} \div \begin{vmatrix} t_2 & t_2^3 & 1 \\ t_3 & t_3^3 & 1 \\ t_4 & t_4^3 & 1 \end{vmatrix} = \frac{1}{16}$$

Hence, the correct answer is (B).

74. $y_1 = \sqrt{2}\sin\left(x_1 + \dfrac{\pi}{4}\right),\ \dfrac{dy}{dx} = \dfrac{y_1}{x_1}$ where (x_1, y_1) is point on the curve

$$\therefore\ \frac{y_1^2}{x_1^2} = 2\cos^2\left(x_1 + \frac{\pi}{4}\right) = 2\left(\frac{-y_1^2}{2} + 1\right)$$

$\Rightarrow$ Locus of (x_1, y_1) is $\dfrac{2}{y^2} - \dfrac{1}{x^2} = 1$

Hence, the correct answer is (D).

75. Clearly,

$y = f(x)$ is discontinuous at $x = 0$

$y = f(x)$ is not differentiable at $x = -1$

$f'(x) = 2(3x - 5);\ x \geq 0$

But $f'(x) = 0$ has a root $x = \dfrac{5}{3}$

Hence, the correct answer is (D).

76. $f(x) = \left[\begin{array}{ll} 3 & x = 0 \\ a - x^2 & 0 < x < 1 \\ mx + b & 1 \leq x \leq 2 \end{array}\right.$

For Lagranges mean value theorem $f(x)$ must be continuous in [0, 2] and differentiable in (0, 2)

Hence, $a = 3$

$$m + b = 2 \qquad \ldots(1)$$

$$f'(x) = \left[\begin{array}{ll} -2x & 0 < x < 1 \\ m & 1 < x \leq 2 \end{array}\right. \Rightarrow m = -2$$

And hence, $b = 4$

Hence, the correct answer is (B).

Multiple Correct Choice Type Questions

1. $\because\quad 2y^3 = ax^2 + x^3$

$$\therefore\ 6y^2\frac{dy}{dx} = 2ax + 3x^2$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{2ax + 3x^2}{6y^2}\right)$$

$$\therefore\ \left.\frac{dy}{dx}\right|_{(a,a)} = \frac{5a^2}{6a^2} = \frac{5}{6}$$

$\Rightarrow$ Equation of tangent is $y - a = \dfrac{5}{6}(x - a)$

or $6y - 6a = 5x - 5a$ or $-5x + 6y = a$

or $\dfrac{x}{\left(-\dfrac{a}{5}\right)} + \dfrac{y}{\left(\dfrac{a}{6}\right)} = 1$

$$\therefore\ \alpha = \frac{a}{5},\ \beta = \frac{a}{6}$$

$$\left(\frac{-a}{5}\right)^2 + \left(\frac{a}{6}\right)^2 = 61 \qquad \text{(Given)}$$

$$\Rightarrow \frac{61a^2}{(30)^2} = 61$$

$$\therefore\ a = \pm30$$

Hence, (A) and (D) are correct.

2. $L_{ST} = \left|\dfrac{y}{m}\right|, L_{SN} = |ym|$

$$L_T = \left|\frac{y\sqrt{1+m^2}}{m}\right|, L_N = \left|y\sqrt{1+m^2}\right|$$

Where $m = \dfrac{dy}{dx}$ at point $P = (x, y)$

On the curve $y = f(x)$

Now, $\dfrac{L_{ST}}{L_{SN}} = \dfrac{1}{m^2} = \left(\dfrac{L_T}{L_N}\right)^2$ and $L_{ST}L_{SN} = y^2$

Hence, (A) and (B) are correct.

3. $x^3 + xy + y^3 = 3; x + y = k\ (k \in \text{R})$

$x^3 + x(k - x) + (k - x)^3 = 3$

$x^3 + kx - x^2 + k^3 - x^3 - 3kx(k - x) = 3$

$kx - x^2 - 3k^2x + 3kx^2 = 3 - k^3$

$(3k - 1)x^2 + x(k - 3k^2) + k^3 - 3 = 0$

$D = k^2(1 - 3k)^2 - 4(k^2 - 3)(3k - 1)$

$D = -(3k - 1)(k^3 + k^2 - 12)$

$D = -(3k - 1)(k^3 - 2k^2 + 3k^2 + 2)$

$= -(3k - 1)(k^2(k - 2) + 3(k^2 - 4))$

$= -(3k - 1)(k - 2)(k^2 + 3(k + 2))$

$= -(3k - 1)(k - 2)(k + 1)(k + 2) = 0.$

Tangent D = 0

$k = \dfrac{1}{3}, 2, -1, -2$

Chord $D > 0$

$k \in (-2, -1) \cup \left(\dfrac{1}{3}, 2\right)$

Hence, (A) and (D) are correct.

4. If $\alpha = 0$, we have $f(x) = \begin{cases} 0 & x = 0 \\ \sin 2\pi x & 0 < x \le 1 \end{cases}$

If $\alpha > 0$, we have $f(x) = \begin{cases} 0 & x = 0 \\ (1-2x)^{\alpha} \sin 2\pi x & 0 < x < \dfrac{1}{2}; \\ (2x-1)^{\alpha} \sin 2\pi x & \dfrac{1}{2} \le x \le 1 \end{cases}$

$$f'(x) = \begin{cases} 0 & x = 0 \\ (1-2x)^{\alpha} 2\pi \cos 2\pi x + \alpha(-2)(1-2x)^{\alpha-1} \sin 2\pi x & 0 < x < \dfrac{1}{2} \\ \alpha(2x-1)^{\alpha-1} 2 \sin 2\pi x + 2\pi(2x-1)^{\alpha} \cos 2\pi x & \dfrac{1}{2} \le x \le 1 \end{cases}$$

Continuous and diff for $\forall\ \alpha \ge 0$

Hence, (A), (B), (C) and (D) are correct.

5. $f(x) = x^5 - 10a^3x^2 + b^4x + c^5 = 0$ has 3 equal roots then $f(x) = f'(x) = f''(x) = 0$ for same value of x

Let $x = \alpha$

$f(\alpha) = \alpha^5 - 10a^3\alpha^2 + b^4\alpha + c^5 = 0$ …(1)

$f'(\alpha) = 5\alpha^4 - 20a^3\alpha + b^4 = 0$ …(2)

$f''(\alpha) = 20\alpha^3 - 20a^3 = 0 \Rightarrow \alpha = a$

From (1), we get $a^5 - 10a^5 + b^4a + c^5 = 0$

$\Rightarrow\ c^5 + ab^4 - 9a^5 = 0$ …(3)

From (2) $f'(\alpha) = 0 \Rightarrow 5a^4 - 20a^4 + b^4 = 0 \Rightarrow b^4 = 15a^4$

From (3) $c^5 + 15a^5 - 9a^5 = 0$

$\Rightarrow\ c^5 + 6a^5 = 0$

$2b^2 - 10a^3b^2 + c^5 \ne 0; 2c^5 - 10a^3b^2 + b^4c^5 \ne 0$

Hence, (B) and (D) are correct.

6. $f(x) = \dfrac{x^3}{3} - \dfrac{5x^2}{2} + 7x - 4$

$f'(x) = x^2 - 5x + 7 = 1 = m = 1$

$x^2 - 5x + 6 = 0$

$x = 2$ and $x = 3$

$x = 2; f(2) = \dfrac{8}{3} - 10 + 14 - 4 = \dfrac{8}{3}$ point $\left(2, \dfrac{8}{3}\right)$

$x = 3; f(3) = 9 - \dfrac{45}{2} + 21 - 4 = \dfrac{7}{2}$ Point $\left(3, \dfrac{7}{2}\right)$

Hence, (A) and (B) are correct.

7. $y = ax^3 + bx^2 + cx$

Point (0, 0)

$\dfrac{dy}{dx} = 3ax^2 + 2bx + c\Big|_{x=0} = \tan 45° = 1$

$\Rightarrow\ c = 1$

Point (1, 0)

$0 = a + b + c \Rightarrow a = -1 - b$

$\dfrac{dy}{dx} = 3ax^2 + 2bx + c\Big|_{(1,0)} = 0$

$\Rightarrow\ 3a + 2b + c = 0$

$\Rightarrow\ -3 - 3b + 2b + 1 = 0$

$\Rightarrow\ b = -2$

$a = -1 - b = 1$

$y = x^3 - 2x^2 + x$

$y' = 3x^2 - 4x + 1$

$y'(1) = 0$

$y'' = 6x - 4$

$y''(2) = 8$

$y''(1) = 2$

$y(1) = 0$

Hence, (A), (B) and (D) are correct.

8. $f'(1) = 1; f(2x) = f(x)\ \forall\ x > 0$

$2f'(2x) = f'(x)$

Put $x = 1$, $2f'(2) = f'(1) \Rightarrow f'(2) = \dfrac{1}{2}$

Put $x = 2$, $2f'(4) = f'(2) = \dfrac{1}{2} \Rightarrow f'(4) = \dfrac{1}{4}$

From LMVT $f''(\text{C}) = \dfrac{f'(4) - f'(2)}{4 - 2} = \dfrac{\frac{1}{4} - \frac{1}{2}}{2} = \dfrac{1}{8}$

Hence, (B) and (C) are correct.

9. $x^3 - y^2 = 0$

$\Rightarrow\ 2y\dfrac{dy}{dx} = 3x^2$

Slope of the tangent at $P = \dfrac{dy}{dx}\Big|_P = \dfrac{3x^2}{2y}\Big|_{\left(4m^2 . 8m^3\right)} = 3m$

Equation of the tangent at P is

$y - 8m^3 = 3m\left(x - 4m^2\right)$ or $y = 3mx - 4m^3$

It cuts the curve again at point Q, solving (1) and (2) we get $x = 4m^2, m^2$

Put $x = m^2$ in equation (2)

$\Rightarrow\ y = 3m\left(m^2\right) - 4m^3 = -3^3$

$\therefore\ Q$ is $\left(m^2, -m^3\right)$

Slope of the tangent at $Q = \dfrac{dy}{dx}\Big|_{\left(m^2 - m^3\right)} = \dfrac{3\left(m^4\right)}{2X\left(-m^3\right)} = \dfrac{-3}{2}m$

Slope of the normal at $Q = \dfrac{1}{(-3/2)m} = \dfrac{2}{3m}.$

Since, tangent at P is normal at $Q \Rightarrow \frac{2}{3m} = 3m$

$\Rightarrow \quad 9m^2 = 2.$

Hence, (A) and (B) are correct.

10. $\left(x^2+1\right)\left(y-3\right) = x$

$$y = 3 + \frac{x}{x^2+1}\ ;\ m = \frac{dy}{dx} = \frac{1-x^2}{\left(1+x^2\right)^2}$$

$$\frac{dm}{dx} = \frac{-2x\left(3-x^2\right)}{\left(1+x^2\right)^2}$$

For extremum $f'\left(x\right) = 0\ ; x = 0,\ x = \pm\sqrt{3}$

$x = 0, y = 3\ (0, 3)$

$x = \sqrt{3}, y = 3 + \frac{\sqrt{3}}{4} \quad \left(\sqrt{3}, 3 + \frac{\sqrt{3}}{4}\right)$

$x = -\sqrt{3}, y = 3 - \frac{\sqrt{3}}{4} \quad \left(-\sqrt{3}, 3 - \frac{\sqrt{3}}{4}\right)$

Hence, (A) and (B) are correct.

11. $g(x)$ is continuous and differentiable $\forall\ x \in R$

$$g'(x) = \begin{cases} 2x\sin\frac{\pi}{x} + x^2\cos\frac{\pi}{x}\left(-\frac{\pi}{x^2}\right) + 2(x-1)\sin\frac{\pi}{x-1} + \\ (x-1)^2\cos\frac{\pi}{(x-1)}\cdot\left(-\frac{\pi}{(x-1)^2}\right) \\ 0 \end{cases}$$

$$= \begin{cases} 2x\sin\frac{\pi}{x} - \pi\cos\frac{\pi}{x} + 2(x-1)\sin\frac{\pi}{x-1} - \pi\cos\frac{\pi}{x-1} & x \neq 0, 1 \\ 0 & x = 0, 1 \end{cases}$$

$g'(x)$ discontinuous at $x = 0, 1$

Hence, (A), (C) and (D) are correct.

12. $f(x) = \begin{cases} (1-x)^{\alpha}x(1-\cos 2\pi x) & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$

$\alpha > 0$ as $f(x)$ is defined at $x = 1$.

$$f'(1^-) = \lim_{h\to 0}\frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h\to 0}\frac{(1-1+h)^{\alpha}(1-h)(1-\cos 2\pi(1-h)) - 0}{-h}$$

$$= \lim_{h\to 0} -h^{\alpha-1}(1-h)\ (1-\cos 2\pi h)$$

$$= \lim_{h\to 0} -h^{\alpha-1}(1-h)\ \frac{2\sin^2\pi h}{(\pi h)^2}\ (\pi h)^2$$

$$= \lim_{h\to 0} -h^{\alpha-1}(1)(1)2\pi^2h^2 = -2\pi^2h^{\alpha+1}$$

$\alpha + 1 > 0, \alpha > -1$ but $\alpha > 0$

Hence, (C) and (D) are correct.

13. Slope of given line = $\frac{a-3}{a} > 0 \Rightarrow a > 3$ or $a < 0$.

Hence, (A) and (B) are correct.

14. Let $P_i\left(x_i, y_i\right)$ $i = 1, 2, 3, \ldots n$

$y = x^3$ …(1)

$\Rightarrow \quad \frac{dy}{dx} = 3x^2$

Equation of tangent at $P_1(x_1, y_1)$

$y - x_1^3 = 3x_1^2\left(x - x_1\right)$ …(2)

Solving (1) and (2), $x = -2x_1$

$\therefore \quad x_2 = -2x_1$ and $y_2 = -8x_1^3$

$P_2 \equiv \left(-2x_1, -8x_1^3\right)$ and like wise $P_3 \equiv \left(-2\left(-2x_1\right), +64x_1^3\right)$

Abscissa of $P_1, P_2, \ldots, P_n$ are given by

$x_1, -2x_1, 4x_1, -8x_1, \ldots$ which is G.P. with common ratio –2.

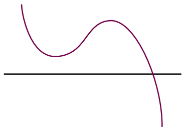
$$\text{Area of } \left(\Delta P_1P_2P_3\right) = \frac{1}{2}\begin{vmatrix} x_1 & x_1^3 & 1 \\ -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \end{vmatrix} = \frac{x_1^4}{2}\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

$$\text{Area of } \left(\Delta P_2P_3P_4\right) = \frac{1}{2}\begin{vmatrix} -2x_1 & -8x_1^3 & 1 \\ 4x_1 & 64x_1^3 & 1 \\ -8x_1 & -512x_1^3 & 1 \end{vmatrix} = 8x_1^4\begin{vmatrix} 1 & 1 & 1 \\ -2 & -8 & 1 \\ 4 & 64 & 1 \end{vmatrix}$$

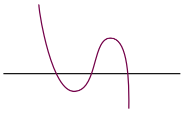
So, area of $\left(\Delta P_1P_2P_3\right) = \frac{1}{16}$ area of $\left(\Delta P_2P_3P_4\right)$

Hence, (B) and (C) are correct.

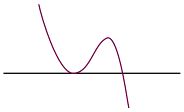
15. $f(x)$ cont. and diff $\forall\ x \in R$



(1 Solution)



(3 Solution)



(2 solution)

Hence, (A), (B) and (C) are correct.

16. Let $f(x) = e^x\cos x - 1$ which is continuous and differentiable for $x \in [\alpha, \beta]$, with α and β are roots

From Rolles theorem $f'(x) = e^x\cos x - e^x\sin x = e^x(\cos x - \sin x) = 0$

$\tan x = +1$ for atleast one x between both roots of $f(x)$

Option (A) is true

Let $g(x) = e^x \sin x - 1$ which is continuous and differentiable for $x \in [\alpha, \beta]$, with α and β are roots

From Rolles theorem $g'(x) = e^x \cos x + e^x \sin x = 0$

We get $\tan x = -1$

Option (B) is true

Let $k(x) = \cos x - e^{-x}$ is continuous and differentiable for $x \in [\alpha, \beta]$ $k(\alpha) = k(\beta) = 0$

From Rolles theorem

$k'(x) = -\sin x + e^{-x} = 0$ for atleast one $x \in (\alpha, \beta)$

Option (C) is true

Let $M(x) = \sin x - e^{-x}$ which is continuous and differentiable for $x \in [\gamma, \delta]$ $M(\gamma) = M(\delta) = 0$

From Rolles theorem we have $M'(x) = \cos x + e^{-x} = 0$

$\Rightarrow \quad e^x\cos x = -1$

Hence, (A), (B) and (C) are correct.

17. (A) Let $g(t) = e^{\frac{t^2}{2}}.f(t)$ for $t \in [a, b]$

Here $g(a) = g(b) = 0$

According to Rolle's theorem

$g'(x) = 0$, for some $x \in (a, b)$

$\Rightarrow \quad xe^{\frac{x^2}{2}}.f(x) + e^{\frac{x^2}{2}}.f'(x) = 0$

$\Rightarrow \quad xf(x) + f'(x) = 0$

(B) Let $g(t) = e^{\frac{-t^2}{2}}.f(t)$, for $t \in [a, b]$

Here $g(a) = g(b) = 0$

$\therefore$ According to Rolle's theorem

$g'(x) = 0$ for some $x \in (a, b)$

$\Rightarrow \quad -xe^{\frac{x^2}{2}}.f(x) + e^{\frac{-x^2}{2}}.f(x) = 0$

$\Rightarrow \quad xf(x) - f'(x) = 0$

(C) Let $g(t) = e^{50t} f(t),\ t \in [a, b]$

Here $g(a) = g(b) = 0$

∴ According to Rolle's theorem

$g'(x) = 0$ for some $x \in (a, b)$

$\Rightarrow \quad 50e^{50x} f(x) + e^{50x} f'(x) = 0$

$\Rightarrow \quad 50\, f(x) + f'(x) = 0$

(D) Let $g(t) = 50^t f(t),\ t \in [a, b]$

Here $g(a) = g(b) = 0$

∴ According to Rolle's theorem

$g'(x) = 0$ for some $x \in (a, b)$

$\Rightarrow \quad 50^x \ln 50\, f(x) + 50^x f'(x) = 0$

$\Rightarrow \quad (\ln 50)\, f(x) + f'(x) = 0$

Hence, (A), (B), (C) and (D) are correct.

18. (A) $f(x) = \begin{cases} x + \sin x, & x \geq 0 \\ -x - \sin x, & x < 0 \end{cases}$

$\Rightarrow f(x)$ is continuous in $[-1, 1]$ but not differentiable at $x = 0$

$f'(0^+) = 2$ and $f'(0^-) = -2$

(C) $f(x) = \begin{cases} 0, & x = 0, \pm 1 \\ 1, & x \neq 0, \pm 1 \end{cases}$

Not continuous at $x = 0, \pm 1$

(B) $\lim\limits_{|x| \to 1} f(x)$

$$\lim_{|x| \to 1} \left(\frac{\tan(x^2 - 1)}{1 - x^2} + \frac{\ln(|x| + 1 - 1)}{|x| - 1} \right)$$

$= -1 + 1 = 0$

$\Rightarrow \quad f(1) = f(-1) = 0$

$\Rightarrow f(x)$ is continuous and differentiable in $[-1, 1]$ and $f(1) = f(-1)$

(D) $0 < x < h$

$f(x) = x - \sin x$

$f'(x) = 1 - \cos x$

$\therefore \quad f'(0^+) = 0$

$-h < x < 0,\ f(x) = -x + \sin x$

$\therefore \quad f'(x) = -1 + \cos x$

$f'(0^-) = 0$

$\Rightarrow f(x)$ is continuous and differentiable in $x \in [-1, 1]$

Hence, (A) and (C) are correct.

19. (A) and (B) Using LMVT for $y = f(x)$ in $[0, 4]$ and $[4, 8]$

$$f'(c_1) = \frac{f(4) - f(0)}{4 - 0} = \frac{1}{4}\quad c_1 \in (0, 4) \qquad \ldots(1)$$

$$f'(c_2) = \frac{f(8) - f(4)}{8 - 4} = 0,\ c_1 \in (4, 8)$$

$$f'(c_2) < \frac{1}{12} < f'(c_1)$$

Let $g(x) = f(x) - \dfrac{x}{12}$

$g'(x) = f'(x) - \dfrac{x}{12} > 0$ at $c = c_1$

$\Rightarrow$ Max. does not occur at $x = c_1$

Similarly, $f'(x) = \dfrac{-1}{12} < 0$ at $x = c_2$

$\Rightarrow$ Min. does not occur at $x = c_2$.

Since, $g(x)$ is continuous on $[c_1, c_2] \Rightarrow$ by extreme value theorem is attains maximum at some point $x = c$, but since g is differentiable $\Rightarrow$

$g'(c) = 0 \Rightarrow f'(c) = \dfrac{1}{12}$.

Now, (C) We have $c_1 \in (0, 4)$ such that $f'(c_1) = \dfrac{1}{4}$.

From (1) $f(0) < \dfrac{1}{2} < f(4)$

By IVT $\exists\, c_2 \in (0, 4)$ such that $f(c_2) = \dfrac{1}{2}$

$\therefore \quad f'(c_1) f(c_2) = \dfrac{1}{8}$ for some $c_1, c_2 \in [0, 8]$

(D) Let $g(x) = \displaystyle\int_0^{x^3} f(t)\,dt$

Also, $g'(x) = 3x^2 f(x^3) \qquad \ldots(1)$

Clearly, domain of g is $[0, 2]$.

Now, applying LMVT to $y = g(x)$ in interval $[0, 1]$ and $[1, 2]$, we get

$$g'(\alpha) = \frac{g(1) - g(0)}{1 - 0},\ \alpha \in (0, 1)$$

And $g'(\beta) = \dfrac{g(2) - g(1)}{2 - 1}\quad \beta \in (1, 2)$

∴ On adding, we get

$g'(\alpha) + g'(\beta) = g(2) - g(0)$

$\Rightarrow \quad g(2) - g(0)$

$$\Rightarrow \quad g(2) = \int_0^8 f(t)\,dt = 3(\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3))$$

Where $0 < \alpha < \beta < 2$.

Hence, (A), (B), (C) and (D) are correct.

20. $x = t^5 - t^3$ and $y = t^2$

$$\frac{dx}{dt} = 5t^4 - 3t^2;\ \frac{dy}{dt} = 2t$$

$$\therefore \quad \frac{dy}{dx} = \frac{2t}{5t^4 - 3t^2} = \frac{2}{5t^3 - 3t}$$

∴ Slope of tangent at (0, 1)

$$m_T = \left.\frac{2}{t(5t^2 - 3)}\right|_{t = \pm 1} = \frac{2}{\pm 2} = \pm 1$$

$\therefore \quad m_N = \pm 1$

∴ Equations of normal are

$y - 1 = 1\,(x - 0) \Rightarrow x - y + 1 = 0$

$y - 1 = -1\,(x - 0) \Rightarrow x + y - 1 = 0$

Hence, (B) and (C) are correct.

21. $x = a(\cos\theta + \theta\sin\theta),\ y = a(\sin\theta - \theta\cos\theta)$

$$\frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta)$$

$$\frac{dy}{d\theta} = a(\cos\theta + \theta\sin\theta + \cos\theta)$$

$$\frac{dx}{d\theta} = a\,\theta\cos\theta,\ \frac{dx}{d\theta} = a\,\theta\sin\theta$$

$$\frac{dy}{dx} = \tan\theta$$

Equation of normal $y - a(\sin\theta - \theta\cos\theta) = -\cot\theta(x - a(\cos\theta + \theta\sin\theta))$

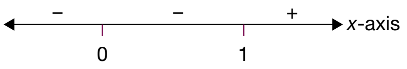
$\Rightarrow \quad \cos\theta\, x - \sin\theta\, y = a$

Distance from origin $= a$

Distance from $(\sin\theta, \cos\theta) = a$

Hence, (B) and (C) are correct.

22.



sign of $f'(x)$

$f(x) = \dfrac{e^x}{x}$;

Domain of $f(x) = (-\infty, 0) \cup (0, \infty)$

Also, $f'(x) = \dfrac{e^x(x-1)}{x^2}$

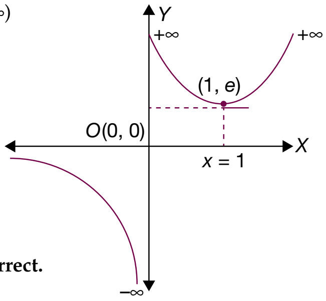
$f'(x) = 0 \Rightarrow x = 1$

Range of $f(x)$ is $(-\infty, 0) \cup [e, \infty)$

Now, verify alternatives

$f(x)$ is increasing in $(1, \infty)$ and decreasing in $(0, 1)$ tangent at $x = 0$ has slope $= 0$

Hence, (A), (B) and (D) are correct.



23. $f(x) = \begin{cases} ax^2 + b, & |x| < 1 \\ 1, & |x| = 1 \\ \dfrac{c}{|x|}, & |x| > 1 \end{cases}$

$$f(x) = \begin{cases} ax^2 + b, & -1 < x < 1 \\ 1, & x = \pm 1 \\ \dfrac{c}{x} & x > 1 \\ -\dfrac{c}{x} & x < -1 \end{cases}$$

As Rolle's theorem is applicable, the function should be continuous and differentiable in $[-3, 3]$

Continuous at $x = 1 \Rightarrow f(1^-) = f(1^+) = f(1)$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\Rightarrow \lim_{x \to 1^-} ax^2 + b = \lim_{x \to 1^+} \frac{c}{x} = f(1)$$

$\Rightarrow a + b + c = 1$

So, at $x = 1$, $a + b = 1 = c$

$f(x)$ is also differentiable at $x = 1$

$f'(1^-) = f(1^+)$

$$\Rightarrow \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

And $\quad 2a = \lim_{h \to 0} \dfrac{\dfrac{c}{1+h} - 1}{h}$ (differentiability)

$$\Rightarrow 2a = \lim_{h \to 0} \frac{1 - (1+h)}{h(1+h)}$$

$\Rightarrow a = \dfrac{-1}{2}, b = \dfrac{3}{2}$ and $c = 1$

Hence, (A), (B), (C) and (D) are correct.

24. $C_1 : y = \dfrac{2x^2}{x^2+1} \quad \Rightarrow \quad y = \dfrac{2(x^2+1) - 2}{x^2+1}$

$g = 2 - \dfrac{2}{x^2+1} \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{4x}{(x^2+1)^2}$

$\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \dfrac{4x_1}{(x_1^2+1)^2}$ of curve C_1

$C_2 : y = \sqrt{r^2 - x^2}$

$\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \dfrac{-x_1}{\sqrt{r^2 - x^2}}$ of curve C_2

$$\Rightarrow \frac{4x_1}{(x_1^2+1)^2} \; \frac{x_1}{\sqrt{r^2 - x_1^2}} = 1$$

$$\Rightarrow \frac{4x_1^2}{(x_1^2+1)^2\sqrt{r^2 - x_1^2}} = 1$$

$$\Rightarrow \frac{4x_1^2\left(x_1^2+1\right)}{(x_1^2+1)^2 - 2x_1^2} = 1 \Rightarrow x_1 = \pm 1$$

Also, (x_1, y_1) satisfy both curve

$\therefore \quad m_1 m_2 = -1 \Rightarrow x_1 = \pm 1$

$\therefore \quad y_1 = 1$

$\therefore \quad r^2 = 2$

Hence, (A), (B) and (C) are correct.

25. $f(x) = \sin x + ax + b$

$f'(x) = \cos x + a$

$f''(x) = -\sin x$

(i) Increasing if $a > 1$;

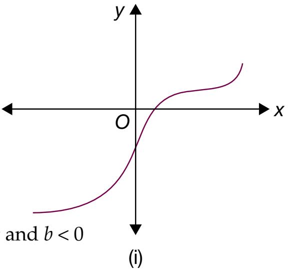
If $b < 0$ then $f(0) < 0$

(ii) If $b > 0$ and $a > 1$, then Increasing and $f(0) > 0$

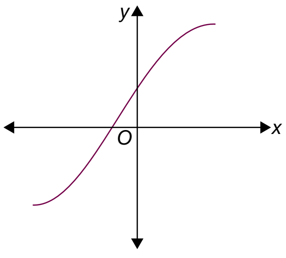
(iii) If $a < -1$, then $f(x)$ is decreasing and $b < 0$

$\Rightarrow \quad f(0) < 0$

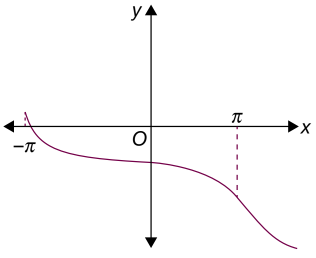
$\Rightarrow \quad$ Root is negative.



(i)



(ii)



(iii)

Hence, (A), (B) and (C) are correct.

26. By IVP, $\dfrac{f(0) + f(1)}{2} = f(c), 0 < c < 1$

By LMVT, $f\left(\dfrac{1}{2}\right) - f(0) = \dfrac{1}{2} f'(c_1), 0 < c_1 < \dfrac{1}{2}$

$f(1) - f\left(\dfrac{1}{2}\right) = \dfrac{1}{2} f'(c_2), \dfrac{1}{2} < c_1 < 1$

Subtracting, we get,

$$f(1) + f(0) - 2f\left(\frac{1}{2}\right) = \frac{f'(c_2) - f'(c_1)}{2} = \frac{c_2 - c_1}{2} f''(c) > 0$$

(Using LMVT)

$$\Rightarrow \quad f(1) + f(0) > 2f\left(\frac{1}{2}\right)$$

Hence, (A) and (C) are correct.

27. (A) $\quad f(x) = x^{4/3}, x \in [-1, 1]$

$f(x)$ is continuous in $[-1, 1]$ and differentiable in $(-1, 1)$.

Also, $f(-1) = f(1) \Rightarrow$ Rolle's theorem is applicable.

(B) $K(x) = |x| + |x - 2| = \begin{cases} 2 - 2x, & -1 \le x < 0 \\ 2, & 0 \le x \le 2 \end{cases}$

$\Rightarrow K(x)$ is not differentiable at $x = 0$

(C) $h(x) = x + \dfrac{1}{x}, x \in \left[\dfrac{1}{5}, 5\right]$

$\Rightarrow h(x)$ is continuous and differentiable in $\left(\dfrac{1}{5}, 5\right)$ also $f\left(\dfrac{1}{5}\right) = f(5)$

$$\text{(D) } g(x) = \begin{cases} \cos\left(\dfrac{1}{x}\right), & x \in [-1, 0) \cup (0, 1) \\ 0, & x = 0 \end{cases}$$

As, $\lim\limits_{x \to 0} g(x)$ does not exist. So, $g(x)$ is discontinuous at $x = 0$.

So, (B, D) are correct options.

Hence, (B) and (D) are correct.

Linked Comprehension Type Questions

1. $f(x) = ax^3 + bx^2 + cx + 4$

$$f'(x) = 3ax^2 + 2bx + c$$

$$f'\left(-\frac{2}{3}\right) = 3a\left(\frac{4}{9}\right) + 2b\left(-\frac{2}{3}\right) + c = \frac{5}{3}$$

$$\Rightarrow \frac{4a}{3} - \frac{4}{3}b + c = \frac{5}{3} \qquad \ldots(1)$$

$$f''(x) = 6ax + 2b$$

$$f''\left(-\frac{2}{3}\right) = 6a\left(-\frac{2}{3}\right) + 2b = 0$$

$\Rightarrow$ $4a = 2b \Rightarrow b = 2a$; from (1)

$$\frac{4a}{3} - \frac{8a}{3} + c = \frac{5}{3} \Rightarrow c = \frac{5}{3} + \frac{4a}{3};$$

$$f(x) = ax^3 + 2ax^2 + \left(\frac{5}{3} + \frac{4a}{3}\right)x + 4$$

Tangent at $x = -\dfrac{2}{3}$;

$$y = -\frac{8a}{27} + \frac{8a}{9} + \left(\frac{5}{3} + \frac{4a}{3}\right)\left(-\frac{2}{3}\right) + 4 = -\frac{8a}{27} + \frac{26}{9}$$

$$y + \frac{8a}{27} - \frac{26}{9} = \frac{5}{3}\left(x + \frac{2}{3}\right)$$

$$y = \frac{5}{3}x + \frac{10}{9} + \frac{26}{9} - \frac{8a}{27}$$

Given $y = \dfrac{5}{3}x + \dfrac{100}{27}$

$$4 - \frac{8a}{27} = \frac{100}{27} \Rightarrow 108 - 8a = 100 \Rightarrow a = 1$$

$$a = 1;\ b = 2;\ c = \frac{5+4}{3} = 3$$

$$a + b + c = 6$$

Hence, the correct answer is (B).

2. $f(g(x)) = x$

$$\frac{d}{x}(g(x)f(g(x)) = \frac{d}{dx}(g(x) \cdot x) = g'(x)x + g(x)\,|_{x=4} = 4g'(4) + g(4)$$

$$f(x) = x^3 + 2x^2 + 3x + 4$$

$$x = 0$$

$$f(0) = 4$$

$$g(4) = 0$$

$$g(f(x)) = x$$

$$g'(f(x)) = \frac{1}{f'(x)} \Rightarrow g'(4) = \frac{1}{f'(0)} = \frac{1}{(3x^2 + 4x + 3)_{x=0}}$$

$$g'(4) = \frac{1}{3}$$

Now, $\dfrac{d}{dx}(g(x)f(g(x))) = 4g'(4) + g(4) = 4\left(\dfrac{1}{3}\right) + 0 = \dfrac{4}{3}$

Hence, the correct answer is (D).

3. $\dfrac{dx}{dt} = -6t;\ \dfrac{dy}{dt} = 1 - 9t^2$

$$\frac{dy}{dt} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - 9t^2}{-6t}$$

$$\tan\theta = \frac{1 - 9t^2}{-6t};\ \sec\theta = \sqrt{1 + \tan^2\theta} = \frac{1 + 9t^2}{6t}$$

$$\tan\theta + \sec\theta = \frac{1 - 9t^2}{-6t} + \frac{1 + 9t^2}{6t} = 3t$$

Hence, the correct answer is (A).

4. If $P(-2, 2) \Rightarrow t = -1$

$$\left.\frac{dy}{dt}\right|_{(-2,2)} = \frac{1-9}{6} = -\frac{4}{3};\ y - 2 = -\frac{4}{3}(x + 2)$$

$$3(t - 3t^3 - 2) = -4(1 - 3t^2 + 2) \Rightarrow (t + 1)\,(9t^2 + 3t - 6) = 0$$

$$\Rightarrow t = -1 \text{ and } t = \frac{2}{3} \Rightarrow x = -\frac{1}{3} \text{ and } y = -\frac{2}{9}$$

Hence, the correct answer is (D).

5. The equation of the tangent at (x_1, y_1) of $x^3 + y^3 = a^3$ is $xx_1^2 + yy_1^2 = a^3$. If this tangent passes through (h, k) we have

$$hx_1^2 + kx_1^2 = a^3 \qquad \ldots(1)$$

Also $x_1^3 + y_1^3 = a^3$ $\qquad \ldots(2)$

$$h^3 + k^3 = a^3 \qquad \ldots(3)$$

Now, let $A = \dfrac{h}{x_1}$, $B = \dfrac{k}{y_1}$, then the three relations become

$$Ax_1^3 + By_1^3 = a^3,\ x_1^3 + y_1^3 = a^3,\ A^3x_1^3 + B^3y_1^3 = a^3$$

Solving first two for x_1^3 and y_1^3, we get

$$x_1^3 = \frac{a^3(1 - B)}{A - B},\ y_1^3 = -\frac{a^3(1 - A)}{A - B}$$

Hence, the correct answer is (A).

6. On putting in the last relation and canceling a^3 we get

$$S^2(1 - B) - B^3(1 - A) = A - B$$

$$\Rightarrow (A^3 - B^3) - AB(A^2 - B^2) = A - B$$

$$\Rightarrow (A - B)[(A + B)^2 - AB(A + B) - AB - 1] = 0$$

$$\Rightarrow (A + B)^2 - AB(A + B) - AB - 1 = 0 \qquad (\because A \neq B)$$

Hence, the correct answer is (C).

7. Solving the quadratic for $A + B$ we get $A + B = \dfrac{AB \pm (AB + 2)}{2}$

On taking –sign, we get $A + B = -1$

$$\Rightarrow \frac{h}{x_1} + \frac{k}{y_1} = -1 \Rightarrow \text{(B) is true.}$$

We have to rule out $A = B$ and $A + B = AB + 1$

If $A = B$, i.e., $\dfrac{h}{x_1} + \dfrac{k}{y_1} = \alpha$ (say), then the point (h, k) and (x_1, y_1) coincide

If $A + B = AB + 1$, then $(1 - A)(1 - B) = 0 \Rightarrow A = 1, B = 1$

$\Rightarrow$ (x_1, y_1) and (h, k) coincide.

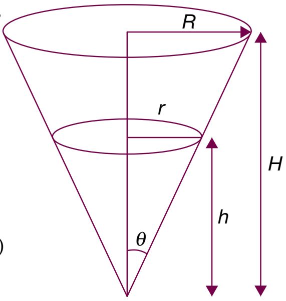
Hence, the correct answer is (B).

Combined solution to 8–10

Let θ be the semi vertical angle of the cone so that $\tan\theta = \dfrac{R}{H}$.

Let the radius and height of water cone at time t be r and h respectively. So

$$\tan\theta = \frac{r}{h}$$



If V is the volume of water and S is the surface of the cone in contact with air at time t, then

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 \cot\theta \text{ and}$$

$$S = \pi r^2$$

We are given that $\frac{dV}{dt} \propto S$

$\Rightarrow \frac{dV}{dt} = -kS$ (V is decreasing)

$\Rightarrow \frac{1}{3}\pi(3r^2)\cot\theta \frac{dr}{dt} = k(\pi r^2)$

$\Rightarrow \frac{dr}{dt} = -k\tan\theta$

Integrating, we get $r = -(k\tan\theta)\, t + C$

When $t = 0$, $r = R$,

$\therefore \quad C = R$

Thus, $r = (-k\tan\theta)t + R$

When cone is empty, $r = 0$. If T is the time taken for the cone to be empty, then

$$0 + (-k\tan\theta)T + R \Rightarrow T = \frac{R}{k\tan\theta} = \frac{R}{k\left(\frac{R}{H}\right)} = \frac{H}{k}$$

Hence, cone will be empty in time $\frac{H}{k}$.

$$r(1) = -k\tan\theta + R = -k\frac{R}{H} + R = R\left(1 - \frac{k}{H}\right)$$

$$\sum_{i=1}^{10} r(i) = 10R - \frac{R}{H}k\sum_{i=1}^{10} i = 10R - \frac{Rk}{H}55$$

8. **Hence, the correct answer is (B).**

9. **Hence, the correct answer is (A).**

10. **Hence, the correct answer is (C).**

11. $f(x) = x^2 - ax + b$

One root must be 2. $F(2) = 0$

$$2a - b = 4 \qquad \ldots(1)$$

$$\sum_{i=1}^{5} f(x) = 20 \Rightarrow \frac{5.6.11}{6} - \frac{a.5.6}{2} + 5b = 20$$

$$\Rightarrow -15a + 5b = -35 \Rightarrow 3a - b = 7 \qquad \ldots(2)$$

From (1) and (2) we get $a = 3, b = 2$

$\therefore \quad f(x) = x^2 - 3x + 2$

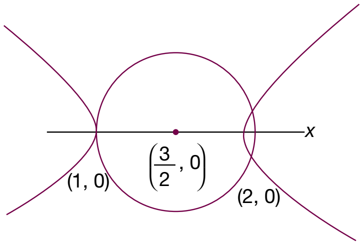
Curve $C_1 : x^2 - y^2 - 3x + 2 = 0$

Rectangular hyperbola with centre $\left(\frac{3}{2}, 0\right)$ and $a^2 = b^2 = \frac{1}{4}$

Curve $C_2 : x^2 + y^2 - 3x + 2 = 0$

Circle with centre $\left(\frac{3}{2}, 0\right)$ and radius is $\frac{1}{2}$

Exactly one normal of C_1 passes through centre of C_2

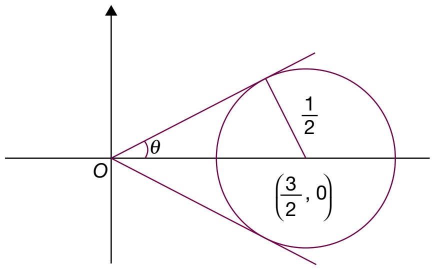


Hence, the correct answer is (A).

12. $\sin\theta = \frac{1}{3}$

$\tan\theta = \frac{1}{\sqrt{9-1}}$

Slope of tangents drawn from origin to the circle is $\pm\frac{1}{2\sqrt{2}}$



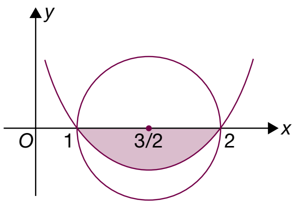
Given that $\frac{1}{t\sqrt{t}} = \frac{1}{2\sqrt{2}} \Rightarrow t = 2$

Hence, the correct answer is (B).

13. $y = f(x)$

$y = x^2 - 3x + 2$

Area bounded by $y = x^2 - 3x + 2$ and x-axis



$$= \left|\int_1^2 (x^2 - 3x + 2)dx\right|$$

$$= \left[\left|\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right|\right]_1^2 = \frac{1}{6}$$

Required Area $= \frac{\pi}{8} - \frac{1}{6}$

$= \frac{3\pi - 4}{24}$

$\therefore \quad \lambda = 4$

Hence, the correct answer is (D).

Combined solution to 14–16

Given $2y + 2x + \frac{1}{2}\pi y = p$

Let amount of light per square meter for the coloured glass be μ. If L is the total light transmitted then

$L = 3\mu \times$ Area of rectangular portion + $\mu \times$ Area of semi circular portion.

$$= 3\mu xy + \frac{1}{8}\mu\pi y^2$$

$$= \mu\left[\frac{3y}{2}\left(p - 2y - \frac{1}{2}\pi y\right) + \frac{1}{8}\pi y^2\right);$$

$$\frac{dL}{dy} = \mu\left[\frac{3}{2}(p - 4y - \pi y) + \frac{\pi}{4}y\right]$$

$$\left(\frac{dL}{dy}\right)_{y=1} = \mu\left[\frac{3}{2}(p - 4 - \pi) + \frac{\pi}{4}\right] = \frac{\mu}{2}\left[3p - 12 - \frac{5\pi}{2}\right]$$

$$\frac{dL}{dy} = 0 \Rightarrow y = \frac{3p}{12 + \frac{5\pi}{2}}; \quad \frac{d^2L}{dy^2} = \frac{\mu}{2}\left(-12 - \frac{5\pi}{2}\right) < 0$$

L is maximum for $y = \frac{3p}{12 + \frac{5\pi}{2}}$;

$$\frac{y}{x}=\frac{2.\dfrac{3p}{12+\dfrac{5\pi}{2}}}{p-\left(2+\dfrac{\pi}{2}\right)\dfrac{3p}{12+\dfrac{5\pi}{2}}}=\frac{6}{6+\pi}$$

14. Hence, the correct answer is (B).

15. Hence, the correct answer is (A).

16. Hence, the correct answer is (C).

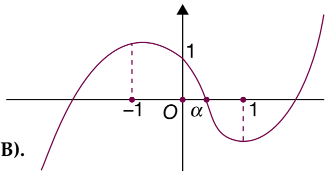
17. Apply Rolle’s theorem

Let $f(x) = x^3 - 3x + 1$

$f'(x) = 3(x^2 - 1)$

$\alpha \in (0, 1)$

Hence, the correct answer is (B).



18. Refer theory

Hence, the correct answer is (C).

19. $f(x) = ax^4 + bx^3 + cx^2 + 2009x + 2010$

$f'(x) = 4ax^3 + 3bx^2 + 2cx + 2009$

$f''(x) = 12ax^2 + 6bx + 2c$

$6ax^2 + 3bx + c > 0$

$\because\ 3b^2 - 8ac < 0$

$f''(x)$ does and has any real root.

$\Rightarrow$ $f'(x)$ has only one real root.

$\Rightarrow$ $f(x)$ has maximum two real roots.

Hence, the correct answer is (C).

20. $\phi(x) = f(b) - f(x) - (b - x)f'(x) - (b - x)^2 A$

$\phi(b) = 0$

$\phi(a) = f(b) - f(a) - (b - a)f'(a) - (b - a)^2 A,$

Since, Rolle’s theorem is applicable

$$\therefore\ A=\frac{f(b)-f(a)-(b-a)f'(a)}{(b-a)^2} \qquad \ldots(1)$$

$\phi'(x) = -f'(x) + f'(x) - (b - x)f''(x) + 2(b - x)A$

$\therefore$ there exists some number $c \in (a, b)$ such that

$0 = \phi'(c) = -(b - c)f''(c) + 2(b - c)A$

$$\text{i.e. } A=\frac{1}{2}f''(c) \qquad \ldots(2)$$

From (1) and (2), $f(b) - f(a) - (b - a)f'(a) = \dfrac{1}{2}(b - a)^2 f''(c)$

i.e. $f(b) = f(a) + (b - a)f'(a) + \dfrac{1}{2}(b - a)^2 f''(c)$

$\therefore\ \lambda = \dfrac{1}{2}$

Hence, the correct answer is (C).

21. $f(1 + h) = f(1) - hf'(1) + \dfrac{1}{2}h^2 f''(c)$

$f(1 + h) = 1 + 3h^2 c,\ \dfrac{f(1+h)-f(1)}{h^2} = 3c$

$\therefore\ \lambda = 3$

Hence, the correct answer is (C).

22. $\sin(\alpha + h) = \sin\alpha + h\cos\alpha + \dfrac{1}{2}h^2(-\sin(\alpha + th))$

$\therefore\ \dfrac{\sin(\alpha+h)-\sin\alpha-h\cos\alpha}{h^2} = -\dfrac{1}{2}\sin(\alpha + th)$

$\therefore\ \lambda = -\dfrac{1}{2}$

Hence, the correct answer is (B).

23. $\dfrac{g'(c)}{f'(c)}=\dfrac{g(b)-g(a)}{f(b)-f(a)}=\dfrac{e^{-b}-e^{-a}}{e^{b}-e^{a}}=\dfrac{-1}{e^{a+b}}$

$-\dfrac{e^{-C}}{e^{C}}=-\dfrac{1}{e^{a+b}} \Rightarrow C=\dfrac{a+b}{2}$

Hence, the correct answer is (A).

24. $\dfrac{g'(c)}{f'(e)}=\dfrac{\dfrac{1}{b}-\dfrac{1}{a}}{\dfrac{1}{b^2}-\dfrac{1}{a^2}}=\dfrac{ab}{a+b} \Rightarrow \dfrac{\dfrac{-1}{c^2}}{\dfrac{-2}{c^3}}=\dfrac{ab}{a+b} \Rightarrow \dfrac{c}{2}=\dfrac{ab}{a+b}$

$\Rightarrow\ c=\dfrac{2ab}{a+b}$

Hence, the correct answer is (C).

25. $\dfrac{g'(c)}{f'(c)}=\dfrac{\dfrac{1}{\sqrt{b}}-\dfrac{1}{\sqrt{a}}}{\sqrt{b}-\sqrt{a}}=-\dfrac{1}{\sqrt{ab}}=\dfrac{-1}{c}$

$\Rightarrow\ c=\sqrt{ab}$

Hence, the correct answer is (B).

Matrix Match/Column Match Type Questions

1. A → q; B → r; C → p; D → s

(A) $r = 6$ cm

$\Delta r = 0.06$ cm

Area $= A = \pi r^2$

$\dfrac{dA}{dr} = 2\pi r$

$\Delta A = \left(\dfrac{dA}{dr}\right)\Delta r = 2\pi r(0.06) = 2\pi 6(0.06) = 0.72\pi$

(B) $\dfrac{\Delta x}{x} = \dfrac{2}{100}$

$V = x^3$

$\dfrac{dV}{dx} = 3x^2$

$\Delta V = \left(\dfrac{dV}{dx}\right)\Delta V = 3x^3\,\Delta x$

$\dfrac{\Delta V}{V}\times 100 = \dfrac{3x^2\Delta x}{x^3} = 3\,\dfrac{(\Delta x)}{x} = 6\%$

(C) $\dfrac{d}{dx}\left(\dfrac{x^2}{2}-2x+5\right) = 3\,\dfrac{dx}{dx}$

$\Rightarrow x - 2 = 3 \Rightarrow x = 5$

(D) $A = \dfrac{\sqrt{3}}{4}a^2$

$\Rightarrow\ \dfrac{dA}{dt} = \dfrac{\sqrt{3}}{4}\cdot 2a\cdot\dfrac{da}{dt} = \dfrac{\sqrt{3}}{2}(30)(0.1) = \dfrac{3\sqrt{3}}{2}$

2. A → p; B → r; C → s; D → q

$y - y_i = f'(x_i, y_i)\cdot(x - x_i)$

(A) $xy^2 = 1$

$y = \dfrac{1}{\sqrt{x}} \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{2x^{3/2}}$

Tangent $y - \dfrac{1}{\sqrt{x_i}} = -\dfrac{1}{2x_i^{3/2}}(x - x_i)$

$y = 0,\ x = x_i + 2x_{\mathrm{i}} = 3x_{\mathrm{i}}$

$x_1, x_2, x_3 \ldots$ G.P.

(B) $y = e^{-3x}$ then $\frac{dy}{dx} = -3e^{-3x}$

Tangent $y - e^{-3xi} - 3e^{-3xi}(x - x_i)$

$y = 0$

$x_{i+1} = \frac{1}{3} + x_i$

$x_{i+2} = \frac{1}{3} + \left(\frac{1}{3} + x_i\right)$

$x_1, x_2, x_3 \ldots$ Increasing A.P.

(C) $y = -\cot^{-1}x = -\frac{\pi}{2} + \tan^{-1}x$ then $\frac{dy}{dx} = \frac{1}{1+x^2}$

Tangent $y + \cot^{-1}x_i = \frac{1}{1+x_i^2}(x - x_i)$

$y = 0,\ x_{i+1} = x_i + \left(1+x_i^2\right)\cot^{-1}x_i$

$$\frac{dx_{i+1}}{dx_i} = 1 + \left(1+x_i^2\right)\left(\frac{-1}{1+x_i^2}\right) + 2x_i\cot^{-1}x_i = 2x_i\cot^{-1}x_i > 0$$

$x_1, x_2, x_3 \ldots$ Increasing

(D) $y^2 = x \quad y = \sqrt{x}$

$y - \sqrt{x_i} = \frac{1}{2\sqrt{x_i}}(x - x_i)$

$y = 0$

$x_{i+1} = x_i - 2x_i = -x_i$

$x_{i+2} = -x_{i+1} = x_i$

Periodic $(T = 2)$

Hence, the correct answer is (B).

3. **A → s; B → r; C → q, D → p**

(A) We first determine the value of t corresponding to the given values of x and y.

From $t^2 - 3t + 1 = -1$

$\Rightarrow t^2 - 3t + 2 = 0 = (t - 1)(t - 2)$

$\therefore t = 1, 2$

And from $2t^2 + 3t - 4 = 10$

$\Rightarrow 2t^2 + 3t - 14 = 0\ (2t + 7)(t - 2)$

$\therefore t = \frac{-7}{2}, 2$

Hence, to the given point $M(-1, 10)$, there corresponds the value $t = 2$.

$\therefore$ The slope of tangent at $M(-1, 10)$ is

$$\left.\frac{dy}{dx}\right|_{t=2} = \left.\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right|_{t=2} = \left.\frac{4t+3}{2t-3}\right|_{t=2} = 11$$

(B) We have $f(x) = x^3 + 6x^2 + px + 7$

$\therefore f'(x) = 3x^2 + 12x + p < 0$

$\Rightarrow 3x^2 + 12x + p = k(x + 3)(x + 1)$

$\therefore$ On comparing coefficients, we get

$k = 3$ and $p = 3k$

Hence, $p = 9$

(C) Since, $f(\theta) = \left(\frac{7}{3}\right) 3\cos 3\theta + 2a\cos\theta = 0$

$\therefore f'\left(\frac{\pi}{3}\right) = 7\cos\pi + 2a\cos\frac{\pi}{3} = -7 + a = 0$

Hence, $a = 7$

(D) Equation of tangent

$y - (t^2 - t + 2) = \left(\frac{2t-1}{2t+1}\right)(x - (t^2 + t + 2))$

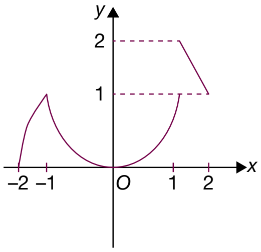
Put $x = 2$, $y = 2$ we get $t = 0$

$\therefore$ Tangent $x + y = 4$

Area of triangle $= \frac{1}{2}\cdot 4\cdot 4 = 8$

4. **A → q; B → q; C → p; D → q**

$$f(x) = \begin{cases} -x^3 : -1 \le x < 0 \\ x^2 : 0 \le x < 1 \\ 3 - x : 1 \le x \le 2 \end{cases}$$



Critical points are $x = -1, 1$

$\because$ $f(x)$ is continuous in $[-1, 1]$ and differentiable in $(-1, 1)$.

Also $f(-1) = f(1)$

$\Rightarrow$ Rolle's theorem is applicable in $[-1, 1]$

$f'(x) = 0 \Rightarrow C = 0$

$f(x)$ is maximum when $x = 1$

$\Rightarrow f(1) = 2$

5. **A → r; B → q; C → t; D → s**

(A) $\frac{dy}{dx} = \frac{4t}{3}$

Tangent is $y - at^4 = \frac{4t}{3}\left(x - at^3\right)$;

x-intercept $= \frac{at^3}{4}$

y-intercept $= -\frac{at^4}{3}$

The point of intersection of tangent with the axes are

$\left(\frac{at^3}{4}, 0\right)$ and $\left(0, \frac{-at^4}{3}\right)$

$\therefore \quad \frac{m}{n} = -\frac{3}{4}$

$m = 3, n = 4$

$m + n = 7$

(B) $\frac{dx}{dy} = e^{\sin y}\cos y$

Slope of normal $= -1$

Equation of normal is $x + y = 1$

Area $= \frac{1}{2}$

(C) $y = \frac{1}{x^2} : \frac{dy}{dx} = -\frac{2}{x^3}$

Slope of tangent $= -2$

$y = e^{2-2x} : \frac{dy}{dx} = e^{2-2x}\cdot(-2)$

Slope of tangent $= -2$

$\therefore \quad \tan\theta = 0$

(D) $\frac{dy}{dx} = \frac{be^{x/3}}{3}$

Length of subtangent $= \left|\frac{y}{y'}\right| = \left|\frac{be^{x/3}}{b\frac{1}{3}e^{x/3}}\right| = 3$

6. **$A \to r; B \to q; C \to p, q, r, s; D \to r$**

(A) By LMVT there exists $a \in (0,4)$ such that $f(4) - f(0) = 4f'(a)$

By Intermediate value theorem $\Im\, b \in (0,4)$ such that

$$\frac{f(4)+f(0)}{2} = f(b)$$

$$\frac{4[(f(4))^2-(f(0))^2].2}{[f(4)-f(0)][f(4)+f(0)]}$$

(B) $F'(x) = fgh' + fg'h + f'gh$ by product rule at $x = x_0$

$21F = kfgh - 7fgh + 4fgh$

$\therefore 21fgh = (k-3)fgh \Rightarrow k = 24$

(C) $\left|\lim\limits_{x_1 \to x_2} \frac{f(x_1)-f(x_2)}{x_1 - x_2}\right| \le 0 \Rightarrow f(x) = c$ for all $x \in R$ Normal is $x =$

1. Since it passes through (1, 2)

$\therefore l = 1, m = 0, n = 1$

(D) Normal at "t_1" is $y + xt_1 = 2at_1 + at_1^3$

It passes through "t_2" $\Rightarrow 2a(t_2 - t_1) = at_1\left(t_1^2 - t_2^2\right)$

$$\Rightarrow t_2 + t_1 = \frac{-2}{t_1}$$

$$\Rightarrow t_2 = -t_1 - \frac{2}{t_1} \Rightarrow |t_2| \ge 2\sqrt{2} \Rightarrow t_2^2 \ge 8$$

7. **$A \to p, q; B \to r; C \to p, r; D \to q$**

(A) $y = 2e^{2x}$ intersect y-axis at (0, 2)

$$\left(\frac{dy}{dx}\right)_{\text{at } x=0} = 4$$

$\therefore$ Angle of intersection with y-axis

$$= \frac{\pi}{2} - \tan^{-1} 4 = \cot^{-1} 4$$

$$\Rightarrow \left|\frac{8n-4}{3}\right| = 4 \Rightarrow n = 2 \text{ or } -1$$

(B) $x = e^{\sin y}$

$$1 = e^{\sin y} \cos y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{e^{\sin y} \cos y}$$

$$\left(\frac{dy}{dx}\right)_{(1,0)} = \frac{1}{1} = 1$$

Slope of the normal at (1, 0) = –1 equation of normal is $x + y = 1$

Area $= \frac{1}{2} \Rightarrow t = 1, -2$

(C) $x^2y^2 = 16 \Rightarrow xy = \pm 4$

$$L_{S.T} = \left|\frac{y}{\frac{dy}{dx}}\right| = |x| \Rightarrow L_{S.T} = 2 \Rightarrow K = \pm 2$$

(D) Here, $y > 0$, putting $y = x$ in $y = \sqrt{4 - x^2}$, we get $x = \sqrt{2}, -\sqrt{2}$

So, the point $\left(\sqrt{2}, \sqrt{2}\right)$

Differentiating w.r.t x, we get

$$\frac{dy}{dx} = \frac{-x}{y}$$

At $\left(\sqrt{2}, \sqrt{2}\right)$, $\frac{dy}{dx} = -1$

Integer/Numerical Answer Type Questions

1. $2y\frac{dy}{dx} - 6x^2 - 4\frac{dy}{dx} = 0$

$$\frac{dy}{dx} = \frac{3x^2}{y-2}$$

$$\frac{dy}{dx} \text{ at } (h,k) = \frac{6h^2}{2k-4} = \frac{k-2}{h-1} \text{ or } (k-2)^2 = 3h^3 - 3h^2 \quad \ldots(1)$$

Since, (h, k) lies on the curve $k^2 - 4k = 2h^3 - 8$ …(2)

Hence, $h^3 - 3h^2 + 4 = 0$ or $(h+1)\left(h^2 - 4h + 4\right) = 0$

$h = 2; k = 2 + \sqrt{12}$

$\therefore$ $h = -1, 2, 2$; But when $h = -1$, k becomes imaginary.

$\therefore$ Hence, there is two tangent.

Hence, the correct answer is (2).

2. $x^2 + y^2 = a^2\sqrt{2}$ and $x^2 - y^2 = a^2$

$$\frac{dy}{dx} = \frac{-x}{y}; \frac{dy}{dx} = \frac{x}{y}$$

$$x^2 = \frac{a^2\left(\sqrt{2}+1\right)}{2}, y^2 = \frac{a^2\left(\sqrt{2}-1\right)}{2}$$

$$\theta = \tan^{-1}\left(\frac{\frac{x}{y} - \left(-\frac{x}{y}\right)}{1 + \left(\frac{x}{y}\right)\left(-\frac{x}{y}\right)}\right) = \tan^{-1}\left(\frac{\frac{2x}{y}}{1 - \frac{x^2}{y^2}}\right)$$

$$= \tan^{-1} \frac{2\left(\sqrt{2}+1\right)}{1 - \left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)} = \tan^{-1} \frac{2(2-1)}{-2}$$

$$\left|\tan^{-1}(-1)\right| = \frac{\pi}{4}$$

Hence, the correct answer is (4).

3. $v = \frac{4}{3}\pi r^3$

$$\frac{dv}{dt} = \frac{4}{3}\pi.3r^2\frac{dr}{dt}$$

$$\frac{dv}{dt} = \frac{dr}{dt}; \; r^2 = \frac{1}{4\pi} \Rightarrow r = \frac{1}{2\sqrt{\pi}} \Rightarrow k = 2$$

Hence, the correct answer is (2).

4. $x^4 + y^4 = a^4$

$$4x^3 + 4y^3\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x_1^3}{y_1^3} \Rightarrow y - y_1 = \frac{-x_1^3}{y_1^3}(x - x_1)y = 0$$

$$x = \frac{a^4}{x_1^3} \text{ put } x = 0; y = \frac{a^4}{y_1^3}; \; y - y_1 : \frac{x_1^4}{y_1^3} y = \frac{a^4}{y_1^3}$$

$$P = \frac{a^4}{x_1^3} \Rightarrow P^{-4/3} = \frac{a^{-16/3}}{x_1^{-4}}; \; q^{-4/3} = \frac{a^{-16/3}}{y_1^{-4}}$$

$$p^{-4/3} + q^{-4/3} = a^{-16/3}.a^4$$

$$= a^{-4/3} \Rightarrow k = 4$$

Hence, the correct answer is (4).

5. $2y\frac{dy}{dx} = 1 - \cos x$

$$\frac{dy}{dx} = \frac{1 - \cos x}{2\sqrt{x - \sin x}} = 0 \Rightarrow \cos x = 1 \Rightarrow x = 0, \pm 2\pi, \pm 4\pi \ldots$$

$\therefore$ $x \in [-10, 10] \Rightarrow x = 0, \pm 2\pi$

For $x=0, y=0$; $\left.\begin{array}{l} x=0, y=0 \\ x=2\pi, y=\pm\sqrt{2\pi} \end{array}\right\}$ they satisfy $-3 \le y \le 3$

$x=-2\pi, y^2=-2\pi$ (not possible)

Also, slope at $(0,0)$ is undefined hence points are $\left(2\pi, \sqrt{2\pi}\right)$ and $\left(2\pi, -\sqrt{2\pi}\right)$

Hence, the correct answer is (2).

6. $y=3x+3$ but $f'(0)=-\frac{1}{3}$

$$\lim_{x\to 0} \frac{2x}{2xf'\left(x^2\right)-40xf'\left(4x^2\right)+56xf'\left(7x^2\right)} \quad \text{(Using L'Hospital Rule)}$$

$$=\frac{1}{f'(0)-20f'(0)+28f'(0)}$$

$$=\frac{1}{9f'(0)}=\frac{1}{9\left(-\frac{1}{3}\right)} \Rightarrow k=3$$

Hence, the correct answer is (3).

7. Given curve is $x^3+y^3=8$

$$\therefore \quad \frac{dy}{dx}=-\frac{x^2}{y^2}$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{(a,b)}=-\frac{a^2}{b^2} \qquad \ldots(1)$$

And tangent at (a, b) cuts the curve again at (a_1, b_1)

$\therefore$ Slope of tangent $=\frac{b_1-b}{a_1-a}=-\frac{a^2}{b^2}$ {From (1)} …(1)

Also, $a^3+b^3=8$ …(2)

And $a_1^3+b_1^3=8$ …(3)

Subtracting (3) from (2), we get

$$\left(a^3-a_1^3\right)+\left(b^3-b_1^3\right)=0$$

$$\Rightarrow \quad \left(a-a_1\right)\left(a^2+aa_1+a_1^2\right)+\left(b-b_1\right)\left(b^2+bb_1+b_1^2\right)=0$$

$$\Rightarrow \quad \frac{b_1-b}{a_1-a}=-\frac{a^2+aa_1+a_1^2}{b^2+bb_1+b_1^2} \qquad \ldots(4)$$

From (1) and (4) $-\frac{a^2}{b^2}=-\frac{a^2+aa_1+a_1^2}{b^2+bb_1+b_1^2}$

$$\Rightarrow \quad a^2b^2+a^2bb_1+a^2b_1^2=a^2b^2+ab^2a_1+a_1^2b^2$$

$$\Rightarrow \quad ab\left(ab_1-ba_1\right)+a^2b_1^2-a_1^2b^2=0$$

$$\Rightarrow \quad ab\left(ab_1-ba_1\right)+\left(ab_1+a_1b\right)\left(ab_1-a_1b\right)=0$$

$$\Rightarrow \quad \left(ab_1-a_1b\right)-\left(ab+ab_1+a_1b\right)=0$$

If $ab_1-a_1b=0$

Then $\frac{a}{a_1}-\frac{b}{b_1}=\frac{\sqrt[3]{a^3+b^3}}{\sqrt[3]{a_1^3+b_1^3}}=\frac{2}{2}=1$ (Law of proportion)

$\therefore$ $a=a_1$ and $b=b_1$ which is impossible

$\therefore$ $ab_1 - a_1b \ne 0$

Hence, $ab+ab_1+a_1b=0$ or $\frac{a_1}{a}+\frac{b_1}{b}=-1$

Hence, the correct answer is (1).

8. $x^{2/3}+y^{2/3}=(64)^{2/3}=16$

$(x_1, y_1) \equiv (64\cos^3\theta, 64\sin^3\theta)$

Tangent $x_1^{-1/3}x+\ y_1^{-1/3}y=(64)^{2/3}=16$

$$\frac{x}{16x_1^{1/3}}+\frac{y}{16y_1^{1/3}}=1$$

$$\ell=\sqrt{(16x_1^{1/3})^2+(16y_1^{1/3})^2}$$

$$\ell=16\sqrt{(x_1^{2/3}+y_1^{2/3})}=16(4)=64$$

$$\sqrt{\ell}=8$$

Hence, the correct answer is (8).

9. We have, $f(x)=\begin{cases} 5, & x=0 \\ x^2+5x+\alpha, & 0<x<2 \\ \beta x+\gamma, & 2\le x\le 3 \end{cases}$

As, $f(x)$ satisfies all the conditions of LMVT in [0, 3]

So, $f(x)$ must be continuous in [0, 3] and derivable in (0, 3)

Now, $f(0)=f(0^+) \Rightarrow 5=\alpha$

$$f'(x)=\begin{cases} 2x+5, & 0<x<2 \\ \beta, & 2<x<3 \end{cases}$$

Also, $f(2^-)=f(2)=f(2^+) \Rightarrow 19=2\beta+\gamma$ …(1)

And $f'(2^-)=f'(2^+) \Rightarrow 9=\beta \Rightarrow \gamma=1$ (using (1))

So, $f(x)=\begin{cases} 5, & x=0 \\ x^2+5x+5, & 0<x<2 \\ 9x+1, & 2\le x\le 3 \end{cases}$

$\therefore$ By LMVT, there exist some $c \in (0, 3)$ such that

$$f'(c)=\frac{f(3)-f(0)}{3-0}$$

$$\Rightarrow \quad 2c+5=\frac{28-5}{3} \Rightarrow 2c+5=\frac{23}{3} \quad \therefore c=\frac{4}{3}$$

Hence, $(\alpha+2\beta+3\gamma+6c)=5+2(9)+3(1)+6\left(\frac{4}{3}\right)=34$

Here, graph is incomplete and unnecessary

Hence, the correct answer is (34).

10. $\frac{dy}{dx}=\left(ke^{kx}\right)_{(0,1)}=k$

Equation of tangent is $y-1=kx \Rightarrow ka=-1$

$$\because \quad a\in[-2,-1] \Rightarrow -2\le\frac{-1}{k}\le -1 \quad \therefore \quad \frac{1}{2}\le k\le 1$$

$$\therefore \quad \frac{\alpha}{8}=\frac{1}{2} \text{ or } \alpha=4$$

Hence, the correct answer is (4).

11. Let tangent is at $P(h, k)$

$$2xy+x^2\frac{dy}{dx}=0 \text{ is } \frac{dy}{dx}=\frac{-2y}{x}=\frac{-2k}{h}$$

Equation of tangent $y-K=\frac{-2K}{h}(x-h)$

x–intercept $\Rightarrow$ $a=\frac{3h}{2}$, y–intercept $\Rightarrow$ $3k=b$

$$\frac{a^2b}{c^3}=\frac{9h^2.3k}{4c^3}=\frac{27.h^2k}{4c^3}=\frac{27}{4} \qquad \left(\text{as } h^2K=C^3\right)$$

$$[\mu]=\left[\frac{27}{4}\right]=6$$

Hence, the correct answer is (6).

12. Let $y = f(x)$, $f'(-2) = 0$, $f(-2) = 0$

$$f'(0) = 3,\ f'(x) = 3ax^2 + 2bx + c$$

$$c = 3$$

$$12a - 4b + 3 = 0$$

$$-8a + 4b - 2c + 5 = 0$$

Solving $a = -\frac{1}{2}, b = -\frac{3}{4}$ and $c = 3$

$\Rightarrow\ 4b - 2a + c = 1$

Hence, the correct answer is (1).

13. $f(x) = \sqrt{8x - x^2} - \sqrt{14x - x^2 - 48}$

We have $f(x) = \sqrt{16-(x-4)^2} - \sqrt{1-(x-7)^2}$

Now, consider $y = \sqrt{16-(x-4)^2}$

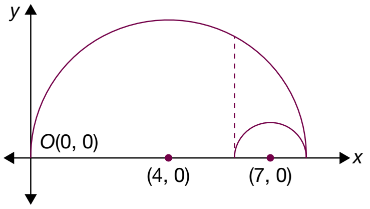
$\Rightarrow\ (x-4)^2 + y^2 = 16, y > 0$

Thus, it is a semi circle with centre (4, 0) and radius = 4

Similarly, $y = \sqrt{1-(x-7)^2}$

$\Rightarrow\ (x-7)^2 + y^2 = 1, y > 0$

Thus, it is a semi circle with centre (7, 0) and radius = 1



Now, on combining the 2 figures, we have

$f(x)_{max.}$ = maximum vertical distance between the curves which occurs when $x = 6$.

Hence, $f(x)_{max} = \sqrt{16-(1-4)^2} - 0 = \sqrt{7}$.

Hence, the correct answer is (8).

14. $g(x) = (f'(x))^2 + f''(x)f(x) = \frac{d}{dx}(f(x)f'(x))$

Let $h(x) = f(x)f'(x)$

Then, $f(x) = 0$ has four roots namely a, α, β, e

Where $b < \alpha < c$ and $c < \beta < d$.

And, $f'(x) = 0$ at three points k_1, k_2, k_3 where $a < k_1 < \alpha, \alpha < k_2 < \beta$, $\beta < k_3 < e$

[$\because$ Between any two roots of a polynomial function $f(x) = 0$ there lies atleast one root of $f'(x) = 0$]

$\therefore$ There are atleast 7 roots of $f(x) \cdot f'(x) = 0$

$\Rightarrow$ There are atleast 6 roots of $\frac{d}{dx}(f(x)f'(x)) = 0$ i.e. of $g(x) = 0$

Hence, the correct answer is (6).

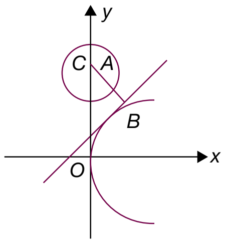
15. $A\left(x_1, 12 + \sqrt{4 - x_1^2}\right)$ $B\left(x_2, \sqrt{16x_2}\right)$

$$y_1^2 + x_1^2 - 24y_1 + 144 = 4$$

$$y_2^2 = 16x_2$$

$$(y_1 - 12)^2 + x_1^2 = 4$$

$$y_2^2 = 4 \times 4x$$



$B(4t^2, 8t)$

$$BC^2 = 16t^4 + (8t - 12)^2 = f(t)$$

$$f'(t) = \left(64t^3 + 16t - 12\right);\ f'(t) = 0 \Rightarrow t = 1$$

$B(4, 8)$; $BC^2 = 16 + 16 = 32$; $BC = 4\sqrt{2}$

$$AB = 4\sqrt{2} - 2 \text{ (least distance)}$$

$$AB = 32 + 4 - 16\sqrt{2}$$

$$= 36 - 16\sqrt{2} \Rightarrow k = 2$$

Hence, the correct answer is (2).

16. $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 6^{\frac{2}{3}}$

$$\left.\begin{aligned} x &= 6\cos^3\theta \\ y &= 6\sin^3\theta \end{aligned}\right]$$

$$\frac{dx}{d\theta} = 6.3\cos^3\theta(-\sin\theta)$$

$$\frac{dy}{d\theta} = 6.3\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = -\frac{6 \cdot 3\sin^2\theta \cdot \cos\theta}{6 \cdot 3\cos^2\theta \cdot \sin\theta}$$

$$\frac{dy}{dx} = -\tan\theta$$

Equation of tangent

$T : (y - 6\sin^3\theta) = -\tan\theta(x - 6\cos^3\theta)$

Equation of normal $N : (y - 6\sin^3\theta) = \cot\theta(x - 6\cos^3\theta)$

$T : x\sin\theta + y\cos\theta = 6\sin\theta\cos\theta)$

$N : x\cos\theta - y\sin\theta = 6\cos 2\theta$

Perpendicular distance from (0, 0) to T is

$$P_1 = \left|\frac{0 + 0 - 6\sin\theta\cos\theta}{\sqrt{\sin^2\theta + \cos^2\theta}}\right|$$

$\Rightarrow\ p_1 = 6\sin\theta\cos\theta \Rightarrow 3\sin 2\theta \Rightarrow 2p_1 = 6\sin 2\theta$ …(1)

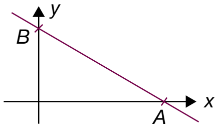
Similarly, $p_2 = 6\cos 2\theta = 6\sin 2\theta$ …(2)

$\therefore$ On squaring and adding, (1) and (2), we get

$\Rightarrow\ 4{p_1}^2 + {p_2}^2 = 36$

Hence, the correct answer is (36).

17. $x = at^3, y = at^4$



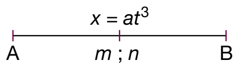
$\therefore\ \frac{dx}{dt} = 3at^2;\ \frac{dy}{dt} = 4at^3;$

$$\frac{dy}{dx} = \frac{4at^3}{3at^2} = \frac{4}{3}t$$

Tangent $y - at^4 = \frac{4}{3}t(x - at^3)$

$$y - at^4 = \frac{4t}{3}x - \frac{4}{3}at^4;\ y = \frac{4t}{3}x - \frac{a}{3}t^4$$

$$A\left(\frac{a}{4}t^3, 0\right)$$



$$B\left(0, -\frac{a}{3}t^4\right)$$

$$\frac{m \cdot 0 - n\left(\frac{a}{4}t^3\right)}{m - n} = at^3 \Rightarrow \frac{-n}{4(m-n)} = 1$$

$\Rightarrow\ -n = 4m - 4n$

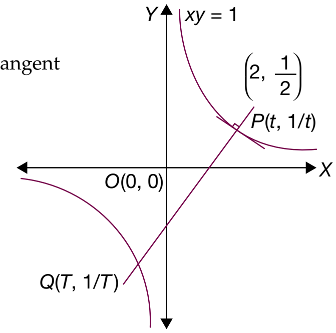
$\Rightarrow\ 3n = 4m$

$m : n = 3 : 4$

$|n + m| = 7$

Hence, the correct answer is (7).

18. We have $y = \dfrac{1}{x}$



$$\frac{dy}{dx} = \frac{-1}{x^2} = \text{slope of the tangent}$$

Slope of normal at $P = x^2 = t^2$

$$\therefore \quad t^2 = \frac{\frac{1}{t} - \frac{1}{T}}{t - T} = \frac{T - t}{tT} \cdot \frac{1}{t - T}$$

$$= \frac{-1}{tT}$$

$$T = \frac{-1}{t^3}.$$

Hence, slope the curve at $Q = \dfrac{-1}{x^2} = \dfrac{-1}{T^2} = -\left(\dfrac{1}{\left(\frac{-1}{t^3}\right)^2}\right) = t^6$

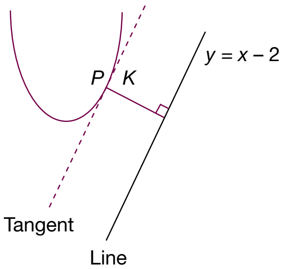
Let $t = 2$

Slope of the curve at $Q = -2^6 = -64 = m$ (Given)

Hence, $|m| = |-64| = 64$

Hence, the correct answer is (64).

19. $P(t_1, t^2 + 3t + 2)$



$$\frac{dy}{dx} = 2x + 3\Big|_t = 2t + 3$$

For shortest distance tangent is parallel to given line

$m_T = m_L \Rightarrow 2t + 3 = 1 \Rightarrow t = -1$

$P(-1, 1 - 3 + 2)$

$x - y - 2 = 0$

$P(-1, 0)$

min^n distance $= K = \dfrac{|-1 - 0 - 2|}{\sqrt{2}} = \dfrac{3}{\sqrt{2}}$

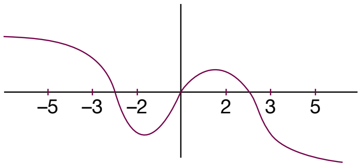
$2k^2 = 9$

Hence, the correct answer is (9).

20. $f(P) = (-1)^P P$

$f(x) \rightarrow$ odd, continuous

$P \rightarrow$ Prime no.



$\therefore \quad P(0) = 0$

$f(2) = 2 \qquad f(-2) = -2$

$f(3) = -3 \qquad f(-3) = 3$

$f(5) = -5 \qquad f(-5) = 5$

$f(7) = -7 \qquad f(-7) = 7$

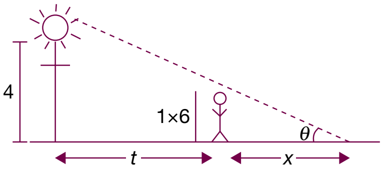
From IMVT $f(x) = 0$ for atleast 3 real roots.

Hence, the correct answer is (3).

21. $\dfrac{1 \cdot 6}{x} = \dfrac{4}{t + x} \Rightarrow 0 \cdot 4t + 0 \cdot 4x = x$

$\Rightarrow \quad 0 \cdot 4t = 0 \cdot 6x \Rightarrow 2t = 3x$

$\Rightarrow \quad \dfrac{2dt}{dT} = \dfrac{3dx}{dT}$



$\Rightarrow \quad 2(30) = \dfrac{3dx}{dT}$

$\Rightarrow \quad \dfrac{dx}{dT} = 20 = k$

$\dfrac{k}{4} = 5$

Hence, the correct answer is (5).

22. $f(x) = x^3 - 6x^2 + ax + b$

$x \in [1, 3]$

$f(1) = f(3) \Rightarrow 1 - 6 + a + b = 27 - 54 + 3a + b$

$a - 5 = 3a - 27$

$a = 11; a - 4 = 7$

Hence, the correct answer is (7).

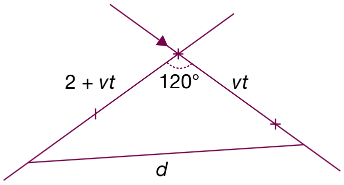
23. $ay + x^2 = 7$ and $x^3 = y$ intersect orthogonally.

Therefore, $\left(\dfrac{dy}{dx}\right) = -\dfrac{2x}{a}$ and $\left(\dfrac{dy}{dx}\right) = 3x^2$

$$\Rightarrow \left[\left(-\frac{2x}{a}\right)\left(3x^2\right)\right]_{(1,1)} = -1 \Rightarrow -\frac{2}{a} \times 3 = -1 \Rightarrow a = 6$$

Hence, the correct answer is (6).

24. Let v is speed



$$\frac{-1}{2} = \cos 120^\circ$$

$$= \frac{v^2t^2 + (2 + vt)^2 - d^2}{2(vt)(2 + vt)}$$

$-2vt - (vt)^2$

$= (vt)^2 + 4 + (vt)^2 + 4vt - d^2$

$d^2 = 3(vt)^2 + 6vt + 4$

$\dfrac{d(d^2)}{d(vt)} = 6(vt) + 6 = 0$

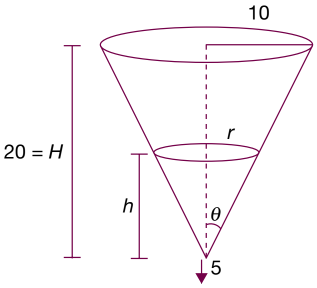
$vt = -1$

$d^2 = 3 - 6 + 4 = 1$

$d = \sqrt{1} = 1 \text{ min}^\text{m}$

Hence, the correct answer is (1).

25. $\tan\theta = \dfrac{r}{h} = \dfrac{10}{20}$



$r = \dfrac{h}{2}$

$V = \dfrac{1}{3}\pi r^2 h = \dfrac{\pi h^3}{12}$

$\dfrac{dV}{dt} = \dfrac{\pi}{4} h^2 \dfrac{dh}{dt} = 5$

$\dfrac{\pi}{4}(15)^2 \dfrac{dh}{dt} = 5$

$\Rightarrow \quad \dfrac{dh}{dt} = \dfrac{4 \times 5}{\pi \times 15 \times 15} = \dfrac{4}{45\pi}$ we get $\lambda = 4$

Hence, the correct answer is (4).

26. $y = \cos x$

$y' = -\sin x$

Hence, the correct answer is (9).

27. The given equation can be written as $\dfrac{x^2}{\alpha^2} + \dfrac{y^2}{4} = 1$

Any point on the curve is $P(a \cos\theta, 2 \sin\theta)$

$\therefore \quad PQ = \sqrt{\alpha^2 \cos^2\theta + 4(1 + \sin\theta)^2}$

Let $S(\theta) = \alpha^2 \cos^2\theta + 4(1 + \sin\theta)^2 = (4 + \alpha^2) + 8\sin\theta + (4 - \alpha^2)\sin^2\theta$

Now, $\dfrac{dS}{d\theta} = 2\cos\theta\,[4 + (4 - \alpha^2)\sin\theta]$

$$\Rightarrow \frac{dS}{d\theta} = 0 \text{ gives } \theta = \frac{\pi}{2}, \frac{3\pi}{2} \text{ or } \sin\theta = \frac{4}{\alpha^2 - 4}.$$

But $0 < a^2 < 8$ and $|\sin\theta| \le 1$, so $\sin\theta = \frac{4}{\alpha^2 - 4}$ is not possible.

Also, at $\theta = \frac{3\pi}{2}$, P becomes (0, –2) which is same as point Q. So, $PQ = 0$.

Also, $\left.\frac{d^2S}{d\theta^2}\right]_{\theta=\frac{\pi}{2}} = -8\sin\theta + 2(4-\alpha^2)\cos 2\theta$

$= -8 + 2(4-\alpha^2)(-1)$

$= -2(8-\alpha^2) < 0$ (As $0 < \alpha^2 < 8$)

Hence, $\theta = \frac{\pi}{2}$ corresponds to the maximum values of S.

So, P is (0, 2)
Clearly, reflection of $P(0, 2)$ in the x-axis is $R(0, -2)$.
So, least distance of $R(0, -2)$ from the line $3x - 4y + 7 = 0$, is

$$\frac{|3(0) - 4(-2) + 7|}{\sqrt{(3)^2 + (-4)^2}} = \frac{15}{5} = 3$$

Hence, the correct answer is (3).

28. We have $f(0) = 2$

Now, $y - f(a) = f'(a)[x - a]$

For x intercept $y = 0$, so

$$x = a - \frac{f(a)}{f'(a)} = a - 2 \Rightarrow \frac{f(a)}{f'(a)} = 2 \Rightarrow \frac{f'(a)}{f(a)} = \frac{1}{2}$$

$\therefore$ On integrating both sides w.r.t. a, we get

$\ln f(a) = \frac{a}{2} + C$

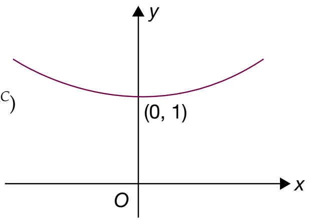
$\Rightarrow f(a) = e^{a/2} e^{c}$

$f(a) = C_1 e^{a/2}$, (where $C_1 = e^C$)

$f(x) = C_1 e^{x/2}$

$f(0) = C \Rightarrow C_1 = 2$

$\therefore f(x) = 2e^{x/2}$



Hence, $k = 2,\ p = \frac{1}{2} \Rightarrow \frac{k}{p} = 4$

Hence, the correct answer is (4).

29. Curve $y = e^{x^3 - 3x^2 - 9x + 1}$

$$\frac{dy}{dx} = e^{x^3 - 3x^2 - 9x + 1}(3x^2 - 6x - 9) = 0$$

$x^2 - 2x - 3 = 0 \Rightarrow (x - 3)(x + 1) = 0 \Rightarrow x = -1, 3$

2 points

Hence, the correct answer is (2).

30. $\frac{dy}{dx} = 3ax^2 - 2ax + 1 \ge 0\ \forall\ x \in R$

$\theta \in [0, 90°)$

$a > 0$ and $D \le 0 \Rightarrow 4a^2 - 4(3a) \le 0$

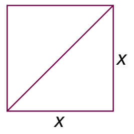
$\Rightarrow a^2 - 3a \le 0 \Rightarrow a(a - 3) \le 0$

We get $a \in (0, 3]$

$a = 1, 2, 3$

Hence, the correct answer is (3).

31. $\frac{d(\sqrt{2}x)}{dt} = 0.2$ cm/sec



Area $= x^2$

$$\frac{dA}{dt} = \frac{2xdx}{dt} = 2\left(\frac{30}{\sqrt{2}}\right)\left(\frac{0\cdot 2}{\sqrt{2}}\right) = 6$$

Hence, the correct answer is (6).

32. We have $x^3 - 3xy^2 + 2 = 0$ …(1)

And $3x^2y - y^3 - 2 = 0$ …(2)

Differentiating (1) and (2) w.r.t. x, we get

$$\left(\frac{dy}{dx}\right)_{c_1} = \frac{x^2 - y^2}{2xy} \text{ and } \left(\frac{dy}{dx}\right)_{c_2} = \frac{-2xy}{x^2 - y^2}$$

Therefore, $\left(\frac{dy}{dx}\right)_{c_1} \times \left(\frac{dy}{dx}\right)_{c_2} = -1$

Hence, the two curves intersect at right angles
Hence, the correct answer is (1).

33. $\because$ Rolle's theorem is applicable

$\therefore$ $g(x)$ is continuous in [0, 1] and differentiable in (0, 1)

$g(0) = g(0^+)$

$$\Rightarrow 2 = \lim_{x\to 0^+} \frac{e^{px} - qx - 1}{x} \Rightarrow 2 = \lim_{x\to 0^+}\left(\frac{p(e^{px} - 1)}{xp} - q\right)$$

$\Rightarrow p - q = 2$ …(1)

And $f(0) = f(1)$ gives $e^p - q = 3$ …(2)

$\Rightarrow p = 0, q = -2$

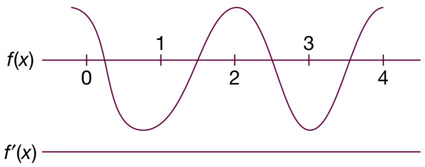
Hence, $p^2 + q^2 = 4$

Hence, the correct answer is (4).

34. $f(0) = f(2) = f(4) = 3; f(1) = f(3) = f(5) = -2$

$f(x) = 0$ has atleast 4 zeroes in [0,4]

$$g(x) = x\left(\frac{d}{dx}\left(f(x)f'(x)\right) + f(x)f'(x)\right) = \frac{d}{dx}\underbrace{(x(f(x)f'(x)))}_{8\ zeroes}$$



From Rolle's theorem $g(x) = 0$ has atleast 7 zeroes
Hence, the correct answer is (7).

35. $f(e^a) = 0$

$$\lim_{x\to e^a} \frac{\ln(1 + 7f(x))}{7f(x)} \cdot \frac{7}{3} - \frac{\sin f(x)}{3f(x)} = (1)\frac{7}{3} - \frac{1}{3}(1) = 2$$

Hence, the correct answer is (2).

36. $x^2 = 9k(9 - y)$

$2x = 0 - 9k\frac{dy}{dx}$

$m_1 = \frac{dy}{dx} = -\frac{2x}{9k}$

$x^2 = k(y + 1)$

$2x = k\frac{dy}{dx}$

$m_2 = \frac{dy}{dx} = \frac{2x}{k}$

$$m_1 m_2 = -1 \Rightarrow -\frac{4x^2}{9k^2} = -1 \Rightarrow 4x_1^2 = 9k^2$$

$x_1^2 = ky_1 + k = 81k - 9ky_1$

$10ky_1 = 80k$

$y_1 = 8$

$x_1^2 = 81k - 9ky_1$

$x_1^2 = 81k - 72k = 9k$

$4x_1^2 = 9k^2 \Rightarrow (9k)4 = 9k^2$ we get $k = 4$

Hence, the correct answer is (4).

37. $P(a, 4a - a^2)$

$A(-1, 4)$

$PA^2 = (a+1)^2 + (4a - a^2 - 4)^2 = a^2 + 1 + 2a + 16a^2 + a^4 + 16 - 8a^3 + 8a^2 - 32a$

$T = a^4 + 25a^2 - 8a^3 - 30a + 17$

$\frac{dT}{da} = 4a^3 + 50a - 24a^2 - 30 = 0$

$2a^3 + 25a - 12a^2 - 15 = 0 \Rightarrow \text{a} = 1$

$2a^3 - 2a^2 - 10a^2 + 10a + 15a - 15 = 0$

$(a-1)(2a^2 - 10a + 15) = 0$

$a = 1$; quadratic $D < OC$ imaginary roots

$b = 4 - 1 = 3$

$a + b = 4$

Hence, the correct answer is (4).

38. $f(x) = x(x^2 + mx + n) + 2$

$f(1) = f(2)$ gives

$3m + n + 7 = 0 \qquad \ldots(1)$

$f(1) = m + n + 3$

$f(2) = 4m + 2n + 10$

$f(x) = x^3 + mx^2 + nx + 2$

$f'(x) = 3x^3 + 2mx + n$

And $f'(x)\left(\text{at } x = \frac{4}{3}\right) = 0$ gives

$8m + 3n + 16 = 0 \qquad \ldots(2)$

Solving equation (1) and (2)

$\Rightarrow \; m = -5, n = 8$

Hence, the correct answer is (3).

39. We have $f(3 + x) = f(3 - x)$

On replacing x by $x - 3$, we get

$f(x) = f(6 - x) \qquad \ldots(1)$

On differentiating (1) w.r.t. x, we get

$f'(x) = -f'(6 - x) \qquad \ldots(2)$

Putting $x = \frac{3}{2}, 2, 3, 5$ in (2), we get

$f'\left(\frac{3}{2}\right) = 0 = f'(2) = f'(3) = f'(5) = f'\left(\frac{9}{2}\right) = f'(4)$

$= f'(1)$

$\therefore \; f'(x) = 0$ has minimum 7 roots in $[1, 6]$

Now, consider a function $y = f'(x)$

As $f'(x)$ satisfy Rolle's theorem in intervals $\left[1, \frac{3}{2}\right], \left[\frac{3}{2}, 3\right]$, $[2, 3]$, $[3, 4]$, $\left[4, \frac{9}{2}\right]$ and $\left[\frac{9}{2}, 5\right]$ respectively.

So, by Rolle's theorem, the equation $f''(x) = 0$ has minimum 6 roots.

Now, $g(x) = (f''(x)^2 + f'(x)f'''(x) = \frac{d}{dx}(f'(x)f'(x))$

$= h'(x)$, where $h(x) = f'(x)f''(x)$

Clearly, $h(x) = 0$ has minimum 13 roots in $[1, 6]$

Hence, again by Rolle's theorem, $g(x) = h'(x)$ has minimum 12 zeroes in $[1, 6]$.

Hence, the correct answer is (12).

40. $f(x) = x^3 - 3x^2 + 3x$

$f'(x) = 3x^2_1 - 6x_1 + 3$

$\therefore \; \frac{y_1 - 1}{x_1 - 0} = 3x^2_1 - 6x_1 + 3$

$y_1 = 3x^3_1 - 6x^2_1 + 3x_1 + 1$

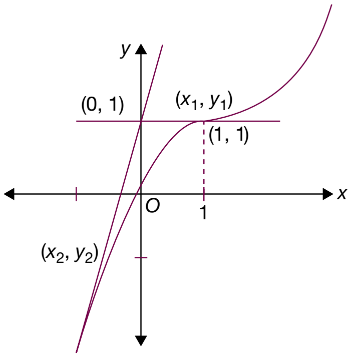
$x^3_1 - 3x^2_1 + 3x_1$

$= 3x^3_1 - 6x^2_1 + 3x_1 + 1$

$2x^3_1 - 3x^2_1 + 1 = 0$

$(x_1 - 1)^2 \cdot (2x_1 + 1) = 0$

$x_1 = 1$ or $x_1 = \frac{-1}{2}$



Hence, $m_1 = 0$ or $m_2 = \frac{27}{4}$

Angle $= \tan^{-1}\left(\frac{27}{4}\right) \Rightarrow m + n = 27 + 4 = 31.$

Point of contact is (1, 1) and $\left(-\frac{1}{2}, -\frac{19}{8}\right)$

Hence, the correct answer is (31).

41. Let $g(x) = f(x) - (ax^2 + bx + c)$.

Since, $g(x) = 0$ has 10 distinct real roots.

$\Rightarrow \; g'(x) = 0$ has minimum 9 distinct real roots,

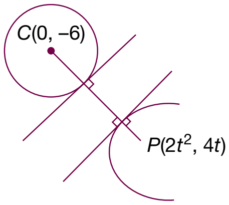
$\Rightarrow \; g''(x) = 0$ has minimum 8 distinct real roots,

$\Rightarrow \; g'''(x) = 0$ has minimum 7 distinct real roots,

$\Rightarrow \; f'''(x) = 0$ has atleast 7 distinct real roots.

Hence, the correct answer is (7).

42. Center C of the circle is $(0, -6)$ and radius $= 1$.



Also, any point on $y^2 = 8x$ is $(2t^2, 4t)$. The distance is minimum, When CP is minimum,

i.e., CP^2 is minimum.

Let $E = (2t^2 - 0)^2 + (4t + 6)^2 = 4t^4 + 16t^2 + 48t + 36$

Now, $\frac{dE}{dt} = 16t^3 + 32t + 48 = 0 \Rightarrow t = -1$ and other roots are imaginary.

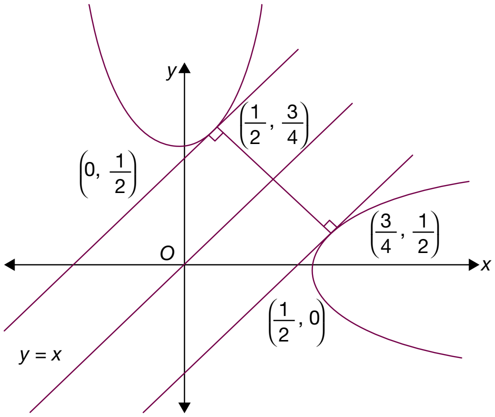
Also, $\left.\frac{d^2E}{dt^2}\right]_{t=-1} = 16(3t^2 + 2) > 0$

$\Rightarrow \; E_{\text{min}}(t = -1) = 4 + 16 + 48(-1) + 36 = 56 - 48 = 8.$

$\therefore \; d = \sqrt{8} - 1 = (2\sqrt{2} - 1) \Rightarrow (d + 1)^2 = (2\sqrt{2})^2 = 8.$

Hence, the correct answer is (8).

43.


y
$\left(\frac{1}{2}, \frac{3}{4}\right)$
$\left(0, \frac{1}{2}\right)$
$\left(\frac{3}{4}, \frac{1}{2}\right)$
O
x
$\left(\frac{1}{2}, 0\right)$
y = x

$\because$ Given parabola's are mirror image of each other about line $y = x$ so corresponding tangent is parallel to $y = x$.

Let point $P(x_1, y_1)$ on $2x^2 - 2y + 1 = 0 \Rightarrow \frac{dy}{dx} = 2x$

Slope at point $P(x_1, y_1) = 2x_1 \Rightarrow 2x_1 = 1$

$$x_1 = \frac{1}{2} \Rightarrow P\left(\frac{1}{2}, \frac{3}{4}\right) \text{ and } Q\left(\frac{3}{4}, \frac{1}{2}\right)$$

Lies on $2y^2 - 2x + 1 = 0$

Clearly, $d = \sqrt{\left(\frac{3}{4} - \frac{1}{2}\right)^2 + \left(\frac{1}{2} - \frac{3}{4}\right)^2}$

$$= \sqrt{\frac{1}{16} + \frac{1}{16}} = \sqrt{\frac{1}{8}}$$

$$= \frac{1}{2\sqrt{2}}$$

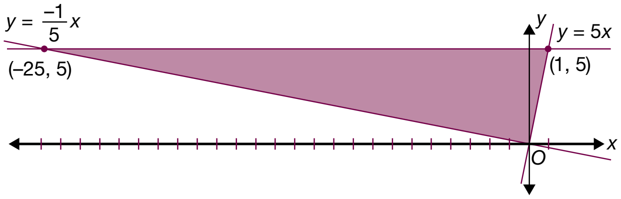
So, $|\sin\theta| = 2\sqrt{2}\,d$

$\Rightarrow |\sin\theta| = 1$

$\therefore \theta = \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}.$

Hence, the correct answer is (3).

44. Given, $x + 5y - y^5 = 0 \Rightarrow \frac{dy}{dx} = \left(\frac{1}{5y^4 - 5}\right)$



Now, when $x = 0$, $y = 0$ and $y = \pm(5)^{\frac{1}{4}}$ (Rejected)

So, slope of tangent at (0, 0) = $\frac{-1}{5}$

$\Rightarrow$ Equation of tangent is $y = \frac{-x}{5}$

Now, equation of normal is $y = 5x$

Also, given line is $y = 5$

Clearly, area of triangle $= \frac{1}{2}(26)(5) = 13(5) = 65$

Hence, the correct answer is (65).

45. $y = \frac{(x-a)^2}{4}$ and $y = e^x$

$$\frac{dy}{dx} = \frac{x-a}{2} \text{ and } \frac{dy}{dx} = e^x$$

$\therefore \frac{x-a}{2} = e^x \Rightarrow$ (Put $x = 0$, we get $-a = 2 \Rightarrow a = -2$) ...(1)

Also, $\frac{(x-a)^2}{4} = e^x$ ...(2)

$\therefore \frac{4e^{2x}}{4} = e^x$ [Using (1)]

$\Rightarrow e^x(e^x - 1) = 0,$

Hence, $e^x = 1 \Rightarrow x = 0$

If $x = 0$, then $y = 1$

$\therefore 1 = \frac{a^2}{4} \Rightarrow a = -2$ or $a = 2$ (rejected)

As, for $a = 2$ curves do not intersect.

Hence, sum of square of all possible values of $a = (-2)^2 = 4$

Hence, the correct answer is (4).

46. Applying LMVT for $f(x)$ on [1, 3] we get

$$f'(c) = \frac{f(3) - f(1)}{3 - 1} \text{ for } c \in (1, 3)$$

$$f'(c) = \frac{9-5}{3-1}$$

$\Rightarrow f'(c) = 2$ $\because f'(x) \geq 2\ \forall\ x \in (1, 3)$

$\Rightarrow f'(x) = 2$

$\Rightarrow f(x) = 2x + c$

$\because f(1) = 5 \Rightarrow c = 3$

$\therefore f(x) = 2x + 3$

Now, $g(x) = \int_0^x f(t)dt = x^2 + 3x$ $\because g'(x) = f(x) > 0\ \forall\ x \in [1, 3]$

$\therefore$ Greatest value $= g(3) = 18$ and least value $= g(1) = 4$

= sum of greatest and least value = 22

Hence, the correct answer is (22).

ARCHIVE: JEE MAIN

1. Slope of tangent at $P(t, t^3) = \left.\frac{dy}{dx}\right]_{(t,t^3)}$

$$= (3x^2)_{x=t} = 3t^2$$

So, equation tangent at $P(t, t^3)$: $y - t^3 = 3t^2(x - t)$

For point of intersection with $y = x^3$

$$x^3 - t^3 = 3t^2x - 3t^3$$

$\Rightarrow (x - t)(x^2 + xt + t^2) = 3t^2(x - t)$

For $x \neq t$

$$x^2 + xt + t^2 = 3t^2$$

$\Rightarrow x^2 + xt - 2t^2 = 0 \Rightarrow (x - t)(x + 2t) = 0$

So, for $Q : x = -2t$, $Q(-2t, -8t^3)$

Ordinate of required point $= \frac{2t^3 - 8t^3}{2+1} = -2t^3$

Hence, the correct answer is (C).

2. $a + b + c = 2$ ...(1)

And $\left.\frac{dy}{dx}\right|_{(0,0)} = 1$

$$2ax + b\big|_{(0,0)} = 1;\ b = 1$$

Curve passes through origin

So, $c = 0$ and $a = 1$

Hence, the correct answer is (A).

3. $f(1) = f(2)$

$\Rightarrow 1 - a + b - 4 = 8 - 4a + 2b - 4$

$\Rightarrow 3a - b = 7$ ...(1)

Also, $f'\left(\frac{4}{3}\right) = 0$ (given)

$\Rightarrow \left(3x^2-2ax+b\right)_{x=\frac{4}{3}}=0\Rightarrow \frac{16}{3}-\frac{8a}{3}+b=0$

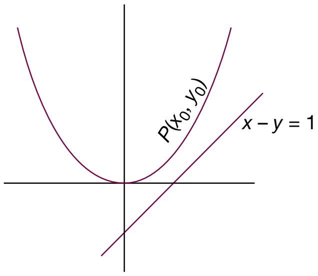
$\Rightarrow 8a-3b-16=0$ …(2)

Solving (1) and (2)

$a=5, b=8$

Hence, the correct answer is (A).

4.



Shortest distance between curves is always along common normal.

$\left.\frac{dy}{dx}\right|_P=$ Slope of line $=1$

$\Rightarrow x_0=1$

$\therefore y_0=\frac{1}{2}$

$\Rightarrow P=\left(1,\frac{1}{2}\right)$

$\therefore$ Shortest distance $=\left|\frac{1-\frac{1}{2}-1}{\sqrt{1^2+1^2}}\right|=\frac{1}{2\sqrt{2}}$

Hence, the correct answer is (B).

5. $x=y^4$ and $xy=k$; for intersection $y^5=k$ …(1)

Also, $x=y^4\Rightarrow 1=4y^3\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{1}{4y^3}$

For $xy=k\Rightarrow x=\frac{k}{y}$

$\Rightarrow 1=-\frac{k}{y^2}\cdot\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{-y^2}{k}$

$\because$ Curve cut orthogonally

$\Rightarrow \frac{1}{4y^3}\times\left(\frac{-y^2}{k}\right)=-1\Rightarrow y=\frac{1}{4k}$

$\therefore$ From (1) $y^5=k$

$\Rightarrow \frac{1}{(4k)^5}=k\Rightarrow 4=(4k)^6$

Hence, the correct answer is (4).

6. $y(x)=\int_0^x(2t^2-15t+10)\,dt$

$y'(x)]_{x=a}=[2x^2-15x+10]_a=2a^2-15a+10$

Slope of normal $=-\frac{1}{3}\Rightarrow 2a^2-15a+10=3\Rightarrow a=7$

And $a=\frac{1}{2}$ (rejected)

$b=y(7)=\int_0^7(2t^2-15t+10)\,dt=\left[\frac{2t^3}{3}-\frac{15t^2}{2}+10t\right]_0^7$

$\Rightarrow 6b=4\times7^3-45\times49+60\times7=-413$

$|a+6b|=406$

Hence, the correct answer is (406).

7. $f(2)=8, f'(2)=5, f'(x)\ge1, f''(x)\ge4, \forall\, x\in(1,6)$

$f''(x)=\frac{f'(5)-f'(2)}{5-2}\ge4\Rightarrow f'(5)\ge17$ …(1)

$f'(x)=\frac{f(5)-f(2)}{5-2}\ge1\Rightarrow f(5)\ge11$ …(2)

Now, $f'(5)+f(5)\ge28$

Hence, the correct answer is (C).

8. $x^4e^y+2\sqrt{y+1}=3$

Differentiate

$x^4e^y\,y'+e^y\,4x^3+\frac{2y'}{2\sqrt{y+1}}=0$

At $P(1,0)$

$y'_p+4+y'_p=0\Rightarrow y'_p=-2$

Tangent at $P(1,0)$ is $y-0=-2(x-1)\Rightarrow 2x+y=2$

$(-2,6)$ lies on it.

Hence, the correct answer is (B).

9. $y=x^2-3x+2$

At x-axis $y=0=x^2-3x+2$

$x=1,2$

$A(1,0)$

$B(2,0)$

$\frac{dy}{dx}=2x-3$

$\left(\frac{dy}{dx}\right)_{x=1}=-1$ so $A(1,0)$ lies on line $x+y=a$, we get $a=1$

$\left(\frac{dy}{dx}\right)_{x=2}=1$ so $B(2,0)$ lies on line $x-y=b$, we get $b=2$,

Now, $\frac{a}{b}=0.50$

Hence, the correct answer is (0.50).

10. Apply Rolles theorem on $y=f(x)$ in $x\in[0,1]$

$f(0)=f(1)=0$

$\Rightarrow f'(\alpha)=0$ where $\alpha\in(0,1)$

Now, apply Rolles theorem on $y=f'(x)$ in $x\in[0,\alpha]$

$f'(0)=f'(\alpha)=0$ and $f'(x)$ is continuous and differentiable

$\Rightarrow f''(\beta)=0$ for some, $\beta\in(0,\alpha)\subset(0,1)$

$\Rightarrow f''(x)=0$ for some, $x\in(0,1)$

Hence, the correct answer is (A).

11. $f(x)=x\log_ex$ then $f'(x)=1+\log_ex$

$f'(x)|_{(c,f(c))}=\frac{e-0}{e-1}=1+\log_ec$

$\log_ec=\frac{e-(e-1)}{e-1}=\frac{1}{e-1}\Rightarrow c=e^{\frac{1}{e-1}}$

Hence, the correct answer is (C).

12. $\frac{f(t_2)-f(t_1)}{t_2-t_1}=2at+b\Rightarrow\frac{a\left(t_2^2-t_1^2\right)+b(t_2-t_1)}{t_2-t_1}=2at+b$

$\Rightarrow a(t_2+t_1)+b=2at+b\Rightarrow t=\frac{t_1+t_2}{2}$

Hence, the correct answer is (D).

13. $y=10-x^2$ or $y=2+x^2$

$10-x^2=2+x^2$ or $8=2x^2$ or $x^2=4$ or $x=2,-2$

Points are $(2,6),(-2,6)$

$\left(\frac{dy}{dx}\right)_{(2,6)}=2x=4$ or $\left(\frac{dy}{dx}\right)_{(-2,6)}=-2x=4$

$$\tan\theta = \left|\frac{4-(-4)}{1+4(-4)}\right| = \left|\frac{8}{1-16}\right| = \frac{8}{15}$$

Hence, the correct answer is (4).

14. $(t, \sqrt{t})\left(\frac{3}{2}, 0\right)$

$$(\text{Distance})^2 = \left(\frac{3}{2}, t\right)^2 + (\sqrt{t})^2 = (t\frac{5}{4} - 1)^2 + \frac{5}{4}$$

Square of minimum (distance) = at (1, 1)

Minimum distance $= \frac{\sqrt{5}}{2}$

Hence, the correct answer is (4).

15. $\frac{dy}{dx} = e^{x^2} + xe^{x^2} . 2x$

At $x = 1$, slope of tangent $m = 3e$

$y - e = 3e(x - 1)$

$\Rightarrow$ $y = 3ex - 2e$

$\left(\frac{4}{3}, 2e\right)$ lies on it

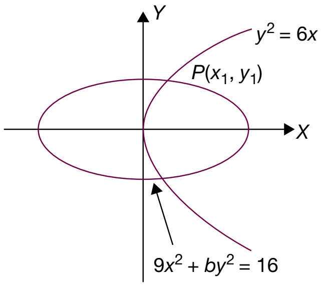
Hence, the correct answer is (1).

16. Let curve intersect each other at point $P(x_1, y_1)$

Since, point of intersection is on both the curves, then

$$y_1^2 = 6x_1 \qquad \ldots(1)$$

And, $9x_1^2 + by_1^2 = 16$ …(2)



Now, find the slope of tangent to both the curves at the point of intersection $P(x_1, y_1)$

For slope of curves:

Curve (1):

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = m_1 = \frac{3}{y_1}$$

Curve (2):

And $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = m_2 = -\frac{9x_1}{by_1}$

Since, both the curves intersect each other at right angle then,

$$m_1 m_2 = -1 \Rightarrow \frac{27x_1}{by_1^2} = 1 \Rightarrow b = 27\frac{x_1}{y_1^2}$$

$\therefore$ From equation (1), $b = 24 \times \frac{1}{6} = \frac{9}{2}$

Hence, the correct answer is (3).

17. We have $y = \frac{x+6}{(x-2)(x-3)}$

At y-axis, $x = 0 \Rightarrow y = 1$

On differentiating, we get

$$\frac{dy}{dx} = \frac{(x^2 - 5x + 6)(1) - (x+6)(2x-5)}{(x^2 - 5x + 6)^2}$$

$\frac{dy}{dx} = 1$ at point (0, 1)

$\therefore$ Slope of normal = –1

Now, equation of normal is $y - 1 = -1(x - 0)$

$\Rightarrow$ $y - 1 = -x \Rightarrow x + y = 1$

$\therefore$ $\left(\frac{1}{2}, \frac{1}{2}\right)$ satisfy it.

Hence, the correct answer is (3).

18. $y = f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$

At $x = \frac{\pi}{6}$, $y = \frac{\pi}{3}$

$$f(x) = \tan^{-1}\left(\left|\frac{\cos\frac{x}{2} + \sin\frac{x}{2}}{\cos\frac{x}{2} - \sin\frac{x}{2}}\right|\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$$

$\therefore$ $x \in \left(0, \frac{\pi}{2}\right)$

$\therefore$ $f(x) = \frac{\pi}{4} + \frac{x}{2}$

$\therefore$ $f'(x) = \frac{1}{2}$

$\therefore$ Slope of normal = –2

$\therefore$ Equation of normal is $y - \frac{\pi}{3} = -2\left(x - \frac{\pi}{6}\right)$

or $y = -2x + \frac{2\pi}{3}$

Hence, the correct answer is (3).

19. $x^2 + 2xy - 3y^2 = 0$

$\Rightarrow$ $2x + 2xy' + 2y - 6yy' = 0 \Rightarrow y' = \frac{x+y}{3y-x}$

At $x = 1$, $y = 1$ we have $\frac{dy}{dx} = 1$

$\therefore$ Equation of normal at (1, 1) is

$y - 1 = -(x - 1) \Rightarrow x = 1, 3$

Thus, $P(1, 1)$ and $Q(3, -1)$.

So, normal meets curve again at (3, –1) in fourth quadrant.

Alternate:

$x^2 + 2xy - 3y^2 = 0$; $(x - y)(x + 3y) = 0$

$x - y = 0$ or $x + 3y = 0$

Equation of normal at (1, 1) to line $x - y = 0$ is $x + y - 2 = 0$, which intersects the line $x + 3y = 0$ at (3, –1).

Hence, the correct answer is (2).

20. Let $h(x) = f(x) - 2g(x)$

As $h(0) = h(1) = 2$

Hence, using Rolle's theorem

$h'(c) = 0 \Rightarrow f'(c) = 2g'(c)$

Hence, the correct answer is (4).

21. $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 > 0\ \forall$ real x

$\therefore$ $f(x)$ is increasing

$\therefore$ $f(x) = 0$ has only one real root

Hence, the correct answer is (4).

22. $y = \int_0^x |t|\, dt$

CASE 1: If $x > 0$

$$y = \int_0^x t\, dt = \left[\frac{t^2}{2}\right]_0^x = \frac{x^2}{2}$$

$\Rightarrow$ $\frac{dy}{dx} = x = 2$ (given)

$\Rightarrow$ $x = 2$ and $y = 2$

$\therefore$ Equation of tangent is $(y-2)=2(x-2)$ or $y-2x+2=0$

Hence, x intercept = 1.

CASE 2: $x<0$

$$y=\int_0^x -t\,dt=\left[\frac{-t^2}{2}\right]_0^x=\frac{-x^2}{2}$$

$$\therefore \quad \frac{dy}{dx}=-x=2$$

$\therefore$ $x=-2$,

$\therefore$ $y=-2$,

$\therefore$ Equation of tangent is $y+2=2(x+2)$

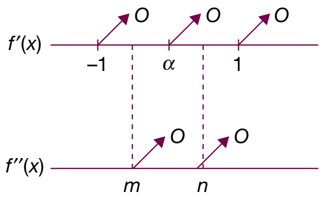
or $2x-y+2=0$

$\therefore$ x intercept $=-1$

Hence, the correct answer is (1).

ARCHIVE: JEE ADVANCED

1.



$f(x)=(x^2-1)^2\,(a_0+a_1x+a_2x^2+a_3x^3)$

$f(1)=0=f(-1)$

By Rolles theorem $f'(\alpha)=0$ for atleast one $\alpha\in(-1,1)$

$f'(x)=2(x^2-1)\,(a_0+a_1x+a_2x^2+a_3x^3)+(x^2-1)^2\,[0+a_1+2a_2x+3a_3x^2]$

$=(x^2-1)\,[2(a_0+a_1x+a_2x^2+a_3x^3+(x^2-1)(a_1+2a_2x+3a_3x^2))]$

It is of degree 4

$f'(x)=0$ for $-1,\alpha,1$

$m_{f'}=3$

Using Rolles theorem $f''(x)=0$ for two values

$m_{f''}=2$

Now, we have $m_{f'}+m_{f''}=5$

Hence, the correct answer is (5).

2. (A) We can always get an interval in which $f(x)$ is an one – one function except when a function is a constant function,

Let $f(x)$ is an constant function then $f'(x)=0$, given condition

$f(0)^2+f'(0)^2=85$

We get $f(0)^2=85$ which is not possible as $|f(x)|\le 2$, it means we can say that $f(x)$ is an one-one function in $x\in(r,s)$ [true]

(B) Using LMVT on $f(x)$ in $x\in[-4,0]$ we get

$$f'(x_0)=\frac{f(0)-f(-4)}{4}$$

$$f'(x_0)|\le\frac{|f(0)|+|f(-4)|}{4}\le 1$$ as we know that $|f(x)|\le 2$ [true]

(C) Let $f(x)=2\sin\dfrac{\sqrt{85}x}{2}$ but $\lim\limits_{x\to\infty} f(x)\ne 1$ [false]

(D) Similarly, $|f'(x_1)|\le 1$ for some $x_1\in(0,4)$

$g(x)=(f(x))^2+(f'(x))^2$ and $g(x_0)\le 5,\ g(x_1)\le 5$

$g(0)=85$ it has a local maximum having value ≥ 85

Say, $x=\alpha$

$g'(\alpha)=0,\ g''(\alpha)\le 0$

$2f(\alpha)f'(\alpha)+2f'(\alpha)f''(\alpha)=0$

$f'(\alpha)(f(\alpha)+f''(\alpha))=0$

As $f'(\alpha)\ne 0$

Hence, (A), (B) and (D) are correct.

3. Using L Hospital's rule $\lim\limits_{t\to x}\cdot\dfrac{f(x)\cos t-f'(t)\sin x}{1}=\sin^2 x$

$\Rightarrow\ f(x)\cos x-f'(x)\sin x=\sin^2 x$

$$\Rightarrow\ \frac{f(x)\cos x-f'(x)\sin x}{\sin^2 x}=1$$

$$\Rightarrow\ \int d\left(\frac{f(x)}{\sin x}\right)=-\int dx\Rightarrow\frac{f(x)}{\sin x}=-x+c$$

$\Rightarrow\ f(x)=-x\sin x+c\sin x$

$$f\left(\frac{\pi}{6}\right)=-\frac{\pi}{12}=-\frac{\pi}{6}\cdot\frac{1}{2}+c\cdot\frac{1}{2}\ \Rightarrow\ c=0$$

$$\Rightarrow\ f(x)=-x\sin x;\ f(\pi/4)=-\frac{\pi}{4\sqrt{2}}$$

(B) Let $g(x)=f(x)-\dfrac{x^4}{6}+x^2=-x\sin x-\dfrac{x^4}{6}+x^2$

$$g(x)=-x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\dots\right)-\frac{x^4}{6}+x^2$$

$$=-\frac{x^6}{5!}\left(1-\frac{x^2}{7!}+\dots\right)<0$$

For all $x\in(0,\pi)$

$$g(x)=f(x)-\frac{x^4}{6}+x^2<0\Rightarrow f(x)<\frac{x^4}{6}-x^2 \text{ for all } x\in(0,\pi)$$

(C) $f(0)=f(\pi)=0$ now from Rolles theorem $f'(x)$ must be zero for atleast one $x\in(0,\pi)$

(D) $f'(x)=-\sin x-x\cos x$

$$f''(x)=-2\cos x+x\sin x;\ f''\left(\frac{\pi}{2}\right)+f\left(\frac{\pi}{2}\right)=0$$

Hence, (B), (C) and (D) are correct.

4. Given curve is $(y-x^5)^2=x(1+x^2)^2$

$$2(y-x^5)\left(\frac{dy}{dx}-5x^4\right)=(1+x^2)^2+2x\,(1+x^2)\,2x$$

At point (1, 3), $x=1$ and $y=3$

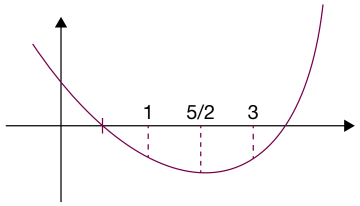
$$\therefore\ 2(3-1)\left(\frac{dy}{dx}-5\right)=4+8\ \Rightarrow\frac{dy}{dx}-5=\frac{12}{4}=3\Rightarrow\frac{dy}{dx}=8$$

Hence, the correct answer is (8).

CHAPTER 7: MONOTONICITY, CONCAVITY AND MAXIMA-MINIMA OF FUNCTIONS

Test Your Concepts-I

1.



$f(1) = 1 - 5 + 2 = -2$

$f(3) = 9 - 15 + 2 = -4$

From the graph of $f(x)$, clearly we can say that

At $x = 1$, $f(x)$ is strictly decreasing function,

At $x = \frac{5}{2}$, $f(x)$ is non-monotonic,

At $x = 3$, $f(x)$ is strictly increasing function

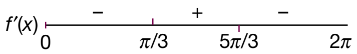
2. $f(0 - h) = \sin(0 - h) = -\sin h < 0$

$f(0) = 0$

$f(0 + h) = \sin^2 h > 0$

At $x = 0$, $f(x)$ is strictly increasing function

3.



$f(x) = x - 2\sin x$

$f'(x) = 1 - 2\cos x = 0 \Rightarrow \cos x = \frac{1}{2}$ we get $x = \frac{\pi}{3}, \frac{5\pi}{3}$

$f'(x) = 1 - 2\cos x$

$f(x)$ is monotonically decreasing in $x \in \left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$

$f(x)$ is monotonically increasing in $x \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$

$f(x)$ is non-monotonic

4. **(i)** $f(x) = \log_{1/2}(x^2 - 2x - 3)$

For $f(x)$ to be increasing, the number $(x^2 - 2x - 3)$ must be positive and decreasing

Now, $x^2 - 2x - 3 > 0 \Rightarrow (x - 3)(x + 1) > 0$

Or $x > 3$ (rejected as for $x > 3$, $x^2 - 2x - 3$ is increasing)

Hence f is increasing for $(-\infty, -1)$

(ii) $f(x) = \sin^4 x + \cos^4 x$

Differentiating w.r.t. x, we get

$f'(x) = 4\sin^3 x \cos x + 4\cos^3 x(-\sin x)$

$= 4\sin x \cos x[\sin^2 x - \cos^2 x] = 2\sin 2x(-\cos 2x)$

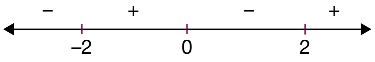
$f'(x) = -\sin 4x$

Since, $f'(x) \geq 0 \Rightarrow -\sin 4x \geq 0 \Rightarrow \sin 4x \leq 0$

$\Rightarrow \quad \pi \leq 4x \leq 2\pi \Rightarrow \frac{\pi}{4} \leq x \leq \frac{\pi}{2}$

5. We have, $f(x) = \int_3^{x^2} e^{-t^2}(t - 4)(t^2 - t + 12)dt$

$f'(x) = 2x \cdot e^{-x^4}(x^2 - 4)(x^4 - x^2 + 12)$



Sign of f′(x)

$= 2xe^{-x^4}(x - 2)(x + 2)(x^4 - x^2 + 12)$

$\therefore \quad f(x) \downarrow$ on $(-\infty, -2) \cup (0, 2)$.

6. $f(x) = \frac{\log_e(\pi + x)}{\log_e(e + x)}$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{(e + x)\log_e(e + x) - (\pi + x)\log_e(\pi + x)}{(\pi + x)(e + x)(\log_e(e + x))^2}$$

Domain $\in (-e, \infty)$ clearly when $x > 0$

Numeration of $f'(x)$ is negative. Since $\pi > e \Rightarrow \pi + x > e + x$

Hence, above inequality only holds good for +ve values of x i.e. $x \in (0, \infty)$

7. $f(x) = x/\log_e x$ has domain $(0, \infty) - \{1\}$

$f(x)$ is decreasing when $f'(x) < 0$

$$\frac{\log_e x - x\frac{1}{x}}{(\log_e x)^2} < 0 \Rightarrow x < e$$

$\therefore \quad f(x)$ is decreasing on $(0, 1) \cup (1, e)$

$\Rightarrow$ No of integer value of x is 1

8. $f(x) = \frac{x}{\sin x}$ and $g(x) = \frac{x}{\tan x}$; $0 < x \leq 1$

Differentiating above functions w.r.t. x, we get

$$f'(x) = \frac{1.\sin x - x.\cos x}{(\sin x)^2} = \frac{\sin x - x\cos x}{(\sin x)^2}$$

Since, $(\sin x)^2$ is positive for $0 < x \leq 1$

Let $h(x) = \sin x - x\cos x$

Differentiating w.r.t. x, we get $h'(x) = \cos x - (\cos x + x(-\sin x))$

$= \cos x - \cos x + x\sin x$

$h'(x) = x\sin x > 0$ for $0 < x \leq 1 \Rightarrow f'(x) > 0$ for $0 < x \leq 1$

$\Rightarrow \quad f(x)$ is increasing function in $0 < x \leq 1$; $g(x) = \frac{x}{\tan x}$

Differentiating w.r.t. x, we get

$$g'(x) = \frac{1.\tan x - x \cdot \sec^2 x}{(\tan x)^2}; \; g'(x) = \frac{\tan x - x \cdot \sec^2 x}{(\tan x)^2}$$

Since, $(\tan x)^2$ + ve for $0 < x \leq 1$

Let $\phi(x) = \tan x - x.\sec^2 x$

Differentiating w.r.t. x, we get

$\phi'(x) = \sec^2 x - [1.\sec^2 x + x.2\sec x.\tan x \sec x]$

$= \sec^2 x - \sec^2 x - 2x\sec^2 x \tan x$

$\phi'(x) = -2x\sec^2 x.\tan x$; $\phi'(x) < 0$ for $0 < x \leq 1$

$\Rightarrow \quad g'(x) < 0$ for $0 < x \leq 1$

$\Rightarrow \quad g(x)$ is decreasing function for $0 < x \leq 1$

9. **(i)** $f(x) = x^3 + 3x^2 - 105x + 25$

Differentiating w.r.t. x, we get

$f'(x) = 3x^3 + 6x - 105 + 0 = 3x^3 + 6x - 105$

Increasing function: $f'(x) = 3x^2 + 6x - 105 > 0$

$\Rightarrow \quad 3(x^2 + 2x - 35) > 0 \Rightarrow 3(x^2 + 7x - 5x - 35) > 0$

$\Rightarrow \quad 3[x(x + 7) - 5(x + 7)] > 0 \Rightarrow 3[(x - 5)(x + 7)] > 0$

$\Rightarrow \quad$ either $x < -7$ or $x > 5 \Rightarrow x \in (-\infty, -7) \cup (5, \infty)$

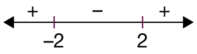
Decreasing function $f'(x) = 3(x + 7)(x - 5) < 0$

$\Rightarrow \quad -7 < x < 5 \Rightarrow x \in (-7, 5)$

(ii) $f(x) = 2x^3 - 24x + 7$

$f'(x) = 6x^2 - 24 = 6(x - 2)(x + 2)$

$f(x) \uparrow (-\infty, -2) \cup [2, \infty) \downarrow [-2, 2]$



10. $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$

Differentiating w.r.t. x, we get $f'(x) = 2e^x + ae^{-x} + (2a+1) - 0$

$f'(x) = 2e^x + ae^{-x} + 2a + 1$

$f'(x) = e^x(2 + e^{-x}) + a(2 + e^{-x})$

$f'(x) = (2 + e^{-x})(e^x + a) \rightarrow$ always +ve

So that $f'(x) > 0$ i.e. $e^x + a > 0 \Rightarrow a \geq 0$

Because e^x is positive for every $x \in R$.

11. $f(x) = \sin 2x - 8(a+1)\sin x + (4a^2 + 8a - 14)x$

$f'(x) = 2\cos 2x - 3(a+1)\cos x + 4a^2 + 8a - 14$

$\Rightarrow 2\cos^2 x - 1 - 4(a+1)\cos x + 2a^2 + 4a - 7 > 0$

$\Rightarrow \cos^2 x - 2(a+1)\cos x + (a+1)^2 - 5 > 0$

$\Rightarrow (\cos x - (a+1))^2 > 5$

$\cos x - (a+1) < -\sqrt{5}$ or $\cos x - (a+1) > -\sqrt{5}$

(1) $a > \cos x - 1 + \sqrt{5}$, $a > \sqrt{5}$

(2) $a < \cos x - 1 - \sqrt{5}$, $a < -2 - \sqrt{5}$

12. $f(x) = \sin(\log_e x) + \cos(\log_e x)$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{\cos(\log_e x)}{x} - \frac{\sin(\log_e x)}{x} < 0; x > 0$$

$$\Rightarrow \frac{\cos(\log_e x)}{x} < \frac{\sin(\log_e x)}{x}; x > 0$$

$\Rightarrow \tan(\log_e x) > 1; x > 0$

$$\Rightarrow 2n\pi + \frac{\pi}{4} < \log_e x < 2n\pi + \frac{5\pi}{4}; n \in I$$

$$\Rightarrow e^{2n\pi + \frac{\pi}{4}} < x < e^{2n\pi + \frac{5\pi}{4}}; n \in I$$

13. $f'(x) = 1 + \ln^2 x + 2\ln x = (1 + \ln x)^2 > 0$

In $x \in (0, \infty)$, $f(x)$ is increasing

14. We have $f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^5 - 3x + \log 5$

$$\Rightarrow f'(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1\right)5x^4 - 3$$

For $f(x)$ to be decreasing for all x, we must have $f'(x) < 0, \forall x$

$$\Rightarrow \left(\frac{\sqrt{a+4}}{1-a} - 1\right)5x^4 - 3 < 0$$

$$\Rightarrow \left(\frac{\sqrt{a+4}}{1-a} - 1\right)x^4 < \frac{3}{5}, \forall x \in R$$

$$\Rightarrow \left(\frac{\sqrt{a+4}}{1-a} - 1\right) \leq 0 \Rightarrow \frac{\sqrt{a+4}}{1-a} \leq 1$$

Which is trivially true for $a > 1$.

When $a < 1$, and $\sqrt{a+4}$ is real, than $-4 \leq a < 1$ and in this case

$\sqrt{a+4} \leq 1 - a \Rightarrow a + 4 \leq 1 - 2a + a^2$

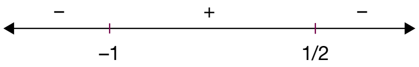
$$\Rightarrow 0 \leq a^2 - 3a - 3 \Rightarrow \frac{-3-\sqrt{21}}{2} \leq a \leq \frac{-3+\sqrt{21}}{2}$$

$$\Rightarrow -4 \leq a \leq \frac{3-\sqrt{21}}{2} \text{ as } a \geq -4.$$

Hence $a \in \left[-4, \frac{3-\sqrt{21}}{2}\right] \cup (1, \infty)$.

15. **(i)** We have $f(x) = x\,e^{-x(x+1)}$

$\therefore f'(x) = x\,e^{-x(x+1)}(-2x-1) + e^{-x(x+1)}$



Sign scheme of *f′*(*x*)

$= e^{-x(x+1)}(x(-2x-1) + 1)$

$= (-2x^2 - x + 1)\,e^{-x(x+1)} = -(2x^2 + x - 1)\,e^{-x(x+1)}$

$\Rightarrow f'(x) = -(2x-1)(x+1)\,e^{-x(x+1)}$

So, $f(x)$ increases in $\left(-1, \frac{1}{2}\right)$ and $f(x)$ decreases in $(-\infty, -1) \cup \left(\frac{1}{2}, \infty\right)$

(ii) $f(x) = (x+2)e^{-x}$

Differentiating w.r.t. x,

we get $f'(x) = 1.e^{-x} + (x+2)e^{-x}(-1)$

$= e^{-x} - (x+2)\,e^{-x} = e^{-x}[1 - x - 2]$

$f'(x) = e^{-x}[-x-1] = -(1+x)\,e^{-x}$

If $(1+x) > 0$ i.e $x > -1$, $f'(x) < 0$

i.e., It decreases for $(-1, \infty)$

If $(1+x) < 0$ i.e. $x < -1 \Rightarrow f'(x) > 0$ i.e. It increases for $(-1, \infty)$ and increases for $(-\infty, -1)$.

(iii) $f(x) = \cos\left(\frac{\pi}{x}\right)$

$$f'(x) = \frac{\pi}{x^2}\sin\left(\frac{\pi}{x}\right) > 0$$

For increasing $\sin\left(\frac{\pi}{x}\right) > 0$

$$\Rightarrow 2n\pi \langle \frac{\pi}{x} \langle (2n+1)\pi \Rightarrow \frac{1}{2n} \rangle x \rangle \frac{1}{2n+1}$$

$$\therefore x \in \left(\frac{1}{2n+1}, \frac{1}{2n}\right), \forall n \in N$$

(iv) $f(x) = |x\log_e x|$

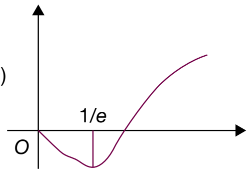
For $g(x) = x\log_e x$,

$$g'(x) = x\frac{1}{x} + \log_e x = 1 + \log_e x$$

$\Rightarrow g(x)$ increases for $\left(\frac{1}{e}, \infty\right)$ and decreases for $\left(0, \frac{1}{e}\right)$

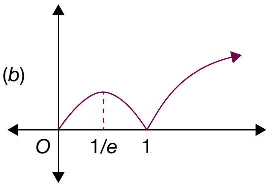
Graph of $y = g(x) = x\log_e x$

(a)



Graph of *y* = *g*(*x*) = *x* log$_e$ *x*

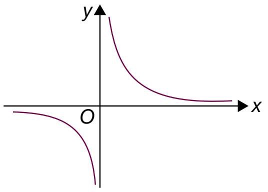
(b)



Graph of *y* = *f*(*x*) = |*x* log$_e$ *x*|

From the graph, $f(x) = |x\log_e x|$ decreases in $\left(\frac{1}{e}, 1\right)$.

16.



$f(x) = 2x^3 + ax^2 + bx - 3\cos^2 x$

$\therefore \quad f'(x) = 6x^2 + 2ax + b + 3\sin 2x$

$\therefore \quad f(x)$ is increasing for all $x \Rightarrow 6x^2 + 2ax + b + 3\sin 2x > 0$

Also, $6x^2 + 2ax + b + 3\sin 2x \geq 6x^2 + 2ax + b - 3$ as $\sin 2x \geq -1$

Hence, $6x^2 + 2ax + b - 3 \geq 0$ for all x

Now, $D = 4a^2 - 4\,.\,6(b-3) \leq 0 \Rightarrow a^2 - 6b + 18 \leq 0$

Test Your Concepts-II

1. For $x_{1,}x_2 \in \left(0, \frac{\pi}{2}\right)$, we know $\frac{x}{\tan x}$ is decreasing function

$\therefore$ for $x_1 < x_2 \Rightarrow \frac{x_1}{\tan x_1} > \frac{x_2}{\tan x_2} \Rightarrow \frac{\tan x_2}{\tan x_1} > \frac{x_2}{x_1}$

2. Given $f''(x) - 3f'(x) > 3$

$\Rightarrow \frac{d}{dx}(e^{-3x} f'(x)) > 3e^{-3x} \Rightarrow \frac{d}{dx}(e^{-3x} f'(x) + e^{-3x}) > 0$

$\Rightarrow e^{-3x}(f'(x) + 1)$ is increasing function

Also $e^{-3x}(f'(x) + 1) > f'(0) + 1\ \forall\ x > 0$

$\Rightarrow f'(x) + 1 > 0 \Rightarrow f(x) + x$ is an increasing function

3. Let $f(x) = \sin(\tan x) - x$

Differentiating w.r.t. x,

We get $f'(x) = \cos(\tan x)\sec^2 x - 1$

$f'(x) = \tan^2 x \cos(\tan x) - (1 - \cos(\tan x))$

Using the relation, $2(1 - \cos x) < x^2$

$f'(x) > \tan^2 x \cos(\tan x) - \frac{1}{2}\tan^2 x;$

$f'(x) > \tan^2 x \left[\cos(\tan x) - \frac{1}{2}\right]$

Again using, $2(1 - \cos x) < x^2 \Rightarrow 1 - \frac{x^2}{2} < \cos x$

$\Rightarrow 1 - \frac{\tan^2 x}{2} < \cos(\tan x)$

$f'(x) > \tan^2 x\left[1 - \frac{\tan^2 x}{2} - \frac{1}{2}\right]; f'(x) > \tan^2 x\left[1 - \frac{\tan^2 x}{2}\right]$

$\Rightarrow f'(x) > 0$ for $x \in \left(0, \frac{\pi}{4}\right)$

Let $x > 0 \Rightarrow f(x) > f(0)$

$\Rightarrow \sin(\tan x) - x > \sin(\tan 0) - 0 \Rightarrow \sin(\tan x) - x > 0$

$\Rightarrow \sin(\tan x) > x$. for $x \in \left(0, \frac{\pi}{4}\right)$

Hence, proved.

4. $\sin x + 2x \geq \frac{3x(x+1)}{\pi}, \forall\ x \in \left[0, \frac{\pi}{4}\right]$

Let $f(x) = \sin x + 2x - \frac{(3x^2 + 3x)}{\pi}$

Differentiating w.r.t. x, we get $f'(x) = \cos x + 2 - \frac{(6x+3)}{\pi}$

$f'(x) = (\cos x - 6x/\pi) + \left(2 - \frac{3}{\pi}\right)$

$f''(x) = -\sin x - 6/\pi < 0$ for $\forall\ x \in \left[0, \frac{\pi}{4}\right]$

It means $f'(x)$ is decreasing

$f'(x)|_{x=\pi/4} = \left(\cos\frac{\pi}{4} - \frac{6}{\pi} \times \frac{\pi}{4}\right) + 2 - \frac{3}{\pi} = \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{3}{\pi} > 0$

$\Rightarrow f'(x)|_{x=\pi/4} > 0$ then $f'(x) > 0$ for $\forall\ x \in \left[0, \frac{\pi}{4}\right]$

$\therefore \quad f(x)$ is increasing function in $x \in \left[0, \frac{\pi}{4}\right]$

Let $x \geq 0 \Rightarrow f(x) \geq f(0)$

$\Rightarrow \sin x + 2x - \frac{(3x^2 + 3x)}{\pi} \geq 0$ Hence, proved.

5. $y' = 1 + \ln x - x$

$y'' = \frac{1}{x} - 1 > 0$ as $x \in (0, 1)$

i.e. $y'(x)\ x \in (0, 1)$

$y'(x) < y'(1) = 0 \Rightarrow y'(x) < 0$

$x \in (0, 1)$ i.e. $y(x)$ is ↓

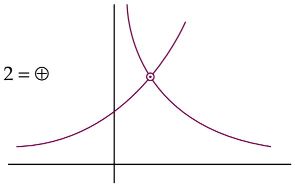
$\Rightarrow y(x) > y(1) = 0 \Rightarrow y(x) > 0$

$\Rightarrow x \ln x > \frac{x^2}{2} - \frac{1}{2}\ \forall\ x \in (0, 1)$ Hence, proved.

6. $f(x) = xe^x - 2$

$f'(x) = xe^x - e^x = e^x(x + 1)$

$f(x)\ x > -1; f(0) = -2; f(1) = e - 2 = \oplus$



Alternate:

Graph $e^x = \frac{2}{x}$

7. $f(x) = x^2 + kx + 1$

For $f(x)$ to be increasing,

$f'(x) > 0 \Rightarrow \frac{d}{dx}(x^2 + kx + 1) > 0$

$\Rightarrow 2x + k > 0 \Rightarrow k > -2x$

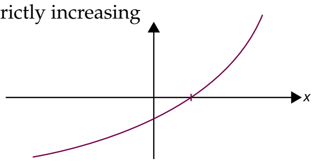
For $x \in (1, 2)$, the least value of k is -2

8. $f(x) = e^x + 4x - 11$

$f'(x) = e^x + 4 - 0 > 0, f(x)$ is strictly increasing

$f(0) = e^{\circ} + 0 - 11 = -10 < 0$

$f(10) = e^{10} + 40 - 11 > 0$



from Intermediate Mean value theorem

$f(x) = 0$ has at least one real solution

As $f(x)$ is strictly increasing function, so graph of $f(x)$ cuts x axis only once

So $f(x) = 0$ has exactly one real roots exactly one real solution

9. **(i)** $f(x) = x^3 - 6x^2 + 12x + 1$

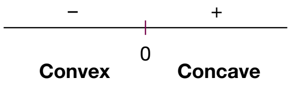
$f'(x) = 3x^3 - 12x + 12$

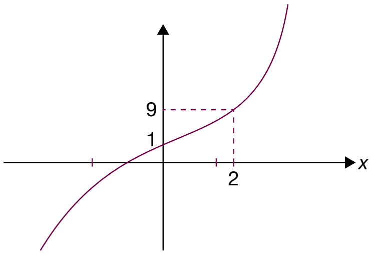
$= 3(x^2 - 4x + 4) = 3(x - 2)^2$

$f''(x) = 6(x - 2)$

$f(2) = 8 - 24 + 24 + 1 = 9; f(0) = 1$

Point (2,9) is point of inflexion



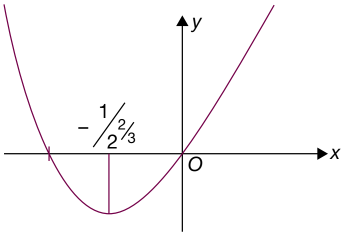


(ii) $f(x) = x^4 + x$

$f'(x) = 4x^3 + 1 = 0 \Rightarrow x^3 = -\frac{1}{4} \Rightarrow x = -\frac{1}{2^{2/3}}$

$f''(x) = 12x^2$ always positive

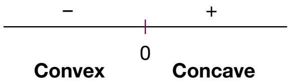
$f(x)$ is always concave

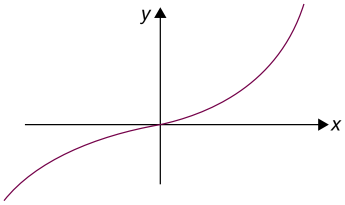


(iii) $f(x) = x + x^{5/3}$ domain $x \in R$

$f(x) = x + x^{5/3}$

$$f'(x) = 1 + \frac{5x^{2/3}}{3} = 1 + \frac{5}{3}\left(x^{\frac{1}{3}}\right)^2 > 0$$





$$f''(x) = 0 + \frac{5}{3}\left(\frac{2}{3}\right) x^{-\frac{1}{3}} = \frac{10}{9}\,\frac{1}{x^{1/3}}$$

Origin is point of inflexion

10. $f(x) = x^4 - 6x^3 + 12x^2 - 8x + 3$

$f'(x) = 4x^3 - 18x^2 + 24x - 8$

$f''(x) = 12x^2 - 36x + 24$

$f''(x) = 0 \Rightarrow x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2$

$f(1) = 1 - 6 + 12 - 8 + 3 = 2$

$f(2) = 16 - 48 + 48 - 16 + 3 = 3$

Point of inflection are (1, 2) and (2, 3)

$f'(1) = 2, f'(2) = 0$

Tangent at (1, 2), we have $y - 2 = 2(x - 1)$ or $y = 2x$

Tangent at (2, 3), we have $y - 3 = 0\,(x - 2)$ or $y = 3$

Test Your Concepts-III

1. [a – ∞; b – ∞; c – 2; d – 1]

(A) $f'(x) = 0 \ \forall x \neq 0$

f has horizontal tangent at $y = 1, y = -1$

Infinite number of critical points

(B) $f'(x) = -\frac{1}{x^2}\cos\frac{1}{x}$, $f'(x) = 0$ where $\frac{1}{x} = (2n+1)\frac{\pi}{2}, n \in I$

$f'(0)$ does not exist

Infinite number of critical points

(C) $f'(x) = 0$ at $x = 4$; $f'(2)$ does not exist

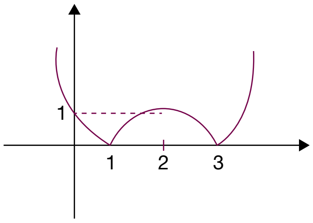
$x = 0$ is not a point in domain. Two critical points

$$f'(x) = \begin{cases} \dfrac{4-x}{x^3} & \text{if} \quad x > 2 \\ \dfrac{x-4}{x^3} & \text{if} \quad x \in (-\infty, 2] \sim \{0\} \end{cases}$$

(D) $f(x) = \frac{1}{x-1} - \frac{1}{x}$

$$f'(x) = \frac{1}{x^2} - \frac{1}{(x-1)^2} = \frac{1-2x}{(x^2-x)^2} \Rightarrow f'(x) = 0 \text{ if } x = \frac{1}{2}$$

2.



$f(x) = e^{|x^2-4x+3|}$ has same extreme as $|x^2 - 4x + 3|$

2 local minimum at $x = 1, 3$

1 local maxima at $x = 2$

3. **(i)** $f(x) = x^2 \log x$

For $f'(x) = x(2\log x + 1) = 0, \Rightarrow x = \frac{1}{\sqrt{e}}$

Which is the point of minima as derivative changes sign from negative to positive

Also, the function decreases in $\left(0, \frac{1}{\sqrt{e}}\right)$

(ii) $y = x \log x$

$\Rightarrow \frac{dy}{dx} = x \times \frac{1}{x} + \log x = 1 + \log x$ and $\frac{d^2y}{dx^2} = \frac{1}{x}$

For $\frac{dy}{dx} = 0 \Rightarrow \log x = -1 \Rightarrow x = \frac{1}{e}$

$\frac{d^2y}{dx^2} = \frac{1}{\frac{1}{e}} = e > 0$ at $x = \frac{1}{e}$ $\Rightarrow y$ is min for $x = \frac{1}{e}$

(iii) $f(x) = \frac{\log x}{x}$

For $f'(x) = \frac{1 - \log x}{x^2} = 0, x = e$. Also, derivative changes sign from positive to negative at $x = e$, hence it is the point of maxima.

(iv) $f(x) = x^{-x}$

$f'(x) = -x^{-x}(1 + \log x) = 0 \Rightarrow x = \frac{1}{e}$, which is clearly point of maxima.

4. $f'(x) = n(x - a)^{n-1} g(x) + (x - a)^n g'(x)$

$= (x - a)^{n-1}(n\,g(x) + (x - a)\,g'(x))$

For $f(x)$ to have local extreme, $f'(x)$ must change its sign across put $x = a$,

$(x - a)^{n-1}$ must change its sign, $n - 1$ must be odd

It means n must be even

5. **(i)** $f(x) = \sin x\,(1 + \cos x)$ where $x \in (0, 2\pi)$

$f'(x) = \cos x\,(1 + \cos x) + \sin x\,(0 - \sin x)$

$= \cos x\,(1 + \cos x) - (1 - \cos^2 x) = (1 + \cos x)\,[\cos x - (1 - \cos x)]$

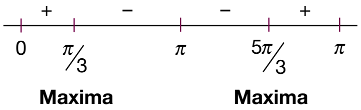
$= (1 + \cos x)\,[2\cos x - 1] = 0$

For critical points, $\cos x = -1 : \cos x = \frac{1}{2}$, we get $x = \pi, \frac{\pi}{3}, \frac{5\pi}{3}$

$f'(x) = (1 + \cos x)\,(2\cos x - 1)$

$f''(x) = -\sin x\,(2\cos x - 1) + (1 + \cos x)\,(-2\sin x)$

$= -\sin x\,[2\cos x - 1 + 2 + 2\cos x] = -\sin x\,[4\cos x + 1]$



$$f''\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}(2+1) < 0; \text{ at } x = \frac{\pi}{3} \text{ is local maxima}$$

$$f''\left(\frac{5\pi}{3}\right) = +\frac{\sqrt{3}}{2}[2+1] > 0; \text{ at } x = \frac{5\pi}{3} \text{ is local minima}$$

$f''(\pi) = 3\sin\pi = 0$
$f''(x) = -\sin x\,(4\cos x + 1)$
$f'''(x) = -\cos x\,(4\cos x + 1) - \sin x(-4\sin x)$
$f'''(\pi) = 1\,(-4+1) + 0 = -3 \neq 0$

At $x = \pi$, no extreme

$f(x)$ has extreme at $x = \frac{\pi}{3}$ (local maxima) and at $x = \frac{5\pi}{3}$ (local minima)

(ii) $f(x) = 10x^6 - 12x^5 + 15x^4 - 20x^3$
$f'(x) = 60x^5 - 60x^4 + 60x^3 - 60x^2$
$= 60x^2[x^3 - x^2 + x - 1]$
$f'(x) = 60\,x^2(x-1)\,(x^2+1)$

At $x = 1$ local minima and at $x = 0$ no extreme

(iii) $f(x) = 2x + 3x^{\frac{2}{3}}$ has domain R.

$$f'(x) = 2 + 2x^{-1/3} = 2\left(1 + x^{-1/3}\right) = 0 \Rightarrow x = -1$$

$$f''(x) = \frac{-2}{3}x^{-\frac{4}{3}} \text{ and } f''(-1) < 0$$

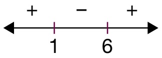
So, $f(x)$ has relative maximum at $x = -1$

But $f(x)$ is not differentiable at $x = 0$

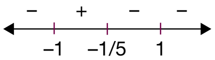
Still, $f(x) > f(0)$ in neighborhood of 0

$\therefore\ f(x)$ has relative minimum at $x = 0$.

6. **(i)** $f'(x) = 6x^2 - 42x + 36$
$= 6(x^2 - 7x + 6) = 6(x-1)(x-6)$
Min. $x = 6$
Max. $x = 1$

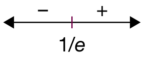


(ii) $f'(x) = -(x-1)^3 = (x+1) - (x+1)^2\,3(x-1)^2$
$= -(x+1)\,(x-1)^2\,(2x - 2 + 3x + 3)$
$= -(x+1)\,(x-1)^2\,(5x+1)$
Min. $x = -1$
Max. $x = -\frac{1}{5}$



(iii) $f'(x) = 1 + \ln x$

Min. $x = \frac{1}{e}$



7. **(i)** $f'(x) = 4 - x$

Max. at $x = 4$

$$f(4) = 16 - \frac{16}{2} = \frac{16}{2};\ f(-2) = -8 - 2 = 10$$

$$f\left(\frac{9}{2}\right) = 18 - \frac{81}{8} = \frac{63}{8}$$

f min. $= -10$
f max. $= 8$

(ii) $f'(x) = 12x^3 - 24x^2 + 24x - 48$
$= 12(x^3 - 2x^2 + 2x - 4) = 12(x-2)\,(x^2+2)$

min. $x = 2$
$f(0) = 25$
$f(3) = 16$
$f(2) = -39$

(iii) $f'(x) = \cos x - \sin 2x$

$$= \cos x(1 - 2\sin x) = -2\cos x\left(\sin x - \frac{1}{2}\right)$$

max. at $x = \frac{\pi}{6}$

$$f\left(\frac{\pi}{6}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

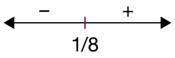
$$f(0) = \frac{1}{2};\ f\left(\frac{\pi}{2}\right) = 1 - \frac{1}{2} = \frac{1}{2}$$

(iv) $f'(x) = 1 + \frac{1}{2\sqrt{x}} > 0$

$$f(x)\left[\begin{array}{ll} f(x)_{\min} & = f(0) \\ & = 0 \\ f(x)_{\max} & = f(4) \\ & = 4 + 2 = 6 \end{array}\right.$$

(v) $f'(x) = 16x^{1/3} - \frac{2x}{x^{2/3}}$

$$= \frac{16}{x^{2/3}}\left(x - \frac{1}{8}\right)$$



min. $x = \frac{1}{8}$; $f(1) = 6$; $f(-1) = 18$

8. Given that $-1 \le p \le 1$.

Consider $f(x) = 4x^3 - 3x - p = 0$

Now, $f\left(\frac{1}{2}\right) = \frac{1}{2} - \frac{3}{2} - p = -1 - p \le 0$ as $(-1 \le p)$

Also $f(1) = 4 - 3 - p = 1 - p \le 0$ as $(-1 \le p)$

$\therefore$ $f(x)$ has at least one real root between [1/2, 1].

Also $f'(x) = 12x^2 - 3 > 0$ on [1/2, 1]

$\Rightarrow$ f is increasing on [1/2]

9. $f(x) = ax + (b/x) \ge c;\ x > 0$

Differentiate w.r.t. x, we get $f'(x) = a - b/x^2 - 0 \ge 0$

$$\Rightarrow a - \frac{b}{x^2} \ge 0 \Rightarrow a \ge \frac{b}{x^2}$$

$$\Rightarrow x^2 \ge \frac{b}{a} \Rightarrow x^2 \ge \sqrt{\frac{b}{a}}$$

Hence, put value of $x \ge \sqrt{\frac{b}{a}}$ in $ax + b/x \ge c$

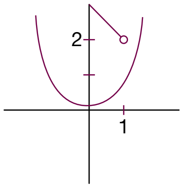
$$\Rightarrow a \times \sqrt{\frac{b}{a}} + b \times \sqrt{\frac{a}{b}} \ge c \Rightarrow \sqrt{ab} + \sqrt{ab} \ge c$$

$$\Rightarrow 2\sqrt{ab} \ge c \Rightarrow \sqrt{ab} \ge \frac{c}{2} \Rightarrow ab \ge \frac{c^2}{4}$$

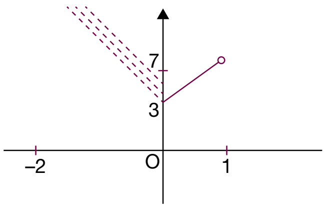
10. $f(1) \le f(1^-) = 2$

$\Rightarrow$ $1 + \ln b \le 2$

$0 < b \le e$



11.



At $x = 0$ to be minima, from graph

we can say that $f(0^-) \ge f(0)$

$\Rightarrow$ $a - 0 \ge 0 + 3$

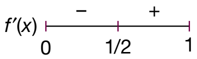
$\Rightarrow$ $a \ge 3$

12. $f(x) = \int_0^1 e^{|t-x|}\,dt \qquad (0 \le x \le 1)$

$$f(x) = \int_0^x e^{x-t}\,dt + \int_x^1 e^{t-x}\,dt = e^x \int_0^x e^{-t}\,dt + e^{-x}\int_x^t e^t\,dt$$

$$= e^x\left(-e^{-x}\right)_0^x + e^{-x}\left(e^t\right)_x^1 = e^x(-e^{-x}+1) + e^{-x}(e^1 - e^x)$$

$f(x) = e^x + e^{1-x} - 2$



$f'(x) = e^x - e^{1-x}$

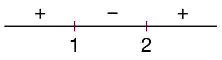
$f(0) = f(1) = e - 1$, $x = 1/2$ has minima

Hence, the correct answer is (C).

13. $f(x) = (x-2)^{2/3}(2x+1)$

$$f'(x) = \frac{2}{3}(x-2)^{-1/3}(2x+1) + (x-2)^{2/3}2$$

$$f'(x) = \frac{5(x-1)}{3(x-2)^{1/3}}$$



$f'(x)$ is zero or not defined then critical point are $x = 1, 2$

14. $f(x) = x + \sqrt{|x|} = \begin{cases} x + \sqrt{x} & x \ge 0 \\ x + \sqrt{-x} & x < 0 \end{cases}$ is continuous

$$f'(x) = \begin{cases} 1 + \dfrac{1}{2\sqrt{x}} & x \ge 0 \\ 1 - \dfrac{1}{2\sqrt{-x}} & x < 0 \end{cases} \quad \text{at } x = 0\ f'(x) \text{ DNE}$$

For $x > 0, f'(x) > 0$;

For $x < 0, f'(x) = 0 \Rightarrow 1 = \dfrac{1}{2\sqrt{-x}}$

$$\sqrt{-x} = \frac{1}{2} \Rightarrow -x = \frac{1}{4} \Rightarrow x = -\frac{1}{4}$$

two critical point

15. $f(x) = \sin(3\sin x - 4\sin^3 x + 4\sin^3 x) = \sin(3 \sin x)$

For all real x, we have $-3 \le 3\sin x \le 3$

$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ is subset of $[-3, 3]$

$f(x)_{\max} = 1$

16. $f(x) = \begin{cases} x^2 + 2x + \dfrac{a^2-1}{a^4-5}; & x < 0 \\ x + 2; & x \ge 0 \end{cases}$

$f'(x) = 2x + 2 + 0 \Rightarrow$ put $f'(x) = 0$

$f''(x) = 2 \Rightarrow x = -1$

For minima, $f(a) \le f(a+h) \Rightarrow f(0) \le f(0+h)$

$f(a) \le f(a-h) \Rightarrow f(0) \le f(0-h) \Rightarrow f(0+h) \le f(0-h)$

$$(-1)^2 + 2(-1) + \frac{a^2-1}{a^4-5} \le 0 + 2;\ a^4 - 5 \ne 0$$

$$1 - 2 + \frac{a^2-1}{a^4-5} \le 2;\ a^4 - 5 \ne 0$$

$$\frac{a^2-1}{a^4-5} \le 3;\ a^4 - 5 \ne 0$$

$\Rightarrow 3a^4 - 15 \ge a^2 - 1;\ a^4 - 5 \ne 0 \Rightarrow 3a^4 - a^2 - 14 \ge 0;\ a^4 - 5 \ne 0$

Now, we have to find roots of $3a^4 - a^2 - 14 = 0$

$$3a^4 - a^2 = 14 = 0 \Rightarrow a^2 = \frac{1 \pm \sqrt{1+168}}{2\times 3}$$

$$a^2 = \frac{1 \pm \sqrt{169}}{2\times 3};\ a^2 = \frac{1 \pm 13}{2 \times 3} = \frac{7}{3};\ a = \pm\sqrt{\frac{7}{3}}$$

$$\Rightarrow \left(a + \sqrt{\frac{7}{3}}\right)\left(a - \sqrt{\frac{7}{3}}\right) \le 0$$

$$a \in \left[-\sqrt{\frac{7}{3}} - 5\frac{1}{4}\right) \cup \left(5\frac{1}{4}, \sqrt{\frac{7}{3}}\right]$$

17. $p(x) = ax^4 + bx^3 + cx^2 + dx + e$

$$\lim_{x\to 0}\left(1 + \frac{p(x)}{x^2}\right) = 2$$

$$\Rightarrow \lim_{x \to 0}\left[1 + \left(\frac{ax^4 + bx^3 + cx^2 + dx + e}{x^2}\right)\right] = 2$$

$d = 0, e = 0; c = 1$

Now, $p'(x) = 4ax^3 + 3bx^2 + 2x \Rightarrow 4a + 3b = -2$ …(1)

$\Rightarrow 32a + 12b = -4$ …(2)

$\Rightarrow 16a = 4;\ a = \dfrac{1}{4};\ b = -1$

$p(x) = \dfrac{1}{4}x^4 - x^3 + x^2;\ p(2) = 0$

18. $f(x) = x^3 + ax^2 + bx + c$

Differentiating above function w.r.t x, we get

$f'(x) = 3x^2 + 2ax + b + 0;\ f'(x) = 3x^2 + 2ax + b$

$f'(x)|_{x=-1} = 3 - 2a + b = 0 \Rightarrow 2a - b = 3$

$f'(x)|_{x=3} = 24 + 6a + b = 0 \Rightarrow 6a + b = -27$

Solving above two equations for a and b.

$2a - b = 3$

$6a + b = -27$

$8a + 0 = -24 \Rightarrow a = -3$

$b = 2a - 3 = -6 - 3 \Rightarrow b = -9$

Hence, $a = -3$, $b = -9$ and $c \in R$

19. $f'(x) = 0 \Rightarrow 6x^2 - 18x + 12 = 0$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow (x-2)(x-1) = 0$

$f''(x) = 12x - 18$

$f''(1) < 0$

$x = 1$ is maxima

$g(x) = 4 + 5x^2 + 6x^4$ has

$g'(x) = -10x - 24x^3 \Rightarrow -2x(5 + 12x^2) = 0$

$\Rightarrow x = 0$

$5 + 12x^2 \ne 0$

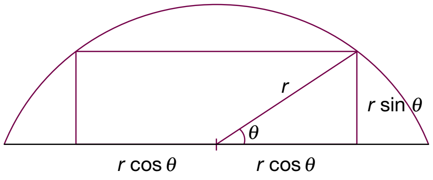
$g''(x) = -10 - 72x^2$

$f_g''(0) = -10 < 0$

$g(x)$is minimum at $x = 0$

Test Your Concepts-IV

1.



$S = (2r\cos\theta)(r\sin\theta)$

$S = r^2 \sin 2\theta$

$S_{\max} = r^2$ when $\theta = \frac{\pi}{4}$

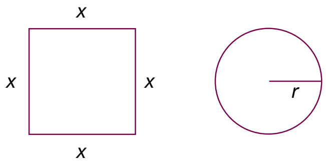
For greatest area of rectangle, sides are $\sqrt{2}\, r, \frac{r}{\sqrt{2}}$

2. Let consider a square of side x and circle of radius r

$4x + 2\pi r = 20 \Rightarrow 2\pi r = 20 - 4x$

$\Rightarrow \; r = \frac{10 - 2x}{\pi}$

$$S = x^2 + \pi r^2 = x^2 + \frac{\pi(5-x)^2 \cdot 4}{\pi^2}$$



$$S = x^2 + \frac{4}{\pi}(5-x)^2$$

$$\frac{dS}{dx} = 2x + \frac{4}{\pi}(2(5-x))(-1)$$

For minimum area, $\frac{dS}{dx} = 0 \Rightarrow 2x = \frac{8}{\pi}(5-x) \Rightarrow x = \frac{20}{\pi} - \frac{4x}{\pi}$

$x\left(1 + \frac{4}{\pi}\right) = \frac{20}{\pi}$ is point of minima as $\frac{d^2S}{dx^2} = 2 + \frac{8}{\pi} > 0$

Square of perimeter $\frac{80}{\pi + 4}$ and circle with circumference $\frac{20\pi}{\pi + 4}$

3. $P(x)$ = selling price – cost = $\left(5 - \frac{x}{100}\right)x - \left(\frac{x}{5} + 500\right)$

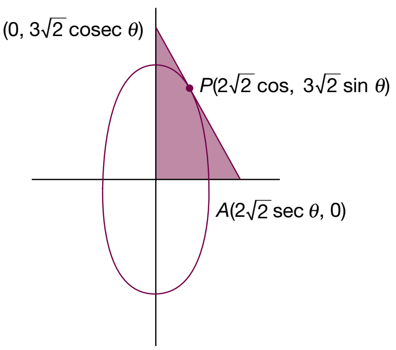
$$P(x) = 5x - \frac{x^2}{100} - \frac{x}{5} - 500 = \frac{24x}{5} - \frac{x^2}{100} - 500$$

$$P'(x) = \frac{24}{5} - \frac{2x}{100} - 0 = 0 \Rightarrow \frac{24}{5} = \frac{2x}{100} \Rightarrow x = 240$$

$P''(x) = -\frac{1}{50} < 0$, $P(x)$ is maximum at $x = 240$

Manufacturer need to sell 240 item to have maximum profit.

4.



Let point $P(2\sqrt{2}\cos, 3\sqrt{2}\sin\theta)$ on given ellipse $\frac{x^2}{8} + \frac{y^2}{18} = 1$

Equation of tangent $T = 0$

$$\Rightarrow \frac{xx_1}{8} + \frac{yy_1}{18} = 1$$

$$\Rightarrow \frac{x}{8}\left(2\sqrt{2}\cos\theta\right) + \frac{y}{18}\left(3\sqrt{2}\sin\theta\right) = 1$$

$$\Rightarrow \frac{x}{2\sqrt{2}\sec\theta} + \frac{y}{3\sqrt{2}\operatorname{cos}ec\theta} = 1$$

$$\text{Area} = A = \frac{1}{2}\left(2\sqrt{2}\sec\theta\right)\left(3\sqrt{2}\operatorname{cosec}\theta\right)$$

$$= \frac{6}{\sin\theta\cos\theta} = \frac{12}{\sin 2\theta}$$

$$A_{\min} = \frac{12}{1}$$

$\theta = 45°$

$P(2, 3)$

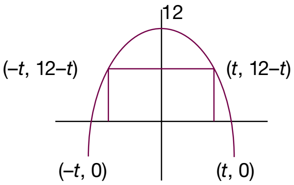
5. Area $= 2t \times 12 - t$

$= 24t - 2t^2$

$\frac{dA}{dt} = 24 - 4t$

max. at $t = 6$

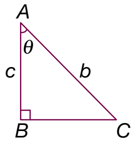
$A_{\max} = (24 \times 6) - 72 = 144 - 72 = 72$



6. $b + c = k$

$b = k - c$

$BC = \sqrt{b^2 - c^2}$; Area $= \frac{1}{2}BC.AB = \frac{1}{2}\sqrt{b^2 - c^2}.c$



$$A(c) = \frac{1}{2}c\sqrt{(k-c)^2 - c^2}$$

$$A(c) = \frac{1}{2}c\sqrt{k^2 - 2kc}$$

$$A'(c) = \frac{1}{2}\left[\frac{k^2 - 3ck}{\left(k^2 - 2kc\right)^{1/2}}\right] A'(c) = 0 \quad k = 3c$$

$$b = 2c \quad \cos\theta = \frac{c}{b} = \frac{c}{2c} \Rightarrow \theta = \frac{\pi}{3}$$

7. $l = 2x + 2r + \pi r$

$$A = 2rx + \frac{1}{2}\pi r^2$$

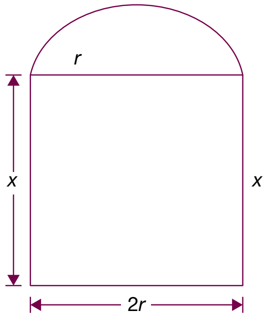
$$A = r(l - 2r - \pi r) + \frac{\pi r^2}{2};$$

$$A = rl - 2r^2 - \frac{\pi r^2}{2}$$

$$\frac{dA}{dr} = l - 4r - \pi r;$$

$$\frac{d^2A}{dr^2} = -4 - \pi < 0$$

$$\frac{dA}{dr} = 0 \Rightarrow r = \frac{l}{4 + \pi}$$



Single Correct Choice Type Questions

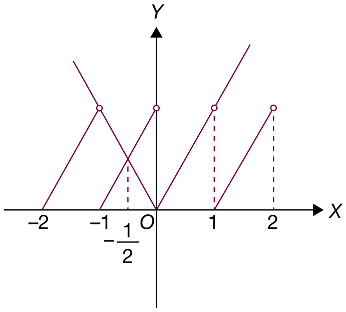
1. $\because \quad f(x) = |x| - \{x\}$

$\therefore \quad f(x)$ is decreasing

$\therefore \quad f'(x) < 0 \Rightarrow |x| - \{x\} < 0$

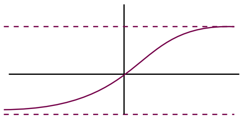
$\Rightarrow \quad |x| < \{x\}$

It is clear from the figure $x \in \left(-\frac{1}{2}, 0\right)$



Hence, the correct answer is (A).

2.



$f(x) = 12x^5 - 15x^4 + 20x^3 - 30x^2 + 60x + 1$

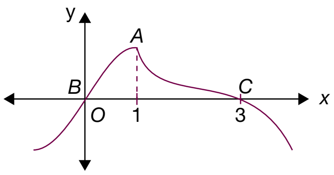
$f'(x) = 12x^5 - 15x^4 + 20x^3 - 30x^2 + 60x + 60$

$= 60(x^4 - x^3 + x^2 - x + 1) > 0\ \forall\ x \in R$

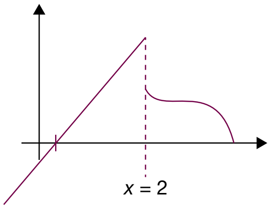
S-2: $y = \tan^{-1}x$ is strictly increasing and Range $\neq R$

Hence, the correct answer is (C).

3. Three critical points *A*, *B*, *C*.



graph is wrong



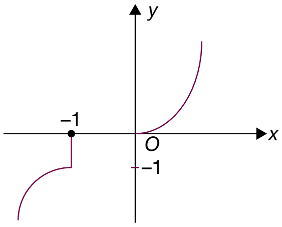
Hence, the correct answer is (C).

4. Note that $m \nleq 0$.

So, $m > 0$.

Also, $m \leq 1$ $m \in (0, 1]$

Hence, the correct answer is (D).



5. $f(x) = \sin(\ln x) - 2\cos(\ln x)$

$$f'(x) = \frac{\cos(\ln x) + 2\sin(\ln x)}{x}$$

$$= \sqrt{5}\left(\cos(\ln x)\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}\sin(\ln x)\right)$$

$$= \sqrt{5}\cdot\sin(\ln x + \alpha) \text{ where } \tan\alpha = \frac{1}{2}$$

For increasing, $f'(x) > 0$

$\Rightarrow\ 0 < (\ln x + \alpha) < \pi$

$\Rightarrow\ -\alpha < \ln x < \pi - \alpha$

$\Rightarrow\ n\pi - \alpha < \ln x < n\pi + \pi - \alpha$

$\Rightarrow\ e^{n\pi - \alpha} < x < e^{n\pi + \pi - \alpha}, n \in I.$

$\therefore\ \sqrt{5}\cos(\mu - \lambda) = \sqrt{5}\cos(n\pi + \pi - \alpha - (n\pi - \alpha)) = -\sqrt{5}$.

Hence, the correct answer is (A).

6. We have

$$g(t) = \frac{\pi}{2} - 2\cot^{-1}(3^{-t})$$

$$\therefore\ g(-t) = \frac{\pi}{2} - 2\cot^{-1}(3^{t})$$

$$= \frac{\pi}{2} - 2\tan^{-1}(3^{-t}) \left(\text{As } \cot^{-1}x = \tan^{-1}\frac{1}{x}, x > 0\right)$$

$$= \frac{\pi}{2} - 2\left(\frac{\pi}{2} - \cot^{-1}(3^{-t})\right) = -\frac{\pi}{2} + 2\cot^{-1}(3^{-t}) = -g(t)$$

$\left(\text{As } \cot^{-1}x + \tan^{-1}x = \frac{\pi}{2}, \forall\ x \in R\right)$

$\Rightarrow g$ is an odd function.

Also $g'(t) = \dfrac{-2.3^{-t}\cdot\ln 3}{1 + (3^{-t})^2}$

$\therefore\ g'(t) < 0, \forall\ x \in R$

$\Rightarrow\ g$ is strictly decreasing in $(-\infty, \infty)$

Hence, the correct answer is (B).

7. $f(x) = \sin^{-1}(x^2 - 3x + 3)$

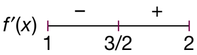
$-1 \leq x^2 - 3x + 3 \leq 1$

$0 \leq x^2 - 3x + 4$ and $x^2 - 3x + 2 \leq 0$

$D = 9 - 16 < 0$ $\quad (x - 1)(x - 2) \leq 0$

$\forall\ x \in R$ $\quad x \in [1, 2]$

$x \in [1, 2]$ domain



$$f'(x) = \frac{1}{\sqrt{1 - (x^2 - 3x + 3)^2}}(2x - 3)$$

In $x \in \left[\frac{3}{2}, 2\right]$ $f(x)$ is increasing

Hence, the correct answer is (C).

8. $f(x) = 2e^x - c\ln x\ \forall\ x \in (0, \infty)$

$f'(x) = 2e^x - \frac{c}{x} \geq 0 \Rightarrow 2e^x \geq \frac{c}{x} \Rightarrow c \leq 2x\,e^x$

$(2xe^x)_{\min} = 0$; for $\forall\ x \in R^+$

Required $c \leq 0$

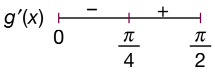
Hence, the correct answer is (D).

9. $f''(x) > 0$

$f'(4) = 0; f'(x)$ is an increasing function; $f'((\cot x - 1)^2 + 4) > 0$

$g(x) = f((\cot x - 1)^2 + 4)$

$0 < x < \frac{\pi}{2}$



$g(x) = f(\cot^2 x - 2\cot x + 5)$

$g'(x) = f'(\cot^2 x - 2\cot x + 5)(2\cot x - 2)(-\operatorname{cosec}^2 x)$

$= -2(\cot x - 1)f'((\cot x - 1)^2 + 4)\operatorname{cosec}^2 x$

$g(x)$ is increasing in $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and decreasing in $\left(0, \frac{\pi}{4}\right)$

Hence, the correct answer is (B).

10. $f(x) = x^5 - 2x^4 + 3x^3 + 100$

$f'(x) = 5x^4 - 8x^3 + 9x^2 = x^2(5x^2 - 8x + 9) \geq 0$

$f(x)$ is strictly increasing $x_1 > x_2 \Rightarrow f(x_1) > f(x_2)$

Hence, the correct answer is (A).

11. $\alpha, \beta \in \left(\frac{\pi}{2}, \pi\right)$

$\alpha < \beta \Rightarrow \frac{\alpha}{\beta} < 1$

$\alpha < \beta \Rightarrow \cos\alpha - \cos\beta > 0 \Rightarrow e^{\cos\alpha - \cos\beta} > e^{\circ} = 1$

$$\Rightarrow\; e^{\cos\alpha-\cos\beta} > 1 > \frac{\alpha}{\beta} \;\Rightarrow\; \frac{\beta}{\alpha} > e^{\cos\beta-\cos\alpha}$$

Hence, the correct answer is (B).

12. $f(x) = (a^2 - 3a + 2)\left(\cos\frac{x}{2}\right) + (a-1)x + \sin 1$

$f'(x) \ne 0$

$f'(x) = (a^2 - 3a + 2)\left(-\frac{1}{2}\sin\frac{x}{2}\right) + (a-1) \ne 0$

$(a-1)\,\frac{(a-2)}{2}\,\sin\frac{x}{2} \ne (a-1)$

$a \ne 1$ and $\sin\frac{x}{2} \ne \frac{2}{a-2}$ then $\frac{2}{a-2} < -1$ or $\frac{2}{a-2} > 1$

$\frac{2}{a-2} + 1 < 0$ or $\frac{2-a+2}{a-2} > 0$

$\frac{a}{a-2} < 0$ or $\frac{a-4}{a-2} < 0$

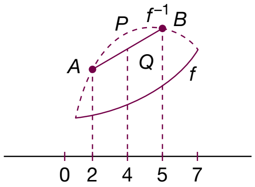
$a \in (0, 2) \cup a \in (2, 4)$

$a \in (0, 2) \cup a \in (2, 4) - \{1\}$

Maximum integral $a = 3$

Hence, the correct answer is (C).

13.



$f(x), f'(x), f''(x) > 0\ \forall\ x \in [0, 7]$

It means $f(x)$ is positive increasing function with concave upward

Let consider two points on curve $f^{-1}(x)$ are $A(2, f^{-1}(2))$, $B(5, f^{-1}(5))$

Now a point which divides AB in 2:1 ratio

$P(4, f^{-1}(4))$ is on curves and $Q\left(4, \frac{f^{-1}(2) + 2f^{-1}(5)}{3}\right)$ is on chord AB

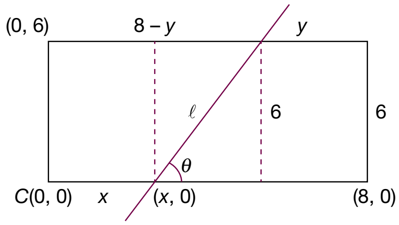
$y_Q < y_P$

$$\Rightarrow\; \frac{f^{-1}(2) + 2f^{-1}(5)}{3} < f^{-1}(4)$$

$$\Rightarrow\; 3f^{-1}(4) - f^{-1}(2) - 2f^{-1}(5) > 0$$

Hence, the correct answer is (A).

14.



Let line cuts x axis at $(x, 0)$ and opposite side at $(8 - y, 0)$

Area of both trapezium are equal

$$\frac{1}{2}(x + 8 - y)\,6 = \frac{1}{2}\,6(y + 8 - x)$$

$$\Rightarrow\; x + 8 - y = y + 8 - x \Rightarrow x = y$$

$$\sin\theta = \frac{6}{\ell} \;\Rightarrow\; \ell = 6 \text{ cosec } \theta$$

sum of perimeter of both parts $S = 2(x + 6 + 8 - y + \ell)$

$= 2(14 + \ell) = 2(14 + 6 \text{ cosec } \theta)$

As $\theta \in \left(\tan^{-1}\frac{3}{4}, \frac{\pi}{2}\right]$

$S_{\min} = 2(14 + 6(1)) = 40;$

$S_{\max} = 2\left(14 + 6\left(\frac{5}{3}\right)\right) = 2(14 + 10) = 48$

$S \in [40, 48]$

Hence, the correct answer is (D).

15. $f(x) = \dfrac{(x^3 + 3x) - |x^3 - x^2 + 2x - 2|}{x^2 + x + 2}$

$$= \frac{(x^3 + 3x) - |(x-1)(x^2+2)|}{x^2 + x + 2}$$

$$= \begin{cases} \dfrac{(x^3+3x) - (x-1)(x^2+2)}{x^2+x+2} & ;\ x \ge 1 \\[2ex] \dfrac{(x^3+3x) + (x-1)(x^2+2)}{x^2+x+2} & ;\ x < 1 \end{cases}$$

$$f(x) = \begin{cases} 1 & ;\ x \ge 1 \\ 2x - 3 + \dfrac{4(x+1)}{x^2+x+2} & ;\ x < 1 \end{cases}$$

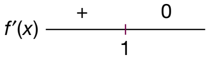
for $x < 1$

$$f'(x) = 2 + \frac{4[(x^2+x+2)1 - (x+1)(2x+1)]}{(x^2+x+2)^2}$$

$$= 2 + 4\frac{(-x^2 - 2x + 1)}{(x^2+x+2)^2} = \frac{2[x^4 + 2x^3 + 3x^2 + 6]}{(x^2+x+2)^2} > 0$$

$f(x)_{\max}$ at $x = 1$

$f(1) = \dfrac{4 - (0)}{4} = 1$



Hence, the correct answer is (D).

16. $f(x) = \sin x + x \cos x$ continuous function

$$f\left(-\frac{\pi}{2}\right) = -1;\ f(0) = 0;\ f\left(\frac{\pi}{2}\right) = 1;\ f(\pi) = -\pi;\ f\left(\frac{3\pi}{2}\right) = -1$$

From intermediate mean value theorem as $f\left(\frac{\pi}{2}\right) f(\pi) < 0$

So $f(x) = 0$ for atleast once in $x \in \left(\frac{\pi}{2}, \pi\right)$

So, $x \in (0, \pi)$ atleast one root

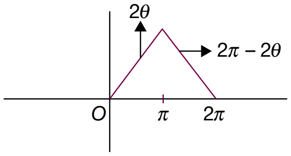
Hence, the correct answer is (B).

17. $f(x) = \cos^{-1}(2x^2 - 1)$

$x = \cos\theta$

$= \cos^{-1}(\cos 2\theta)$

$\theta \in [0, \pi]$



$$f(x) = \begin{cases} 2\theta & ;\ 0 \le 2\theta \le \pi \Rightarrow 0 \le \theta \le \dfrac{\pi}{2} \Rightarrow x \in [0, 1] \\[2ex] 2\pi - 2\theta & ;\ \pi < 2\theta \le 2\pi \Rightarrow \dfrac{\pi}{2} < \theta \le \pi \Rightarrow x \in [-1, 0) \end{cases}$$

$$f(x) = \begin{cases} 2\cos^{-1} x & ;\ x \in [0, 1] \\ 2\pi - 2\cos^{-1} x & ;\ x \in [-1, 0) \end{cases}$$

At $x = 0$, local maxima, non-monotonic, not diff. at $x = 0$, even function

Hence, the correct answer is (B).

18. $f'(x) = 2 - \dfrac{1}{1+x^2} - \dfrac{1}{\sqrt{1+x^2}} = \dfrac{x^2}{1+x^2} + \left(1 - \dfrac{1}{\sqrt{1+x^2}}\right)$

Both terms are positive $\forall\ x \in R$

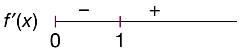
$f(x)$ is increasing in R

Hence, the correct answer is (C).

19. $f(x) = x \ln x - x + 1$

$$f'(x) = \ln x + x\left(\frac{1}{x}\right) - 1$$

$$f'(x) = \ln x$$



At $x = 1$ is minima of $f(x)$

$$f(x) \geq f(1)$$
$$f(x) \geq \ln 1 - 1 + 1$$
$$f(x) \geq 0 \ \forall \ x \in R^+$$
$$f(x) > 0 \text{ for } x \ (0, 1) \cup (1, \infty)$$

Hence, the correct answer is (D).

20. Maximum of $f(x) = \sec^{-1}\left(\frac{7 - 5(x^2 + 3)}{2(x^2 + 2)}\right)$

$$= \sec^{-1}\left(\frac{1}{x^2 + 2} - \frac{5}{2}\right)_{\max.}$$

Minimum of $\left(\frac{1}{x^2 + 2}\right)_{\min.} = \frac{1}{2}$

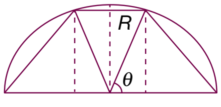
Max of $f(x) = \sec^{-1}\left(\frac{1}{2} - \frac{5}{2}\right) = \sec^{-1}(-2) = \frac{2\pi}{3}$

Hence, the correct answer is (D).

21. Area of trapezium $A = \frac{1}{2}$ (height)(sum of parallel sides)

$$A = \frac{1}{2}(R \sin\theta)(2R\cos\theta + 2R) = R^2(\cos\theta\sin\theta + \sin\theta)$$

$$\frac{dA}{d\theta} = R^2(-\sin^2\theta + \cos^2\theta + \cos\theta)$$



Rcos θ Rcos θ

$\Rightarrow \ 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \ (2\cos\theta - 1)(\cos\theta + 1) = 0,$

But $\cos\theta = -1$ is not possible

$\Rightarrow \ \cos\theta = \frac{1}{2}$, we get $\theta = 60°$

$$\frac{d^2A}{d\theta^2} = R^2(-4\cos\theta\sin\theta - \sin\theta) < 0 \text{ for } \theta = 60°$$

Largest area $A = R^2\sin\theta(\cos\theta + 1) = R^2\frac{\sqrt{3}}{2}\left(\frac{3}{2}\right) = \frac{3\sqrt{3}}{4}R^2$

Hence, the correct answer is (A).

22. $f(x) = \cos|-x| - 2ax + b \ \forall \ x \in R$

$$f(x) = \cos x - 2ax + b$$
$$f'(x) = -\sin x - 2a \geq 0 \Rightarrow \sin x + 2a \leq 0$$
$$2a \leq -\sin x \text{ for all real } x$$
$$2a \leq -1 \Rightarrow a \leq -1/2$$

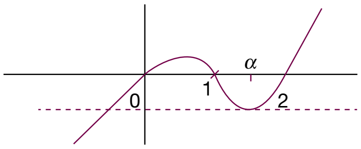
Hence, the correct answer is (A).

23. $f(x) = x^3 - 3x^2 + 2x = x(x - 1)(x - 2)$

$f'(x) = 3x^2 - 6x + 2$

$$x = 1 \pm \frac{1}{\sqrt{3}}$$

$$x = 1 + \frac{1}{\sqrt{3}} = \alpha$$



Equation $f(x) = k$ has exactly one positive and one negative solution

$$k = f(\alpha) = \left(1 + \frac{1}{\sqrt{3}}\right)^3 - 3\left(1 + \frac{1}{\sqrt{3}}\right)^2 + 2\left(1 + \frac{1}{\sqrt{3}}\right)$$

$$K = -\frac{2\sqrt{3}}{9}$$

Hence, the correct answer is (A).

24. $h(x) = -\log_2(x^2 - 2x - 3) \quad (\because (x - 3)(x + 1) > 0) \Rightarrow x \ (-\infty, -1) \cup (3, \infty)$

$$h'(x) = -\frac{1}{\ln 2}\left(\frac{1}{x^2 - 2x - 3}\right) = \frac{-2(x - 1)}{\ln 2\underset{\oplus}{(x^2 - 2x - 3)}}$$

$h(x)$ is increasing in $x \in (-\infty, -1)$

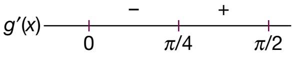
$h(x)$ is decreasing in $x \in (3, \infty)$

Hence, the correct answer is (B).

25. $f''(x) > 0; \ \forall \ x \in R$

It means $f'(x)$ is an increasing function

$$f'(3) = 0; f'(x) \geq 0 \text{ if } x \geq 3$$
$$g(x) = f((\tan x - 1)^2 + 3)$$
$$0 < x < \frac{\pi}{2}$$

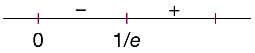


$$g'(x) = f'(\tan^2 x - 2\tan x + 4)(2\tan x \sec^2 x - 2\sec^2 x)$$
$$= f'(\tan x - 1)^2 + 3) \, 2\sec^2 x(\tan x - 1)$$

For $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, $g'(x) > 0$ so $g(x)$ is increasing

Hence, the correct answer is (D).

26. $f(x) = x \ln x$



$$f'(x) = \ln x + x\left(\frac{1}{x}\right) = \ln x + 1$$

$f(x)$ is not Monotonic and increasing for $x \in \left(\frac{1}{e}, \infty\right)$ and decreasing for $x \in (0, 1/e)$

S-2 :If f and g both are increasing

$$y = f(x) \cdot g(x)$$

$$\frac{dy}{dx} = \underset{\oplus}{f'(x)} \ \underset{-,+}{g(x)} + \underset{-|+}{f(x)} \ \underset{\oplus}{g'(x)}$$

we cannot decide over all sign

Let $f(x) = e^x$ and $g(x) = e^x - 1$, both are increasing

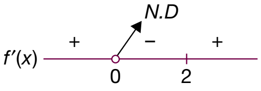
$$h(x) = f(x)g(x) = e^x(e^x - 1)$$

$h'(x) = e^x(e^x - 1) + e^x(e^x) = e^x(2e^x - 1)$ is not always positive

$h(x)$ is non-monotonic

Hence, the correct answer is (C).

27. $f(x) = \frac{e^x}{x^2} \ \ x \in R - \{0\}$



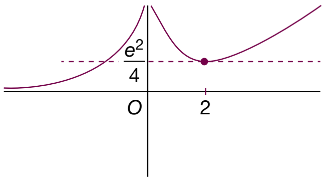
$$f'(x) = \frac{x^2e^x - e^x2x}{x^4} = \frac{e^x(x - 2)}{x^3}$$

As $x \to -\infty, f(x) = \frac{e^x}{x^2} \to 0$

As $x \to 0^+, f(x) = \frac{e^x}{x^2} \to \infty$

As $x \to 0^-, f(x) = \frac{e^x}{x^2} \to \infty;$

$x \to \infty, f(x) = \frac{e^x}{x^2} \to +\infty$



$f(2) = e^2/4$; using above information graph of $f(x)$ is drawn now for equation $f(x) = k$ to have two solution $K = \frac{e^2}{4}$

Hence, the correct answer is (C).

28. $f(x) = 2e^{2x} - 4ae^x - (4a - 5)x$

$$f'(x) = 4e^{2x} - 4ae^x - (4a - 5)$$
$$= 4(e^x)^2 - 4a(e^x) + 5 - 4a \geq 0$$

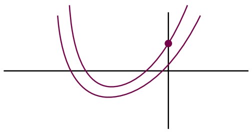
Let $e^x = t$, $g(t) = 4t^2 - 4at + 5 - 4a \geq 0$ where $t \in (0, \infty)$

CASE 1: both roots of $g(t)$ are imaginary or real and equal

$D \leq 0$

$\Rightarrow \ (4a)^2 - 4(4)(5 - 4a) \leq 0 \Rightarrow a^2 - 5 + 4a \leq 0$

$\Rightarrow \ a^2 + 4a - 5 \leq 0 \Rightarrow (a + 5)(a - 1) \leq 0 \Rightarrow a \in [-5, 1]$ …(I)



CASE 2: Both roots of $g(t) = 4t^2 - 4at + 5 - 4a$ are negative

Sum of the roots is negative $= \dfrac{-B}{A} < 0$

$\Rightarrow \quad \dfrac{4a}{4} < 0 \Rightarrow a < 0 \qquad \ldots(1)$

Product of roots is positive $g(0) = f'(0) \geq 0$

$\Rightarrow \quad 5 - 4a \geq 0 \Rightarrow a \leq \dfrac{5}{4} \qquad \ldots(2)$

Real roots of $g(t)$, $D \geq 0 \Rightarrow a\ (-\infty, -5] \cup [1, \infty) \qquad \ldots(3)$

$(1) \cap (2) \cap (3)$, we obtain

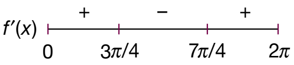
$a \in (-\infty, -5] \qquad \ldots(\text{II})$

Final set of values of a, we have CASE 1 $\cup$ CASE 2

$a \in (-\infty, -5] \cup [-5, 1]$ or $a \leq 1$

Hence, the correct answer is (B).

29. $f(x) = \displaystyle\int_{\cos x}^{\sin x} (1 - t + 2t^3)\, dt$



$x \in [0, 2\pi]$

$f'(x) = (1 - \sin x + 2\sin^3 x)\cos x - (1 - \cos x + 2\cos^3 x)(-\sin x)$

$= \cos x - \sin x \cos x + 2\sin^3 x \cos x + \sin x - \cos x \sin x + 2\cos^3 x \sin x$

$= \cos x + \sin x - 2\sin x \cos x + 2 \sin x \cos x (\sin^2 x + \cos^2 x)$

$f'(x) = \cos x + \sin x$

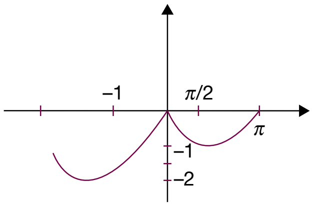
Critical points are $x = \dfrac{3\pi}{4}, \dfrac{7\pi}{4}$

$x = \dfrac{3\pi}{4}$ is point of maxima,

$x = \dfrac{7\pi}{4}$ is point of minima

Hence, the correct answer is (B).

30.



Using the definition of $f(x)$, graph of $f(x)$ is drawn

Global max. $f(x) = 0$ at point $x = 0, \pi$

Global minimum of $f(x)$ at point $(-1, -2)$

non-monotonic

Hence, the correct answer is (D).

31. $f(x) = x^3 - 3x^2 + 6x + 2006$

$f'(x) = 3x^2 - 6x + 6 = 3(x^2 - 2x + 2) = 3((x-1)^2 + 1) > 0\ \forall\ x \in R$

$f(x)$ is strictly increasing (monotonic)

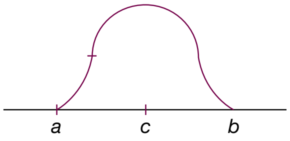
$f(x)$ is one-one, onto

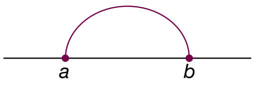
$f''(x) = 6x - 6 = 0;\ x = 1$ (point of inflection)

Slope of tangent $= m_T = f'(1) = 3(1 - 2 + 2) = 3$

Hence, the correct answer is (D).

32. $f(a) = f(b) = 0$





$f(x) > 0\ \forall\ x \in (a, b)$

$f''(c) < 0$ for some $c \in (a, b)$

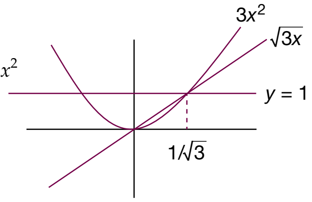
Hence, the correct answer is (A).

33. $g(x) = f(x) - x$

$h(x) = f(x) - x^3$

$g'(x) = f'(x) - 1 \geq 0 \Rightarrow f'(x) \geq 1$

$h'(x) = f'(x) - 3x^2 \geq 0 \Rightarrow f'(x) \geq 3x^2$



$F(x) = f(x) - \dfrac{\sqrt{3}x^2}{2}$

$F'(x) = f'(x) - \sqrt{3}\, x$

If $x < \dfrac{1}{\sqrt{3}} \Rightarrow 1 > \sqrt{3}x$, $F'(x) = f'(x) - \sqrt{3}x > f'(x) - 1 \geq 0$

If $x \geq \sqrt{3} \Rightarrow 3x^2 > \sqrt{3}x$, $F'(x) = f'(x) - \sqrt{3}x > f'(x) - 3x^2 > 0$

$\therefore \quad F(x)$ is SMI for all x

Hence, the correct answer is (A).

34. $f(x) = \ln(2 + x) - \left(\dfrac{2x+2}{x+3}\right) = \ln(2 + x) - \left(\dfrac{2(x+3) - 4}{x+3}\right)$

$f(x) = \ln(2 + x) - 2 + \dfrac{4}{x+3}$

$f'(x) = \dfrac{1}{x+2} - \dfrac{4}{(x+3)^2} = \dfrac{x^2 + 9 + 6x - 4x - 8}{(x+2)(x+3)^2}$

$= \dfrac{(x+1)^2}{(x+2)(x+3)^2} > 0$

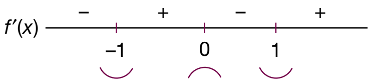
$f(x)$ is increasing and $f(-1) = 0$

S2: it is false if $f(x)$ does not cross x axis

Let $f(x) = 2^x$ defined in [0,5] is SMI and positive but it not zero

Hence, the correct answer is (C).

35. $f'(x) = \dfrac{x^4 - 1}{e^{x^2+1}} \cdot 2x$



$f'(x) = \dfrac{2x(x-1)(x+1)(x^2+1)}{e^{x^2}+1}$

$f(x)$ has 2 local minima at $x = -1, 1$ and one local maxima at $x = 0$

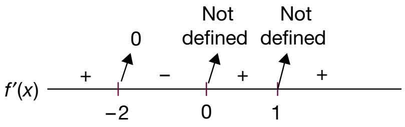
As $f(x)$ is twice differentiable function and $f'(x)$ changes sign around $x = 0$ so it cannot be point inflexion

Hence, the correct answer is (C).

36. $f(x) = \begin{cases} \sqrt{x} & x \geq 1 \\ x^3 & 0 \leq x \leq 1 \\ \dfrac{x^3}{3} - 4x & x < 0 \end{cases}$

$f(x)$ is a continuous function

$f'(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & x > 1 \\ 3x^2 & 0 < x < 1 \\ x^2 - 4 & x < 0 \end{cases}$



$f(x)$ is not differentiable at $x = 0, 1$

Hence, the correct answer is (C).

37. $f'(x) = 12x^3 - 12x^2 + 12x + a = 12x(x^2 - x + 1) + a$ for all $a, b \in R$

$f'(x)$ is cubic, it must have atleast one real root so $f(x)$ has atleast one extrema.

Hence, the correct answer is (B).

38. $$f(x) = \begin{cases} \dfrac{(x-1)(6x-1)}{2x-1} & x \neq \dfrac{1}{2} \\ 0 & x = \dfrac{1}{2} \end{cases}$$

$$f\left(\frac{1}{2}\right) = 0;\ f\left(\frac{1^+}{2}\right) \to -\infty;\ f\left(\frac{1^-}{2}\right) \to +\infty$$

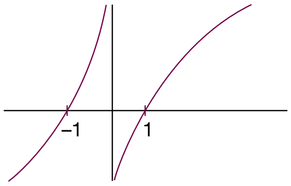
At point $x = \frac{1}{2}$, $f(x)$ is discontinuous (non-removable discontinuity)

neither maxima nor minima

Hence, the correct answer is (C).

39. $f(x) = x - \dfrac{1}{x}$

$$f'(x) = 1 + \frac{1}{x^2} > 0$$



$f(x)$ increasing if $x < 0$ and $x > 0$

$f(x)$ is many-one so not invertible

line $x = 0$ is asymptotes

$$m = \lim_{x\to\infty} \frac{y}{x} = \lim_{x\to\infty} \frac{x - \frac{1}{x}}{x} = \lim_{x\to\infty} 1 - \frac{1}{x^2} = 1$$

$$C = \lim_{x\to\infty} (y - mx) = \lim_{x\to\infty} \left(x - \frac{1}{x} - x\right) = 0$$

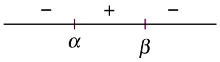
Line $y = x$ and $x = 0$ are two asymptotes

Hence, the correct answer is (D).

40. $f(x) = \dfrac{t + 3x - x^2}{x - 4}$

$$f'(x) = \frac{(x-4)(3-2x) - (t+3x-x^2)(1-0)}{(x-4)^2} = \frac{-x^2 + 8x - 12 - t}{(x-4)^2}$$

For having extrema $f'(x)$ must change its sign, It means quadratic $-x^2 + 8x - 12 - t$



Must have distinct roots so $D > 0$

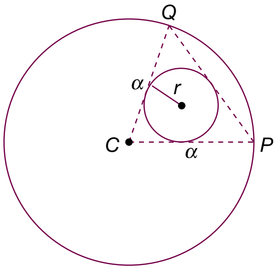
$\Rightarrow\ 64 - 4(-1)(-12 - t) > 0 \Rightarrow 16 - 12 - t > 0$

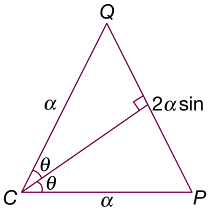
$\Rightarrow\ 4 - t > 0 \Rightarrow t < 4$

Hence, the correct answer is (C).

41. $$r = \frac{\Delta}{S} = \frac{\frac{1}{2}\cdot\alpha\cdot\alpha\sin 2\theta}{\left(\frac{\alpha + \alpha + 2\alpha\sin\theta}{2}\right)}$$

$$\Rightarrow\ r = \frac{\alpha\sin\theta\cos\theta}{1 + \sin\theta}$$





$$\frac{dr}{d\theta} = \alpha\,\frac{(1+\sin\theta)(\cos^2\theta - \sin^2\theta) - \sin\theta\cos\theta\cos\theta}{(1+\sin\theta)^2} = 0$$

$\Rightarrow\ (1 + \sin\theta)\,[\cos^2\theta - \sin^2\theta - \sin\theta\,(1 - \sin\theta)] = 0$

$\Rightarrow\ \cos^2\theta - \sin\theta = 0 \Rightarrow \sin^2\theta + \sin\theta - 1 = 0$

$$\Rightarrow\ \sin\theta = \frac{-1 \pm \sqrt{5}}{2}$$

$$\sin\theta = \frac{\sqrt{5} - 1}{2} \text{ is acceptable}$$

Hence, the correct answer is (B).

42. $f(x) + f''(x) = -x|\sin x|f'(x)$

$$f'(x)f(x) + f'(x)f''(x) = -x|\sin x|(f'(x))^2$$

$$\frac{1}{2}\frac{d}{dx}\left((f(x))^2 + (f'(x))^2\right) = -x|\sin x|(f'(x))^2$$

$$\frac{1}{2}\frac{d}{dx}(F(x)) = -x|\sin x|(f'(x))^2;\ \text{as}\ x \geq 0$$

$F'(x) < 0 \Rightarrow f(x)$ is decreasing f^n

$f \geq 0$

$F(x) \leq F(0);\ (f(x))^2 + (f'(x))^2 \leq (f(0))^2 + (f'(0))^2$

$\leq 9 + 16$

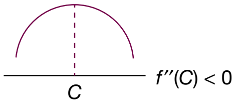
≤ 25

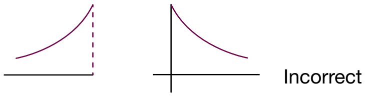
$(f(x))^2 + (f'(x))^2 \leq 25;\ (f(x))^2 \leq 25 - (f'(x))^2$

$\Rightarrow$ Maximum value of $f(x)$ is 5

Hence, the correct answer is (C).

43. (I)





(II) $\sin^2 x$ is periodic function, $\displaystyle\int \sin^2 x\,dx = \int \frac{1 - \cos 2x}{2}\,dx$

$$\Rightarrow\ \frac{x}{2} - \frac{\sin 2x}{4} + C \text{ (aperiodic)}$$

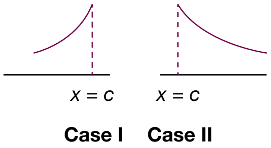
(III) $\displaystyle\int_0^T f(x + a)\,dx$

Put $x + a = t \Rightarrow dx = dt$

$$\Rightarrow\ \int_a^{a+T} f(t)\,dt = \int_0^T f(x)\,d\alpha \quad \text{(True)}$$

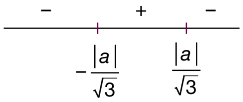
(IV) Maximum can occur at boundary point also, then it can be increasing at CASE I only

It can be decreasing like CASE II only



Hence, the correct answer is (A).

44. $\dfrac{x^2 + x + 2}{(x+2)(x+3)} < 0 \Rightarrow x \in (-3, -2)$



$f(x) = 1 + a^2x - x^3$

$\Rightarrow\ f'(x) = a^2 - 3x^2$

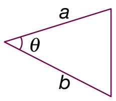
Point of minima is $x = -\dfrac{|a|}{\sqrt{3}}$ is lying between –3 to –2

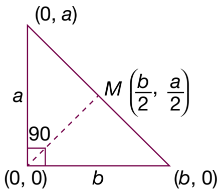
$$-3 < -\frac{|a|}{\sqrt{3}} < -2 \Rightarrow 3\sqrt{3} > |a| > 2\sqrt{3}$$

$$\Rightarrow \; a \in (-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$$

Hence, the correct answer is (D).

45. $A = \frac{1}{2} ab \sin\theta$





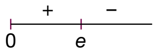
$A_{\max} \Rightarrow \theta = 90°$

$A = \frac{1}{2} ab$

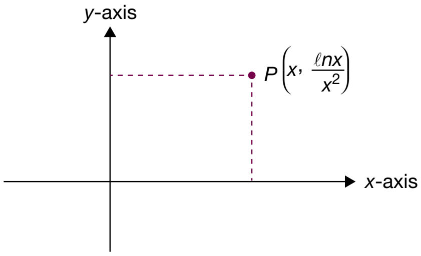
Required median $= \frac{\sqrt{b^2 + a^2}}{2}$

Hence, the correct answer is (A).

46. Area of rectangle $A = x \cdot \frac{\ln x}{x^2} = \frac{\ln x}{x}$



$$\frac{dA}{dx} = \frac{x\left(\frac{1}{x}\right) - \ln x(1)}{x^2} = \frac{1 - \ln x}{x^2}$$



Point $x = e$ is point of maxima

Greatest area $A(e) = \frac{\ln e}{e} = \frac{1}{e}$

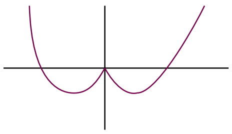
Hence, the correct answer is (A).

47. $f(x) = ax^2 - b|x| = a|x|^2 - b\,|x|$ where a and b constant

$f(x) = 0 \Rightarrow |x| = 0, \frac{b}{a}$

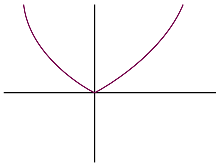
$$f(x) = \begin{cases} ax^2 - bx & x \geq 0 \\ ax^2 - bx & x < 0 \end{cases}$$

If $a > 0$, $b > 0$ then roots are $x = 0, \frac{b}{a}, -\frac{b}{a}$



At $x = 0$ Local maxima

If $a > 0$, $b < 0$ then root is $x = 0$ only

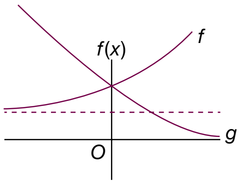


At $x = 0$ Local minima

Hence, the correct answer is (A).

48. $f(x) = 2^x + 1$

$f'(x) = 2^x \ln 2 > 0$



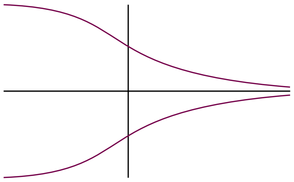
$f(x)$ is strictly increasing

$g(x) = 4^{-x} + 2^{-x}$

$g'(x) = -4^{-x} \ln 4 - 2^{-x} \ln 2$

$= -\ln 2(4^{-x}2 + 2^{-x}) < 0$

$g(x)$ is strictly decreasing



$f(x) = g(x)$ one solution

S-2: $f = \cot^{-1}x$ is decreasing

$g = \tan^{-1}x - \frac{\pi}{2}$ is increasing

$f(x) = g(x)$ has no point of intersection

Hence, the correct answer is (C).

49. (A) $x \in (-\infty, -1]$

$f(x) = 3[x]\{x\} - 1 - [x]^3 - \{x\}^3$

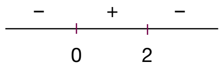
$= -([x]^3 + \{x\}^3 - 3[x]\{x\} + 1)$

$= -([x] + \{x\})^3 - 3[x]\{x\}([x] + \{x\}) - 3[x]\{x\} + 1)$

$= -x^3 + 3[x]\{x\}(x + 1) - 1$

$= 3[x]\{x\}(x + 1) - (x^3 + 1) \geq 0$ as $x \leq -1$

$f(-1) = -(-1 + 0 + 0 + 1) = 0$

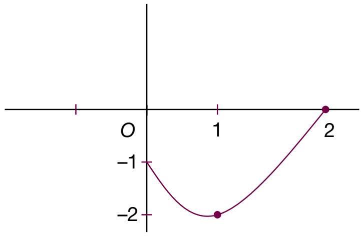


As $x \in (-2, -1]$ then $[x] = -2$

$f(x) = -(-8 + \{x\}^3 + 6\{x\} + 1) > 0$

$f(x) \geq 0$ if $x \in (-\infty, -1]$

(B)



If $x \in (0, 1)$

$[x] = 0$ and $\{x\} = x$

$f(x) = 3[x]\{x\} - 1 - [x]^3 - \{x\}^3$

$f(x) = -1 - x^3$

$f(0^+) = -1, f(1^-) = -2$

then $f(x) \neq 0$ for intermediate values

(C) If $x \in [0, 1)$ then $[x] = 0$ and $\{x\} = x$

$f(x) = -1 - x^3$

Range $(-2, -1]$

(D) When $x \in [1, 2)$ then $[x] = 1$ and $\{x\} = x - 1$

$f(x) = 3[x]\{x\} - 1 - [x]^3 - \{x\}^3$

$= 3\{x\} - 1 - 1 - \{x\}^3 = 3(x - 1) - 2 - (x - 1)^3$

$f(x) = -x^3 + 3x^2 - 4$

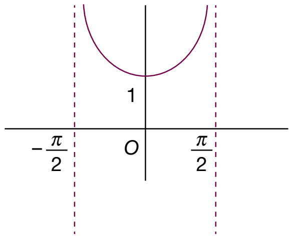
$f'(x) = -3x^2 + 6x = -3x(x - 2)$

$f(1) = -1 + 3 - 4 = -2$ and $f(2) = -8 + 12 - 4 = 0$

Exactly are minima in $x \in [1, 2)$

Hence, the correct answer is (B).

50. **(a)**



$f(x) = \sec x$ has minima at $x = 0$ (False)

(b) True

(c) $f(x) = x^5 - 10x + \sin^5 x$

$f(1) = \sin^5 1 - 9 < 0$

$f(2) = 32 - 20 + \sin^5 2 > 0$

From IMT $f(x) = 0$ for non-zero positive values True

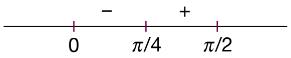
(d) True

(e) depends on higher derivative test (false)

(a, e) → False and bcd → True

Hence, the correct answer is (C).

51. $f(x) = \int_0^x \frac{d\theta}{\cos\theta} + \int_x^{\pi/2} \frac{d\theta}{\sin\theta}$



$x \in \left(0, \frac{\pi}{2}\right)$

$$f'(x) = \left(\frac{1}{\cos x} - 0\right) + \left(0 - \frac{1}{\sin x}\right) = \frac{\sin x - \cos x}{\sin x \cos x}$$

$$f\left(\frac{\pi}{4}\right) = \int_0^{\pi/4} \sec\theta\, d\theta + \int_{\pi/4}^{\pi/2} \operatorname{cosec}\theta\, d\theta$$

$$= \ln(\sec\theta + \tan\theta)\Big|_0^{\pi/4} + \ln(\operatorname{cosec}\theta - \cot\theta)\Big|_{\pi/4}^{\pi/2}$$

$$= \ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right)$$

$$= \ln\left(\frac{(\sqrt{2}+1)\,(\sqrt{2}+1)}{(\sqrt{2}-1)\,(\sqrt{2}+1)}\right) = \ln\left(\frac{(\sqrt{2}+1)^2}{2-1}\right) = 2\ln\left(\sqrt{2}+1\right)$$

Hence, the correct answer is (A).

52. At $x = 1$ is point of minima

So we have $f(1^-) \geq f(1)$

$\Rightarrow\ -1 + \cos^{-1}a \geq 1 \Rightarrow \cos^{-1}a \geq 2$ as $0 \leq \cos^{-1}a \leq \pi$

then $2 \leq \cos^{-1}a \leq \pi$

$\cos 2 \geq a \geq -1$

Hence, the correct answer is (B).

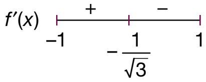
53. $f(x) = \cos^{-1}x^2 + \cos^{-1}x$; domain $x \in [-1, 1]$

$$f'(x) = \frac{-1}{\sqrt{1-x^4}} \cdot 2x - \frac{1}{\sqrt{1-x^2}} = \frac{-(2x+\sqrt{1+x^2})}{\sqrt{1-x^4}}$$

for $x > 0, f'(x) < 0$

for $x < 0$, $-2x = \sqrt{1+x^2}$ or $4x^2 = 1 + x^2$ or $x = -\frac{1}{\sqrt{3}}$

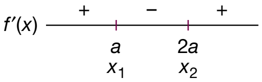
only one local maxima $x = -\frac{1}{\sqrt{3}}$



2 local maxima at $x = -1, 1$

Hence, the correct answer is (C).

54. $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$



$f'(x) = 6x^2 - 18ax + 12a^2$

$= 6(x^2 - 3ax + 2a^2)$

$f'(x) = 6(x - 2a)(x - a)$

$x_1 = a$ (a point of maxima); $x_2 = 2a$ (point of minima)

$x_1^2 = x_2 \Rightarrow a^2 = 2a \Rightarrow a = 2$

Hence, the correct answer is (B).

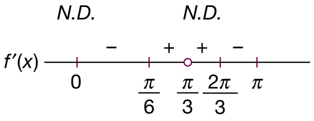
55. Given function is periodic with π

$$f(x) = \frac{\tan\left(x + \frac{\pi}{6}\right)}{\tan x} = \left(\frac{\tan x + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}\tan x}\right)\frac{1}{\tan x}$$

$$= \frac{\sqrt{3}\tan x + 1}{\tan x(\sqrt{3} - \tan x)}$$

$$f'(x) = \frac{\left[(\sqrt{3}\tan x - \tan^2 x)\sqrt{3}\sec^2 x - (\sqrt{3}\tan x + 1)(\sqrt{3}\sec^2 x - 2\tan x \sec^2 x)\right]}{(\sqrt{3}\tan x - \tan^2 x)^2}$$

$$f'(x) = \frac{(\tan x + \sqrt{3})(\sqrt{3}\tan x - 1)}{\tan^2 x(\sqrt{3} - \tan x)^2}\ \sec^2 x$$

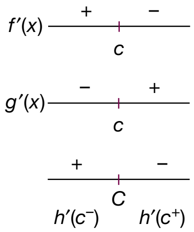


$$f\left(\frac{\pi}{6}\right) = \frac{\tan 60°}{\tan 30°} = 3 \text{ Local minima}$$

$$f\left(\frac{2\pi}{3}\right) = \frac{\tan 150°}{\tan 120°} = \frac{-\frac{1}{\sqrt{3}}}{-\sqrt{3}} = \frac{1}{3} \text{ Local maxima}$$

Hence, the correct answer is (D).

56. $f : R \to R^+$



$g : R \to R^+$

$f(x), g(x) > 0$

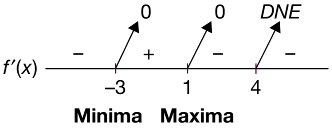
$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

For $h(x)$ at $x = c$, local maxima

Hence, the correct answer is (A).

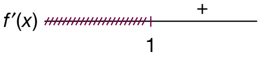
57.



$f(4)$ Not defined

Hence, the correct answer is (D).

58. $f'(x) = x\ln x - \frac{\ln x}{x} \cdot 1 - 0$

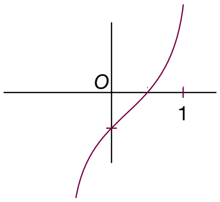


$$= \ln x\left(\frac{(x-1)(x+1)}{x}\right) > 0$$

$f(x)$ is increasing for all $x > 1$, it has no extreme, as it is monotonic

Hence, the correct answer is (D).

59. $P(x) = 2x^3 + x^2 + 3x - 2$



$P'(x) = 6x^2 + 2x + 3 > 0$ for $\forall\, x \in R$

As $D = B^2 - 4AC < 0$ so $P(x)$ is strictly increasing

$P(1) = 2 + 1 + 3 - 2 > 0$

$P(0) = -2 < 0$

$P(x) = 0$ has exactly one positive root.

(i) (iii)

Hence, the correct answer is (C).

60. S-1: by Intermediate Mean value Theorem, it is True (True)

S-2: $f'(x) = g'(x) \Rightarrow f(x) = g(x) + C$ (False)

S-3: It is Rolles theorem conclusion (True)

Hence, the correct answer is (D).

61. $f(\theta) = \frac{\sin\theta}{\theta} + \frac{\theta}{\tan\theta}$, $\theta \in \left(0, \frac{\pi}{2}\right)$

$$f'(\theta) = \frac{-\sin\theta}{\theta^2} + \frac{\cos\theta}{\theta} + \frac{1}{\tan\theta} - \frac{\theta}{\tan^2\theta}\ \sec^2\theta$$

$$= \frac{-\sin\theta}{\theta^2} + \frac{\cos\theta}{\theta} + \frac{\cos\theta}{\sin\theta} - \frac{\theta}{\sin^2\theta} \text{ on simplification}$$

$$= \frac{(\sin\theta+\theta)(\sin\theta\cos\theta\,\overset{-}{\overparen{(\theta-\tan\theta)}}+\theta\,\overset{-}{\overparen{(\theta-\sin\theta)}})}{\theta^2\sin^2\theta} < 0$$

As $\sin\theta < \theta < \tan\theta$

$f(\theta)$ is decreasing

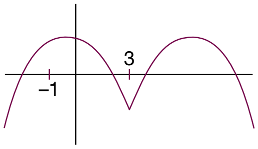
$$f(0^+) = \lim_{\theta\to 0} \frac{\sin\theta}{\theta} + \frac{\theta}{\tan\theta} = 1 + 1 = 2$$

$$f\left(\frac{\pi}{2}^-\right) = \frac{1}{\frac{\pi}{2}} + 0 = \frac{2}{\pi}$$

Range $\in \left(\frac{2}{\pi}, 2\right)$

Hence, the correct answer is (D).

62. As given relative maxima at point (–1, 4) and relative minima at point (3, –2)

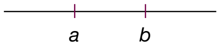


As it is continuous, It definitely cuts both axes or having extrema. Its not necessary to have derivative zero (horizontal tangent) at that point.

Possible situation can be shown in diagram, so it is not necessary that Point of Inflexion does not exist in between –1 and 3.

Hence, the correct answer is (D).

63. $f(x)$ is continuous and differentiable and $a < b$, $f(a) < 0$, $f(b) > 0$



(i) As function is continuous and differentiable at boundary points also so it have finite bound range (True)

(ii) From IMT, $f(x) = 0$ for altleast one x (True)

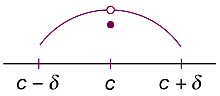
(iii) boundary point may be point of extrema (False)

(iv) As $f(a) < 0$ and $f(b) > 0$, then there exist atleast one point for which $f'(c) > 0$ (True)

(v) In given condition if $f(x)$ is Strictly increasing then $f'(d) < 0$ not possible (False)

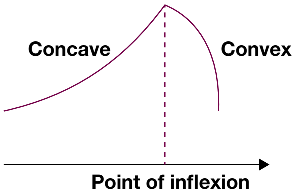
Hence, the correct answer is (D).

64. S-1: False, it depends on value of $f(c)$



As shown in diagram point $x = c$ is minima

S-2: false, from diagram we can see that point of inflection and local maxima are at same point



S-3: $f(x) = \begin{cases} x^3 & x \ge 0 \\ -x^3 & x < 0 \end{cases}$;

$$f'(x) = \begin{cases} 3x^2 & x \ge 0 \\ -3x^2 & x < 0 \end{cases}$$

$$f''(x) = \begin{cases} 6x & x \ge 0 \\ -6x & x < 0 \end{cases}$$

$f''(x) \ge 0$

$f(x)$ is always concave so no point of inflection False

S-4: $f : [c-1, c+1] \to [a, b]$ bijective

$f^{-1} = g$

$gof(x) = x$

$$g'(f(x)) = \frac{1}{f'(x)}$$

f^{-1} is diff if $f'(x) \ne 0$

False

Hence, the correct answer is (A).

65. Since, the given function is continuous but not differentiable at $x = 0$ and

$$Lf'(0) = -3,\ Rf'(0) = 2$$

$\Rightarrow$ $x = 0$ is point of local minima and $f(0) = 1$

Also, there is no other critical point

Now, $f(-2) = 11$, $f(2) = 4 + \cos 2$ on comparing values of function at $x = -2, 0, 2$

Global maximum occurs at $x = -2 \Rightarrow f(-2) = 11$

Hence, global maximum and local minimum values are 11, 1

Hence, the correct answer is (B).

66. $\lim_{x\to k} f(x) = 3 \Rightarrow f(k) = 3$

$$f(k^-) > f(k) \text{ and } f(k^+) > f(k)$$

$\Rightarrow$ $a^2 - 2 + 1 > 3$

$|a| > 2$.

Hence, the correct answer is (C).

67. $f'(x) = x^2 + x + 1 > 0$

$f(x)$ is increasing then check the local minima of $g(x)$.

$g'(x) = x^3 - 6x^2 + 11x - 6$; $g''(x) = 3x^2 - 12x + 11$

$g'(x) = (x - 1)(x - 2)(x - 3)$ $\qquad \because g''(1) > 0$

$g''(2) < 0$

$g''(3) > 0$

So, local minima at $x = 1$ and $x = 3$

Hence, the correct answer is (D).

68. $f(a) = \int_{a-1}^{a+1} \frac{1}{1+x^4}dx$

$$f'(a) = \frac{1}{1+(a+1)^4} - \frac{1}{1+(a-1)^4} = 0 \text{ at } a = 0$$

$a > 1 \Rightarrow a + 1 > a - 1$

$$\frac{1}{1+(a+1)^4} < \frac{1}{1+(a-1)^4}$$

$f'(0+)$ is –ve; $f'(0-)$ is +ve

$\therefore$ $f(a)$ has a local maximum at $a = 0$

Hence, the correct answer is (D).

69. $f(x) + f''(x) = -xg(x)f'(x)$

Let $h(x) = f^2(x) + (f'(x))^2$

$\Rightarrow$ $h'(x) = 2f(x)f'(x) + 2f'(x)f''(x)$

$= 2f'(x)(-x)g(x)f'(x)$

$= -2x(f'(x))^2 g(x)$

$\Rightarrow$ $h'(x) = 0 \Rightarrow x = 0$

$h'(0^-) > 0$ and $h'(0^+) < 0$

$\Rightarrow$ $h(x)$ has local maxima at $x = 0$

Hence, the correct answer is (A).

70. $f'(x)f(x) \ge (f(x))^4 + 1$

$$\Rightarrow \frac{2f'(x)f(x)}{\{(f(x))^2\}^2 + 1} \ge 2$$

$$\Rightarrow \int_0^a \frac{2f'(x)f(x)}{\left\{(f(x))^2\right\}^2+1} \geq 2\int_0^a 1dx$$

$$\Rightarrow \left|\tan^{-1}(f(x))^2\right|_0^a \geq 2a \Rightarrow \frac{\pi}{3}-\frac{\pi}{4} \geq 2a$$

Hence, the correct answer is (B).

71. we have $2a^2+a+1>3a^2-4a+1 \Rightarrow a^2-5a<0 \Rightarrow 0<a<5$ …(A)

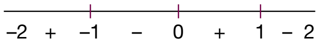
Also $3a^2-4a+1>0 \Rightarrow (3a-1)(a-1)>0$

$\Rightarrow \quad a \in \left(-\infty, \frac{1}{3}\right) \cup (1, \infty)$ …(B)

Intersection of (A) and (B) yields $a \in \left(0, \frac{1}{3}\right) \cup (1, 5)$

Hence, the correct answer is (C).

72.

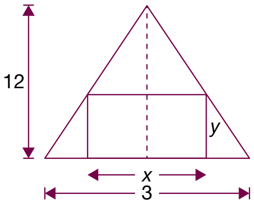


$$f'(x)=\begin{cases} 3(x+1)^2, & -2<x<-1 \\ \frac{2}{3}\times x^{-1/3}, & -1<x<1-\{0\} \\ -2(x-1), & 1<x<2 \end{cases}$$

$f'(x)$ DNE at $x=-1, 0, 1$; Sign of $f'(x)$

Hence, the correct answer is (B).

73.



Area of rectangle $= A = xy$ …(1)

Also $\frac{36}{x}=\frac{12}{12-y} \Rightarrow 3y=(36-x)$ …(2)

$\therefore \quad A=\frac{x}{3}(36-x)=\frac{1}{3}\left(36x-x^2\right)$

Now $A'(x)=0 \Rightarrow 36-2x=0 \Rightarrow x=18$

$$A''(x)=\frac{1}{3}(-2)<0$$

Also $y=\frac{36-x}{3}-\frac{36-18}{3}=6$

$\therefore \quad A_{\text{mas}}\,18\times 6=108$ sq.feet

Hence, the correct answer is (B).

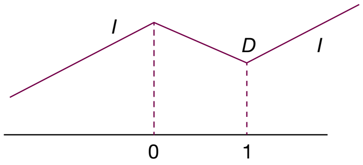
74. Since $-x^2+2x+7$ takes maximum value 8 at $x=1$, so $f(x)$ take maximum value at $x=1$,

If $\lim\limits_{x\to 1} f(x) \leq f(1)$

$\Rightarrow \quad |a^2-4| \leq 5 \Rightarrow a \in [-3, 3]$

Hence, the correct answer is (B).

75.



$f'(x)=kx(x-1)$

$f'(2)=2k=6 \Rightarrow k=3$

$\therefore \quad f'(x)=3x^2-3x \Rightarrow f(x)=x^3-\frac{3x^2}{2}+C$

$f(2)=8-6+C=2+C=2 \Rightarrow C=0$

$\Rightarrow \quad f(x)=x^2\left(x-\frac{3}{2}\right).$

$f(1)=-\frac{1}{2}$

$f\left(\frac{3}{2}\right)=0$

$f(0)=0$

$\tan^{-1}(f(1))+\tan^{-1}\left(f\left(\frac{3}{2}\right)\right)+\tan^{-1}(f(0))$

$=\tan^{-1}\left(-\frac{1}{2}\right)=-\cot^{-1}(2)$

Hence, the correct answer is (D).

76. $f(x)=p^2x-\int 2^{4-x^2}\,dx$

$f'(x)=p^2-2^{4-x^2}$

We must have, $f'(x) \geq 0 \ \forall\, x \in R \Rightarrow p^2 \geq \frac{16}{2^{x^2}} \ \forall\, x \in R$

So, $p^2 \geq 16$ [Because maximum value of $\frac{16}{2^{x^2}}$, $x \in R$ is 16]

Hence, the correct answer is (C).

77. dose $x \rightarrow$ does not passes

We have $f(x)=(b^2-3b+2)(\cos^2x-\sin^2x)+(b-1)x+\sin 2$

$\therefore \quad f'(x)=(b-1)(b-2)(-2\sin 2x)+(b-1)$

Now, $f'(x) \neq 0$ for every $x \in R$

So $(b-1)(1-2(b-2)\sin 2x) \neq 0 \ \forall\, x \in R$

$\therefore \quad b \neq 1$

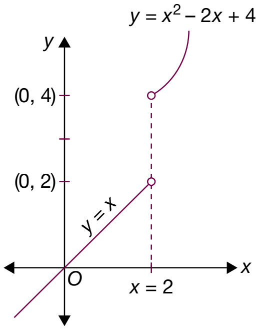
Also, $\left|\frac{1}{2(b-2)}\right|>1 \Rightarrow b \in \left(\frac{3}{2}, 2\right) \cup \left(2, \frac{5}{2}\right)$

Now, when $b=2$, $f(x)=x+\sin 2 \Rightarrow f'(x)=1 \ (\neq 0)$.

Hence, $b \in \left(\frac{3}{2}, \frac{5}{2}\right)$.

Hence, the correct answer is (C).

78.



Let $g(x)=x$ where $x<2$, then $g(x)$ is strictly increasing.

Let $h(x)=x^2-2x+4$, where $x>2$, then $h(x)$ is also strictly increasing.

Since, $f(x)$ is strictly increasing at $x=2$,

So, $\lim\limits_{x\to 2^-} x \leq k \leq \lim\limits_{x\to 2^+}(x^2-2x+4)$

$\Rightarrow \quad 2 \leq k \leq 4$

$\Rightarrow \quad k \in [2, 4]$

Hence, the correct answer is (D).

79. Let $f(x)=2\tan x+(2a+1)\ln|\sec x|+(a-2)x$

$f'(x)=2\sec^2x+\frac{(2a+1)\sec x\tan x}{\sec x}+(a-2)$

$\Rightarrow \; f'(x) = 2\tan^2 x + (2a+1)\tan x + a$

For f to be increasing, $f'(x) \geq 0 \; \forall \; x \in R$

$\Rightarrow \; (2a+1)^2 - 8a \leq 0$

$\Rightarrow \; (2a-1)^2 \leq 0$

$a = \dfrac{1}{2}$

$\therefore \; \dfrac{1}{2} \in (0, 1)$

Hence, the correct answer is (A).

80. Let $f(x) = \left(\dfrac{a^2-4}{a^2+2}\right)x^3 - 3x + \sin 3$

$\therefore \; f'(x) = 3 \cdot \left(\dfrac{a^2-4}{a^2+2}\right)x^2 - 3$

As, f is strictly decreasing on R

So, $\left(\dfrac{a^2-4}{a^2+2}\right) \leq 0 \Rightarrow a^2 - 4 \leq 0 \Rightarrow a \in [-2, 2]$.

Hence, the correct answer is (C).

81. We have $g'(x) = \dfrac{2}{x + g(x)} > 0 \; \forall \; x \in [1, 6]$

$\Rightarrow \; g(x)$ is increasing function on $[1, 6]$

$\therefore \; \displaystyle\int_1^6 g'(x)dx \leq \int_1^6 \frac{2}{x}dx$

$\Rightarrow \; (g(x))_{x=1}^{x=6} \leq 2(\ln x)_{x=1}^{x=6}$

$\Rightarrow \; g(6) - g(1) \leq 2\ln 6$

$\Rightarrow \; g(6) \leq 2\ln 6$ (As $g(1) = 0$)

Hence, the correct answer is (D).

82. $f(x)$ increases from $x = 0$ to $x = 1$ and $x = 3$ to $x = 5$.

$f(x)$ decrease from $x = 1$ to $x = 3$

So, $x = 1$ is point of local maximum

and $x = 3$ is point of local minimum.

Hence, the correct answer is (B).

83. $(x^2 + 4x + 1)^2 + a - 1 = 0$

$\therefore \; (x^2 + 4x + 1)^2 = 1 - a$

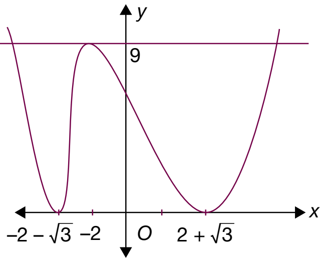
$f(x) = (x^2 + 4x + 1)^2$

Let $g(x) = x^2 + 4x + 1$,

Where $f(x) = (g(x))^2$

$g(x) = 0 \Rightarrow x = -2 \pm \sqrt{3}$

$g'(x) = 2x + 4$



$g(x)$ minimum when $x = -2$; $g(-2) = -3$

So, $f(x)$ has local maximum at $x = -2$ and $f(-2) = 9$

$\therefore$ For four distinct solutions,

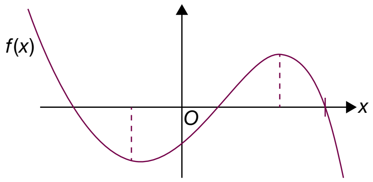
$\Rightarrow \; 1 - a \in (0, 9)$

$\Rightarrow \; -a \in (-1, 8)$

$\therefore \; a \in (-8, 1)$.

Hence, the correct answer is (C).

84.



$f(x) = (4-p)x^3 + (p-2)x^2 + (p^2-25)x + 2$

$f'(x) = 0;$

Product of roots of $f'(x) < 0$ and co-efficient of $x^3 < 0$

$f'(x) = 3(4-p)x^2 + 2(p-2)x + (p^2-25)$

$4 - p < 0$ and $\dfrac{p^2-25}{3(4-p)} < 0 \Rightarrow p \in (-5, 4) \cup (5, \infty)$

Hence, the correct answer is (D).

85. $f(x) = \dfrac{x}{2} + \dfrac{\sin 2x}{4} - (3-c)\sin x - 3cx$

$f'(x) = \dfrac{1}{2} + \dfrac{\cos 2x}{2} - (3-c)\cos x - 3c \geq 0 \; \forall \; x \in R$

$f'(x) = \dfrac{1}{2} + \dfrac{(2\cos^2 x - 1)}{2} + (c-3)\cos x - 3c \geq 0 \; \forall \; x \in R$

$\Rightarrow \; \cos^2 x + (c-3)\cos x - 3c \geq 0 \; \forall \; x \in R$

$\Rightarrow \; \cos^2 x - 3\cos x - c\cos x - 3c \geq 0 \; \forall \; x \in R$

$\Rightarrow \; \underbrace{(\cos x - 3)}_{\text{negative}} (\cos x + c) \geq 0 \; \forall \; x \in R$

$\therefore \; c + \cos x \leq 0 \; \forall \; x \Rightarrow c \leq -\cos x \; \forall \; x \in R$

$\therefore \; c \leq -1 \Rightarrow c_{\text{max.}} = -1$

Hence, the correct answer is (C).

86. If the function $f(x)$ has smallest value at $x = 1$, then

$\lim\limits_{x \to 1^-} f(x) \geq f(1)$

$\Rightarrow \; -1 + \log_2 b \geq 3$

$\Rightarrow \; \log_2 b \geq 4 \Rightarrow b \geq 16$

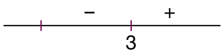
$\therefore \; b \in [16, \infty)$

Hence, the correct answer is (C).

87. $f(x) = x^{3/2} + x^{-3/2} + x^{-3/2} - 4\left(x + \dfrac{1}{x}\right)$

$\therefore f(x) = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^3 - 3\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right) - 4\left[\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2 - 2\right]$

Let $\sqrt{x} + \dfrac{1}{\sqrt{x}} = t \; (x > 0)$



Sign scheme of $g'(t)$

Let $g(t) = t^3 - 3t - 4t^2 + 8$

Now $g(t) = t^3 - 4t^2 - 3t + 8$ where $t \in [2, \infty)$

$\therefore \; g'(t) = 3t^2 - 8t - 3$

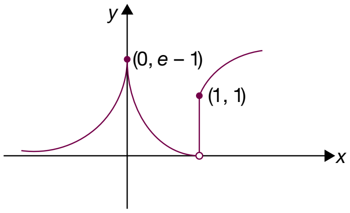
$= (t-3)(3t+1)$; $g'(t) = 0 \Rightarrow t = 3 \; (t \neq -1/3)$

$g''(t) = 6t - 8$

$g''(3) = 27 - 9 - 36 + 8 = -10$

Hence, the correct answer is (A).

88.

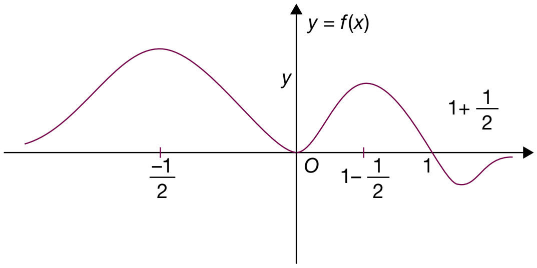


$$f(x) = \begin{cases} e^{1+x} - e^x, & -\infty < x \leq 0 \\ e^{1-x} - 1, & 0 < x < 1 \\ x + \ln x, & 1 \leq x < \infty \end{cases}$$

Hence, the correct answer is (A).

89. Global maximum at $x=-\frac{1}{\sqrt{2}}$

$\because\ f\left(-\frac{1}{\sqrt{2}}\right) > f\left(1-\frac{1}{\sqrt{2}}\right)$



Hence, the correct answer is (C).

90. $f'(0)=0$ and $f''(0)>0$

$y=f(x)+ax+b$ has a relative minimum at $x=0$.

Then $\frac{dy}{dx}=0$ at $x=0$; $f'(x)+a=0 \Rightarrow a=0$

$f''(x)>0 \Rightarrow f''(0)>0$

Hence y has relative minimum at $x=0$ if $a=0$ and $b\in R$.

Hence, the correct answer is (C).

91. $$f(x)=\begin{cases}\cos\frac{\pi}{2}x & ;\quad \forall\, 0\le x<1\\ 5x+3 & ;\quad \forall\, x\ge 1\end{cases}$$

$$f'(x)=\begin{cases}-\frac{\pi}{2}\sin\frac{\pi}{2}x & ;\quad \forall\, 0\le x<1\\ 5 & ;\quad \forall\, x\ge 1\end{cases}$$

$\Rightarrow$ $f'(x)$ changes its sign from –ve to +ve in the immediate neighborhood of $x=1$

$\Rightarrow$ $f(x)$ changes from decreasing function to increasing function

$\Rightarrow$ $f(x)$ has a local minimum value at $x=1$

Hence, the correct answer is (A).

92. Let $f(x)=x^2-x+1+\sin x$

$$=\left(x-\frac{1}{2}\right)^2+\left(\frac{3}{4}+\sin x\right)$$

$$\ge \frac{3}{4}+\sin x \qquad \left(\because \left(x-\frac{1}{2}\right)^2\ge 0\right)$$

$$\ge \frac{3}{4}-1=-\frac{1}{4} \qquad (\because \text{ minimum value of } \sin x=-1)$$

Hence, the correct answer is (C).

93. $f(x)=\frac{ax+b}{x^2-1}$

$$f'(x)=\frac{(x^2-1)a-(ax+b)2x}{(x^2-1)^2}$$

$\therefore$ $f'(3)=0 \Rightarrow 8a=6(3a+b)$

and $f(3)=1 \Rightarrow 3a+b=8$

Hence $a=6$ and $b=-10$

Hence, the correct answer is (C).

94. $f(x)=\frac{\sin x+x}{x^2}\ \ \forall\, x\in\left(0,\frac{\pi}{2}\right]$

$$f'(x)=\frac{x^2(\cos x+1)-(\sin x+x)2x}{x^4}=\frac{x(\cos x-1)-2\sin x}{x^3}\le 0$$

$f(x)$ is deceasing

$$\min f(x)=f\left(\frac{\pi}{2}\right)=\frac{1+\frac{\pi}{2}}{\frac{\pi^2}{4}}=\frac{4+2\pi}{\pi^2}$$

$$\max f(x)=\lim_{x\to 0^+}\frac{\sin x+x}{x^2}\to +\infty$$

$$f(x)=\frac{\sin x+x}{x^2}\ge\frac{4+2\pi}{\pi^2}$$

$\sin x+x\ge |k|\, x^2$

$$|K|=\frac{4+2\pi}{\pi^2}$$

Greatest $K=\frac{2(2+\pi)}{\pi^2}$

Hence, the correct answer is (B).

95. $f(x, y)=x^2-4x+y^2+6y=(x-2)^2+(y+3)^2-13$

$0\le x\le 1$

$0\le y\le 1$

$1\le (x-2)^2\le 4$

$9\le (y+3)^2\le 16$

$1+9\le (x-2)^2+(y+3)^2\le 4+16$

$10\le (x-2)^2+(y+3)^2\le 20$

$-3\le (x-2)^2+(y+3)^2-13\le 7$

max. → 7

min $=-3=m$

$|m|=3$

Hence, the correct answer is (C).

96. $f(-1)=b+2$

$f(-1^+)=2(-1\ )+3=1$

$f(-1)\le f(-1^+)$

$\Rightarrow$ $b+2\le 1\Rightarrow b\le -1$

Hence, the correct answer is (C).

97. $f(x)=\cot x-\sqrt{2}\operatorname{cosec} x$

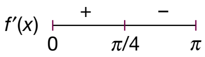
$x\in(0,\pi)$

$f'(x)=-\operatorname{cosec}^2 x+\sqrt{2}\operatorname{cosec} x\cot x$

$=\operatorname{cosec} x\,(\sqrt{2}\cot x-\operatorname{cosec} x)=0$

or $\sqrt{2}\cot x=\operatorname{cosec} x$, we get $x=\frac{\pi}{4}$

$$f\left(\frac{\pi}{4}\right)=1-\sqrt{2}\left(\frac{\sqrt{2}}{1}\right)=-1$$



Hence, the correct answer is (B).

98. $f(1)=-6$

For maximum at $x=1$

$$\lim_{x\to 1^-} f(x)=\tan^{-1}\alpha-5<-6$$

$\Leftrightarrow$ $\tan^{-1}\alpha<-1\Leftrightarrow \alpha<-\tan 1$

Hence, the correct answer is (D).

99. $$f(x)=\left|\frac{x-1}{x^2}\right|,\ f(x)=\begin{cases}\frac{1}{x}-\frac{1}{x^2} & x\ge 1\\ -\frac{1}{x}+\frac{1}{x^2} & x<1\end{cases}$$

$$f'(x)=\begin{cases}-\frac{1}{x^2}+\frac{2}{x^3} & x\ge 1\\ \frac{1}{x^2}-\frac{2}{x^3} & x<1\end{cases}$$

$f'(x)=0$ at $x=2$ and $f'(x)$ does not exist at $x=1$.

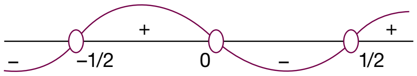
Two critical points.

Hence, the correct answer is (B).

Multiple Correct Choice Type Questions

1. $\because \quad f(x) = 2x^2 - \ln|x|$

$\therefore \quad f'(x) = 4x - \frac{1}{x} = \frac{(2x+1)(2x-1)}{x}$



For increasing, $f'(x) > 0$

$\therefore \quad x \in \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$

And for decreasing, $f'(x) < 0$

$\therefore \quad x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$

Hence, (A) and (D) are correct.

2. $g(x)$ is increasing and $f(x)$ is decreasing

$\Rightarrow \quad g(x+1) > g(x-1)$ and $f(x+1) < f(x-1)$

$\Rightarrow \quad f(g(x+1)) < f(g(x-1))$ and $g(f(x+1)) < g(f(x-1))$

Hence, (B) and (D) are correct.

3. $f(x) = x^3 - x^2 + 100x + 2002$

$f'(x) = 3x^2 - 2x + 100 > 0 \ \forall \ x \in R \qquad (\because$ Discriminant $< 0)$

$\therefore \quad f(x)$ is increasing (strictly)

$\therefore \quad f\left(\frac{1}{2000}\right) > f\left(\frac{1}{2001}\right)$

Also $f(x-1) > f(x-2)$ as $x - 1 > x - 2$ for $\forall \ x$

Hence, (B) and (C) are correct.

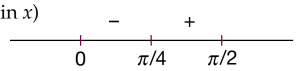
4. $f(x) = \sin^5 x + \cos^5 x - 1 \qquad x \in \left[0, \frac{\pi}{2}\right]$

$f'(x) = 5\sin^4 x \cos x + 5\cos^4 x(-\sin x)$

$= 5\sin x \cos x(\sin^3 x - \cos^3 x)$

$= \sin x \cos^4 x(\tan^3 x - 1)$

$= \sin x \cos^4 x(\tan x - 1)(\tan^2 x + \tan x + 1)$



for $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$f(x)$ is increasing; $x \in \left(0, \frac{\pi}{4}\right) \qquad f(x)$ is decreasing

$f\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}\right)^5 2 - 1 = \frac{1}{2\sqrt{2}} - 1 < 0$

$f\left(\frac{\pi}{2}\right) = 0$ and $f(0) = 0$ for $x \in \left(0, \frac{\pi}{2}\right]$ $f(x) \le 0$

$f(x) = 0$ has only two solutions $x = 0, \frac{\pi}{2}$

Hence, (A), (B), (C) and (D) are correct.

5. $f'(x) = \cos x + f'(x)\left(2\sin x - \sin^2 x\right)$

$\Rightarrow \quad \left(1 - 2\sin x + \sin^2 x\right) f'(x) = \cos x$

$\Rightarrow \quad f'(x) = \frac{\cos x}{(\sin x - 1)^2} \Rightarrow f(x) = \frac{1}{1 - \sin x} + c$

Since $f(o) = 0$ we have $C = -1$

$\therefore \quad f(x) = \frac{\sin x}{1 - \sin x}$

$f\left(\frac{\pi}{6}\right) = \frac{\frac{1}{2}}{\frac{1}{2}} = 1; g'(x) = f(x)$

$g'(x) = \frac{\sin x}{1 - \sin x} > 0 \ \forall \ x \in (0, \pi) \Rightarrow g(x)$ is increasing

$f(0) = 0; f'(x) = \frac{\cos x}{(1 - \sin x)^2}$

Hence, (A), (B) and (C) are correct.

6. Given $f(x) = \left(x^2 - 4\right)^n \left(x^2 - x + 1\right)$, $n \in m$ has a local minimum at $x = 2$ and $f(2) = 0$, $x^2 - x + 1 > 0 \ \ \forall x \in R$

So, by first derivative test, $f'(x)$ should change sign from negative to positive
at $x = 2$, So x must be even integer

$\therefore \quad n = 2010 \ (\text{or}) \ 2008$.

Hence, (A) and (D) are correct.

7. $g'(x) = \lim\limits_{h \to 0} \frac{g(x+h) - g(x)}{h}$

$= \lim\limits_{h \to 0} \frac{g(x) + g(h) + 3xh(x+h) - g(x)}{h}$

$= \lim\limits_{h \to 0} \frac{g'(h) + 3x^2 + 6xh}{1} \Rightarrow g'(x) = 3x^2 - 4$

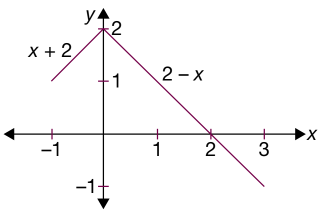
$\Rightarrow \quad g(x) = x^3 - 4x$

Here $g(x)$ is increasing on $\left(-\infty, -\frac{2}{\sqrt{3}}\right) \cup \left(\frac{2}{\sqrt{3}}, \infty\right)$ and

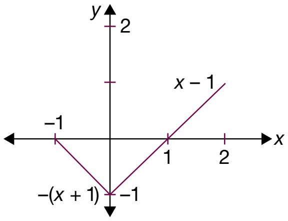
decreasing on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$

Hence, (A), (B) and (C) are correct.

8. We have, $f(x) = \begin{cases} 1 - x, & 0 \le x \le 2 \\ x - 3, & 2 \le x \le 4 \end{cases}$ and $g(x) = \begin{cases} 2 + x, & 0 \le x \le 2 \\ 2 - x, & 0 < x \le 3 \end{cases}$



Graph of *g*(*x*)



Graph of *fog*(*x*)

$\therefore \quad (fog)(x) = \begin{cases} 1 - g(x), & 0 \le g(x) \le 2 \\ g(x) - 3, & 2 < g(x) \le 4 \end{cases}$

$= \begin{cases} -1 + x, & -1 \le x \le 0 \\ -1 + x, & 0 < x \le 2 \end{cases}$

Now, verify alternatives.

NOTE:

The point $x = 0$ is an extremum point (local minimum) of $y = f(g(x))$.

Hence, (A), (B), (C) and (D) are correct.

9. The range of $\sin x + \cos x$ is $\left[-\sqrt{2}, \sqrt{2}\right]$ on $\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$

The range of $\frac{4}{\pi}x + \frac{1}{2}$ is $\left(\frac{3}{2}, \frac{9}{2}\right]$ on $\left(\frac{\pi}{4}, \pi\right]$

$f'(x) = \cos x - \sin x > 0$ on $\left[-\frac{3\pi}{4}, \frac{\pi}{4}\right]$

and $f'(x) = \frac{4}{\pi} > 0$ on $\left(\frac{\pi}{4}, \pi\right]$

$\therefore \quad f(x)$ is increasing

Hence $f(x)$ is one-one

Range $f(x)$ is $\left[-\sqrt{2},\sqrt{2}\right]\cup\left(\frac{3}{2},\frac{9}{2}\right]$ which is not equal to the codomain. So not onto clearly $f(x)$ is discontinuous at $x=\frac{\pi}{4}$.

Hence, (A) and (B) are correct.

10. When $f(x)$ is continuous at $x=2$

$f'(x)$ DNE at $x=2$ and $f'(x)$ changes sign from + to –

$\Rightarrow$ $f(x)$ attains max. At $x=2$ if $\frac{k^3(k-1)^2}{k^2-k-2}=0\Rightarrow k=0,1$

When $f(x)$ is discontinuous at $x=2$, $f'(x)$ changes its sign from + to – . $f(x)$ will attain maximum if $\lim\limits_{x\to 2^+} f(x)<f(2)$ as

$$\lim_{x\to 2^-} f(x)=f(2) \text{ i.e if } k\in(-\infty,-1)\cup(0,1)\cup(1,2)$$

$\Rightarrow$ $k\in(-\infty,-1)\cup[0,2)$

Hence, (A), (C) and (D) are correct.

11. $f(x)=\frac{\pi^e}{x-e}+\frac{e^\pi}{x-\pi}+\frac{\pi^\pi+e^e}{x-\pi-e}$ is $f(x)$ is discontinuous at $x=\pi, e, \pi+e$

$x\in(e,\pi)$ $f(x)$ is continuous

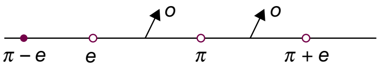
$f(e^+)\to+\infty; f(\pi^-)\to-\infty$

$f(x)=0$

$$f'(x)=\frac{-\pi^e}{(x-e)^2}-\frac{e^\pi}{(x-\pi)^2}-\frac{(\pi^\pi+e^e)}{(x-\pi-e)^2}$$

$f'(x)<0$ then $f(x)$ is decreasing

$$f(\pi-e^+)=\frac{\pi^e}{\pi-2e}+\frac{e^\pi}{-e}-\frac{\pi^\pi e^e}{2e}$$



$$\left.\begin{array}{l} f(\pi^+)\to+\infty \\ f(\pi+e^-)\to-\infty \end{array}\right\} f(x)=0 \text{ for one value in } (\pi,\pi+e)$$

Hence, (B) and (C) are correct.

12. $f(x)$ is increasing

$f(2a^2+a+1)>f(3a^2-4a+1)$

domain $x>0$

$\Rightarrow$ $2a^2+a+1>3a^2-4a+1$

$\Rightarrow$ $a^2-5a<0\Rightarrow a(a-5)<0\Rightarrow a\in(0,5)$ …(1)

Domain of $f(x)$ is $x>0$

$2a^2+a+1>0$ for all a

$3a^2-4a+1>0\Rightarrow(3a-1)(a-1)>0$

we get $a\in\left(-\infty,\frac{1}{3}\right)\cup(1,\infty)$ …(2)

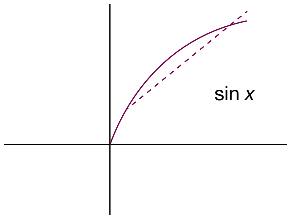
Common $a\in\left(0,\frac{1}{3}\right)\cup(1,5)$

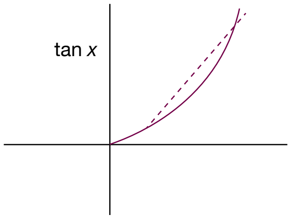
Integral a are 2, 3, 4

Hence, (B), (C) and (D) are correct.

13. $x\in(0,1)$

$f(x)=\sin x$



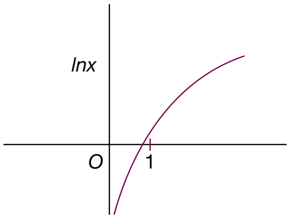


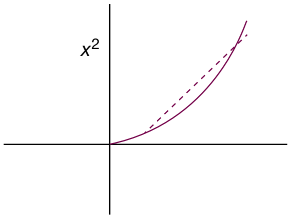
$$\sin\left(\frac{x_1+x_2}{2}\right)>\frac{\sin x_1+\sin x_2}{2}$$

$$\tan\left(\frac{x_1+x_2}{2}\right)\le\frac{\tan x_1+\tan x_2}{2}$$

$$\ln\left(\frac{x_1+x_2}{2}\right)\ge\frac{\ln x_1+\ln x_2}{2}$$

$$\frac{(x_1+x_2)^2}{2}\le\frac{x_1^2+x_2^2}{2}$$





Hence, (B) and (D) are correct.

14. $h(x)=x(1+\ln x)$

$h:[1,\infty)\to[1,\infty)$

$h'(\alpha)=(1+\ln x)+x(1/x)=\ln x+2>0$ for $x\in[1,\infty)$

$h(x)$ has inverse *fn*.

$h(x)=x(1+\ln x)=\lambda$

$f(\lambda)$ is inverse of $h(x)$

$x\to\infty; h(x)\to\infty$

$h(x)$ is increasing then $f(\lambda)$is also increasing

$\lambda\to\infty; f(\lambda)\to\infty$

Solution of equation $x(1+\ln x)=\lambda$ is $x=f(\lambda)$, as $x\to\infty, \lambda\to\infty$

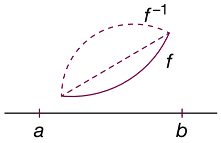
$$\lim_{\lambda\to\infty}\frac{f(\lambda)\ln\lambda}{\lambda}=\lim_{x\to\infty}\frac{x\ln x(1+\ln x)}{x(1+\ln x)}=\lim_{x\to\infty}\frac{\ln(x+x\ln x)}{(1+\ln x)}$$

$$=\lim_{x\to\infty}\frac{\frac{1}{x(1+\ln x)}(1+1+\ln x)}{0+\frac{1}{x}}=\lim_{x\to\infty}\frac{2+\ln x}{1+\ln x}=1$$

Hence, (A), (B), (C) and (D) are correct.

15. $f'(x)>0$ and $f''(x)>0$

It means $f(x)$ is increasing and concave



$$f\left(\frac{a+b}{2}\right)<\frac{f(a)+f(b)}{2}$$

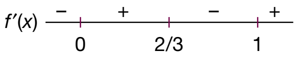
$f^{-1}(x)$ is increasing and convex

$(f^{-1})''<0, (f^{-1})'>0$

$$f^{-1}\left(\frac{a+b}{2}\right)>\frac{f^{-1}(a)+f^{-1}(b)}{2}$$

Hence, (A) and (D) are correct.

16. $f(x)=x^4(1-x)^2$



Neither Even Nor Odd

$f'(x)=4x^3(1-x)^2+x^4 2(1-x)(-1)$

$=2x^3(1-x)(2-3x)$

$f'(2/3)=0$

$f''(x)=6x^2(1-x)(2-3x)+2x^3(-1)(2-3x)+2x^3(1-x)(-3)$

$=2x^2[15x^2-20x+6]=0$ for three times

Hence, (A) and (D) are correct.

17. $f(x) = e^x + \int_0^1 e^x f(t)\,dt = e^x + e^x \int_0^1 f(t)\,dt$

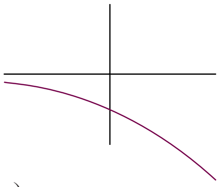
$$f(x) = e^x + e^x K = (K+1)\,e^x \text{ where } K = \int_0^1 f(t)\,dt$$

$$K = \int_0^1 (K+1)e^t\,dt = (K+1)\,e^t\,|_0^1 = (K+1)\,(e^1 - e^0)$$

$$K = (K+1)e - K - 1$$

$$2K + 1 = Ke + e$$

$$K(2-e) = e - 1$$



$$\Rightarrow \quad K = \int_0^1 f(t)\,dt = \frac{e-1}{2-e} < 0$$

$$f(x) = e^x(1+K) = e^x\left(1 + \frac{e-1}{2-e}\right) = e^x\left(\frac{1}{2-e}\right)$$

$$f(x) = \frac{e^x}{2-e} \text{ as } f(0) = \frac{1}{2-e} < 0$$

$f(x)$ is decreasing

Hence, (A) and (B) are correct.

18. $g(x) = f_1(f_2(f_3(\ldots.(f_n(x)))))$

$$g'(x) > 0$$

$$\left.\begin{matrix} r & fi\text{'s are decreasing} \\ n-r & fi\text{'s are increasing} \end{matrix}\right\} \quad g(x) \text{ increasing}$$

$(\therefore\ r \to \text{even})$

CASE 1: (n is type $4k$)

maximum value of $r(n-r) = rn - r^2$ when $r = \frac{n}{2} = 2k$ (even)

$$= \frac{n}{2}\left(n - \frac{n}{2}\right) = \frac{n^2}{4}$$

CASE 2: (n is $4k+1$)

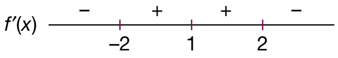
Maximum value of $r(n-r) = rn - r^2$ when $r = \frac{n-1}{2} = 2k$ (even)

$$= \left(\frac{n-1}{2}\right)\left(n - \frac{(n-1)}{2}\right) = \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) = \frac{n^2-1}{4}$$

Hence, (C) and (D) are correct.

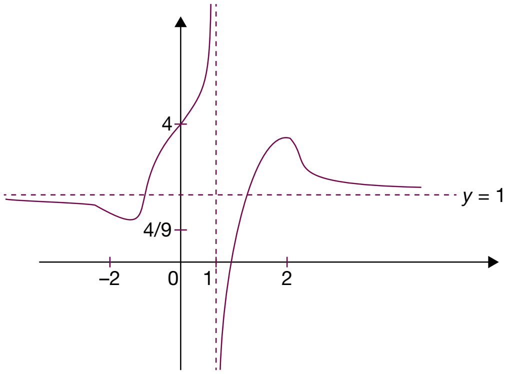
19. (A) $\lim\limits_{x\to-\infty} \dfrac{x^3-4}{(x-1)^3} = \lim\limits_{x\to-\infty} \dfrac{1-4/x^3}{(1-1/x)^3} = \dfrac{1-0}{1-0} = 1$

$$f(x) = \frac{x^3-4}{(x-1)^3}$$



$$f'(x) = \frac{(x-1)^3 3x^2 - (x^3-4)3(x-1)^2}{(x-1)^6}$$

$$= \frac{3(2-x)(2+x)}{(x-1)^4}$$



$$f(-2) = \frac{-8-4}{-27} = \frac{4}{3}$$

$$f(2) = 4$$

$y = 1$ cuts two points

$$c \in \left(\frac{4}{9}, 1\right) \cup (1, 4)$$

$$f(x) = C$$

3 real and distinct

$$\frac{x^3-4}{(x-1)^3} = 1 \Rightarrow x^3 - 4 = x^3 - 1 - 3x(x-1)$$

$$x^2 - x - 1 = 0 \quad \Rightarrow \quad x = \frac{1 \pm \sqrt{5}}{2}$$

Hence, (B) and (C) are correct.

20. $f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + tf(t))\,dt$

$$f(x) = \sin x + \sin x \cdot \pi + \int_{-\pi/2}^{\pi/2} tf(t)\,dt;\ f(x) = (\pi+1)\sin x + \alpha$$

$$\alpha = \int_{-\pi/2}^{\pi/2} t((\pi+1)\sin t + \alpha)\,dt = (\pi+1)\int_{-0}^{\pi/2} 2t\sin t\,dt + 0$$

$$= 2(\pi+1)\left(t(-\cos t)\Big|_0^{\pi/2} - \int_0^{\pi/2} 1(-\cos t)dt\right)$$

$$\alpha = 2(\pi+1)\left((0-0) + \sin t\Big|_0^{\pi/2}\right) = 2(\pi+1)$$

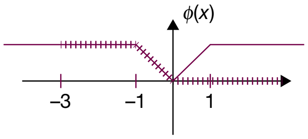
$$f(x) = (\pi+1)\sin x + 2(\pi+1) = (\pi+1)\,(\sin x + 2)$$

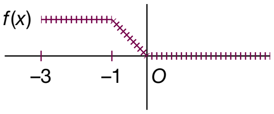
$$M = 3(\pi+1) \text{ and } m = (\pi+1)$$

$$\frac{M}{m} = 3;\ M - m = 2\pi + 2;\ M + m = 4(\pi+1);\ Mm = 3(\pi+1)^2$$

Hence, (A) and (C) are correct.

21. $f(x) = \min(\phi(t), -3 \le t \le x)\ \forall\ x \in [-3, \infty)$





$$\phi(x) = \big||x-1| - |x+1|\big| = \begin{cases} 2 & x \ge 1 \\ 2|x| & -1 < x < 1 \\ 2 & x \le -1 \end{cases}$$

$f(x)$ is not differentiable at $x = -1, 0$; $f(100) = 0$

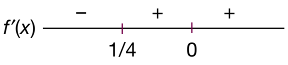
$$\int_3^{10} f(x)\,dx = \int_{-3}^{-1} 2\,dx + \int_{-1}^{0} -2x\,dx + \int_0^{10} 0\,dx$$

$$= 2x\Big|_{-3}^{-1} + -x^2\Big|_{-1}^{0} + 0 = -2 + 6 + 0 + 1 = 5$$

Hence, (A), (C) and (D) are correct.

22. $f(x) = x^{1/3}(x-1)$

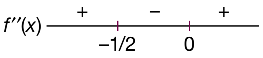
$$f'(x) = x^{1/3} + \frac{1}{3}x^{-2/3}(x-1) = \frac{4x-1}{3x^{2/3}}$$



$$f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^{\frac{1}{3}}\left(-\frac{3}{4}\right) = -\frac{3}{4^{4/3}} \text{ minimum}$$

$$f''(x) = \frac{1}{3}x^{-2/3} - \frac{2}{9}x^{-5/3}(x-1) + \frac{1}{3}x^{-2/3}(1-0)$$

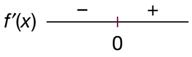
$$= \frac{2}{3}x^{-2/3} - \frac{2}{9x^{5/3}}(x-1) = \frac{4x+2}{9x^{5/3}}$$



two point of inflection

Hence, (A), (B), (C) and (D) are correct.

23. $f(x) = x^4 - 6bx^2 + b^2$



$x \in [-2, 1]$

$f'(x) = 4x^3 - 12bx = 4x(x^2 - 3b)$

If $b < 0$

$f(0) = b^2; f(-2) = 16 - 24b + b^2; f(1) = b^2 - 6b + 1$

greatest of $f(x) = b^2 - 24b + 16$

If $b \geq 1$

$f'(x) = 4x(x^2 - 3b) = 0$

Critical point $x = 0$ lies in given set but $x = \pm\sqrt{3b} \notin [-2, 1]$

If $b = 2015, 2016$

$f(0) = b^2; f(1) = b^2 - 6b + 1; f(-2) = b^2 - 24b + 16$

Greatest of $f(x) = b^2$

Hence, (A) and (C) are correct.

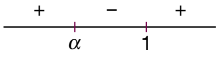
24. $f(x) = \dfrac{e^x}{1+x^2}$

$$g(x) = f'(x) = \frac{e^x}{(1+x^2)} - \frac{e^x}{(1+x^2)^2}2x$$

$$g(x) = \frac{e^x}{(1+x^2)}(1 + x^2 - 2x) = e^x\frac{(1-x)^2}{(1+x^2)^2} > 0$$

$$g'(x) = e^x\frac{(1-x)^2}{(1+x^2)^2} + \frac{e^x 2(1-x)(0-1)}{(1+x^2)^2} + e^x(1-x)^2\left(\frac{-2}{(1+x^2)^3}\cdot 2x\right)$$

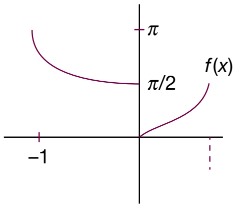
$$= \frac{e^x(x-1)(x^3-3x^2+5x+1)}{(x^2+1)^3}$$

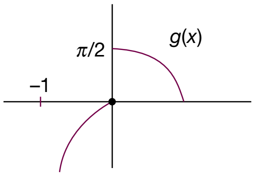


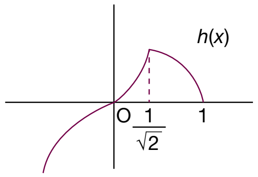
$x^3 - 3x^2 + 5x + 1$ has only one real root α, where $\alpha \in (-1, 0)$

Hence, (B) and (D) are correct.

25. Graph of $f(x)$ and $g(x)$ are as shown







$h(x) = \min\{f(x).\ g(x)\}$

$$= \begin{cases} \sin^{-1}x & x \leq \dfrac{1}{\sqrt{2}} \\ \cos^{-1}x & \dfrac{1}{\sqrt{2}} \leq x \leq 1 \end{cases} \quad \text{not diff at } x = \frac{1}{\sqrt{2}}$$

$h(x)$ has $x = \dfrac{1}{\sqrt{2}}$ local maximum and it is not monotonic

$h(x)$ is non-differentiable at $x = \dfrac{1}{\sqrt{2}}$

$x \in \left(0, \dfrac{1}{\sqrt{2}}\right)$ concave $h''(x) > 0$

Hence, (A), (C) and (D) are correct.

26. $f(x) = \sin(\cos x) + \cos(\sin x)$

$f'(x) = -[\sin x \cdot \cos(\cos x) + \cos x \cdot \sin(\sin x)] > 0$ in $\left[\dfrac{-\pi}{2}, 0\right]$

$\Rightarrow f(x)$ is maximum when $x = 0$ and minimum when $x = -\dfrac{\pi}{2}$

$\therefore\ f_{\max} - f_{\min} = f(0) - f\left(\dfrac{-\pi}{2}\right) = 1 + \sin 1 - \cos 1 > 1$

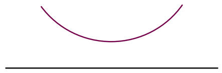
f has local maxima at $x = 0$ then $f(0) \geq f(0^+)$

$\Rightarrow\ 1 + \sin 1 \geq \sin a + 1 \Rightarrow \sin 1 \geq \sin a$

$\Rightarrow\ a = 1, 3$ are two possible values.

Hence, (A) and (D) are correct.

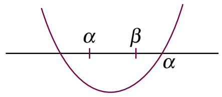
27. Let $f(x) = ax^2 + bx + c$ where $a, b, c \in N$



given $\displaystyle\int_\alpha^\beta f(x)dx > 0$ for $\forall\ \alpha, \beta \in R$

It means $f(x)$ always lies above x axis, it has imaginary roots

$D < 0 \Rightarrow b^2 - 4ac < 0$ …(1)



$g(t) = f''(t)\,f(t) = 2a(at^2 + bt + c)$

$g(0) = 12 = 2ac = 12 \Rightarrow ac = 6$

from (1), we have $b^2 < 24$

$ac = 6$

$b^2 < 24$

$a, b, c \in N$

b can have values 1, 2, 3, 4

(A, C) ordered pairs are (1, 6)(6, 1)(2, 3)(3, 2)

No. of triplet of a, b, c are $= 4 \times 4 = 16$

Minimum value of minima of $f(t) = -\dfrac{D}{4a} = \dfrac{24 - b^2}{4a}$

$= \dfrac{24-16}{24} = 1/3$ (for $b = 4$; $a = 6$)

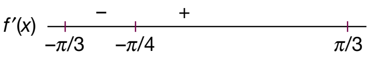
Maximum value of minima of $f(t) = \dfrac{24 - b^2}{4a} = \dfrac{24-1}{4}$

$= \dfrac{23}{4}$ (for $b = 1$; $a = 1$)

Hence, (A) and (C) are correct.

28. $f(x) = e^x\sec x - \sqrt{2}\cos x + x$

$x \in \left[-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right]$



$f'(x) = e^x\sec x + e^x\sec x\tan x + \sqrt{2}\sin x + 1$

$= e^x\sec x(1 + \tan x) + (\sqrt{2}\sin x + 1)$

$f(x)_{\min} = f\left(-\dfrac{\pi}{4}\right) = \sqrt{2}\,e^{-\pi/4} - 1 - \pi/4$

$f(x)_{\max} = f\left(\dfrac{\pi}{3}\right)$

$f''(x) = e^x\sec x(1 + \tan x) + e^x\sec x\tan x(1 + \tan x) + e^x\sec x(\sec^2 x)$

$+ \sqrt{2}\cos x$

$= e^x\sec x(1 + \tan x + \tan x + \tan^2 x + \sec^2 x) + \sqrt{2}\cos x$

$= e^x\sec x(2\sec^2 x + 2\tan x) + \sqrt{2}\cos x$

$f''(x) = 2e^x\sec x(\tan^2 x + \tan x + 1) + \sqrt{2}\cos x > 0$

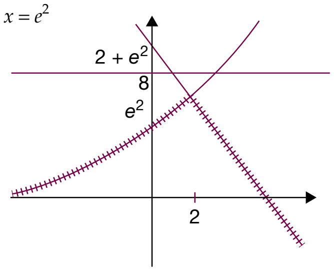
For all $x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$

$f''(x) > 0, f'(x)$ is increasing so $f'(x) \le f'\left(\frac{\pi}{3}\right)$

Hence, (A) and (C) are correct.

29. $f(x) = \min(e^x, 2 + e^2 - x, 8)$ is drawn in diagram

Maximum value of $f(x)$ is 8 at $x = e^2$



$\alpha = e^2$

$[\alpha] = 7$

$$\frac{x(x-7)}{x^2 - 7x + 12} < 0$$

$$\Rightarrow \frac{x(x-7)}{(x-4)(x-3)} < 0$$

$\Rightarrow x \in (0, 3) \cup (4, 7)$

Integral x are 1, 2, 5, 6

Hence, (A), (C) and (D) are correct.

30. $f(x) = 1 + x\ln(x + \sqrt{x^2+1}) - \sqrt{1+x^2}$

$$f'(x) = x\frac{\left(1 + \frac{x}{\sqrt{1+x^2}}\right)}{x + \sqrt{1+x^2}} + \ln(x + \sqrt{x^2+1}) - \frac{x}{\sqrt{x^2+1}}$$

$$= \frac{x}{\sqrt{x^2+1}} + \ln(x + \sqrt{x^2+1}) - \frac{x}{\sqrt{x^2+1}}$$

$$= \ln(x + \sqrt{x^2+1})$$

$$f'(x) = \begin{cases} > 0, & x > 0 \\ < 0, & x < 0 \end{cases}$$

$h(x) = f(x) - f^2(x) + f^3(x)$

$\therefore$ $h'(x) = (1 - 2f(x) + 3f^2(x))\,f'(x)$

$h'(x) = f'(x)\,(1 - 2f(x) + 3f^2(x))$; $\quad (\because 3f^2(x) - 2f(x) + 1 > 0)$

$\Rightarrow$ $h(x)$ is increasing when f is increasing and decreasing when f is decreasing

$\therefore$ $h(x)$ is increasing in $(0, \infty)$ and $h(x)$ decreasing in $(0, \infty)$

Hence, (A) and (B) are correct.

31. Let Degree of polynomial = n

$n = n - 1 + n - 2 \Rightarrow n = 3$

$\therefore$ $f(x) = ax^3 + bx^2 + cx + d, a \ne 0$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f'''(x) = 6a$

$18(ax^3 + bx^2 + cx + d) = (3ax^2 + 2bx + c - 6)\,(6ax + 2b) + 3(6a)$

Comparing $18a = 18a^2 \Rightarrow a = 1$

$18b = 12ab + 6ab \Rightarrow b = 0$

$18c = 6a(c - 6) + 4b^2 \Rightarrow 18c = 6c - 36 \Rightarrow c = -3$

$18d = -12b + 18a \Rightarrow d = 1$

$\therefore$ $f(x) = x^3 - 3x + 1$

Now $f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x + 1)\,(x - 1)$

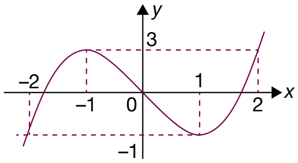
$f''(x) = 6x$

$f(-1) = -1 + 3 + 1 = 3$

$f(1) = 1 - 3 + 1 = -1$

$f(3) = 27 - 9 + 1 = 19$

$f(-3) = -27 + 9 + 1 = -17$



$\Rightarrow$ minimum value of $f(x)$ in $x \in [-1, 1]$ equal to -1 and absolute maximum value = 19 in $[-3, 3]$

(D) $I = \int_{-2}^{2} (x^3 - 3x + 1)\,dx$

$= 2 + 2 = 4$

Hence, (A), (B), (C) and (D) are correct.

32. $f(x) = \sin^{-1}\left(1 - 2\sqrt{x}\right) + \cos^{-1}\left(2\sqrt{\sqrt{x} - x}\right) + \tan^{-1}\left(\frac{\sqrt{2} - 1 - \sqrt{x}}{1 + \sqrt{2x} - \sqrt{x}}\right)$

Domain : $x \in (0, 1)$

$f(x) = \sin^{-1}\left(1 - 2\sqrt{x}\right) + \sin^{-1}\left|2\sqrt{x} - 1\right| + \tan^{-1}\left(\sqrt{2} - 1\right) - \tan^{-1}\left(\sqrt{x}\right)$

$$\therefore \quad f(x) = \begin{cases} 2\sin^{-1}\left(1 - 2\sqrt{x}\right) + \frac{\pi}{8} - \tan^{-1}\sqrt{x}; & x \in \left(0, \frac{1}{4}\right] \\ \frac{\pi}{8} - \tan^{-1}\sqrt{x}; & x \in \left(\frac{1}{4}, 1\right] \end{cases}$$

$$f'(x) = \begin{cases} 2\frac{1}{\sqrt{1 - (1 - 2\sqrt{x})^2}}\left(\frac{-2}{2\sqrt{x}}\right) - \frac{1}{1+x}\cdot\frac{1}{2\sqrt{x}}; \left(0, \frac{1}{4}\right) \\ \frac{-1}{1+x}\cdot\frac{1}{2\sqrt{x}}; \left(\frac{1}{4}, 1\right) \end{cases}$$

and $f'\left(\frac{1^+}{4}\right) = \frac{2}{\sqrt{1-0}}(-2) - \frac{4}{5} = \frac{-24}{5}$

Also $f'(x) < 0$

$\therefore$ $f(x)$ is decreasing in (0, 1).

Also minimum value of $f(x)$ does not exist.

Hence, (A), (C) and (D) are correct.

33. $f(x) = 4\sin^3 x + 9\cos^2 x + 6\sin x - 3$ for $0 \le x \le \pi$

$f'(x) = (12\sin^2 x - 18\sin x + 6)\cos x = 0$

Hence $\cos x = 0$

$\Rightarrow x = \frac{\pi}{2}$

or $6(2\sin^2 x - 3\sin x + 1) = 0$

or $2\sin^2 x - 2\sin x - \sin x + 1 = 0$

or $2\sin x(\sin x - 1) - (\sin x - 1) = 0$

$\Rightarrow$ $\sin x = 1$ or $\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$ or $x = \frac{5\pi}{6}$

Critical points in [0, p] are $\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$

Points at boundary $f(0) = f(\pi) = 6$

$f\left(\frac{\pi}{2}\right) = 4 + 6 - 3 = 7$

$f\left(\frac{\pi}{6}\right) = \frac{4}{8} + 9\cdot\frac{3}{4} + 3 - 3 = \frac{1}{2} + \frac{27}{4} = \frac{29}{4}$

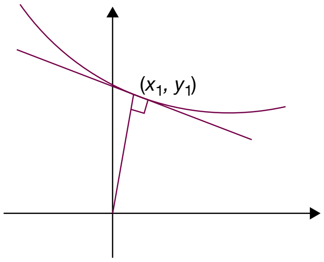
$f\left(\frac{5\pi}{6}\right) = 4\cdot\frac{1}{8} + 9\cdot\frac{3}{4} + 3 - 3 = \frac{29}{4}$

Hence maximum occurs at $x = \frac{5\pi}{6}$ or $\frac{\pi}{6}$

Hence, (B) and (C) are correct.

34. Let volume $v = \frac{4}{3}\pi r^3$, $\frac{dv}{dt} = 4\pi r^2\frac{dr}{dt}$

$$\frac{dv}{dt} = -k\frac{4}{3}\pi r^3 \Rightarrow 4\pi r^2\frac{dr}{dt} = -k\frac{4}{3}\pi r^3 \Rightarrow \frac{dr}{dt} = -k\frac{r}{3}$$



$\Rightarrow \quad r = r_0 e^{-\frac{kt}{3}}$

At $t = 0, r = r_0 = 1, \ \frac{1}{e} = e^{-\frac{k}{3}}$

$= \frac{1}{e^{k/3}} \Rightarrow k = 3$

$r = e^{-t} \Rightarrow$ Let $f(t) = t^2 r = t^2 e^{-t}$

$f'(t) = 2te^{-t} - t^2 e^{-t}$

$= t\, e^{-t}\,(2 - t) > 0 \rightarrow t \in (0, 2)$

$\int_0^\infty e^{-t}dt = -\left[e^{-t}\right]_0^\infty = -[0 - 1] = 1$ (unity)

$y = e^{-x}, \ \left.\frac{dy}{dx}\right|_{x_1 y_1} = -e^{x_1}$

Equation of normal, $y = \frac{1}{e^{-x_1}}\, x \Rightarrow y_1 = \frac{x_1}{e^{-x_1}}$ and $y_1 = e^{-x_1}$

$y_1 = \frac{x_1}{y_1} \Rightarrow y_1^2 = x_1\ p = \sqrt{(x_1^2 + y_1^2)} = \sqrt{(x_1^2 + x_1)} = \sqrt{(x_1^2 + x_1)}$

$p = \sqrt{t_0(t_0 + 1)}$ put $x_1 = t_0, r = e^{-t}\ [r] = [e^{-t}] = 0, t > 0$

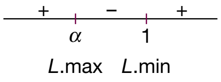
$[0] \rightarrow$ GIF in opt D

Hence, (A), (B), (C) and (D) are correct.

35. $f(x) = \frac{(x-1)^2 e^x}{(1+x^2)^2}$

$$f'(x) = \frac{(1+x^2)^2(2(x-1)e^x + (x-1)^2 e^x) - (x-1)^2 e^x\, 2(1+x^2)2x}{(1+x^2)^4}$$

$$= \frac{e^x (x-1)(x^3 - 3x^2 + 5x + 1)}{(1+x^2)^3}$$



$x^3 - 3x^2 + 5x + 1$ has only one real root α, where $\alpha \in (-1, 0)$

Hence, (A) and (C) are correct.

36. $f(x) = x^3 + x^2 - 10x = x(x^2 + x - 10)$ for $-1 \le x < 0$

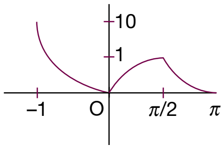
Graph of $f(x)$ is drawn as shown

Absolute maximum at $x = -1$

Local minima at $x = 0, \pi$

Local maxima at $x = -1, \pi/2$

Hence, (A) and (D) are correct.



37. $f(x) = x^3 - 3x + 1 = 0$

$x_1 < x_2 < x_3$

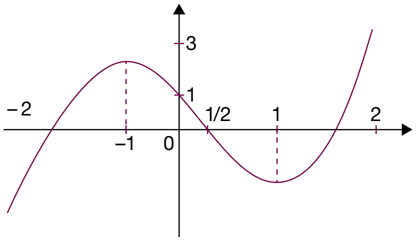
$f'(x) = 3x^3 - 3 = 3(x - 1)(x + 1)$

Local maxima and local minima are respectively $(-1, 3)$ and $(1, -1)$

Distance $= \sqrt{4 + 16} = 2\sqrt{5}$

$f(-1) = -1 + 3 + 1 = 3; f(1) = 1 - 3 + 1 = -1$

$$\left.\begin{aligned} f(-2) &= -8 + 6 + 1 = -1 \\ f(-1) &= -1 + 3 + 1 = 3 \end{aligned}\right\} x_1 \text{ root lies between } (-2, -1)$$



$$\left.\begin{aligned} f(0) &= 1 \\ f\left(\frac{1}{2}\right) &= \frac{1}{8} - \frac{3}{2} + 1 = -\frac{3}{8} \end{aligned}\right\} x_2 \text{ roots lies between } (0, 1/2)$$

$$\left.\begin{aligned} f(1) &= -1 \\ f(2) &= 8 - 6 + 1 = 3 \end{aligned}\right\} x_3 \text{ root lies between } (1, 2)$$

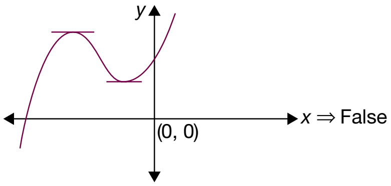
$[x_1] = -2; [x_2] = 0\ ; [x_3] = 1$

$[x_1] + [x_2] + [x_3] = -1$

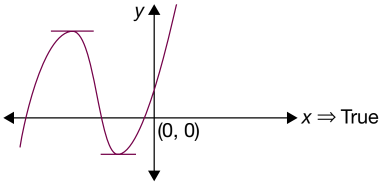
Hence, (A), (B) and (D) are correct.

38. $f(x) = x^3 + ax^2 + bx + c$

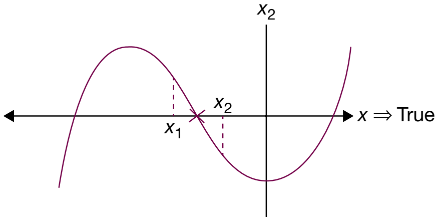
(A)



(B)



(C)



$f'(x) = 3x^2 + 2ax + b$

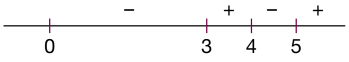
$f''(x) = 6x + 2a$

(D) $f''(x) = 2(3x + a) = 0 \Rightarrow x = \frac{-a}{3}$

$\Rightarrow$ True and $f''(x)$ changes sign also.

Hence, (B), (C) and (D) are correct.

39. We have $f(x) = \int_0^x e^{-t^2}\,(t - 5)\,(t^2 - 7t + 12)\,dt$



Sign of $f'(x)$

$\therefore \ f'(x) = e^{-x^2}\,(x - 5)\,(x^2 - 7x + 12) = e^{-x^2}\,(x - 3)\,(x - 4)\,(x - 5).$

Now, verify alternatives.

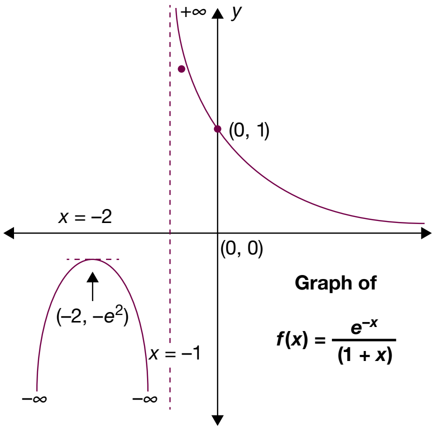
$f(x)$ has local maximum at $x = 4$ and local minimum at $x = 3$.

$f(x)$ is decreasing on $(0, 3) \cup (4, 5)$ and increasing on $(3, 4) \cup (5, \infty)$

$\because \ f'(x)$ has 3 real roots; $f''(x)$ has atleast two real roots and $f'''(x)$ has one real root.

Hence, (A), (C) and (D) are correct.

40.



Graph of $f(x) = \dfrac{e^{-x}}{(1 + x)}$

$$f'(x) = \frac{(1+x)(-e^{-x}) - e^{-x}}{(1+x)^2}$$

$$f'(x) = \frac{-e^{-x}(1+x+1)}{(1+x)^2}$$

Let $f(x) = \dfrac{e^{-x}}{1+x}$

$$\Rightarrow \; f'(x) = \frac{-e^{-x}(x+2)}{(1+x)}$$

$f'(x) = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2$

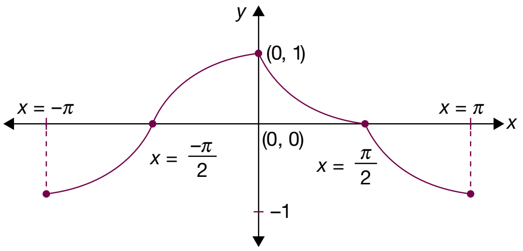
$\because \; x = -2$ is point of local maxima

$f(-2) = -e^2$

Now verify options.

Hence, (B), (C) and (D) are correct.

41.



Graph of *f*(*x*)

$f(x) = \min. (1, \cos x, 1 - \sin x), -\pi \le x \le \pi$

We have

$$f(x) = \begin{cases} \cos x \; ; & -\pi \le x \le 0 \\ 1 - \sin x ; & 0 < x \le \dfrac{\pi}{2} \\ \cos x \; ; & \dfrac{\pi}{2} < x \le \pi \end{cases}$$

$$f'(x) = \begin{cases} -\sin x \; ; & -\pi < x < 0 \\ -\cos x \; ; & 0 < x < \dfrac{\pi}{2} \\ -\sin x \; ; & \dfrac{\pi}{2} < x < \pi \end{cases}$$

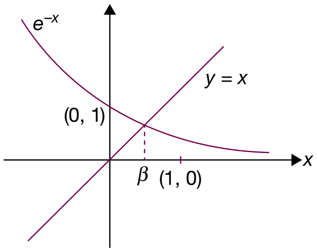
$f'(0^-) = 0$ and $f'(0^+) = -1 \Rightarrow f(x)$ is not differentiable at $x = 0$.

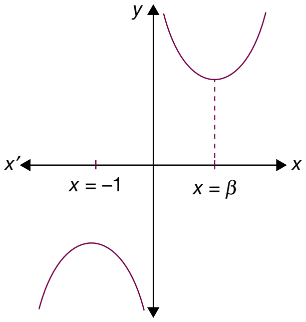
Clearly $f(x)$ has local maximum at $x = 0$.

$f(x)$ is differentiable at $x = \dfrac{\pi}{4}$ also $f(x)$ is continuous in $[-\pi, \pi]$ and $f(x) \in [-1, 1]$

Hence, (A), (B), (C) and (D) are correct.

42. $f(x) = xe^x + \dfrac{1}{xe^x}$ where $x \ne 0$



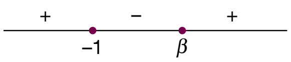


$$f'(x) = (xe^x + e^x) - \frac{1}{(xe^x)^2}(xe^x + e^x)$$

$$= \frac{(xe^x + e^x)(xe^x - 1)(xe^x + 1)}{(xe^x)^2}$$

$$= \frac{e^{3x}(x+1)(x+e^{-x})(x-e^{-x})}{(xe^x)^2}$$

Sign scheme of *f′*(*x*)



$$f'(x) = 0; \; \frac{e^{3x}(x+1)(x+e^{-x})(x-e^{-x})}{(xe^x)^2} = 0$$

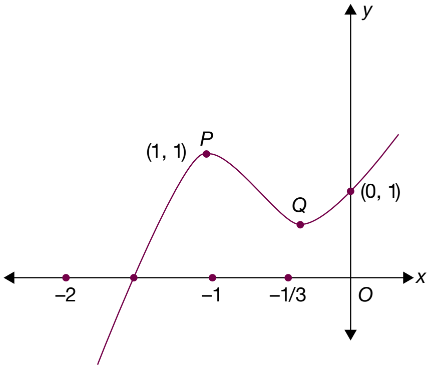
$x = -1, e^{-x} = -x$ (No solution)

$e^{-x} = x$ (exactly one solution, let it is β)

at $x = -1$, local maxima at $x = \beta \in (0, 1)$ local minima.

Hence, (A), (B) and (D) are correct.

43.



$f(x) = 1 + x(x + 1)^2$

$f'(x) = 3x^2 + 4x + 1 = (3x + 1)(x + 1)$

$x = -1, x = \dfrac{-1}{3}$

$$f(-1) = 1, \; f\left(\frac{-1}{3}\right) = 1 - \frac{1}{3} . \left(\frac{4}{9}\right) = 1 - \frac{4}{27} = \frac{23}{27}$$

Point P (–1, 1)

Point $Q \left(-\dfrac{1}{3}, \dfrac{23}{27}\right)$

$M_{OP} = -1, m_{OQ} = \dfrac{-23}{9}$

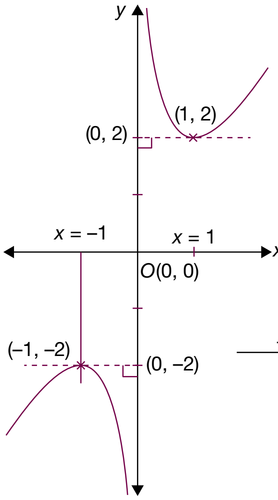
$$\tan\theta = \left| \frac{-1 + \dfrac{23}{9}}{1 + \dfrac{23}{9}} \right|$$

$$\tan\theta = \left| \frac{14}{32} \right|$$

$\tan\theta = \dfrac{7}{16}$

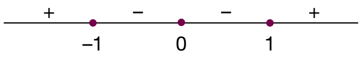
Hence, (A), (B), (C) and (D) are correct.

44.



Graph of $f(x) = x + \dfrac{1}{x}$

Sign scheme of $f'(x)$



$$(f(x) - 1)(x^2 + x + 1)^2 - (f(x) + 1)(x^4 + x^2 + 1) = 0$$

We have, $\dfrac{f(x)+1}{f(x)-1} = \dfrac{(x^2+x+1)^2}{(x^4+x^2+1)}$

$$\frac{f(x)+1}{f(x)-1} = \frac{(x^2+x+1)^2}{(x^2+x+1)(x^2-x+1)}$$

$$\frac{f(x)+1}{f(x)-1} = \frac{x^2+x+1}{x^2-x+1}$$

$\Rightarrow\ f(x) = x + \dfrac{1}{x} \Rightarrow |f(x)| \geq 2$

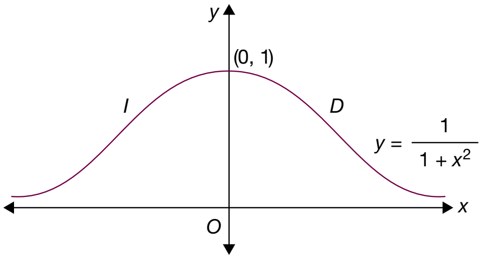
Also, $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{(x-1)(x+1)}{x^2}$

$\therefore$ Local minimum at $x = 1$ and local maximum at $x = -1$

Also, $f(x)$ is an odd function $\Rightarrow \displaystyle\int_{-\pi}^{\pi} (\cos x)\cdot f(x)\,dx = 0$.

Hence, (A), (B), (C) and (D) are correct.

45.



$f(x) > 0$ for all x

f is increasing for $x < 0$ and decreasing the $x > 0$

$\therefore\ f'(x) > 0$ for $x < 0$ …(1)

and $f'(x) > 0$ for $x < 0$ …(2)

Now $g(x) = \dfrac{1}{f\left(\dfrac{1}{x}\right)}$

$$g'(x) = \frac{1}{f^2\left(\dfrac{1}{x}\right)} \cdot f'(x)\left(\frac{1}{x}\right)\cdot\left(\frac{1}{x^2}\right)$$

for $x < 0$, $g'(x) =$ (positive) (positive) (positive) > 0

$\Rightarrow\ g(x)$ is increasing for $x < 0$.

for $x > 0$ $g'(x) =$ (positive) (negative) (positive) < 0

$\therefore\ g(x)$ is decreasing for $x > 0$

Alternative:

Consider the graph of $f(x) = \dfrac{1}{1+x^2}$ and interpret

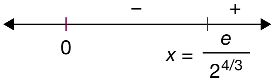
Hence, (B) and (D) are correct.

46. $f(x) = \displaystyle\int_x^{2x} (t\ln t - t)dt$ for all $x > 0$

$$f'(x) = 2(2x\ln(2x) - 2x) - (x\ln x - x)$$
$$= 4x\ln(2x) - x\ln x - 3x$$
$$= x[\ln(16x^4) - \ln x - 3]$$

$\therefore\ f'(x) = x[\ln(16x^3) - 3]$

Now, $f'(x) > 0$



$\Rightarrow\ 16x^3 > e^3 \Rightarrow 2^{\frac{4}{3}}x > e$

$\therefore\ x > \dfrac{e}{2^{\frac{4}{3}}}$

Clearly, $f(x)$ has a local minima at $x = \dfrac{e}{2^{\frac{4}{3}}}$.

$$\lim_{x\to\infty}\frac{f'(x)}{x^2} = \lim_{x\to\infty}\frac{x[\ln 16x^3 - 3]}{x^2}, \qquad \left(\frac{\infty}{\infty}\text{ Form}\right)$$

$$= \lim_{x\to\infty}\frac{\dfrac{x}{16x^3}\cdot 16.3x^2 + \left[\ln(16x^3) - 3\right]}{2x}$$

$$= \lim_{x\to\infty}\frac{\ell n(16x^3) - 3}{2x}, \qquad \left(\frac{\infty}{\infty}\text{ Form}\right)$$

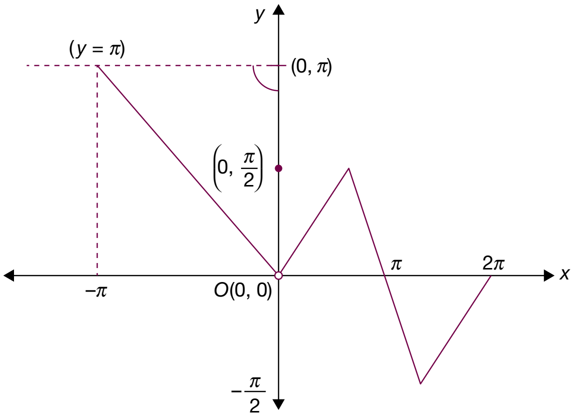
$$= \lim_{x\to\infty}\left(\frac{1}{16x^3}\,\frac{16.3x^2}{2}\right) = \lim_{x\to 0}\frac{2}{2x} = 0$$

Hence, (B), (C) and (D) are correct.

47. From graph we can observe $f(x)$ has local maxima at $x = 0$.

$f(x)$ is discontinuous at $x = 0$

$f(x)$ has maximum value $= \pi$ and minimum value $-\dfrac{\pi}{2}$



Hence, (A) and (C) are correct.

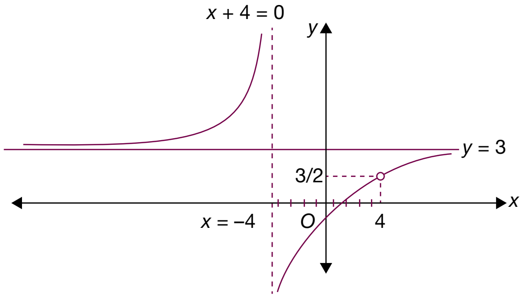
48. $f(x) = \dfrac{3x(x-4)}{(x-4)(x+4)}$

Domain of $f(x)$ is $R - \{-4, 4\}$

$$f(x) = \frac{3x(x-4)}{(x-4)(x+4)}$$

$$f(x) = \frac{3x}{x+4}, x \neq 4$$

$$f(x) = 3\left[1 - \frac{4}{x+4}\right]$$



(A) $\lim\limits_{x\to4} f(x) = \lim\limits_{x\to4} \dfrac{3x}{x+4} = \dfrac{12}{8} = \dfrac{3}{2}$

(B) $f(x)$ is monotonic in domain

(C) $f(x) \neq \dfrac{3}{2}, 3$ no solution for $k = \dfrac{3}{2}, 3$

(D) discontinuous at $x = -4$ and $x = 4$

NOTE:
The graph of $f(x) = \dfrac{3x^2 - 12x}{x^2 - 16}$ passes through $O(0, 0)$.

Hence, (B) and (C) are correct.

49. $x f'(x) + f(x) = 1$

$d(xf(x)) = 1$ integrating both side

$$x f(x) = x + C$$

$$f(1) = 1 + C \Rightarrow C = 1$$

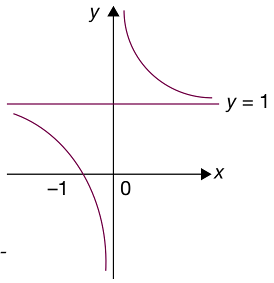
$$f(x) = \frac{x+1}{x}$$

$$f(x) = 1 + \frac{1}{x}; f'(x) = -\frac{1}{x^2}$$

$\Rightarrow$ f is always derivable and decreasing in its domain

$\Rightarrow$ monotonic also f is not bounded.

The graph of $y = f(x)$ is as shown



$y = 1$ and $x = 0$ are the two asymptotes and range is $R - \{1\}$

Only C and D are correct.

Hence, (C) and (D) are correct.

50. $f(x) = \displaystyle\int_{-4\pi^2}^{x^2} \frac{\sin\sqrt{\theta}}{1+\cos^2\sqrt{\theta}}\, d\theta$

$$f'(x) = 0$$

$$f'(x) = \frac{\sin|x|}{(1+\cos^2 x)} . 2x = 0$$

$\Rightarrow$ $x = 0, \sin|x| = 0$

$$x = -\pi, 0, \pi, 2\pi$$

$\lim\limits_{x\to2\pi^+} f'(x) =$ positive quantity.

$\lim\limits_{x\to2\pi^-} f'(x) =$ Negative, so minima at $x = 2\pi$.

$\lim\limits_{x\to\pi^-} f'(x) = +\text{ve}$, $\lim\limits_{x\to\pi^+} f'(x) = -\text{ve}$, $\Rightarrow$ maxima at $x = \pi$.

$\lim\limits_{x\to-\pi^-} f'(x) = +\text{ ve}$, $\lim\limits_{x\to-\pi^+} f'(x) = -\text{ve}$, $\Rightarrow$ maxima at $x = -\pi$.

$\lim\limits_{x\to0^-} f'(x) =$ Negative

$\lim\limits_{x\to0^+} f'(x) =$ positive

$\Rightarrow$ $f(x)$ has local minima at $x = 0$

Hence, (A), (B), (C) and (D) are correct.

Linked Comprehension Type Questions

1. $f(x) = 2\dfrac{(1+\cos 4x)}{2} + \dfrac{3}{4}\sin 4x + ax$

$$= 1 + ax + \frac{1}{4}(4\cos 4x + 3\sin 4x)$$

$$f(x) = 1 + ax + \frac{5}{4}\sin(4x + \alpha)$$

$$f'(x) = a + 5\cos(4x + \alpha) \geq 0 \ \forall\ x \in R$$

$$a \geq 5$$

Hence, the correct answer is (B).

2. $f'(x) \neq 0$

$$a \in (-\infty, -5) \cup (5, \infty)$$

Hence, the correct answer is (D).

3. $\dfrac{dy}{dx} = f'(x) = a + \dfrac{1}{4}(-16\sin 4x + 12\cos 4x)_{x=0} = \tan 45°$

$a + 3 = 1$ we get $a = -2$

Hence, the correct answer is (C).

4. $f(x) = [x] + \sqrt{\{x\}}$

$$= [x] + \sqrt{x - [x]}$$

$f(x)$ is Convex, Continuous and Strictly increasing

Hence, the correct answer is (C).

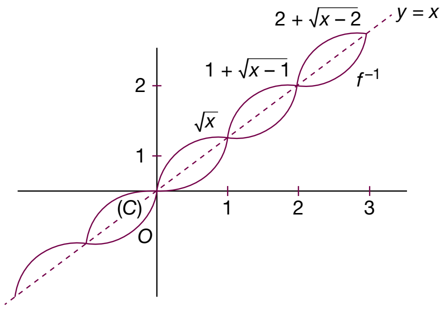
5. f^{-1} is image of $f(x)$ about line $y = x$

So it is concave and increasing

$$\frac{f^{-1}(1\cdot2) + f^{-1}(1\cdot8)}{2} > f^{-1}(1\cdot5)$$

$$\frac{f(1\cdot2) + f(1\cdot8)}{2} < f(1\cdot5)$$

$$f(-1\cdot3) > f^{-1}(-1\cdot3)$$



Hence, the correct answer is (C).

6. $6 < x < 7$

$$f(x) = 6 + \sqrt{x-6} = y$$

$$x - 6 = (y-6)^2 \Rightarrow x = (y-6)^2 + 6$$

$$f^{-1}(x) = (x-6)^2 + 6\backslash$$

$$m_T = \frac{d}{d_x}(f^{-1}(\alpha)) = 2(x-6)\big|_{x=6.5} = 1$$

$x = 6{\cdot}5$ and $y = (x-6)^2 + 6 = 6\cdot 25$

tangent $y - 6{\cdot}25 = 1(x - 6{\cdot}5)$

$x - y - 0{\cdot}25 = 0$, we get $a = -1$, $b = -0{\cdot}25$

$a + [b] = -1 + [-0{\cdot}25] = -1 + (-1) = -2$

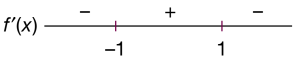
Hence, the correct answer is (A).

7. $f : R \to R$

$$\int_3^x f(t)\,dt = \frac{x^2}{2} + \int_x^3 t^2 f(t)\,dt - \frac{9}{2}$$

Diff.

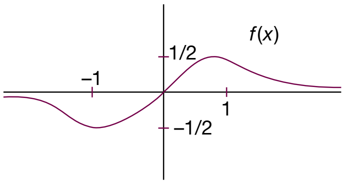
$f(x) - 0 = x + 0 - x^2 f(x) - 0;$



$$f(x) = \frac{x}{1+x^2}$$

$$f'(x) = \frac{(1+x^2)\cdot 1 - x2x}{(1+x^2)^2} = \frac{(1-x)(1+x)}{(1+x^2)^2}$$

Clearly $f(x)$ has local maxima and minima so non-monotonic (Manyone)



$f(-1) = -1/2;\ f(1) = 1/2$

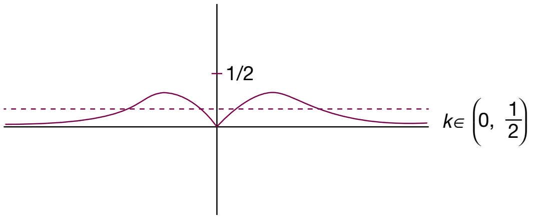
$\int_{-1}^{1} f(x)\,dx = 0$ as $f(x)$ is odd function

Range $\left[-\frac{1}{2}, \frac{1}{2}\right]$ is not R so $f(x)$ is many-one into

(A,C,D) correct; incorrect (B)

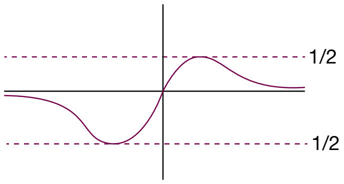
Hence, the correct answer is (B).

8. $f(|x|) = k$ to have four solutions, we can conclude from graph that $k \in \left(0, \frac{1}{2}\right)$



Hence, the correct answer is (C).

9. Required distance between parallel horizontal tangent = 1



Hence, the correct answer is (A).

10. $f(x) = a(x-2)(x-b)$

$a, b \in R, a \neq 0$

$$f(f(x)) = a^3\left(x^2 - (2+b)x + 2b - \frac{2}{a}\right)\left(x^2 - (2+b)x + 2b - \frac{b}{a}\right)$$

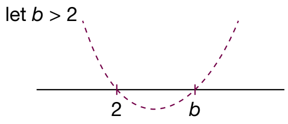
$a \neq 0$

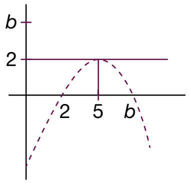
$= a(f(x) - 2)(f(x) - b) = 0$

$f(x) = 2;\ f(x) = b$

for one real zero 5.

$f(x)$ has roots 2 and b and $f(f(x)) = 0$ for only one $x = 5$





max. of $f(x) = 2$ at $x = 5$

$\frac{b+2}{2} = 5$ we get $b = 8$

$f(5) = a(5-2)(5-b) = 2$

$a = -2/9$

Hence, the correct answer is (B).

11. $f(x) = a(x^2 - (2+b)x + 2b)$ let point (x_1, y_1) on curve then

$y_1 = a(x_1^2 - (2+b)x_1 + 2b)$ …(1)

Slope of tangent $= f'(x) = a(2x - 2 - b)|_{x_1}$

Equation of tangent $y - y_1 = a(2x_1 - 2 - b)(x - x_1)$

It passing through (0, 0); $0 - y_1 = a(2x_1 - 2 - b)(0 - x_1)$

$a(x_1^2 - (2+b)x_1 + 2b) = ax_1(2x_1 - 2 - b)$

$a x_1^2 - ax_1(2+b) + 2ab = 2a x_1^2 - 2ax_1 - abx_1$

$a x_1^2 - abx_1 + abx_1 - 2ab = 0$

$x_1^2 = 2b = 16;\ x_1 = \pm 4$

$f(x) = -\frac{2}{9}(x-2)(x-8)$

for $x_1 = +4$; slope of tangent $= f'(4) = 4/9$

for $x_1 = -4$; slope of tangent $= f'(-4) = 4$

Hence, the correct answer is (C).

12. Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e$

$$\lim_{x\to 0}\left(3 - \frac{P(x)}{x}\right) = 27 \Rightarrow \lim_{x\to 0}\frac{P(x)}{x} = -24$$

We get $d = -24;\ e = 0$

$P(1) = a + b + c + d = -9$

Or $a + b + c = 15$ …(1)

$P'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$P'(2) = 32a + 12b + 4c + d = 0$

Or $8a + 3b + c - 6 = 0$ …(2)

$P''(x) = 12ax^2 + 6bx + 2c$

$P'''(x) = 24ax + 6b$

$P'''(2) = 48a + 6b = 0$ or $b = -8a$ …(3)

Solving (1)(2)(3) we get $a = 1$, $b = -8$, $c = 22$, $d = -24$

Set A: $x^2 + 12 \le 7x$ or $(x-3)(x-4) \le 0$ or $3 \le x \le 4$

$P'(x) = 4ax^3 + 3bx^2 + 2cx + d = 4x^3 - 24x^2 + 44x - 24$

$P''(x) = 12x^2 - 48x + 44 = 4(3x^2 - 12x + 11) > 0$ where $x \in [3, 4]$

$P'(x)$ is increasing

minimum at $x = 3$

$P'(3) = 4(x^3 - 6x^2 + 11x - 6) = 4(27 - 54 + 33 - 6)$

$P'(3) = 0$

Hence, the correct answer is (B).

13. $P''(x) = 12ax^2 + 6bx + 2c = 12x^2 - 48x + 44$

$m = P'''(x) = 24x - 48\,|_{x=3/2} = -12$

$x = \frac{3}{2}$

$$P''\left(\frac{3}{2}\right)=12\left(\frac{9}{4}\right)-48\left(\frac{3}{2}\right)+44=-1$$

Tangent $y+1=-12\left(x-\frac{3}{2}\right)$

or $y+12x=17$

Hence, the correct answer is (D).

14. $P'(x)=\alpha(x-1)(x-2)(x-3)=\alpha(x^3-6x^2+11x-6)$

$$P(x)=\alpha\left(\frac{x^4}{4}-\frac{6x^3}{3}+\frac{11x^2}{2}-6x\right)+k$$

$$P(0)=0=k$$

$$P(-1)=\alpha\left(\frac{1}{4}+2+\frac{11}{2}+6\right)=55$$

$$\alpha=4$$

$$P(x)=4\left(\frac{x^4}{4}-\frac{6x^3}{3}+\frac{11x^2}{2}-6x\right)$$

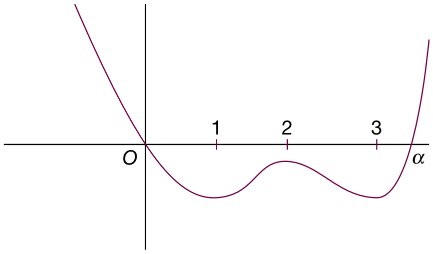
$$P(x)=x^4-8x^3+22x^2-24x$$

Extrema points are (1, –9) (2, –8) (3, –9)

$$\text{Area of triangle}=\frac{1}{2}\begin{vmatrix}1 & -9\\ 2 & -8\\ 3 & -9\\ 1 & -9\end{vmatrix}=1$$

Hence, the correct answer is (D).

15.



$$P(x)=x^4-8x^3+22x^2-24x$$

Extrema (1, –9) (2, –8) (3 –9)

Range [–9, ∞)

Sum of roots = 8

Sum of real roots = 0 + α

Number of negative integers in range are 9

Two point of inflection, two relative minima and one local maxima

Hence, the correct answer is (B).

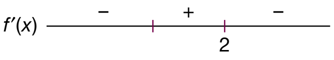
16. $f: R \to R$

$f(x)=x^2e^{-x}$ is Neither Even Nor Odd

$$f'(x)=2xe^{-x}+x^2(-e^{-x})$$

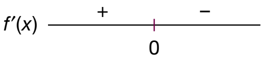
$$=e^{-x}x(2-x)$$

$$\lim_{x\to\infty}\frac{x^2}{e^x}\to 0$$



Hence, the correct answer is (B).

17. $g(x)=\int_0^{2e^x}\frac{f'(t)}{1+t^2}\,dt$



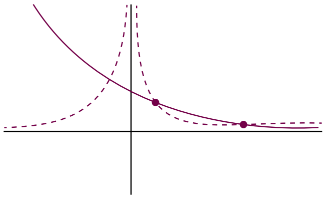
$$\Rightarrow\ g'(x)=\frac{f'(2e^x)}{1+4e^{2x}}\cdot 2e^x$$

$$g'(x)=\frac{2e^x}{1+4e^{2x}}\cdot 2e^x\cdot e^{-2e^x}\cdot 2(1-e^x)$$

$g(x)$ increases on (–∞, 0) and decreases on (0, ∞)

Hence, the correct answer is (A).

18.



$$e^{-x}=\frac{1}{4x^2}$$

At $x\to\infty$; $e^{-x} <<<< \frac{1}{4x^2}$

3 Solution

Hence, the correct answer is (D).

19. $h(x)=f(x)-a(f(x))^2+a(f(x))^3$

$$\Rightarrow\ h'(x)=f'(x)-2af(x)f'(x)+3a(f(x))^2f'(x)$$

$$\Rightarrow\ h'(x)=f'\left[3a(f(x))^2-2af(x)+1\right]$$

Now $h(x)$ increases if $f(x)$ increases and $3a(f(x))^2$

$-2af(x)+1>0$ for all $x\in R$

$\Rightarrow$ $3a>0$ and $4a^2-12a\le 0$

$\Rightarrow$ $a>0$ and $a\in[0,3]$

$\Rightarrow$ $a\in[0,3]$.

Hence, the correct answer is (A).

20. $h(x)$ increases as $f(x)$ decreases for all real values of x if

$3a(f(x))^2-2af(x)+1\le 0$ for all $x\in R$

$\Rightarrow$ $3a<0$ and $4a^2-12a\le 0$

$\Rightarrow$ $a<0$ and $a\in[0,3]$

$\Rightarrow$ no such a is possible.

Hence, the correct answer is (D).

21. $h(x)$ is non- monotonic functions if

$3a(f(x))^2-2af(x)+1$ changes

sign for which $D\ge 0$ or $4a^2-12a\ge 0$

$\Rightarrow$ $a\in(-\infty,0]\cup[3,\infty)$.

Hence, the correct answer is (D).

Combined solution to 22–24

If function satisfies

$$(x-y)f(x+y)-(x+y)f(x-y)=2y(x^2-y^2)$$

$$Af(B)-Bf(A)=(B-A)AB$$

Now $B=1$

$A=X$

$Xf(1)-1f(X)=(1-X)X$

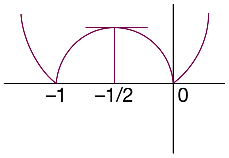
$2X-f(X)=X-X^2$

$X^2+X=f(X)$

$f'(x)=2x+1$

(a) $f'(x)>0\ x>-\frac{1}{2}$

(b)



(c) $y=1/4$

22. **Hence, the correct answer is (C).**

23. **Hence, the correct answer is (A).**

24. **Hence, the correct answer is (B).**

25. If $\alpha = 1 \Rightarrow f(x) = 8x^3 + 4x^2 + 2\beta x + 1$

$$\Rightarrow f'(x) = 24x^2 + 8x + 2\beta \text{ for increasing function,}$$

$$f'(x) > 0 \ \forall x \in R$$

$$\Rightarrow D < 0$$

$$\Rightarrow 16 - 4.12.\beta < 0 \Rightarrow \beta > \frac{1}{3}$$

Hence, the correct answer is (B).

26. If $\beta = 1 \Rightarrow f(x) = 8x^3 + 4\alpha x^2 + 2x + \alpha$

$\Rightarrow f'(x) = 24x^2 + 8\alpha x + 2$ for non-monotonic, $f'(x) = 0$ must have distinct roots

$$\Rightarrow D > 0 \Rightarrow \alpha^2 > 3 \Rightarrow \alpha > \sqrt{3} \text{ (or) } \alpha < -\sqrt{3}$$

sum of integral values of α in [1, 100] is 5049. ($a \in \{2, 3, 4, \ldots, 100\}$)

Hence, the correct answer is (D).

27. If a, b, c are the roots of $f(x)$ the $\log_2 a + \log_2 b + \log_2 c$

$$\Rightarrow abc = 32 \Rightarrow \frac{-\alpha}{8} = 32 \Rightarrow \alpha = -256$$

Hence, the correct answer is (A).

Combined solution to 28–30

$$f(x) = x^3 - 3(7 - a)x^2 - 3(9 - a^2)x + 2$$

$$f'(x) = 3x^3 - 6(7 - a)x - 3(9 - a^2)$$

For real root $D \geq 0$; $4(7 - a)^2 + 4(9 - a^2) \geq 0$

$$49 + a^2 - 14a + 9 - a^2 \geq 0$$

$$14a \leq 58$$

$$a \leq \frac{58}{14} \qquad \ldots(1)$$

Let $f'(x)$ has two roots x_1 and x_2 $(x_1 > x_2)$

Minima at $x = x_2$

28. Both roots –ve $\Rightarrow 2(7 - a) < 0 \Rightarrow a > 7$

Not possible

Hence, the correct answer is (A).

29. Both roots are +ve

Sum of roots $> 0 \Rightarrow a < 7$ …(2)

Product of roots $> 0 \Rightarrow a \in (-\infty, -3)\ (3, \infty)$ …(3)

From 1, 2, 3, we get $(-\infty, -3) \cup \left(\frac{58}{14}, \infty\right)$

Hence, the correct answer is (B).

30. For points of opposite sign,

Product of roots < 0

$-(9 - a^2) < 0 \Rightarrow a^2 - 9 < 0$

$a \in (-3, 3)$

Hence, the correct answer is (B).

31. $f'(x) = 6x^2 - 6(a - 3)x + 6a$

f has no point of extrema $\Rightarrow f'(x) = 0$ has no real root or $f'(x) = 0$ has a double root

$$D \leq 0 \Rightarrow (a - 3)^2 - 4a \leq 0$$

$$\Rightarrow (a - 1)(a - 9) \leq 0 \Rightarrow a \in [1, 9]$$

Hence, the correct answer is (B).

32. $f'(x) = 6(x^2 - (a - 3)x + a)$.

so $f'(x) = 0$ has a pair of real and distinct roots α and $\beta(\alpha < \beta)$ if and only if

$(a - 3)^2 > 4a$ i.e. $(a - 1)(a - 9) > 0$ (or) if $a < 1$ or $a > 9$

Hence, the correct answer is (A).

33. The roots of $f'(x) = 0$ are given by

$$\alpha = \frac{(a-3) - \sqrt{(a-1)(a-9)}}{2} \text{ and } \beta = \frac{(a-3) + \sqrt{(a-1)(a-9)}}{2}$$

Since $\alpha < \beta$, f has a local maximum at $\alpha(< 0)$ and a local minimum at $\beta(< 0)$ only if

$$\alpha\beta = \frac{(a-3)^2 - (a-1)(a-9)}{4} < 0 \text{ i.e. only if } a < 0$$

Hence, the correct answer is (D).

34. $f''(x) > 0 \Rightarrow f'(x)$ is an increasing function

$f'(x_1) > f'(x_2) \Rightarrow x_1 > x_2,\ f'(x_1) = f'(x_2) \Rightarrow x_1 = x_2$

$h'(x) = \sin 2x(f'(\sin^2 x) - f'(\cos^2 x))$

$f'(x) = 0 \Rightarrow \sin 2x = 0 \Rightarrow x = 0$

or $f'(\sin^2 x) = f'(\cos^2 x) \Rightarrow \sin^2 x = \cos^2 x$

$$\Rightarrow \tan^2 x = 1 \Rightarrow x = \pm\frac{\pi}{4}$$

Hence, the correct answer is (C).

35. $f'(\sin^2 x) < f'(\cos^2 x) \Rightarrow \sin^2 x < \cos^2 x \Rightarrow \tan^2 x < 1$

$$\Rightarrow x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

Hence, the correct answer is (A).

36. $h(x)$ is increasing $\Rightarrow h'(x) > 0$

CASE 1:

$$\left.\begin{array}{l} \text{(i) } \sin 2x > 0 \Rightarrow x \in \left(0, \frac{\pi}{2}\right) \\ \text{(ii) } f'(\sin^2 x) > f'(\cos^2 x) \Rightarrow \tan^2 x > 1 \end{array}\right\}$$

$$\Rightarrow x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$$

CASE 2:

$$\left.\begin{array}{l} \text{(i) } \sin 2x < 0 \Rightarrow x \in \left(-\frac{\pi}{2}, 0\right) \\ \text{(ii) } f'(\sin^2 x) < f'(\cos^2 x) \Rightarrow \tan^2 x < 1 \end{array}\right\}$$

$$\Rightarrow x \in \left(\frac{-\pi}{4}, 0\right)$$

Hence, the correct answer is (C).

37. $y' = 4x^3 + 3x^2 - 36x + 24$

$$y'' = 12\left(x^2 + \frac{x}{2} - 3\right)$$

$y'' > 0$ on $(-\infty, -2) \cup (3/2, \infty)$ $y'' < 0$ on $(-2, 3/2)$

$y'' = 0$ at $x = -2$ and $x = 3/2$

$\therefore$ $(-2, -124)$ and $\left(\frac{3}{2}, -8\frac{1}{16}\right)$ are points of inflection

Hence, the correct answer is (B).

38. $y' = \sin(\log x) + \cos(\log x)$

$$y'' = \frac{\sqrt{2}\sin(\pi/4 - \log x)}{x}$$

$y'' = 0$ at $x_k = e^{\pi/4 + K\pi}$ $K = 0, +1, +2, \ldots$

y'' changes sign when passing through each point X_k

$\therefore$ $f(x)$ has infinite no. of points of inflection

Hence, the correct answer is (D).

39. $y'' = 12x^2 + 6ax + 3$

$y'' \geq 0\ \forall \in R$

$\therefore$ y is concave on entire number scale

$4x^2 + 2ax + 1 \geq 0\ \forall x \in R$

$\therefore$ $4a^2 - 16 \leq 0$

$\Rightarrow$ $|a| \leq 2$

Hence, the correct answer is (C).

Combined solution to 40–42

$$f(x+y)=f(x)+f(y)-8xy$$

diff partially w.r.t x and y $f'(x+y)=f'(x)-8y$

$$f'(x+y)=f'(y)-8x$$

$\therefore$ $f'(x)+8x$ is constant has

$$f'(x)=k-8x$$

$\therefore$ $f(x)=8x-4x^2$ maximum at (1, 4)

Similarly $g(x) = x^3 - 4x$

$$g'(x)=3x^2-4=3\left(x-\frac{2}{\sqrt{3}}\right)\left(x+\frac{2}{\sqrt{3}}\right)$$

$$\alpha=\beta=\gamma=1$$

Required area $= \int_0^2 (f(x)-g(x))\,dx=\frac{28}{3}$

40. **Hence, the correct answer is (A).**

41. **Hence, the correct answer is (D).**

42. **Hence, the correct answer is (A).**

43. The equation $P(x) = 0$ has 2 distinct roots and the equation $P'(x) = 0$ has 3 distinct roots.

Also, $h(x) = (P'(x))^2 + P(x)\,P''(x) = \frac{d}{dx}(P(x)\cdot P'(x))$

So, by Rolle's theorem to $y = P(x)\,P''(x)$, we get $h(x) = 0$ has minimum 4 real roots.

Hence, the correct answer is (B).

44. Consider, $P(x) = -k$ has 4 distinct real roots, $k \in (-2, -1)$

$\therefore$ $[\alpha] = -2$, $[\beta] = -1$, $[\gamma] = 0$, $[\delta] = 2$

Hence, the correct answer is (A).

Combined solution to 45–47

(i) $f(x) = e^{(p+1)x} - e^x$

$$f'(x) = (p+1)\cdot e^{(p+1)x} - e^x = 0$$

$$e^x[(p+1)e^{px} - 1] = 0 \qquad (e^x \neq 0)$$

$$\Rightarrow \quad e^{px} = \frac{1}{p+1}$$

$$\Rightarrow \quad px = -\ln(p+1)$$

$$x = \frac{-\ln(p+1)}{p}$$ [verify that $f''(x) > 0$]

Hence $x = s_p = \dfrac{-\ln(p+1)}{p}$

(ii) $g(t) = \int_t^{t+1} (e^{px}\cdot e^x - e^x)e^t e^{-x}dx = e^t \int_t^{t+1} (e^{px}-1)dx$

On integrating

$$g(t) = e^t\left[\frac{e^{px}}{p}-x\right]_t^{t+1} = e^t\left[\left\{\frac{e^{P(t+1)}}{P}-t-1\right\}-\left\{\frac{e^{Pt}}{P}-t\right\}\right]$$

$$= e^t\left[\frac{e^{pt}(e^p-1)}{p}-1\right]$$

$$g(t) = \frac{(e^p-1)}{p}\,e^{(p+1)t} - e^t$$

$$g'(t) = (p+1)\,\frac{(e^p-1)}{p}\,e^{(p+1)t} - e^t = 0$$

$$\Rightarrow \quad \frac{(p+1)(e^p-1)}{p}\,e^{pt} = 1 \Rightarrow e^{pt} = \frac{p}{(p+1)(e^p-1)}$$

$$\Rightarrow \quad pt = \ln\frac{p}{(p+1)(e^p-1)}$$

$$\therefore \quad t = \frac{-1}{p}\ln\left(\frac{(p+1)(e^p-1)}{p}\right)$$

$$\Rightarrow \quad t_p = \frac{-1}{p}\ln\left(\frac{(p+1)(e^p-1)}{p}\right)$$

(iii) $s_p - t_p = \dfrac{1}{p}\ln\left(\dfrac{(p+1)(e^p-1)}{p}\right) - \dfrac{\ln(p+1)}{p}$

$$= \frac{1}{p}\ln\left(\frac{(p+1)(e^p-1)}{p}\cdot\frac{1}{(p+1)}\right) = \frac{1}{p}\ln\left(\frac{e^p-1}{p}\right)$$

Hence $s_p - t_p = \lim\limits_{p\to 0}\left(\dfrac{1}{p}\ln\dfrac{(e^p-1)}{p}\right)$

$$= \lim_{p\to 0}\ln\left(\frac{e^p-1}{p}\right)^{1/p} \to 1^{\infty} \text{ from}$$

$$\lambda = \lim_{p\to 0}\frac{1}{p}\left(\frac{e^p-1}{p}-1\right)\lim_{p\to 0}\left(\frac{e^p-1-p}{p}\right) = \frac{1}{2}$$

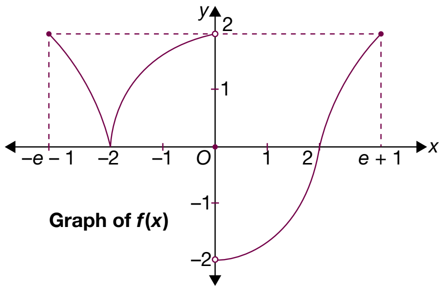
45. **Hence, the correct answer is (A).**

46. **Hence, the correct answer is (C).**

47. **Hence, the correct answer is (B).**

Combined solution to 48–50

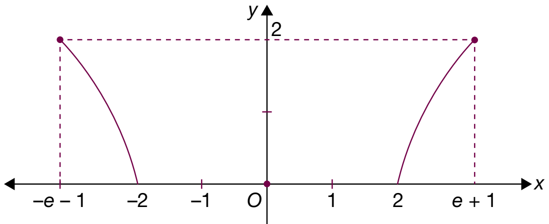
$$\text{Given, } f(x) = \begin{cases} 2\ln(-x-1), & -e-1 \le x \le -2 \\ \frac{1}{2}(4-x^2), & -2 < x < 0 \\ 0, & x = 0 \\ \frac{1}{2}(x^2-4), & 0 < x < 2 \\ 2\ell n(x-1), & 2 \le x \le e+1 \end{cases}$$



Graph of *f*(*x*)

$$\text{Clearly, } g(x) = \begin{cases} 2\ln(-x-1), & -e-1 \le x < -2 \\ 0, & -2 \le x < 2 \\ 2\ln(x-1), & 2 < x \le e+1 \end{cases}$$

(i) From above graph of $g(x)$, range of $g(x)$ is [0, 2] and $g(x) = 0$ in interval [–2, 2]. Also $g(x)$ is an even continuous function.



Graph of *f*(*x*)

(ii) Clearly, $g(x)$ is non-differentiable at $x = -2, 2$.

(iii) Clearly, $g(x) = k$ has exactly two distinct solutions, then integral value of k is either 1 or 2. So, sum of all possible integral values of $k = 1 + 2 = 3$.

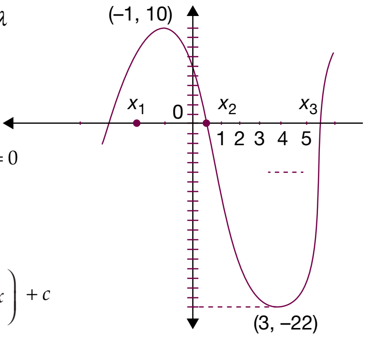
48. **Hence, the correct answer is (D).**

49. **Hence, the correct answer is (C).**

50. **Hence, the correct answer is (D).**

Combined solution to 51–53

$f''(x) = 6k(x-1);\ k > 0$ $\quad (\because\ f'(x)$ has local minimum at $x = 1$.)



$$\int f''(x)\,dx = \int 6k(x-1)\,dx$$

$$f'(x) = 6k\left(\frac{x^2}{2} - x\right) + \lambda$$

$f'(x) = 3k(x^2 - 2x) + \lambda$

$f'(-1) = 0$

$\Rightarrow\ 9k + \lambda = 0$

$\Rightarrow\ f'(x) = 3k(x^2 - 2x - 3) = 0$

$f'(x) = 0$

$\Rightarrow\ x = -1, 3$

Integrate $f'(x)$ to get

$$\therefore\ f'(x) = 3k\left(\frac{x^3}{3} - x^2 - 3x\right) + c$$

Given $f(-1) = 10$ and $f(0) = 5$

$c = 5$ and $k = 1$

$\therefore\ f(x) = x^3 - 3x^2 - 9x + 5$

Distance between horizontal tangents $|10 - (-22)| = 32$.

$[x_1] = -3$

$[x_2] = 0$

$[x_3] = 4$

$[x_1]^2 + [x_2]^2 + [x_3]^2 = 3^2 + 0^2 + 4^2 = 25$.

51. **Hence, the correct answer is (C).**

52. **Hence, the correct answer is (B).**

53. **Hence, the correct answer is (D).**

Matrix Match/Column Match Type Questions

1. **A → (p, q, r, s); B → (q); C → (q, r, t); D → (q, t)**

(A) $f'(x) = x^4 + (b+1)x^3 + (b+2)x^2 + (b+1)x + 1$

$= x^4 + x^3 + 2x^2 + x + 1 + b(x^3 + x^2 + x)$

$= (x^2 + x + 1)x^2 + (x^2 + x + 1) + xb(x^2 + x + 1)$

$f'(x) = (x^2 + x + 1)(x^2 + bx + 1)$

$b^2 - 4 \le 0 \Rightarrow b \in [-2, 2]$ (p, q, r, s)

(B) $f'(x) = 1 - 2\cos x \ge 0 \Rightarrow \cos x \le \frac{1}{2}$ for $x \in \left[\frac{\pi}{3}, \frac{5\pi}{3}\right]$

$a + b = \frac{1}{3} + \frac{5}{3} = 2$ (q)

(C) $f(x) = x^3 + x^2 + bx + c$

$a, b \in N$

$f'(x) = 3x^2 + 2ax + b > 0$

$D \le 0 \Rightarrow 4a^2 - 4(3)b \le 0$

$a^2 - 3b \le 0;\ a^2 \le 3b$

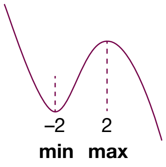
$\frac{a^2}{b} \le 3$ (q, r, t)

(D) $f(x) = \dfrac{a\sin x + 2\cos x}{\sin x + \cos x}$

$$f'(x) = \frac{(\sin x + \cos x)(a\cos x - 2\sin x) - (a\sin x + 2\cos x)(\cos x - \sin x)}{(\sin x + \cos x)^2} = \frac{a-2}{(\sin x + \cos x)^2} \ge 0$$

$a \ge 2$ (q, t)

2. **A → (s); B → (p); C → (q); D → (r, t)]**



$f(x) = \sqrt{ax^3 + bx^2 + cx + d}$;

$P(x) = ax^3 + bx^2 + cx + d$

$P'(x) = k(x+2)(x-2) = k(x^2 - 4)$

$$P(x) = k\left(\frac{x^3}{3} - 4x\right) + c$$

$x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = 3$ and $x = -2$

Only $a = -2$ is acceptable, due to possible $p(x)$ can be of type shown in diagram

$P(x) = -2x^3 + bx^2 + cx + d;$

$P'(x) = -6x^3 + 2bx^2 + c$

$P'(2) = 0 \Rightarrow -24 + 4b + c = 0$

$P'(-2) = 0 \Rightarrow -24 - 4b + c = 0$

We get $b = 0$ and $c = 24$

$P(x) = -2x^3 + 24x + d \ge 0$

$P(x) = -2x(x^2 - 12) + d$

$P(-2) = 4(-8) + d \ge 0$

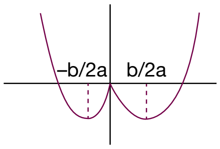
$P(2) = -4(-8) + d \ge 0$

We have $d \ge 32$

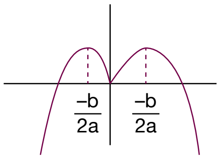
3. **A → (p, s); B → (q, r); C → (p, r); D → (p, q, r, s)**

$f(x) = ax^2 - b|x| = |x|\ (a|x| - b)$

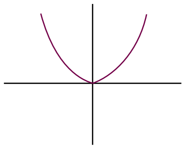
When $a, b > 0$; zeroes of $f(x)$ are $|x| = 0$, $|x| = b/a$



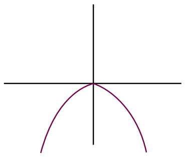
When $a, b < 0$; zeroes of $f(x)$ are $|x| = 0$, $|x| = b/a$



When $a > 0, b < 0$; zeroes of $f(x)$ is $x = 0$; $f(x) = |x|\ (a|x| - b) \ge 0$



When $a < 0, b > 0$; zeroes of $f(x)$ is $x = 0$; $|x|\ f(x) = (a|x| - b) \le 0$



4. **A → (q); B → (t); C → (p, q, r, s, t); D → (p)**

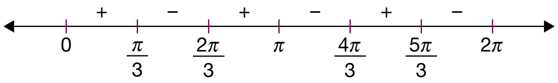
(A) $f(x) = 64(\sin^3 x)(\cos^3 x) = 8.\ \sin^3 2x \le 8$.

$\therefore$ Maximum value of $f(x)$ is 8.

(B) We have, $f(x) = \sin x \cdot e^{\sin 2x} + \int_0^x e^{\sin 2t}(\sin t - \cos t)dt$

$\therefore\ f'(x) = \sin x\,(2\cos 2x)\cdot e^{\sin 2x} + (\cos x)\,e^{\sin 2x} + e^{\sin 2x}(\sin x - \cos x)$

$= e^{\sin 2x}\cdot \sin x\,(2\cos 2x + 1)$



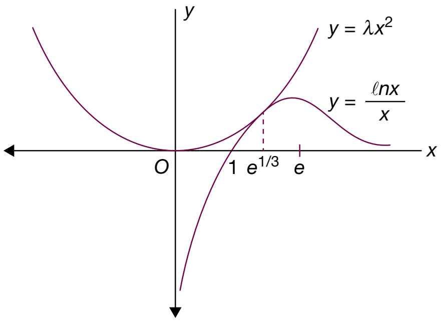
$= 2.e^{\sin 2x}\cdot \sin x \cdot 2\left(\cos 2x + \frac{1}{2}\right)$

$= 2.e^{\sin 2x}\,\sin x \cdot 2\left(\cos^2 x - \frac{1}{4}\right)$

$4.e^{\sin 2x}\,\sin x \cdot \left(\cos x + \frac{1}{2}\right)\cdot\left(\cos x - \frac{1}{2}\right)$

$\therefore$ Points of local minima are $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ of $f(x)$ in $x \in (0, 2\pi)$.

(C)



Since, C_1 and C_2 touches each other.

$\Rightarrow\ \frac{\ln x}{x} = \lambda x^2$

$\Rightarrow\ \ln x = \lambda x^3$ …(1)

And $\frac{1 - \ln x}{x^2} = 2\lambda x \Rightarrow 1 - \lambda x^3 = 2\lambda x^3$

$\Rightarrow\ 3\lambda x^3 = 1 \Rightarrow x = \left(\frac{1}{3\lambda}\right)^{\frac{1}{3}} = (3\lambda)^{\frac{-1}{3}}$

From (1), $\frac{-1}{3}\ \ln(3\lambda) = 1\cdot\frac{1}{3\lambda} \Rightarrow \ln(3\lambda) = -1$.

$\Rightarrow\ \lambda = \frac{1}{3e}$.

(D) $\lim\limits_{x\to 0}\dfrac{2\int_0^x (t\sin(10t) + 5)dt}{x}\ \left(\frac{0}{0}\right)$

$\Rightarrow\ \lim\limits_{x\to 0}\dfrac{2(x\sin(10x) + 5)}{1} = 10$

Option D is from limits (solution is correct)

5. **A → (p); B → (s, t); C → (r, t); D → (r, s, t)**

(A) Here, $\sin x > 0$ and $\sin 3x > 0$.

Now, $\log_2(2\cdot\sin x\cdot\sin 3x) \geq 0$

$\Rightarrow\ 2\sin x\cdot\sin 3x \geq 1$

$\Rightarrow\ \cos 2x - \cos 4x \geq 1$

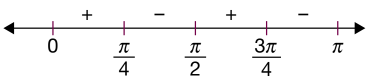
$\Rightarrow\ \cos 2x \geq 2\cos^2 2x$

$\Rightarrow\ \cos 2x\cdot\left(\cos 2x - \frac{1}{2}\right) \leq 0 \Rightarrow x \in \left(\frac{\pi}{6}, \frac{\pi}{4}\right)$.

(B) $f'(x) = 2\sin\left(x + \frac{\pi}{3}\right) \leq 0 \Rightarrow x \in \left(\frac{2\pi}{3}, \frac{3\pi}{4}\right)$ or $\left(\frac{2\pi}{3}, \pi\right)$

(C) $g'(x) = 6\cos x\cdot(1 - \sqrt{2}\sin x)\cdot(1 + \sqrt{2}\sin x)$

sign of $g'(x)$



$\therefore\ x = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ are extremum points of $g(x)$.

(D) (A) Given, $\int_x^{3x} f(t)dt = \int_5^x f(t)dt\ \ \forall\ x > 0$

(using Leibnitz's rule)

$\Rightarrow\ 3f(3x) - f(x) = f(x)$

$\Rightarrow\ 3f(3x) = 2f(x)$

As, $f(3) = 3$

$\therefore\ f(9) = \frac{2}{3}f(3) = 2$

$f(27) = \frac{2f(9)}{3} = \frac{4}{3}$

$f(81) = \frac{2f(27)}{3} = \frac{2}{3}\left(\frac{4}{3}\right) = \frac{8}{9}$.

$\therefore$ Using LMVT to $f(x)$ in [27, 81], we get $f'(c)$

$= \frac{f(81) - f(27)}{81 - 27} = \frac{\frac{8}{9} - \frac{12}{9}}{54}$

$\therefore\ f'(c) = \frac{-4}{9\times 54} = \frac{-2}{9\times 27} \Rightarrow |243 f'(c)| = 2$

6. **A → (r); B → (s); C → (t); D → (q)**

(A) $f(x) = \begin{cases} x^{3/5} & x \leq 1 \\ -(x-2)^3 & x > 1\end{cases}$

$f'(x) = \begin{cases} \frac{3}{5}x^{2/5} & x \leq 1 \\ -3(x-2)^2 & x > 1\end{cases}$

N.D ↗ $x = 0$ $x = 1$
$f'(x)$
O ↘ $x = 2$

Three Critical point (r)

(B) $\log_2^2 x + (x - 1)\log_2 x + 2x - 6 = 0$

$\log_2 x = \frac{1 - x \pm\sqrt{(x-1)^2 - 4(2x - 6)}}{2}$

$\log_2 x = \frac{1 - x \pm |x - 5|}{2}$

$\log_2 x = \frac{1 - x + x - 5}{2}$

$\log_2 x = -2$

$x = \frac{1}{4}$

$\log_2 x = \frac{1 - x + (-(x - 5))}{2}$

$\log_2 x = \frac{1 - x - x + 5}{2}$

$x = 2$

Two solution (s)

(C) St. line $3x + 4y = c$

Curve $\frac{x^4}{2} = x + y \Rightarrow y = \frac{x^4}{2} - x$

$$\frac{dy}{dx} = 2x^3 - 1\Big|_{x_1} = -\frac{3}{4}$$

$$2{x_1}^3 = \frac{1}{4} \Rightarrow x_1 = \frac{1}{2}$$

$$y_1 = \frac{x_1^4}{2} - x_1 = \frac{1}{32} - \frac{1}{2} = \frac{-15}{32}$$

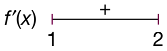
$$3x_1 + 4y_1 = c$$

$$c = 3\left(\frac{1}{2}\right) + 4\left(\frac{-15}{32}\right) = -\frac{3}{8} \qquad \text{1 value} \qquad \text{(t)}$$

(D) $f'(x) = (x^2 - 1)2x - (x - 1).1$

$= (x - 1)\,((x + 1)\,2x - 1)$

$= (x - 1)\,(2x^2 + 2x - 1)$



$$\text{global max. } f(2) = \int_2^4 (t-1)\,dt = \frac{(t-1)^2}{2}\Big|_2^4 = 4 \qquad \text{(q)}$$

7. A → (r, t); B → (p, r); C → (p, r); D → (p)

(a) $f'(x) = 4x^3 - 24x^2 + 44x - 24$

$= 4\left(x^3 - 6x^2 + 11x - 6\right) = 4(x-1)(x-2)(x-3)$

for $f'(x) > 0, x \in (1,2) \cup (3,\infty)$

and for $f'(x) < 0, x \in (-\infty,1) \cup (2,3)$

(b) $f'(x) = \dfrac{2}{(x-2)} - 2x + 4$

$$= \frac{2 - 2x^2 + 4x + 4x - 8}{(x-2)} = \frac{\left(-2x^2 + 8x - 6\right)}{(x-2)}$$

$$= \frac{-2\left(x^2 - 4x + 3\right)}{(x-2)} = \frac{-2(x-1)(x-3)}{(x-2)}$$

If $f'(x) > 0$

$$\Rightarrow \frac{-2(x-1)(x-3)}{(x-2)} > 0 \Rightarrow \frac{(x-1)(x-3)}{(x-2)} < 0$$

$\therefore \quad x \in (-\infty,1) \cup (2,3)$ but $x > 2$ and if $f'(x) < 0$

$$\Rightarrow \frac{(x-1)(x-3)}{(x-2)} > 0$$

$\therefore \quad \therefore x \in (1,2) \cup (2,3)$

(c) $\because \; f(x) = \begin{cases} \dfrac{1-x}{x^2}, x < 1, \, x \neq 0 \\ \dfrac{x-1}{x^2}, x \geq 1, \; x \neq 0 \end{cases}$

CASE 1: When $x < 1$

$$f'(x) = \frac{-2}{x^3} + \frac{1}{x^2} = \frac{(x-2)}{x^3}$$

For increasing $x \in (-\infty,0) \cup (2,\infty)$

But $x < 1$

$\therefore \quad x \in (-\infty,0)$

For decreasing $x \in (0,2)$ but $x < 1$

$\therefore \quad x \in (0, 1)$

CASE 2: When $x \geq 1$

$$f'(x) = -\frac{1}{x^2} + \frac{2}{x^3} = -\frac{(x-2)}{x^3}$$

For increasing $f'(x) > 0 \;\Rightarrow\; \dfrac{-(x-2)}{x^3} > 0$

$\Rightarrow \quad \dfrac{(x-2)}{x^3} < 0 \Rightarrow x \in (0,2)$, but $x \geq 1, \;\; \therefore x \in [1,2)$

For decreasing, $f'(x) < 0$

$$\Rightarrow \quad -\frac{(x-2)}{x^3} < 0 \Rightarrow \frac{(x-2)}{x^3} > 0$$

$\Rightarrow \quad x \in (-\infty,0) \cup [2,\infty)$

But $x \geq 1$

$\therefore \quad x \in (2,\infty)$

From CASE 1 and CASE 2

$f(x)$ is increasing on $x \in (-\infty,0) \cup [1,2)$

and decreasing on $x \in (0,1) \cup (2,\infty)$ (Q)

(d) $f(x) = -\left(x^3 - \dfrac{15}{2}x^2 + 18x + 2\right)$

$\Rightarrow \quad f'(x) = -\left(3x^2 - 15x + 18\right) > 0 \Rightarrow x \in (2,3)$

$f(x)$ is increasing in (2, 3) decreasing in $(-\infty, 2) \cup (3, \infty)$

8. A → p, q; B → r, s, t; C → p, q, r; D → p, q, r, s, t

(A) $g'(x) = 2\cos x - 2\sin 2x > 0$

$2\cos x - 2.2\sin x\cos x > 0$

$= 2\cos x(1 - 2\sin x) > 0$

$\cos x > 0, \; \sin x < \dfrac{1}{2} \;\;\Rightarrow\;\; x \in \left(-\pi, \dfrac{\pi}{6}\right)$

or $\cos x < 0, \; \sin x > \dfrac{1}{2} \;\;\Rightarrow\;\; x \in \left(\dfrac{\pi}{2}, \dfrac{5\pi}{6}\right)$

$\Rightarrow$ p, q

$a \rightarrow$ p, q

(B) $f(x) = \cos^2 x + \sin x$

$\Rightarrow \; f'(x) = 0 \;\;\Rightarrow\;\; x = \dfrac{\pi}{6}, \dfrac{\pi}{2}$

At $x = \dfrac{\pi}{6}$ ‘f’ attains local maximum

$\Rightarrow \;\; b \rightarrow r,s,t$

(C) $f'(x) = \dfrac{\cos x - \sin x}{1 + \sin x\cos x} > 0$

$\Rightarrow \;\; c \rightarrow p,q,r$

(D) $f'(x) = 99x^{88} + \cos x > 0$

$\Rightarrow \;\; d \rightarrow p,q,r,s,t$

9. A → (q), B → (r), C → (q), D → (q)

$y = f(x) + \dfrac{1}{x}$ so $y > 0$

Now $f(y)f\left(f(y) + \dfrac{1}{y}\right) = 1$ also $f(x)f(y) = 1$

$$\therefore \quad f(x) = f\left(f(y) + \frac{1}{y}\right) = f\left(\frac{1}{f(x)} + \frac{1}{f(x) + 1/x}\right)$$

Also $f(x)$ is increasing

$$\therefore \quad x = \frac{1}{f(x)} + \frac{1}{f(x) + 1/x}$$

$$\Rightarrow \quad f(x) = \frac{1 \pm \sqrt{5}}{2x}$$

Now $f(x) = \dfrac{1 + \sqrt{5}}{2x}$ is decreasing so discarding it

$$f(x)=\frac{1-\sqrt{5}}{2x}\,.$$

Maximum value $f(2)=\dfrac{1-\sqrt{5}}{4}$; Minimum value $f(1)=\dfrac{1-\sqrt{5}}{2}$

$$\int_e^{e^2}\left(\frac{1-\sqrt{5}}{2x}\right)dx=\frac{1-\sqrt{5}}{2}\qquad [\ln x]_e^{e^2}=\frac{1-\sqrt{5}}{2}$$

10. A → (q); B → (s); C → (p); D → (r)

(A) F is decreasing for $x < 3$ and increasing for $x > 3$

$\therefore$ for $x = 3$ to be point of local maximum, $f(3^-)$

$\Rightarrow -15 \geq 12 - 27 + \log_e (a^2 - 3a + 3)$

$\Rightarrow 0 < a^2 - 3a + 3 \leq 1 \qquad (\because\ a^2 - 3a + 3 > 0 \quad \forall\, a \in R)$

$a^2 - 3a + 2 \leq 0$

$(a - 2)(a - 1) \leq 0$

$a \in [1, 2]$

(B) $f(x) = x + \cos x - a \Rightarrow f'(x) = 1 - \sin x \geq 0\ \forall\ x \in R$

$\Rightarrow f$ is increasing on R

For positive root, $f(0) = 1 - a < 0$

(C) $f(k) < f(k^+)$

$3 \leq a^2 - 2 + 1$

$a^2 \geq 4$

$a \in (-\infty, -2] \cup [2, \infty)$

(D) $\dfrac{dy}{dx} = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8 = 3(x - 2)(x - 4)$

For real and distinct roots $f(2) \,.\, f(4) < 0$

11. A → p; B → p; C → s; D → s

(A) $f(x)=\ell n(\sin x)$; $f'(x)=\dfrac{\cos x}{\sin x}>0$

$\cos x > 0$ in $\left(0,\dfrac{\pi}{2}\right)$

Required number of values of x is 0

(B) $f'(x)=3x^2-3\leq 0 \Rightarrow 3(x^2 - 1) \leq 0 \Rightarrow x \in [-1, 1]$

$\therefore\ a = -1, b = 1$

$\therefore\ a + b = 0$

(C)
$$f(x)=\begin{cases} x^2+2; & 1\leq x<2\\ \dfrac{x^2+2}{2}; & 2\leq x<3\\ \dfrac{11}{3}; & x=3\end{cases}$$

$$f'(x)=\begin{cases} 2x\,, & 1<x<2\\ x\,, & 2<x<3\end{cases}$$

$\therefore$ least value of $f(x)$ in $[1, 2)$ is 3

least value of $f(x)$ in $[2, 3)$ is 3

$$f(3)=\frac{11}{3}$$

$\therefore$ least value of $f(x)$ is 3

(D) $f(x) = e^{2x} - (a + 1)\, e^x + 2x$

$f'(x) = 2e^{2x} - (a + 1)\, e^x + 2$

Now, $2e^{2x} - (a + 1)\, e^x + 2 \geq 0$ for all $x \in R$

i.e. $2\left(e^x+\dfrac{1}{e^x}\right)-(a+1)\geq 0$ for all $x\in R$

i.e. $4 - (a + 1) \geq 0$

i.e. $a \leq 3$

$\therefore\ a = 3$

12. A → (q); B → (p); C → (s); D → (s)

(A) $f(x) = 3x^2 - 12x + 14$

$f'(x) = 0 \Rightarrow 6x - 12 = 0 \Rightarrow x = 2$

(B) Put $3^x = t \Rightarrow f(t) = 2t^3 - 4t^2 + 2t,\ t \in [1/33]$

$f'(t) < 0$ in $(1/3, 1)$ and $f'(t) > 0$ in $(1, 3)$ $f(t)$ minimum $= f(1) = 0$

(C) $f'(x) = 4(x^2 - 1) \Rightarrow f$ is decreasing for $0 < x < 1$ and f is increasing for $1 < x \leq 2$

$\therefore\ f$ minimum is $f(1) = 4/3 - 4 = -8/3$

(D) $f'(x) = -12x + 18x - 6x^2 = -6(x - 1)(x - 2)$

F is increasing in $(1, 2)$ and decreasing in $(2, 4)$

$\therefore$ absolute maximum $=$ maximum $\{f(1), f(2)\}$

$=$ maximum $\{1, 2\} = 2$

Integer/Numerical Answer Type Questions

1. $f'(x)=6x^2-6(\lambda+2)x+2\lambda$

Here disc $=36(\lambda+2)^2-48\lambda$

$=12\left(3\lambda^2+12\lambda+12-4\lambda\right)$

$=12\left(3\lambda^2+8\lambda+12\right)>0\ \forall \lambda\in R$

$\therefore\ f'(x)=0$ has two real and distinct roots $\therefore \alpha+\beta=2$

Hence, the correct answer is (2).

2. $\lim\limits_{x\to 0-} f(x)=k$ & $\lim\limits_{x\to 0+} f(x)=5$

For $f(x)$ to have a local maximum at $x = 0$, clearly $k\geq 5$

Hence, the correct answer is (5).

3. Here $f'(x)=\dfrac{5x-7}{(x+1)^3}>0$

$\Rightarrow x\in(-\infty,-1)\cup\left(\dfrac{7}{5},\infty\right)$ and $f'(x)<0\Rightarrow x\in\left(-1,\dfrac{7}{5}\right)$

But $\lim\limits_{x\to -1} f(x)=\infty$ and $f(-1)=0$

Hence $f(x)$ has a local minimum at $x = -1$ and at $x=\dfrac{7}{5}$

Hence, the correct answer is (2).

4. $\because$ If $y = ax^2 + bx + c$

$\Rightarrow y_{\min}=\dfrac{-(b^2-4ac)}{4a}$

$m(b)=\dfrac{4\left(1+b^2\right)-4b^2}{4\left(1+b^2\right)}=\dfrac{1}{1+b^2}$ whose range is (0, 1].

Hence, the correct answer is (1).

5. $f(x) = Kx^3 - 9x^2 + 9x + 3$

$f'(x) = 3Kx^2 - 18x + 9$

$= 3(Kx^2 - 6x + 3) \geq 0$ for all real x

$D \leq 0 \Rightarrow 36 - 12K \leq 0 \Rightarrow K \geq 3$

$K \in [-5, 100]$

$K \in \{3, 4, 5, \ldots 100\}$

$n = 98$

Now $100 - n = 100 - 98 = 2$

Hence, the correct answer is (2).

6. $\lim\limits_{x\to 0}\dfrac{f(x)-f(x^2)}{f(x)-f(0)}=\lim\limits_{x\to 0}\dfrac{f'(x)-2xf'(x^2)}{f'(x)}$

$=\lim\limits_{x\to 0} 1-2x\dfrac{f'(x^2)}{f'(x)}=1$

$f'(x)=0$

Hence, the correct answer is (1).

7. $f(x) = \int_0^x (bt^2 + b + \sin t)\, dt \quad \forall x \in R$

$f'(x) = bx^2 + b + \sin x - 0$

(i) $f(x)$ MI

$f'(x) = bx^2 + b + \sin x \geq 0$

$bx^2 + b - 1 \geq 0 \quad \forall x \in R$

$b > 0 \quad D \leq 0$

$b - 1 \geq 0$

$b \geq 1$

(ii) $f(x)$ MD

$f'(x) = bx^2 + b + \sin x \leq 0$

$bx^2 + b + 1 \leq 0 \quad \forall x \in R$

$b < 0$ and $(b + 1) \leq 0$

$b \leq -1$

$b \in (-\infty, -1] \cup [1, \infty]$

we get $a_1 = -1$ and $a_2 = 1$

$a_1{}^4 + a_2{}^4 + 3a_2 = 1 + 1 + 3 = 5$

Hence, the correct answer is (5).

8. $K(x) = \ln\left(\dfrac{x^2 - x + 1}{x^2 + x + 1}\right) \quad x \in \left(\dfrac{-t}{7}, \dfrac{t}{7}\right)$

$$K'(x) = \frac{x^2 + x + 1}{x^2 - x + 1}\left[\frac{(x^2 + x + 1)(2x - 1) - (x^2 - x + 1)(2x + 1)}{(x^2 + x + 1)^2}\right]$$

$$= \frac{2(x - 1)(x + 1)}{(x^2 + x + 1)(x^2 - x + 1)} < 0$$

for $x \in (-1, 1) = \left(\dfrac{-t}{7}, \dfrac{t}{7}\right)$ we have $t = 7$

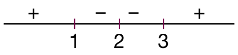
Hence, the correct answer is (7).

9. $P'(x) = 5(x - 1)\,(x - 3)\,(x - 2)^2$

$P(x) = x^5 + \ldots$

$P'(6) = 5 \cdot 5 \cdot (3) \cdot (4)^2$

$= 25 \times 3 \times 4 \times 4 = 1200$



Hence, the correct answer is (1200).

10. $f(x) = 2x^3 + 3(1 - 3a)x^2 + 6(a^2 - a)x + b$

$f'(x) = 6x^2 + 6(1 - 3a)x + 6(a^2 - a) = 6(x^2 + (1 - 3a)x + a^2 - a)$ has two distinct real roots to have extremum

As coefficient of x^3 is positive, smaller root is for local maxima and larger root is for local minima

If we are looking for positive point of local maxima then it says both roots of $f'(x)$ are positive

(i) sum of both roots is positive

$-B/2A > 0 \Rightarrow (3a - 1) > 0 \Rightarrow a > 1/3$

(ii) product of both the roots $f(0) > 0 \Rightarrow a^2 - a > 0$

$\Rightarrow \quad a(a - 1) > 0$ we get $a \in (-\infty, 0) \cup (1, \infty)$

(iii) for real and distinct roots, $D > 0 \Rightarrow (1 - 3a)^2 - 4(a^2 - a) > 0$

$\Rightarrow \quad 5a^2 - 2a + 1 > 0$ true for all real R

Final common set of values of a $a \in (1, \infty)$

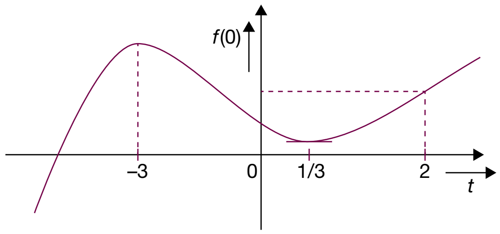
least integral value of $a = 2$ (C)

Hence, the correct answer is (2).

11. $f(x) = \left(3 - \sqrt{4 - x^2}\right)^2 + \left(1 + \sqrt{4 - x^2}\right)^3$

Let $t = \sqrt{4 - x^2}$, clearly $0 \leq t \leq 2$

$\Rightarrow \quad F(t) = (3 - t)^2 + (1 + t)^3$



For maxima and minima,

$\Rightarrow \quad F'(t) = 0$

$\Rightarrow \quad -2(3 - t) + 3(1 + t)\,2 = 0$

$\Rightarrow \quad 3t^2 + 8t - 3 = 0$

$\Rightarrow \quad (3t - 1)(t + 3) = 0$

$\Rightarrow \quad t = -3, \dfrac{1}{3}$

Also $F''(t)|_{t=-3} = -10 \Rightarrow$ maxima

And $F''(t)|_{t=1/3} = +10 \Rightarrow$ minima

As $t \neq -3$, hence maximum value of $F(t)$ will occur at the end points for which

$F(2) = 10$, $F(0) = 28$

Hence, maximum value of $F(t) = 28$ for $t = 2$

$\Rightarrow$ Maximum value of $f(x) = 28$ for $x = 0$.

Hence, the correct answer is (4).

12. $x_1 = 2R\cos A,\ x_2 = 2R\cos B,\ x_3 = 2R\cos C$

$\dfrac{dx_1}{dA} = -2R\sin A$ also $a = 2R\sin A \Rightarrow \dfrac{da}{dA} = 2R\cos A$

So, $\dfrac{dx_1}{da} = -\tan A, \dfrac{dx_2}{db} = -\tan B, \dfrac{dx_3}{dc} = -\tan C$

$\tan A + \tan B + \tan C \geq 3\sqrt{3}$

So, $\left(\dfrac{dx_1}{da} + \dfrac{dx_2}{db} + \dfrac{dx_3}{dc}\right) \leq -3\sqrt{3}$

$\Rightarrow \quad \sqrt{3}\left(\dfrac{dx_1}{da} + \dfrac{dx_2}{db} + \dfrac{dx_3}{dc}\right) + 11 \leq 2$

Hence, the correct answer is (2).

13. $f'(x) = -\pi\sin(\pi x) + 10 + 6x + 3x^2$

$= 3(x + 1)^2 + 7 - \pi\sin(\pi x) > 0$ for all x

$f(x)$ is monotonic increasing in $-2 \leq x \leq 3$ so the absolute minimum $= f(-2) = -15$

Hence, the correct answer is (3).

14. $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$

$\therefore \quad f'(x) = 6x^2 - 18ax + 12a^2$ and $f''(x) = 12x - 18a$

For max/min, $6x^2 - 18ax + 12a^2 = 0$

$\Rightarrow \quad x^2 - 3ax + 2a^2 = 0$

$\Rightarrow \quad (x - a)(x - 2a) = 0$

$\Rightarrow \quad x = a$ or $x = 2a$

Now, $f''(a) = 12a - 18a = -6a < 0$

and $f''(2a) = 24a - 18a = 6a > 0$

$\therefore \quad f(x)$ is max. at $x = a$ and minimum at $x = 2a$

$\Rightarrow \quad p = a$ and $q = 2a$

Given that $p^2 = q \Rightarrow a^2 = 2a \Rightarrow a(a - 2) = 0 \Rightarrow a = 2.$

Hence, the correct answer is (2).

15. $f\left(\frac{3}{2}\right) = 0 \Rightarrow \lim\limits_{x\to\frac{3}{2}} |x^2 - 3x| + a \le 0$

$a \le -\frac{9}{4}$

Hence, $|4k| = 9$

Hence, the correct answer is (9).

16. Local minimum $= f\left(\frac{1}{2}\right) = \frac{4}{3}$

Global maximum $= f(0) = f(1) = \frac{3}{2}k = \frac{4}{3} + \frac{3}{2} = \frac{17}{6}$

Hence, the correct answer is (8).

17. Local extremum does not occur at any value of $x \ne 0$. But global minimum $= f(0) = 0$

$\therefore \quad n_1 = 0, n_2 = 0$ then $n_1 + n_2 = 0$

Hence, the correct answer is (0).

18. Let $f(x) = \sqrt{x-1} + \sqrt{2-x}$

$$f'(x) = \frac{1}{2\sqrt{x-1}} + \frac{-1}{2\sqrt{2-x}} = 0$$

$$\sqrt{2-x} = \sqrt{x-1}$$

$$\Rightarrow \quad x = \frac{3}{2} \Rightarrow f'(x) = 0$$

$f(x)$ maximum at $x = \frac{3}{2}$

$$g(x) = x^2 - bx + c$$

$$g'(x) = 2x - b$$

$$g'(x) = 0 \quad b = +3 \Rightarrow 2x - b = 0 \Rightarrow x = \frac{b}{2}$$

and $\frac{b}{2} = \frac{3}{2} \Rightarrow b = 3$

Hence, the correct answer is (3).

19. $f(x) = \left(\sin^{-1}\frac{x}{2} + \cos^{-1}\frac{x}{2}\right)\left(\left(\sin^{-1}\frac{x}{2}\right)^2 + \left(\cos^{-1}\frac{x}{2}\right)^2 - \sin^{-1}\frac{x}{2}\cos^{-1}\frac{x}{2}\right)$

$$f(x) = \frac{\pi}{2}\left(\left(\frac{\pi}{2}\right)^2 - 3\sin^{-1}\frac{x}{2}\cos^{-1}\frac{x}{2}\right)$$

$$f(x) = \frac{\pi}{2}\left(\left(\frac{\pi}{2}\right)^2 - 3\sin^{-1}\frac{x}{2}\left(\frac{\pi}{2} - \sin^{-1}\frac{x}{2}\right)\right)$$

$$f(x) = \frac{1}{3}\,\frac{\pi}{2}\left[\left(\sin^{-1}\frac{x}{2}\right)^2 - \frac{\pi}{2}\sin^{-1}\frac{x}{2} + \left(\frac{\pi}{2}\right)^2\right]$$

$$f(x) = \frac{\pi}{6}\left[\left(\sin^{-1}\frac{x}{2} - \frac{\pi}{4}\right)^2 + \frac{3\pi^2}{16}\right]$$

For min of $f(x)$ $x = \sqrt{2}$

For max of $f(x)$ $x = -2$

Hence, the correct answer is (6).

20. Given $f : [1,10] \Rightarrow [1,10]$ is a non-decreasing function

$g : [1,10] \to [1,10]$ is a non-increasing function.

$h(x) = fog(x) : [1, 10] \to [1, 10]$ is a decreasing function.

$\therefore \quad h(1) \ge 1 \to 1$

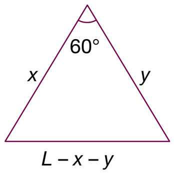
But $x \ge 1 \Rightarrow h(x) \le h(1) \Rightarrow h(x) \le 1 \to 2$

From 1, 2, $h(x) = 1 \;\; \forall x \in [1,10]$

So $h(7) = 1$.

Hence, the correct answer is (1).

21. For area max, triangle must be equilateral



$$x = y = \frac{L}{3}$$

$$a = b = c$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3$$

Hence, the correct answer is (3).

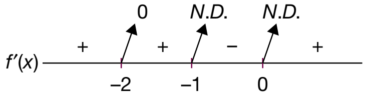
22. $g(x) = \int_{-3}^{x} f(t)\,dt$

$$-3 < x < 2$$

$$g'(x) = f(x)$$

$$g''(x) = f'(x) = \begin{cases} 3(x+2)^2 & -3 < x < -1 \\ \frac{3}{2}\frac{1}{x^{1/3}} & -1 < x < 2 \end{cases}$$

$f'(x) = 0 \quad$ at $x = -2$



$f'(x)$ is not defined at $x = -1, 0$

2 extremum only at $x = -1, 0$

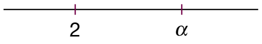
Hence, the correct answer is (2).

23. $f(x) = 2x^3 - 3(2 + \alpha)x^2 + 12\alpha x$

$$f'(x) = 6x^2 - 6(2 + \alpha)x + 12\alpha$$

$$= 6(x^2 - (2 + \alpha)x + 2\alpha)$$

$$= 6(x - 2)(x - \alpha)$$



For one local maximum and one local minimum, $f'(x)$ must change its sign across critical points

So $\alpha \ne 2$ and as given $\alpha \in [1, 100]$

Required sum $= 1 + 3 + 4 + \ldots + 100 = \frac{100.101}{2} - 2$

$= 5000 - 2 = 5048$

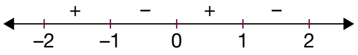
Hence, the correct answer is (5048).

24. $f'(x) = \begin{cases} 3(x+1)^2, & -2 < x < -1 \\ \frac{2}{3}x^{\frac{-1}{3}}, & -1 < x < 1 - \{0\} \\ -2(x-1), & 1 < x < 2 \end{cases}$

$\because \quad f'(-1^-) = 3, f'(-1^+) = -\frac{2}{3}$

$f'(0) =$ Not exist.

$f'(1^-) = \frac{2}{3}; f(1^+) = 0$

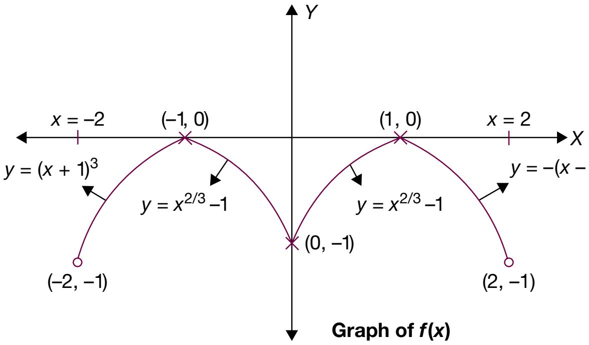


Sign of $f'(x)$

$\therefore \quad f'(x)$ does not exist at $x = -1, 0, 1$.

sign scheme of $f'(x)$

Here, points of local maximum are $x = -1, 1$ and point of local minimum is $x = 0$.



Graph of $f(x)$

NOTE:

If $f(x) = \begin{cases} -(x+1)^2, & -2 < x \le -1 \\ x^{\frac{-1}{3}} - 1, & -1 < x \le 1 \\ -(x-1)^2, & 1 < x < 2 \end{cases}$ then $f(x)$ is even function.

Hence, the correct answer is (3).

25. $f(x) = 30 - 2x - x^3$

$f(x) = -2 - 3x^2 < 0 \Rightarrow f(x)$ is decreasing function

Hence $f(f(f(x))) > f(f(-x)) \Rightarrow f(f(x)) < f(-x)$

$\Rightarrow \quad f(x) > -x$

$\Rightarrow \quad 30 - 2x - x^3 > -x \Rightarrow x^3 + x - 30 < 0 \Rightarrow (x-3)(x^2+3x+10) < 0$

$\Rightarrow \quad 30 - 2x - x^3 > -x \Rightarrow x^3 + x - 30 < 0 \Rightarrow (x-3)(x^2+3x+10) < 0$

$\Rightarrow \quad x < 3$

Hence, the correct answer is (2).

26. We have

$$\lim_{x\to 0}\left(3 - \frac{P(x)}{x}\right) = 27 \Rightarrow \lim_{x\to 0}\frac{P(x)}{x} = -24$$

$\therefore$ Let $P(x) = ax^4 + bx^3 + cx^2 - 24x$ …(1)

Now, $P(1) = -9 \Rightarrow a + b + c = 15$ …(2)

Also, $P'(2) = 0 \Rightarrow 8a + 3b + c = 6$ …(3)

and $P'''(2) = 0 \Rightarrow b = -8a$ …(4)

$\Rightarrow$ On solving (2), (3) and (4), we get

$a = 1$, $b = -8$ and $c = 22$.

Hence, $P(x) = x^4 - 8x^3 + 22x^2 - 24x$

$\therefore \quad P'(x) = 4x^3 - 24x^2 + 44x - 24$

$= 4(x^3 - 6x^2 + 11x - 6) = 4(x-1)(x-2)(x-3)$

$\Rightarrow \quad P''(x) = 4(3x^2 - 12x + 11)$

As $P''(x) > 0 \ \forall \ x \in [3, 4]$

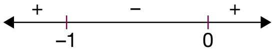
$\therefore \quad P'(x)$ is increasing function on $[3, 4]$

$\therefore \quad P'_{\text{minimum}}(x = 3) = 0$

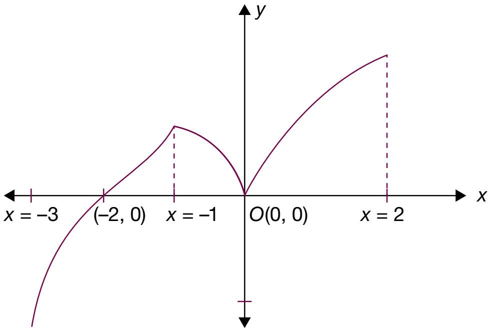
Hence, the correct answer is (0).

27. Clearly,

$g(x) = \int_3^x f(t)\,dt, -3 < x < 2$



Sign scheme of $g''(x)$



Graph of $g'(x) = f(x)$

Differentiation w.r. to x. $g'(x) = f(x)$

$$g'(x) = f(x) = \begin{cases} (x+2)^3, & -3 < x \le -1 \\ x^{\frac{2}{3}}, & -1 < x < 2 \end{cases}$$

$\Rightarrow \quad g'(x)$ is a continuous function.

$$g''(x) = \begin{cases} 3(x+2)^2, & -3 < x < -1 \\ \frac{2}{3}x^{-\frac{1}{3}}, & -1 < x < 2 \end{cases}$$

$\Rightarrow \quad g'(x)$ has a local maximum at $x = -1$ and local minimum at $x = 0$.

Hence, $g'(x)$ has two extremum points.

Hence, the correct answer is (2).

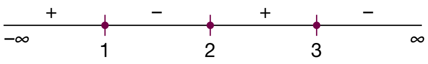
28. As $y = f(x)$ is polynomial hence it is differentiable.

$\therefore \quad f'(x) = k(x-1)(x-2)(x-3)$

$f'(0) = -6k$

$6 = -6k \Rightarrow k = -1$

Sign scheme of $f'(x)$



Hence $f'(x) = -(x-1)(x-2)(x-3)$

$= -(x^3 - 6x^2 + 11x - 6)$

$$f(x) = -\left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{11x^2}{2} - 6x\right] + 1; \qquad (\because f(0) = 1)$$

Hence, $f(2) = 1 - 4 + 16 - 22 + 12 = 3$

Hence, the correct answer is (3).

29. We have, $f(g(x)) = 4 - x + 5x^{1/3} - x^{2/3}$

Also, $g(x) = 2 - x^{1/3} \Rightarrow x^{1/3} = (2 - g(x))$

$\therefore \quad f(x) = 4 - (2-x)^3 + 5(2-x) - (2-x)^2$

$\Rightarrow \quad f(x) = x^3 - 7x^2 + 11x + 2$

Now, $f'(x) = 3x^2 - 14x + 11 = 0$

$\therefore \quad x = 1, \frac{11}{3}$

$f''(x) = 6x - 14$

Also, $f''(1) < 0$

$\Rightarrow \quad f_{\max}(x = 1) = (1)^3 - 7(1)^2 + 11(1) + 2 = 7$

Alternative:

Put $g(x) = t$

So, $t = 2 - x^{1/3} \Rightarrow x^{1/3} = (2 - t)$

Now, $f(g(x)) = 4 - x + 5x^{1/3} - x^{2/3}$

becomes

$f(t) = 4 - (2-t)^3 + 5(2-t) - (2-t)^2$

$\Rightarrow \quad f(t) = (t^3 - 7t^2 + 11t + 2)$

Hence, the correct answer is (7).

30. We have $F(x) = \begin{cases} -2x + \log_{\frac{1}{2}}(k^2 - 6k + 8), & -2 \le x < -1 \\ x^3 + 3x^2 + 4x + 1, & -1 \le x \le 3 \end{cases}$

$$F'(x) = \begin{cases} -2, & -2 \le x < -1 \\ 3x^2 + 6x + 4, & -1 \le x \le 3 \end{cases}$$

$\because \quad 3x^2 + 6x + 4 > 0$

Also $F(x)$ is increasing on $[-1, 3]$ because $F'(x) > 0 \ \forall \ x \in [-1, 3]$.

And $F'(x) = -2 \ \forall \ x \in [-2, -1)$, so $F(x)$ is decreasing on $[-2, -1)$.

$\therefore \quad$ If $F(x)$ has smallest value at $x = -1$, then we must have

$\lim_{h\to 0} F(-1 - h) \ge F(-1)$

$\lim_{h\to 0} [-2(-1-h) + \log_{1/2}(k^2 - 6k + 8)] \ge (-1)^3 + 3(-1)^2 + 4(-1) + 1$

$\Rightarrow \quad 2 + \log_{1/2}(k^2 - 6k + 8) \ge -1 + 3 - 4 + 1$

$\Rightarrow \quad 2 + \log_{1/2}(k^2 - 6k + 8) \ge -1$

$\Rightarrow \quad \log_{1/2}(k^2 - 6k + 8) \ge -3$

$\Rightarrow \quad k^2 - 6k + 8 \le 8$

$\Rightarrow \quad k^2 - 6k \le 0$

$\Rightarrow \quad k \in [0, 6]$ …(1)

But in order to define $\log_{1/2}(k^2 - 6k + 8)$, we must have $k^2 - 6k + 8 > 0$

$\Rightarrow$ $(k-2)(k-4) > 0 \Rightarrow k < 2$ or $k > 4$ …(2)

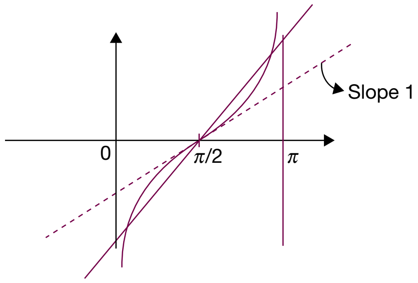
$\therefore$ From (1) and (2), we get $k \in [0, 2) \cup (4, 6]$

$\Rightarrow$ Possible integer(*s*) in the range of k are 0, 1, 5, 6

Hence the sum of all possible integer(*s*) in the range of $k = 1 + 5 + 6 = 12$

Hence, the correct answer is (12).

31. $$f(x) = \frac{\pi}{2} + \left|\operatorname{sgn}\left(\tan^{-1}\left(\frac{x}{1+x^2}\right)\right)\right| \tan^{-1}x = \begin{cases} \frac{\pi}{2} + \tan^{-1}x; & x > 0 \\ \frac{\pi}{2} & ; \; x = 0 \\ \frac{\pi}{2} + \tan^{-1}x; & x < 0 \end{cases}$$



$\therefore$ $f(x) = \frac{\pi}{2} + \tan^{-1}x \;\forall\; x \in R$

For finding inverse of $f(x)$.

$$\frac{\pi}{2} + \tan^{-1}x = y \Rightarrow x = \tan\left(y - \frac{\pi}{2}\right)$$

$\therefore$ $g(x) = \tan\left(x - \frac{\pi}{2}\right)$

The given equation is

$$g(x) = k\left(x - \frac{\pi}{2}\right)$$

$\Rightarrow$ $\tan\left(x - \frac{\pi}{2}\right) = k\left(x - \frac{\pi}{2}\right)$

Straight line $y = k\left(x - \frac{\pi}{2}\right)$ is passing through $\left(\frac{\pi}{2}, 0\right)$.

The line is tangent to the curve $y = g(x)$ at $\left(\frac{\pi}{2}, 0\right)$ then slope is equal to 1. i.e. $k = 1$ but if slope of the line is greater than 1 then line intersects the curve $y = g(x)$ at three distinct point

$\therefore$ $k \in (1, \infty)$

$\therefore$ $a = 1$

Hence, the correct answer is (1).

32. $$f'(x) = \begin{bmatrix} 3x^2 - 2x + 10, & x \le 1 \\ -2, & x > 1 \end{bmatrix}$$

Since for the quadratic $3x^2 - 2x + 10$, $D < 0$ hence it is always positive

$\Rightarrow$ function $f(x)$ increases in $(-\infty, 1)$ and decreasing in $(1, \infty)$

$\therefore$ for max at $x = 1$

$f(1) \ge f(1^+)$

$\Rightarrow$ $1 - 1 + 10 - 5 \ge -2 + \log_2(a^2 - 2)$

$\log_2(a^2 - 2) \le 7$

$\Rightarrow$ $a^2 - 2 \le 2^7$

$a^2 \le 130$

Hence, $2 < a^2 \le 130$

$\Rightarrow$ $a \in \{\pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8, \pm 9, \pm 10, \pm 11\}$

$\therefore$ No. of integral values of $a = 20$

Hence, the correct answer is (20).

33. We have, $f(x)\ 2x^3 - 3(2+p)x^2 + 12px + \ln(16 - p^2)$

Now, $f'(x) = 6(x^2 - (2+p)x + 2p)$

$$\therefore \quad f'(x) = 0 \begin{cases} x = 2 \\ x = p \end{cases}$$

As, $f(x)$ has exactly one local maxima and one local minima so, $p \ne 2$ …(1)

Also, $16 - p^2 > 0 \Rightarrow p^2 - 16 < 0$

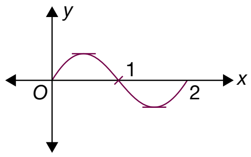
$\Rightarrow$ $(p+4)(p-4) < 0 \Rightarrow -4 < p < 4$ …(2)

$\therefore$ The possible integral values of p are –3, –2, –1, 0, 1, 3

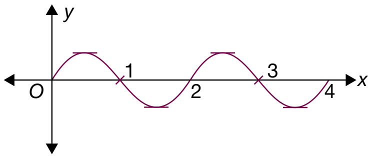
$\Rightarrow$ The total number of integral values of p are 6.

Hence, the correct answer is (6).

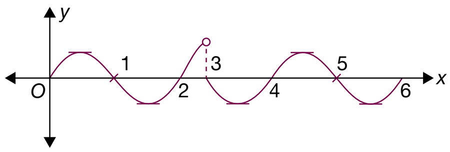
34. For $n = 1$, graph of $f(x)$



For $n = 2$, graph of $f(x)$



For $n = 3$, graph of $f(x)$



Hence $n = 3$ is the least value for which $f(x)$ has more minima than maxima

Hence, the correct answer is (3).

35. The polynomial is an everywhere differentiable function. Therefore, the points of extremum can only be roots of the derivative. Furthermore, the derivative of a polynomial is a polynomial. The polynomial of the least degree with roots $x_1 = 1$ and $x_2 = 3$ has the from $a(x-1)(x-3)$. Hence, $P'(x) = a(x-1)(x-3) = a(x^2 - 4x + 3)$ since at the point $x = 1$, there must be $P(1) = 6$, we have

$$P(x) = \int_1^x P'(x)dx + 6 = a\int_1^x (x^2 - 4x + 3)dx + 6$$

$$= a\left[\frac{x^3}{3} - 4\frac{x^2}{2} + 3x\right]_1^x + 6$$

$$= a\left(\frac{x^3}{3} - 2x^2 + 3x - \frac{4}{3}\right) + 6$$

The Coefficient '*a*' is determined from the condition $P(3) = 2$, hence $a = 3$.

Hence $P(x) = x^3 - 6x^2 + 9x + 2$

Now $P(2) = 8 - 24 + 18 + 2 = 28 - 24 = 4$

Also $P'(x) = 3(x^2 - 4x + 3) \Rightarrow P'(0) = 9$

$\therefore$ $P(2) + P'(0) = 4 + 9 = 13$

Hence, the correct answer is (13).

36. Let x_1, x_2 and x_3 be roots of $f(x)$.

Given $x_1 x_2 x_3 = 64$

Hence $f(x) = x^3 - x^2(x_1 + x_2 + x_3) + x\Sigma x_1x_2 - 64$

$f(-1) = -1 - (\Sigma x_1) - \Sigma x_1x_2 - 64$

Now $\frac{x_1 + x_2 + x_3}{3} \geq (x_1x_2x_3)^{1/3} \Rightarrow \frac{x_1 + x_2 + x_3}{3} \geq (64)^{1/3}$

$\therefore\ x_1 + x_2 + x_3 \geq 12$

and $\frac{\sum x_1x_2}{3} \geq (x_1x_2x_3)^{2/3}$

$\frac{\sum x_1x_2}{3} \geq (64)^{2/3} \Rightarrow \sum x_1x_2 \geq 3 \times 16$

$\therefore\ \Sigma x_1x_2 \geq 48$

$\therefore\ f(-1) = -65 - [\Sigma x_1 + \Sigma x_1x_2]$

$f(-1)_{\max} = -65 - [12 + 48] = -125$

$|f(-1)| = 125$

Hence, the correct answer is (125).

37. $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + x + 2$

$f'(x) = x^2 + x + 1 > 0\ \forall\ x \in R$

$g'(x) = (x^2 - 9)(x^2 - 4x + 3)(x^2 - 3x + 2)(x^2 - 2x + 2)$

$g'(x) = (x - 3)(x + 3)(x - 1)(x - 3)(x - 1)(x - 2)(x - 3)(x + 1)$

$= (x - 3)^3 (x + 3)(x - 1)^2 (x - 2)(x + 1)$

Let $h(x) = f(g(x))$

$h'(x) = f'(g(x))\, g'(x)$

$\because\ f'(g(x)) > (0)$

Point of local minima are $x = -1$ and 3

Point of local maxima are $x = -3$ and 2 and point of inflection is $x = 1$

$\therefore\ n_1 = 2, n_2 = 2, n_3 = 1 \Rightarrow n_1 + n_2 + n_3 = 5$

Hence, the correct answer is (5).

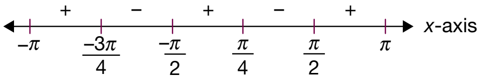
38. Given $f'(x) = 2f(x)$

$\therefore\ \frac{f'(x)}{f(x)} = 2 \Rightarrow f(x) = Ae^{2x}$

$f(0) = 1 = A$ sign chart of $f'(x)$

$\therefore\ f(x) = e^{2x}$

Sign chart of $f'(x)$



Now, $g(x) = e^{2x} \cdot \cos^2 x$

$g'(x) = e^{2x}(-2\cos x \sin x + 2\cos^2 x)$

$g'(x) = 2 \cdot \cos x \cdot e^{2x} (\cos x - \sin x)$

$g'(x) = 0 \Rightarrow x = \frac{-3\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{-\pi}{2}$

$\because\ g(x) \geq 0$

$\therefore$ Points of maxima are $\frac{-3\pi}{4}, \frac{\pi}{4}$ and points of minima are $-, \frac{-\pi}{2}, \frac{\pi}{2}$ and global minimum value occurs at $\frac{\pm\pi}{2}$ which is zero.

Hence, $n_1 = 3, n_3 = 2 \Rightarrow n_1 + n_2 + n_3 = 8$

Hence, the correct answer is ($n_1 = 3, n_2 = 3, n_3 = 2, n_1 + n_2 + n_3 = 8$).

39. $f(x) = \alpha\cos 2x - 4\cos^3 x$

$f'(x) = -2\alpha\sin 2x + 12\cos^2 x \sin x$

$f'(x) = -4\alpha \sin x \cos x + 12\cos^2 x \sin x$

$f'(x) = 4\sin x \cos x\,(-\alpha + 3\cos x)$

$f(x) = \alpha\cos 2x - 4\cos^3 x$

$f'(x) = 2\sin 2x \cdot (-\alpha + 3\cos x)$

$\therefore\ x = \frac{\pi}{2}$ is the only critical point in $(0, \pi)$. So $\alpha = 3$

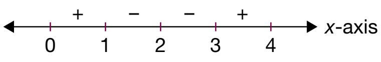
(Smallest positive integer).

Hence, the correct answer is (3).

40. Given, $f(x) = \left(\frac{2 - x}{\pi}\right)\cos\pi(x + 3) + \frac{1}{\pi^2}\sin\pi(x + 3)$

$= \left(\frac{x - 2}{\pi}\right)\cos\pi x - \frac{1}{\pi^2}\sin\pi x$

($\because\ \cos(3\pi + \theta) = -\cos\theta$ and $\sin(3\pi + \theta) = -\sin\theta$)



Now, $f'(x) = \left(\frac{x - 2}{\pi}\right)(-\pi\sin\pi x) + \frac{1}{\pi}\cos\pi x - \frac{1}{\pi^2}(\pi\cos\pi x)$

$\Rightarrow\ f'(x) = (2 - x)\sin(\pi x)$

$\because\ f'(x) = 0 \Rightarrow 2 - x = 0$ and $\sin\pi x = 0$

$\Rightarrow\ x = 2$ and $x = 1, 2, 3, x \in (0, 4)$

Clearly, $x = 1$ is the point of local maximum. Also $x = 3$ is the point of local minimum.

NOTE:

$x = 2$ is inflection point of function.

Hence, the correct answer is (2).

41. $f(x) = xe^{-x} - (1 - x)^{\frac{2}{3}} + 2x - x^2 + \int_0^x 2^t \log_{e^{-t}} t\, dt$

We have $f(x) = xe^{-x} - (1 - x)^{\frac{2}{3}} + 2x - x^2 - \int_0^x 2^t \log t\, dt$ ...(1)

Clearly domain of $f(x)$ is R^+.

Differentiation of $f(x)$ with respect to x

Now $f'(x) = e^{-x}(1 - x) + \frac{2}{3(1 - x)^{\frac{1}{3}}} + 2(1 - x) - 2^x \ln x$

For $0 < x < 1, f'(x) > 0$ and for $x > 1\ f'(x) < 0$.

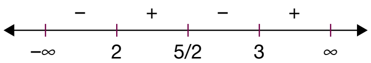
Also at $x = 1, f'(x)$ does not exist. So the only critical point is $x = 1$, which is the point of maxima

(As $f'(x)$ changes sign from positive to negative in the neighborhood of $x = 1$)

Also minimum value does not exist.

Hence, the correct answer is (1).

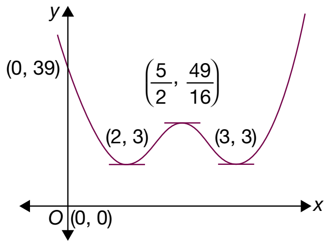
42. Let $f(x) = k(x - 2)^2 (x - 3)^2 + 3$



Sign of $f'(x)$

As, $f(1) = 4k + 3 = 7$

$\therefore\ k = 1 \Rightarrow f(x) = (x - 2)^2 (x - 3)^2 + 3$



Now, $f'(x) = 2(x-2)(x-3)(2x-5)$

$\Rightarrow$ $f(x)$ has local maximum at $x = \frac{5}{2}$

$\therefore$ $f_{\max}\left(x = \frac{5}{2}\right) = \frac{49}{16} \equiv \frac{p}{q}$

$\Rightarrow$ $(p+q) = 65$

Hence, the correct answer is (65).

43. $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$

$f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9)$

$f'(x) = 3[x^2 + 2(a-7)x + (a^2-9)]$ ...(1)

For positive point of maxima,

Both roots of $f'(x) = 0$ be must be positive.

$\therefore$ sum $> 0 \Rightarrow -2(a-7) > 0$

i.e., $a - 7 < 0$

$a < 7$ and

Product, $P > 0 \Rightarrow a^2 - 9 > 0 \Rightarrow a > 3$

or $a < -3$ and $D > 0$

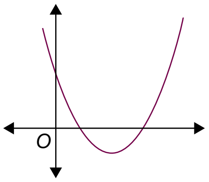
$4(a-7)^2 - 4(a^2-9) > 0$

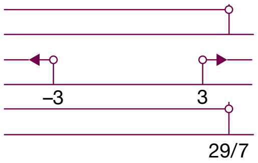
$a^2 - 14a + 49 - a^2 + 9 > 0$

$58 - 14a > 0$

$29 - 7a > 0$

$a < \frac{29}{7}$.





Hence $a \in (-\infty, -3) \cup \left(3, \frac{29}{7}\right)$

$\therefore$ Largest natural value of 'a' = 4

Hence, the correct answer is (0).

44. $\because$ $p(1+x) = p(1-x)\ \forall\ x \in R$

$\therefore$ $P(x)$ is symmetrical about the line $x = 1$ and it must be of degree 4.

Let $P(x) = (x-1)^4 + b(x-1)^2 + c$

$P(2) = 4 \Rightarrow 1 + b + c = 4 \Rightarrow b + c = 3$ ...(1)

$p'(x) = 4(x-1)^3 + 2b(x-1)$

$\Rightarrow$ $(x-1)(4(x-1)^2 + 2b) = 0$

$x = 1$ is point of local maximum

$p(1) = 5 \Rightarrow c = 5$ ...(2)

From (1) and (2) $= b = -2$

$\therefore$ $p(x) = (x-1)^4 - 2(x-1)^2 + 5 = ((x-1)^2 - 1)^2 + 4$

$P(6) = ((6-1)^2 - 1)^2 + 4 = 580$

Hence, the correct answer is (580).

45. $f(x) = \frac{x^3}{3} - (m-3)\frac{x^2}{2} + mx - 2013$

$\therefore$ $f'(x) = (x^2 - (m-3)x + m) \geq 0,\ \forall\ x \in [0, \infty)$

CASE 1: When $D \leq 0 \Rightarrow (m-3)^2 - 4m \leq 0 \Rightarrow m \in [1, 9]$ ...(1)

CASE 2: When $D \geq 0 \Rightarrow m \in [9, \infty)$ ...(2)

$\frac{-b}{2a} \leq 0 \Rightarrow (m-3) \leq 0 \Rightarrow m \leq 3$ ...(3)

and $f'(0) \geq 0 \Rightarrow m \geq 0$ ...(4)

$\therefore$ $(2) \cap (3) \cap (4)$

$\Rightarrow$ $m \in [0, 1]$ ...(5)

So, finally $(1) \cup (5) \Rightarrow m \in [0, 9]$

$\therefore$ $k = 9$

Hence, the correct answer is (9).

46. $f(x) = 1 - 3\sqrt{e}\ x - ex^3$

$f'(x) = -3\sqrt{e} - 3ex^2 < 0 \Rightarrow f(x)$ is decreasing.

$\therefore$ $f\left(\frac{e^{\frac{1}{x}}}{x-2}\right) < f\left(\frac{e^{\frac{1}{10}}}{8} \times 1\right)$

$\Rightarrow$ $\frac{e^{\frac{1}{10}}}{x-2} > \frac{e^{\frac{1}{10}}}{8}$

$\Rightarrow$ $x \in (2, 10)$

$\therefore$ least integral solution = 3

Hence, the correct answer is (3).

47. $f(x) = \left\{\frac{x - \sin x}{5}\right\}$

Let $h(x) = x - \sin x$; $h'(x) = 1 - \cos x \geq 0$

Now, $(x - \sin x)$ is an increasing function and range of $\left(\frac{x - \sin x}{5}\right)$ in $(0, 20\pi)$ is $(0, 4\pi)$

Hence, number of points where $f(x)$ is non-derivable are 12.

Hence, $c_1 : c_2 \ldots c_{12}$ are the values of c where $g'(c)$ is same

$\therefore$ Using Rolle's Theorem minimum number of points where g'' vanishes is $11 = n$

$\therefore$ $\left[\frac{n}{2}\right] = 5$

Hence, the correct answer is (5).

48. Let $f_n(x) = \int_x^{2x} e^{-t^n}\,dt$

$f'_n(x) = 2 \cdot e^{-(2x)^n} - e^{-x^n}$

For maxima and minima $f'_n(x) = 0 \Rightarrow 2\,e^{-(2x)^n} = e^{-x^n}$

$2 \cdot e^{-(2^n x^n)} = e^{-x^n}$

Taking log on both sides

$\ln 2 - 2^n x^n = -x^n \Rightarrow \ln 2 - x^n(2^n - 1) = 0$

$$x^n = \frac{\ln 2}{2^n - 1} \Rightarrow x = \left(\frac{\ln 2}{2^n - 1}\right)^{\frac{1}{n}}$$

$$f''_n(x) = 2e^{-(2x)^n}(-n(2x)^{n-1} \cdot 2) \Rightarrow f''_n(x) = -4ne^{-(2x)^n}(2x)^{n-1}$$

Also, $f''\left(x = \left(\frac{\ln 2}{2^n - 1}\right)^{\frac{1}{n}}\right) < 0$

$\Rightarrow$ $f_n(x)$ is maximum at $x = \left(\frac{\ln 2}{2^n - 1}\right)^{\frac{1}{n}}$

$$\ln a_n = \frac{\ln\left(\frac{\ln 2}{2^n - 1}\right)}{n} = \frac{\ln(\ln 2) - \ln(2^n - 1)}{n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\ln(\ln 2)}{n} - \frac{\ln\left(2^n\left(1 - \frac{1}{2^n}\right)\right)}{n}\right)$$

$$= \lim_{n\to\infty}\left(0 - \frac{n \cdot \ln 2 + \ln\left(1 - \frac{1}{2^n}\right)}{n}\right)$$

$$L = \lim_{n\to\infty}\left(0 - \ln 2 - \frac{\ln\left(1 - \frac{1}{2^n}\right)}{n}\right) = -\ln 2$$

Hence $e^{-L} = e^{-(-\ln 2)} = 2$

Hence, the correct answer is (2).

49. Clearly, domain of function is (0, 1).

Also, $f'(x) = k + \frac{(k-4)}{x} + \frac{36}{1-x} > 0\ \forall\ x \in (0, 1)$

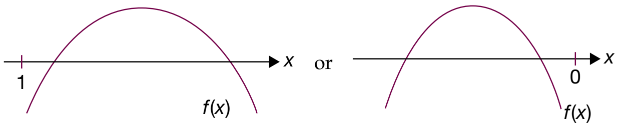
$\Rightarrow$ $kx(1-x) + (k-4)(1-x) + 36x > 0\ \forall\ x \in (0, 1)$

$\Rightarrow$ $kx^2 - 40x + (4-k) < 0\ \forall\ x \in (0, 1)$

Disc. $= D = 1600 + 4k(k-4) = 4k^2 - 16k + 1600 > 0\ \forall\ x \in R$.

Now, we shall consider following cases:

CASE 1: When $k < 0$;



$\frac{-b}{2a} > 1 \Rightarrow \frac{40}{2k} > 0 \quad f(0) < 0 \Rightarrow 4 - k < 0 \Rightarrow k > 4$

(Not possible as $k < 0$) (Not possible $k < 0$)

CASE 2: When $k = 0$, so

$-40x + 4 < 0$

$\Rightarrow \quad x > \frac{1}{10}$, which does no cover the complete domain so rejected.

CASE 3: When $k > 0$,

So, $f(0) \leq 0 \Rightarrow 4 - k \leq 0 \Rightarrow k \geq 4$ ...(1)

and $f(1) \leq 0 \Rightarrow k - 40 + 4 - k \leq 0 = -36 \leq 0$ (True) ...(2)

must be satisfied.

Hence, $k \in [4, \infty)$

So, $\alpha = 4$

Hence, the correct answer is (4).

50. Given, $f(x) = \cos^3 x - a\cos^2 x = \cos^2 x(\cos x - a),\ 0 < x < \pi$.

$\therefore \quad f'(x) = \cos^2 x(-\sin x) + (\cos x - a)\cdot 2\cos x(-\sin x) = \cos x \sin x\,(-\cos x + 2a - 2\cos x)$

$\therefore \quad f'(x) = \cos x \sin x\,(2a - 3\cos x),\ 0 < x < \pi$

So, $f'(x) = 0$, gives $\cos x = 0 \Rightarrow x = \frac{\pi}{2}$ or $\cos x = \frac{2a}{3}$

As, $0 < x < \pi \Rightarrow -1 < \cos x < 1 \Rightarrow -1 < \frac{2a}{3} < 1$

$\Rightarrow \quad \frac{-3}{2} < a < \frac{3}{2}$.

But $a = 0$ (Rejected)

for $a = 0$ there is only one point of extrema

So, $a \in \left(\frac{-3}{2}, 0\right) \cup \left(0, \frac{3}{2}\right)$

Hence, two integral value of a exist

Hence, the correct answer is (2).

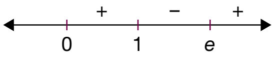
51. Given, $f(x) = (x^2 - 2x)\cdot \ln x - \frac{3}{2}x^2 + 4x + \frac{5}{2},\ x > 0$

$\therefore \quad f'(x) = (2x - 2)\ln x + (x^2 - 2x).\ \frac{1}{x} - \frac{3}{2}(2x) + 4.$

$= 2(x - 1)\ln x + (x - 2) - 3x + 4$

$= 2(x - 1)\ln x - 2(x - 1)$

$= 2(x - 1)\cdot(\ln x - 1)$



Sign scheme of $f'(x)$

So, $f'(x) = 0 \Rightarrow x = 1, e$.

$\therefore \quad f(x)$ has local maximum at $x = 1$.

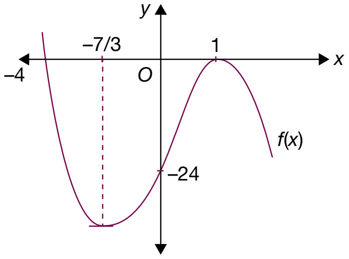
$\Rightarrow \quad f_{\text{local maximum}}(x = 1) = 0 - \frac{3}{2} + 4 + \frac{5}{2} = 1 + 4 = 5.$

Hence, the correct answer is (5).

52. $\Delta(x) = \begin{vmatrix} 0 & 2x-2 & 2x+8 \\ x-1 & 4 & x^2+7 \\ 0 & 0 & x+4 \end{vmatrix}$

$\Delta(x) = -(x - 1)(2(x - 1)(x + 4) - 0)$

$\Delta(x) = -2(x - 1)^2(x + 4)$

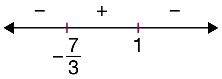


$f(x) = \sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}c_{ij} = 3\Delta(x)$

$= -6(x - 1)^2(x + 4)$

$f'(x) = -6(x - 1)(3x + 7) = -6(x - 1)(3x + 7)$

Sign chart of $f'(x)$



From the graph it can be easily observed that greatest value of $f(x)$ in $[-3, 18]$ is 0 at $x = 1$

Hence, the correct answer is (0).

53. $f(x) = 2x^3 + bx^2 + 4x + \int_2^x (2\sin^2 t + 3\cos^2 t)dt \quad \forall\ x \in R$

Clearly, $f'(x) = 6x^2 + 2bx + 4 + 2\sin^2 x + 3\cos^2 x$

Now, $f'(x) \geq 0\ \forall\ x \in R$.

$\Rightarrow \quad 6x^2 + 2bx + 4 \geq -(2\sin^2 x + 3\cos^2 x)$

$\Rightarrow \quad 6x^2 + 2bx + 4 \geq -(2 + \cos^2 x)$

$\Rightarrow \quad 6x^2 + 2bx + 4 \geq -2\ \forall\ x \in R$

$\Rightarrow \quad 3x^2 + bx + 3 \geq 0\ \forall\ x \in R$

$\Rightarrow \quad b^2 - 4\cdot 3\cdot 3 \leq 0$

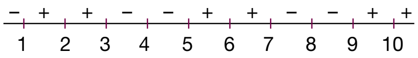
$\Rightarrow \quad -6 \leq b \leq 6$

$\Rightarrow \quad b = \pm 6, \pm 5, \pm 4, \pm 3, \pm 2, \pm 1, 0$

Hence, the correct answer is (13).

54. $f'_{10}(x) = (x - 1)(x - 2)^2(x - 3)^3 \ldots (x - 10)^{10}$

Sign chart of $f'_{10}(x)$



$f'_{10}(x) > 0$

$\Rightarrow \quad (x - 1)(x - 3)(x - 5)(x - 7)(x - 9) > 0$

$\Rightarrow \quad$ Points of maxima = 3, 7 $\qquad n_1 = 10$

Points of minima = 1, 5, 9 $\qquad n_2 = 15$

Points of inflexion = 2, 4, 6, 8, 10 $\qquad n_3 = 30$

= value of $\frac{6(n_1 + n_2)}{n_3} = \frac{6 \times 25}{30} = 5$

Hence, the correct answer is (5).

ARCHIVE: JEE MAIN

1. $f(x) = \frac{4x^3 - 3x^2}{6} - 2\sin x + (2x - 1)\cos x$

$f'(x) = (2x^2 - x) - 2\cos x + 2\cos x - \sin x\,(2x - 1)$

$= (2x - 1)(x - \sin x)$

For $x > 0,\ x - \sin x > 0$

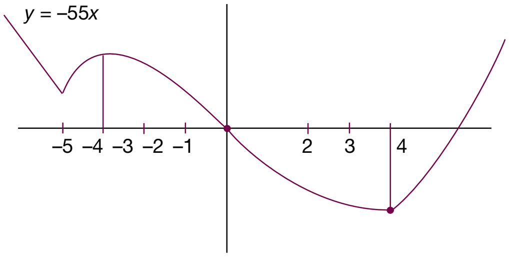
$x < 0,\ x - \sin x < 0$

$\Rightarrow \quad f(x)$ is increasing in $(-\infty, 0] \cup \left[\frac{1}{2}, \infty\right)$

and $f(x)$ is decreasing in $\left(0, \frac{1}{2}\right)$

Hence, the correct answer is (A).

2.



$$f'(x) = \begin{cases} -55; & x < -5 \\ 6(x-5)(x+4); & -5 < x < 4 \\ 6(x-3)(x+2); & x > 4 \end{cases}$$

$f(x)$ is increasing in

$x \in (-5, -4) \cup (4, \infty)$

Hence, the correct answer is (B).

3. Let $f(x) = x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ as $\lim\limits_{x \to 0} \dfrac{f(x)}{x^3}$

$= 1$ non-zero finite

So, $d = e = f = 0$ and $f(x) = x^3(x^3 + ax^2 + bx + c)$

Hence, $\lim\limits_{x \to 0} \dfrac{f(x)}{x^3} = c = 1$

Now, as $f(x) = x^6 + ax^5 + bx^4 + x^3$ and $f'(x) = 0$ at $x = 1$ and $x = -1$

i.e., $f'(x) = 6x^5 + 5ax^4 + 4bx^3 + 3x^2$

$f'(1) = 0 \Rightarrow 6 + 5a + 4b + 3 = 0$

$\Rightarrow\ 5a + 4b = -9$ and $f'(-1) = 0 \Rightarrow -6 + 5a - 4b + 3 = 0$

$\Rightarrow\ 5a - 4b = 3$

Solving both we get, $a = \dfrac{-6}{10} = \dfrac{-3}{5}$; $b = \dfrac{-3}{2}$

$\therefore\ f(x) = x^6 - \dfrac{3}{5}x^5 - \dfrac{3}{2}x^4 + x^3$

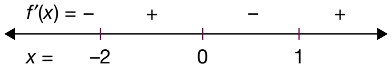
$\therefore\ 5f(2) = 5\left[64 - \dfrac{3}{5}\cdot 32 - \dfrac{3}{2}\cdot 16 + 8\right]$

$= 320 - 96 - 120 + 40$

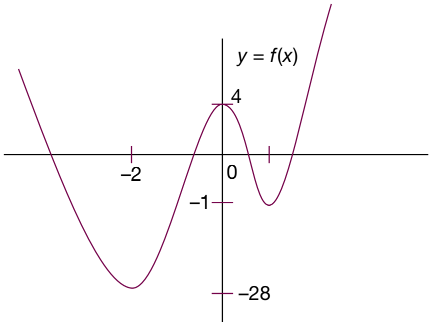
$= 144$

Hence, the correct answer is (144).

4. $3x^4 + 4x^3 - 12x^2 + 4 = 0$



So,



Let $f(x) = 3x^4 + 4x^3 - 12x^2 + 4$

$\therefore\ f'(x) = 12x(x^2 + x - 2)$

$= 12x\,(x + 2)\,(x - 1)$

Hence, the correct answer is (4).

5. Let $g(x) = \sin x + \cos x = \sqrt{2}\quad \sin\left(x + \dfrac{\pi}{4}\right)$

$g(x) \in \left[1, \sqrt{2}\right]$ for $x \in [0, \pi/2]$

$f(x) = \tan^{-1}(\sin x + \cos x) \in \left[\dfrac{\pi}{4}, \tan^{-1}\sqrt{2}\right]$

$\tan\left(\tan^{-1}\sqrt{2} - \dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}-1}{1+\sqrt{2}} \times \dfrac{\sqrt{2}-1}{\sqrt{2}-1} = 3 - 2\sqrt{2}$

Hence, the correct answer is (D).

6. $V = l.b.h = (a - 2x)\,(b - 2x)\,x$

$\Rightarrow\ V(x) = (2x - a)\,(2x - b)\,x$

$\Rightarrow\ V(x) = 4x^3 - 2\,(a + b)\,x^2 + abx$

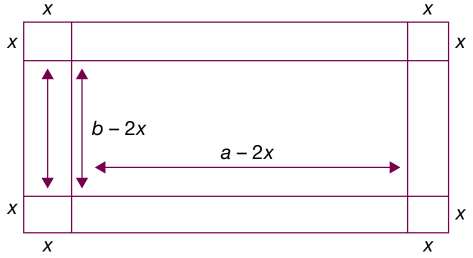
$\Rightarrow\ \dfrac{d}{dx}v(x) = 12x^2 - 4\,(a + b)\,x + ab$

$\dfrac{d}{dx}(v(x)) = 0$

$\Rightarrow\ 12x^2 - 4\,(a + b)\,x + ab = 0\ (\alpha, \beta)$

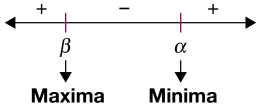
$\Rightarrow\ x = \dfrac{4(a+b) \pm \sqrt{16(a+b)^2 - 48ab}}{2(12)}$

$= \dfrac{(a+b) \pm \sqrt{a^2 + b^2 - ab}}{6}$



Let $x = \alpha = \dfrac{(a+b) + \sqrt{a^2 + b^2 - ab}}{6}$;

$\beta = \dfrac{(a+b) - \sqrt{a^2 + b^2 - ab}}{6}$



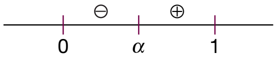
Now, $12(x - \alpha)\,(x - \beta) = 0$

$\therefore\ x = \beta$

$= \dfrac{a + b - \sqrt{a^2 + b^2 - ab}}{b}$

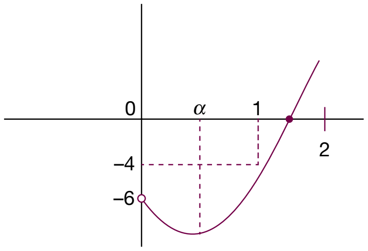
Hence, the correct answer is (C).

7. Let $e^x = t > 0$



$f(t) = t^4 + 2t^3 - t - 6 = 0$

$f'(\mathrm{t}) = 4t^4 + 6t^3 - 1$



$f''(t) = 12t^4 + 12t > 0$

$f(0) = -6, f(1) = -4, f(2) = 24$

$\Rightarrow$ Number of real roots $= 1$

Hence, the correct answer is (C).

8. $f(x) = x^2 + ax + 1$

$f'(x) = 2x + a$

When $f(x)$ is increasing on [1, 2]

$2x + a \geq 0 \quad \forall\, x \in [1, 2]$

$a \geq -2x \;\forall\, x \in [1, 2]$

$R = -4$

When $f(x)$ is decreasing on [1, 2]

$2x + a \leq 0 \;\forall\, x \in [1, 2]$

$a \leq -2 \;\forall\, x \in [1, 2]$

$S = -2$

$|R - S| = |-4 + 2| = 2$

Hence, the correct answer is (2).

9. $F'(x) = a(x - 1)(x + 3)$

$F''(x) = 6a(x + 1)$

$F'(x) = 3a(x + 1)^2 + b$

$F'(1) = 0$

$\Rightarrow \quad b = -12a$

$F(x) = a(x + 1)^3 - 12ax + c$

$= (x + 1)^3 - 12x - 6$

$F(3) = 64 - 36 - 6 = 22$

Hence, the correct answer is (22).

10. $f(x) = x^3 - 6x^2 + ax + b$

$f(2) = 8 - 24 + 2a + b = 0$

$2a + b = 16$ …(1)

$f(4) = 64 - 96 + 4a + b = 0$

$4a + b = 32$ …(2)

Solving (1) and (2) $\quad a = 8, b = 0$

$f(x) = x^3 - 6x^2 + 8x; f(x) = x^3 - 6x^2 + 8x$

$f'(x) = 3x^2 - 12x + 8; f''(x) = 6x - 12$

$\Rightarrow$ $f'(x)$ is ↑ for $x > 2$, and $f'(x)$ is ↓ for $x < 2$

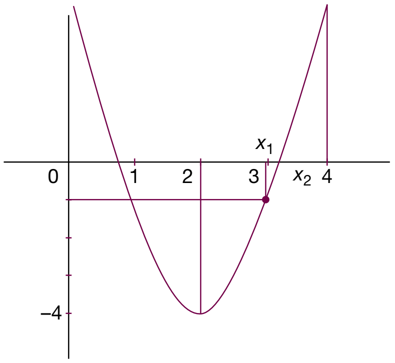
$f'(2) = 12 - 24 + 8 = -4; f'(4) = 48 - 48 + 8 = 8$

$f'(x) = 3x^2 - 12x + 8$

Vertex (2, –4) $\quad f'(2) = -4, f'(4) = 8, f'(3) = 27 - 36 + 8$

$f'(x_1) = -1$, then $x_1 = 3$

$f'(x_2) = 0$



Again $f'(x) < 0$ for $x \in (2, x_4)$

$f'(x) > 0$ for $x \in (x_4, 4)$

$x_4 \in (3, 4)$

$f(x) = x^3 - 6x^2 + 8x; f(3) = 27 - 54 + 24 = -3$

$f(4) = 64 - 96 + 32 = 0$

For $x_4(3, 4)$

$f(x_4) < -\;3\sqrt{3}$ and $f'(x_3) > -4$

$2f'(x_3) > -8$

So, $2f'(x_3) = \sqrt{3}\; f(x_4)$

Hence, the correct answer is (A).

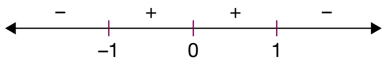
11. $\lim\limits_{x\to 0}\left(2 + \dfrac{f(x)}{x^3}\right) = 4$

$\Rightarrow \; f(x) = 2x^3 + ax^4 + bx^5$

$f'(x) = 6x^2 + 4ax^3 + 5bx^4$ and $f'(1) = 0, f'(-1) = 0$

$a = 0, b = \dfrac{-6}{5} \Rightarrow f(x) = f(x) = 2x^3 - \dfrac{6}{5}x^5$

$f(x) = 6x^2 - 6x^4 = 6x^2(1 - x)(1 + x)$

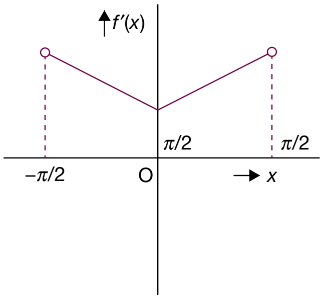


Sign scheme for $f'(x)$

Minima at $x = -1$, Maxima at $= 1$

Hence, the correct answer is (B).

12. $f(x)$ is an odd function.



Now, if $x \geq 0$, then $f(x) = x\cos^{-1}(-\sin x)$

$$= x\left(\frac{\pi}{2} - \sin^{-1}(-\sin x)\right) = x\left(\frac{\pi}{2} + x\right)$$

$$\text{Hence, } f(x) = \begin{cases} x\left(\dfrac{\pi}{2} + x\right) & ; \quad x \in \left[0, \dfrac{\pi}{2}\right] \\ x\left(\dfrac{\pi}{2} - x\right) & ; \quad x \in \left(-\dfrac{\pi}{2}, 0\right) \end{cases}$$

$$\text{So, } f'(x) = \begin{cases} \dfrac{\pi}{2} + 2x & ; \quad x \in \left[0, \dfrac{\pi}{2}\right] \\ \dfrac{\pi}{2} - 2x & ; \quad x \in \left(-\dfrac{\pi}{2}, 0\right) \end{cases}$$

Hence, the correct answer is (A).

13. $f''(x) = \lambda(x - 1)$

$f''(x) = \dfrac{\lambda x^2}{2} - \mathrm{l}x + C$

$\Rightarrow \; f'(-1) = 0 \Rightarrow c = \dfrac{-3\lambda}{2}$

$f(x) = \dfrac{\lambda x^3}{6} - \dfrac{\lambda x^2}{2} - \dfrac{3\lambda}{2}x + d$

$f(1) = -6 \Rightarrow -11\lambda + 6d = -36$ …(1)

$f(-1) = 10 \Rightarrow 5\lambda + 6d = 60$ …(2)

From (1) and (2) $\lambda = 6$ and $d = 5$

$f(x) = x^3 - 3x^2 - 9x + 5$

which has minima at $x = 3$

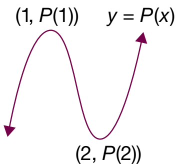
Hence, the correct answer is (3).

14. Since $p(x)$ has relative extreme at $x = 1$ and 2

So $p'(x) = 0$ at $x = 1$ and 2

$\Rightarrow \; p'(x) = A(x - 1)(x - 2)$

$\Rightarrow \; p(x) = \displaystyle\int A(x^2 - 3x + 2)\, dx$



$$p(x) = A\left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right) + C \qquad \ldots(1)$$

$$P(1) = 8 = A\left(\frac{1}{3} - \frac{3}{2} + 2\right) + C \Rightarrow 8 = \frac{5A}{6} + C$$

$$\Rightarrow \; 48 = 5A + 6C \qquad \ldots(2)$$

$$P(2) = 4 = A\left(\frac{8}{3} - 6 + 4\right) + C \Rightarrow 4 = \frac{2A}{3} + C$$

$$\Rightarrow \; 12 = 2A + 3C \qquad \ldots(3)$$

From (2) and (3) we have $A = 24$, $C = -12$

Now $P(0) = C = -12$

Hence, the correct answer is (D).

15. $$f'(x) = \frac{\frac{x}{1+x} - \ln(1+x)}{x^2} = \frac{x - (1+x)\ln(1+x)}{x^2(1+x)}$$

Suppose $h(x) = x - (1 + x)\, ln(1 + x)$

$\Rightarrow \; h'(x) = 1 - ln(1 + x) - 1 = - ln(1 + x)$

$h'(x) > 0, \forall\, x \in (-1, 0)$ and $h'(x) < 0, \forall\, x \in (0, \infty)$

$h(0) = 0 \Rightarrow h'(x) < 0 \;\forall\, x \in (-1, \infty)$

$\Rightarrow \; f'(x) < 0 \;\forall\, x \in (-1, \infty)$

Hence, $f(x)$ is a decreasing function for all $x \in (-1, \infty)$

Hence, the correct answer is (A).

16. $f'(x) = x(x + 1)\,(x - 1) = x^3 - x$

$$\int df(x) = \int x^3 - x\, dx \Rightarrow f(x) = \frac{x^4}{4} - \frac{x^2}{2} + f(0)$$

Now $f(x) = f(0) \Rightarrow \dfrac{x^4}{4} - \dfrac{x^2}{2} = 0 \Rightarrow x^2(x^2 - 2) = 0$

$\Rightarrow \; x = 0, 0, \sqrt{2}, -\sqrt{2}$

$x_1^2 + x_2^2 + x_3^2 = 0 + 2 + 2 = 4$

Hence, the correct answer is (C).

17. $f(x) = (3x - 7)x^{2/3} = 3x^{5/3} - 7x^{2/3}$

$$\Rightarrow \; f'(x) = 5x^{2/3} - \frac{14}{3x^{1/3}} = \frac{15x - 14}{3x^{1/3}} > 0$$

$$\therefore \; f'(x) > 0 \quad \forall\, x \in (-\infty, 0) \cup \left(\frac{14}{15}, \infty\right)$$

Hence, the correct answer is (B).

18. $h(x) = f(g(x))$

$h'(x) = f'(g(x))\, g'(x)$ and $f'(x) = e^x - 1$

$h'(x) = (e^{g(x)} - 1)\, g'(x)$

$h'(x) = (e^{x^2 - x} - 1)\,(2x - 1) \geq 0$

CASE 1: $e^{x^2 - x} \leq 1$ and $2x - 1 \leq 0 \Rightarrow x \in \left[0, \frac{1}{2}\right]$ …(1)

CASE 2: $e^{x^2 - x} \geq 1$ and $2x - 1 \geq 0 \Rightarrow x \in [1, \infty]$ …(2)

From (1) and (2)

$$x \in \left[0, \frac{1}{2}\right] \cup [1, \infty)$$

Hence, the correct answer is (D).

19. $f(x)$ is increasing $f'(x) \geq 0$ and $f(x_1) \leq f(x_2)$, $x \in [0, 3]$

$f(0) \leq f(3) \Rightarrow 0 \leq 3\sqrt{3k - 9} \Rightarrow k \geq 3$

$$f'(x) = \frac{x(k - 2x)}{2\sqrt{kx - x^2}} + \sqrt{kx - x^2}$$

$$= \frac{x(k - 2x) + 2(kx - x^2)}{2\sqrt{kx - x^2}} = \frac{3kx - 4x^2}{2\sqrt{kx - x^2}} \geq 0$$

Now $kx - x^2 \geq 0$ $x \in [0, k]$

$3kx - 4x^2 \geq 0 \Rightarrow x(3k - 4x) \geq 0$ $x \in [0, 3k/4]$ given interval is $x \in [0, 3]$

$k \geq 4$ hence minimum value of $k = 4 = m$

$f(3) = M = 3\sqrt{4(3) - 9} = 3\sqrt{3}$

$(m, M) = (4, 3\sqrt{3})$

Hence, the correct answer is (B).

20. $f(x) = 5 - |x - 2|$ $f(x)$ attains maximum value of $x = +2 = \alpha$

$g(x)$ attains minimum value of $x = -1 = \beta \Rightarrow \alpha\beta = -2$

$$\lim_{x \to -\alpha\beta} \frac{(x - 1)(x^2 - 5x + 6)}{x^2 - 6x + 8} = \lim_{x \to 2} \frac{(x - 1)(x - 2)(x - 3)}{(x - 2)(x - 4)}$$

$$= \frac{(2 - 1)(2 - 3)}{(2 - 4)} = \frac{1}{2}$$

Hence, the correct answer is (A).

21. Here, $$h(x) = \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}}$$

When $x - \dfrac{1}{x} < 0$ $\quad \therefore x - \dfrac{1}{x} + \dfrac{2}{x - \frac{1}{x}} \geq 2\sqrt{2}$

Hence, $2\sqrt{2}$ will be local minimum value of $h(x)$.

Hence, the correct answer is (C).

22. We have

Total length $= r + r + r\theta = 20$

$\Rightarrow \; 2r + r\theta = 20$

$$\Rightarrow \; \theta = \frac{20 - 2r}{r} \qquad \ldots(1)$$

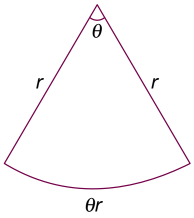
$$A = \text{Area} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2} r^2 \theta = \frac{1}{2} r^2 \left(\frac{20 - 2r}{r}\right) \qquad A = 10r - r^2$$

For A to be maximum

$\dfrac{dA}{dr} = 0 \Rightarrow 10 - 2r = 0 \;\Rightarrow r = 5$

$\dfrac{d^2A}{dr^2} = -2 < 0$

$\therefore$ For $r = 5$ A is maximum



From (1) $\theta = \dfrac{20 - 2(5)}{5} = \dfrac{10}{5} = 2$

$$A = \frac{2}{2\pi} \times \pi(5)^2 = 25 \text{ sq.m.}$$

Hence, the correct answer is (D).

23. We have $4x + 2\pi r = 2$

Now sum of areas of square and circle is $x^2 + \pi r^2$

$$\Rightarrow \; f(r) = \left(\frac{1 - \pi r}{2}\right)^2 + \pi r^2$$

$$\Rightarrow \; \frac{df(r)}{dr} = 2\left(\frac{\pi r - 1}{2}\right)\left(\frac{\pi}{2}\right) + 2\pi r$$

$$\frac{df(r)}{dr} = 0 \Rightarrow r = \frac{1}{\pi + 4}$$

From (1), $2x = \dfrac{4}{\pi + 4} \;\Rightarrow x = 2r$

Hence, the correct answer is (D).

24. $f'(x) = 0$ at $x = 1$ and $x = 2$ or $\qquad f'(1) = 0$ and $f'(2) = 0$

Also, $\displaystyle\lim_{x \to 0} \frac{f(x)}{2} = 2$ $\qquad \therefore\; f(x) = ax^4 + bx^3 + 2x^2$

$\Rightarrow \; f'(1) = 4ax^4 + bx^3 + 2x^2$ $\qquad \therefore\; f'(1) = 4a + 3b + 4$

and, $f'(2) = 32a + 12b + 8$

Thus, we have

$8a + 3b + 2 = 0$ …(1)

$4a + 3b + 4 = 0$ …(2)

On solving (1) and (2), we get

$4a = 2 \Rightarrow a = \frac{1}{2}$

$\therefore\ b = -2$

$\therefore\ f(x) = \frac{x^4}{2} - 2x^3 + 2x^2$ Hence, $f(2) = 8 - 16 + 8 = 0$

Hence, the correct answer is (A).

25. $f'(x) = \frac{\alpha}{x} + 2\beta x + 1$

$\Rightarrow\ 2\beta x^2 + x + \alpha = 0$ has roots –1 and 2

$\Rightarrow\ \alpha = 2, \beta = \frac{-1}{2}$

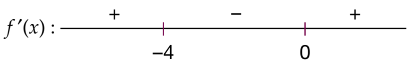
Hence, the correct answer is (C).

ARCHIVE: JEE ADVANCED

1. $f(x) = \frac{x^2 - 3x - 6}{x^2 + 2x + 4}$

$$f'(x) = \frac{(x^2 + 2x + 4)(2x - 3) - (x^2 - 3x - 6)(2x + 2)}{(x^2 + 2x + 4)^2}$$

$$f'(x) = \frac{5x(x + 4)}{(x^2 + 2x + 4)^2}$$



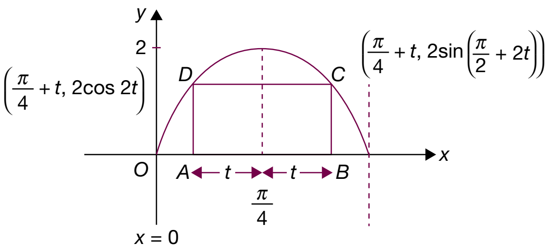
$f(-4) = \frac{11}{6}, f(0) = -\frac{3}{2}, \lim_{x \to \pm\infty} f(x) = 1$

Range : $\left[-\frac{3}{2}, \frac{11}{6}\right]$, clearly $f(x)$ is into

Hence, (A) and (B) are correct.

2. Let AB is the side on x axis

$A\left(\frac{\pi}{4} - t, 0\right)\ \ B\left(\frac{\pi}{4} + t, 0\right)$


$\left(\frac{\pi}{4} + t, 2\cos 2t\right)$
$\left(\frac{\pi}{4} + t, 2\sin\left(\frac{\pi}{2} + 2t\right)\right)$
y
2
D
C
x
O
A
t
t
B
$\frac{\pi}{4}$
x = 0

Perimeter of rectangle $P(t) = 2(2t + 2\cos 2t)$ where $t \in \left(0, \frac{\pi}{4}\right)$

$P'(t) = 4(1 - 2\sin 2t) = 0 \Rightarrow \sin 2t = \frac{1}{2} \Rightarrow 2t = \frac{\pi}{6} \Rightarrow t = \frac{\pi}{12}$

$P''(t) = -16\cos 2t < 0$ at $t = \frac{\pi}{12}$, $P(t)$ is maximum

Area of rectangle $= AB \cdot BC = 2t \cdot 2\cos 2t = 4t\cos 2t$

$$= 4\left(\frac{\pi}{12}\right)\cos\frac{\pi}{6} = \frac{\pi}{3} \times \frac{\sqrt{3}}{2} = \frac{\pi}{2\sqrt{3}}$$

Hence, the correct answer is (C).

3. $b \neq 0$

$$f'(x) = \frac{f(x)}{b^2 + x^2} \Rightarrow \int \frac{f'(x)}{f(x)}\,dx = \int \frac{1}{b^2 + x^2}\,dx$$

$$\Rightarrow\ \ln f(x) = \frac{1}{b}\tan^{-1}\frac{x}{b} + C \Rightarrow f(x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b} + C}$$

Given $f(0) = 1 = e^c$ we get $f(x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}$

and $f(-x) = e^{\frac{1}{b}\tan^{-1}\left(\frac{-x}{b}\right)} = e^{-\frac{1}{b}\tan^{-1}\frac{x}{b}}$

Now $f(x)f(-x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}\, e^{-\frac{1}{b}\tan^{-1}\frac{x}{b}} = 1$

$f(x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}\quad \forall\, x \in R$

$$f'(x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}\left(\frac{1}{b} \cdot \frac{1}{1 + \left(\frac{x}{b}\right)^2} \times \frac{1}{b}\right)$$

$$= e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}\left[\frac{1}{b^2 + x^2}\right] > 0 \text{ for } \forall\, b \in R; \forall\, x \in R$$

Hence, (A) and (C) are correct.

4.
$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1 & x < 0 \\ x^2 - x + 1 & 0 \le x < 1 \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3} & 1 \le x < 3 \\ (x - 2)\ell n(x - 2) - x + \frac{10}{3} & x \ge 3 \end{cases}$$

$$f'(x) = \begin{cases} 5(x^4 + 4x^3 + 6x^2 + 4x + 1) - 2 & x < 0 \\ 2x - 1 & 0 \le x < 1 \\ 2x^2 - 8x + 7 & 1 \le x < 3 \\ \ell n(x - 2) & x \ge 3 \end{cases}$$

$f'(x)$ is continuous at $x = 1$

L.H.D.$= f''(1^-) = 2$ and R.H.D.$= f''(1^+) = -4$,

f' is not differentiable at $x = 1$ and $f'(x)$ has local max. at $x = 1$

$f'(x) = 5(x^4 + 4x^3 + 6x^2 + 4x + 1) - 2 = 5(x + 1)^4 - 2$ is changing sign in $(-\infty, 0)$,

so if is not increasing

Range of f is R, so f is onto

Hence, (B), (C) and (D) are correct.

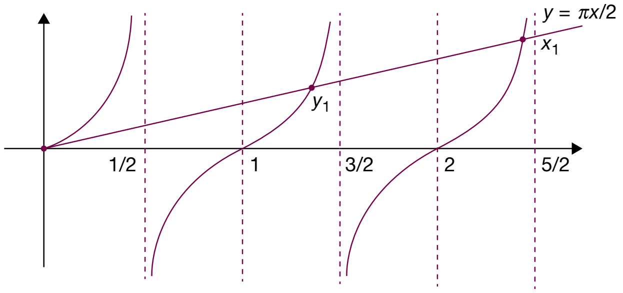
5. $f(x) = \frac{\sin\pi x}{x^2}$

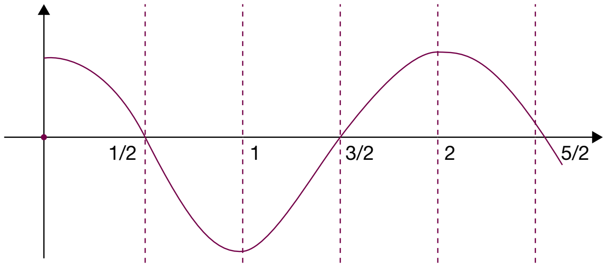
$$\Rightarrow\ f'(x) = \frac{\pi x^2\cos\pi x - 2x\sin\pi x}{x^4} = \frac{2\cos\pi x\left(\frac{\pi x}{2} - \tan\pi x\right)}{x^3}$$

$f'(x) = 0 \Rightarrow \cos\pi x = 0$ or $\frac{\pi x}{2} = \tan\pi x$

$\Rightarrow\ \pi x = (2n + 1)\frac{\pi}{2}$ or $\frac{\pi x}{2} = \tan\pi x$

$x = \frac{(2n + 1)}{2}, n \in W$





$f'(x)$ does not change its sign about these points hence neither maxima nor minima

$$\frac{\pi x}{2} = \tan \pi x$$

Where $y_n \in (2n-1, 2n-1/2)\ \forall\ n = 1, 2, 3, \ldots$ point of minima and $x_n \in (2n, 2n+1/2)\ \forall\ n = 1, 2, 3 \ldots$ point of maxima first diagram explains about point of intersection of $\tan\pi x$ with $\pi x/2$ and second diagram tells about $\cos p x$ clearly $y_1 < x_1$

$$x_n - y_n > 1$$

$$x_{n+1} - y_{n+1} > 1 \text{ and } y_{n+1} - x_n > 1 \Rightarrow x_{n+1} - x_n > 2.$$

Hence, (B), (C) and (D) are correct.

Combined solution to 6–7

6. **Hence, the correct answer is (B).**
7. **Hence, the correct answer is (B).**

$f(x) = 0 \Rightarrow \sin(\pi\cos x) = 0$

$\Rightarrow\ \pi\cos x = n\pi, n \in I$

$\Rightarrow\ \cos x = -1, \cos x = 0, \cos x = 1$

$\Rightarrow\ x = \frac{n\pi}{2}, n \in I$

$\Rightarrow\ X = \left\{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \ldots\right\}$ I matches with P, Q

$f'(x) = 0 \Rightarrow -\cos(\pi\cos x)\ \pi\sin x = 0$

$\Rightarrow\ \sin x = 0$ or $\cos(\pi\cos x) = 0$

$\Rightarrow\ x = n\pi$ or $\pi\cos x = (2k+1)\pi/2$

$\Rightarrow\ \cos x = -1/2, 1/2 \Rightarrow x = m\pi \pm \pi/3$

$Y = \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi\ldots\right\}$ II matches with Q, T

$g(x) = 0 \Rightarrow \cos(2\pi\sin x) = 0$

$\Rightarrow\ 2\pi\sin x = (2n+1)\frac{\pi}{2}$

$\Rightarrow\ \sin x = \frac{2n+1}{4} = -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}$

$\Rightarrow\ Z = \left\{x \middle| \sin x = \pm\frac{1}{4}, \pm\frac{3}{4}, x > 0\right\}$ III matches with R

$g'(x) = 0 \Rightarrow -2\pi\cos x \sin(2\pi\sin x) = 0$

$\cos x = 0$ or $\sin(2\pi\sin x) = 0 \Rightarrow 2\pi\sin x = k\pi \Rightarrow \sin x = k/2$

$\Rightarrow\ \sin x = -1, -\frac{1}{2}, 0, \frac{1}{2}, 1$

$W = \left\{x \middle| \sin x = \pm 1, \pm\frac{1}{2}, 0,\ x > 0\right\}$ IV matches with P, R, S

Final correct matches

I – (P), (Q), II – (Q), (T), III – (R), IV – (P), (R), (S)

Combined solution to 8–10

8. **Hence, the correct answer is (B).**
9. **Hence, the correct answer is (B).**
10. **Hence, the correct answer is (C).**

For $x \in R^+$

$f(x) = x + \ln x - x\ln x\ ;\ f'(x) = \frac{1}{x} - \ln x\ ;$

$f''(x) = -\frac{1}{x^2} - \frac{1}{x} = -\frac{(1+x)}{x^2}$

$f(1) \cdot f(e^2) = 2 - e^2 < 0$ then $f(x) = 0$ for some $x \in (1, e^2)$

$f'(1) \cdot f'(e) = \left(\frac{1}{e} - 1\right) < 0 \Rightarrow f'(x) = 0$ for some $x \in (1, e)$

$f''(x) = -\frac{(1+x)}{x^2} < 0$ for all $x \in R^+$ hence $f''(x) \neq 0$

Column I has I and II correct options

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 1 - (x-1)(\ln x - 1) = -\infty$$

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\left(\frac{1}{x} - \ln x\right) = -\infty$$

$$\lim_{x\to\infty} f''(x) = \lim_{x\to\infty} -\frac{(1+x)}{x^2} = 0$$

Column 2 has (ii)(iii)(iv) correct options

$f'(x) = \frac{1}{x} - \ln x\quad f'(e) = \frac{1}{e} - 1 < 0, f'(e^2) = \frac{1}{e^2} - 2 < 0$

$f'(x) > 0$ for $x \in (0, 1)$

$f(x)$ is decreasing for $x \in (e, e^2)$

$f''(x) = -\frac{(1+x)}{x^2} < 0$ $f'(x)$ is decreasing in $(0, \infty)$

Correct options in column 3 are PQS

11. Let $h(x) = f(x) - x$ continuous and differentiable in $[1/2, 1]$

$$h\left(\frac{1}{2}\right) = 0 = h(1)$$

by Rolles theorem, $h'(\alpha) = 0$ for some $\alpha \in (1/2, 1)$
$f'(\alpha) = 1$ for some $\alpha \in (1/2, 1)$
given $f''(x) > 0$, $f'(x)$ is increasing
$1 > \alpha\quad \therefore f'(1) > f'(\alpha)$
$f'(1) > 1$
Hence, the correct answer is (D).

12. $f'(x) - 2f(x) > 0$

$\Rightarrow\ \frac{d}{dx}\left(f(x) \cdot e^{-2x}\right) > 0$

$f(x) \cdot e^{-2x}$ is increasing function for all $x > 0$

$\Rightarrow\ f(x) \cdot e^{-2x} > f(0)1$

$f(x) > e^{2x}$ for $\forall\ x > 0$

Given $f'(x) > 2f(x) \Rightarrow f'(x) > 2f(x) > 2e^{2x} > 0\ \forall\ x > 0.$

$f'(x) > 0$, $f(x)$ is increasing function
Hence, (A) and (C) are correct.

13. $f(x)=\begin{vmatrix}\cos 2x & \cos 2x & \sin 2x\\ -\cos x & \cos x & -\sin x\\ \sin x & \sin x & \cos x\end{vmatrix}=\cos 4x+\cos 2x$

Now $f'(x) = -2\sin 2x - 4\sin 4x = 0$

$\Rightarrow\ f'(x) = 2\sin 2x\,(1 + 4\cos 2x) = 0$

then $\sin 2x = 0$ or $\cos 2x=-\dfrac{1}{4}$

For $\sin 2x = 0$; $x = 0, \pi/2, -\pi/2$

For $\cos 2x = -1/4$ there are four solutions

$f'(x) = 0$ has more than three solutions

Again $f''(x) = -(4\cos 2x + 16\cos 4x)$

$f''(0) < 0$ at $x = 0$ maximum

Hence, (B) and (C) are correct.

14. $4\alpha x^2+\dfrac{1}{x}\ge 1\ \Rightarrow\ y=4\alpha x^2+\dfrac{1}{x}$

$\Rightarrow\quad y=\dfrac{dy}{dx}=8\alpha x-\dfrac{1}{x^2}=0\Rightarrow x=\left(\dfrac{1}{8\alpha}\right)^{1/3}$

$\Rightarrow\quad f(x)=\dfrac{4\alpha x^3+1}{x}=\dfrac{1/2+1}{1/(8\alpha)^{1/3}}\Rightarrow\dfrac{3}{2}\times(8\alpha)^{1/3}\ge 1$

$\Rightarrow\quad \alpha^{1/3}\ge 1/3\Rightarrow\alpha\ge\dfrac{1}{27}$

Alternate:

Using $AM \ge GM$

$$\frac{4\alpha x^2+\frac{1}{2x}+\frac{1}{2x}}{3}\ge\left(4\alpha x^2.\frac{1}{2x}.\frac{1}{2x}\right)^{1/3}=\alpha^{1/3}$$

$4\alpha x^2+\dfrac{1}{x}\ge 3\alpha^{1/3}$ for all positive x

Then $3\alpha^{1/3}=1$ we get $\alpha=1/27$

Hence, the correct answer is (C).

15. $\lim\limits_{x\to 2}\dfrac{f(x)g(x)}{f'(x)g'(x)}=1$

$\because\quad \lim\limits_{x\to 2}\dfrac{f(x)g(x)}{f'(x)g'(x)}\ \left(\dfrac{0}{0}\right)$ form as $f'(2) = 0$, $g(2) = 0$

$\because$ Using L - Hospital rule

$$=\lim_{x\to 2}\frac{f'(x)g(x)+g'(x)f(x)}{f''(x)g'(x)+g''(x)f'(x)}$$

$$=\frac{f'(2)g(2)+g'(2)f(2)}{f''(2)g'(2)+g''(2)f'(2)}$$

$$=\frac{g'(2)f(2)}{f''(2)g'(2)}=1\Rightarrow f''(2)=f(2)$$

and $f'(2) = 0$ and range of $f(x) \in (0, \infty)$

so $f''(2) = f(2) > 0$ so $f(x)$ has point of minima at $x = 2$

and $f(2) = f''(2)$ so $f(x) = f''(x)$ have atleast one solution in $x \in R$

Hence, (A) and (D) are correct.

16. Let inner radius be r and inner length be l

$\pi r^2 l = V$

Volume of material be M

$M = \pi(r + 2)^2(l + 2) - \pi r^2 l$

$= \pi(4 + 4r)l + 2\pi(r + 2)^2 = (4 + 4r)V/r^2 + 2\pi(r + 2)^2$

$\dfrac{dM}{dr}=-\dfrac{4V}{r^2}-\dfrac{8V}{r^3}+8\pi+0+4\pi r$

$\dfrac{dM}{dr}=0$ when $r = 10$

$\Rightarrow\quad V=1000\pi\Rightarrow\dfrac{V}{250\pi}=4$

Hence, the correct answer is (4).

17. Tangent at $P(x_1, y_1)$is $xx_1 - yy_1 = 1$ intersects x axis at $M\left(\dfrac{1}{x_1},0\right)$

Slope of normal at $P=-\dfrac{y_1}{x_1}=\dfrac{y_1-0}{x_1-x_2}$ (slope of line PN)

$\Rightarrow\quad x_2=2x_1\Rightarrow N\equiv(2x_1,0)$

For centroid $\ell=\dfrac{3x_1+\frac{1}{x_1}}{3}$, $m=\dfrac{y_1}{3}$,

$\dfrac{d\ell}{dx_1}=1-\dfrac{1}{3x_1^2}$; $\dfrac{dm}{dy_1}=\dfrac{1}{3}$,

$P(x_1, y_1)$ lies on hyperbola $x_1^2 - y_1^2 = 1$ then $y_1=\sqrt{x_1^2-1}$

$\dfrac{dm}{dx_1}=\dfrac{1}{3}\dfrac{dy_1}{dx_1}=\dfrac{x_1}{3\sqrt{x_1^2-1}}$

Hence, (A), (B) and (D) are correct.

18. Let $H(x) = f(x) - 3g(x)$

$H(-1) = H(0) = H(2) = 3.$

Applying Rolle's Theorem in the interval $[-1, 0]$

$H'(x) = f'(x) - 3g'(x) = 0$ for atleast one $c \in (-1, 0)$.

As $H''(x)$ never vanishes in the interval

$\Rightarrow$ Exactly one $c \in (-1, 0)$ for which $H'(x) = 0$

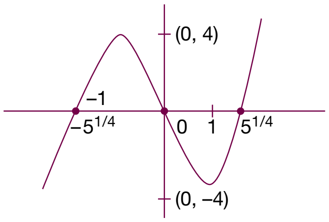
Similarly, apply Rolle's Theorem in the interval $[0, 2]$ and $H''(x)$ never vanishes in the interval

$\Rightarrow$ $H'(x) = 0$ has exactly one solution in $(0, 2)$

Hence, (B) and (C) are correct.

19. $f(x) = x^5 - 5x + a = 0 \Rightarrow x^5 - 5x = -a$

$\Rightarrow\ x(x^4 - 5) = -a$



$\Rightarrow\ x(x^2 - \sqrt{5})(x^2 + \sqrt{5}) = -a$

$\Rightarrow\ x(x - 5^{1/4})(x + 5^{1/4})(x^2 + \sqrt{5}) = -a$

$f'(x) = 5x^4 - 5 = 0 \Rightarrow (x^2 - 1)(x^2 + 1) = 0$

$\Rightarrow\ (x - 1)(x + 1)(x^2 + 1) = 0$

$f(x)$ has maxima and minima at $x = -1$, and $x = 1$ respectively

from the graph, we can conclude that for $-4 < a < 4$, there are 3 solution.

For $a < -4$ or $a > 4$, there is only one solution

Hence, (B) and (D) are correct.

20. given function $f(x) = (\log(\sec x + \tan x))^3$ for $x \in (-\pi/2, \pi/2)$

$f(-x) = -f(x)$ so it is odd function

$f'(x)=3(\log(\sec x+\tan x))^2\ \dfrac{1}{(\sec x+\tan x)}(\sec x\tan x+\sec^2 x)$

$= 3(\log(\sec x + \tan x))^2 \sec x > 0$

$f(x)$ is an increasing function

$f\left(-\dfrac{\pi^+}{2}\right)\to-\infty$ and $f\left(\dfrac{\pi^-}{2}\right)\to\infty$

Range of $f(x)$ is R hence it is onto function

Hence, (A), (B) and (C) are correct.

CHAPTER 8: INDEFINITE INTEGRATION

Test Your Concepts-I

1. $\int (x^2-1)(x+1)\,dx$

$= \int x^3 - x + x^2 - 1\,dx$

$= \frac{x^4}{4} - \frac{x^2}{2} + \frac{x^3}{3} - x + C$

2. $\int \frac{x^3-1}{x-1}\,dx \quad = \int \frac{(x-1)(x^2+x+1)}{(x-1)}\,dx$

$= \int x^2+x+1\,dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C$

3. $I = \int \cos 6x \cos 3x\,dx$

$I = \frac{1}{2}\int \cos 3x + \cos 9x\,dx$

(using 2cos A cos B = cos(A – B) + cos(A + B))

$= \frac{1}{2}\left[\frac{\sin 3x}{3} + \frac{\sin 9x}{9}\right] + C$

4. $I = \int 20 \sin 2x \sin 6x\,dx$

$= 10\int \cos(-4x) - \cos(8x)\,dx$

(using 2sin A sin B = cos(A – B) – cos(A + B))

$= 10\int \cos 4x - \cos 8x\,dx$

$= 10\left[\frac{\sin 4x}{4} - \frac{\sin 8x}{8}\right] + C$

5. $I = \int \sec^2 5x - \text{cosec}^2 7x + \text{cosec}3x \cot 3x\,dx$

$= \frac{1}{5}\tan 5x + \frac{1}{7}\cot 7x - \frac{1}{3}\text{cosec}\,3x + C$

6. $I = \int \frac{1+\cos^2 x}{1+\cos 2x}\,dx = \int \frac{1+\cos^2 x}{1+2\cos^2 x - 1}\,dx$

$= \frac{1}{2}\int \frac{1+\cos^2 x}{\cos^2 x}\,dx = \frac{1}{2}\int \sec^2 x + 1\,dx = \frac{1}{2}[\tan x + x] + C$

7. $I = \int \frac{dx}{\sin(x-\alpha)\cos(x-\beta)}$

$I = \frac{1}{\cos(\alpha-\beta)}\int \frac{\cos((x-\beta)-(x-\alpha))dx}{\sin(x-\alpha)\cos(x-\beta)}$

$= \sec(\alpha-\beta)\int \frac{\cos(x-\beta)\cos(x-\alpha)+\sin(x-\beta)\sin(x-\alpha)}{\sin(x-\alpha)\cos(x-\beta)}\,dx$

$= \sec(\alpha-\beta)\int \cot(x-\alpha) + \tan(x-\beta)\,dx$

$= \sec(\alpha-\beta)[\ln \sin(x-\alpha) + \ln \sec(x-\beta)] + C$

$= \sec(\alpha-\beta) \ln |\sin(x-\alpha) . \sec(x-\beta)| + C$

8. $I = \int \frac{2-3\sin x}{\cos^2 x}\,dx$

$= \int 2\sec^2 x - 3\tan x \sec x\,dx = 2\tan x - 3\sec x + c$

9. $I = \int (1-\tan 3x)^2\,dx = \int 1 + \tan^2 3x - 2\tan 3x\,dx$

$I = \int \sec^2 3x - 2\tan 3x\,dx$

$= \frac{1}{3}\tan 3x - \frac{2}{3}\ln|\sec 3x| + C$

$= \frac{1}{3}[\tan 3x + \ln(\cos^2 3x)] + C$

10. $I = \int \sin\alpha \sin(x-\alpha) + \frac{1-\cos(x-2\alpha)}{2}\,dx$

$I = -\sin\alpha\cos(x-\alpha) + \frac{x}{2} - \frac{\sin(x-2\alpha)}{2} + C$

$I = \frac{x}{2} - \left[\frac{\sin(x-2\alpha)+2\sin\alpha\cos(x-\alpha)}{2}\right] + C$

$I = \frac{x}{2} - \frac{1}{2}[\sin(x-\alpha)\cos\alpha - \cos(x-\alpha)\sin\alpha +$

$2\sin\alpha\cos(x-\alpha)] + C$

$I = \frac{x}{2} - \frac{1}{2}[\sin(x-\alpha)\cos\alpha + \cos(x-\alpha)\sin\alpha] + C$

$I = \frac{x}{2} - \frac{1}{2}\sin(x-\alpha+\alpha) + C$

$I = \frac{x}{2} - \frac{\sin x}{2} + C$

11. $I = \int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos\alpha}\,dx$

$I = \int \frac{2\cos^2 x - 1 - 2\cos^2\alpha + 1}{\cos x - \cos\alpha}\,dx$

$I = 2\int (\cos x + \cos\alpha)\,dx$

$I = 2(\sin x + x\cos\alpha) + C$

12. $I = \int \frac{1+\cos^2 x}{1+\cos 2x}\,dx = \int \frac{1+\cos^2 x}{1+2\cos^2 x - 1}\,dx$

$I = \frac{1}{2}\int \sec^2 x\,dx + \frac{1}{2}\int dx$

$I = \frac{1}{2}(\tan x + x) + c$

13. $I = \int 2\cos x\left(2\sin\frac{21x}{2}\cos\frac{x}{2}\right)dx$

$I = \int 2\cos x(\sin 11x + \sin 10x)\,dx$

$I = \int 2\sin 11x\cos x\,dx + \int 2\sin 10x\cos x\,dx$

$I = \int \sin 12x + \sin 10x\,dx + \int \sin 11x + \sin 9x\,dx$

$I = -\frac{\cos 12x}{12} - \frac{\cos 10x}{10} - \frac{\cos 11x}{11} - \frac{\cos 9x}{9} + C$

14. $I = \int \frac{x^6-1}{x^2+1}\,dx = \int \frac{x^6+x^4-x^4-x^2+x^2-1}{x^2+1}\,dx$

$I = \int \frac{x^4(x^2+1)-x^2(x^2+1)}{x^2+1} + \frac{x^2-1}{x^2+1}\,dx$

$$I = \int x^4 - x^2 \, dx + \int\left(1 - \frac{2}{x^2+1}\right)dx$$

$$I = \frac{x^5}{5} - \frac{x^3}{3} + x - 2\tan^{-1}x + C$$

15. $I = \int \frac{\sin 2x + \sin 5x - \sin 3x}{\cos x + 1 - 2\sin^2 2x}\, dx$

$$= \int \frac{2\sin x \cos x + 2\cos 4x \sin x}{\cos x + \cos 4x}\, dx$$

$$= \int \frac{2\sin x(\cos x + \cos 4x)}{(\cos x + \cos 4x)}\, dx = \int 2\sin x \, dx$$

$= -2 \cos x + C$

16. $\int \frac{1+\sin x}{1-\sin x}\, dx = \int \frac{1+\cos\left(\frac{\pi}{2}-x\right)}{1-\cos\left(\frac{\pi}{2}-x\right)}\, dx$

$$= \int \frac{1+2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)-1}{1-\left(1-2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)}\, dx$$

$$= \int \cot^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx = \int \operatorname{cosec}^2\left(\frac{\pi}{4}-\frac{x}{2}\right)-1 \, dx$$

$$= -2\left(-\cot\left(\frac{\pi}{4}-\frac{x}{2}\right)\right) - x + C$$

$$= 2 \cot\left(\frac{\pi}{4}-\frac{x}{2}\right) - x + C = 2\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) - x + C$$

Alternate:

$$I = \int \frac{1+\sin x}{1-\sin x} \times \frac{1+\sin x}{1+\sin x}\, dx = \int \frac{1+\sin^2 x + 2\sin x}{1-\sin^2 x}\, dx$$

$$= \int \frac{1+2\sin x+\sin^2 x}{\cos^2 x}\, dx = \int \sec^2 x + 2\tan x \sec x + \tan^2 x \, dx$$

$$= \int 2\sec^2 x + 2\sec x \tan x - 1 \, dx = 2 \tan x + 2 \sec x - x + C$$

17. $I = \int \frac{dx}{(x+\sqrt{x^2-1})^2} = \int (x-\sqrt{x^2-1})^2 \, dx$

$$I = \int (x^2 - 2x\sqrt{x^2-1} + x^2 - 1)\, dx$$

$$= \int (2x^2 - 1 - 2x\sqrt{x^2-1}) + c$$

$$I = \frac{2}{3}x^3 - x - \frac{2}{3}(x^2-1)^{3/2} + C$$

18. $I = \int \{1 + 2\tan^2 x + 2\tan x \sec x\}^{1/2}\, dx$

$$I = \int \{\sec^2 x + \tan^2 x + 2\tan x \sec x\}^{1/2}\, dx$$

$$I = \int (\sec x + \tan x)\, dx$$

$I = \ln(\sec x + \tan x) + \ln(\sec x) + C$

$= \ln \sec x.(\sec x + \tan x) + C$

Test Your Concepts-II

1. $I = \int \frac{\ln^2 x}{x}\, dx$

Put $\ln x = t \Rightarrow \frac{1}{x}dx = dt$

$$I = \int t^2 \, dt = \frac{t^3}{3} + C = \frac{\ln^3 x}{3} + C$$

2. $I = \int e^{-x^3} x^2 \, dx$

Put $-x^3 = t \Rightarrow 3x^2 \, dx = -dt$

$$I = \int e^t\left(-\frac{dt}{3}\right) = -\frac{1}{3}e^t + C = -\frac{e^{-x^3}}{3} + C$$

3. $\int \frac{x^3 dx}{\sqrt{1-x^8}}$

Put $x^4 = t \Rightarrow 4x^3 \, dx = dt$

$$= \int \frac{dt}{4\sqrt{1-t^2}} = \frac{1}{4}\sin^{-1}t + C = \frac{1}{4}\sin^{-1}x^4 + C$$

4. $I = \int \frac{\cos x}{2\sin x - 1}\, dx$

Put $2\sin x - 1 = t$

On diff. $2\cos x \, dx = dt$

$$I = \int \frac{dt}{2(t)} = \frac{1}{2}\ln t + c$$

$$I = \frac{1}{2}\ln|2\sin x - 1| + c$$

5. $I = \int e^x \operatorname{cosec}(e^x)\cot(e^x)\, dx$

Put $e^x = t \Rightarrow e^x \, dx = dt$

$$I = \int \operatorname{cosec} t . \cot t \, .dt$$

$I = -\operatorname{cosec} t + c = -\operatorname{cosec}(e^x) + C$

6. $\int \sin x d(\sin x) + \int \tan^3 x d(\tan x) = \frac{\sin^2 x}{2} + \frac{\tan^4 x}{4} + C$

7. $I = \int (4x + 2)\sqrt{x^2+x+1}dx$

Put $x^2 + x + 1 = t$, $(2x + 1)dx = dt$

$$I = \int 2\sqrt{t}\, dt = \frac{2t^{3/2}}{\frac{3}{2}} + C$$

$$I = \frac{4t^{3/2}}{3} + C = \frac{4}{3}\left(x^2+x+1\right)^{3/2} + C$$

8. $I = \int \frac{1}{x-\sqrt{x}}\, dx$

$$I = \int \frac{1}{(\sqrt{x}-1)} \frac{dx}{\sqrt{x}}$$

Put $\sqrt{x} - 1 = t$; $\frac{1}{2\sqrt{x}}\, dx = dt$

$$I = \int \frac{2}{t} dt\ I$$

$$I = 2\ln|t| + C = 2\ln\left|\sqrt{x} - 1\right| + C$$

9. $I = \int \frac{\sin^2 x \cos x}{1 + \sin^2 x} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$= \int \frac{t^2}{1+t^2} dt = \int 1 - \frac{1}{1+t^2} dt = t - \tan^{-1} t + C$$

$$= \sin x - \tan^{-1}(\sin x) + C$$

10. $I = \int \frac{\ell n(x+1) - \ell n\, x}{x(x+1)} dx$

Put $\ln(x + 1) - \ln x = t \Rightarrow \left(\frac{1}{x+1} - \frac{1}{x}\right) dx = dt$

$$\Rightarrow \quad \frac{dx}{x(x+1)} = -dt$$

$$I = \int t(-dt) = -\frac{t^2}{2} + c = -\frac{1}{2}(\ln(x+1) - \ln x)^2 + C$$

$$= \frac{1}{2}\ln^2\left(\frac{x}{x+1}\right) + C$$

11. $I = \int \frac{d(\ln x)}{\cos^2(1 + \ln x)}$

Put $\ln x = t;\ d(\ln x) = dt$

$$I = \int \frac{dt}{\cos^2(1+t)}$$

$$I = \int \sec^2(1+t)\, dt = \tan(1+t) + C = \tan(1 + \ln x) + C$$

12. $I = \int \sqrt{e^x - 1}\, dx$

Put $\sqrt{e^x - 1} = t \Rightarrow e^x - 1 = t^2$

$$\Rightarrow \quad e^x\, dx = 2t\, dt \Rightarrow dx = \frac{2t\, dt}{1+t^2}$$

$$I = \int t \cdot \frac{2t}{1+t^2} dt = 2\int \frac{t^2 + 1 - 1}{t^2 + 1} dt = 2\int \left(1 - \frac{1}{t^2+1}\right) dt$$

$$I = 2t - 2\tan^{-1} t + C$$

$$I = 2\sqrt{e^x - 1} - 2\tan^{-1}\sqrt{e^x - 1} + C$$

13. $\int \sqrt{\frac{\cos x - \cos^3 x}{1 - \cos^3 x}}\, dx$

$$= \int \sqrt{\frac{\cos x}{1 - \cos^3 x}} \sin x\, dx$$

Let $\cos x = t \Rightarrow -\sin x \,.\, dx = dt$

$$= -\int \sqrt{\frac{t}{1-t^3}}\, dt = -\int \frac{\sqrt{t}}{\sqrt{1 - (t^{3/2})^2}} dt,$$

$$= -\frac{2}{3}\int \frac{\frac{3}{2}\sqrt{t}}{\sqrt{1 - (t^{3/2})^2}} dt = \frac{2}{3}\cos^{-1}(t^{3/2}) + C$$

$$= \frac{2}{3}\cos^{-1}(\cos^{3/2} x) + C$$

14. $\int \frac{ax^2 - b}{x\sqrt{c^2x^2 - (ax^2 + b)^2}} dx$

$$= \int \frac{\frac{a-b}{x^2}}{\sqrt{c^2 - \left(ax + \frac{b}{x}\right)^2}} dx$$

Put $ax + b/x = t$

$$\Rightarrow \quad \left(a - \frac{b}{x^2}\right) dx = dt$$

$$= \int \frac{1}{\sqrt{c^2 - \left(ax + \frac{b}{x}\right)^2}} d\left(ax + \frac{b}{x}\right)$$

$$= \sin^{-1}\left(\frac{ax + \frac{b}{x}}{c}\right) + k$$

Test Your Concepts-III

1. $\int \frac{dx}{\sqrt{x^2 - x - 2}} = \int \frac{dx}{\sqrt{\left(x - \frac{1}{2}\right)^2 - \left(\frac{3}{2}\right)^2}}$

$$= \ln\left(x - \frac{1}{2} + \sqrt{x^2 - x - 2}\right) + C$$

2. $I = \int \frac{dx}{\sqrt{x^2 - 3x + 2}}$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} = \ln\left(x - \frac{3}{2} + \sqrt{x^2 - 3x + 2}\right) + C$$

3. $I = \int \frac{dx}{\sqrt{-4x^2 - 12x - 8}}$

$$I = \int \frac{dx}{\sqrt{1 - (2x+3)^2}} = \frac{1}{2}\sin^{-1}(2x + 3) + C$$

4. $\int \frac{(1+x)^2}{x(1+x^2)} dx = \int \frac{1 + x^2 + 2x}{x(1+x^2)} dx$

$$= \int \frac{1}{x} + \frac{2}{1+x^2} dx = \ln|x| + 2\tan^{-1} x + c$$

5. $I = \int \frac{x}{x^4 - 1} dx$

Put $x^2 = t,\ 2x\, dx = dt$

$$I = \int \frac{dt}{2(t^2 - 1)} = \frac{1}{2}\int \frac{dt}{t^2 - 1} = \frac{1}{2}\left(\frac{1}{2}\ell n\left|\frac{t-1}{t+1}\right|\right) + c$$

$$= \frac{1}{4}\ln\left|\frac{x^2 - 1}{x^2 + 1}\right| + C$$

6. $I = \int \frac{dx}{20 - 6x - x^2}$

$$= -\int \frac{dx}{x^2 + 6x - 20} = -\int \frac{dx}{(x+3)^2 - 29}$$

$$= -\int \frac{dx}{(x+3)^2 - (\sqrt{29})^2}$$

$$= -\frac{1}{2\sqrt{29}}\ln\left|\frac{x + 3 - \sqrt{29}}{x + 3 + \sqrt{29}}\right| + C$$

7. $\int \frac{xdx}{x^4 - 3x^2 + 2}$

Put $x^2 = t \Rightarrow 2x\,dx = dt$

$$I = \int \frac{dt}{2(t^2 - 3t + 2)} = \frac{1}{2}\int \frac{dt}{(t-2)(t-1)}$$

$$= \frac{1}{2}\int\left(\frac{1}{t-2} - \frac{1}{t-1}\right)dt$$

$$= \frac{1}{2}(\ln|t-2| - \ln|t-1|) + C = \frac{1}{2}\ln\left|\frac{x^2-2}{x^2-1}\right| + C$$

8. $\int \frac{(2x^2-5)dx}{x^4 - 5x^2 + 6} = \int \frac{(x^2-2)+(x^2-3)}{(x^2-2)(x^2-3)}dx = \int \frac{dx}{x^2-3} + \int \frac{dx}{x^2-2}$

$$= \frac{1}{2\sqrt{3}}\ln\left(\frac{x-\sqrt{3}}{x+\sqrt{3}}\right) + \frac{1}{2\sqrt{2}}\ln\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right) + C$$

9. Let $e^x = t \Rightarrow e^x\,.\,dx = dt$

$$\int \frac{e^x\,.\,dx}{\sqrt{e^{2x} - 3e^x + 1}} = \int \frac{dt}{\sqrt{t^2 - 3t + 1}}$$

$$= \int \frac{dt}{\sqrt{t^2 - 3t + \frac{9}{4} + 1 - \frac{-9}{4}}}$$

$$= \int \frac{dt}{\sqrt{\left(t - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2}}$$

$$= \log\left|\left(t - \frac{3}{2}\right) + \sqrt{\left(t-\frac{3}{2}\right)^2 - \left(\frac{5}{2}\right)^2}\right| + c$$

$$= \log\left|2t - 3\sqrt{t^2 - 3t + 1}\right| + c$$

$$= \log\left|2e^x - 3 + \sqrt{e^{2x} - 3e^x + 1}\right| + c$$

Test Your Concepts-IV

1. $I = \int \underset{II}{1}.\underset{I}{\ln x}\,dx$

(by taking 1 as second function and ln x as the first function)

$$= x\ln x - \int x\left(\frac{1}{x}\right)dx$$

$$= x\ln x - \int dx = x\ln x - x + C$$

2. $I = \int \underset{II}{1}.\underset{I}{\sin^{-1}}x\,dx$

$$I = x\sin^{-1}x - \int x\left(\frac{1}{\sqrt{1-x^2}}\right)dx$$

$$I = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}dx$$

Put $\sqrt{1-x^2} = t$

$\Rightarrow\ 1 - x^2 = t^2 \Rightarrow -x\,dx = tdt$

$$I = x\sin^{-1}x + \int \frac{tdt}{t}$$

$$I = x\sin^{-1}x + t + C = x\sin^{-1}x + \sqrt{1-x^2} + c$$

3. $I = \int \underset{II}{x}\,\underset{I}{\ln^2 x}\,dx$

$$I = \frac{x^2}{2}\ln^2 x - \int \frac{x^2}{2}\cdot 2\ln x\cdot\frac{1}{x}dx = \frac{x^2}{2}\ln^2 x - \int \underset{II}{x}\,\underset{I}{\ln x}\,dx$$

$$I = \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \int \frac{x^2}{2}.\frac{1}{x}dx$$

$$I = \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C$$

4. $I = \int \cos(\ln x)\,dx$

Put $\ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$$I = \int \cos t.e^t\,dt$$

$$= e^t\cos t - \int e^t(-\sin t)\,dt = e^t\cos t + \int e^t\sin t\,dt$$

$$I = e^t\cos t + e^t\sin t - \int e^t\cos t + C$$

$$2I = e^t(\text{cost} + \sin t) + C$$

$$I = \frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C$$

5. $I = \int e^{\sqrt{x}}\,dx$

Put $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow dx = 2t\,dt$

$$I = \int e^t.2t\,dt = e^t(2t) - \int e^t(2)\,dt$$

$$= 2te^t - 2e^t + C = 2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$$

$$= 2e^{\sqrt{x}}(\sqrt{x} - 1) + C$$

6. $I = \int \underset{II}{1}\cdot\underset{I}{\tan^{-1}}x\,dx$

$$= x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{1+x^2}dx = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$$

7. $I = \int \underset{II}{x}\,\underset{I}{\tan^{-1}}x\,dx$

$$= \frac{x^2}{2}\tan^{-1}x - \int\frac{x^2}{2}\left(\frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}dx\right)$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}[x - \tan^{-1}x] + C$$

8. $\int x^3\sin x\,dx = x^3(-\cos x) - \int 3x^2(-\cos x)\,dx$

$$= -x^3\cos x + 3\int x^2\cos x\,dx$$

$$= -x^3\cos x + 3\left(x^2\sin x - \int 2x\sin x\,dx\right)$$

$$= -x^3\cos x + 3x^2\sin x - 6\int x\sin x\,dx$$

$$= -x^3\cos x + 3x^2\sin x - 6\left(x(-\cos x) - \int(-\cos x)dx\right)$$

$$= -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$$

9. $\int x^n\ln x\,dx = \left(\frac{x^{n+1}}{n+1}\right)\ln x - \int\left(\frac{x^{n+1}}{n+1}\right)\frac{1}{x}dx$

$$= \left(\frac{x^{n+1}}{n+1}\right)\ln x - \frac{1}{n+1}\int x^n\,dx$$

$$= \left(\frac{x^{n+1}}{n+1}\right)\ln x - \frac{x^{n+1}}{(n+1)^2} + C$$

10. $$I = \int\left(\frac{x}{2}\right)\left(\frac{2x}{(1+x^2)^2}\right)dx$$

$$= \left(\frac{x}{2}\right)\left(-\frac{1}{1+x^2}\right) - \int\frac{1}{2}\left(-\frac{1}{1+x^2}\right)dx$$

$$= -\frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1}x + C$$

11. $$I = \int\frac{\sin^{-1}\sqrt{x}}{\sqrt{1-x}}\,dx$$

Put $\sin^{-1}\sqrt{x} = t \Rightarrow \sqrt{x} = \sin t \Rightarrow \frac{1}{2\sqrt{x}}\,dx = \cos t\,dt$

$\Rightarrow \; dx = 2\text{ sint cost } dt$

$$I = \int\frac{t}{\sqrt{1-\sin^2 t}}(2\sin t\cos t\,dt)$$

$$= 2\int t\sin t\;dt = 2\left(t(-\cos t) + \int\cos t\,dt\right)$$

$$= -2t\text{ cost} + 2\text{ sint} + C$$

$$= -2\sqrt{1-x}\,\sin^{-1}\sqrt{x} + 2\sqrt{x} + C$$

12. $$I = \int\frac{(x+\sin x)}{1+\cos x}\,dx = \int\frac{x+2\sin\frac{x}{2}\cos\frac{x}{2}}{1+2\cos^2\frac{x}{2}-1}\,dx$$

$$I = \int\frac{x}{2}\sec^2\frac{x}{2}\,dx + \int\tan\frac{x}{2}\,dx$$

$$= x\left(\tan\frac{x}{2}\right) - \int\tan\frac{x}{2}\,dx + \int\tan\frac{x}{2}\,dx + C = x\tan\frac{x}{2} + C$$

13. $$I = \int e^{3x}(\sin 2x - \cos 2x)\,dx$$

$$I = \frac{e^{3x}}{3}(\sin 2x - \cos 2x) - \int\frac{e^{3x}}{3}(2\cos 2x + 2\sin 2x)\,dx$$

$$I = \frac{e^{3x}}{3}(\sin 2x - \cos 2x) - \frac{e^{3x}}{9}(2\cos 2x + 2\sin 2x) + \int\frac{e^{3x}}{9}(-4\sin 2x + 4\cos 2x).dx$$

$$I = \frac{e^{3x}}{9}[3\sin 2x - 3\cos 2x - 2\cos 2x - 2\sin 2x] - \frac{4}{9}\int e^{3x}(\sin 2x - \cos 2x)\,dx$$

$$I + \frac{4}{9}I = \frac{e^{3x}}{9}[\sin 2x - 5\cos 2x] + C$$

$$I = \frac{e^{3x}}{13}[\sin 2x - 5\cos 2x] + C$$

14. $$I = \int\frac{\ln^2 x}{\sqrt{x^5}}\,dx = \int x^{-5/2}\ln^2 x\,dx$$

$$= \left(\frac{x^{-\frac{3}{2}}}{-\frac{3}{2}}\right)\ln^2 x - \int\left(-\frac{2}{3}x^{-\frac{3}{2}}\right)2\ln x.\frac{1}{x}\,dx$$

$$= -\frac{2}{3}\frac{\ln^2 x}{x^{3/2}} + \frac{4}{3}\int x^{-5/2}\ln x\,dx$$

$$= -\frac{2}{3}\frac{\ln^2 x}{x^{3/2}} + \frac{4}{3}\left(\frac{x^{-\frac{3}{2}}}{-\frac{3}{2}}\right)\ln x - \frac{4}{3}\int\left(\frac{-2}{3}x^{-\frac{3}{2}}\right)\frac{1}{x}\,dx$$

$$I = -\frac{2}{3}\frac{\ln^2 x}{x^{3/2}} - \frac{8}{9}\frac{\ln x}{x^{3/2}} + \frac{8}{9}\int x^{-\frac{5}{2}}\,dx$$

$$= -\frac{2}{3}\frac{\ln^2 x}{x^{3/2}} - \frac{8}{9}\frac{\ln x}{x^{3/2}} + \frac{8}{9}\left(\frac{x^{-\frac{3}{2}}}{-\frac{3}{2}}\right) + C$$

$$= \frac{-8}{27\sqrt{x^3}}\left(\frac{9}{4}\ln^2 x + 3\ln x + 2\right) + C$$

15. $$I = \int e^x(\underset{f}{x^2} + \underset{f'}{2x})\,dx + \int e^x(-\underset{f'}{\sin x} + \underset{f}{\cos x})\,dx$$

$$I = e^x x^2 + e^x\cos x\;C$$

16. $$I = \int e^x(\underset{f}{\sec x} + \underset{f'}{\sec x\tan x})\,dx + \int e^x\left(\underset{f}{\tan x} + \underset{f'}{\sec^2 x}\right)dx$$

$$= e^x\sec x + e^x\tan x + C$$

17. $$I = \int e^{\tan^2 x}(2\sin 2x\cos 2x)\,dx$$

$$I = \int e^{\tan^2 x}(4\sin x\cos x\,(2\cos^2 x - 1))\,dx$$

$$I = \int e^{\tan^2 x}\,8\sin x\cos^3 x\,dx - \int e^{\tan^2 x}\,4\sin x\cos x\,dx$$

$$\left(\because \int 8\sin x\cos^3 x\,dx = -2\cos^4 x\right)$$

$$I = e^{\tan^2 x}(-2\cos^4 x) - \int e^{\tan^2 x}(2\tan x\sec^2 x)(-2\cos^4 x)dx - \int e^{\tan^2 x}\,4\sin x\cos x\,dx$$

$$I = -2e^{\tan^2 x}\cos^4 x + 4\int e^{\tan^2 x}\sin x\cos x\,dx - \int e^{\tan^2 x}\,4\sin x\cos x\,dx + C$$

$$I = -2e^{\tan^2 x}\cos^4 x + C$$

18. $$I = \int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right)dx$$

$$I = \int xe^{\sin x}\cos x\,dx - \int e^{\sin x}\tan x\sec x\,dx$$

$$= x(e^{\sin x}) - \int 1.e^{\sin x}\,dx - \left(e^{\sin x}\sec x - \int(e^{\sin x}\cos x)\sec x\,dx\right)$$

$$= e^{\sin x}(x - \sec x) - \int e^{\sin x}\,dx + \int e^{\sin x}\,dx + C$$

$$= e^{\sin x}(x - \sec x) + C$$

19. $$\int\frac{x}{\sqrt{1+x^2}}.\tan^{-1}x\,dx = \sqrt{1+x^2}\,\tan^{-1}x - \int\sqrt{1+x^2}\,\frac{1}{1+x^2}\,dx$$

$$I = \sqrt{1+x^2}\,\tan^{-1}x - \int\frac{1}{\sqrt{1+x^2}}\,dx$$

$$I = \sqrt{1+x^2}\,\tan^{-1}x - \log(x + \sqrt{1+x^2}) + C$$

We get $K = -1$

20. $$I = \int e^x\left(\frac{x^3 - x + 2}{(x^2+1)^2}\right)dx$$

$$I = \int e^x\left(\frac{(x^3 + x^2 + x - x^2 - 2x + 2)}{(x^2+1)^2}\right)dx$$

$$= \int e^x\left(\frac{(x^2+1)(x+1)}{(x^2+1)^2} - \frac{(x^2+2x-1)}{(x^2+1)^2}\right)dx$$

$$I = \int e^x\left(\frac{x+1}{(x^2+1)} - \frac{(x^2+2x-1)}{(x^2+1)^2}\right)dx$$

$$= \int e^x \left(\underset{\downarrow f(x)}{\frac{x+1}{(x^2+1)}} - \underset{\downarrow f'(x)}{\frac{(x^2+2x-1)}{(x^2+1)^2}} \right) dx$$

$$I = e^x \left(\frac{x+1}{x^2+1} \right) + C$$

Test Your Concepts-V

1. $L = \alpha Q' + \beta$

$x + 3 = \alpha(2x - 2) + \beta;\ x + 3 = 2\alpha x - 2\alpha + \beta$

$2\alpha = 1 \Rightarrow \alpha = \frac{1}{2};\ -2\alpha + \beta = 3 \Rightarrow \beta = 4$

$$I = \int \frac{\alpha Q' + \beta}{Q}\, dx$$

$$= \frac{1}{2} \int \frac{Q'}{Q} dx + 4 \int \frac{dx}{Q} = \frac{1}{2} \ln|Q| + 4 \int \frac{dx}{x^2 - 2x - 5}$$

$$= \frac{1}{2} \ln|x^2 - 2x - 5| + 4 \int \frac{dx}{(x-1)^2 - 6}$$

$$= \frac{1}{2} \ln|x^2 - 2x - 5| + \frac{2}{\sqrt{6}} \ln\left|\frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}}\right| + C$$

2. $\dfrac{L}{Q} = \dfrac{x+2}{\sqrt{x^2-1}}$

Let $L = \alpha Q' + \beta$

$x + 2 = \alpha(2x) + \beta;$

$\alpha = \frac{1}{2};\ \beta = 2$

$$I = \frac{1}{2} \int \frac{2x\, dx}{\sqrt{x^2-1}} + \int \frac{2}{\sqrt{x^2-1}}\, dx$$

$$I = \sqrt{x^2-1} + 2 \ln\left|x + \sqrt{x^2-1}\right| + c$$

3. $I = \displaystyle\int \frac{x \ell n\, x}{(x^2-1)^{3/2}}\, dx$

$$= \ln x \int \frac{x}{(x^2-1)^{\frac{3}{2}}}\, dx - \int \frac{1}{x} \left(\int \frac{x\, dx}{(x^2-1)^{\frac{3}{2}}} \right) dx$$

$\because \displaystyle\int \frac{x\, dx}{(x^2-1)^{3/2}}$

Put $x^2 - 1 = t^2 \Rightarrow x\, dx = t\, dt$

$$= \int \frac{t\, dt}{t^3} = \int \frac{dt}{t^2} = -\frac{1}{t} = -\frac{1}{\sqrt{x^2-1}}$$

$$I = \ln x \left(\frac{-1}{\sqrt{x^2-1}} \right) + \int \frac{dx}{x\sqrt{x^2-1}} = -\frac{\ell n\, x}{\sqrt{x^2-1}} + \sec^{-1} x + C$$

4. $\displaystyle\int \frac{(6x-5)dx}{2\sqrt{3x^2-5x+6}}\, dx$

Put $3x^2 - 5x + 6 = t \Rightarrow (6x - 5)dx = dt$

$$= \int \frac{dt}{2\sqrt{t}} = \sqrt{t} + C = \sqrt{3x^2 - 5x + 6} + C$$

5. $I = \displaystyle\int \frac{e^x(1+e^x)}{\sqrt{1-e^{2x}}}\, dx$

Put $e^x = t \Rightarrow e^x\, dx = dt$

$$= \int \frac{(1+t)}{\sqrt{1-t^2}}\, dt = \int \frac{dt}{\sqrt{1-t^2}} + \int \frac{t\, dt}{\sqrt{1-t^2}}$$

$$= \sin^{-1} t + \int \frac{t\, dt}{\sqrt{1-t^2}}$$

Put $\sqrt{1-t^2} = p \Rightarrow 1 - t^2 = p^2 \Rightarrow t\, dt = -p dp$

$$= \sin^{-1} t - \int \frac{p dp}{p} = \sin^{-1} t - p + C$$

$$= \sin^{-1} t - \sqrt{1-t^2} + C = \sin^{-1} e^x - \sqrt{1-e^{2x}} + C$$

6. $I = \displaystyle\int \frac{\sqrt{1+\ln x}}{x \ln x}\, dx$

Put $\sqrt{1 + \ell n\, x} = t \Rightarrow 1 + \ln x = t^2 \Rightarrow \dfrac{dx}{x} = 2t\, dt$

$$I = \int \frac{t(2t\, dt)}{t^2-1} = 2 \int \frac{t^2-1+1}{t^2-1}\, dt = 2\left(\int \left(1 + \frac{1}{t^2-1} \right) dt \right)$$

$$= 2t + \ln\left|\frac{t-1}{t+1}\right| + C$$

$$= 2\sqrt{1+\ln x} + \ln\left|\frac{\sqrt{1+\ln x} - 1}{\sqrt{1+\ln x} + 1}\right| + C$$

7. $\displaystyle\int \frac{x dx}{(x+1)(2x+1)} = \int \frac{1}{x+1} - \frac{1}{2x+1}\, dx$

$$= \ln|x+1| - \frac{1}{2} \ln|2x+1| + C$$

$$= \ln\left|\frac{x+1}{\sqrt{2x+1}}\right| + C$$

8. $I = \displaystyle\int \frac{2x^2 + 41x - 91}{(x+1)(x+3)(x-4)}\, dx$

$$\frac{2x^2 + 41x - 91}{(x-1)(x+3)(x-4)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x-4}$$

$\Rightarrow\ 2x^2 + 41x - 91 = A(x + 3)\,(x - 4) + B(x - 1)\,(x - 4) + C(x - 1)\,(x + 3)$

Put $x = 1 : 2 + 41 - 91 = A(4)\,(-3) + 0 + 0 \quad \Rightarrow A = 4$

Put $x = -3 : 18 - 123 - 91 = 0 + B(-4)\,(-7) + 0 \quad \Rightarrow B = -7$

Put $x = 4 : 32 + 164 - 91 = 0 + C(3)\,7 \quad \Rightarrow C = 5$

$$I = \int \frac{4}{x-1} - \frac{7}{x+3} + \frac{5}{x-4}\, dx$$

$$= 4 \ln|x-1| - 7 \ln|x+3| + 5 \ln|x-4| + C$$

$$= \ln\left|\frac{(x-1)^4(x-4)^5}{(x+3)^7}\right| + C$$

9. $\dfrac{x^3 - 6x^2 + 9x + 7}{(x-2)^3(x-5)} = \dfrac{A}{x-5} + \dfrac{B}{x-2} + \dfrac{C}{(x-2)^2} + \dfrac{D}{(x-2)^3}$

$\Rightarrow\ x^3 - 6x^2 + 9x + 7 = A(x-2)^3 + B(x-2)^2\,(x-5) + C(x-2)\,(x-5) + D(x-5)$

Put $x = 2,\ 8 - 24 + 18 + 7 = 9 = 0 + D(-3) \quad \Rightarrow D = -3$

Put $x = 5,\ 125 - 150 + 45 + 7 = 27 = A(27) \quad \Rightarrow A = 1$

Put $x = 0,\ 7 = -8A - 20B + 10C - 5D = -8 - 20B + 10C + 15 \Rightarrow C = 2B$

Put $x = 1,\ 11 = -A - 4B + 4C - 4D = -1 - 4B + 4C + 12 \Rightarrow B = C$

$\therefore\ B = C = 0$

$$I = \int \frac{x^3 - 6x^2 + 9x + 7}{(x-2)^3(x-5)}\, dx = \int \frac{1}{x-5} - \frac{3}{(x-2)^3}\, dx$$

$$= \ln|x-5| + \frac{3}{2(x-2)^2} + C$$

10. $I = \displaystyle\int \frac{(3+x^2)^2 x^3}{(1+x^2)^3}\, dx$

Put $1 + x^2 = t \Rightarrow 2x\, dx = dt$

$$I = \int \frac{(2+t)^2(t-1)\,dt}{2t^3} = \frac{1}{2}\int \frac{(t^2+4t+4)(t-1)}{t^3}\,dt$$

$$= \frac{1}{2}\int \frac{t^3+4t^2+4t-t^2-4t-4}{t^3}\,dt = \frac{1}{2}\int \frac{t^3+3t^2-4}{t^3}\,dt$$

$$= \frac{1}{2}\int 1+\frac{3}{t}-\frac{4}{t^3}\,dt = \frac{1}{2}\left(t+3\ln t+\frac{2}{t^2}\right)+C$$

$$= \frac{1}{2}(1+x^2) + \frac{3}{2}\ln(1+x^2) + \frac{1}{(1+x^2)^2} + C$$

11. $I = \int \frac{(x^2+3)dx}{x^6(x^2+1)}$

Let $x^2 = t$

$$\frac{t+3}{t^3(t+1)} = \frac{A}{t+1} + \frac{B}{t} + \frac{C}{t^2} + \frac{D}{t^3}$$

$\Rightarrow$ $t+3 = At^3 + Bt^2(t+1) + Ct(t+1) + D(t+1)$
We get $D = 3$, $A = -2$, $B = 2$, $C = -2$

$$I = \int \frac{(x^2+3)}{x^6(x^2+1)}\,dx = \int \frac{-2}{x^2+1} + \frac{2}{x^2} - \frac{2}{x^4} + \frac{3}{x^6}\,dx$$

$$= -2\tan^{-1}x - \frac{2}{x} + \frac{2}{3x^3} - \frac{3}{5x^5} + C$$

12. $I = \int \frac{e^x(1+x)}{xe^x(1+xe^x)^2}\,dx$

Put $1 + xe^x = t \Rightarrow e^x(1+x)dx = dt$

$$I = \int \frac{dt}{(t-1)t^2} = \int\left(\frac{1}{t-1}-\frac{t+1}{t^2}\right)dt = \int\left(\frac{1}{t-1}-\frac{1}{t}-\frac{1}{t^2}\right)dt$$

$$I = \ln(1-t) - \ln t + \frac{1}{t} + C = \ln\left(\frac{1-t}{t}\right) + \frac{1}{t} + C$$

$$I = \ln\left|\frac{xe^x}{1+xe^x}\right| + \frac{1}{1+xe^x} + C$$

13. Let $\frac{1}{(x-1)\,(1+x^2)} \equiv \frac{A}{x-1} + \frac{Bx+C}{1+x^2}$

$\Rightarrow$ $1 = A\,(1+x^2) + (Bx+C)\;(x-1)$ …(1)

Putting $x = 1$ we get, $1 = A\,(2)$; $\therefore$ $A = \frac{1}{2}$

Equating coefficients on both sides, we get:
$0 = A + B$ and $1 = A - C$

Thus, $A = 1/2$, $B = -1/2$ and $C = -1/2$

$$\therefore \quad I = \int\left(\frac{A}{x-1} + \frac{Bx+C}{1+x^2}\right)dx$$

$$= \int\left(\frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{x+1}{1+x^2}\right)dx$$

$$= \frac{1}{2}\int\frac{dx}{x-1} - \frac{1}{2}\int\frac{xdx}{1+x^2} - \frac{1}{2}\int\frac{dx}{1+x^2}$$

$$= \frac{1}{2}\log\,(x-1) - \frac{1}{4}\log\,(1+x^2) - \frac{1}{2}\tan^{-1}\,x + c$$

14. Let $\frac{3x+1}{(x-1)^3\cdot(x+1)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}$

$\Rightarrow$ $3x+1 \equiv A\,(x-1)^2\,(x+1) + B(x-1)\,(x+1) + C(x+1) + D\,(x-1)^3$

Putting $x = -1$, $-2 = -D\,(-2)^3$; $\therefore$ $D = \frac{1}{4}$

Putting $x = 1$; we get $C = 2$.

Equating coefficients of x^3 and x^2 on both sides we get:
$0 = A + D$ and $0 = -A + B - 3D$

Thus, $A = -1/4$, $B = 1/2$, $C = 2$ and $D = 1/4$

$$\therefore \quad I = \int\left\{\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+1}\right\}dx$$

$$= \int -\frac{1}{4}\cdot\frac{dx}{x-1} + \int 2\cdot\frac{dx}{(x-1)^3} + \int\frac{1}{2}\cdot\frac{dx}{(x-1)^2} + \int\frac{1}{4}\,\frac{dx}{x+1}$$

$$= -\frac{1}{4}\log\,(x-1) - \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{1}{4}\log\,(x+1) + c$$

$$= \frac{1}{4}\log\,\frac{x+1}{x-1} - \frac{1}{2\,(x-1)} - \frac{1}{(x-1)^2} + c$$

Test Your Concepts-VI

1. $I = \int \frac{x^7dx}{(1-x^2)^5} = \int \frac{x^7dx}{x^{10}\left(\frac{1}{x^2}-1\right)^5}$

Put $\frac{1}{x^2} - 1 = t \Rightarrow \frac{-2}{x^3}\,dx = dt$

$$= \int\frac{-dt}{2(t)^5} = -\frac{1}{2}\int t^{-5}\,dt$$

$$= -\frac{1}{2}\left(\frac{t^{-4}}{-4}\right) + C$$

$$= \frac{1}{8}\left(\frac{1}{x^2}-1\right)^{-4} = \frac{x^8}{8(1-x^2)^4} + C$$

2. $I = \int \frac{5x^4+4x^5}{(x^5+x+1)^2}\,dx = \int \frac{(5x^4+4x^5)dx}{x^{10}\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2}$

Put $1 + \frac{1}{x^4} + \frac{1}{x^5} = t \Rightarrow \left(-\frac{4}{x^5}-\frac{5}{x^6}\right)dx = dt$

$$I = \int\frac{\left(\frac{5}{x^6}+\frac{4}{x^5}\right)dx}{\left(1+\frac{1}{x^4}+\frac{1}{x^5}\right)^2} = \int\frac{-dt}{(t)^2}$$

$$= \frac{1}{t} + K = \frac{1}{1+\frac{1}{x^4}+\frac{1}{x^5}} + K = \frac{x^5}{x^5+x+1} + K$$

$$= \frac{x^5+x+1-x-1}{x^5+x+1} + K = -\frac{(x+1)}{x^5+x+1} + C$$

3. $I = \int \frac{1}{x^{11}\sqrt{1+x^4}}\,dx = \int \frac{1}{x^{13}\sqrt{1+\frac{1}{x^4}}}\,dx$

$$I = \int\left(\frac{1}{x^4}\right)^2\frac{dx}{x^5\sqrt{1+\frac{1}{x^4}}}$$

Put $1 + \frac{1}{x^4} = t^2 \Rightarrow -\frac{4}{x^5}\,dx = 2t\,dt \Rightarrow \frac{1}{x^5}\,dx = -\frac{t}{2}\,dt$

$$I = -\frac{1}{2}\int(t^2-1)^2\,dt$$

$$I = -\frac{1}{2}\int(t^4-2t^2+1)\,dt = -\frac{1}{2}\left[\frac{t^5}{5}-\frac{2t^3}{3}+t\right] + C$$

where $t^2 = 1 + \frac{1}{x^4}$.

4. $I = \int \frac{(x^3-1)dx}{x^4\sqrt{6x^6-4x^3+2}} = \int \frac{(x^3-1)dx}{x^7\left(6-\frac{4}{x^3}+\frac{2}{x^6}\right)^{1/2}}$

$$I = \int \frac{\frac{1}{x^4}-\frac{1}{x^7}}{\left(6-\frac{4}{x^3}+\frac{2}{x^6}\right)^{1/2}}\, dx$$

Put $6 - \frac{4}{x^3} + \frac{2}{x^6} = t$

$$\Rightarrow \left(\frac{12}{x^4}-\frac{12}{x^7}\right) dx = dt$$

$$I = \frac{1}{12}\int \frac{dt}{\sqrt{t}} = \frac{\sqrt{t}}{6} + C$$

$$I = \frac{1}{6}\sqrt{\frac{6x^6-4x^3+2}{x^6}} + C = \frac{\sqrt{6x^6-4x^3+2}}{6x^3} + C$$

5. $I = \int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\,dx = \int \frac{x^4-1}{x^3\sqrt{x^2+1+\frac{1}{x^2}}}\,dx$

$$I = \int \frac{x^4-1}{x^3\sqrt{\left(x+\frac{1}{x}\right)^2-1}}\,dx = \int \frac{x-\frac{1}{x^3}}{\sqrt{\left(x+\frac{1}{x}\right)^2-1}}\,dx$$

Put $\left(x+\frac{1}{x}\right)^2 - 1 = t$

$$\Rightarrow 2\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right)dx = 2\left(x-\frac{1}{x^3}\right)dx = dt$$

$$I = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \sqrt{t} + c = \sqrt{\left(x+\frac{1}{x}\right)^2-1} + C$$

6. Let $I = \int \frac{1}{(x-1)\sqrt{x^2+4}}\,dx$

Put $(x-1) = 1/z$

$dx = -1/z^2\, dz$

$$I = \int \frac{-\frac{1}{z^2}\,dz}{\frac{1}{z}\sqrt{\left(\frac{1}{z}+1\right)^2+4}} = \int \frac{dz}{\sqrt{5z^2+2z+1}}$$

$$= -\frac{1}{\sqrt{5}}\log\left|\left(\frac{1}{x-1}+\frac{1}{5}\right)+\sqrt{\frac{1}{(x+1)^2}+\frac{2}{5(x-1)}+\frac{1}{5}}\right| + c$$

7. Let $x^2+4x+7 = A\,(x^2+x+1) + B\,(2x+1) + C$

Comparing the coefficient of x^2, x and constant term, we get:

$A = 1,\ A+2B = 4,\ A + B + C = 7$

$\Rightarrow\ A = 1,\ B = \frac{3}{2},\ C = \frac{9}{2}$

So, $I = \int\sqrt{x^2+x+1}\,dx + \frac{3}{2}\int\frac{(2x+1)\,dx}{\sqrt{x^2+x+1}} + \frac{9}{2}\int\frac{dx}{\sqrt{x^2+x+1}}$

So, $\left(\frac{x+\frac{1}{2}}{2}\right)\sqrt{x^2+x+1} +$

$$3\sqrt{x^2+x+1} + \frac{9}{2}\ln\left(x+\frac{1}{2}+\sqrt{x^2+x+1}\right) + c$$

8. Divide Numerator and Denominator by x^2, we get:

$$\int\frac{1+\frac{1}{x^2}}{x^2+2+\frac{1}{x^2}}\,dx = \int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+4}\,dx$$

Put $x - 1/x = t \Rightarrow (1+1/x^2)\,dx = dt$

$$= \int\frac{dt}{t^2+(2)^2} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{x^2-1}{2x}\right) + C$$

9. $\int\sqrt{2x^2+3x+4}\,dx = \int\sqrt{2\left(x^2+\frac{3}{2}x+2\right)}dx$

$$\Rightarrow\ I = \sqrt{2}\int\sqrt{\left(x+\frac{3}{4}\right)^2+\frac{23}{16}}\,dx$$

Let $x + 3/4 = z,\ dx = dz$

$$I = \sqrt{2}\int\sqrt{z^2+a^2}\,dz$$

$$= \frac{z}{2}\sqrt{z^2+a^2} + \frac{a^2}{2}\log\left(z+\sqrt{z^2+a^2}\right) + c$$

$$= \frac{4x+3}{8\sqrt{2}}\sqrt{2x^2+3x+4} + \frac{23}{32\sqrt{2}}\log\left[\frac{4x+3}{4}+\frac{\sqrt{2x^2+3x+4}}{\sqrt{2}}\right] + c$$

10. Here, $I = \frac{1}{2}\int\frac{(x^2+1)+(1-x^2)}{1+x^4}\,dx$

$$= \frac{1}{2}\int\frac{x^2+1}{x^4+1}\,dx + \frac{1}{2}\int\frac{1-x^2}{1+x^4}\,dx$$

$= I_1 + I_2$ ...(1)

Now, $I_1 = \frac{1}{2}\int\frac{x^2+1}{x^4+1}\,dx = \frac{1}{2}\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$

Put, $x - \frac{1}{x} = z$; then $\left(1+\frac{1}{x^2}\right)dx = dz$

$$\therefore\ I_1 = \frac{1}{2}\int\frac{dz}{\left(x-\frac{1}{x}\right)^2+2}$$

$$= \frac{1}{2}\int\frac{dz}{z^2+(\sqrt{2})^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z}{\sqrt{2}}\right) + C_1$$

$= \frac{1}{2\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right) + c_1$ ...(2)

Next, $I_2 = \frac{1}{2}\int\frac{1-x^2}{1+x^4}\,dx = \frac{1}{2}\int\frac{\frac{1}{x^2}-1}{\frac{1}{x^2}+x^2}\,dx$

Put, $x + \frac{1}{x} = t$; then $\left(1-\frac{1}{x^2}\right)dx = dt$

$$\therefore\ I_2 = \frac{1}{2}\int\frac{-dt}{\left(x+\frac{1}{x}\right)^2-2}$$

$$= -\frac{1}{2}\int\frac{dt}{t^2-(\sqrt{2})^2}$$

$$= -\frac{1}{2}\cdot\frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C_2$$

$$= -\frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + C_2 \quad \ldots(3)$$

$$\therefore \quad I = \frac{1}{2\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\left(x-\frac{1}{x}\right) - \frac{1}{4\sqrt{2}}\log\left|\frac{x^2-\sqrt{2}+1}{x^2+\sqrt{2}x+1}\right| + C$$

Test Your Concepts-VII

1. $I = \int \frac{dx}{1+\cot x}$

$$I = \int \frac{\sin x\, dx}{\sin x+\cos x}$$

$$= \frac{1}{2}\int \frac{\sin x+\cos x+\sin x-\cos x}{\sin x+\cos x}dx$$

$$= \frac{1}{2}\left[\int 1+\frac{\sin x-\cos x}{\sin x+\cos x}dx\right]$$

$$= \frac{1}{2}\left[x-\int\frac{\cos x-\sin x}{\sin x+\cos x}dx\right] = \frac{1}{2}\left[x-\ln(\sin x+\cos x)\right] + c$$

2. $I = \int \frac{2\cos x-3\sin x}{6\cos x+4\sin x}dx$

Put, $6\cos x + 4\sin x = t$, $(-6\sin x + 4\cos x)dx = dt$

$$I = \int\frac{dt}{2(t)} = \frac{1}{2}\ln t + C = \frac{1}{2}\ln|6\cos x+4\sin x| + C$$

$$= \frac{1}{2}(\ln 2 + \ln|3\cos x+2\sin x|) + C$$

$$= \frac{1}{2}\ln|3\cos x+2\sin x| + \lambda$$

3. $I = \int\frac{2-\sin x}{2+\cos x}dx$

$$= \int\frac{2}{2+\cos x}dx + \int\frac{-\sin x}{2+\cos x}dx$$

$$= \int\frac{2\sec^2\frac{x}{2}}{2\left(1+\tan^2\frac{x}{2}\right)+1-\tan^2\frac{x}{2}}dx + \ln|2+\cos x| + C$$

$$= \int\frac{2\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2}+3} + \ln|2+\cos x| + C$$

$$= \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + \ln|2+\cos x| + C$$

4. $I = \int\frac{5\cos x+6}{2\cos x+\sin x+3}dx$

Rewriting, $5\cos x + 6 = A(2\cos x + \sin x + 3) + B(-2\sin x + \cos x)$

This gives: $2A + B = 5$, $A - 2B = 0$, $3A = 6$
we get, $A = 2$, $B = 1$

So, $I = \int 2\,dx + \int\frac{-2\sin x+\cos x}{2\cos x+\sin x+3}dx$

$$I = 2x + \ln(2\cos x + \sin x + 3) + C$$

5. $I = \int\frac{\sin x}{(1-\cos^2 x)(2+\cos x)}dx$

Put $\cos x = t \Rightarrow -\sin x\, dx = dt$

$$I = \int\frac{-dt}{(1-t)(1+t)(2+t)}$$

Using partial fraction we get:

$$I = \frac{-1}{6}\int\frac{1}{1-t}dt - \frac{1}{2}\int\frac{dt}{1+t} + \frac{1}{3}\int\frac{1}{2+t}dt$$

$$= \frac{1}{6}\ln|1-\cos x| - \frac{1}{2}\ln|1+\cos x| + \frac{1}{3}\ln|2+\cos x| + C$$

6. $I = \int\frac{\cos^2 x}{\sin x\cdot\cos 3x}dx = \int\frac{\cos^2 x\, dx}{\sin x(4\cos^3 x-3\cos x)}$

$$= \int\frac{\cos x\, dx}{\sin x(4\cos^2 x-3)} = \int\frac{\cos x\, dx}{\sin x(1-4\sin^2 x)}$$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$

$$= \int\frac{dt}{t(1-4t^2)} = \int\frac{dt}{t^3\left(\frac{1}{t^2}-4\right)}$$

Put $\frac{1}{t^2} - 4 = \alpha \Rightarrow \frac{-2}{t^3}dt = d\alpha$

$$= \int\frac{-d\alpha}{2(\alpha)} = -\frac{1}{2}\int\frac{d\alpha}{\alpha} = -\frac{1}{2}\ln\alpha + C$$

$$= -\frac{1}{2}\ln\left|\frac{1}{\sin^2 x}-4\right| + \ln C = \ln\frac{|C\sin x|}{\sqrt{1-4\sin^2 x}}$$

7. $I = \int\frac{dx}{1+\tan x} = \int\frac{\cos x}{\cos x+\sin x}dx$

$$I = \int\frac{(\cos x+\sin x)+(-\sin x+\cos x)}{2(\cos x+\sin x)}dx$$

$$= \frac{1}{2}\int\left(1+\frac{-\sin x+\cos x}{\cos x+\sin x}\right)dx$$

$$= \frac{x}{2} + \frac{1}{2}\ln|\cos x+\sin x| + C$$

8. $I = \int\frac{dx}{\tan x\cos 2x} = \int\frac{(1+\tan^2 x)}{\tan x(1-\tan^2 x)}dx$

Put, $\tan x = t \Rightarrow \sec^2 x\, dx = dt$.

$$I = \int\frac{dt}{t(1-t^2)} = \int\left(\frac{1}{t}+\frac{t}{1-t^2}\right)dt$$

$$I = \ln|t| - \frac{1}{2}\ln|1-t^2| + C$$

$$I = \ln|\tan x| - \frac{1}{2}\ln|1-\tan^2 x| + \ln C$$

$$= \ln\frac{|C\sin x/\cos x|\cos x}{\sqrt{\cos^2 x-\sin^2 x}} = \ln\frac{|C\sin x|}{\sqrt{\cos 2x}}$$

9. $I = \int\frac{dx}{\sin^2 x+\tan^2 x} = \int\frac{dx}{\sin^2 x+\frac{\sin^2 x}{\cos^2 x}}$

$$= \int\frac{\mathrm{cosec}^2 x\, dx}{1+\sec^2 x} = \int\frac{\mathrm{cosec}^2 x\, dx}{2+\frac{1}{\cot^2 x}}$$

CHAPTER 8

Put $\cot x = t \Rightarrow -\operatorname{cosec}^2 x\, dx = dt$

$$= \int \frac{-dt}{2+\frac{1}{t^2}} = -\int \frac{t^2}{2t^2+1}\, dt = -\frac{1}{2}\int\left(1-\frac{1}{2t^2+1}\right)dt$$

$$= -\frac{t}{2} + \frac{1}{4}\int \frac{dt}{t^2+\frac{1}{2}}$$

$$= -\frac{t}{2} + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}t\right) + K$$

$$= -\frac{\cot x}{2} + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\sqrt{2}\cot x\right) + K$$

$$= -\frac{\cot x}{2} + \frac{1}{2\sqrt{2}}\left(\frac{\pi}{2} - \cot^{-1}\left(\sqrt{2}\cot x\right)\right) + K$$

$$= -\frac{\cot x}{2} - \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) + C$$

10. $I = \int \frac{\sin 2x\, dx}{\cos^4 x + \sin^4 x} = \int \frac{2\sin x \cos x}{\cos^4 x(1+\tan^4 x)}\, dx$

$$= \int \frac{2\tan x \sec^2 x\, dx}{(1+(\tan^2 x)^2)}$$

Put $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x\, dx = dt$

$$I = \int \frac{dt}{1+t^2} = \tan^{-1}t + C = \tan^{-1}(\tan^2 x) + C$$

11. $I = \int \frac{\tan x\, dx}{1+\tan x+\tan^2 x}$

Put $\tan x = t$

$\Rightarrow \sec^2 x\, dx = dt \Rightarrow dx = \frac{dt}{1+t^2}$

$$I = \int \frac{t}{(1+t+t^2)}\, \frac{dt}{(1+t^2)}$$

$$I = \int\left(\frac{1}{t^2+1} - \frac{1}{t^2+t+1}\right)dt = \tan^{-1}t - \int \frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \tan^{-1}(\tan x) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + C$$

$$= \tan^{-1}(\tan x) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan x+1}{\sqrt{3}}\right) + C$$

12. We can also use profile $N = \alpha D + \beta D'$ in case of exponential

$3e^x + 4e^{-x} = \alpha(2e^x - 3e^{-x}) + \beta(2e^x + 3e^{-x})$

$3e^x + 4e^{-x} = (2\alpha + 2\beta)e^x + 3(\beta - \alpha)e^{-x}$

$2(\alpha + \beta) = 3$ and $3(\beta - \alpha) = 4$

We get, $\alpha = \frac{1}{12}$, $\beta = \frac{17}{12}$

$$I = \int \frac{3e^x + 4e^{-x}}{2e^x - 3e^{-x}}\, dx = \int \frac{\alpha D + \beta D'}{D}\, dx = \alpha x + \beta \ln(D) + C$$

$$I = \frac{x}{12} + \frac{17}{12}\ln|2e^x - 3e^{-x}| + C$$

13. $I = \int \frac{\cos 5x + \cos 4x}{1-2\cos 3x}\, dx = \int \frac{\cos(3x+2x)+\cos(3x+x)}{1-2\cos 3x}\, dx$

$$= \int \frac{\cos 3x\cos 2x - \sin 2x \sin 3x + \cos 3x \cos x - \sin 3x \sin x}{1-2\cos 3x}\, dx$$

$$= \int \frac{\cos 3x(\cos 2x + \cos x) - \sin 3x \sin x(2\cos x + 1)dx}{1-2\cos 3x}$$

$\sin 3x \sin x = \frac{1}{2}[\cos 2x - \cos 4x]$

$= \frac{1}{2}[2\cos^2 x - 2\cos^2 2x] = (\cos 2x + \cos x)(\cos x - \cos 2x)$

$$I = \int \frac{(\cos 2x + \cos x)(\cos 3x - (\cos x - \cos 2x)(2\cos x + 1))dx}{1-2\cos 3x}$$

$$= \int \frac{(\cos 2x+\cos x)(\cos 3x - 2\cos^2 x - \cos x + 2\cos 2x\cos x + \cos 2x))dx}{1-2\cos 3x}$$

$$= \int \frac{(\cos 2x+\cos x)(\cos 3x - 2\cos^2 x - \cos x + \cos x + \cos 3x + 2\cos^2 x - 1))dx}{1-2\cos 3x}$$

$$= \int -(\cos 2x + \cos x)\, dx = -\left[\sin x + \frac{1}{2}\sin 2x\right] + c$$

14. $I = \int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\, dx$

Put, $x = \cos^2\theta$, $dx = -2\sin\theta\cos\theta\, d\theta$,

We get, $I = \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\,.\,(-2\sin\theta\cos\theta)\, d\theta$

$$I = -\int 2\tan\frac{\theta}{2}.\sin\theta\cos\theta\, d\theta = -4\int \sin^2\frac{\theta}{2}.\cos\theta\, d\theta$$

$$I = -2\int(1-\cos\theta)\cos\theta\, d\theta = -2\int(\cos\theta - \cos^2\theta)\, d\theta$$

$$I = -2\int\cos\theta\, d\theta + \int(1+\cos 2\theta) = -2\sin\theta + \theta + \frac{\sin 2\theta}{2} + C$$

$$I = -2\sqrt{1-x} + \cos^{-1}\sqrt{x} + \sqrt{x(1-x)} + C$$

15. Let

$3 + 2\cos x + \sin x$

$= A\frac{d}{dx}(6 + 3\cos x + 4\sin x) + B(6 + 3\cos x + 4\sin x) + C$

$\Rightarrow 3 + 2\cos x + \sin x = A(-3\sin x + 4\cos x) + B(6 + 3\cos x + 4\sin x) + C$

Equating coefficients, we get:

$A = \frac{1}{5}$, $B = \frac{2}{5}$ and $C = \frac{3}{5}$

$$\therefore\quad I = \int \frac{\left\{A\frac{d}{dx}(6+3\cos x+4\sin x)+B(6+3\cos x+4\sin x)+C\right\}}{6+3\cos x+4\sin x}\, dx$$

$$= \frac{1}{5}\int \frac{d\,(6+3\cos x+4\sin x)}{6+3\cos x+4\sin x}.dx + \frac{2}{5}\int dx + \frac{3}{5}\int \frac{dx}{6+3\cos x+4\sin x}$$

$$= \frac{1}{5}\log(6+3\cos x+4\sin x) + \frac{2}{5}x + \frac{3}{5}\int \frac{dx}{6+3\cos x+4\sin x}$$

Now,

$$\int \frac{dx}{6+3\cos x+4\sin x} = \int \frac{dx}{6+3\cdot\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+4\cdot\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$= \int \frac{\sec^2\frac{x}{2}\, dx}{6\left(1+\tan^2\frac{x}{2}\right)+3\left(1-\tan^2\frac{x}{2}\right)+8\tan\frac{x}{2}}$$

$\left\{\text{Put } \tan\frac{x}{2} = z, \text{ we get } \frac{1}{2}\sec^2\frac{x}{2}\, dx = dz\right\}$

$$= \int \frac{2dz}{6\,(1+z^2) + 3\,(1-z^2) + 8z}$$

$$= \int \frac{2dz}{3z^2 + 8z + 9} = \frac{2}{3}\int \frac{dz}{z^2 + \frac{8}{3}z + 3}$$

$$= \frac{2}{3}\int \frac{dz}{\left(z+\frac{4}{3}\right)^2 + 3 - \frac{16}{9}}$$

$$= \frac{2}{3}\int \frac{dz}{\left(z+\frac{4}{3}\right)^2 + \left(\frac{\sqrt{11}}{3}\right)^2}$$

$$= \frac{2}{3}\cdot\frac{1}{\frac{\sqrt{11}}{3}}\tan^{-1}\left(\frac{z+\frac{4}{3}}{\frac{\sqrt{11}}{3}}\right) + C$$

$$= \frac{2}{\sqrt{11}}\tan^{-1}\left(\frac{3\tan\frac{x}{2}+4}{\sqrt{11}}\right) + C$$

$$\therefore\quad I = \frac{1}{5}\log(6+3\cos x + 4\sin x) + \frac{2}{5}x + \frac{5}{5\sqrt{11}}\tan^{-1}\left(\frac{3\tan\frac{x}{2}+4}{\sqrt{11}}\right) + C$$

Single Correct Choice Type Questions

1. $\int \tan^{-1}\left(\frac{\tan x - 2\cot x}{3}\right)dx = \int \tan^{-1}\left(\frac{\tan x - 2\cot x}{1+2\tan x\cot x}\right)dx$

$$\int\left(\tan^{-1}(\tan x) - \tan^{-1}(2\cot x)\right)dx$$

$$= \frac{x^2}{2} - \int\left(\frac{\pi}{2} - \cot^{-1}\left(\frac{2}{\tan x}\right)\right)dx$$

$$= \frac{x^2}{2} - \frac{\pi x}{2} + \int\left(\tan^{-1}\left(\frac{\tan x}{2}\right)\right)dx$$

Given $\int \tan^{-1}\left(\frac{\tan x}{2}\right)dx = \alpha$

$$= \frac{x^2}{2} - \frac{\pi x}{2} + \alpha + c$$

Hence, the correct answer is (B).

2. Let $I = \int \frac{1}{(1-x^8)}\left\{\sin^{-1}\left(\frac{2x}{1+x^2}\right) + \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}dx$

$$= \int \frac{1}{(1-x^8)}\left\{\left(\pi - 2\tan^{-1}x\right) + \left(-\pi + 2\tan^{-1}x\right)\right\}dx$$

$$= \int 0dx = 0 + c = c$$

Hence, the correct answer is (A).

3. $\int\left(\int e^x\left(\ln x + \frac{2}{x} - \frac{1}{x^2}\right)dx\right)dx$

$$= \int\int e^x\left(\left(\ln x + \frac{1}{x}\right) + \left(\frac{1}{x} - \frac{1}{x^2}\right)\right)dx\,dx$$

Using $\int e^x\left(f(x) + f'(x)\right)dx = e^x f(x) + c$

$$= \int\left(e^x\left(\ln x + \frac{1}{x}\right) + C_1\right)dx = e^x\ln x + C_1x + C_2$$

Hence, the correct answer is (A).

4. $A = \begin{bmatrix} x^2 & 6 & 0 \\ 1 & -5 & 1 \\ 2 & 0 & x \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}$

$$\text{AB} = \begin{bmatrix} 4x^2 & \ldots & \ldots \\ \ldots & -5 & \ldots \\ \ldots & \ldots & 8x \end{bmatrix}$$

$f(x)$ = trace of AB $= 4x^2 - 5 + 8x$

$$\int \frac{3dx}{f(x)} = \frac{1}{4}\int \frac{3dx}{x^2 + 2x - \frac{5}{4}} = \frac{3}{4}\int \frac{dx}{(x+1)^2 - \left(\frac{3}{2}\right)^2}$$

$$= \frac{3}{4}\,\frac{1}{2\left(\frac{3}{2}\right)}\ln\left|\frac{x+1-\frac{3}{2}}{x+1+\frac{3}{2}}\right| + C = \frac{1}{4}\ln\left|\frac{2x-1}{2x+5}\right| + C$$

Hence, the correct answer is (A).

5. $\cos 2x = 2\cos^2 x - 3$

$$\int \frac{x^3-1}{(x^4+1)(x+1)}dx = \int \frac{(x^4+x^3)-(x^4+1)}{(x^4+1)(x+1)}dx$$

$$= \int \frac{x^3}{x^4+1}dx - \int \frac{1}{x+1}dx$$

$$= \frac{1}{4}\ln(x^4+1) - \ln(x+1) + C$$

Hence, the correct answer is (C).

6. $4\cos^4 x - 2\cos 2x - \frac{1}{2}\cos 4x = (1+\cos 2x)^2 - 2\cos 2x - \frac{\cos 4x}{2}$

$$= 1 + 2\cos 2x + \frac{1+\cos 4x}{2} - 2\cos 2x - \frac{\cos 4x}{2} = \frac{3}{2}$$

So $g(x) = \left(\frac{3}{2} - x^7\right)^{\frac{1}{7}}$

$\therefore\quad g(g(x)) = x$

$$\int g(g(x))dx = \int x dx = \frac{x^2}{2} + c$$

Hence, the correct answer is (B).

7. Put $t = xe^x$, $dt = e^x(1+x)dx$

$$I = \int \frac{(x+1)}{x(1+xe^x)^2}dx = \int \frac{e^x(1+x)}{xe^x(1+xe^x)^2}dx$$

$$= \int \frac{dt}{t(1+t)^2} = \int\left(\frac{1}{t} - \frac{1}{1+t} - \frac{1}{(1+t)^2}\right)dt$$

$$= \log|t| - \log|1+t| + \frac{1}{1+t}$$

$$= \log\left(1 - \frac{1}{1+t}\right) + \frac{1}{1+t} + C$$

Hence, the correct answer is (B).

8. $\int_0^1 \frac{2}{(x^3+6x^2+11x+6)}dx$

$$= \int_0^1 \frac{2dx}{(x+1)(x+2)(x+3)} = \int_0^1\left(\frac{1}{x+1} - \frac{2}{x+2} + \frac{1}{x+3}\right)dx$$

$$= \ln(x+1) - 2\ln(x+2) + \ln(x+3)\Big|_0^1$$

$$= \ln 2 - 2\ln 3 + \ln 4 - \ln 1 + 2\ln 2 - \ln 3$$

$$= 5\ln 2 - 3\ln 3$$

Hence, the correct answer is (A).

9. $\displaystyle\int \frac{dx}{(\tan x+1)\sin^2 x} = \int \frac{1}{(\tan x+1)}\frac{\sec^2 x}{\tan^2 x}dx$

Put $\tan x = \frac{1}{t} \Rightarrow \sec^2 x\, dx = -\frac{1}{t^2}\, dt$

$$I = \int \frac{1}{\left(\frac{1}{t}+1\right)\left(\frac{1}{t}\right)^2}\left(\frac{-1}{t^2}\right)dt = -\int \frac{t}{t+1}\, dt$$

$$I = \int \frac{t}{t+1} - 1\, dt = \ln(t+1) - t + C;$$

$$I = \ln(\cot x + 1) - \cot x + C$$

$$f(x) = \ln(\cot x + 1) - \cot x;\ f\left(\frac{\pi}{2}\right) = 0,\ f\left(\frac{\pi}{4}\right) = \ln 2 - 1$$

Hence, the correct answer is (B).

10. $\displaystyle I = \int \frac{(x^3-1)dx}{\sqrt{x^6+2x^3}}$

$$= \int \frac{x - \frac{1}{x^2}}{\sqrt{x^2 + \frac{2}{x}}}dx$$

Put $x^2 + \frac{2}{x} = t \Rightarrow 2x - \frac{2}{x^2} = dt$

$$I = \int \frac{dt}{2\sqrt{t}} = \sqrt{t} + C = \sqrt{x^2 + \frac{2}{x}} + C = \sqrt{\frac{x^3+2}{x}} + C$$

Hence, the correct answer is (D).

11. Put, $x^{-4} + 1 = t^2$

$\Rightarrow\ -4x^{-5}dx = 2t\, dt \Rightarrow \displaystyle\int -\frac{t(t^2-1)}{t}\frac{dt}{2}$

$$= \frac{t}{2} - \frac{t^3}{6} + C = \frac{\sqrt{x^4+1}(2x^4-1)}{6x^6} + C$$

Hence, the correct answer is (A).

12. $\displaystyle\int \frac{1+\cos 8x}{\cot 2x - \tan 2x}\, dx = A\cos(8x) + k;$

Differentiate both sides w.r.t. *x*,

we get $\dfrac{1+\cos 8x}{\left(\dfrac{\cos 2x}{\sin 2x} - \dfrac{\sin 2x}{\cos 2x}\right)} = -8A \cdot \sin(8x)$

$\Rightarrow\ \dfrac{(2\cos^2 4x)(\sin 2x \cdot \cos 2x)}{(\cos 4x)} = -8A \cdot \sin 8x$

$\Rightarrow\ \frac{1}{2}\sin 8x = -8A \cdot \sin 8x \Rightarrow A = \frac{-1}{16}$

Hence, the correct answer is (C).

13. Differentiate both sides with respect to *x*, we get

$$\frac{(x^2+1)-x}{x^2+1} = \frac{(x^2+1)A'(x) - A(x)}{x^2+1}$$

$\therefore\ \ A(x) = x$

Hence, the correct answer is (C).

14. $\displaystyle\int \frac{dx}{x.\ell(x)\ell^2(x).\ell^3(x)\ldots\ldots.\ell^n(x)}$

Let $\ell^n(x) = \log\left(\log\left(\log\left(\overset{n\text{ times}}{\ldots\ldots\ldots\ldots}\log(x)\right)\right)\right) = t$

$$\frac{dx}{x \cdot \ell(x)\ell^2(x)\ldots\ldots\ldots.\ell^{n-1}(x)} = dt$$

$$\int \frac{dt}{t} = \ln t + c = \ln(\ell^n(x)) + C = \ell^{n+1}(x) + C$$

Hence, the correct answer is (B).

15. Let $\displaystyle I = \int \frac{dx}{x(x+1)(\ln(x+1)-\ln x)^{11}}$

Put, $\ln(x+1) - \ln x = t \Rightarrow \dfrac{dx}{x(1+x)} = -dt$

So, $I = -\displaystyle\int \frac{dt}{t^{11}} = \frac{1}{10}\left(\frac{1}{t^{10}}\right)$

$$= \frac{1}{10(\ln(x+1) - \ln x)^{10}} + C$$

Hence, the correct answer is (A).

16. $\displaystyle I = 2\int\left\{\cos(2x+\pi) + \cos\frac{2\pi}{3}\right\}\cos 2x\, dx$

$$= 2\int\left(-\cos 2x - \frac{1}{2}\right)\cos 2x\, dx$$

$$= -\int(2\cos^2 2x + \cos 2x)dx$$

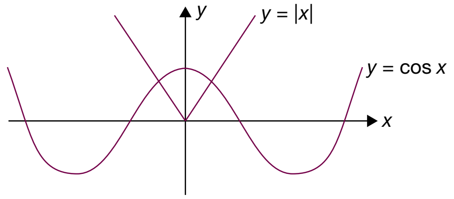
$$= -\int(1\cos 4x + \cos 2x)dx = -\left(x + \frac{\sin 4x}{4} + \frac{\sin 2x}{2}\right) + C$$

Hence, the correct answer is (A).

17. $\displaystyle\int e^{1-\cos x}(1 + x\sin x)dx = \int e^{1-\cos x}dx + \int e^{1-\cos x}x\sin x dx$

$$= xe^{1-\cos x} - \int x\left(e^{1-\cos x}\sin x\right)dx + \int e^{1-\cos x}x\sin x dx + c$$

$$= xe^{1-\cos x} + c = f(x).\ e^{1-g(x)} + c$$



$f(x) = x, g(x) = \cos x$

$|f(x)| = g(x)$

$\Rightarrow\ |x| = \cos x$ has two solutions

Hence, the correct answer is (B).

18. $\displaystyle\int \frac{(\cos x + x\sin x)}{x^2\left(\frac{\cos x}{x} - 1\right)}dx$

Put, $\dfrac{\cos x}{x} - 1 = t \Rightarrow \left(\dfrac{-x\sin x - \cos x}{x^2}\right)dx = dt$

$$I = \int \frac{-dt}{t} = -\ln t + K = \ln\left|\frac{cx}{\cos x - x}\right|$$

Hence, the correct answer is (A).

19. $f(x^2+1) = (x^2+1)^2 + 3x^2 + 1$

Put $x^2 + 1 = t$

$f(t) = t^2 + 3(t-1) + 1$

$f(t) = t^2 + 3t - 2$

$\therefore\ \int f(x)\, dx = \dfrac{x^3}{2} + \dfrac{3x^2}{2} - 2x + C$

Hence, the correct answer is (A).

20. $\int \sec^2 x(\sec\theta + \tan\theta)^2 \, d\theta$

$\sec\theta + \tan\theta = y$ …(1)

$\Rightarrow$ $\sec\theta(\tan\theta + \sec\theta)\,d\theta = dy$

Now, $\sec\theta - \tan\theta = \frac{1}{y}$ …(2)

$\therefore$ From (1) and (2),

$2\sec\theta = y + \frac{1}{y}$

$$\therefore \quad I = \frac{1}{2}\int y\left(y + \frac{1}{y}\right) dy = \frac{1}{2}\left[\frac{y^3}{3} + y\right] + C$$

$$= \frac{1}{2}\left[\frac{(\sec\theta + \tan\theta)^3}{3} + (\sec\theta + \tan\theta)\right]$$

$$= \frac{(\sec\theta + \tan\theta)}{6}\,[(\sec\theta + \tan\theta)^2 + 3] + C$$

Hence, the correct answer is (C).

21. $I = \int e^x\left(\frac{1 - x^6 - 6x^2}{(1+x^3)^2}\right)dx = \int e^x\left(\left(\frac{1-x^3}{1+x^3}\right) - \frac{6x^2}{(1+x^3)^2}\right)dx$

Using $\int e^x\left(f(x) + f'(x)\right)dx = e^x f(x) + c$

$$I = e^x\left(\frac{1-x^3}{1+x^3}\right) + c$$

Hence, the correct answer is (B).

22. $\int \frac{x - \sin x\cos x}{x^2\cos^2 x}dx = \int \frac{1}{x\cos^2 x} - \frac{\sin x}{x^2\cos x}dx$

$$= \int \frac{\sec^2 x}{x} - \frac{\tan x}{x^2}dx$$

$$= \frac{1}{x}\tan x + \int \frac{\tan x}{x^2}dx - \int \frac{\tan x}{x^2}dx + c$$

$$= \frac{\tan x}{x} + c$$

$$\lim_{x\to 0} f(x) = 1$$

Hence, the correct answer is (A).

23. Let $= \int e^x\left(\frac{2\tan x}{1 + \tan x} + \tan^2\left(x - \frac{\pi}{4}\right)\right) dx$

$$= \int e^x\left(\frac{2\tan x}{1+\tan x} + \sec^2\left(x - \frac{\pi}{4}\right) - 1\right)dx$$

$$= \int e^x\left(\frac{\tan x - 1}{1 + \tan x} + \sec^2\left(x - \frac{\pi}{4}\right)\right)$$

$$= \int e^x\left(\tan\left(x - \frac{\pi}{4}\right) + \sec^2\left(x - \frac{\pi}{4}\right)\right)dx$$

$$= e^x\tan\left(x - \frac{\pi}{4}\right) + C$$

Hence, the correct answer is (B).

24. $I = \int \ln x + \ln(\ln(x\ln x))\,dx + \int\left(\frac{1 + \ln x}{\ln x \cdot \ln(x\ln x)}\right)dx$

$$I = x\ln x - x + \int\left(\underset{\text{II}}{1}\cdot\underbrace{\ln(\ln(x\ln x))}_{\text{I}}\right)dx + \int\left(\frac{1 + \ln x}{\ln x\cdot\ln(x\ln x)}\right)dx$$

$$= x\ln x - x + x\cdot\ln(\ln(x\ln x)) + C$$

Hence, the correct answer is (C).

25. $I = \int\left(x^{7m} + x^{2m} + x^m\right)\left(2x^{6m} + 7x^m + 14\right)^{1/m} dx$

$$I = \int\left(\frac{x^{7m} + x^{2m} + x^m}{x}\right)\left(2x^{7m} + 7x^{2m} + 14x^m\right)^{1/m} dx$$

(multiply and divide by x)

Let $2x^{7m} + 7x^{2m} + 14x^m = t \Rightarrow 14m(x^{7m-1} + x^{2m-1} + x^{m-1}) = dt$

$$I = \frac{1}{14m}\int t^{1/m}\cdot dt = \frac{t^{\frac{m+1}{m}}}{14m\left(\frac{m+1}{m}\right)} + c$$

$$= \frac{\left(2x^{7m} + 7x^{2m} + 14x^m\right)^{\frac{m+1}{m}}}{14(m+1)} + c$$

Hence, the correct answer is (C).

26. $\int \frac{e^x(x-1)(x-\ln x)}{x^2}dx = \int e^x\left(\frac{1}{x} - \frac{1}{x^2}\right)\ln\left(\frac{e^x}{x}\right)dx$

Put, $t = \frac{e^x}{x}$

$$I = \int \ln t\,dt = (t\ln t - t) + c = \frac{e^x}{x}\left(\ln\left(\frac{e^x}{x}\right) - 1\right) + c$$

$$I = \frac{e^x(x - \ln x - 1)}{x} + c$$

Hence, the correct answer is (D).

27. $I = \int \frac{\cot x}{\sqrt{5 + 9\cot^2 x}}dx = \int \frac{\cos x}{\sqrt{5\sin^2 x + 9\cos^2 x}}dx$

$$I = \int \frac{\cos x}{\sqrt{9 - 4\sin^2 x}}dx$$

Put, $\sin x = t$;

$$I = \int \frac{dt}{\sqrt{9 - 4t^2}} = \frac{1}{2}\int \frac{dt}{\sqrt{\frac{9}{4} - t^2}}$$

$$I = \frac{1}{2}\sin^{-1}\left(\frac{2t}{3}\right) + C = \frac{1}{2}\sin^{-1}\left(\frac{2\sin x}{3}\right) + C.$$

Hence, the correct answer is (A).

28. $\int \frac{3x^4 - 1}{(x^4 + x + 1)^2}dx = \int \frac{3x^2 - 1/x^2}{(x^3 + 1/x + 1)^2}dx$

Put $x^3 + 1/x + 1 = t \Rightarrow (3x^2 - 1/x^2)dx = dt$

$$= \int \frac{dt}{t^2} = -\frac{1}{t} + c = -\frac{x}{x^4 + x + 1} + C$$

Hence, the correct answer is (B).

29. $f'(x) = \frac{1}{1 + \cos x}$

$$f(x) = \int \frac{1 - \cos x}{\sin^2 x}dx = \int \operatorname{cosec}^2 x - \operatorname{cosec} x\cot x\,dx$$

$$f(x) = -\cot x + \operatorname{cosec} x + C$$

$$f(x) = \frac{1 - \cos x}{\sin x} = \frac{2\sin^2 x/2}{2\sin x/2\cos x/2} + C$$

$f(x) = \tan x/2 + 3$ as $f(0) = 3$

$f(\pi/2) = 1 + 3 = 4$

Hence, the correct answer is (C).

30. $A = 2^{\frac{-\log_{10}\frac{100-x}{x}}{\log_{10} 2}} \Rightarrow A = 2^{\left(\log 2\left(\frac{x}{100-x}\right)\right)} \Rightarrow A = \frac{x}{100 - x}$

$\Rightarrow \frac{1}{A} = \frac{100 - x}{x}$

$$I = \int \log_e 10 \cdot \log_{10}\left(\frac{100-x}{x}\right) dx = \int (\ln(100-x) - \ln x)\, dx$$

$$I = (x-100)\ln(100-x) + (100-x) - x\ln x + x + C$$
$$I = (x-100)\ln(100-x) - x\ln x + C.$$

Hence, the correct answer is (A).

31. Let $I = \int \frac{x\,dx}{2-x^2+\sqrt{2-x^2}}$

Put, $x = \sqrt{2}\sin\theta \Rightarrow dx = \sqrt{2}\cos\theta\, d\theta$

So, $I = \int \frac{2\sin\theta\cos\theta\, d\theta}{2\cos\theta+\sqrt{2}\cos\theta} = \int \frac{\sqrt{2}\sin\theta\, d\theta}{\sqrt{2}\cos\theta+1}$

$$= -\ln\left|\sqrt{2}\cos\theta+1\right| + C$$

$$= -\ln\left|1+\sqrt{2}\cdot\sqrt{1-\frac{x^2}{2}}\right| + c = -\ln\left|1+\sqrt{2-x^2}\right| + c$$

Hence, the correct answer is (C).

32. Let $I = \frac{1}{2}\int e^{\sin^2 x}(1+\cos^2 x)\sin 2x\, dx$

Put, $\sin^2 x = t \Rightarrow \sin 2x\, dx = dt$,

So $I = \frac{1}{2}\int e^t(1+1-t)\,dt = \frac{1}{2}\int \underbrace{e^t}_{\text{II}}\underbrace{(2-t)}_{\text{I}}\,dt$

$$\frac{1}{2}\left[(2t)e^t + \int e^t dt\right] (2-1)$$

$$= \frac{1}{2}\left[(2-1)e^t + \int e^t dt\right] = \frac{1}{2}[(2-t)e^t + e^t] + C$$

$$= \frac{1}{2}(3-t)e^t + C = \frac{1}{2}(3-\sin^2 x)\, e^{\sin^2 x} + C$$

Hence, the correct answer is (A).

33. $I = \int x2^{\ln(x^2+1)}dx$ Let $x^2+1 = t \Rightarrow 2xdx = dt$

$$I = \frac{1}{2}\int 2^{\ln t}dt \text{ using } 2^{\ln t} = t^{\ln 2}$$

$$= \frac{1}{2}\int t^{\ln 2}\cdot dt = \frac{1}{2}\left(\frac{t^{\ln 2+1}}{\ln 2+1}\right) + C = \frac{(x^2+1)^{\ln 2+1}}{2(\ln 2+1)} + C$$

Hence, the correct answer is (C).

34. $I = \int \frac{\sin 2x+\sin 4x-\sin 6x}{1+\cos 2x+\cos 4x+\cos 6x}\,dx$

$$= \int \frac{2\sin 3x\cos x - 2\sin 3x\cos 3x}{2\cos^2 x+2\cos x\cos 5x}\,dx$$

$$= \int \frac{\sin 3x(\cos x-\sin 3x)}{\cos x(\cos x+\cos 5x)}\,dx$$

$$= \int \frac{2\sin 3x\sin 2x\sin x}{2\cos x\cos 2x\cos 3x}\,dx = \int \tan(x)\tan(2x)\tan(3x)\,dx$$

$\because$ $3x = 2x + x \Rightarrow \tan 3x = \frac{\tan 2x+\tan x}{1-\tan 2x+\tan x}$

$\Rightarrow$ $\tan 3x - \tan 3x\tan 2x\tan x = \tan 2x + \tan x$

$\Rightarrow$ $\tan 3x\tan 2x\tan x = \tan 3x - \tan 2x - \tan x$

$\because$ $I = \int(\tan 3x - \tan 2x - \tan x)dx$

$$I = \frac{1}{3}\ln|\sec 3x| - \frac{1}{2}\ln|\sec 2x| - \ln|\sec x| + C$$

Hence, the correct answer is (B).

35. $\int \frac{\tan x\cdot\sec x}{\sqrt{2\cos^2 x-1}}\,dx = \int \frac{\sec x\cdot\tan x\cdot\sec x}{\sqrt{2-\sec^2 x}}\,dx$

$\therefore$ $I = C - \sqrt{2-\sec^2 x}$

$$= C - \sqrt{2-(1+\tan^2 x)}$$

Hence, the correct answer is (A).

36. Use, $\int(f(x) + xf'(x))\,dx = xf(x) + C$

Hence, the correct answer is (D).

37. $\int e^x\left(\frac{x^2+4x+4}{(x+4)^2}\right)dx$

$$= \int e^x\left(\frac{x}{x+4} + \frac{x}{(x+4)^2}\right)dx$$

$$= \frac{e^x x}{(x+4)} + C \qquad \therefore f(x) = x$$

Required sum $= \frac{f(x)}{1-f(x)} = \frac{x}{1-x} = 1$

Hence, the correct answer is (C).

38. $\int \frac{(2x+1)}{(x^2+4x+1)^{3/2}}\,dx = \int \frac{\left(\frac{2}{x^2}+\frac{1}{x^3}\right)dx}{\left(1+\frac{4}{x}+\frac{1}{x^2}\right)^{3/2}}$;

Put $1+\frac{4}{x}+\frac{1}{x^2} = t \Rightarrow \left(-\frac{4}{x^2}-\frac{2}{x^3}\right)dx = dt$

$$= -\int \frac{dt}{2t^{3/2}} = \frac{1}{\sqrt{t}} + c = \frac{1}{\left(1+\frac{4}{x}+\frac{1}{x^2}\right)^{1/2}} + C$$

$$= \frac{x}{(x^2+4x+1)^{1/2}} + C$$

Hence, the correct answer is (B).

39. $I = \int \frac{px^{p+2q-1}-qx^{q-1}}{x^{2p+2q}+2x^{p+q}+1}\,dx = \int \frac{x^{2q}(px^{p-1}-qx^{-q-1})dx}{\left(x^{p+q}+1\right)^2}$

$$= \int \frac{px^{p-1}-qx^{-q-1}}{\left(x^p+x^{-q}\right)^2}\,dx$$

put $x^p + x^{-q} = t \Rightarrow \left(px^{p-1}-qx^{-q-1}\right)dx = dt$

$$= \int \frac{dt}{t^2} = -\frac{1}{t} + c$$

$$I = -\frac{1}{x^p+x^{-q}} + c = -\frac{x^q}{x^{p+q}+1} + C$$

Hence, the correct answer is (C).

40. $f(x) = \int 2e^x\frac{(1+\tan x-\tan^2 x)}{1+\tan^2 x}\,dx$

$$= \int e^x\left[\underbrace{2\cos 2x}_{f'(x)} + \underbrace{\sin 2x}_{f(x)}\right]dx$$

$f(x) = e^x\sin 2x + C$

(As, $f(\pi) = 0$)

$f(x) = e^x\sin 2x$

$\Rightarrow$ $f'(x) = e^x\, 2\cos 2x + \sin 2x\cdot e^x$

$\therefore$ $f(0) = 0$ and $f'(0) = 2$

$\Rightarrow$ $f(0) + f'(0) = 2$

Hence, the correct answer is (C).

41. $I = \int\left(\frac{fg'-f'g}{fg}\right)\ln\left(\frac{g}{f}\right)dx$

Let, $\ln\left(\frac{g}{f}\right) = t \Rightarrow \left(\frac{f}{g}\right)\left(\frac{fg'-gf'}{f^2}\right)dx = dt$

$$\Rightarrow \quad \frac{fg' - f'g}{fg}\,dx = dt$$

$$\therefore \quad I = \int t\,dt = \frac{t^2}{2} + C = \frac{1}{2}\left(\ln\frac{g(x)}{f(x)}\right)^2 + C$$

Hence, the correct answer is (B).

42. $$\underset{I_1}{\int e^{\sin x}.x\cos x dx} - \underset{I_2}{\int e^{\sin x}\tan x \sec x dx}$$

$$\therefore \quad I_1 = \int e^{\sin x}.x.\cos x dx = \int \underset{I}{(x)}(\underset{II}{e^{\sin x}\cos x})\,dx$$

Integrating by parts,

$$\left(x\int e^{\sin x}.\cos x dx - \int e^{\sin x} dx\right) = x.e^{\sin x} - \int e^{\sin x} dx$$

$$I_2 = \int \underset{\text{I}}{e^{\sin x}}\,\underset{\text{II}}{\tan x \sec x}\,dx$$

Integrating by parts,

$$= e^{\sin x}\int \tan x \sec x dx - \int e^{\sin x}.\cos x.\sec x.dx$$

$$= e^{\sin x}\sec x - \int e^{\sin x} dx$$

$$\therefore \quad I_1 - I_2 = xe^{\sin x} - e^{\sin x}\sec x + C$$

Hence, the correct answer is (C).

43. $$I = \int \frac{dx}{x^{1/2} - x^{1/3}}$$

Put, $x = t^6$, $I = \int \frac{6t^5}{t^3 - t^2} = 6\int \frac{t^3 - 1 + 1}{t - 1}\,dt$

$$= 6\int \frac{t^3 - 1}{t - 1}\,dt + 6\int \frac{1}{t - 1}\,dt = 6\left[\int (t^2 + t + 1)dt + 6\ln(t - 1)\right]$$

$$= 6\left\{\frac{t^3}{3} + \frac{t^2}{2} + t + \ln(t - 1)\right\}$$

$$= 6\left\{\frac{\sqrt{x}}{3} + \frac{\sqrt[3]{x}}{2} + \sqrt[6]{x} + \ln|\sqrt[6]{x} - 1|\right\} + C$$

$$= 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln|\sqrt[6]{x} - 1| + C$$

Hence, the correct answer is (A).

44. $$I = \int x^2 e^{-2x} dx$$

Put $-2x = t \Rightarrow -2dx = dt$

$$= -\frac{1}{4}\int t^2 e^t dt = -\frac{1}{8}\left[t^2\cdot e^t - \int 2t\cdot e^t dt\right]$$

$$= -\frac{1}{8}\left[t^2\cdot e^t - 2t\cdot e^t + 2e^t\right] + c = e^t\left[-\frac{t^2}{8} + \frac{2t}{8} - \frac{1}{4}\right] + C$$

where $t = -2x$

$$= e^{-2x}\left[-\frac{x^2}{2} - \frac{x}{2} - \frac{1}{4}\right];\ a = -\frac{1}{2}, b = -\frac{1}{2}, c = -\frac{1}{4}$$

Hence, the correct answer is (A).

45. Put, $\sec\theta + \tan\theta = t \Rightarrow (\sec\theta\tan\theta + \sec^2\theta)d\theta = dt$

$$\Rightarrow \quad \sec\theta(\tan\theta + \sec\theta)d\theta = dt \Rightarrow \sec\theta\,d\theta = \frac{dt}{t}$$

$$\sec\theta + \tan\theta = t \text{ and } \sec\theta - \tan\theta = \frac{1}{t} \Rightarrow \sec\theta = \frac{1}{2}\left(t + \frac{1}{t}\right)$$

$$I = \int \sec\theta(\sec\theta + \tan\theta)^2\sec\theta\,d\theta$$

$$= \int \frac{1}{2}\left(t + \frac{1}{t}\right)t^2\,\frac{dt}{t} = \frac{1}{2}\int t^2 + 1\,dt$$

$$= \frac{1}{2}\left(\frac{t^3}{3} + t\right) + C = \frac{t}{6}(t^2 + 3) + C$$

$$= \left(\frac{\sec\theta + \tan\theta}{6}\right)(\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta + 3) + C$$

$$= \left(\frac{\sec\theta + \tan\theta}{3}\right)(2 + \tan^2\theta + \sec\theta\tan\theta) + C$$

$$= \frac{(\sec\theta + \tan\theta)}{3}(1 + \sec\theta(\sec\theta + \tan\theta)) + C$$

Hence, the correct answer is (C).

46. $$I = \int e^x\frac{(1 + n.x^{n-1} - x^{2n})}{(1 - x^n)\sqrt{1 - x^{2n}}}\,dx = \int e^x\left[\frac{(1 - x^{2n}) + nx^{n-1}}{(1 - x^n)\sqrt{1 - x^{2n}}}\right]dx$$

$$= \int e^x\left[\frac{\sqrt{1 + x^n}}{\sqrt{1 - x^n}} + \frac{nx^{n-1}}{(1 - x^n)\sqrt{1 - x^{2n}}}\right]dx$$

$$f(x) = \sqrt{\frac{1 + x^n}{1 - x^n}};\ f'(x) = \frac{nx^{n-1}}{(1 - x^n)\sqrt{1 - x^{2n}}}$$

$$\therefore \quad I = e^x\sqrt{\frac{1 + x^n}{1 - x^n}} + C$$

Hence, the correct answer is (B).

47. $$I = \int \frac{(\cos^2 x + \cos^4 x)\cos x}{\sin^2 x + \sin^4 x}\,dx$$

Put, $\sin x = t \Rightarrow \cos x\,dx = dt$

$$I = \int \frac{(1 - t^2) + (1 - t^2)^2}{t^2 + t^4}\,dt$$

$$I = \int \frac{t^4 - 3t^2 + 2}{t^4 + t^2}\,dt = \int\left[1 + \frac{-4t^2 + 2}{t^4 + t^2}\right]dt$$

$$I = \int dt + \int \frac{2}{t^2}\,dt - 6\int \frac{1}{1 + t^2}\,dt$$

$$I = t - \frac{2}{t} - 6\tan^{-1}t + C$$

$$I = \sin x - 2\,\mathrm{cosec}\,x - 6\tan^{-1}(\sin x) + C$$

Hence, the correct answer is (C).

48. Put, $1 - x^{2013} = t^2$

$$\Rightarrow \quad -2013x^{2012}dx = 2tdt \Rightarrow dx = \frac{-2}{2013}\,\frac{tdt}{x^{2012}}$$

$$\therefore \quad = \int \frac{-2}{2013}\,\frac{tdt}{(1 - t^2)t} = \frac{-2}{2013}\int \frac{dt}{1 - t^2} = \frac{1}{2013}\log\left|\frac{t - 1}{t + 1}\right| + C$$

$$= \frac{1}{2013}\log\left|\frac{\sqrt{1 - x^{2013}} - 1}{\sqrt{1 - x^{2013}} + 1}\right| + C$$

Hence, the correct answer is (D).

49. $$\int \frac{x^3 - 8}{x^3\sqrt{x^4 - x^3 + 2}}\,dx$$

$$= \int \frac{\frac{1}{x^2} - \frac{8}{x^5}}{\sqrt{1 - \frac{1}{x} + \frac{2}{x^4}}}\,dx = \frac{2}{x^2}\sqrt{x^4 - x^3 + 2} + c$$

$\therefore \quad k = 2$

Hence, the correct answer is (B).

50. $\displaystyle\int \frac{\cos^6 x}{\sin^4 x(\cos^7 x+\sin^7 x)^{4/7}}\,dx$

$$= \int \frac{\cos^6 x}{\sin^4 x.\cos^4 x(1+\tan^7 x)^{4/7}}\,dx$$

$\therefore$ Integral $\displaystyle= \int \frac{\sec^2 x.dx}{\tan^4 x.(1+\tan^7 x)^{4/7}}$

$$= \int \frac{\sec^2 x.dx}{\tan^8 x.(1+\tan^{-7} x)^{4/7}} = \int \frac{\sec^2 x.(\tan x)^{-8}}{(1+(\tan x)^{-7})^{4/7}}\,dx$$

Let $1 + (\tan x)^{-7} = t$

$\Rightarrow -7(\tan x)^{-8} \sec^2 x\, dx = dt \Rightarrow (\tan x)^{-8}.\sec^2 x\, dx = \dfrac{-1}{7}\, dt$

$\therefore\ I = \dfrac{-1}{7}\displaystyle\int \frac{dt}{t^{4/7}} = \frac{-1}{7}\left(\frac{t^{1-\frac{4}{7}}}{1-\frac{4}{7}}\right) + C$

$$= -\frac{1}{7}\cdot\frac{7}{3}\,t^{3/7} = \frac{-1}{3}.(1 + (\tan x)^{-7})^{3/7} + C$$

$\therefore\ I = \dfrac{-1}{3}(1 + (\cot x)^7)^{3/7} + C$

Hence, the correct answer is (A).

51. $I = \displaystyle\int \frac{\sec x \tan x}{\sqrt{-6\sec^2 x+9+5\tan^2 x}}\,dx$

$$= \int \frac{\sec x \tan x}{\sqrt{4-\sec^2 x}}\,dx$$

Put, $\sec x = t$

$\Rightarrow\ I = \displaystyle\int \frac{dt}{\sqrt{4-t^2}} = \sin^{-1}\left(\frac{t}{2}\right) + C$

Hence, the correct answer is (A).

52. $f(x) = \displaystyle\int\left(\sin\sqrt{x} + \cos\sqrt{x}\right)dx$; let $\sqrt{x} = y \Rightarrow dx = 2y\,dy$

$\Rightarrow\ 2\displaystyle\int y\sin y\,dy + 2\int y\cos y\,dy$

$= 2(-y\cos y + \sin y) + 2(y\sin y + \cos y)$

$f(x) = 2(-\sqrt{x}\cos\sqrt{x} + \sin\sqrt{x} + \sqrt{x}\sin\sqrt{x} + \cos\sqrt{x})$

$\Rightarrow\ f(\pi^2) + f\left(\dfrac{\pi^2}{4}\right) = 3\pi - 4$

Hence, the correct answer is (B).

53. Let $\sec x - \tan x = t$

$\Rightarrow\ \sec x + \tan x = \dfrac{1}{t} \Rightarrow 2\tan x = \dfrac{1}{t} - t$

Also $\sec x(\tan x - \sec x)dx = dt$

$\Rightarrow\ \displaystyle\int \frac{2\sec x\tan x}{(\sec x-\tan x)^{10}}\,dx = \int \frac{\left(t-\frac{1}{t}\right)}{t^{11}}\,dt$

$$= \frac{t^{-9}}{-9} + \frac{t^{-11}}{11} + C$$

$$= (\sec x - \tan x)^{-11}\left(\frac{(\sec x-\tan x)^2}{-9} + \frac{1}{11}\right) + C$$

$$= (\sec x - \tan x)^{-11}\left(\frac{(\sec x-\tan x)^2}{-9} + \frac{1}{11}\right) + C$$

$\Rightarrow\ m = -\dfrac{1}{9}$ and $n = \dfrac{1}{11}$

Hence, the correct answer is (D).

54. $I = \displaystyle\int e^{x^2}\cdot e^x(2x^2+x+1)\,dx$

$$= \int e^x\left(xe^{x^2} + (2x^2+1)\,e^{x^2}\right)dx$$

$= e^x\cdot xe^{x^2} + c \qquad \left(\because \displaystyle\int e^x(f(x)+f'(x))dx = e^x f(x) + c\right)$

Hence, the correct answer is (A).

55. $\displaystyle\int (x^{23}+x^{15}+x^7)(2x^{24}+3x^{16}+6x^8)^{1/8}\,dx$

Put, $2x^{24} + 3x^{16} + 6x^8 = t$

$$= \int t^{1/8}\,\frac{dt}{48} = \frac{1}{48}\,\frac{t^{9/8}}{\left(\frac{9}{8}\right)} + C$$

$$= \frac{t^{9/8}}{54} + C = \frac{(2x^{24}+3x^{16}+6x^8)^{9/8}}{54} + C$$

$\therefore\ \alpha = 54, \beta = 9, \gamma = 8.$

Hence, the correct answer is (C).

56. Let $\tan^{-1}x^3 = t$

$\Rightarrow\ \dfrac{3x^2dx}{1+x^6} = dt \Rightarrow I = \dfrac{1}{3}\displaystyle\int t\,dt \Rightarrow I = \frac{t^2}{6}$

$\Rightarrow\ I = \dfrac{(\tan^{-1}(x^3))^2}{6} + C$

$\Rightarrow\ (\alpha, \beta, k) = (3, 2, 6)$

Hence, the correct answer is (D).

57. Let $x^2e^{-x} = t \Rightarrow (2xe^{-x} - x^2e^{-x})\,dx = dt$

$\Rightarrow\ xe^{-x}(2 - x)\,dx = dt$

$\therefore\ I = \displaystyle\int \frac{dt}{(1+t)^2} = -\frac{1}{1+t} + C = -\frac{1}{1+x^2e^{-x}} + C = \frac{-e^x}{e^x+x^2} + C$

Hence, the correct answer is (D).

58. Put, $\tan x = t$; $\displaystyle\int \frac{1-t}{1+t}\,\frac{dt}{\sqrt{t^3+t^2+t}}$

$$= \int \frac{(1-t^2)dt}{(1+t)^2\sqrt{t^3+t^2+t}} = \int \frac{(1-t^2)dt}{(1+t^2+2t)\sqrt{t^3+t^2+t}}$$

$$= \int \frac{\left(\frac{1}{t^2}-1\right)dt}{\left(2+t+\frac{1}{t}\right)\sqrt{t+\frac{1}{t}+1}}$$

Put $t + \dfrac{1}{t} + 1 = y^2 \Rightarrow \left(1-\dfrac{1}{t^2}\right)dt = 2y\,dy$

$$= -\int \frac{2y\,dy}{(1+y^2)y} = -\int \frac{2\,dy}{1+y^2}$$

$$= -2\tan^{-1}y + c = -2\tan^{-1}\left(\sqrt{\tan x+\frac{1}{\tan x}+1}\right) + c$$

Hence, the correct answer is (A).

59. Put, $\tan^2 x - x = t$

$\Rightarrow\ (2\tan x\sec^2 x - 1)dx = dt$

$\therefore\ \displaystyle\int \sec t\,dt = \ln|\sec t + \tan t| + c$

Hence, the correct answer is (C).

60. $\displaystyle\int e^x\left(\underbrace{xe^{x^3+\sin x}}_{f(x)} + \underbrace{e^{x^3+\sin x} + x\cos x e^{x^3+\sin x} + 3x^3e^{x^3+\sin x}}_{f'(x)}\right)$

$$= xe^x\cdot e^{x^3+\sin x} + C$$

Hence, the correct answer is (D).

61. $(\tan x)^{1/3} = t$

$$I = \int \frac{2^{1/3} d((\tan x)^{1/3})}{\dfrac{\sin^{2/3} x + \cos^{2/3} x}{\sin^{1/3} x . \cos^{1/3} x}} = 2^{1/3} \int \frac{dt}{t + \dfrac{1}{t}}$$

$$= 2^{1/3} \int \frac{t}{1+t^2} dt = \frac{2^{1/3}}{2} . \ln(1 + t^2) + c$$

Hence, the correct answer is (D).

62. $$I = \int \frac{\left(1 - \dfrac{\sin^2 x}{\cos^2 x}\right)}{\dfrac{1}{\cos^2 x}} \frac{dx}{\sin 2x}$$

$$= \int \cot 2x dx = \frac{\ln|\sin 2x|}{2} + C$$

Hence, the correct answer is (A).

63. $$\int (2 - x \tan x)\sqrt{\cos x}\, dx$$

$$= \int \left(2\sqrt{\cos x} - x \tan x \sqrt{\cos x}\right) dx$$

$$= \int \left(f(x) + x f'(x)\right) dx$$

$$= x f(x) = 2x\sqrt{\cos x} + C$$

Hence, the correct answer is (C).

64. $$\frac{d}{dx}(x \sin x + \cos x) = x \cos x$$

$f'(x) = x \cos x$

Let $f(x) = x \sin x + \cos x$, $f''(x) = -x \sin x + \cos x$

$$\int e^{f(x)} \left(x\ f'(x) + \frac{f''(x)}{(f'(x))^2} \right) dx$$

$$= \int x e^{f(x)} f'(x) dx + e^{f(x)} \frac{f''(x)}{(f'(x))^2} dx$$

$$= x e^{f(x)} - \int e^{f(x)} dx + e^{f(x)} \left(-\frac{1}{f'(x)}\right) - \int e^{f(x)} f'(x) \left(\frac{-1}{f'(x)}\right) dx$$

$$= x e^{f(x)} - \frac{e^{f(x)}}{f'(x)} + c = e^{f(x)} \left(x - \frac{1}{f'(x)} \right) + c$$

Hence, the correct answer is (B).

65. We have, $$\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}} dx$$

$$= \int \frac{x(x^2-1)}{(x^2+1)(x+1)^2 \sqrt{x^3+x^2+x}} dx$$

$$= \int \frac{x^3\left(1 - \dfrac{1}{x^2}\right)}{x^3\left(x + \dfrac{1}{x}\right)\left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2 \sqrt{x + \dfrac{1}{x} + 1}} dx$$

$$= \int \frac{\left(1 - \dfrac{1}{x^2}\right)}{\left(x + \dfrac{1}{x}\right)\left(x + \dfrac{1}{x} + 2\right)\sqrt{x + \dfrac{1}{x} + 1}} dx$$

Put, $x + \dfrac{1}{x} + 1 = t^2 \Rightarrow \left(1 - \dfrac{1}{x^2}\right) dx = 2t\, dt$

$$= \int \frac{2}{(t^2-1)(t^2+1)} dt$$

$$I = \int \frac{2t}{(t^2-1)(t^2+1)\sqrt{t^2}} dt,$$

$$= \int \left(\frac{1}{t^2-1} - \frac{1}{t^2+1} \right) dt = \int \frac{1}{t^2-1} dt - \int \frac{1}{t^2+1} dt$$

$$= \frac{1}{2} \log\left|\frac{t-1}{t+1}\right| - \tan^{-1} t + c$$

$$= \frac{1}{2} \log\left|\frac{\sqrt{x + \dfrac{1}{x} + 1} - 1}{\sqrt{x + \dfrac{1}{x} + 1} + 1}\right| - \tan^{-1} \sqrt{x + \frac{1}{x} + 1} + c$$

Hence, the correct answer is (B).

66. $$\int \frac{\cos^3 x(1 + \cos^2 x)}{\sin^2 x(1 + \sin^2 x)}$$

$$I = \int \frac{\cot^3 x(1 + \cos^2 x)}{\sin^2 x(1 + \sin^2 x)} = \int \frac{\cos x(\cos^2 x + \cos^4 x)}{\sin^2 x + \sin^4 x} dx$$

Put, $\sin x = t \Rightarrow \cos x\, dx = dt$

$$\Rightarrow \quad I = \int \frac{1 - t^2 + (1 - t^2)^2}{t^2 + t^4} dt \Rightarrow I = \int \frac{2 - 3t^2 + t^4}{t^2 + t^4} dt$$

$$\Rightarrow \quad I = \int \left(1 + \frac{2}{t^2} - \frac{6}{t^2 + 1} \right) dt$$ (using partial fractions)

$$\Rightarrow \quad I = t - \frac{2}{t} - 6 \tan^{-1} t + C$$

$$= \sin x - \frac{2}{\sin x} - 6 \tan^{-1}(\sin x) + C$$

Hence, the correct answer is (B).

67. Put, $x^{-4} + 1 = t^2$

$$\Rightarrow \quad -4x^{-5} dx = 2t\ dt \Rightarrow \int -\frac{t(t^2-1)}{t} \frac{dt}{2}$$

$$\frac{t}{2} - \frac{t^3}{6} + C = \frac{\sqrt{x^4+1}(2x^4-1)}{6x^6} + C$$

Hence, the correct answer is (A).

68. $$\int \frac{\cos 9x + \cos 6x}{2\cos 5x - 1} dx = \int \frac{2\cos\left(\dfrac{15x}{2}\right).\cos\left(\dfrac{3x}{2}\right)}{-1 + 2\left(2\cos^2 \dfrac{5x}{2} - 1\right)} dx$$

$$= \int \frac{2\cos\left(\dfrac{15x}{2}\right).\cos\left(\dfrac{3x}{2}\right).\cos\left(\dfrac{5x}{2}\right)}{\left(4\cos^2 \dfrac{5x}{2} - 3\right).\cos\left(\dfrac{5x}{2}\right)}$$

$$= \int \frac{2\left(\cos \dfrac{5x}{2} . \cos \dfrac{3x}{2}\right).\cos\left(\dfrac{15x}{2}\right)}{\cos\left(\dfrac{15x}{2}\right)} dx$$

$$= \int (\cos 4x + \cos x) dx = \frac{1}{4} \sin 4x + \sin x + C$$

So, on comparing we get

$$A = \frac{1}{4}, B = 1 \Rightarrow (A + B) = \frac{1}{4} + 1 = \frac{5}{4}$$

Hence, the correct answer is (C).

69. $$I = \int \frac{1 + x + \sqrt{x + x^2}}{\sqrt{x} + \sqrt{1+x}} dx$$

$$= \int \frac{\sqrt{(1+x)^2} + \sqrt{x}\sqrt{1+x}}{\sqrt{x} + \sqrt{1+x}} dx$$

$$= \int \sqrt{1+x}\ dx = \frac{2}{3}(1+x)^{3/2} + c$$

Hence, the correct answer is (B).

70. $\displaystyle\int \frac{\sec^2 x\, dx}{\tan^{101} x\left(1+(\tan x)^{-100}\right)}$

Let 1 + $(\tan)^{-100} = t$

$-100(\tan x)^{-101} \sec^2 x\, dx = dt$

Let $I + (\tan x)^{-100} = t$

$-100(\tan x)^{-101} \sec^2 x\, dx = dt$

$$-\int \frac{dt}{100t} = -\frac{1}{100} \ln t + c$$

$$= -\frac{1}{100} \ln\left(1+\frac{1}{(\tan x)^{100}}\right) + c$$

$$g(x) = -\frac{1}{100} \ln\left(1+\frac{1}{(\tan x)^{100}}\right)$$

$$\lim_{x\to\frac{\pi}{2}} g(x) = -\frac{1}{100} \ln(1) = 0$$

Hence, the correct answer is (A).

71. $\displaystyle\int\left(\log_e(\log_e x)+\frac{1}{\log_e x}\right)dx\cdot$

Let $\log_e x = t \Rightarrow x = e^t$

$$\int\left(\log_e t+\frac{1}{t}\right) e^t\ dt = e^t \log_e(t)+c = x \log_e(\log_e x) + c$$

Hence, the correct answer is (A).

72. $\displaystyle\int \frac{\sin[7x-2x]}{\cos 7x \cos 2x} dx$

$$= \int \frac{[\sin 7x \cos 2x - \cos 7x \sin 2x]}{\cos 7x \cos 2x} dx$$

$$= \int \tan 7x dx - \int \tan 2x\, dx$$

$$= \frac{1}{7} \ln \sec 7x - \frac{1}{2} \ln \sec 2x = \ln\left(\frac{\cos^{\frac{1}{2}} 2x}{\cos^{\frac{1}{7}} 7x}\right) + C$$

Hence, the correct answer is (D).

73. $\displaystyle I = \int\left(\text{cosec}^{2010} x \sec^2 x - 2010\, \text{co}\sec^{2010} x\right) dx$

$$= \text{cosec}^{2010} x.\tan x + \int 2010.\text{cosec}^{2009} x.\cot x.\text{cosec}\, x.\tan x\ dx$$

$$- \int 2010.\text{cosec}^{2010} x\, dx + c = \frac{\tan x}{\sin^{2010} x} + c$$

$P(x) = \tan x$

$$P\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Hence, the correct answer is (C).

74. $\displaystyle J - I = \int\left(\frac{x+y}{xy} dy - \frac{x+y}{x^2} dx\right)$

$$= \int\left(\frac{x+y}{x^2 y}\right)(xdy - ydx) = \int\left(\frac{x+y}{y}\right)\left(\frac{xdy - ydx}{x^2}\right)$$

$$= \int\left(\frac{x}{y}+1\right) d\left(\frac{y}{x}\right)$$

$$g(x) = \frac{y}{x} + \ln\left(\frac{y}{x}\right) + c$$

Put $x = 1 \Rightarrow 1 + c \Rightarrow c = 0$

$$\therefore\ g(x) = \frac{y}{x} + \ln\left(\frac{y}{x}\right)$$

$g(e) = e + 1$

Hence, the correct answer is (B).

75. $\displaystyle\int \frac{\sqrt{x^2+8x+15}}{(x+4)} \cdot \frac{2(x+4)}{2(x+4)} dx$

$x^2 + 8x + 15 = t^2$

$(2x + 8)dx = 2t\ dt$

$$\frac{1}{2} \int \frac{t(2t)dt}{t^2+1} = \int \frac{t^2}{t^2+1} = t - \tan^{-1} t + c$$

$$= \sqrt{x^2+8x+15} - \tan^{-1} \sqrt{x^2+8x+15} + c$$

Hence, the correct answer is (B).

76. $\displaystyle\int \frac{3+\sqrt{x}}{(x+2\sqrt{x}+3)^2}\ dx \Rightarrow \int \frac{\frac{3}{x^2}+\frac{1}{x\sqrt{x}}}{\left(1+\frac{2}{\sqrt{x}}+\frac{3}{x}\right)^2} dx$

$$1 + \frac{2}{\sqrt{x}} + \frac{3}{x} = t;\ \left(-\frac{1}{x\sqrt{x}} - \frac{3}{x^2}\right) dx = dt$$

$$\int -\frac{dt}{t^2} = \frac{1}{t} + C = \frac{1}{1+\frac{2}{\sqrt{x}}+\frac{3}{x}} + C$$

$$= \frac{x}{(x+2\sqrt{x}+3)} + C$$

Hence, the correct answer is (D).

77. $\displaystyle\int \frac{dx}{(2+\cos x)} = \int \frac{dx}{\left(2+\frac{1-\tan^2 \frac{x}{2}}{1+\tan^2 \frac{x}{2}}\right)} = \int \frac{\sec^2 \frac{x}{2} dx}{\left(3+\tan^2 \frac{x}{2}\right)}$

$$\tan \frac{x}{2} = t, \frac{1}{2} \sec^2 \frac{x}{2} dx = dt = \sec^2 \frac{x}{2} dx = 2dt$$

$$= 2\int \frac{dt}{\left(t^2+\left(\sqrt{3}\right)^2\right)}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan \frac{x}{2}}{\sqrt{3}}\right) + c$$

Hence, the correct answer is (C).

78. $\displaystyle -\int e^{\tan^{-1} x}. \frac{(1+x+x^2)}{1+x^2} dx$

Put $\tan^{-1} x = t$ $\displaystyle = -\int e^t[\sec^2 t + \tan t]dt = -e^t \tan t + c$

Hence, the correct answer is (C).

79. $\displaystyle\int \frac{\cos x - \sin x}{e^{-x} + \cos x} dx = \int \frac{e^x(\cos x - \sin x)}{1 + e^x \cos x} dx$

Put $1 + e^x \cos x = t$

$(-e^x \sin x + \cos x.\ e^x)\ dx = dt$

$$\int \frac{dt}{t} = \ln |\ t\ | + c = \ln |1 + e^x \cos x)| + C$$

$$= \ln |e^x(e^x + \cos x)| + C$$

$$f(x) = x + \ln\,|e^{-x} + \cos x| + C$$
$$f(0) = \ln 2 + C \Rightarrow C = 0$$

Hence, the correct answer is (B).

80. Let $I = \int \underset{\text{I}}{e^{\tan x}}\,(\underset{\text{II}}{3x^2e^{x^3} + 2xe^{x^2}})\,dx + \int e^{\tan x}\cdot\sec^2 x(e^{x^3} + e^{x^2})\,dx$

Integrating by parts.

$$I = e^{\tan x}(e^{x^3} + e^{x^2}) - \int e^{\tan x}\cdot\sec^2 x(e^{x^3} + e^{x^2})\,dx + \int e^{\tan x}\sec^2 x(e^{x^3} + e^{x^2})\,dx$$
$$= e^{\tan x}\left(e^{x^3} + e^{x^2}\right) + c$$

Hence, the correct answer is (C).

81. $\int\left(\sqrt{1+\sin x} + \sqrt{1-\sin x}\right)dx$

$$= \int\left(\left|\sin\frac{x}{2} + \cos\frac{x}{2}\right| + \left|\sin\frac{x}{2} - \cos\frac{x}{2}\right|\right)$$
$$= \int 2\cos\frac{x}{2}\,dx \qquad \left\{\cos\frac{x}{2} > \sin\frac{x}{2}\right\}$$
$$= 4\sin\frac{x}{2} + C$$

Hence, the correct answer is (B).

82. $f'(x)\cdot f'(x) - f(x)\cdot f''(x) = 0$ or $\dfrac{(f'(x))^2 - f(x)f''(x)}{(f'(x))^2} = 0$

$$\frac{d}{dx}\left[\frac{f(x)}{f'(x)}\right] = 0$$

Integrating, $\dfrac{f(x)}{f'(x)} = C$

Put, $x = 0$, $\dfrac{f(0)}{f'(0)} = C \Rightarrow C = \dfrac{1}{2}$, hence $\dfrac{f(x)}{f'(x)} = \dfrac{1}{2}$

From (1) $2f(x) = f'(x)$

$$\therefore \quad \frac{f'(x)}{f(x)} = 2$$

Again integrating, $\ln[f(x)] = 2x + k$

Put, $x = 0$ to get, $k = 0$

$$f(x) = e^{2x}$$

Hence, the correct answer is (C).

83. $\int(\sec^{2005} x\,\text{cosec}^2 x - 2005\sec^{2005} x)\,dx$

$$= \sec^{2005}x(-\cot x) - \int 2005\sec^{2004} x.\sec x\tan x.(-\cot x)\,dx - \int 2005\sec^{2004} x\,dx + C$$
$$= -\sec^{2005}x\cot x + \int 2005\sec^{2005} x\,dx - \int 2005\sec^{2005} x\,dx + C$$
$$= -\sec^{2005}x\cot x + C$$

Hence, the correct answer is (D).

84. $f(x) = \int\dfrac{x^2dx}{(1+x^2)(1+\sqrt{1+x^2})}$, $f(0) = 0$

$$= \int\frac{x^2\left(\sqrt{1+x^2} - 1\right)}{(1+x^2).x^2}\,dx = \int\left(\frac{1}{\sqrt{1+x^2}} - \frac{1}{1+x^2}\right)dx$$
$$f(x) = \ln\left|x + \sqrt{1+x^2}\right| - \tan^{-1}x + C$$
$$f(0) = 0 \Rightarrow C = 0$$
$$f(x) = \ln\left|x + \sqrt{1+x^2}\right| - \tan^{-1}x$$
$$f(1) = \ln(1 + \sqrt{2}) - \frac{\pi}{4}$$

Hence, the correct answer is (B).

85. $\dfrac{\cos x(1+2\sin x)}{1+\sin x} - \dfrac{\cos^2 x - \sin^2 x}{\cos x}$

$$= \frac{\cos^2 x(1+\sin x)(\cos^2 x - \sin^2 x)}{\cos x(1+\sin x)}$$
$$= \frac{\sin x\cos^2 x + \sin^3 x + \sin^2 x}{\cos x(1+\sin x)}$$
$$= \frac{\sin x(\cos^2 x + \sin x) + \sin^2 x}{\cos x(1+\sin x)} = \tan x$$

Hence, the correct answer is (A).

86. $I = \int\dfrac{dx}{x\sqrt{1-x^3}} = \int\dfrac{x^2dx}{x^3\sqrt{1-x^3}}$

Put, $1 - x^3 = t^2 \Rightarrow -3x^2dx = 2t\,dt$

$$I = -\int\frac{2t\,dt}{3(1-t^2)t} = \frac{2}{3}\int\frac{dt}{t^2-1}$$
$$= \frac{2}{3}\cdot\frac{1}{2}\ln\left(\frac{t-1}{t+1}\right) + C = \frac{1}{3}\ln\left(\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right) + C$$

Hence, the correct answer is (B).

87. Multiply and divide by x^8

$$I = \int\frac{x^{10} + 20x^8dx}{\left(x^5\sin x + 5x^4\cos x\right)^2}$$
$$= \int\frac{\left(x^5\sec x\right)\left(x^5\cos x + 20x^3\cos x\right)}{\left(x^5\sin x + 5x^4\cos x\right)^2}$$
$$= \left(x^5\sec x\right)\left(\frac{-1}{x^5\sin x + 5x^4\cos x}\right) + \int\frac{\left(5x^4\sec x + x^5\sec x\tan x\right)dx}{\left(x^5\sin x + 5x^4\cos x\right)}$$
$$= \frac{-x}{\cos x\left(x\sin x + 5\cos x\right)} + \int\sec^2 x dx$$
$$= -\frac{x}{\cos x\left(x\sin x + 5\cos x\right)} + \tan x + c$$

Hence, the correct answer is (A).

88. Let $x = \sin\theta$; $dx = \cos\theta d\theta$

$$\int\frac{(\cos\theta - \sin\theta)\cos\theta\,d\theta}{\cos\theta(1+\sin\theta\cos\theta)}$$
$$= \int\frac{2(\cos\theta - \sin\theta)}{2+\sin 2\theta}\,d\theta = \int\frac{2(\cos\theta - \sin\theta)}{1+(\sin\theta+\cos\theta)^2}\,d\theta$$

$\sin\theta + \cos\theta = t \Rightarrow (\cos\theta - \sin\theta)\,d\theta = dt$

$$= 2\int\frac{dt}{1+t^2}$$
$$= 2\tan^{-1}t + c$$
$$= 2\tan^{-1}\left(x + \sqrt{1-x^2}\right) + c$$

Hence, the correct answer is (A).

89. $I = \int\dfrac{dx}{x(x^{2007}+1)} = \int\dfrac{x^{2007} + 1 - x^{2007}}{x(x^{2007}+1)}\,dx$

$$= \int\left(\frac{1}{x} - \frac{x^{2006}}{1+x^{2007}}\right)dx = \ln x - \frac{1}{2007}\ln(1 + x^{2007})$$

$$= \frac{\ell n x^{2007} - \ell n(1+x^{2007})}{2007} = \frac{1}{2007} \ln\left(\frac{x^{2007}}{1+x^{2007}}\right) + C$$

$p + q + r = 6021$

Alternate:

$$I = \int \frac{dx}{x^{2008}[1+x^{-2007}]} = \int \frac{x^{-2008}dx}{1+x^{-2007}}$$

Hence, the correct answer is (C).

90. $I = \int \left(\sqrt{\sin x} + \sqrt{\cos x}\right)^{-4} dx = \int \frac{dx}{\left[\sqrt{\cos x}\left(\sqrt{\tan x}+1\right)\right]^4}$

$$= \int \frac{dx}{\cos^2 x\left(\sqrt{\tan x}+1\right)^4}$$

$$= \int \frac{\sec^2 x dx}{\left(\sqrt{\tan x}+1\right)^4}$$

Put $\sqrt{\tan}\ x + 1 = y \Rightarrow \frac{1}{2\sqrt{\tan x}} \sec^2 x\, dx$

$= 2\sqrt{\tan}\ x dy = 2(y-1)dy$

$$I = \int \frac{1}{y^4} \,.\, 2(y-1)\, dy = 2\int\left(\frac{1}{y^3} - \frac{1}{y^4}\right) dy$$

$$= 2\left[\frac{-1}{2y^2} + \frac{1}{3y^3}\right] + c = \frac{2}{y^2}\left[\frac{1}{3y} - \frac{1}{2}\right]$$

$$= \frac{2}{\left(\sqrt{\tan x}+1\right)^2}\left[\frac{1}{3\left(\sqrt{\tan x}+1\right)} - \frac{1}{2}\right] + c$$

Hence, the correct answer is (A).

91. $I = \int \left(\frac{x-1}{x+1}\right) \frac{dx}{x\sqrt{x+1+\frac{1}{x}}}$

So, $I = \int \frac{(x-1)dx}{(x+1)x\sqrt{x+1+\frac{1}{x}}} = \int \frac{\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x}\right)dx}{(x+1)\left(1+\frac{1}{x}\right)\sqrt{x+1+\frac{1}{x}}}$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(1+\frac{1}{x}+2\right)\sqrt{x+\frac{1}{x}+1}}$$

Put, $x + 1 + \frac{1}{x} = t^2$

$\left(1-\frac{1}{x^2}\right)dx = 2t\, dt$

$$= \frac{2t\, dt}{(t^2+1)t} = 2\tan^{-1}t + c = 2\tan^{-1}\left(\sqrt{x+\frac{1}{x}+1}\right) + c$$

$f(x) = x + \frac{1}{x} + 1$

Hence, the correct answer is (A).

92. We have, $\int x\sqrt{\frac{2\sin(x^2+1) - \sin 2(x^2+1)}{2\sin(x^2+1) + \sin 2(x^2+1)}}\, dx$

$$= \int x\sqrt{\frac{2\sin(x^2+1) - \sin 2(x^2+1)\cos(x^2+1)}{2\sin(x^2+1) + 2\sin(x^2+1)\cos(x^2+1)}}\, dx$$

$$= \int x\sqrt{\frac{1-\cos(x^2+1)}{1+\cos(x^2+1)}}\, dx = \int x\tan\left(\frac{x^2+1}{2}\right)dx$$

$$= \int \tan\left(\frac{x^2+1}{2}\right) d\left(\frac{x^2+1}{2}\right)$$

$$= \log\left|\sec\left(\frac{x^2+1}{2}\right)\right| + C$$

Hence, the correct answer is (B).

93. Note that $\sec^{-1}\sqrt{1+x^2} = \tan^{-1}x$; $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$= 2\tan^{-1}x$ for $x > 0$

$$I = \int \frac{e^{\tan^{-1} x}}{1+x^2}\left((\tan^{-1} x)^2 + 2\tan^{-1} x\right) dx$$

Put, $\tan^{-1} x = t$

$$= \int e^t(t^2+2t)\, dt = e^t \,.\, t^2 = e^{\tan^{-1} x}(\tan^{-1}x)^2 + C$$

Hence, the correct answer is (C).

94. $y(x+y)^2 = x \Rightarrow y'(x+y)^2\, y \,.\, 2(x+y) \,.\, (1+y') = 1$

$$\Rightarrow \frac{dy}{dx} = \frac{1-2y(x+y)}{(x+y)(x+3y)} \Rightarrow 1 + \frac{dy}{dx} = \frac{(x+y)^2+1}{(x+y)(x+3y)}$$

$$\frac{(x+y)(dx+dy)}{(x+y)^2+1} = \frac{dx}{x+3y}$$

$$\therefore \int \frac{dx}{x+3y} = \frac{1}{2}\ln((x+y)^2+1) + c$$

Hence, the correct answer is (D).

95. Put, $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$I = \int\left(\frac{1+\sqrt{t}}{2t}\right)dt = \frac{1}{2}[\ln t + 2\sqrt{t}\ ] + C$$

$$= \frac{1}{2}\left[\ln\tan x + 2\sqrt{\tan x}\right] + C$$

Hence, the correct answer is (A).

96. Put $t = \frac{x-3}{x+1} \Rightarrow dx = \frac{(x+1)^2 dt}{4}$

$$= \int \frac{dx}{\left(\frac{x-3}{x-1}\right)^{4/5}(x+1)^2} = \frac{1}{4}\int\frac{dt}{t^{4/5}} = \frac{5}{4}t^{1/5}$$

$$= \frac{5}{4}\left(\frac{x-3}{x-1}\right)^{1/5} + C$$

Hence, the correct answer is (B).

97. $M(x) = \int \frac{e^x(e^{2x}-2)}{e^{4x}+8e^{2x}+4}\, dx$; Put $e^x = t$

$$= \int \frac{(t^2-2)dt}{t^4+8t^2+4}$$

$$= \int \frac{\left(1-\frac{2}{t^2}\right)dt}{t^2+\frac{4}{t^2}+8} = \int \frac{\left(1-\frac{2}{t^2}\right)dt}{\left(t+\frac{2}{t}\right)^2+2^2}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{t+2/t}{2}\right) + c$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{e^x+2e^{-x}}{2}\right) + c$$

Hence, the correct answer is (A).

98. $$\int \frac{2-3\sin^2 x}{\sqrt{\cos x}}\,dx = \int \frac{2\cos^2 x-\sin^2 x}{\sqrt{\cos x}}\,dx$$

$$= 2\int (\cos x)\sqrt{\cos x}\,dx - \int \frac{\sin^2 x}{\sqrt{\cos x}}\,dx \qquad \text{(Using I.B.P.)}$$

$$= 2\sqrt{\cos x}\ \sin x + 2\int \frac{\sin^2 x}{2\sqrt{\cos x}}\,dx - \int \frac{\sin^2 x}{\sqrt{\cos x}}\,dx + C$$

$$= 2\sin x\sqrt{\cos x}\ + c$$

$\Rightarrow$ $f(x) = \sin x$, $g(x) = \cos x$

Hence, the correct answer is (A).

99. $$\int \frac{x}{(1+x^2)^2\left(\frac{2+x^2}{1+x^2}\right)^{\frac{3012}{2021}}}\,dx$$

$$I = -\int \frac{dt}{2t^{\frac{3012}{2012}}}$$

Let $\frac{2+x2}{1+x^2} = t$ then $\frac{(1+x^2)2x-(2+x^2)2x}{(1+x^2)^2}\,dx = dt$

$\Rightarrow$ $\frac{-2x}{(1+x^2)^2}\,dx = dt$

$$I = -\frac{1}{2}\left[\frac{t^{-\frac{3012}{2012}+1}}{-\frac{3012}{2012}+1}\right] + C;\ I = \frac{1}{2}\left(\frac{1+x^2}{2+x^2}\right)^{\frac{1000}{2012}}\frac{2012}{1000} + C$$

$$= \frac{1006}{1000}\left(\frac{1+x^2}{2+x^2}\right)^{\frac{1000}{2012}} + C \Rightarrow \alpha = 503,\ \beta = 500$$

$$f(x) = \frac{1}{2+x^2} \Rightarrow f(1) = \frac{1}{3}$$

Hence, the correct answer is (C).

100. On solving $\frac{\sin\theta}{\cos 3\theta} = \frac{1}{2}[\tan 3\theta - \tan\theta]$

$$= \frac{1}{2}\int (\tan 27\theta - \tan\theta)\,d\theta$$

$$\frac{\sin 3\theta}{\cos 9\theta} = \frac{1}{2}[\tan 9\theta - \tan 3\theta] = \frac{1}{2}\left[\frac{\ln\sec 27\theta}{27} - \ln\sec\theta\right] + C$$

$$\frac{\sin 9\theta}{\cos 27\theta} = \frac{1}{2}[\tan 27\theta - \tan 9\theta] = \frac{1}{2}\left[\ln\left\{\frac{\sec(27\theta)^{1/27}}{\sec\theta}\right\}\right]$$

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1. $$I = \int \sin 6x\,dx = -\frac{1}{6}\cos 6x + c$$

$$= -\frac{1}{6}(1-2\sin^2 3x) = \frac{1}{3}\sin^2 3x + C \qquad \text{option (C) is correct}$$

$$= -\frac{1}{6}(2\cos^2 3x - 1) = -\frac{1}{3}\cos^2 3x + C \qquad \text{option (B) is correct}$$

$$\frac{1}{3}\sin\left(3x+\frac{\pi}{7}\right)\sin\left(3x-\frac{\pi}{7}\right) + c$$

$$= \frac{1}{3}\left(\sin^2 3x - \sin^2\frac{\pi}{7}\right) + c \qquad \text{(D) is also correct}$$

Hence, (B), (C) and (D) are correct.

2. $$f(x) = \lim_{n\to\infty} e^{x\tan(1/n)\ln(1/n)}$$

$$f(x) = e^{\lim\limits_{n\to\infty}\frac{x\ln 1/n}{\cot 1/n}} = e^{\lim\limits_{n\to\infty}\frac{x\,n(-1/n^2)}{-\operatorname{cosec}^2(1/n)(-1/n^2)}}$$

$$f(x) = e^{\lim\limits_{n\to\infty} x\frac{\sin(1/n)}{1/n}\sin(1/n)} = e^{x.1.0} = 1$$

$$\int \frac{f(x)}{\sqrt[3]{(\sin^{11} x\cos x)}}\,dx = \int \frac{dx}{\sin^{11/3} x\cos^{1/3} x} = \int \frac{\sec^4 x}{\tan^{11/3} x}\,dx$$

$$I = \int \frac{(1+\tan^2 x)\sec^2 x}{\tan^{11/3} x}\,dx$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$I = \int \frac{(1+t^2)}{t^{11/3}}\,dt = \int t^{-11/3} + t^{-5/3}\,dt$$

$$= -\frac{3}{8}t^{-8/3} - \frac{3}{2}t^{-2/3} + C$$

$$I = \frac{-3}{8(\tan x)^{8/3}} - \frac{3}{2(\tan x)^{2/3}} + C$$

$$g(x) = \frac{-3}{8(\tan x)^{8/3}} - \frac{3}{2(\tan x)^{2/3}} + C$$

$$g\left(\frac{\pi}{4}\right) = -\frac{3}{8} - \frac{3}{2} = -\frac{15}{8}$$

$g(x)$ is discontinuous and non-differentiable at $x = n\pi$

Hence, (C) and (D) are correct.

3. $$I = \int \sin(\ln x)\,dx$$

Put $\ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$$I = \int e^t \sin t\,dt = \sin t\cdot e^t - \int e^t\cos t\,dt$$

$$I = e^t\sin t - \left[e^t\cos t + \int e^t\sin t\,dt\right]$$

$$2I = (\sin t - \cos t)e^t\ + c$$

$$I = \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C = f(x)\{\sin g(x) - \cos h(x)\} + c$$

$$f(x) = \frac{x}{2};\ g(x) = h(x) = \ln x$$

(A) $\lim\limits_{x\to 2} f(x) = 1$ (B) $\lim\limits_{x\to 1}\frac{g(x)}{h(x)} = 1$

(C) $g(e^3) = \ln e^3 = 3$ (D) $n(e^s) = \ln e^5 = 5$

Hence, (A), (B), (C) and (D) are correct.

4. $$J + K = \int \frac{1+\sin x+\cos x}{1+\sin x+\cos x}\,dx = x + c$$

$$J - K = \int \frac{\sin^2 x + \sin x - \cos^2 x - \cos x}{1+\sin x+\cos x}\,dx$$

$$= \int \frac{(\sin x - \cos x)(1+\sin x+\cos x)}{1+\sin x+\cos x}\,dx$$

$$= \int (\sin x - \cos x)\,dx = -\cos x - \sin x + C$$

$$2J = x - \sin x - \cos x + C$$

$$2K = x + \cos x + \sin x + C$$

Hence, (B) and (C) are correct.

5. Let $f(x) = \tan x + 2\tan 2x + 4\tan 4x + 8\cot 8x$

Using formula, $\tan x - \cot x = -2\cot 2x$

$\therefore$ $f(x)\cot x + \{\tan x - \cot x + 2\tan 2x + 4\tan^4 x + 8\cot 8x\}$

$= \cot x + \{-2\cot 2x + 2\tan 2x + 4\tan 4x + 8\cot 8x\}$

$= \cot x + \{-4\cot 4x + 4\tan 4x + 8\cot 8x\}$

$$= \cot x + \{-8\cot 8x + 8\cot 8x\}$$

$$= \cot x$$

$$\therefore \int \cot x\,dx = \ln(\sin x) + C$$

Hence, (A), (B) and (D) are correct.

6. $$\int\left(1+\tan\frac{x}{2}\cdot\tan x\right)dx = \int\frac{\cos\frac{x}{2}\cos x+\sin\frac{x}{2}\sin x}{\cos x\cos\frac{x}{2}}\,dx$$

$$= \int \sec x\,dx = -\ln|\sec x - \tan x| + C$$

$$\int\left(\tan x\cdot\cot\frac{x}{2}-1\right) = \int\frac{\sin\frac{x}{2}dx}{\cos x\sin\frac{x}{2}}$$

$$= \int \sec x\,dx$$

$$= \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| = -\ln\left|\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right| + C$$

Hence, (A), (B), (C) and (D) are correct.

7. Put, $x = t^6$

$$\therefore\quad I = 6\int\frac{t^6+t^4+t}{t(1+t^2)}\,dt = 6\int t^3dt + \int\frac{dt}{1+t^2}$$

$$= \frac{3}{2}x^{2/3} + 6\tan^{-1}(x^{1/6}) + \lambda$$

$$\therefore\quad \alpha = \frac{3}{2},\ \beta = \frac{2}{3},\ \gamma = 6,\ \delta = \frac{1}{6}$$

Sum of all coefficients of even power of $x = \frac{(1+6)^6+(1-6)^6}{2}$

Hence, (B), (C) and (D) are correct.

8. $f(x) = \int\frac{e^{\sin x}}{2\sin^3 x}(\sin x - 2)\cos x\,dx$ put $\sin x = t \Rightarrow \cos x\,dx = dt$

$$= \int\frac{e^t}{2t^3}(t-2)\,dt = \int e^t\left(\frac{e^t}{2t^2}-\frac{e^t}{t^3}\right)dt = \frac{e^t}{2t^2}+C$$

$$\Rightarrow\quad f(x) = \frac{e^{\sin x}}{2\sin^2 x} + C$$

$$f\left(\frac{\pi}{2}\right) = \frac{e}{2} \Rightarrow \frac{e}{2} = \frac{e}{2} + C \Rightarrow C = 0$$

$$\lim_{x\to 0}\frac{2\sin^2 x(f(x))-1}{\tan\left(\frac{x}{2}\right)}$$

$$= \lim_{x\to 0}\frac{2\sin^2 x\frac{e^{\sin x}}{2\sin^2 x}-1}{\tan\frac{x}{2}} = \lim_{x\to 0}\frac{e^{\sin x}-1}{\sin x}\frac{\sin x}{\tan\frac{x}{2}} = 2$$

$g'(x) = f(x) = \frac{e^{\sin x}}{2\sin^2 x} > 0$, $g(x)$ is injective

$$\lim_{x\to 0} x\,f(x) = \lim_{x\to 0}\frac{xe^{\sin x}}{2\sin^2 x} = \lim_{x\to 0}\frac{e^{\sin x}}{2}\left(\frac{x}{\sin x}\right)^2\frac{1}{x} = \text{DNE}$$

Hence, (A), (B) and (C) are correct.

9. $$I = \int\frac{(3\tan x - \tan 3x)dx}{\tan x\tan 3x(\tan x - 3\tan 3x)}$$

$$I = \int\frac{3\tan x - \left(\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)}{\tan x\left(\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)\left(\tan x - 3\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}\right)}$$

$$I = \int\frac{(1-3\tan^2 x)(-8\tan^3 x)dx}{\tan^2 x(3-\tan^2 x)(-8\tan x)}$$

Simplifying gives $I = \int\frac{(1-3\tan^2 x)\sec^2 x}{(3-\tan^2 x)(1+\tan^2 x)}dx$ put $\tan x = t$

$$\Rightarrow\quad I = \int\frac{1-3t^2}{(3-t^2)(1+t^2)}\,dt$$

$$= \int\left(\frac{1}{1+t^2}-\frac{2}{3-t^2}\right)dt$$

$$= \tan^{-1}t + \frac{1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + c$$

Hence, (A) and (D) are correct.

10. $$\int\frac{\left(1+\frac{3}{x^2}\right)dx}{\left(x-\frac{3}{x}\right)^2+x-\frac{3}{x}+5} = \int\frac{dy}{y^2+y+5}$$ where $y = x - \frac{3}{x}$

$$= \int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\left(\frac{\sqrt{19}}{2}\right)^2} = \frac{2}{\sqrt{19}}\tan^{-1}\left(\frac{2y+1}{\sqrt{19}}\right)+C$$

$$= \frac{2}{\sqrt{19}}\tan^{-1}\left(\frac{2(x+2)\left(x-\frac{3}{2}\right)}{\sqrt{19}x}\right)+C$$

$\alpha = \frac{3}{2}$ and $\beta = -2$ and $m = 2$ and $n = 19$

Hence, (A), (C) and (D) are correct.

11. $f(x) = x^2 + x\sec x$

(A) $\lim_{x\to 0}\frac{x^2+x\sec x}{x} = 1$

(B) $\lim_{x\to\frac{\pi}{2}}\frac{x\sec x(\pi-2x)}{x} = \lim_{x\to\frac{\pi}{2}}\frac{2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)} = 2$

(D) $\lim_{x\to 0}\frac{f(x)\cos x - x}{x} = \lim_{x\to 0}\frac{x^2\cos x}{x} = 0$

Hence, (A) and (B) are correct.

12. $$I = \int\frac{x\cos x+1}{x\sqrt{2xe^{\sin x}+1}}\,dx$$

Put $2xe^{\sin x}+1 = t^2 \Rightarrow 2e^{\sin x}(x\cos x+1)\,dx = 2tdt$

$$I = \int\frac{2tdt}{(t^2-1)t} = \log\left|\frac{t-1}{t+1}\right| + c$$

Hence, (A) and (B) are correct.

13. $$\int\ln(\ln x)dx + \int(\ln x)^{-2}dx$$

$$= x.\ln(\ln x) - \int\frac{x}{\ln x}.\frac{1}{x}dx + \int(\ln x)^{-2}dx$$

$$= x.\ln\ln x - \left[\int\left(\frac{1}{\ln x}-\frac{1}{(\ln x)^2}\right)dx\right]$$

$$= x\ln(\ln x) - x(\ln x)^{-1} + C$$

Hence, (A) and (C) are correct.

14. $$\int\frac{\cos^{-1}\frac{x}{a}}{x^2}dx$$

Let $\cos^{-1}\frac{x}{a} = \theta$; $dx = -a\sin\theta d\theta$

$$= -\int \frac{\theta}{a^2\cos^2\theta}(a\sin\theta d\theta);$$

$$= -\frac{1}{a}\int \theta\tan\theta.\sec\theta d\theta \ = -\frac{1}{a}\left[\theta.\sec\theta - \int \sec\theta d\theta\right]$$

$$= -\frac{1}{a}\left[\cos^{-1}\frac{x}{2}.\frac{a}{x} - \ln\left|\frac{a}{x} - \frac{\sqrt{a^2-x^2}}{x}\right|\right]$$

$$= \frac{1}{a}\ln\left(a+\sqrt{a^2-x^2}\right) - \frac{1}{a}\ln x - \frac{1}{x}\cos^{-1}\frac{x}{a} + C$$

Hence, (A) and (B) are correct.

15. $I = \int \frac{x\cos x + \sin x}{\sqrt{1-(x\sin x)^2}}dx = \sin^{-1}(x\sin x) + c$

$\therefore$ $f(x) = \sin^{-1}(x \sin x) \Rightarrow f(1) = \sin^{-1}(\sin 1) = 1$

Again $\lim\limits_{x\to 0}\frac{f(x)}{|x|\tan x} = 1$

(one-sided limit $\therefore$ cosec $x > 0$).

Hence, (B), (C) and (D) are correct.

16. $\int \frac{x^2\sin^2 x + x^2\cos^2 x + x\sin x\cos x - x\sin x\cos x}{x^2\sin 2x + 2x(\cos^2 x - \sin^2 x) - \sin 2x}$

$$= \frac{1}{2}\frac{x\sin x(x\sin x + \cos x) + x\cos x(x\cos x - \sin x)}{x^2\sin x\cos x + x\cos^2 x - \sin^2 x - \sin x\cos x}$$

$$= \frac{1}{2}\int \frac{x\sin x(x\sin x + \cos x) + x\cos x(x\cos x - \sin x)}{x\sin x(x\cos x - \sin x) + \cos x(x\cos x - \sin x)}$$

$$= \frac{1}{2}\int \frac{x\sin x(x\sin x + \cos x) + x\cos x(x\cos x - \sin x)}{(x\sin x + \cos x)(x\cos x - \sin x)}$$

$$= \frac{1}{2}\int \frac{x\sin x}{x\cos x - \sin x}dx + \frac{1}{2}\int \frac{x\cos x}{x\sin x + \cos x}dx$$

$\because$ $\frac{d}{dx}(x\cos x - \sin x)$

$= -x \sin x$ and $\frac{d}{dx}(x\sin x + \cos x) = x\cos x$

$$= \frac{-1}{2}\ln|x\cos x - \sin x| + \frac{1}{2}\ln|x\sin x + \cos x| + C$$

$$= \frac{1}{2}\ln\left|\frac{x\sin x + \cos x}{x\cos x - \sin x}\right| + C$$

Hence, (C) and (D) are correct.

17. $I_n = \int \frac{dx}{x^{n+1}(1+x^{-n})}$

$1 + x^{-n} = u$

$-nx^{-n-1}\,dx = du$

$\frac{-n}{x^{n+1}}dx = du$

$I_n = \frac{-1}{n}\int\frac{du}{u}$

$= \ln|x| - \frac{1}{n}\ln|1+x^n| + c$

Hence, (A), (B), (C) and (D) are correct.

18. Put, $t = k^2 \Rightarrow I = \int \frac{2kdk}{(1+k)^8}$

$$= \int \frac{(2k+2)-2}{(1+k)^8}dk = \frac{2}{7(1+\sqrt{t})^7} - \frac{1}{3(1+\sqrt{t})^6}$$

$\therefore$ $p_2 = 7, p_1 = 6$

Hence, (B) and (C) are correct.

19. $\int \frac{3e^x + 5e^{-x}}{4e^x - 5e^{-x}}dx$

$(3e^x + 5e^{-x}) = P(4e^x - 5e^{-x}) + Q(4e^x + 5e^{-x})$

$4P + 4Q = 3$ and $5Q - 5P = 5 \Rightarrow P = \frac{-1}{8}, Q = \frac{7}{8}$

$$\int\left(\frac{-1}{8} + \left(\frac{7}{8}\right)\frac{4e^x + 5e^{-x}}{4e^x - 5e^{-x}}\right)dx$$

$$= -\frac{x}{8} + \frac{7}{8}\ln|4e^x - 5e^{-x}| + C$$

$$= -\frac{x}{8} + \frac{7}{8}\ln|4e^{2x} - 5| - \frac{7}{8}x + C$$

$$= -x + \frac{7}{8}\ln|4e^{2x} - 5| + C\ ;\ A = -1,\ B = \frac{7}{8}$$

C is an arbitrary constant .

Hence, (B), (C) and (D) are correct.

20. $\frac{f'(x)}{f(x)} = 2 \Rightarrow \ln(f(x)) = 2x + C$

At $x = 0$, $f(0) = 3$

$\therefore$ $C = \ln 3$

$\Rightarrow$ $f(x) = 3e^{2x}$

Now $\int \frac{3e^{2x}}{12 + 12e^{2x}}dx = \frac{1}{4}\int\frac{e^{2x}}{1+e^{2x}}dx$

$= \frac{1}{8}\ln(1+e^{2x}) + C \Rightarrow k = 8, a = 1, b = 2$

Hence, (A), (B) and (C) are correct.

21. $I = \int e^{x\sin x + \cos x}\left(x^2\cos x + \frac{\cos x - x\sin x}{(x\cos x)^2}\right)dx$

$$= \int x(e^{x\sin x + \cos x}.x\cos x)dx + \int e^{x\sin x + \cos x}d\left(-\frac{1}{x\cos x}\right)$$

$$= x.e^{x\sin x + \cos x} - e^{x\sin x + \cos x}\frac{1}{x\cos x} + c$$

Hence, (A) and (D) are correct.

22. $I = \int \frac{-2\sin\frac{15x}{2}\sin\frac{x}{2}}{3 - 4\sin^2\frac{5x}{2}}dx = \int -2\sin\frac{5x}{2}\sin\frac{x}{2}dx = \int(\cos 3x - \cos 2x)dx$

$I = \int \frac{-2\sin\frac{15x}{2}\sin\frac{x}{2}}{3 - 4\sin^2\frac{5x}{2}}dx = \int -2\sin\frac{5x}{2}\sin\frac{x}{2}dx = \int(\cos 3x - \cos 2x)dx$

$$= \frac{\sin 3x}{3} - \frac{\sin 2x}{2} + c$$

Hence, (B) and (C) are correct.

23. $I = \int e^{-5t}\sin 4t\frac{dt}{2}(t = x^2)$

$$= \frac{e^{-5t}}{82}(-5\sin 4t - 4\cos 4t) + c$$

$$= -\frac{e^{-5x^2}}{82}(5\sin 4x^2 + 4\cos 4x^2) + c$$

Hence, (A), (C) and (D) are correct.

24. $f''(x) = f(x)$

$f''(x)f'(x) = f'(x)f(x)$

$$\Rightarrow \int f''(x)f'(x)dx = \int f'(x)\, f(x)dx$$

$$\Rightarrow (f'(x))^2 = (f(x))^2$$

$$\Rightarrow f'(x) = f(x) \text{ or } f'(x) = -f(x) \quad (\because f(0) = f'(0) = 0)$$

$$\Rightarrow \int \frac{f(x)}{4-f(x)} = \int \frac{\pm f'(x)}{4-f(x)}dx$$

$$= \log_e(4-f(x)) + c \text{ or } -\log_e(4-f(x)) + c$$

Hence, (A) and (B) are correct.

25. $I = \int \frac{dx}{x^{2011}(1+x^{-2010})^{1/2010}} = \int \frac{x^{-2011}dx}{(1+x^{-2010})^{\frac{1}{2010}}}$

Put, $1 + x^{-2010} = t^{2010}$

$$-2010\, x^{-2011}\, dx = 2010\, t^{2009}\, dt$$

$$= -\int \frac{t^{2009}}{t} dt = -\frac{t^{2009}}{2009} + k = -\frac{1}{2009}(1+x^{-2010})^{\frac{2011-2}{2010}} + k$$

$a = 2009, b = 2010, c = 2011$

Hence, (A) and (B) are correct.

26. $\int x^5 \cos^2 \log \sqrt{x} dx$

$$\int x^5 \cos^2 \frac{\log x}{2} dx = \int x^5 \left(\frac{1+\cos\log x}{2}\right)dx$$

$$= \frac{1}{2}\int x^5 dx + \frac{1}{2}\int x^5 \cos\log x\ dx$$

$$= \frac{x^6}{12} + \frac{1}{2}\int e^{6t}\cos t dt \text{, where } t = \log x$$

$$I = \int e^{6t}\cot t\, dt = \cos t\frac{e^{6t}}{6} + \int \sin t\frac{e^{6t}}{6}dt$$

$$= \cos t\frac{e^{6t}}{6} + \sin t\frac{e^{6t}}{36} - \int \cos t\frac{e^{6t}}{36}dt$$

$$I = \frac{e^{6t}}{36}(6\cos t + \sin t) - \frac{1}{36}I$$

$\therefore \quad I = \frac{36}{37}\frac{e^{6t}}{36}$ (6 cos t + sin t)

$\therefore$ The given integral $= \frac{x^6}{12} + \frac{x^6}{74}$ (6 cos log x + sin log x) + c

If you replace c by $C - \frac{1}{12}$, we get option (D).

Hence, (A) and (D) are correct.

27. $x + \sqrt{x^2+2} = t \Rightarrow x^2 + 2 = t^2 + x^2 - 2tx$

$$\Rightarrow x = \frac{1}{2}\left(t - \frac{2}{t}\right)$$

So, $\int \sqrt{x + \sqrt{x^2+2}}\, dx = \frac{1}{2}\int t^{1/2}\left(1 + \frac{2}{t^2}\right)dt$

$$I = \frac{1}{2}\int t^{1/2}\, dt + \int t^{-3/2}\, dt; = \frac{1}{3}\int t^{3/2} - \frac{2}{\sqrt{t}} + c$$

Hence, (A) and (B) are correct.

28. $\int \frac{dx}{\cos^3 x - \sin^3 x} = \int \frac{dx}{(\cos x - \sin x)\left(1 + \frac{\sin 2x}{2}\right)}$

$$= \int \frac{(\cos x - \sin x)dx}{(1-\sin 2x)\left(1+\frac{\sin 2x}{2}\right)}$$

Let cos x + sin $x = t \Rightarrow$ (cos x – sin x)$dx = dt$

$$= \int \frac{2dt}{(2-t^2)(t^2+1)}$$

$$= \frac{2}{3}\int \frac{dt}{2-t^2} + \frac{2}{3}\int \frac{dt}{t^2+1}$$

$$= \frac{2}{3}\frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}+t}{\sqrt{2}-t}\right| + \frac{2}{3}\tan^{-1} t$$

$$\frac{2}{3}\tan^{-1}(\sin x + \cos x) - \frac{1}{3\sqrt{2}} \ln\left|\frac{\cos x + \sin x - \sqrt{2}}{\cos x + \sin x + \sqrt{2}}\right| + C$$

Hence, (A) and (D) are correct.

29. $I = \int \frac{1}{3\sin x} \frac{2.(2\sin x + \cos x) - (\sin x + 2\cos x)}{(\sin x + 2\cos x)(\cos x + 2\sin x)} dx$

$$= \int\left(\frac{2\sec^2 x}{3\tan^2 x + 6\tan x} - \frac{\sec^2 x}{3\tan x + 6\tan^2 x}\right)dx \qquad (\because \tan x = t)$$

$$= \frac{2}{3}\int \frac{dt}{t^2+2t} - \frac{1}{3}\int \frac{dt}{t+2t^2}$$

$$= \frac{2}{3}\int \frac{dt}{(t+1)^2-1} - \frac{1}{6}\int \frac{dt}{t^2 + \frac{t}{2} + \frac{1}{16} - \frac{1}{16}}$$

$$= \frac{2}{3}\int \frac{dt}{(t+1)^2-1} - \frac{1}{6}\int \frac{dt}{\left(t+\frac{1}{4}\right)^2 - \frac{1}{4^2}}$$

$$= \frac{2}{3}\frac{1}{2}\log\left|\frac{t+1-1}{t+1+1}\right| - \frac{1}{3}\ln\left|\frac{t+\frac{1}{4}-\frac{1}{4}}{t+\frac{1}{4}+\frac{1}{4}}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{\tan x}{\tan x + 2}\right| - \frac{1}{3}\ln\left|\frac{\tan x}{\tan x + \frac{1}{2}}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{2\tan x + 1}{\tan x + 2}\right| + C$$

$$= \frac{1}{3}\ln\left|\frac{2\sin x + \cos x}{\sin x + 2\cos x}\right| + C$$

Hence, (A) and (D) are correct.

Linked Comprehension Type Questions

1. $I = \int \frac{x^2(x\sec^2 x + \tan x)}{(x\tan x + 1)^2} dx$

Let x tan x +1 = $t \Rightarrow$ (x sec^2x + tan x)$dx = dt$

$$I = x^2\frac{-1}{x\tan x + 1} + \int \frac{2xdx}{x\tan x + 1}$$

$$I = \frac{-x^2}{x\tan x + 1} + \int \frac{2x\cos x}{x\sin x + \cos x}dx$$

Let $x\sin x + \cos x = u \Rightarrow x\cos x = du$

$$I = -\frac{x^2}{x\tan x + 1} + 2\ln(x\sin x + \cos x) + C$$

$$= 2\ln|f(x)| - \frac{x^2}{g(x)} + k$$

$$f(x) = x\sin x + \cos x, g(x) = x\tan x + 1$$

$$f'(x) = \sin x + x\cos x - \sin x = x\cos x,$$

$f(x)$ is non-monotonic, unbounded

$$f(-x) = -x\sin(-x) + \cos(-x) = f(x) \text{ even function}$$

Hence, the correct answer is (B).

2. $\lim\limits_{x\to 0} \dfrac{g(x)-1}{f(x)-1}$

$$= \lim_{x\to 0} \frac{x\tan x}{(x\sin x + \cos x) - 1} = \lim_{x\to 0} \frac{\tan x + x\sec^2 x}{\sin x + x\cos x - \sin x - 0}$$

$$= \lim_{x\to 0} \frac{\tan \frac{x}{x} + \sec^2 x}{\cos x} = 1 + 1 = 2$$

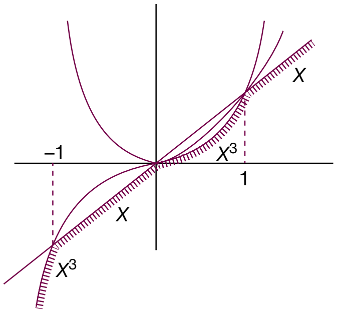
Hence, the correct answer is (C).

3. $\int (f(x) + g(x))\,dx = \dfrac{x^3}{3} + C_1$, on differentiation

$$f(x) + g(x) = x^2 \qquad \ldots(1)$$

$\int (f(x) - g(x))\,dx = x^2 - \dfrac{x^3}{3} + C_2$, on differentiation

$$f(x) - g(x) = 2x - x^2 \qquad \ldots(2)$$



From (1) and (2), we get

$f(x) = x,\ g(x) = x^2 - x$

$\int \dfrac{f(x)}{h(x)}\,dx = -\dfrac{1}{x} + C_3$ on differentiation

$$\frac{f(x)}{h(x)} = \frac{1}{x^2} \Rightarrow h(x) = x^3$$

$\phi(x) = \min\,\{f(x), f(x) + g(x), h(x)\}$

$\phi(x) = \min\left(x, x^2, x^3\right)$

$$\phi(x) = \begin{cases} x^3 & x < -1 \\ x & -1 \le x \le 0 \\ x^3 & 0 < x < 1 \\ x & x \ge 1 \end{cases}$$

$\phi(x)$ not differentiable at $x = -1, 0, 1$

Hence, the correct answer is (D).

4. $\left((1 + f(x)) + \left(\dfrac{f(x) + g(x)}{h(x)}\right)\right)^{\Sigma n}$

$$= \left(1 + x + \frac{1}{x}\right)^{\Sigma n} = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{2\Sigma n}$$

Total term = $2\Sigma n + 1 = 31$

$\Rightarrow \quad \Sigma n = 15 \Rightarrow n(n+1) = 30$

we get $n = 5$

Hence, the correct answer is (B).

Combined solution to 5–6

5. $\int (2x^6 + 15x^4 + 2x^2 + 3)\cos 2x\ dx$

$$= \int \left(\underbrace{2x^6 + 2x^2}_{\text{I}}\right)\underbrace{\cos 2x\,dx}_{\text{II}} + \int \underbrace{(15x^4 + 3)}_{\text{II}}\underbrace{\cos 2x\,dx}_{\text{I}}$$

$$= (2x^6 + 2x^2)\ \frac{\sin 2x}{2} - \int (12x^5 + 4x)\frac{\sin 2x}{2}\ dx + \cos 2x\left(\frac{15x^5}{5} + 3x\right) + \int 2\sin 2x \cdot \left(\frac{15x^5}{5} + 3x\right) dx$$

$$= (x^6 + x^2)\sin 2x - \int (6x^5 + 2x)\sin 2x\ dx + (3x^5 + 3x)\cos 2x + \int 6\sin 2x(x^5 + x)\ dx$$

$$= (x^6 + x^2)\sin 2x + 3(x^5 + x)\cos 2x + 4\int x\sin 2x$$

$$= (x^6 + x^2)\sin 2x + 3(x^5 + x)\cos 2x + 4\left[x\left(\frac{-\cos 2x}{2}\right) + \int \frac{\cos 2x}{2}\,dx\right]$$

$$= \underbrace{(x^6 + x^2 + 1)}_{f(x)}\ \sin 2x + \underbrace{(3x^5 + x)}_{g(x)}\ \cos 2x + k$$

$$\int \frac{3x^5 + x}{x^6 + x^2 + 1}\,dx = \frac{1}{2}\ln(x^6 + x^2 + 1) + C$$

Hence, the correct answer is (A).

6. $f(x)$ and $g(x)$ are respectively even and odd.
Hence, the correct answer is (B).

7. $\int f(x).g(x)dx = \dfrac{x^4}{4} + C$

Differentiating $f(x).g(x) = x^3$ $\qquad \ldots(1)$

$$\int \frac{f(x)}{g(x)}\,dx = \ell n\,|x| + C$$

Differentiating

$$\frac{f(x)}{g(x)} = \frac{1}{x} \qquad \ldots(2)$$

(1) × (2), we get:

$f^2(x) = x^2$

$f(x) = x \qquad (\because\ f'(x) > 0) \Rightarrow g(x) = x^2$ and $g(x)/h(x) = \dfrac{1}{x}$

$\Rightarrow \quad h(x) = g(x).x$ $\qquad \ldots(3)$

From (3) $h(x) = x^3$;

Now, $\int f(x).g(x), h(x)dx = \int x^6 dx = \dfrac{x^7}{7} + C$.

Hence, the correct answer is (C).

8. $f(x)$, $g(x)$, $h(x)$ are in G.P.

Hence, the correct answer is (B).

Combined solution to 9–10

$$3I_1 + 3I_2 = \int \frac{e^x + e^{-x}}{(e^x - e^{-x}) + 2}\,dx$$

Put $e^x - e^{-x} = t \Rightarrow (e^x + e^{-x})dx = dt$

$$\therefore\ 3I_1 + 3I_2 = \int \frac{dt}{t^2 + 2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$\therefore \quad I_1 + I_2 = \frac{1}{3\sqrt{2}} \tan^{-1}\left(\frac{e^x - e^{-x}}{\sqrt{2}}\right) + C \qquad \ldots(1)$$

And $I_1 - I_2 = \int \frac{e^x - e^{-x}}{(e^x + e^{-x})^2 - 2} dx$

Put, $e^x + e^{-x} = z \Rightarrow (e^x - e^{-x})dx = dz$

$$\therefore \quad I_1 - I_2 = \int \frac{dz}{z^2 - 2} = \frac{1}{2\sqrt{2}} \ln\left|\frac{z - \sqrt{2}}{z + \sqrt{2}}\right| + C \qquad \ldots(2)$$

So, From (1) to (2), we get:

$$I_1 = \frac{1}{6\sqrt{2}} \tan^{-1}\left(\frac{e^x - e^{-x}}{\sqrt{2}}\right) + \frac{1}{4\sqrt{2}} \ln\left|\frac{e^x + e^{-x} - \sqrt{2}}{e^x + e^{-x} + \sqrt{2}}\right| + C$$

And $I_2 = \frac{1}{6\sqrt{2}} \tan^{-1}\left(\frac{e^x - e^{-x}}{\sqrt{2}}\right) - \frac{1}{4\sqrt{2}} \ln\left|\frac{e^x + e^{-x} - \sqrt{2}}{e^x + e^{-x} + \sqrt{2}}\right| + C$

9. **Hence, the correct answer is (A).**

10. **Hence, the correct answer is (B).**

11. Let $P = \sin^{m-1} x \cos^{(n+1)} x$

$$\therefore \quad \frac{dp}{dx} = \sin^{m-1} x (n+1) \cos^n x (-\sin x) + \cos^{n+1} x.(m-1)\sin^{m-2} x.\cos x$$

$$= -(n+1)\sin^m x \cos^n x + (m-1)\sin^{m-2} x \cos^{n+2} x$$

On integrating both sides, we get:

$$(n+1)I_{m,n} = -\sin^{m-1} x \cos^{n+1} x + (m-1) I_{m-2,n+2}$$

$$I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{(n+1)} + \frac{(m-1)}{(n+1)} I_{m-2,n+2}$$

Here, $f(m,n) = \frac{(m-1)}{(n+1)}$

$$\therefore \quad f(2,3) = \frac{1}{4}$$

Hence, the correct answer is (C).

12. Let $p = x^{m+1}(a + bx^n)^p$

$$\therefore \quad \frac{dp}{dx} = x^{m+1} p (a + bx^n)^{p-1}.(bnx^{n-1}) + (a + bx^n)^p.(m+1)x^m$$

$$= bpnx^{m+1}.(a + bx^n)^{p-1} + (m+1)x^m (a + bx^n)^p$$

on integration both sides, we get

$$= bpnI_{m+1,p-1} + (m+1)_{m,p}$$

or $I_{m,p} = \frac{x^{m+1}.(a + bx^n)^p}{(m+1)} - \frac{bnp}{(m+1)} I_{m+n,p-1}$

$$\therefore \quad f(m,n,p,b) = \frac{bnp}{(m+1)} \Rightarrow f(1,2,3,4) = \frac{2x3x4}{(1+1)} = 12$$

Hence, the correct answer is (C).

13. $I_{m,n} = \int x^m.(\ln x)^n = (\ln x)^n \frac{x^{m+1}}{m+1} - \int \frac{n(\ln x)^{n-1}}{x}.\frac{x^{m+1}}{m+1} dx$

$$= \frac{(\ln x)^n x^{m+1}}{m+1} - \frac{n}{m+1} \int (\ln x)^{n-1}.x^m \, dx$$

$$I_{m,n} - \frac{x^{m+1}}{m+1} (\ln x)^n = -\frac{n}{m+1} I_{m,n-1}$$

Hence, the correct answer is (C).

14. $I_{n-2,\,n} = \int (\cos x)^{n-3}.(\cos x.\sin nx)dx$

$$I_{n-2,\,n} = \frac{1}{2} \int (\cos x)^{n-3} \{\sin(n+1)x + \sin(n-1)\} dx$$

$$\Rightarrow I_{n-2,\,n} = \frac{1}{2} [I_{n-3,\,n+1} + I_{n-3,\,n-1}] \qquad \ldots(1)$$

$$I_{n-2,\,n} = -\cos^{n-2}x \frac{\cos nx}{n} - \int (n-2)\cos^{n-3} x \sin x.\frac{\cos nx}{n} dx$$

$$I_{n-2,\,n} = -\frac{\cos^{n-2} x \cos nx}{n} - \frac{n-2}{n} \int \cos^{n-3} x \sin x \cos nx \, dx$$

$$= -\frac{\cos^{n-2} x \cos nx}{n} - \frac{n-2}{2n} \int \cos^{n-3} x.\{\sin(n+1)x - \sin(n-1)x\} dx$$

$$= -\frac{\cos^{n-2} x \cos nx}{n} - \frac{n-2}{2n} [I_{n-3,\,n+1}] + \frac{n-2}{2n} .I_{n-3,\,n-1} \qquad \ldots(2)$$

Adding (1) and (2), we get:

$$2I_{n-2,n} = -\frac{\cos^{n-2} x \cos nx}{n} + I_{n-3,\,n+1}\left\{\frac{1}{2} - \frac{n-2}{2n}\right\} + I_{n-3,\,n-1}\left\{\frac{1}{2} + \frac{n-2}{2n}\right\}$$

$$= \frac{-\cos^{n-2} x \cos nx}{n} + \frac{I_{n-3,n+1}}{n} + I_{n-3,\,n-1}\left(\frac{n-1}{n}\right)$$

$$nI_{n-2,n} = -\frac{\cos^{n-2} x \cos nx}{2} + \frac{I_{n-3,n+1}}{2} + \frac{(n-1)I_{n-3,n-1}}{2} \qquad \ldots(3)$$

Now, (3) – (1) , we get:

$$I_{n-2,\,n}(n-1) = -\frac{\cos^{n-2} x.\cos nx}{2} + I_{n-3,\,n-1}\left(\frac{n-1}{2} - \frac{1}{2}\right)$$

$$(n-1)I_{n-2,n} = -\frac{\cos^{n-2} x.\cos nx}{2} + \left(\frac{n-2}{2}\right) I_{n-3,\,n-1}$$

$$I_{n-2,n} = -\frac{\cos^{n-2} x \cos nx}{2(n-1)} + \frac{n-2}{2(n-1)} I_{n-3,\,n-1}$$

Hence, the correct answer is (B).

Combined solution to 15–16

15. $I_1(x) = \frac{1}{4} \tan^{-1}\left(\frac{x^2}{2}\right) + C$

$$I_1(\sqrt{2}) = \frac{1}{4} \tan^{-1}(1) + C = \frac{9\pi}{16} \Rightarrow C = \frac{\pi}{12}$$

$$\Rightarrow I_1(x) = \frac{1}{4} \tan^{-1}\left(\frac{x^2}{2}\right) + \frac{\pi}{2}$$

$$\lim_{x\to 0} \frac{\cos(I_1(x))}{x^2} = \lim_{x\to 0} \frac{\cos\left(\frac{\pi}{2} + \frac{1}{4}\tan^{-1}\left(\frac{x^2}{2}\right)\right)}{x^2}$$

$$= \lim_{x\to 0} \left(\frac{-\sin\left(\frac{1}{4}\tan^{-1}\left(\frac{x^2}{2}\right)\right)}{x^2}\right)$$

$$\Rightarrow -\lim_{x\to 0} \frac{\sin\left(\frac{1}{4}\tan^{-1}\frac{x^2}{2}\right)}{\frac{1}{4}\tan^{-1}\frac{x^2}{2}} \cdot \frac{\frac{1}{4}\cdot\tan^{-1}\frac{x^2}{2}}{2\cdot\frac{x^2}{2}} = -\frac{1}{8}$$

Hence, the correct answer is (B).

16. $I_2(x) + 2I_0(x) = \int \frac{x^2+2}{x^4+4}\,dx$

$$= \int \frac{1+\frac{2}{x^2}}{\left(x-\frac{2}{x}\right)^2+4}\,dx = \frac{1}{2}\tan\left(\frac{x^2}{2}\right) + C$$

Hence, the correct answer is (C).

17. We have, $f_n(x) = \int n\sin 2x(\sin^{2n-2}x - \cos^{2n-2}x)\ dx - \frac{1}{2^{n-1}}$

$$= 2n\int(\cos x\cdot\sin^{2n-1}x - \sin x\cos^{2n-1}x)\ dx - \frac{1}{2^{n-1}}$$

$$= \sin^{2n}x + \cos^{2n}x - \frac{1}{2^{n-1}} + C$$

As, $f_n\left(\frac{\pi}{4}\right) = \frac{1}{2^{n-1}}$

$$\Rightarrow \frac{1}{2^n} + \frac{1}{2^n} - \frac{1}{2^{n-1}} + C = \frac{1}{2^{n-1}}$$

$$\Rightarrow C = \frac{1}{2^{n-1}}$$

$\therefore\ f_n(x) = \sin^{2n}x + \cos^{2n}x$

We have, $g(x) = \lim\limits_{n\to\infty}\left(\sum_{n=1}^{n}(\sin^{2n}x + \cos^{2n}x)\right)$

$$= (\sin^2x + \sin^4x + \sin^6x + \ldots\infty) + (\cos^2x + \cos^4x + \cos^6x + \ldots\infty)$$

$$= \frac{\sin^2x}{1-\sin^2x} + \frac{\cos^2x}{1-\cos^2x} = \tan^2x + \cot^2x$$

$$= (\tan x - \cot x)^2 + 2$$

Hence, the minimum value of $g(x)$ is 2.

Hence, the correct answer is (B).

18. Clearly, $n = \lim\limits_{t\to-\infty}[\cot^{-1}t] = 3$

Now, $\int \frac{f_3(x)}{\sin^2(x).\cos^2(x)}\,dx = \int \frac{\sin^6x+\cos^6x}{\sin^2(x).\cos^2(x)}\,dx$

$$= \int \frac{1-3\sin^2x\cos^2x}{\sin^2x\cos^2x}\,dx$$

$$= \int(\sec^2x\cdot\text{cosec}^2x - 3)dx = \int(\sec^2x + \text{cosec}^2x - 3)dx$$

$$= \tan x - \cot x - 3x + C.$$

Hence, the correct answer is (D).

Matrix Match/Column Match Type Questions

1. **A → (s); B → (p, s); C → (r, s); D → (q, s)**

$$\int\frac{dx}{\cot^2x-1} = \int\frac{\sin^2x\,dx}{\cot 2x}$$

(A) $\int\frac{dx}{x^{1/3}+2}$

Let $x = t^3$

$dx = 3t^2dt$

$$I = 3\int\frac{t^2dt}{t+2}$$

$$\Rightarrow \frac{I}{3} = \int\frac{(t^2-4+4)dt}{t+2} \Rightarrow \frac{I}{3} = \int\left((t-2)+\frac{4}{t+2}\right)dt$$

$$\Rightarrow \frac{I}{3} = \frac{t^2}{2} - 2t + 4\ln|t+2|$$

$$\Rightarrow I = \frac{3}{2}x^{2/3} - 6x^{1/3} + 12\ln|x^{1/3}+2| + C$$

$$g(x) = \frac{3}{2}x^{2/3} - 6x^{1/3} + 12\ln|x^{1/3}+2|$$

$$g(-1) = \frac{3}{2} + 6 = \frac{15}{2}$$

(B) $I = \int(\sec x)^{9/5}(\text{cosec})^{1/5}\,dx = \int\frac{\sec^2x\,dx}{(\tan x)^{1/5}}$

Let $\tan x = t^5$

$\sec^2x\,dx = 5t^4\,dt$

$$\Rightarrow I = 5\int\frac{t^4dt}{t} = \frac{5t^4}{4} = \frac{5}{4}(\tan x)^{4/5} + C$$

$$\Rightarrow km = 1$$

(C) $\int\frac{dx}{\cot^2x-1} = \int\frac{\sin^2x\,dx}{\cos 2x}$

$$2I = \int\frac{(1-\cos 2x)dx}{\cos 2x} = \int\sec 2x\,dx - \int dx$$

$$2I = \frac{1}{2}\ln|\sec 2x + \tan 2x| - x$$

$$\Rightarrow I = \frac{1}{4}\ln|\sec 2x + \tan 2x| - \frac{x}{2} + C \Rightarrow l + m = 6$$

(D) $I = \int\frac{(x+1)e^x dx}{xe^x\sqrt{1+xe^x}}$

Let $1 + xe^x = t^2 \Rightarrow (x+1)e^x dx = 2t\,dt$

$$\Rightarrow I = 2\int\frac{t\,dt}{(t^2-1)t} = \ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \ln\left|\frac{\sqrt{1+xe^x}-1}{\sqrt{1+xe^x}+1}\right| + C$$

$$\Rightarrow g^2(x) = 1 + xe^x \Rightarrow g^2(1) = 1 + e$$

$$\Rightarrow g^2(1) = 3.71828 \Rightarrow [g^2(1)] = 3$$

2. **A → (q); B → (p); C → (r); D → (s)**

(A) $\int\tan\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}}\right)dx$ rationalization

$$= \int\tan\left(2\tan^{-1}\frac{\sqrt{1+\sqrt{x}}-1}{x^{1/4}}\right)dx$$

$$= \int\tan\left(\tan^{-1}\left(\frac{\frac{2\left(\sqrt{1+\sqrt{x}}-1\right)}{x^{1/4}}}{1-\frac{1+\sqrt{x}+1-2\sqrt{1+\sqrt{x}}}{x^{1/2}}}\right)\right)dx$$

$$I = \int x^{1/4}dx = \frac{4}{5}x^{5/4} + C \qquad \text{(Q)}$$

(B) $f(x) = \int\cot\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+\sqrt{x}}-\sqrt{\sqrt{x}}}{\sqrt{1+\sqrt{x}}+\sqrt{\sqrt{x}}}}\right)\cdot\frac{dx}{\sqrt{x}}$

Put $\sqrt{x} = t^2 \Rightarrow \frac{dx}{2\sqrt{x}} = 2t\,dt \Rightarrow \frac{dx}{\sqrt{x}} = 4t\,dt$

$$\therefore \quad f(x) = \int \cot\left(2\tan^{-1}\sqrt{\frac{\sqrt{1+t^2}-t}{\sqrt{1+t^2}+t}}\right) 4t\,dt$$

$$= 4\int t\cot\left(2\tan^{-1}\left(\sqrt{1+t^2}-t\right)\right)dt = 2\int t\frac{1-\left(\sqrt{1+t^2}-t\right)^2}{\sqrt{1+t^2}-t}dt$$

$$= 2\int t\left(\frac{-2t^2+2t\sqrt{1+t^2}}{\sqrt{1+t^2}-t}\right)dt = 4\int t^2dt = \frac{4t^3}{3}+c$$

$$= \frac{4}{3}x^{3/4}+c \qquad \text{(p)}$$

(C) $$f(x) = \int\left[\frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}\right]\frac{dx}{2\sqrt{x}}$$

Put $\sqrt{x} = t, \dfrac{dx}{2\sqrt{x}} = dt$

$$\therefore \quad f(x) = \int\frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-t}{1+t}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-t}{1+t}\right)\right)}dt$$

$$= \int\tan\left(\frac{\pi}{4}-\frac{1}{2}\sin^{-1}\left(\frac{1-t}{1+t}\right)\right)dt$$

$$= \int\tan\frac{1}{2}\left(\frac{\pi}{2}-\sin^{-1}\left(\frac{1-t}{1+t}\right)\right)dt = \int\tan\frac{1}{2}\left(\cos^{-1}\left(\frac{1-t}{1+t}\right)\right)dt$$

$$= \int\tan\frac{1}{2}\left(\cos^{-1}(\cos 2\tan^{-1}(\sqrt{t})\right)dt = \int\tan\frac{1}{2}\left(2\tan^{-1}(\sqrt{t})\right)dt$$

$$I = \int\sqrt{t}dt = \frac{t^{3/2}}{3/2}+c = \frac{2x^{3/4}}{3}+c \qquad \text{(r)}$$

(D) $$f(x) = \int\frac{\sqrt{x}}{2}\tan\left(2\tan^{-1}\left[\frac{\sqrt{\sqrt{1+\sqrt{x}}+1}-\sqrt{\sqrt{1+\sqrt{x}}-1}}{\sqrt{\sqrt{1+\sqrt{x}}+1}+\sqrt{\sqrt{1+\sqrt{x}}-1}}\right]\right)dx$$

On rationalising

$$f(x) = \int\frac{\sqrt{x}}{2}\tan\left(2\tan^{-1}\frac{\left(\sqrt{1+\sqrt{x}}+1\right)+\left(\sqrt{1+\sqrt{x}}-1\right)-2x^{1/4}}{\left(\sqrt{1+\sqrt{x}}+1-\sqrt{1+\sqrt{x}}+1\right)}\right)dx$$

$$= \int\frac{\sqrt{x}}{2}\cdot\tan\left(2\tan^{-1}\left(\sqrt{1+\sqrt{x}}-x^{1/4}\right)\right)dx$$

$$= \int\frac{\sqrt{x}}{2}\tan\left(\tan^{-1}\left(\frac{2\left(\sqrt{1+\sqrt{x}}-x^{1/4}\right)}{1-\left[1+\sqrt{x}+x^{1/2}-2\sqrt{1+\sqrt{x}}.\sqrt{\sqrt{x}}\right)}\right]$$

$$= \int\sqrt{x}\frac{\left(\sqrt{1+\sqrt{x}}-\sqrt{\sqrt{x}}\right)}{-2\sqrt{x}+2\sqrt{\sqrt{x}}\sqrt{1+\sqrt{x}}}dx$$

$$= \int\left(\frac{x^{1/4}}{2}\right)\times\left(\frac{\sqrt{(1+\sqrt{x})}-x^{1/4}}{\sqrt{(1+\sqrt{x})}-x^{1/4}}\right).\,dx = \frac{1}{2}\int x^{1/4}\,dx$$

$$= \frac{2}{5}x^{5/4}+C \qquad \text{(s)}$$

3. **A → (s); B → (r); C → (q); D → (p)**

(A) $$\int e^x\left(\log_e(3x+2)+\frac{9}{(3x+2)^2}\right)dx$$

$$= \int e^x\left(\log_e(3x+2)+\frac{3}{3x+2}-\frac{3}{3x+2}+\frac{9}{(3x+2)^2}\right)dx$$

$$= e^x\left(\log_e(3x+2)-\frac{3}{3x+2}\right)+c$$

$$= e^x\left(\log_e(3x+2)+\frac{m}{nx+r}\right)+C$$

$m = –3, n = 3, r = 2$

$m+n+r = 2$ (s)

(B) $$\int\frac{\cos 9x+\cos 6x}{2\cos 5x-1}dx\,dx$$

$$= \int\frac{2\cos\frac{15x}{2}\cos\frac{3x}{2}}{2\cos 5x-1}dx = \int\frac{2\cos\frac{15n}{2}\cos 3\frac{x}{2}}{4\cos^2\frac{5x}{2}-3}\frac{\cos\frac{5x}{2}}{\cos\frac{5x}{2}}dx$$

$$= \int\frac{2\cos\frac{15x}{2}\cos\frac{3x}{2}\cos\frac{5x}{2}}{\cos\frac{15x}{2}}dx = \int 2\cos\frac{3x}{2}\cos\frac{5x}{2}dx$$

$$I = \int\cos 4x+\cos x dx = \sin x+\frac{\sin 4x}{4}+C$$

$I = \text{A}\sin 4x+B\sin x+C$

$A = \frac{1}{4}; B = 1$, we get $A+B = \frac{5}{4}$ (r)

(C) $$\int\sqrt[3]{\frac{\sin^{2016}x}{\cos^{2022}x}}dx = \int(\tan x)^{2016/3}\cdot\sec^2 x dx$$

$$= \frac{3}{2019}\tan^{2019/3}x+C = \frac{1}{p}(\tan x)^q+C;$$

$$p = \frac{2019}{3} = q \Rightarrow \frac{p}{q} = 1 \qquad \text{(q)}$$

(D) $$\int\frac{(x^2+1)^{3/2}.(x^2-1)}{x^{7/2}}dx = \int\left(x+\frac{1}{x}\right)^{3/2}\left(1-\frac{1}{x^2}\right)dx$$

$$= \frac{2}{5}\left(x+\frac{1}{x}\right)^{5/2}+c = \frac{1}{m}\left(x+\frac{1}{x}\right)^n+c$$

$$m+n = \frac{5}{2}+\frac{5}{2} = 5 \qquad \text{(p)}$$

4. **A → (q); B → (p); C → (r); D → (s)**

(A) $$I = \int\frac{f'(x)}{f(x).\log(f(x))}dx$$

put $\log f(x) = t \Rightarrow \dfrac{f'(x)}{f(x)}dx = dt$

$$I = \int\frac{dt}{t} = \ln t++c$$

$I = \log(\log(f(x)))+C; K = 1$ (q)

(B) $$\int\frac{3x^3-2x^2}{\sqrt{(x^2-3x+1)^5}}dx = \int\frac{\frac{3}{x^2}-\frac{2}{x^3}}{\left(1-\frac{3}{x}+\frac{1}{x^2}\right)^{5/2}}dx$$

Put $1-\dfrac{3}{x}+\dfrac{1}{x^2} = t \Rightarrow \left(\dfrac{3}{x^2}-\dfrac{2}{x^3}\right)dx = dt$

$$I = \int \frac{dt}{t^{5/2}} = -\frac{2}{3t^{3/2}} + c$$

$$I = -\frac{2}{3}\frac{1}{\left(1 - \frac{3}{x} + \frac{1}{x^2}\right)^{3/2}} + c = -\frac{2}{3}\frac{x^3}{3\left(x^2 - 3x + 1\right)^{3/2}} + c$$

$$= \frac{nx^3}{m\sqrt{(x^2 - 3x + 1)^3}} + C$$

$m = 3$

$n = -2$

$m - n = 5$ (p)

(C) $$f(x) = \int \frac{1 + 2^x \ln(e.2^{1-x})}{(x + 2^x)^2} dx$$

$$f(x) = \int \frac{1 + 2^x(1 + \ln 2^{1-x})}{\left(x + 2^x\right)^2} dx = \int \frac{1 + 2^x + 2^x\left(1 - x\right)\ln 2}{\left(x + 2^x\right)^2} dx$$

$$f(x) = \int \frac{1 + 2^x \ln 2}{\left(x + 2^x\right)^2} dx + \frac{2^x\left(1 - x\ln 2\right)}{\left(x + 2^x\right)^2} dx$$

$$f(x) = -\frac{1}{\left(x + 2^x\right)} + \int \frac{(x + 2^x)\left(-2^x \ln 2\right) + 2^x(1 + 2^x \ln 2)}{\left(x + 2^x\right)^2} dx$$

$$f(x) = -\frac{1}{\left(x + 2^x\right)} + \int d\left(\frac{-2^x}{x + 2^x}\right)$$

$$f(x) = -\frac{1}{x + 2^x} - \frac{2^x}{x + 2^x} + c = -\frac{1 + 2^x}{x + 2^x} + C$$

$f(0) = 2 = -2 + C \Rightarrow C = 4$

$f(1) = -1 + C = 3$ (r)

(D) $$\int \frac{x^2}{(x^2 - 1)\sin 2x + 2x\cos 2x} dx$$

$$= \frac{1}{2}\int \frac{x^2 \sec^2 x dx}{\left(x^2 - 1\right)\tan x + x\left(1 - \tan^2 x\right)}$$

$$= \frac{1}{2}\int \frac{x^2 \sec^2 x \; dx}{x^2 \tan x - \tan x + x - x\tan^2 x}$$

$$= \frac{1}{2}\int \frac{x^2 \sec^2 x dx}{x\left(x\tan x + 1\right) - \tan x\left(x\tan x + 1\right)}$$

$$= \frac{1}{2}\int \frac{x^2 \sec^2 dx}{\left(x\tan x + 1\right)\left(x - \tan x\right)}$$

$$= \frac{1}{2}\int -\left(\frac{1 - \sec^2 x}{x - \tan x}\right) + \left(\frac{x\sec^2 x + \tan x}{x\tan x + 1}\right) dx$$

$$= \frac{1}{2}\left[-\ln\left(x - \tan x\right) + \ln\left(x\tan x + 1\right)\right] + c$$

$$= \frac{1}{2}\ln\left|\frac{x\sin x + \cos x}{x\cos x - \sin x}\right| + C$$

$$= = \frac{1}{\lambda}\ln\left|\frac{x\sin x + \cos x}{x\cos x - \sin x}\right| + C$$

We get $\lambda = 2$ (s)

5. A → (r); B → (s); C → (q); D → (p)

(A) $$I = \int \frac{x^2(x^6 + x^5 - 1)}{(2x^6 + 3x^5 + 2)^2} dx = \int \frac{x^6\left(x^2 + x - \frac{1}{x^4}\right)dx}{x^6\left(2x^3 + 3x^2 + \frac{2}{x^3}\right)^2}$$

Put $2x^3 + 3x^2 + \frac{2}{x^3} = t \Rightarrow \left(6x^2 + 6x - \frac{6}{x^4}\right)dx = dt$

$$I = \int \frac{dt}{6t^2} = -\frac{1}{6t} + C = -\frac{1}{6}\left(2x^3 + 3x^2 + \frac{2}{x^3}\right) + C$$ (r)

(B) $$I = \int \frac{(x^5 + x^4 + x^2)}{\sqrt{4x^7 + 5x^6 + 10x^4}} dx$$

$$I = \int \frac{(x^4 + x^3 + x)dx}{\sqrt{4x^5 + 5x^4 + 10x^2}}$$

Put $4x^5 + 5x^4 + 10x^2 = t \Rightarrow 20(x^4 + x^3 + x)dx = dt$

$$= \int \frac{dt}{20\sqrt{t}} = \frac{1}{10}\sqrt{t} + c = \frac{1}{10}\sqrt{4x^5 + 5x^4 + 10x^2} + C$$

$$= x\left(\frac{x^3}{25} + \frac{x^2}{20} + \frac{1}{10}\right)^{1/2} + c$$ (s)

(C) $$\int \frac{(2x^{12} + 5x^9)}{(x^5 + x^3 + 1)^3} dx = \int \frac{x^{15}(2x^{-3} + 5x^{-6})}{x^{15}\left(1 + x^{-2} + x^{-5}\right)^3} dx$$

$$= \int \frac{(2x^{-3} + 5x^{-6})dx}{\left(1 + x^{-2} + x^{-5}\right)^3}$$

Put $1 + x^{-2} + x^{-5} = t \Rightarrow -(2x^{-3} + 5x^{-6})dx = dt$

$$= -\int \frac{dt}{t^3} = \frac{1}{2t^2} + c = \frac{1}{2}\left(1 + x^{-2} + x^{-5}\right)^{-2} + C$$ (q)

(D) $$I = \int \frac{(2x + 1)}{(x^2 + 4x + 1)^{3/2}} dx = \int \frac{\frac{2}{x^2} + \frac{1}{x^3}}{\left(1 + \frac{4}{x} + \frac{1}{x^2}\right)^{3/2}} dx$$

Put $1 + \frac{4}{x} + \frac{1}{x^2} = t \Rightarrow \left(-\frac{4}{x^2} - \frac{2}{x^3}\right)dx = dt$

$$I = \int -\frac{dt}{2t^{3/2}} = \frac{1}{t^{1/2}} + c = \frac{1}{\left(1 + \frac{4}{x} + \frac{1}{x}\right)^{1/2}} + c$$

$$I = \frac{x}{(x^2 + 4x + 1)^{1/2}} + c$$ (p)

6. A → (r); B → (p); C → (q); D → (s)

$$I = \int \frac{e^x}{e^{4x} + 1} dx \text{ and } J = \int \frac{e^{-x}}{e^{-4x} + 1} dx$$

$$I + J = \int \frac{e^x}{e^{4x} + 1} dx + \int \frac{e^{3x}}{e^{4x} + 1} dx$$

$$I + J = \int \frac{e^x\left(e^{2x} + 1\right)}{e^{4x} + 1} dx$$

put $e^x = t \Rightarrow e^x dx = dt$

$$I + J = \int \frac{t^2 + 1}{t^4 + 1} dt$$

$$I+J=\int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+c$$

$$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2-1}{t\sqrt{2}}\right)+C=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)+C \qquad \text{(p)}$$

$$J-I=\int \frac{e^{-x}}{e^{-4x}+1}-\frac{e^x}{e^{4x}+1}dx=\int \frac{\left(e^{2x}-1\right)e^x}{e^{4x}+1}dx$$

Put $e^x=t \Rightarrow e^x dx=dt$

$$J-I=\int \frac{(t^2-1)dt}{t^4+1}=\int \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)^2-2}dt$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{t+\frac{1}{t}-\sqrt{2}}{t+\frac{1}{t}+\sqrt{2}}\right|+c$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{e^{2x}+1-\sqrt{2}e^x}{e^{2x}+1+\sqrt{2}e^x}\right|+C \qquad \text{(q)}$$

$$J+I=\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)+C\,;$$

$$J-I==\frac{1}{2\sqrt{2}}\ln\left|\frac{e^{2x}+1-\sqrt{2}e^x}{e^{2x}+1+\sqrt{2}e^x}\right|+C$$

Add $J=\frac{1}{2}\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)+\frac{1}{2\sqrt{2}}\ln\left(\frac{e^{2x}+1-\sqrt{2}e^x}{e^{2x}+1+1\sqrt{2}e^x}\right)\right)$ (s)

Sub $I=\frac{1}{2}\left(\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{e^{2x}-1}{\sqrt{2}e^x}\right)-\frac{1}{2\sqrt{2}}\ln\left(\frac{e^{2x}+1-\sqrt{2}e^x}{e^{2x}+1+\sqrt{2}e^x}\right)\right)$ (r)

7. A → (r); B → (q); C → (s); D → (p)

(A) $\int \frac{e^{2x}-2e^x}{e^{2x}+1}dx$

$$=\int \frac{e^x\left(e^x-2\right)}{(e^x)^2+1}dx \ (e^x=t)$$

$$=\int \frac{t-2}{t^2+1}dt=\frac{1}{2}\int \frac{2t}{t^2+1}dt-2\int \frac{dt}{t^2+1}$$

$$=\frac{1}{2}\ln(t^2+1)-2\tan^{-1}t+C=\frac{1}{2}\ln(e^{2x}+1)-2\tan^{-1}e^x+C$$

$$A=\frac{1}{2}\,;B=-2$$

(B) $\int \sqrt{x+\sqrt{x^2+2}}\,dx$

Let $x+\sqrt{x^2+2}=t$ …(1)

$$\left(1+\frac{2x}{2\sqrt{x^2+2}}\right)dx=dt \Rightarrow \frac{\left(x+\sqrt{x^2+2}\right)}{\sqrt{x^2+2}}dx=dt$$

$$\frac{x^2+2-x^2}{\sqrt{x^2+2}-x}=t$$

$\sqrt{x^2+2}-x=\frac{2}{t}$ …(2)

$$\sqrt{x^2+2}=\frac{1}{2}\left(t+\frac{2}{t}\right);\ dx=\frac{dt}{t}\ \frac{1}{2}\left(t+\frac{2}{t}\right);$$

$$dx=\frac{1}{2}\left(1+\frac{2}{t^2}\right)dt$$

$$I=\int \frac{\sqrt{t}\left(1+\frac{2}{t}\right)dt}{2}\,;I=\int \frac{\sqrt{t}}{2}dt+\int t^{-3/2}\,dt$$

$$=\frac{t^{3/2}}{3}+\frac{t^{-\frac{1}{2}}}{-\frac{1}{2}}+C$$

$$=\frac{\left(x+\sqrt{x^2+2}\right)^{3/2}}{3}-2\frac{1}{\sqrt{x+\sqrt{x^2+2}}}+C$$

$$A=\frac{1}{3}\,;B=-2$$

(C) $\int\left(\frac{\cos 8x-\cos 7x}{1+2\cos 5x}\right)dx$

$$=\int \frac{-2\sin\frac{15x}{2}\sin\frac{x}{2}}{1+2\left[1-2\sin^2\frac{5x}{2}\right]}dx=\int \frac{2\sin\left(\frac{15x}{2}\right)\sin\frac{x}{2}}{3-4\sin^2\frac{5x}{2}}dx$$

$$=\int \frac{2\sin\frac{15x}{2}\sin\frac{x}{2}\sin\frac{5x}{2}}{3\sin\frac{5x}{2}-4\sin^3\frac{5x}{2}}=\int \frac{-2\sin\frac{15x}{2}\sin\frac{5x}{2}\sin\frac{x}{2}}{\sin\frac{15x}{2}}dx$$

$$=\int(\cos 3x-\cos 2x)\,dx$$

$$=\frac{\sin 3x}{3}-\frac{\sin 2x}{2}+C$$

$$\Rightarrow\ A=\frac{1}{3}\,;B=-\frac{1}{2}$$

(D) $\int \frac{\ln x}{x^3}dx$

$$=\ln x\left(-\frac{1}{2x^2}\right)+\int \frac{1}{2x^3}dx$$

$$=\frac{-\ln x}{2x^2}-\frac{1}{4x^2}+C$$

$$A=-\frac{1}{2}\,;B=-\frac{1}{4}$$

Integer/Numerical Answer Type Questions

1. $f(x)=\int(x\cos x+1)e^{\sin x}dx$

$f(x)=\int d\left(x\cdot e^{\sin x}\right)=x\cdot e^{\sin x}+c$ $f(0)=0$, we have $f(x)=x\cdot e^{\sin x}$

$g(x)=\int(x\sin x-1)e^{\cos x}dx=\int d\left(-xe^{\cos x}\right)$

$g(x)=-xe^{\cos x}+C$, $g(0)=0$, we have $g(x)=-xe^{\cos x}$

Now equation $f(x).\,g(x)=0$

$$xe^{\sin x}\left(-xe^{\cos x}\right)=0 \Rightarrow x^2e^{\sin x+\cos x}=0$$

As $e^{\sin x+\cos x}\neq 0$, then only $x=0$ is the solution

Hence, the correct answer is (1).

2. $I = \int \frac{(1+x^{1/4})^{1/3}}{\sqrt{x}} dx \qquad \text{put } x = t^4 \Rightarrow dx = 4t^3 dt$

$$I = \int \frac{(1+t)^{1/3}}{t^2} 4t^3 dt = 4\int (1+t)^{1/3} t dt$$

$$I = 4\frac{(1+t)^{4/3}}{4/3} t - 3\int (1+t)^{4/3} dt$$

$$= 3t(1+t)^{4/3} - 3\frac{(1+t)^{7/3}}{\frac{7}{3}} + c$$

$$= 3t(1+t)^{4/3} - \frac{9}{7}(1+t)^{7/3} + c = 3(1+t)^{4/3}(t - \frac{3}{7}(1+t)) + c$$

$$I = 3(1+t)^{4/3}\left(\frac{4}{7}(t+1) - 1\right) + c = \frac{12}{7}(t+1)^{7/3} - 3(1+t)^{4/3} + c$$

$$I = \frac{12}{7}(1+x^{1/4})^{7/3} - 3(1+x^{1/4})^{4/3} + c$$

$a = 12, b = 7, c = 3$

$a - b + c = 12 - 7 + 3 = 8$

Hence, the correct answer is (8).

3. $I = \int \frac{(3x^3+2x)}{(x^6+2x^5+x^4+2x^3+2x^2+5)} dx$

$$I = \int \frac{3x^2+2x}{(x^3+x^2+1)^2+4} dx$$

put $x^3 + x^2 + 1 = t \Rightarrow (3x^2+2x)dx = dt$

$$I = \int \frac{dt}{t^2+4} = \frac{1}{2}\tan^{-1}\frac{(x^3+x^2+1)}{2} + c = Af\left(\frac{g(x)}{B}\right) + c$$

$A = 1/2;\ B = 2,\ f(x) = \tan^{-1}(x),\ g(x) = x^3 + x^2 + 1$

Now $B/A = 4$

Hence, the correct answer is (4).

4. $f(x) = \lim_{n\to\infty}\left(\frac{x^{\frac{1}{n}} - 1}{\frac{1}{n}}\right) = \ln x$

$$\int x^x (f(x)+1)\, dx$$

Let $x^x = t \Rightarrow x \ln x = \ln t \Rightarrow (\ln x + 1)dx = \frac{dt}{t}$

$$= \int t \frac{dt}{t} = t = x^x + C$$

$g(x) = x$ and $f(x) = x$

$\lim_{x\to 0} (x+4)^{(x+4)} = 4^4 = 256$

Hence, the correct answer is (256).

5. $\int \frac{1}{\ln(ex)} \frac{1}{\ln(e^2x)} \frac{1}{\ln(e^3x)} \frac{dx}{x}$

$$= \int\left(\frac{1}{1+\ln x} \frac{1}{2+\ln x} \frac{1}{3+\ln x}\right)\frac{dx}{x}$$

Put $\ln x = t \Rightarrow \frac{1}{x}.dx = dt$

$$\int \frac{1}{(1+t)} \cdot \frac{1}{(2+t)} \cdot \frac{1}{(3+t)} dt$$

$$= \frac{1}{2}\int \frac{1}{1+t}dt - \int \frac{1}{2+t}dt + \frac{1}{2}\int \frac{1}{3+t}dt \qquad \text{(using partial fraction)}$$

$$= \frac{1}{2}\ln(1+t) - \ln(2+t) - \frac{1}{2}\ln(3+t)$$

$$= \frac{1}{2}\ln(1+tnx) - \ln(2+\ln x) + \frac{1}{2}\ln(3+\ln x)$$

$\Rightarrow\ A + C = 1$

Hence, the correct answer is (1).

6. Let $I = \int 2\cdot\sin 2\theta \cdot \cos^2\theta \times e^{\sin\theta} d\theta$

$$= \int 4\sin\theta \cdot \cos^3\theta \cdot e^{\sin\theta} \cdot d\theta$$

Put $\sin\theta = t$, so

$$I = \int 4t(1-t^2)e^t\ dt = 4\int e^t(t-t^3)\ dt$$

$$= 4(e^t(t-t^3) - \int e^t(1-3t^2)\ dt)$$

$$= 4(e^t(t-t^3) - e^t + 3e^t t^2 - \int 3e^t 2t\ dt)$$

$$= 4[e^t(3t^2 + t - t^3 - 1) - 6(e^t t - \int e^t\ dt)]$$

$$= 4e^t(-t^3 + 3t^2 - 5t + 5) + C$$

$$= e^{\sin\theta}(-4\sin^3\theta - 12\cos^2\theta - 20\sin\theta + 32) + k$$

$\therefore\ A = -4, B = -12, C = -20, D = 0, E = 32$

Now $B/A + D/E = 3$

Hence, the correct answer is (3).

7. $I = \int\left(\frac{\sin x}{\cos x}\right)^{\frac{-3}{4}} (\cos x)^{-6}\ dx$

$$= \int (\tan x)^{\frac{-3}{4}} (\sec^6 x)\ dx$$

Put, $\tan x = t$

$$I = \int t^{-3/4}(1+t^2)^2\ dt = \int (t^{-3/4} + 2t^{5/4} + t^{13/4}) dt$$

$$= \frac{t^{1/4}}{\frac{1}{4}} + \frac{2t^{9/4}}{\frac{9}{4}} + \frac{t^{17/4}}{\frac{17}{4}} + C$$

$$= \frac{(\tan x)^{1/4}}{\frac{1}{4}} + \frac{(\tan x)^{9/4}}{\frac{9}{8}} + \frac{(\tan x)^{17/4}}{\frac{17}{4}} + C$$

$\therefore\ a_1 = 1/4\ b_1 = 1/4\ a_2 = 9/4\ b_2 = 9/8\ a_3 = 17/4\ b_3 = 17/4$

$\therefore\ a_1 + a_2 + a_3 + b_1 + b_2 + b_3 = \frac{99}{8}$

$$\Rightarrow \left[\frac{a_1 + a_2 + a_3 + b_1 + b_2 + b_3}{2}\right] = \left[\frac{99}{16}\right] = 6$$

Hence, the correct answer is (6).

8. Let $I = \int \frac{(1 + x\cos x - \sin x)dx}{\sqrt{\frac{x+\sin x+\cos x}{x-\sin x+\cos x}} \cdot (x - \sin x + \cos x)^2}$

Put, $\frac{x+\sin x+\cos x}{x-\sin x+\cos x} = t^2$

$$\Rightarrow \frac{2(1+x+\cos x - \sin x)dx}{(x-\sin x+\cos x)^2} = 2tdt$$

$$\Rightarrow\ I = \int \frac{tdt}{t} = t = \sqrt{\frac{x+\sin x+\cos x}{x-\sin x+\cos x}} + C$$

$l = 1, m = 1, n = 1/2; p = -1, q = 1, r = -1/2$

$\left(\frac{\ell + m}{n}\right)(p-q)r = 4(-2)\ (-1/2) = 4$

Hence, the correct answer is (4).

9. $\int\left(\sqrt{\sec x+\tan x}+\sqrt{\sec x-\tan x}\right)dx$

$= \sqrt{2}\int\sqrt{(\sec x)+1}\,dx = \sqrt{2}\int\frac{\sin x}{\sqrt{\cos x-\cos^2 x}}dx$

$= \sqrt{2}\sin^{-1}(1-2\cos x)+c$

$\therefore\ g(x) = 2\cos x \Rightarrow \left[g\left(\frac{\pi}{6}\right)\right] = 1$

Hence, the correct answer is (1).

10. $\int\frac{1+\sin 2x}{\sqrt{\tan x}+\sqrt{\cot x}}dx = \frac{1}{\sqrt{2}}\int(\sin x+\cos x)\ \sqrt{\sin 2x}\ dx$

$= \frac{1}{\sqrt{2}}\int\sqrt{1-t^2}\ dt\ (t = \sin x - \cos x)$

$= \frac{1}{2\sqrt{2}}\left[t\sqrt{1-t^2}+\sin^{-1}(t)\right]+c$

$f(x) = \sin x - \cos x \Rightarrow \left[f\left(\frac{3\pi}{4}\right)\right] = 1$

Hence, the correct answer is (1).

11. $\int\frac{\tan x\sqrt{\sec x}(1+\cos^2 x)}{\sqrt{\cos x+\sin^2 x}}dx$

$= \int\frac{\tan x\sqrt{\sec x}(1+\cos^2 x)}{\sqrt{1-\cos^2 x+\cos x}}dx = \int\frac{\sec x\tan x(1+\cos^2 x)}{\sqrt{\sec x-\cos x+1}}dx$

$= \int\frac{\sec x\tan x+\sin x}{\sqrt{\sec x-\cos x+1}}dx$

$= 2\sqrt{\sec x-\cos x+1}\ +c$

$\Rightarrow\ f(x) = \sec x - \cos x\ \left[f\left(\frac{\pi}{4}\right)\right] = 0$

Hence, the correct answer is (0).

12. $I = \int\frac{1+\ln x+x.\ln x+x\ln^2 x}{1+x\ln x(2+x\ln x)}dx$

$= \int\frac{(1+\ln x)(1+x\ln x)}{(1+x\ln x)^2}dx = \int\frac{1+\ln x}{(1+x\ln x)}dx$

$= \ln(1 + x\ln x) + C$ $\qquad\{\because\ (1 + x\ln x)' = 1 + \ln x\}$

$\Rightarrow\ f(x) = \ln(1 + x\ln x)$

$\Rightarrow\ e^{f(2)} = 1 + 2\ln 2 \Rightarrow e^{f(2)} - 1 = \ln 4$

Hence, the correct answer is (4).

13. $\int\{(x^6+6x^5)+(x^5+5x^4)+(x^4+4x^3)+(x^3+3x^2)+(x^2+2x)+(x+1)\}e^x\,dx$

$= (x^6+x^5+x^4+x^3+x^2+x)e^x + C$

$\because\ \int e^x(f(x)+f'(x))dx = e^x f(x) + C$

$= \sum_{k=1}^{6}x^k e^x\ + C$

$\Rightarrow\ \alpha = 6, \beta = 1 \Rightarrow (\alpha+\beta) = 7$

Hence, the correct answer is (7).

14. $\int\frac{e^{7x/2}}{\sqrt[4]{e^{2x}+e^{-2x}}}dx$

$\int\frac{e^{4x}}{\sqrt[4]{e^{4x}+1}}$, let $(e^{4x}+1) = t \Rightarrow e^{4x}\,.\,4\,dx = dt$

$\frac{1}{4}\int\frac{dt}{t^{1/4}} = \frac{1}{4}\frac{t^{3/4}}{\left(\frac{3}{4}\right)}+C = \frac{1}{3}(e^{4x}+1)^{3/4}+C$

We get $k = 3, \alpha = 4, \beta = 3, \gamma = 4$

$\Rightarrow\ \left(\frac{k}{\beta}+\frac{\gamma}{\alpha}\right) = 2$

Hence, the correct answer is (2).

15. We have $\int\frac{1-7\cos^2 x}{\sin^7 x.\cos^2 x}dx = \int\frac{\sec^2 x}{\sin^7 x}dx - 7\int\frac{1}{\sin^7 x}dx$

$= I_1 - I_2$

Now, $I_1 = \int\underset{(I)}{\left(\frac{1}{\sin^7 x}\right)}\ \underset{(II)}{\sec^2 x}\ dx$

(By Parts)

$= \frac{\tan x}{\sin^7 x}+7\int\frac{\tan x}{\sin^8 x}\cos x\,dx = \frac{\tan x}{\sin^7 x}+I_2$

$\therefore\ I_1 - I_2 = \frac{\tan x}{\sin^7 x}+C$, where C is constant of integration.

Hence, $g(x) = \tan x$

So, $g'(x) = \sec^2 x$ and $g''(x) = 2\sec^2 x\tan x$

$\therefore\ g'(0) = 1$ and $g''\left(\frac{\pi}{4}\right) = 4$

Hence, $g'(0)+g''\left(\frac{\pi}{4}\right) = 1+4 = 5.$

Hence, the correct answer is (5).

16. Let $f(x) = \int\frac{(\sin^{12}x-\cos^{12}x)}{(\sin^2 x-\cos^2 x)((\sin^2 x+\cos^2 x)-3\sin^2 x\cos^2 x)}dx$

$\int\frac{\sin^{12}x-\cos^{12}x}{(\sin^2 x-\cos^2 x)((\sin^2 x+\cos^2 x)^3-3\sin^2 x\cos^{2x})}$

$= \int\frac{(\sin^6 x-\cos^6 x)(\sin^6 x+\cos^6 x)}{(\sin^2 x-\cos^2 x)(\sin^6 x+\cos^6 x)}dx$

$= \int\frac{(\sin^2 x-\cos^2 x)(\sin^4 x+\cos^4 x+\sin^2 x\cos^2 x)}{(\sin^2 x-\cos^2 x)}dx$

$\Rightarrow\ f(x) = \int(\sin^4 x+\cos^4 x+\sin^2 x\cos^2 x)\,dx$

$= \int(1-\sin^2 x\cos^2 x)\,dx = \int(1-\frac{1}{4}\sin^2 2x)\,dx$

$= \int\left(1-\frac{1}{8}(1-\cos 4x)\right)dx = \int\left(\frac{7}{8}+\frac{1}{8}\cos 4x\right)dx$

$f(x) = \frac{7}{8}x+\frac{1}{32}\sin 4x + C$

As $f(0) = 0 \Rightarrow C = 0$

$\therefore\ f(x) = \frac{7}{8}\ x+\frac{1}{32}\sin 4x$

Hence, $f\left(\frac{\pi}{2}\right) = \frac{7}{16}\pi \equiv \frac{k\pi}{16}$, so $k = 7$.

Hence, the correct answer is (7).

17. $y(x+y) = x \Rightarrow (x+2y)\ \frac{dy}{dx}\ = 1-y$

$\Rightarrow\ \frac{dx}{x+2y}\ = \frac{dy}{1-y}$

$\Rightarrow\ \int\frac{dx}{x+2y}\ = -\ln(1-y)+c \Rightarrow \text{k} = 1$

Hence, the correct answer is (1).

18. $I = \int\frac{(2\sin x\cos x+2\tan x)}{\cos^6 x+6\cos^2 x+4}dx$

$= \int\frac{(2\tan x.\sec^4 x+2\tan x\sec^6 x)}{1+6\cos^4 x+4\cos^6 x}dx$

Let $1 + 6\sec^4 x + 4\sec^6 x = t$

$\Rightarrow$ $(24\sec^4 x\tan x + 24\sec^6 x\tan x)dx = dt \Rightarrow I = \dfrac{1}{12}\displaystyle\int \dfrac{dt}{t}$

$$= \frac{1}{12}\cdot \ln\left(1+\frac{6}{\cos^4 x}+\frac{4}{\cos^6 x}\right)+C \Rightarrow a = 12, b = 6, d = 4$$

$\Rightarrow$ $a - b - d = 2$

Hence, the correct answer is (2).

19. Let $I = \displaystyle\int \frac{(x^6-4)dx}{(x^6+2)^{1/4}x^4} = \int \frac{\left(x-\frac{4}{x^5}\right)dx}{\left(x^2+\frac{2}{x^4}\right)^{1/4}}$

Let $x^2 + \dfrac{2}{x^4} = t^4 \Rightarrow \left(x-\dfrac{4}{x^5}\right)dx = 2t^3\,dt$

$$\Rightarrow\quad I = 2\int \frac{t^3dt}{t} = \frac{2t^3}{3} = \frac{2(x^6+2)^{3/4}}{3x^3}+C$$

We get, $l = 2/3$, $m = 3/4$, $n = 3$; $\dfrac{n}{\ell m} = 6$

Hence, the correct answer is (6).

20. Put, $1 - x = t^2$, so that $I = \displaystyle\int \frac{-2t\ dt}{2\left(1-t^2\right)t\sqrt{1+t^2+t}}$

$$= \frac{1}{2}(-I_1 - I_2)$$

Where $I_1 = \displaystyle\int \frac{1}{t-1}\frac{dt}{\sqrt{1+t^2+t}}$ and $I_2 = \displaystyle\int \frac{1}{t+1}\frac{dt}{\sqrt{1+t^2+t}}$

In I_1, put, $t - 1 = \dfrac{1}{u}$; In I_2 put, $t + 1 = \dfrac{1}{v}$

Thus,

$$I = -\frac{1}{2\sqrt{3}}\log\left|u+\frac{1}{2}+\sqrt{u^2+u+\frac{1}{3}}\right| + \frac{1}{2}\log\left|v-\frac{1}{2}+\sqrt{v^2-v+1}\right|+C$$

where, $u = \dfrac{1}{\sqrt{1-x}-1}$ and $v = \dfrac{1}{\sqrt{1-x}+1}$

Hence, the correct answer is (8).

21. $\displaystyle\int \frac{\cos x}{3+3\sin x-\cos^2 x}dx = \int \frac{\cos x}{\sin^2 x+3\sin x+2}dx$

$$= \int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$$

Let $\sin x = t$, $\cos x\ dx = dt$

$$= \int \frac{dt}{(1+t)(2+t)} = \int\left(\frac{1}{(1+t)}-\frac{1}{(2+t)}\right)dt$$

$$= \ln\left(\frac{1+t}{2+t}\right)+C = \ln\left(1-\frac{1}{2+\sin x}\right)+C$$

$\Rightarrow$ $f(x) = \dfrac{1}{2+\sin x}$

Range of $f(x)$ is $\left[\dfrac{1}{3},1\right)$ because $1 + \sin x \neq 0$

number integral value of $f(x)$ is '0'.

Hence, the correct answer is (0).

22. $f(x) = \displaystyle\int x\left(1-{}^{10}C_1x+{}^{10}C_2x^2-{}^{10}C_3x^3+\ldots+{}^{10}C_{10}x^{10}\right)dx$

$$= \int x(1-x)^{10}\ dx$$

Let $1 - x = t$; $-dx = dt$

$$= -\int (1-t)t^{10}\ dt = \frac{t^{12}}{12}-\frac{t^{11}}{11}+C$$

$$f(x) = \frac{(1-x)^{12}}{12}-\frac{(1-x)^{11}}{11}+C$$

$$f(0) = \frac{1}{12}-\frac{1}{11}+C = -\frac{1}{132}+C$$

$$C = \frac{263}{132}+\frac{1}{132} = 2$$

$$f(x) = \frac{(1-x)^{12}}{12}-\frac{(1-x)^{11}}{11}+2$$

$\Rightarrow$ $f(1) = 2.$

Hence, the correct answer is (2).

23. $F(x) = \displaystyle\int \frac{(1+x)((1-x+x^2)(1+x+x^2)+x^2)}{1+2x+3x^2+4x^3+3x^4+2x^5+x^6}dx$

$$= \int \frac{(1+x)((1+x^2)^2-x^2+x^2)}{(1+x)^2(x^2+1)^2}dx$$

$$= \int \frac{(1+x^2)^2}{(1+x)(x^2+1)^2}dx = \ln(x+1)+c\ ;$$

$$[\,F(99)-F(3)\,] = \left[\ln\left(\frac{100}{4}\right)\right] = [\ln(25)] = 3$$

Hence, the correct answer is (3).

24. $I = \displaystyle\int (x^{12}+x^8+x^4)(2x^8+3x^4+6)^{1/4}\ dx$

$$= \int\left(x^{11}+x^7+x^3\right)(2x^{12}+3x^8+6x^4)^{1/4}\ dx$$

Put, $2x^{12} + 3x^8 + 6x^4 = t$

$\Rightarrow$ $24(x^{11} + x^7 + x^3)dx = dt$

$$\Rightarrow\quad I = \int \frac{t^{1/4}}{24}dt = \frac{t^{5/4}}{24\cdot\frac{5}{4}}+C$$

$$= \frac{1}{30}(2x^{12}+3x^8+6x^4)^{5/4}+C$$

$\Rightarrow$ $(60a + 1) = (2 + 1) = 3$

Hence, the correct answer is (3).

25. $I = \displaystyle\int \tan^2 x\sin^2 x dx$

$$= \int(\tan^2 x-\sin^2 x)dx = (\tan x - x) - \frac{1}{2}\int(1-\cos 2x)\,dx$$

$$= \tan x - x - \frac{x}{2} + \frac{1}{4}\sin 2x + C,$$

where C is indefinite integration constant

$$= \tan x + \underbrace{\frac{1}{4}\sin 2x - \frac{3x}{2}}_{g(x)} + C$$

$\therefore$ $g(x) = \dfrac{1}{4}\sin 2x - \dfrac{3x}{2}$

Now, $g\left(\dfrac{\pi}{12}\right) = \dfrac{1}{4}\left(\dfrac{1}{2}\right) - \dfrac{\pi}{8} \Rightarrow \pi + 8\,.\,g\left(\dfrac{\pi}{12}\right) = 1.$

Hence, $8\left(\dfrac{\pi}{8}+g\left(\dfrac{\pi}{12}\right)\right) = 1.$

Hence, the correct answer is (1).

26. $I = \displaystyle\int \frac{\{f'(x)g(x)-g'(x)f(x)\}dx}{\{f(x)+g(x)\}\sqrt{g(x)(f(x)-g(x))}}$

$$I = \int \frac{\left\{\dfrac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}\right\}}{\left(\dfrac{f(x)}{g(x)}+1\right)\sqrt{\dfrac{f(x)}{g(x)}-1}}dx$$

Let $\frac{f(x)}{g(x)} - 1 = t^2 \Rightarrow \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} dx = 2tdt$

$$\therefore \quad I = \int \frac{2tdt}{(t^2+2)\cdot t} = \frac{2}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$$

$$= \sqrt{2} \tan^{-1}\left\{\sqrt{\frac{f(x)}{2g(x)} - \frac{1}{2}}\right\} + C$$

$\Rightarrow \quad m = 2, n = 2$

$\therefore \quad m^2 + n^2 = 8$

Hence, the correct answer is (8).

27. $I = \int \frac{x^4+4}{x^4+x^3-7x^2-4x+16} dx = \int \frac{\left(1+\frac{4}{x^2}\right)dx}{x^2+\frac{16}{x^2}-7-\frac{4}{x}+x}$

$$I = \int \frac{\left(1+\frac{4}{x^2}\right)dx}{\left(x-\frac{4}{x}+\frac{1}{2}\right)^2+\frac{3}{4}} dx$$

Put $x - \frac{4}{x} + \frac{1}{2} = t \Rightarrow (1 + 4/x^2)dx = dt$

$$I = \int \frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2t}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x^2-8+x}{\sqrt{3}x}\right) + \lambda$$

$$I = \frac{A}{\sqrt{3}} \tan^{-1}\left(\frac{Ax^2+Bx+2C}{\sqrt{3}x}\right) + \lambda$$

We get $A = 2, B = 1, C = -4$ then $|A + B + C| = 1$

Hence, the correct answer is (1).

28. $\int \frac{(2x+3)}{x(x+1)(x+2)(x+3)+1} dx = \int \frac{2n+3}{(x^2+3x)(x^2+3x+2)+1} dx$

$$= \int \frac{2x+3}{\left(x^2+3x+1\right)^2} dx$$

Put $x^2 + 3x + 1 = t$, then $(2x + 3)dx = dt$

$$I = \int \frac{dt}{t^2} = -\frac{1}{t} + C = -\frac{1}{(x^2+3x+1)} + C$$

$f(x) = x^2 + 3x + 1, a + b + c = 5.$

Hence, the correct answer is (5).

29. $F(x) = \int e^{\sin^{-1} x}\left(1 - \frac{x}{\sqrt{1-x^2}}\right) dx$

Put $\sin^{-1} x = t, x = \sin t \Rightarrow dx = \cos t dt$

$$F(x) = \int e^t \left(1 - \frac{\sin t}{\cos t}\right) \cos t\, dt = \int e^t (\cos t - \sin t)\, dt$$

$$F(x) = e^t \cos t + c = e^{\sin^{-1} x}\sqrt{1-x^2} + C$$

$F(0) = 1, C = 0$

$$F\left(\frac{1}{2}\right) = e^{\pi/6} \frac{\sqrt{3}}{2} = \frac{k\sqrt{3}e^{\pi/6}}{\pi}$$

$k = \pi/2$

Hence, the correct answer is (2).

30. $I_{10} = \int (1-x^5)^{10} dx = x(1-x^5)^{10} - \int x10(1-x^5)^9(-5x^4)dx$

$$= x(1-x^5)^{10} + 50I_9 - 50I_{10} \Rightarrow I_{10} = \frac{x(1-x^5)}{51} + \frac{50}{51} I_9$$

$\Rightarrow \quad n = 50; m = 51$

Hence, the correct answer is (8).

31. $\int \frac{2t^4-4t^2+2}{(t^2+4)(t^2-1)^2} + \int \frac{t^3+4t}{(t^2+4)(t^2-1)^2} dt$

$$= \tan^{-1} \frac{t}{2} - \frac{1}{2(t^2-1)} + C = \tan^{-1}\left(\frac{e^x}{2}\right) - \frac{1}{2}\frac{1}{e^{2x}-1} + C$$

Hence, the correct answer is (2).

32. $\int \frac{dx}{x-2} - \int \frac{2dx}{(x-1)(x+1)(x-2)}$

$$= \int \frac{dx}{x-2} + \int \frac{dx}{x-1} - \frac{1}{3} \int \frac{dx}{x+1} - \frac{2}{3} \int \frac{dx}{x-2}$$

$$= \frac{1}{3} \ln \left|\frac{C.(x-2)(x-1)^3}{(x+1)}\right|$$

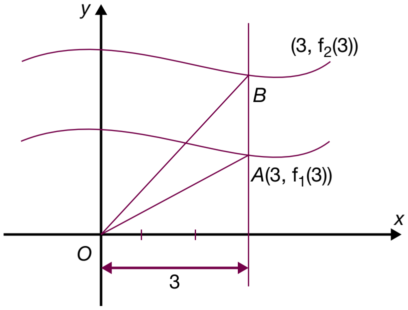
Hence, the correct answer is (3).

33. $\int f(x)\, dx = \int (ax + b + \sin x)\, dx$

$$= \frac{ax^2}{2} + bx - \cos x + C$$

$\therefore \quad f_1(x) = \frac{ax^2}{2} + bx - \cos x + C_1$ and $f_2(x)$

$$= \frac{ax^2}{2} + bx - \cos x + C_2$$



Now, $\int_2^4 \left(f_1(x) - f_2(x)\right) dx = 8$

$$\Rightarrow \int_2^4 (C_1 - C_2)\, dx = 8$$

$\Rightarrow \quad C_1 - C_2 = 4 \quad \ldots(1)$

Area of the triangle $OAB = \frac{1}{2} \times 3 \times |f_2(3) - f_1(3)|$

As, $f_2(3) - f_1(3) = f_2(0) - f_1(0)$

$= C_2 - 1 - (C_1 - 1)$

$= C_2 - C_1$

$\therefore \quad \text{Area} = \frac{1}{2} \times 3 \times 4 = 6$

Hence, the correct answer is (6).

34. Let $I = \int \frac{\left(x^6-4\right)dx}{\left(x^6+2\right)^{1/4} x^4} = \int \frac{\left(x-\frac{4}{x^5}\right)dx}{\left(x^2+\frac{2}{x^4}\right)^{1/4}}$

Let $x^2 + \frac{2}{x^4} = t^4 \Rightarrow \left(x - \frac{4}{x^5}\right)dx = 2t^3dt$

$$\Rightarrow \quad I = 2\int \frac{t^3 dt}{t} = \frac{2t^3}{3}$$

$$= \frac{2}{3}\frac{(x^6+2)^{3/4}}{x^3} + C; \frac{n}{\ell m} = 6$$

Hence, the correct answer is (6).

35. $I = \int \frac{dx}{x(x^{2007}+1)} = \int \frac{x^{2007}+1-x^{2007}}{x(x^{2007}+1)}dx = \int\left(\frac{1}{x} - \frac{x^{2006}}{1+x^{2007}}\right)dx$

$$= \log x - \frac{1}{2007}\log(1+x^{2007}) = \frac{\log x^{2007} - \log(1+x^{2007})}{2007}$$

$$= \frac{1}{2007}\log\left(\frac{x^{2007}}{1+x^{2007}}\right) + c$$

$$p+q+r = 6021.$$

The sum of digits in $p+q+r$ is $6+0+2+1=9$.

Hence, the correct answer is (9).

36. $\int \frac{(1-\cos x)^{2/7}}{(1+\cos x)^{9/7}}dx = \int \frac{\left(2\sin^2\frac{x}{2}\right)^{2/7}}{\left(2\cos^2\frac{x}{2}\right)^{9/7}}dx$

$$= \frac{1}{2}\int \frac{\sin^{4/7}\left(\frac{x}{2}\right)}{\cos^{4/7}\left(\frac{x}{2}\right)\cos^{14/7}\left(\frac{x}{2}\right)}dx = \frac{1}{2}\int\left(\tan\frac{x}{2}\right)^{4/7}.\sec^2\frac{x}{2}dx$$

$$= \frac{7}{11}\left(\tan\left(\frac{x}{2}\right)\right)^{11/7} + c \Rightarrow ab = 1$$

Hence, the correct answer is (1).

37. Put $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

$$\therefore \quad \int \frac{dx}{(1+x^2)^2} = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \cos^2\theta d\theta = \int \frac{1+\cos 2\theta}{2}d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{4}\frac{2\tan\theta}{1+\tan^2\theta} + c = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\left(\frac{x}{1+x^2}\right) + c$$

$$\therefore \quad a = \frac{1}{2}; b = \frac{1}{2}$$

$$\therefore \quad a+b = 1$$

Hence, the correct answer is (1).

38. $f''(x) = f(x)$

$$\Rightarrow \quad 2f'(x)f''(x) = 2f(x).f'(x)$$

$$\Rightarrow \quad \frac{d}{dx}\left[(f'(x))^2\right] = \frac{d}{dx}\left[(f(x))^2\right]$$

$$\Rightarrow \quad [f'(x)]^2 = [f(x)]^2 + C \text{ give } f(0)=2, f'(0)=3$$

$$C = 5 \qquad \therefore (f'(x))^2 = (f(x))^2 + 5$$

$$f'(x) = \sqrt{5+(f(x))^2} \Rightarrow \frac{f'(x)}{\sqrt{5+(f(x))^2}} = 1 \text{ integrating}$$

$$\Rightarrow \quad \int \frac{d(f(x))}{(\sqrt{5})^2 + (f(x))^2} = x + C_1$$

$$\Rightarrow \quad \log\left(f(x) + \sqrt{5+(f(x))^2}\right) = x + C_1 \; f(0) = 2$$

$$\Rightarrow \quad C_1 = \log 5$$

$$\Rightarrow \quad \log\left(f(x) + \sqrt{5+(f(x))^2}\right) = x + \log 5$$

$$\Rightarrow \quad f(x) + \sqrt{5+(f(x))^2} = 5e^x$$

$$\therefore \quad 2f(x) = 5\left(e^x - e^{-x}\right)$$

$$\Rightarrow \quad f(4) = \frac{5}{2}\left(e^4 - e^{-4}\right) = \frac{5}{2}\frac{\left(e^8-1\right)}{e^4}$$

$$\therefore \quad k = \frac{5}{2} \Rightarrow [K] = 2$$

Hence, the correct answer is (2).

39. $\frac{d}{dx}(x\sin x + \cos x) = x\cos x$

$$I = \int\left(\frac{x}{\cos x}\right)\left(\frac{x\cos x}{(x\sin x + \cos x)^2}\right)dx$$

Integrate by parts.

$$= \left(\frac{x}{\cos x}\right)\left(\frac{-1}{x\sin x + \cos x}\right) + \int \frac{1}{\cos^2 x}dx$$

$$= \frac{-x}{\cos x(x\sin x + \cos x)} + \tan x + c$$

$$= \frac{\sin x - x\cos x}{x\sin x + \cos x} + c \quad = l + p = 2$$

Hence, the correct answer is (2).

40. $I = \int \frac{\sin^2 x\cos^2 x}{(\sin x + \cos x + 1)^2}dx$

$$= \frac{1}{4}\int \frac{\left((\sin x + \cos x)^2 - 1\right)^2}{(\sin x + \cos x + 1)^2} = \frac{1}{4}\int(\sin x + \cos x - 1)^2 dx$$

$$= \frac{1}{4}\int(\sin^2 x + \cos^2 x + 1 - 2\sin x - 2\cos x + \sin 2x)\, dx$$

$$= \frac{1}{4}\int(2 - 2\sin x - 2\cos x + \sin 2x)\, dx$$

$$= \frac{1}{4}\left[2x + 2\cos x - 2\sin x - \frac{\cos 2x}{2}\right] + dx$$

$$= \frac{x}{2} + \frac{\sqrt{2}}{2}\cos\left(x + \frac{\pi}{4}\right) - \frac{\cos 2x}{8} + dx$$

$$\Rightarrow \quad a = 2, b = 2; c = -8$$

$$|a+b+c| = 4$$

Hence, the correct answer is (4).

41. $f(x) = \lim_{n\to\infty}\sin^n\left(\frac{2\pi n}{3n+1}\right) + \lim_{n\to\infty}\frac{1-x^{-2n}}{1+x^{-2n}} = 0+1 = 1$

$$I = \int x\frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$$

$$\therefore \quad = \ln\left(x+\sqrt{1+x^2}\right).\sqrt{1+x^2} - \int dx$$

$$= \sqrt{1+x^2}\ln\left(x+\sqrt{1+x^2}\right) - x + C$$

Hence, the correct answer is (1).

42. $f'(x) = f(x), f(0) = 2 \Rightarrow f(x) = 2e^x$

$$\therefore \quad \int \frac{f(x)}{3+4f(x)}dx = 2\int \frac{e^x}{3+8e^x} = \frac{1}{4}\ln(3+8e^x) + C$$

$$\therefore \quad K = 4$$

Hence, the correct answer is (4).

43. $$\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$

Let $x+1=\frac{1}{t}$

$$\Rightarrow \quad dx=-\frac{dt}{t^2}=-\int \frac{dt}{t\sqrt{\frac{(1-t)}{t^2}+1}}=-\int \frac{dt}{\sqrt{t^2-t+1}}$$

$$\Rightarrow \quad -\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$f(x)=-\ln\left(\left(t-\frac{1}{2}\right)+\sqrt{t^2-t+1}\right)+C$$

$$\Rightarrow \quad f\left(\frac{1-t}{t}\right)=-\ln\left(\left(t-\frac{1}{2}\right)+\sqrt{t^2-t+1}\right)+C \qquad \ldots(1)$$

Put $t=1$

$$\Rightarrow \quad f(0)=-\ln\frac{3}{2}+C=-\ln 2$$

$$C=\ln\frac{3}{4}$$

In equation (1), put $t=2$

$$\Rightarrow \quad f\left(-\frac{1}{2}\right)=-\ln\left(\frac{3}{2}+\sqrt{3}\right)+\ln\left(\frac{3}{4}\right)$$

$$=\ln\left(\frac{3}{4\left(\frac{3}{2}+\sqrt{3}\right)}\right)=\ln\left(\frac{3}{2(3+2\sqrt{3})}\right)$$

$$=\ln\frac{3(2\sqrt{3}-3)}{2\times 3}=\ln\left(\sqrt{3}-\frac{3}{2}\right)$$

$\Rightarrow \quad k=3$

Hence, the correct answer is (3).

44. $$\int \frac{(4+3x)dx}{(x^3-2x-4)}$$

Using partial fractions

$$\text{Let } \frac{4+3x}{(x-2)(x^2+2x+2)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+2x+2}$$

$\Rightarrow \quad 4+3x=A(x^2+2x+2)+(Bx+C)(x-2)$

Put, $x=0, 2, -1$ to get:

$A=1, B=C=-1$

$$\Rightarrow \quad I=\ln|x-2|-\frac{1}{2}\int \frac{(2x+2)dx}{x^2+2x+2} \Rightarrow I$$

$$=\ln\left|\frac{x-2}{\sqrt{x^2+2x+2}}\right|+C \Rightarrow k^2+\ell^2=8$$

Hence, the correct answer is (8).

45. $$\text{I}=\int \frac{\cos^4 x\, dx}{\sin^3 x(\sin^5 x+\cos^5 x)^{3/5}}=\int \frac{\cos^4 x\, dx}{\sin^6 x(1+\cot^5 x)^{3/5}}$$

$$=\int \frac{\cot^4 x\operatorname{cosec}^2 x\, dx}{(1+\cot^5 x)^{3/5}}$$

Put $1+\cot^5 x=t$; $5\cot^4 x\operatorname{cosec}^2 x\, dx=-dt$

$$=-\frac{1}{5}\int \frac{dt}{t^{3/5}}=-\frac{1}{5}\frac{t^{\frac{-3}{5}+1}}{\left(-\frac{3}{5}+1\right)}+c=-\frac{1}{2}(1+\cot^5 x)^{2/5}+c$$

$\therefore \quad A=5, B=\frac{2}{5}$ and $AB=2$

Hence, the correct answer is (2).

46. Put $x=t^6$; $$\text{I}=\int \frac{(t^6+t^4+t)6t^5dt}{t^6(1+t^2)}$$

$$=6\int \frac{t^5+t^3+1}{1+t^2}dt=\frac{3}{2}t^4+6\tan^{-1}t+c_1$$

$$=\frac{3}{2}x^{2/3}+6\tan^{-1}(x^{1/6})+C_1$$

$\Rightarrow \quad \alpha=3, b=2, c=6 \Rightarrow 100a+10b+C-320=6$

Hence, the correct answer is (6).

47. $$I=\int \operatorname{cosec}^2 x(\cos x)^{-2009}x\, dx-2009\int \frac{1}{\cos^{2009}x}dx$$

$=I_1-I_2$

Applying by parts on I

$$I=\frac{-\cot x}{(\cos x)^{2009}}+C$$

$\therefore \quad A(x)=\cot x, B(x)=\cos x$

$$\Rightarrow \quad \frac{A(x)}{B(x)}=\operatorname{cosec} x\neq\{x\} \text{ no solution}$$

Hence, the correct answer is (0).

48. $f(x)=x\,e^{\sin x}+k_1, k=0$

$g(x)=-xe^{\cos x}+k_2, f(x).\, g(x)=-x^2e^{\sin x+\cos x}$

$f(x).\, g(x)=0$

$x=0$

Hence, the correct answer is (1).

49. $$I=\int \frac{d\left(\frac{f}{g}\right)}{\left(\frac{f}{g}+1\right)\sqrt{\frac{f}{g}-1}}$$

$$\text{Let } \frac{f}{g}-1=t^2 \Rightarrow d\left(\frac{f}{g}\right)=2t\,dt$$

$$I=\int \frac{2dt}{t^2+2}=\frac{2}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}=\sqrt{2}\tan^{-1}\sqrt{\frac{f-g}{2g}}+C$$

Hence, the correct answer is (8).

ARCHIVE: JEE MAIN

1. $$\int \frac{\cos x-\sin x}{\sqrt{8-\sin 2x}}dx=\int \frac{\cos x-\sin x}{\sqrt{9-(\sin x+\cos x)^2}}dx$$

Let $\sin x+\cos x=t$

$$\int \frac{dt}{\sqrt{9-t^2}}=\sin^{-1}\frac{t}{3}+c=\sin^{-1}\left(\frac{\sin x+\cos x}{3}\right)+c$$

So, $a=1, b=3$.

Hence, the correct answer is (D).

2. $$I = \int \frac{\sin\theta.\sin 2\theta(\sin^6\theta + \sin^4\theta + \sin^2\theta)\sqrt{2\sin^4\theta + 3\sin^2\theta + 6}}{1 - \cos 2\theta}\, d\theta$$

$$\Rightarrow \quad I = \int \frac{\sin\theta.2\sin\theta\cos\theta.\sin^2\theta(\sin^4\theta + \sin^2\theta + 1)(2\sin^4\theta + 3\sin^2\theta + 6)^{1/2}}{2\sin^2\theta}\, d\theta$$

$$= \int \sin^2\theta.\cos\theta(\sin^4\theta + \sin^2\theta + 1)(2\sin^4\theta + 3\sin^2\theta + 6)^{1/2}\ d\theta$$

Let $\sin\theta = t$

$\Rightarrow \quad \cos\theta\, d\theta = dt$

$$\therefore \quad I = \int t^2(t^4 + t^2 + 1)(2t^4 + 3t^2 + 6)^{1/2}\ dt$$

$$= \int (t^5 + t^3 + t)(2t^6 + 3t^4 + 6t^2)^{1/2}\ dt$$

Let $2t^6 + 3t^4 + 6t^2 = u^2 \Rightarrow 12(t^5 + t^3 + t)\ dt = 2u\, du$

$$\therefore \quad I = \int (u^2)^{1/2} \cdot \frac{2u\, du}{12}$$

$$= \int \frac{u^2}{6} du = \frac{u^3}{18} + C = \frac{(2t^6 + 3t^4 + 6t^2)^{3/2}}{18} + C$$

When, $t = \sin\theta$ and $t^2 = 1 - \cos^2\theta$ will give option (D)

Hence, the correct answer is (D).

3. $$\int \frac{e^{3\log_e 2x} + 5e^{2\log_e 2x}}{e^{4\log_e x} + 5e^{3\log_e x} - 7e^{2\log_e x}}\, dx,\ x > 0$$

$$= \int \frac{(2x)^3 + 5(2x)^2}{x^4 + 5x^3 - 7x^2}\, dx = \int \frac{4x^2(2x + 5)}{x^2(x^2 + 5x - 7)}\, dx$$

$$= 4\int \frac{d(x^2 + 5x - 7)}{(x^2 + 5x - 7)} = 4\log_e|x^2 + 5x - 7| + c$$

Hence, the correct answer is (B).

4. $$\int \frac{(x^2 - 1)dx}{(x^4 + 3x^2 + 1)\tan^{-1}\left(x + \frac{1}{x}\right)} + \int \frac{dx}{x^4 + 3x^2 + 1}$$

$$\int \frac{\left(1 - \frac{1}{x^2}\right)dx}{\left(\left(x + \frac{1}{x}\right)^2 + 1\right)\tan^{-1}\left(x + \frac{1}{x}\right)} + \frac{1}{2}\int \frac{(x^2 + 1) - (x^2 - 1)dx}{x^4 + 3x^2 + 1}$$

Put, $\tan^{-1}\left(x + \frac{1}{x}\right) = t;$

$$\int \frac{dt}{t} + \frac{1}{2}\int \frac{\left(1 - \frac{1}{x^2}\right)dx}{\left(x - \frac{1}{x}\right)^2 + 5} - \frac{1}{2}\int \frac{\left(1 - \frac{1}{x^2}\right)dx}{\left(x + \frac{1}{x}\right)^2 + 1}$$

Put, $x - \frac{1}{x} = y,\ x + \frac{1}{x} = z;$

$$= \log_c t + \frac{1}{2}\int \frac{dy}{y^2 + 5} - \frac{1}{2}\int \frac{dz}{z^2 + 1}$$

$$= \log_c \tan^{-1}\left(x + \frac{1}{x}\right) + \frac{1}{2\sqrt{5}}\tan^{-1}\left(\frac{x^2 - 1}{\sqrt{5}x}\right) - \frac{1}{2}\tan^{-1}\left(\frac{x^2 + 1}{x}\right) + C$$

$$\alpha = 1,\ \beta = \frac{1}{2\sqrt{5}},\ \gamma = \frac{1}{\sqrt{5}},\ \delta = \frac{-1}{2}$$

or $\alpha = 1,\ \beta = \frac{-1}{2\sqrt{5}},\ \gamma = \frac{-1}{\sqrt{5}},\ \delta = \frac{-1}{2}$

$$10(\alpha + \beta\gamma + \delta) = 10\left(1 + \frac{1}{10} - \frac{1}{2}\right) = 6$$

Hence, the correct answer is (6).

5. $$\int \frac{(2x - 1^2)\cos\sqrt{2x(-1^2 + 5)}}{\sqrt{(2x - 1^2) + 5}}\, dx$$

$(2x - 1)^2 + 5 = t^2;\ 2(2x - 1)\ 2dx = 2t\ dt$

$2\sqrt{t^2 - 5}\ dx = t\ dt$

So, $$\int \frac{\sqrt{t^2 - 5}\cos t}{2\sqrt{t^2 - 5}}\, dt = \frac{1}{2}\sin t + c$$

$$= \frac{1}{2}\sin\sqrt{(2x - 1^2) + 5} + c$$

Hence, the correct answer is (A).

6. $$f(x) = \int \frac{(5x^8 + 7x^6)dx}{x^{14}(x^{-5} + x^{-7} + 2)^2}\frac{1}{K}$$

Let $x^{-5} + x^{-7} + 2 = t$

$$(-5x^{-6} - 7x^{-8})dx = dt \Rightarrow f(x) = \int -\frac{dt}{t^2} = \frac{1}{t} + c$$

$$f(x) = \frac{x^7}{x^2 + 1 + 2x^7};\ f(1) = \frac{1}{4}$$

Hence, the correct answer is (4).

7. $$I = \int \frac{dx}{\left[\left(x + \frac{1}{2}\right)^2 + \frac{3}{4}\right]^2}$$

$$\int \frac{dt}{\left(t^2 + \frac{3}{4}\right)^2} \qquad \left(\text{Put } x + \frac{1}{2} = t\right)$$

$$= \frac{\sqrt{3}}{2}\int \frac{\sec^2\theta d\theta}{\frac{9}{16}\sec^4\theta} \qquad \left(\text{Put } t = \frac{\sqrt{3}}{2}\tan\theta\right)$$

$$= \frac{4\sqrt{3}}{9}\int (1 + \cos 2\theta)\, d\theta = \frac{4\sqrt{3}}{9}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{4\sqrt{3}}{9}\left[\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + \frac{\sqrt{3}(2x + 1)}{3 + (2x + 1)^2}\right] + c$$

$$= \frac{4\sqrt{3}}{9}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right) + \frac{1}{3}\left(\frac{2x + 1}{x^2 + x + 1}\right) + c$$

Hence, $9\left(\sqrt{3}a + b\right) = 15$

Hence, the correct answer is (15).

8. $$I = \int \frac{d\theta}{\cos^2\theta(\tan 2\theta + \sec 2\theta)} = \int \frac{\sec^2\theta d\theta}{\frac{2\tan\theta}{1 - \tan^2\theta} + \frac{1 + \tan^2\theta}{1 - \tan^2\theta}}$$

$$= \int \frac{(1 - \tan^2\theta)\sec^2\theta d\theta}{(1 + \tan\theta)^2}$$

Let $\tan\theta = t \Rightarrow \sec^2\theta\, d\theta = dt$

$$I = \int \frac{1 - t^2}{(1 + t)^2}\, dt = \int \frac{(1 - t)(1 + t)}{(1 + t)^2}\, dt$$

$$= \int \frac{dt}{1+t} - \int \frac{t}{1+t}\, dt$$

$$= \ln|1+t| - \int\left(\frac{1+t}{1+t} - \frac{1}{1+t}\right) dt$$

$$= \ln|1+t| - t + \ln|1+t|$$

$$= 2\ln|1+t| - t + C$$

$\lambda = -1, f(\theta) = 1 + \tan\theta$

Hence, the correct answer is (4).

9. Put, $x = \tan^2\theta \Rightarrow dx = 2\tan\theta\sec^2\theta\, d\theta$

$$\int \underset{\text{I}}{\underset{\downarrow}{\theta}}.(2\underset{\text{II}}{\underset{\downarrow}{\tan}}\theta\cdot\sec^2\theta)\, d\theta = \theta.\tan^2\theta - \int \tan^2\theta\, d\theta$$

$$= \theta.\tan^2\theta - \int(\sec^2\theta - 1)\, d\theta$$

$$= \theta(1+\tan^2\theta) - \tan\theta + C$$

$$= \tan^{-1}\left(\sqrt{x}\right)(1+x) - \sqrt{x} + C$$

Hence, the correct answer is (C).

10. $f(x) = \int_1^3 \frac{\sqrt{x}\, dx}{(1+x)^2} = \int_1^{\sqrt{3}} \frac{t.2t\, dt}{(1+t^2)^2}$ (put $\sqrt{x} = t$)

$$= \left(-\frac{t}{1+t^2}\right)_1^{\sqrt{3}} + \left(\tan^{-1} t\right)_1^{\sqrt{3}}$$ [Appling by parts]

$$= -\left(\frac{\sqrt{3}}{4} - \frac{1}{2}\right) + \frac{\pi}{3} - \frac{\pi}{4} = \frac{1}{2} - \frac{\sqrt{3}}{4} + \frac{\pi}{12}$$

Hence, the correct answer is (D).

11. $$\int\left(\frac{x}{x\sin x + \cos x}\right)^2 dx = \int\left(\frac{x}{\cos x}\right)\cdot\frac{x\cos x\, dx}{(x\sin x+\cos x)^2}$$

$$= \frac{x}{\cos x}\left(-\frac{1}{x\sin x + \cos x}\right) + \int\left(\frac{\cos x + x\sin x}{\cos^2 x}\right)\left(\frac{1}{x\sin x+\cos x}\right) dx$$

$$= -\frac{x\sec x}{x\sin x + \cos x} + \int \sec^2 x\, dx$$

$$= -\frac{x\sec x}{x\sin x+\cos x} + \tan x + C$$

Hence, the correct answer is (D).

12. $$\int\frac{\cos\theta\, d\theta}{5 + 7\sin\theta - 2\cos^2\theta} = \int\frac{\cos\theta\, d\theta}{3 + 7\sin\theta + 2\sin^2\theta}$$

Put $\sin\theta = t \Rightarrow \cos\theta d\theta = dt$

$$= \int\frac{dt}{2t^2 + 7t + 3} = \int\frac{dt}{(2t+1)(t+3)}$$

$$= \frac{1}{5}\int\left(\frac{2}{2t+1} - \frac{1}{t+3}\right)dt = \frac{1}{5}\ln\left|\frac{2t+1}{t+3}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{2\sin\theta+1}{\sin\theta+3}\right| + C$$

We have $A = \frac{1}{5}$ and $B(\theta) = \frac{2\sin\theta+1}{\sin\theta+3}$

Hence, the correct answer is (D).

13. $e^{2x} + 2e^x - e^{-x} - 1$

$$= e^x(e^x+1) - e^{-x}(e^x+1) + e^x = [(e^x+1)(e^x - e^{-x}) + e^x]$$

so $I = \int(e^x+1)(e^x - e^{-x})e^{e^x+e^{-x}}\, dx + \int e^x\cdot e^{e^x+e^{-x}}\, dx$

$$= (e^x+1)\; e^{e^x+e^{-x}} - \int e^x\cdot e^{e^x+e^{-x}}\, dx + \int e^x\cdot e^{e^x+e^{-x}}\, dx$$

$$= (e^x+1)\; e^{e^x+e^{-x}} + C$$

$\therefore\quad g(x) = e^x + 1 \Rightarrow g(0) = 2$

Hence, the correct answer is (A).

14. $I = \int\sec^{2/3} x\,\text{cosec}^{4/3} x\, dx$

$$= \int\frac{dx}{(\sin x)^{4/3}(\cos x)^{2/3}}\, dx$$

$$= \int\frac{dx}{\left(\frac{\sin x}{\cos x}\right)^{4/3}\cdot\cos^2 x} = \int\frac{\sec^2 x dx}{(\tan x)^{4/3}}$$

Put, $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$I = \int\frac{dt}{t^{4/3}} = \frac{t^{-1/3}}{(-1/3)} + C = \frac{-3}{(\tan x)^{1/3}} + C$$

Hence, the correct answer is (1).

15. $$\int e^{\sec x}\left(\sec x\tan x f(x) + (\sec x\tan x + \sec^2 x)\right)dx = e^{\sec x} f(x) + C$$

Diff both side w.r.t. x

$\Rightarrow\; e^{\sec x}(\text{secx}\tan x\, f(x) + (\sec x\tan x + \sec^2 x))$

$= e^{\sec x}\sec x\tan x\, f(x) + e^{\sec x} f'(x)$

$\Rightarrow\; f'(x) = \sec^2 x + \tan x\sec x$

$\Rightarrow\; f(x) = \tan x + \sec + C$

Hence, the correct answer is (3).

16. $$\int\frac{dx}{(x^2 - 2x + 10)^2} = \int\frac{dx}{\left((x-1)^2 + 9\right)^2}$$

Let $x - 1 = 3\tan\theta\, dx = 3\sec^2\theta\, d\theta$

$$\therefore\quad \frac{1}{27}\int\cos^2\theta\, d\theta = \frac{1}{54}\int(1 + \cos 2\theta)d\theta$$

$$= \frac{1}{54}\left(\theta + \frac{\sin 2\theta}{2}\right)$$

$$= \frac{1}{54}\left(\tan^{-1}\left(\frac{x-1}{3}\right) + \frac{3(x-1)}{x^2 - 2x + 10}\right) + C$$

Hence, the correct answer is (2).

17. Let $x^2 = t \Rightarrow 2xdx = dt$

$$\frac{1}{2}\int t^2 e^{-t}\, dt = \frac{1}{2}[-t^2e^{-t} + \int 2te^{-t}\, dt]$$

$$= \frac{-t^2e^{-t}}{2} - te^{-t} - e^{-t} + c$$

$$= \left(-\frac{x^4}{2} - x^2 - 1\right)e^{-x^2} + c$$

$$g(x) = -\frac{x^4}{2} - x^2 - 1;\; g(-1) = -\frac{5}{2}$$

Hence, the correct answer is (2).

18. $$\int \frac{2x^3-1}{x^4+x}\,dx = \int \frac{(4x^3+1)-(2x^3+2)}{x^4+x}\,dx$$

$$= \int \frac{4x^3+1}{x^4+x}\,dx - 2\int \frac{1}{x}\,dx$$

Put, $x^4 + x = t \Rightarrow (4x^3 + 1)dx = dt$

$$= \int \frac{dt}{t} - 2\int \frac{1}{x}\,dx = \ln|t| - 2\ln x + C$$

$$= \ln\left|\frac{x^4+x}{x^2}\right| + C = \ln\left|\frac{x^3+1}{x}\right| + C$$

Hence, the correct answer is (3).

19. $$\int \frac{\sin x\cos\alpha + \cos x\sin\alpha}{\sin x\cos\alpha - \cos x\sin\alpha}\,dx = \int \frac{\sin(x+\alpha)}{\sin(x-\alpha)}\,dx$$

$$= \int \frac{\sin(x-\alpha+2\alpha)}{\sin(x-\alpha)}\,dx$$

$$= \int \frac{\sin(x-\alpha)\cos 2\alpha}{\sin(x-\alpha)}\,dx + \int \frac{\cos(x-\alpha)\sin 2\alpha}{\sin(x-\alpha)}\,dx$$

$$= (x-\alpha)\cos 2\alpha + \sin 2\alpha \log|\sin(x-\alpha)| + C$$

Hence, the correct answer is (4).

20. Let $$I = \int \frac{\sin^2 x\cos^2 x}{(\sin^5 x + \cos^3 x\sin^2 x + \sin^3 x\cos^2 x + \cos^5 x)^2}\,dx$$

$$= \int \frac{\sin^2 x\cos^2 x}{[(\sin^2 x + \cos^2 x)(\sin^2 x + \cos^3 x)]^2}\,dx$$

$$= \int \frac{\sin^2 x\cos^2 x}{(\sin^3 x + \cos^3 x)^2}\,dx = \int \frac{\tan^2 x\cdot\sec^2 x}{(1+\tan^3 x)^2}\,dx$$

Now, put $(1 + \tan^3 x) = t \Rightarrow 3\tan^2 x\sec^2 x\,dx = dt$

$$\therefore\quad 1 = \frac{1}{3}\int \frac{dt}{t^2} = -\frac{1}{3t} + C = \frac{-1}{3(1+\tan^3 x)} + C$$

Hence, the correct answer is (1).

21. $$l_n = \int \tan^n x\,dx, n > 1$$

Let $I_4 + I_6 = \int (\tan^4 x + \tan^6 x)\,dx = \int \tan^4 x\sec^2 x\,dx$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\therefore\quad I = \int t^4 dt = \frac{t^5}{5} + C = \frac{1}{5}\tan^5 x + C$$

On comparing, we have

$$a = \frac{1}{5},\ b = 0$$

Hence, the correct answer is (C).

22. $$I = \int \frac{dx}{x^2(x^4+1)^{3/4}};\qquad \int \frac{dx}{x^5\left(1+\frac{1}{x^4}\right)^{3/4}}$$

Let $1 + \frac{1}{x^4} = t^4$

$$\Rightarrow \frac{-4}{x^5}\,dx = 4t^3 dt \quad\text{or}\quad \frac{dx}{x^5} = -t^3 dt$$

$$\therefore\quad I = \int \frac{-t^3 dt}{t^3} = -t + c = -\left(1+\frac{1}{x^4}\right)^{1/4} + c$$

Hence, the correct answer is (2).

23. $$\int \left(1 + x - \frac{1}{x}\right)e^{\left(x+\frac{1}{x}\right)}\,dx$$

$$= \int e^{\left(x+\frac{1}{x}\right)}\,dx + \int x\left(1-\frac{1}{x^2}\right)e^{\left(x+\frac{1}{x}\right)}\,dx + c$$

$$= \int e^{\left(x+\frac{1}{x}\right)}\,dx + xe^{\left(x+\frac{1}{x}\right)} - \int e^{\left(x+\frac{1}{x}\right)}\,dx + c$$

(Integrating second integral using by parts)

$$= xe^{\left(x+\frac{1}{x}\right)} + c$$

Hence, the correct answer is (2).

ARCHIVE: JEE ADVANCED

1. Put, $\sec x + \tan x = t$, so that

$$\frac{1}{t} = \frac{1}{\sec x + \tan x} = \frac{\sec x - \tan x}{\sec^2 x - \tan^2 x} = \sec x - \tan x$$

Also, $dt = \sec x(\sec x + \tan x)\,dx \Rightarrow \sec x\,dx = \frac{dt}{t}$

We have, $2\sec x = t + \frac{1}{t}$

Now, $$I = \frac{1}{2}\int \frac{(t+1/t)}{t^{9/2}}\frac{dt}{t} = \frac{1}{2}\int \left[t^{-9/2} + t^{-13/2}\right]dt$$

$$= \frac{1}{2}\left[-\frac{2}{7}t^{-7/2} - \frac{2}{11}t^{-11/2}\right] + K; = \frac{-1}{t^{11/2}}\left[\frac{1}{7}t^2 + \frac{1}{11}\right] + K$$

$$= -\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$$

Hence, the correct answer is (C).

2. $$J - I = \int \frac{e^{3x}}{1 + e^{2x} + e^{4x}}\,dx - \int \frac{e^x}{e^{4x} + e^{2x} + 1}\,dx$$

$$= -\int \frac{e^x\left(1 - e^{2x}\right)}{e^{4x} + e^{2x} + 1}\,dx$$

Put, $e^x = t$, $e^x\,dx = dt$, to obtain

$$J - I = -\int \frac{(1-t^2)}{t^4 + t^2 + 1}\,dt$$

$$= -\int \frac{\frac{1}{t^2} - 1}{t^2 + \frac{1}{t^2} + 1}\,dt$$

Put, $t + \frac{1}{t} = u \Rightarrow \left(1 - \frac{1}{t^2}\right)dt = du$

Thus, $$J - I = -\int \frac{(-1)du}{(u^2-2)+1} = \int \frac{du}{u^2 - 1}$$

$$= \frac{1}{2}\log\left|\frac{1-u}{1+u}\right| + C;$$

$$I - J = \frac{1}{2}\log\left(\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right) + C$$

Hence, the correct answer is (C).

CHAPTER 9: DEFINITE INTEGRATION

Test Your Concepts-I

1. $0 \le x < \frac{\pi}{4}$ then $[x] = 0$

Then $\displaystyle\int_0^{\pi/4} \sin x\, dx = -\{\cos x\}_0^{\pi/4} = \left(1 - \frac{1}{\sqrt{2}}\right).$

2. If $F(x) = \begin{cases} e^{\cos x} \sin x & \text{for } |x| \le 2 \\ 2 & \text{otherwise} \end{cases}$

$$\int_{-2}^{3} F(x)\, dx = \int_{-2}^{2} F(x)\, dx + \int_{2}^{3} F(x)$$

$$= \int_{-2}^{2} e^{\cos x} \sin x\, dx + \int_{2}^{3} 2\, dx$$

$$= 0 + 2\,[x]_2^3 \qquad (\because\ e^{\cos x} \sin x \text{ is an odd function})$$

$$= 2[3 - 2] = 2$$

3. $k = \displaystyle\int_0^1 \frac{e^t}{1+t}\, dt$

$$\int_0^1 e^t \ln(1+t)\, dt = \left(\ln(1+t) e^t\right)_0^1 - \int_0^1 \frac{e^t}{1+t}\, dt$$

$$= e \ln 2 - k$$

4. $\displaystyle\int_0^{\pi} \cos^{-1}(\sin x)\, dx = 2\left[\int_0^{\pi} \cos^{-1}(\sin x) dx\right]$

$$= 2\left[\int_0^{\pi/2} \left[\frac{\pi}{2} - \sin^{-1}(\sin x)\right]\right] dx = 2\left[\frac{\pi^2}{4} - \frac{\pi^2}{8}\right] = \frac{\pi^2}{4}$$

5. $\displaystyle\int_1^3 \left[\tan^{-1}\frac{x}{x^2+1} + \cot^{-1}\left(\frac{x}{x^2+1}\right)\right] dx$

$$I = \int_1^3 \frac{\pi}{2} dx = \pi$$

6. $\displaystyle\int_0^{1.5} [x^2]\, dx = \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{1.5} 2\, dx$

$$= 0 + [x]_1^{\sqrt{2}} + 2[x]_{\sqrt{2}}^{1.5}$$

$$= (\sqrt{2} - 1) + 2(1.5 - \sqrt{2}) = \sqrt{2} - 1 + 3 - 2\sqrt{2} = 2 - \sqrt{2}$$

7. $\displaystyle\int_{-4}^{3} |x^2 - 4|\, dx = \int_{-4}^{-2} (x^2 - 4)\, dx + \int_{-2}^{2} (4 - x^2)\, dx + \int_{2}^{3} (x^2 - 4)\, dx$

$$= \left(\frac{x^3}{3} - 4x\right)_{-4}^{-2} + \left(4x - \frac{x^3}{3}\right)_{-2}^{2} + \left(\frac{x^3}{3} - 4x\right)_{2}^{3} = \frac{71}{3}.$$

8. $\displaystyle\int_{-2}^{2} |1 - x^2|\, dx = \int_{-2}^{-1} (x^2 - 1)\, dx + \int_{-1}^{1} (1 - x^2)\, dx + \int_{1}^{2} (x^2 - 1)\, dx$

$$= \left[\frac{x^3}{3} - x\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^3}{3} - x\right]_{1}^{2}$$

$$= \left(-\frac{1}{3} + 1 + \frac{8}{3} - 2\right) + \left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) + \left(\frac{8}{3} - 2 - \frac{1}{3} + 1\right)$$

$$= 4$$

9. $\displaystyle\int_0^{\pi} \sqrt{\frac{1 + \cos 2x}{2}}\, dx = \int_0^{\pi} \sqrt{\cos^2 x}\, dx = \int_0^{\pi} |\cos x|\, dx$

$$= \int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = 1 - (-1) = 2$$

10. $I = \displaystyle\int_{0.3}^{4.5} [x] dx$

Here, $[x]$ is the greatest integer function.

$$\text{We have } f(x) = [x] = \left|\begin{array}{ll} 0; & 0.3 \le x < 1 \\ 1; & 1 \le x < 2 \\ 2; & 2 \le x < 3 \\ 3; & 3 \le x < 4 \\ 4; & 4 \le x \le 4.5 \end{array}\right.$$

$$I = \int_{0.3}^{1} 0\, dx + \int_1^2 dx + \int_2^3 2\, dx + \int_3^4 3\, dx + \int_4^{4.5} 4\, dx$$

$$= 0 + 1 + 2 + 3 + 4(0.5) = 8$$

11. $I = \displaystyle\int_0^{n^2} \left[\sqrt{x}\right] dx$

$$I = \int_0^1 \left[\sqrt{x}\right] dx + \int_1^4 \left[\sqrt{x}\right] dx + \int_4^9 \left[\sqrt{x}\right] dx + \ldots + \int_{(n-1)^2}^{n^2} \left[\sqrt{x}\right] dx$$

$$= \int_0^1 0.\, dx + \int_1^4 1.\, dx + \int_4^9 2.\, dx + \ldots + \int_{(n-1)^2}^{n^2} (n-1)\, dx$$

$$= 0.1 + 1.3 + 2.5 + 3.7 + \ldots + (n-1)(2n-1)$$

$$= \sum_{n=1}^{n} (n-1)(2n-1) = \sum_{n=1}^{n} (2n^2 - 3n + 1)$$

$$= 2\,\frac{n(n+1)(2n+1)}{6} - 3\,\frac{n(n+1)}{2} + n$$

$$= n\left[\frac{2(n+1)(2n+1) - 9(n+1) + 6}{6}\right]$$

$$= n\left[\frac{4n^2 - 3n - 1}{6}\right] = \frac{n(n-1)(4n+1)}{6}$$

12. Let $f(x) = \log \sin x$, then $f'(x) = \cot x$

We have $\displaystyle\int e^x (f + f')\, dx = f \,.\, e^x + C$

$\therefore \quad I = [e^x \log \sin x]_{\pi/4}^{\pi/2} = e^{\pi/4} \log \sqrt{2}$

13. $f(x) = \dfrac{2}{e^x}$

As $x \in (0, \log_e 2)$, $\dfrac{2}{e^x}$ varies from 1 to 2 and hence $\left[\dfrac{2}{e^x}\right] = 1$

As $x \in (\log_e 2, \infty)$, $\dfrac{2}{e^x}$ varies from 0 to 1 and hence $\left[\dfrac{2}{e^x}\right] = 0$

$$\therefore \quad \int_0^{\infty} \left[\frac{2}{e^x}\right] dx = \int_0^{\log_e 2} 1\, dx + \int_{\log_e 2}^{\infty} 0\, dx = (x)_0^{\log_e 2} + 0 = \log_e 2$$

14. $\because \quad \sqrt{x+\left(2\sqrt{(x-1)}\right)} = \sqrt{\sqrt{(x-1)}^2+1^2+2\sqrt{(x-1)}}$

$$= \sqrt{(x-1)}\ +1$$

And $\sqrt{x-2\sqrt{(x-1)}} = \sqrt{\left(\sqrt{(x-1)}\right)^2+1^2-2\sqrt{(x-1)}}$

$$= \left|\sqrt{(x-1)}-1\right|$$

$$\int_1^5 \sqrt{x+2\sqrt{(x-1)}}+\sqrt{x-2\sqrt{(x-1)}}$$

$$= \int_1^5\left(\sqrt{(x-1)}+1\right) + \int_1^5\left|\sqrt{(x-1)}-1\right| dx$$

$$= \int_1^5\left(\sqrt{(x-1)}+1\right) dx + \int_1^2\left(1-\sqrt{(x-1)}\right) dx + \int_2^5\left(\sqrt{(x-1)}-1\right) dx$$

$$= \int_0^4\left(\sqrt{x}+1\right) dx + \int_0^1\left(1-\sqrt{x}\right) dx + \int_1^4\left(\sqrt{x}-1\right) dx$$

$$= \left[\frac{2}{3}(x^{3/2})+x\right]_0^4 + \left[x-\frac{2}{3}x^{3/2}\right]_0^1 + \left[\frac{2}{3}x^{3/2}-x\right]_1^4$$

$$= \left(\frac{16}{3}+4\right) + \left(1-\frac{2}{3}\right) + \left(\frac{16}{3}-4\right) - \left(\frac{2}{3}-1\right) = \frac{34}{3}$$

15. $\int\limits_{-1}^{1} f(x)\,dx = \int\limits_{-1}^{0}(x+1)\,dx + \int\limits_{0}^{1}\cos(\pi x)\,dx$

$$= \frac{1}{2}(x+1)^2\Big|_{-1}^{0} + \frac{1}{\pi}\sin(\pi x)\Big|_{0}^{1}$$

$$= \frac{1}{2} + \frac{1}{\pi}(\sin\pi - \sin 0) = \frac{1}{2}.$$

16. We know that

$x - [x] = \{x\}, x - [x + 1] = \{x\} - 1$

$\therefore \quad \int\limits_{-2}^{4} f(x)\,dx = 6\times\frac{1}{2}\times(1\times1) = 3$

17. $I_n = \int\limits_0^1 (x\ln x)^n\,dx$

$$I_n = \left((\ln x)^n\cdot\frac{x^{n+1}}{n+1}\right)\Bigg|_0^1 - \frac{n}{n+1}\int\limits_0^1 x^n(\ln x)^{n-1}\,dx$$

$\therefore \quad I_4 = \frac{-4}{5}\int\limits_0^1 x^4(\ln x^3)\,dx = \frac{-4}{5}$

18. $f(x) = \frac{1}{x+1}$

$\therefore \quad f(g(x)) = \frac{1}{g(x)+1} = \frac{x-1}{x}$

$\therefore \quad g(x)+1 = \frac{x}{x-1} \Rightarrow g(x) = \frac{x}{x-1} - 1 = \frac{1}{x-1}$

$\Rightarrow \quad g(f(x)) = \frac{1}{f(x)-1} = \frac{1}{\frac{1}{x+1}-1} = -\frac{x+1}{x}$

$$I = \int\limits_2^1 -\left(\frac{x+1}{x}\right)dx = \int\limits_1^2\left(1+\frac{1}{x}\right)dx$$

$$= [x + \ln x]_1^2 = (2 + \ln 2) - (1) = 1 + \ln 2.$$

19. Let $I = \int\limits_{3\pi/4}^{4\pi/3}(-\sin x)\,dx = (\cos x)_{3\pi/4}^{\pi/3}$

$$= \left(\frac{-1}{2}\right) - \left(\frac{-1}{\sqrt{2}}\right) = \frac{\sqrt{2}-1}{2}$$

Test Your Concepts-II

1. $\int\limits_{b-1}^{b}\frac{e^{-t}dt}{t-b-1}$,

Put $t = b - y, dt = - dy$

$$= -e^{-b}\int\limits_0^1\frac{e^y dy}{y+1} = -ae^{-b} \qquad \left(\text{as } \because\ a = \int\limits_0^1\frac{e^t dt}{t+1}\right)$$

2. $I = \int\limits_{\pi}^{5\pi/4}\frac{\sin 2x\,dx}{\sin^4 x+\cos^4 x}$

$$I = \int\limits_{\pi}^{5\pi/4}\frac{2\sin x\cos x\,dx}{\sin^4 x+\cos^4 x} = \int\limits_{\pi}^{5\pi/4}\frac{2\sin x\sec^2 x\,dx}{\sin^4 x+1}$$

Put $\tan^2 x = t$; $2\tan x\sec^2 x\,dx = dt$

$$\left.\begin{array}{l}\text{When } x = \frac{5\pi}{4}\ ;\quad t = 1\\ \text{When } x = \pi\quad ;\quad t = 0\end{array}\right]$$

$$I = \int\limits_0^1\frac{dt}{t^2+1} = \tan^{-1}t\Big|_0^1 = \tan^{-1}1 - 0 = \frac{\pi}{4}$$

3. $I = \int\limits_2^3\frac{dx}{x\sqrt{4x^2+1}}$

Put $x = \frac{1}{t} \Rightarrow dx = -\frac{dt}{t^2}$

$$I = \int\limits_{1/2}^{1/3}\frac{-dt}{t^2\left(\frac{1}{t}\right)\sqrt{\frac{4}{t^2}+1}} = -\int\limits_{1/2}^{1/3}\frac{dt}{\sqrt{4+t^2}}$$

$$= \int\limits_{1/3}^{1/2}\frac{dt}{\sqrt{4+t^2}} = \left[\ln\left(t+\sqrt{4+t^2}\right)\right]_{1/3}^{1/2} = \ln\left(\frac{3}{2}\left(\frac{\sqrt{17}+1}{\sqrt{37}+1}\right)\right)$$

4. Put $e^x - 1 = t^2 \Rightarrow e^x\,.\,dx = 2t\,dt$

At $x = 0, t = 0$ and at $x = \log 5, t = 2$

$$\int\limits_0^{\log 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}dx = 2\int\limits_0^2\frac{t^2}{t^2+4}dt = 2\left(\int\limits_0^2 1\,dt - 4\int\limits_0^2\frac{dt}{t^2+4}\right)$$

$$= 2\left[\left(t-2\tan^{-1}\left(\frac{t}{2}\right)\right)_0^2\right]$$

$$= 2\left[\left(2-2\times\frac{\pi}{4}\right)\right] = 4 - \pi$$

5. Putting $t = \frac{1}{u}$ in I_1, we get:

$$I_1 = \int\limits_{1/x}^{1}\frac{1}{1+\frac{1}{u^2}}\left(-\frac{1}{u^2}\right)du$$

$$= -\int\limits_{1/x}^{1}\frac{du}{u^2+1} = \int\limits_{1}^{1/x}\frac{1}{1+u^2}du = I_2$$

6. $I=\int_{1}^{e^{37}}\frac{\pi\sin(\pi\log x)}{x}dx$

Put $\pi\log x=t$, $\frac{\pi}{x}dx=dt$

$$I=\int_{0}^{37\pi}\sin t\,dt$$

$$I=-\left[\cos t\right]_{0}^{37\pi}=2$$

7. The value of $\int_{e^{-1}}^{e^{2}}\left|\frac{\log x}{x}\right|dx$ is

$\log x=0$ when $x=1$ and hence negative in $\left(\frac{1}{e},1\right)$ and positive in $\left(1,e^{2}\right]$

$$\therefore\quad I=\int_{\frac{1}{e}}^{1}\left|\frac{\log x}{x}\right|dx+\int_{1}^{e^{2}}\left|\frac{\log x}{x}\right|dx$$

$$=\int_{1/e}^{1}\frac{-\log x}{x}dx+\int_{1}^{e^{2}}\frac{\log x}{x}dx$$

$$=\left[-\frac{1}{2}(\log x)^{2}\right]_{\frac{1}{e}}^{1}+\left[\frac{1}{2}(\log x)^{2}\right]_{1}^{e^{2}}$$

$$=\frac{1}{2}(-1)^{2}+\frac{4}{2}-0=\frac{5}{2}$$

8. $\int_{0}^{\pi/2}\frac{4x\sin x}{2\sqrt{\sin x}}dx+\int_{0}^{\pi/2}\frac{x^{2}\cos x}{2\sqrt{\sin x}}dx$

$$=\int_{0}^{\pi/2}(2x)\sqrt{\sin x}\,dx+\int_{0}^{\pi/2}\frac{x^{2}\cos x}{2\sqrt{\sin x}}dx$$

$$=x^{2}\sqrt{\sin x}\Big|_{0}^{\pi/2}-\int_{0}^{\pi/2}x^{2}\left(\frac{1}{2\sqrt{\sin x}}\cos x\right)dx+\int_{0}^{\pi/2}\frac{x^{2}\cos x}{2\sqrt{\sin x}}dx$$

$$=\frac{\pi^{2}}{4}$$

9. $I=\int_{1}^{e}\frac{dx}{x\sqrt{\ln x+(1+\ln x)}}$

Put $1+\ln x=t^{2}$

$$I=\int_{1}^{\sqrt{2}}\frac{2t\,dt}{t\sqrt{t^{2}-1}}=2\ln(\sqrt{2}+1)=\ln(3+2\sqrt{2}).$$

10. Put $x-1=t$

$$I=\int_{1}^{2}\frac{(t+1)^{2}-2}{t^{3}}dt=\int_{1}^{2}\frac{(t^{2}+2t-1)}{t^{3}}dt=\int_{1}^{2}\left(\frac{1}{t}+\frac{2}{t^{2}}-\frac{1}{t^{3}}\right)dt$$

$$=\ln t-\frac{2}{t}+\frac{1}{2t^{2}}\Big]_{1}^{2}$$

$$I=\ln 2+1-\frac{3}{8}$$

11. $I=\int_{0}^{1}\frac{dx}{\sqrt{(x+1)^{3}\cdot(3x+1)}}$

Put $x+1=u\Rightarrow dx=du\Rightarrow 3x+1=3(u-1)+1=3u-2$

$$I=\int_{1}^{2}\frac{du}{\sqrt{u^{3}\cdot(3u-2)}}=\int_{1}^{2}\frac{du}{u^{2}\sqrt{\left(3-\frac{2}{u}\right)}}$$

Let $3-\frac{2}{u}=t^{2}\Rightarrow\frac{2}{u^{2}}du=2t\,dt$

$$I=\int_{1}^{\sqrt{2}}\frac{tdt}{t}=t\Big]_{1}^{\sqrt{2}}=\sqrt{2}-1.$$

12. Let $I_{1}=\int_{0}^{\infty}\frac{x^{2}\sqrt{x}}{(1+x)^{6}}dx$

Put $x=\frac{1}{t}\Rightarrow dx=\frac{-1}{t^{2}}dt,$

$$\text{So, }I=\int_{\infty}^{0}\frac{\frac{1}{t^{2}}\cdot\frac{1}{\sqrt{t}}\left(\frac{-1}{t^{2}}\right)dt}{\frac{(1+t)^{6}}{t^{6}}}=\int_{0}^{\infty}\frac{t\sqrt{t}}{(1+t)^{6}}dt=I_{2}$$

$\therefore\quad I_{1}=I_{2}.$

13. $I=\int_{\cos\theta}^{\sec\theta}f(x)\,dx$ …(1)

Put $x=\frac{1}{t}$

$$\text{So, }I=\int_{\sec\theta}^{\cos\theta}\frac{f\left(\frac{1}{t}\right)}{-t^{2}}dt=\int_{\cos\theta}^{\sec\theta}\frac{f\left(\frac{1}{t}\right)}{t^{2}}dt=\int_{\cos\theta}^{\sec\theta}\frac{f\left(\frac{1}{x}\right)}{x^{2}}dx\quad\ldots(2)$$

According to given relation, $f(x)=\frac{-f\left(\frac{1}{x}\right)}{x^{2}}$

$$\therefore\quad I=\int_{\cos\theta}^{\sec\theta}-f(x)\,dx=-I\Rightarrow 2I=0\Rightarrow I=0.$$

14. $\int_{1}^{3}\frac{dx}{(x+1)^{3/2}\sqrt{x}}$

Let $x=\tan^{2}\theta$

$$\int_{\pi/4}^{\pi/3}\frac{2\tan\theta\sec^{2}\theta}{\sec^{3}\theta\cdot\tan\theta}d\theta=2\int_{\pi/4}^{\pi/3}\cos\theta=2\sin\theta\Big]_{\pi/4}^{\pi/3}$$

$$=2\left[\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\right]=\sqrt{3}-\sqrt{2}\equiv\sqrt{a}-\sqrt{b}$$

$\therefore\quad(a-b)=1$

15. $\int_{0}^{\pi/2}(x\cos x+1)e^{\sin x}dx=\int_{0}^{\pi/2}x\cos xe^{\sin x}dx+\int_{0}^{\pi/2}e^{\sin x}dx$

$$=\int_{0}^{\pi/2}x\cos xe^{\sin x}dx+xe^{\sin x}\Big|_{0}^{\pi/2}-\int_{0}^{\pi/2}xe^{\sin x}\cos x\,dx$$

$$= \frac{\pi}{2}\, e^{\sin\pi/2} - 0 = \frac{\pi e}{2}$$

$$\int_0^{\pi/2} (x\sin x - 1)e^{\cos x}\, dx = \int_0^{\pi/2} x\sin x e^{\cos x}\, dx - \int_0^{\pi/2} e^{\cos x}\, dx$$

$$= \int_0^{\pi/2} x\sin x e^{\cos x}\, dx - \left. xe^{\cos x}\right|_0^{\pi/2} + \int_0^{\pi/2} xe^{\cos x}(-\sin x)\, dx$$

$$= -\frac{\pi}{2}\, e^{o} = -\frac{\pi}{2}$$

Required term $= \left|\dfrac{\pi\dfrac{e}{2}}{-\dfrac{\pi}{2}}\right| = e$

Test Your Concepts-III

1. $I = \displaystyle\int_0^a x\sqrt{a^2 - x^2}\, dx + (-x)\ \sqrt{a^2 - (-x)^2}\, dx$

$$I = \int_0^a 0\, dx = 0$$

2. Let $I = \displaystyle\int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin x} + 1}$ …(1)

$$I = \int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin\left(\frac{\pi}{2} - \frac{\pi}{2} - x\right)} + 1}\, dx = \int_{-\pi/2}^{\pi/2} \frac{dx}{e^{-\sin x} + 1}\, dx$$

$$I = \int_{-\pi/2}^{\pi/2} \frac{e^{\sin x}}{e^{\sin x} + 1}\, dx \qquad \text{…(2)}$$

Adding Eqs. (1) and (2) then we get:

$$2I = \int_{-\pi/2}^{\pi/2} 1\ .\, dx = \pi$$

$$\therefore\quad I = \frac{\pi}{2}$$

3. $I = \displaystyle\int_0^1 x(1 - x)^{98}\, dx = \int_0^1 (1 - x)x^{98}\, dx$

$$\therefore\quad f(x)\, dx = \int_0^a f(a - x)\, dx$$

$$= \frac{1}{99} - \frac{1}{100} = \frac{1}{9900}\,.$$

4. $\displaystyle\int_0^1 \tan^{-1}(1 - x + x^2)\, dx = \int_0^1 \left(\frac{\pi}{2} - \cot^{-1}(1 - x + x^2)\right) dx$

$$= \int_0^1 \left(\frac{\pi}{2} - \tan^{-1}\left(\frac{x + (1 - x)}{1 - x(1 - x)}\right)\right) dx$$

$$= \frac{\pi}{2} - \int_0^1 \left(\tan^{-1}(1 - x) + \tan^{-1} x\right) dx$$

Replacing x by $1 - x$ in first integral

$$= \frac{\pi}{2} - 2\int_0^1 \tan^{-1}(x)\, dx$$

$$= \frac{\pi}{2} - 2\left[\tan^{-1} x \cdot x\Big|_0^1 - \frac{1}{2}\int_0^1 \frac{2x}{1 + x^2}\, dx\right] = \ln 2$$

5. As, $f(x) = 1 + \ln(x + \sqrt{x^2 + 1}\,) + 5x^3 - 4x^4$

Clearly, $\ln(\sqrt{x^2 + 1}\ + x)$, $5x^3$ are odd functions, so

$$I = \int_{-1}^1 (1 - 4x^4)e^{-x^4}\, dx = 2\int_0^1 (1 - 4x^4)e^{-x^4}\, dx$$

$$= 2\int_0^1 \frac{d}{dx}(x \cdot e^{-x^4})\, dx$$

$$= 2\left(\frac{x}{e^{x^4}}\right)_0^1 = 2\left(\frac{1}{e} - 0\right) = \frac{2}{e}$$

6. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta - \sin\theta}{(1 + \cos\theta)(1 + \sin\theta)}\, d\theta = \int_0^{\pi/2} \frac{\sin\theta - \cos\theta}{(1 + \sin\theta)(1 + \cos\theta)}\, d\theta$

(using King)

$$2I = 0 \Rightarrow I = 0$$

7. $\displaystyle\int_{-1/2}^{1/2} \sin\left(\tan^{-1}\left(\frac{x^2}{\sqrt{1 - x^4}}\right)\right) dx$

$$\int_{-1/2}^{1/2} \sin(\sin^{-1} x^2)\, dx = \int_{-1/2}^{1/2} x^2\, dx = 2 \cdot \int_0^{1/2} x^2\, dx$$

$$= 2 \cdot \left(\frac{x^3}{3}\right)_0^{1/2} = \frac{2}{3} \cdot \frac{1}{8} = \frac{1}{12}$$

8. $I = \displaystyle\int_0^a \left(\frac{x^2\cos(x) + e^x}{e^x + 1}\, dx + \frac{x^2\cos(x) + e^{-x}}{e^{-x} + 1}\right) dx$

$$\left[\int_{-a}^a f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)dx\right]$$

$$= \int_0^a \left(\frac{x^2\cos(x) + e^x}{e^x + 1} + \frac{x^2\cos(x)e^x + 1}{e^x + 1}\right) dx$$

$$= \int_0^a \left(\frac{x^2\cos(x) + e^x + x^2\cos(x)e^x + 1}{e^x + 1}\right) dx$$

$$= \int_0^a \left(\frac{(x^2\cos(x) + 1)(e^x + 1)}{e^x + 1}\right) dx$$

$$= \int_0^a (x^2\cos(x) + 1)\, dx$$

$$= \int_0^a \underset{\text{(I.B.P.)}}{\underset{\text{(I)}}{\underbrace{x^2}}\,\underset{\text{(II)}}{\underbrace{\cos x}}}\, dx + \int_0^a 1\, dx \qquad \text{…(A)}$$

$$= x^2\sin x\Big]_0^a - 2\int_0^a \underbrace{x\sin x}_{\substack{\text{(I)(II)}\\ \text{(I.B.P.)}}}\,dx + \int_0^a dx$$

$$= (a^2\sin a + a) - 2\left[-x\cos x\right]_0^a + \int_0^a \cos x\,dx$$

$$= a^2\sin a + a + 2a\cos a - 2\sin a$$

Hence, $I = 2a\cos(a) + a + (a^2 - 2)\sin(a)$

9. Using King x by $\dfrac{\pi}{2} - x$

$$I = \int_{-3\pi/4}^{5\pi/4} \frac{\sin x + \cos x}{1 + e^{\frac{\pi}{2} - x - \frac{\pi}{4}}}\,dx$$

$$I = \int_{-3\pi/4}^{5\pi/4} \frac{\cos x + \sin x}{1 + e^{\left(x - \frac{\pi}{4}\right)}}\,e^{\left(x - \frac{\pi}{4}\right)}\,dx$$

$$\therefore\quad 2I = \int_{-3\pi/4}^{5\pi/4} (\cos x + \sin x)dx$$

$$= (\sin x - \cos x)\Big]_{-3\pi/4}^{5\pi/4} = 0 - \left(\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)$$

$$2I = 0 \Rightarrow I = 0$$

10. $I = \displaystyle\int_{-\pi}^{\pi} \frac{\cos^2 x}{1 + a^x}\,dx$

$$\Rightarrow\quad I = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1 + a^{-x}}\,dx = \int_{-\pi}^{\pi} \frac{a^x\cos^2 x}{1 + a^x}\,dx$$

$$\Rightarrow\quad 2I = \int_{-\pi}^{\pi} \cos^2 x\,dx = 2\int_{-\pi}^{\pi} \cos^2 x\,dx$$

$$\Rightarrow\quad 2I = \int_0^{\pi} (1 + \cos 2x)\,dx = \left[x + \frac{\sin 2x}{2}\right]_0^{\pi} = \pi$$

$$\Rightarrow\quad I = \frac{\pi}{2}$$

11. Let $I = \displaystyle\int_{-\pi/4}^{\pi/4} \frac{e^x\sec^2 x\,dx}{e^{2x} - 1}$

If $f(x) = \dfrac{e^x\sec^2 x}{e^{2x} - 1}$

$$\therefore\quad f(-x) = \frac{e^{-x}\sec^2 x}{e^{-2x} - 1}$$

$$= \frac{e^x\sec^2 x}{1 - e^{2x}} = -\frac{e^x\sec^2 x}{e^{2x} - 1}$$

$$= -f(x)$$

$$\therefore\quad I = 0$$

($f(x)$ is odd function)

12. $I = \displaystyle\int_0^{\pi} x\,f(\sin x)\,dx$

Using $\displaystyle\int_a^b f(x)\,dx = \int_0^b f(a + b - x)\,dx$

$$I = \int_0^{\pi} (\pi - x)\,f(\sin(\pi - x))\,dx = \int_0^{\pi} (\pi - x)\,f(\sin x)\,dx$$

$$I = \int_0^{\pi} \pi\,f(\sin x)\,dx - \int_0^{\pi} x\,f(\sin x)\,dx$$

$$I = \pi\int_0^{\pi} f(\sin x)\,dx - I$$

$$\Rightarrow\quad 2I = \pi\int_0^{\pi} f(\sin x)\,dx \Rightarrow I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$$

13. Let $I = \displaystyle\int_0^{\pi} \theta\sin^2\theta\cos^2\theta\,d\theta$

$$= \int_0^{\pi} (\pi - \theta)\sin^2(\pi - \theta)\cos^2(\pi - \theta)\,d\theta$$

$$= \int_0^{\pi} (\pi - \theta)\sin^2\theta\cos^2\theta\,d\theta$$

$$= \pi\int_0^{\pi} \sin^2\theta\cos^2\theta\,d\theta - \int_0^{\pi} \theta\sin^2\theta\cos^2\theta\,d\theta$$

$$= \pi\int_0^{\pi} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta - I$$

$$2I = \frac{\pi}{4}\int_0^{\pi} \sin^2 2\theta = \frac{\pi}{4}\int_0^{\pi} \left(\frac{1 - \cos 4\theta}{2}\right)d\theta$$

$$2I = \frac{\pi}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi} = \frac{\pi^2}{8} \Rightarrow I = \frac{\pi^2}{16}$$

14. $I_n = \displaystyle\int_0^1 1(1 - x^5)^n\,dx$

$$= x(1 - x^5)^n\Big|_0^1 - \int_0^1 x(1 - x^5)^{n-1}\,(0 - 5x^4)dx$$

$$= 0 + 5n\int_0^1 (1 - x^5)^{n-1}x^5\,dx$$

$$= 5n\int_0^1 (1 - x^5)^{n-1}(1 - (1 - x^5))\,dx$$

$$= 5n\int_0^1 (1 - x^5)^{n-1} - (1 - x^5)^n\,dx$$

$I_n = 5n\,I_{n-1} - 5nI_n \Rightarrow (1 + 5n)I_n = 5n\,I_{n-1}$

$n = 11$

$(1 + 55)\,I_{11} = 55I_{10}$

$$\frac{I_{10}}{I_{11}} = \frac{56}{55}$$

15. $I_2 = \int\limits_0^1 (1-x^{50})(1-x^{50})^{100}\ dx$

$$= \int\limits_0^1 (1-x^{50})^{100}\ dx - \int\limits_0^1 x^{50}(1-x^{50})^{100}\ dx$$

$$= I_1 - \int\limits_0^1 x^{50}(1-x^{50})^{100}\ dx$$

$$= I_1 - \int\limits_0^1 x\cdot(1-x^{50})^{100}\ x^{49}\ dx$$

$$= I_1 + \frac{1}{50}\ \left(\frac{x(1-x^{50})^{101}}{101}\right)_0^1 - \frac{1}{50}\int\limits_0^1 \frac{(1-x^{50})^{101}}{101}\ dx$$

$$= I_1 - \frac{1}{5050}\int\limits_0^1 (1-x^{50})^{101}\ dx = I_1 - \frac{1}{5050}\ I_2$$

$$I_2\left(1+\frac{1}{5050}\right) = I_1 \Rightarrow \frac{I_1}{I_2} = \frac{5051}{5050}$$

16. $I = \int\limits_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-7x^3-12x^2+x+1}{x^2+2}\,dx$

Using $\int\limits_{-a}^{a} f(x)\,dx = \int\limits_0^a (f(x)+f(-x))dx$

$$I = \int\limits_0^{\sqrt{2}} \frac{2x^7+3x^6-7x^3-12x^2+x+1}{x^2+2} + \frac{-2x^7+3x^6+7x^3-12x^2-x+1}{x^2+2}\,dx$$

$$I = 2\int\limits_0^{\sqrt{2}} \frac{3x^6-12x^2+1}{x^2+2}\ dx$$

$$I = 2\int\limits_0^{\sqrt{2}} \frac{3x^2(x^4-4)+1}{x^2+2}\,dx$$

$$= 2\int\limits_0^{\sqrt{2}} \left(3x^2(x^2-2)+\frac{1}{x^2+2}\right)dx$$

$$I = 6\int\limits_0^{\sqrt{2}} (x^4-2x^2)\,dx + 2\int\limits_0^{\sqrt{2}} \frac{1}{x^2+2}\,dx$$

$$= \frac{6}{5}\ (\sqrt{2}\,)^5 - 4(\sqrt{2}\,)^3 + \frac{2}{\sqrt{2}}\left[\tan^{-1}\frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}}$$

$$I = \frac{6}{5}\,(2^{5/2}) - 4(2^{3/2}) + \sqrt{2}\ \frac{\pi}{4} = \frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}$$

17. $I = \int\limits_0^{\pi/2} \frac{x\sin x.\cos x}{\cos^4 x+\sin^4 x}\,dx$

$$I = \int\limits_0^{\pi/4} \frac{x\sin x\cdot\cos x}{\cos^4 x+\sin^4 x} + \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cdot\cos\left(\frac{\pi}{2}-x\right)}{\cos^4\left(\frac{\pi}{2}-x\right)+\sin^4\left(\frac{\pi}{2}-x\right)}\,dx$$

$$\therefore \quad I = \frac{\pi}{2}\int_0^{\pi/4} \frac{\tan x.\sec^2 x}{\tan^4 x+1}\,dx$$

$$\Rightarrow \quad I = \frac{\pi}{4}\cdot\int_0^{\pi/4} \frac{1}{1+(\tan^2 x)^2}\cdot d(\tan^2 x),\ \text{where } t=\tan^2 x$$

$$\therefore \quad I = \frac{\pi}{4}\cdot\left[\tan^{-1} t\right]_0^1 = \frac{\pi}{4}\,(\tan^{-1} 1 - \tan^{-1} 0)$$

$$\Rightarrow \quad I = \frac{\pi^2}{16}$$

18. $I = \int\limits_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right)dx$

$$\Rightarrow \quad I = \int\limits_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right) + \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{-2x}{1+x^2}\right)dx$$

$$I = \int\limits_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}\cos^{-1}\left(\frac{2x}{1+x^2}\right) + \frac{x^4}{1-x^4}\left(\pi-\cos^{-1}\left(\frac{2x}{1+x^2}\right)\right)dx$$

$$I = \int\limits_0^{1/\sqrt{3}} \frac{\pi x^4}{1-x^4}dx = \pi\int\limits_0^{1/\sqrt{3}} -1+\frac{1}{2}\left(\frac{1}{1-x^2}+\frac{1}{1+x^2}\right)dx$$

$$I = \pi\left[-\frac{1}{\sqrt{3}}+\frac{1}{4}\left(\ln\frac{1+x}{1-x}\right)_0^{1/\sqrt{3}} + \frac{1}{2}\left(\tan^{-1} x\right]_0^{1/\sqrt{3}}\right]$$

$$I = \frac{\pi}{2}\left[\frac{\pi}{6}+\frac{1}{2}\ln\frac{\sqrt{3}+1}{\sqrt{3}-1}-\frac{2}{\sqrt{3}}\right]$$

19. $I = \int_0^{\pi} \frac{x\sin(2x).\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}\,dx \qquad \ldots(1)$

Using $\int\limits_a^b f(x)\,dx = \int\limits_0^b f(a+b-x)\,dx$

$$I = \int_0^{\pi} \frac{(\pi-x).\sin(2\pi-2x).\sin\left(\frac{\pi}{2}\cos(\pi-x)\right)}{2(\pi-x)-\pi}\,dx \qquad \ldots(2)$$

$$I = \int_0^{\pi} \frac{(x-\pi)\sin 2x\cdot\sin\left(\frac{\pi}{2}\cos x\right)}{(2x-\pi)}\,dx \qquad \ldots(3)$$

Adding (1) and (3), we get,

$$2I = \int_0^{\pi} \sin 2x.\sin\left(\frac{\pi}{2}\cos x\right)dx$$

$$\Rightarrow \quad I = \int_0^{\pi} \sin x\cos x\cdot\sin\left(\frac{\pi}{2}\cos x\right)dx$$

$$\left\{\text{Put, } \frac{\pi}{2}\cos x = t \Rightarrow -\frac{\pi}{2}\sin x\,dx = dt \Rightarrow \sin x\,dx = -\frac{\pi}{2}\,dt\right\}$$

$$\Rightarrow \quad I = \frac{-2}{\pi}\int_{\pi/2}^{-\pi/2} \frac{2}{\pi}\,.\sin t\cdot dt = \frac{4}{\pi^2}\int_{-\pi/2}^{\pi/2} t\sin t\ dt$$

$$\Rightarrow \quad I = \frac{4}{\pi^2}\ \left[-t\cos t+\sin t\right]_{-\pi/2}^{\pi/2} = \frac{4}{\pi^2}\times 2 = \frac{8}{\pi^2}$$

20. $I = \int_0^{\pi/2} \frac{\sin 8x\log\cot x}{\cos 2x}\,dx \quad \int\limits_a^b f(x)\,dx = \int\limits_0^b f(a+b-x)\,dx$

$$I = \int_0^{\pi/2} \frac{\sin(4\pi-8x)\left(\log\cot\left(\frac{\pi}{2}-x\right)\right)}{\cos(\pi-2x)}\,dx$$

$$= \int_0^{\pi/2} \frac{-\sin 8x(-\log\cot x)}{-\cos 2x}\,dx$$

$$I = -\int_0^{\pi/2} \frac{\sin 8x \log\cot x}{\cos 2x}\,dx = -I$$

$$\Rightarrow \quad 2I = 0 \Rightarrow I = 0$$

Test Your Concepts-IV

1. $$\int_0^1 xt^2\,dt = x\int_0^1 t^2\,dt = x\int_0^1 u^2 du$$

$$= x\int_0^1 x^2 dx \neq \int_0^1 x^3 dx$$

2. $$\int_0^x xt\,dt = x\int_0^x t\,dt = x\int_0^x u\,du = x\int_0^x x\,dx$$

3. (i) $\int_0^x \sin x\,dx = -\cos x\big|_0^x = -\cos x + 1 \neq 0$ False

(ii) $$\int_0^x \sin(t-x)\,dx = -\left.\frac{\cos(t-x)}{-1}\right|_0^x$$

$= \cos(t-x) - \cos t \neq \int_0^x \sin t dx$ False

(iii) $\int_0^x xt\,dx = t\int_0^x x\,dx = t\left.\frac{x^2}{2}\right|_0^x = \frac{tx^2}{2} \neq 0$ False

(iv) $$\int_0^x xt\,dt = x\int_0^x t\,dt = x\int_0^x (x+0-t)\,dt$$

$= x\int_0^x (x-t)\,dt = \int_0^x x(x-t)\,dt$ True

(v) $$\int_0^x xt\,dx = t\int_0^x x\,dx = t\int_0^x u\,du$$

$= t\int_0^x (x-u)\,du = \int_0^x t(x-u)\,du$ True

(vi) $\int_0^x \sin(x-t)\,dt = \int_0^x \sin(x-(x-t))\,dt = \int_0^x \sin t\,dt$ True

4. Since, $|\cos x|$ is an even function

$$\Rightarrow \quad I = \int_{-20\pi}^{20\pi} |\cos x|dx = 2\int_0^{20\pi} |\cos x|dx$$

Now, $|\cos x|$ is a periodic function of the period π, because $|\cos x| = |\cos(\pi + x)|$

$$\Rightarrow \quad I = 2\times 20\int_0^{\pi} |\cos x|dx$$

$$= 40\left\{\int_0^{\pi/2} |\cos x|\,dx + \int_{\pi/2}^{\pi} |\cos x|\,dx\right\}$$

$$= 40\left\{\int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{\pi} -\cos x\,dx\right\}$$

$$= 40\left\{\left[\sin x\right]_0^{\pi/2} - \left[\sin x\right]_{\pi/2}^{\pi}\right\} = 40 + 40 = 80$$

5. $$I = \int_0^{50\pi} \sqrt{(1-\cos 2x)}\,dx = \sqrt{2}\int_0^{50\pi} |\sin x|\,dx$$

$|\sin x|$ has period π

$I = 50\sqrt{2}\int_0^{\pi} |\sin x|\,dx = 50\sqrt{2}\int_0^{\pi} \sin x\,dx$ (by property)

$I = 100\sqrt{2}$

6. $$I = \int_0^{\pi/4} (1-2\sin^2 x)^{3/2}\cos x\,dx$$

Put $\sqrt{2}\sin x = \sin\theta$; $\sqrt{2}\cos x = \cos\theta\,d\theta$

$$I = \int_0^{\pi/2} (1-\sin^2\theta)^{3/2}\frac{\cos\theta}{\sqrt{2}}\,d\theta$$

$$I = \frac{1}{\sqrt{2}}\int_0^{\pi/2} \cos^4\theta\,d\theta = \frac{1}{\sqrt{2}}\,\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}$$

$$I = \frac{3\pi}{16\sqrt{2}}$$

7. $T = 1$

$$I = (37-19)\int_0^1 (\{x\}^2 + 3\sin 2\pi x)\,dx = 18\int_0^1 (x^2 + 3\sin 2\pi x)\,dx$$

$$I = 18\left(\frac{x^3}{3} + 3\frac{(-\cos 2\pi x)}{2\pi}\right)_0^1 = 18\left(\frac{1}{3} - \frac{3}{2\pi} - \frac{(0-3)}{2\pi}\right) = 6$$

8. $$\int_{-\pi}^{\pi} \left(x^3 + 3x + 2\right)\sin^4 x.\cos^2 x dx$$

$$= 2\int_{-\pi}^{\pi} \sin^4 x\cos^2 x dx$$

$$= 8\int_o^{\pi/2} \sin^4 x\cos^2 x\,dx = 8\left(\frac{3}{6}.\frac{1}{4}.\frac{1}{2}.\frac{\pi}{2}\right) = \frac{\pi}{4}$$

9. Substituting $x = a\sin\theta$, we get $dx = a\cos\theta\,d\theta$.

when $x = 0$, $\theta = 0$

when $x = a$, $\theta = \pi/2$.

$$I = \int_0^{\pi/2} \frac{a^4\sin^4\theta.a\cos\theta\,d\theta}{\sqrt{a^2 - a^2\sin^2\theta}}$$

$$I = a^4\int_0^{\pi/2} \frac{\sin^4\theta.\cos\theta\,d\theta}{\cos\theta}$$

$I = a^4\int_0^{\pi/2} \sin^4\theta\,d\theta$ (Using Walli's formula)

$$I = a^4 . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2} = \frac{3\pi}{16}a^4$$

10. $g(x) = \int_0^x \cos^4 t\, dt$ (given)

$$g(x+\pi) = \int_0^{\pi+x} \cos^4 t\, dt$$

$$g(x+\pi) = \int_0^{\pi} \cos^4 t\, dt + \int_{\pi}^{\pi+x} \cos^4 t\, dt = I_1 + I_2$$

$$\Rightarrow \quad I_1 = g(\pi);\ I_2 = \int_{\pi}^{\pi+x} \cos^4 t\, dt$$

Put $t = \pi + n\, y \Rightarrow dt = dy$

$$I_2 = \int_0^x \cos^4(y+\pi)\, dy = \int_0^x (-\cos y)^4\, dy$$

$$= \int_0^x \cos^4 y\, dy = g(x)$$

$$\therefore \quad g(x+\pi) = g(x) + g(\pi)$$

11. $\int\limits_0^{n\pi+v} |\sin x|\, dx = \int_0^{\pi} |\sin x|\, dx + \int_0^{2\pi} |\sin x|\, dx + \ldots +$

$$\int_{(n-1)\pi}^{n\pi} |\sin x|\, dx + \int_{n\pi}^{n\pi+v} |\sin x|\, dx$$

$$= \sum_{r=1}^{\pi} \int_{(r-1)\pi}^{\pi} |\sin x|\, dx + \int_{n\pi}^{n\pi+v} |\sin x|\, dx$$

Now to solve, $\int_{(r-1)\pi}^{\pi} |\sin x|\, dx$, we have $x = (r-1)\pi + t$

$$\Rightarrow \quad \sin x = \sin[(r-1)\,\pi + t] = (-1)^{r-1} \sin t$$

and when $x = (r-1)\pi$, $t = \pi$

$$\Rightarrow \quad \int_{(r-1)\pi}^{\pi} |\sin x|\, dx = \int_0^{\pi} |(-1)^{r-1} \sin t|\, dt$$

$$= \int_0^{\pi} |\sin t|\, dt = \int_0^{\pi} \sin t\, dt$$

$$= [-\cos t]_0^{\pi} = -\cos\pi + \cos 0 = 2$$

Again $\int_{n\pi}^{n\pi+v} |\sin x|\, dx$, putting $x = n\pi + t$

Then $\int_{n\pi}^{n\pi+v} |\sin x|\, dx$

$$= \int_0^{v} |(-1)^n \sin t|\, dt = \int_0^{v} \sin t\, dt$$

$$= [-\cos t]_0^{v} = -\cos v + \cos 0 = 1 - \cos v$$

Therefore, $\int_0^{n\pi+v} |\sin x|\, dx$

$$= \sum_{r=1}^{\pi} \int_{(r-1)\pi}^{n} |\sin x|\, dx + \int_{n\pi}^{n\pi+v} |\sin x|\, dx$$

$$= \sum_{r=1}^{\pi} 2 + \int_{n\pi}^{n\pi+v} |\sin x|\, dx$$

$$= 2n + 1 - \cos v$$

12. Put $y = 0 \Rightarrow f(x+1) = f(x)$ ($\because$ $f(x)$ has period 1)

$$I = \int_0^{10} f(10-x)\, dx$$

Put $10 - x = t \Rightarrow dx = -dt$

$$I = \int_0^{10} f(t)\, dx = 10 \int_0^{1} f(t)\, dt \qquad (\because f(x) \text{ has period } 1)$$

13. Since, function is periodic

$$\Rightarrow \quad f(x) = f(x+8) \text{ with period } 8$$

Hence, $\int\limits_0^{200} f(x) dx = 25 \int\limits_0^{8} f(x) dx = 125$.

14. Let $\phi(x) = 2x + \int\limits_0^{x} f(t) dt - a$

As given, $\phi(x) = 0$ have at least two roots of opposite signs. then $\phi(1) = 2+5-a > 0$ and $\phi(-1) = -2+5-a > 0$ and $\phi(0) = 0+0-a < 0$, This is possible when $a \in (0, 3)$

15. $f(x) = \int\limits_0^1 \underset{\text{I}}{t}\, \underbrace{\sin(x+\pi t)}_{\text{II}}\, dt$

(I.B.P)

$$f(x) = \frac{-t}{\pi} \cos(x+\pi t) \Big]_0^1 + \frac{1}{\pi} \int\limits_0^1 \cos(x+\pi t)\, dt$$

$$= \left(\frac{1}{\pi} \cos x - 0\right) + \frac{1}{\pi^2} \sin(x+\pi t) \Big]_0^1$$

$$f(x) = \frac{1}{\pi} \cos x - \frac{2}{\pi^2} \sin x$$

$$\therefore \quad f(x)\,|_{\max} = \sqrt{\frac{1}{x^2} + \frac{4}{x^4}} = \frac{\sqrt{\pi^2+4}}{\pi^2}$$

16. Given $\int\limits_{-2T}^{a+5T} f(x)\, dx = 19$

$$\Rightarrow \quad \int\limits_{-2T}^{5T} f(x)\, dx + \int\limits_{5T}^{a+5T} f(x)\, dx = 19$$

$$\Rightarrow \quad 7 \int\limits_0^{T} f(x)\, dx + \int\limits_0^{a} f(x)\, dx = 19$$

$$\Rightarrow \quad \int\limits_0^{a} f(x)\, dx = 19 - 7 \times 2 = 19 - 14 = 5$$

17. $I = \int\limits_0^1 (e-1)\sqrt{\ln(1+(e-1)x)}\, dx + \int\limits_0^1 e^{x^2}\, dx$

Put $1 + (e-1)x = t$

$$I = \int\limits_1^{e} \sqrt{\ln t}\, dt + \int\limits_0^{1} e^{x^2}\, dx = \int\limits_1^{e} f(x)\, dx + \int\limits_0^{1} g(y)\, dy,$$

Where f is the inverse of g

$$\therefore \quad I = e \times 1 - 1 \times 0 = e.$$

Test Your Concepts-V

1. Using L.H. Rule

$$\lim_{t\to a}\frac{f(t)-\left(\frac{t-a}{2}\right)f'(t)-\{f(t)+f(a)\}\frac{1}{2}}{3(t-a)^2}$$

Using L.H. Rule

$$=\lim_{t\to a}\frac{f'(t)-\left(\frac{t-a}{2}\right)f''(t)-\frac{1}{2}f'(t)-\frac{1}{2}f'(t)}{6(t-a)}$$

$$=-\frac{1}{12}f''(a)=3a \qquad \text{(given)}$$

$$\Rightarrow \quad f''(a)=-36a\Rightarrow f'(a)=-18a^2+60$$

$$\Rightarrow \quad f(a)=-6a^3+60a-60$$

$$\Rightarrow \quad f(0)=-60$$

2. $$\lim_{x\to 0}\int_0^{x^2}\frac{\cos t^2dt}{x\sin x}=\lim_{x\to 0}\frac{\int_0^{x^2}\cos t^2dt}{x\sin x}$$

Using LH rule $\left(\frac{0}{0}\text{ form}\right)$

$$=\lim_{x\to 0}\frac{\cos x^4\cdot 2x-0}{x\cos x+\sin x}=\frac{2(1)}{1+1}=1.$$

3. $I=\int_x^{xy}f(t)\,dt;\ \frac{dI}{dx}=0=f(xy)\,y-f(x)=0$

$f(xy)\,y=f(x)$ put $x=1$

$f(y)\,y=f(1)=2$

$f(y)=\frac{2}{y}\Rightarrow f(x)=\frac{2}{x}$

4. $$\lim_{x\to 0}\frac{\int_0^x(1-\cos 2t)dt}{x\int_0^x\tan t\,dt}=\lim_{x\to 0}\frac{(1-\cos 2x)}{\int_0^x\tan t\,dt+x\tan x} \qquad \text{(Using LH Rule)}$$

$$=\lim_{x\to 0}\frac{2\sin 2x}{\tan x+\tan x+x\sec^2 x} \qquad \text{(Using LH Rule)}$$

$$=\lim_{x\to 0}\frac{4\cos x}{\frac{1}{\cos x}+\frac{1}{\cos x}+\frac{x}{\sin x}\sec^2 x}=\frac{4(1)}{1+1+1}=\frac{4}{3}$$

5. $f'(x)=x(x-1)(x-2)$

$\therefore$ $f(x)$ is decreasing on (1, 2) and increasing on (2, 3).

Here, $f(x)=\frac{x^4}{4}-x^3+x^2-\frac{1}{4}$

$$f(1)=0,f(2)=-\frac{1}{4},f(3)=2$$

Minimum of $f(x)$ is at $x=2$ and maximum at $x=3$ in [1, 3]

Hence, the range is $\left[-\frac{1}{4},2\right]$.

6. $f'(x)>0\Rightarrow\log_{0.5}\left(\frac{2x-8}{x-2}\right)>0$

$$\Rightarrow \quad \frac{2x-8}{x-2}<1\Rightarrow\frac{2x-8}{x-2}-1<0$$

$$\Rightarrow \quad x\in(4,6)$$

7. $f'(x)=\frac{1}{x}f(x)-\frac{1}{x^2}\int_1^x f(t)dt$

$$\Rightarrow \quad f'(x)=\frac{f(x)}{x}-\frac{1}{x}[f(x)-1]$$

$$\Rightarrow \quad f'(x)=\frac{1}{x}\Rightarrow f(x)=\log_e x+c$$

Given $f(1)=1$ $\quad\therefore C=1$ $\quad\therefore f(x)=1+\log_e x$

8. Differentiate the given equation with respect to x,

$$\cos y^2\,\frac{dy}{dx}-0=\frac{\sin x^2}{x^2}\cdot(2x)-0$$

or $$\frac{dy}{dx}=\frac{2\sin x^2}{x}\cdot\sec y^2 \qquad \ldots(1)$$

Differentiating (1) with respect to x,

$$\frac{d^2y}{dx^2}=\frac{2\sin x^2}{x^2}\cdot\sec y^2\tan y^2\cdot 2y\,\frac{dy}{dx}+2\sec y^2\left[\frac{x\cos x^2(2x)-\sin x^2}{x^2}\right]$$

$$\frac{d^2y}{dx^2}=\frac{8\sin^2 x^2}{x^3}\cdot y\sec^2 y^2\tan y^2+2\sec y^2\left(2\cos x^2-\frac{\sin x^2}{x^2}\right)$$

9. Using *L* Hospital's rule, we get:

$$L=\lim_{x\to 0^+}\frac{-x\sin x}{2-2\cos 2x}=\lim_{x\to 0}\frac{-x\sin x}{2(2\sin^2 x)}$$

$$=\lim_{x\to 0}\frac{-1}{4\frac{\sin x}{x}}=-\frac{1}{4}$$

10. Use L'Hospital rule, we have

$$\lim_{x\to a}\frac{1}{x-a}\int_{\sqrt{a}}^{\sqrt{x}}te^t\sin t\,dt\ (a>0)\left(\frac{0}{0}\text{ form}\right)=\lim_{x\to a}\sqrt{x}\ e^{\sqrt{x}}$$

$$\sin\sqrt{x}\ \frac{1}{2\sqrt{x}}=\frac{1}{2}\,e^{\sqrt{a}}\sin\sqrt{a}$$

11. We have, $$\lim_{x\to 0}\frac{\int_0^{x^2}\sin(t^2)dt}{x^6}\left(\frac{0}{0}\text{ form}\right)=\lim_{x\to 0}\frac{2x\sin(x^4)}{6x^5}$$

(Using Leibnitz's rule and L'Hospital's rule)

$$=\frac{1}{3}\lim_{x\to 0}\frac{\sin(x^4)}{x^4}=\frac{1}{3}$$

12. $f(x)=x^2+e^{-x}\cdot\int_0^x e^t\cdot f(t)\,dt$ (Using King property) …(1)

$\therefore$ On differentiating both sides with respect to x, we get

$f'(x)=2x+x^2$ (using (1))

$\Rightarrow$ $f(x)=x^2+\frac{x^3}{3}$ (As, $f(0)=0$)

13. $$\lim_{t\to\frac{\pi}{3}} \frac{\displaystyle\int_{\pi/3}^{t} \frac{3\tan x-\tan^3 x}{\cos\left(x+\frac{\pi}{6}\right)}}{3t-\pi}\left(\frac{0}{0}\right);$$

Apply L.H. rule, we get

$$=\lim_{t\to\frac{\pi}{3}}\frac{3\tan t-\tan^3 t}{3\cos\left(t+\frac{\pi}{6}\right)}\left(\frac{0}{0}\right)$$

$$=\lim_{t\to\frac{\pi}{3}}\frac{3\sec^2 t-3\tan^2 t\sec^2 t}{-3\sin\left(t+\frac{\pi}{6}\right)}=8$$

14. $$\lim_{x\to 0}\frac{(x^4-1)e^{-x^2}\cdot 2x}{x(x-1)}=2$$

15. We have:

$$\lim_{x\to 1}\frac{\displaystyle\int_1^x (t-1)\sin\pi t\,dt}{(x-1)^3} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 1}\frac{(x-1)\sin\pi x}{3(x-1)^2}=\lim_{x\to 1}\frac{\sin(\pi x)}{3(x-1)}\left(\frac{0}{0}\right)$$

$$=\lim_{x\to 1}\frac{\sin(\pi-\pi x)}{3(x-1)}$$

$$=-\frac{\pi}{3}\lim_{x\to 1}\frac{\sin\pi(x-1)}{\pi(x-1)}=-\frac{\pi}{3}\,(1)=-\frac{\pi}{3}$$

16. $$\lim_{x\to-\pi}\frac{\displaystyle\int_0^{\sin x}\sin^{-1}t\,dt}{(x+\pi)^2}=\lim_{x\to-\pi}\frac{(\sin^{-1}(\sin x))\cdot\cos x}{2(x+\pi)} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to-\pi}\frac{-(x+\pi)}{2(x+\pi)}\cos x=\frac{1}{2}$$

17. $$f(x)=\int_2^x\frac{dt}{\sqrt{1+t^4}}$$

Let $g(x)=f^{-1}(x)$

$$g(f(x))=x\Rightarrow g'(f(x))\,f'(x)=1$$

$$\Rightarrow\quad g'(f(x))=\frac{1}{f'(x)}=\sqrt{1+x^4} \qquad \ldots(1)$$

$$f'(x)=\frac{1}{\sqrt{1+x^4}};\ f(2)=0$$

Put $x=2$ in (1) $g'(f(2))=\sqrt{1+16}=\sqrt{17}$

$$\Rightarrow\quad g'(0)=\sqrt{17}$$

18. Using L'Hospital's rule

$$l=\lim_{x\to\infty}\frac{2e^{2x}}{\sqrt{x+e^{4x}}}=\lim_{x\to\infty}\frac{2}{\sqrt{xe^{-4x}+1}}=2.$$

19. $$f'(t)=\frac{\sqrt{1+t^8}}{t^2}.2t=\frac{2\sqrt{1+t^8}}{t^2}$$

$$F'(x)=f(x)$$

$$F''(x)=f'(x)=\left.\frac{2\sqrt{1+x^8}}{x}\right|_{x=2}=\sqrt{1+2^8}=\sqrt{257}$$

Test Your Concepts-VI

1. Since, $f(x)=px+q$, we get:

$$I=\int_a^b (px+q)dx$$

$$I=\lim_{h\to 0}h[f(a)+f(a+h)+\ldots+f(a+(n-1)h)]$$

$$I=\lim_{h\to 0}h[(pa+q)+\{p(a+h)+q\}+\ldots+\{p(a+(n-1)h)+q\}]$$

$$I=\lim_{h\to 0}h[p(a+a+\ldots+a)+ph(1+2+\ldots+(n-1))+q(1+1+\ldots+1)]$$

$$I=\lim_{h\to 0}h\left[pna+\frac{1}{2}pnh(n-1)+qn\right]$$

$$I=\lim_{h\to 0}\left[hpna+\frac{1}{2}pnh(hn-h)+qnh\right] \qquad \ldots(1)$$

Since, $h=\dfrac{(b-a)}{n}$, or $nh=b-a$,

We obtain from (1)

$$I=\lim_{h\to 0}\left[(pa+q)(b-a)+\frac{p}{2}(b-a)(b-a-h)\right]$$

$$=(pa+q)(b-a)+\frac{p}{2}(b-a)^2$$

$$=\frac{p}{2}(b-a)(2a+b-a)+q(b-a)$$

$$=\frac{p}{2}(b^2-a^2)+q(b-a)$$

2. $$L=\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\frac{1}{\sqrt{4n^2-9}}+\ldots\frac{1}{\sqrt{3n^2}}\right]$$

$$L=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\frac{1}{\sqrt{4-(r/n)^2}}$$

Here, $b=1$, $a=0$ and $f(x)=\dfrac{1}{\sqrt{4-x^2}}$

So $L=\displaystyle\int_0^1\frac{dx}{\sqrt{4-x^2}}=\sin^{-1}\frac{x}{2}\Big|_0^1=\frac{\pi}{6}$

3. $$L=\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots+\frac{1}{64n}\right]$$

$$L=\lim_{n\to\infty}\sum_{r=1}^{3n}\frac{n^2}{(n+r)^3}=\lim_{n\to\infty}\sum_{r=1}^{3n}\frac{1}{n\left(1+\frac{r}{n}\right)^3}$$

$$L=\int_0^3\frac{dx}{(1+x)^3}=\left.\frac{-1}{2(1+x)^2}\right|_0^3=\frac{1}{2}\left[-\frac{1}{16}+1\right]$$

$$L=\frac{1}{2}\left[\frac{15}{16}\right]=\frac{15}{32}$$

4. $$\lim_{n\to\infty}\left\{\frac{1}{1-n^2}+\frac{2}{1-n^2}+\frac{3}{1-n^2}+\ldots+\frac{n}{1-n^2}\right\}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{r}{1-n^2}=\lim_{n\to\infty}\frac{1}{\left(\frac{1}{n^2}-1\right)}\sum_{r=1}^{n}\frac{1}{n}\left(\frac{r}{n}\right)$$

Replace $\dfrac{r}{n}$ by x $\dfrac{1}{n}$ by dx

$$=\frac{1}{(0-1)}\int_0^1 x\,dx=\lim_{n\to\infty}\frac{1}{2}\times\frac{1}{(0-1)}=-\frac{1}{2}$$

5. $A = \lim\limits_{n\to\infty} \frac{1}{n}[(n+1)\,(n+2)\,(n+3)\ldots(n+n)]^{1/n}$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\left(\frac{n+3}{n}\right)\ldots\left(\frac{n+n}{n}\right)\right]^{1/n}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\left(1+\frac{1}{n}\right)\right]^{1/n}$$

$$\Rightarrow \quad \log A = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$= \int_0^1 \log(1+x)dx = [x\log(1+x)]_0^1 - \int_0^1 \frac{x}{x+1}dx$$

$$= \log 2 - \int_0^1\left(1-\frac{1}{x+1}\right)dx$$

$$= \log 2 - 1 + \log 2 = \log\left(\frac{4}{e}\right)$$

$$\therefore \quad A = \frac{4}{e}.$$

6. Let $S = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{n}{n^2+r^2}$

$$S = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\left[\frac{1}{1+\left(\frac{r}{n}\right)^2}\right] = \int_0^1 f(x)\,dx\,,$$

we get $S = \int\limits_0^1 \frac{dx}{1+x^2} = [\tan^{-1}x]_0^1 = \frac{\pi}{4}$

7. $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n-1}\frac{1}{n}\sqrt{\left(\frac{n+r}{n-r}\right)}$

$$= \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{n}\sqrt{\frac{1+\frac{r}{n}}{1-\frac{r}{n}}}\cdot\frac{1}{n} = \int_0^1\sqrt{\frac{1+x}{1-x}}\,dx = \int_0^1\frac{1+x}{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1\frac{dx}{\sqrt{1-x^2}} + \int_0^1\frac{x\,dx}{\sqrt{1-x^2}}$$

$$= \left[\sin^{-1}x - \sqrt{1-x^2}\right]_0^1 = [\sin^{-1}1 - 0] - [\sin^{-1}0 - 1] = \frac{\pi}{2} + 1.$$

8. $\lim\limits_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \ldots + \sin\frac{n\pi}{2n}\right)$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\sin\left(\frac{\pi}{2}\frac{r}{n}\right)$$

$$= \int_0^1 \sin\frac{\pi}{2}x\,dx = -\frac{2}{\pi}\left[\cos\frac{\pi}{2}x\right]_0^1$$

$$= -\frac{2}{\pi}\left[\cos\frac{\pi}{2} - \cos 0\right] = \frac{2}{\pi}$$

9. $\sum\limits_{k=1}^{n}\frac{2}{n}\left(2+\frac{2k}{n}\right)^5 = 2\int\limits_0^1(2+2x)^5\,dx$

$$= 64\int_0^1(1+x)^5\,dx = 64\cdot\left.\frac{(1+x)^6}{6}\right]_0^1 = \frac{64\cdot 63}{6} = 672$$

10. $S_n = \frac{1}{n}\sum\limits_{k=1}^{n}\sqrt{1-\left(\frac{k}{n}\right)^2} = \int\limits_0^1\sqrt{1-x^2}\,dx.$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\,d\theta$

$$= \int_0^{\pi/2}\cos^2\theta\,d\theta = \frac{\pi}{4}$$

11. We have,

$$\lim_{n\to\infty}\frac{1}{n^4}\left(\sum_{r=1}^{2n}(3nr^2+2n^2r)\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\left(3\left(\frac{r}{n}\right)^2 + 2\left(\frac{r}{n}\right)\right)$$

$$= \int_0^2(3x^2+2x)\,dx = (x^3+x^2)_0^2 = 8+4 = 12$$

12. Let $L = \lim\limits_{n\to\infty}\frac{1}{n^3}\left(\sqrt{n^2} + 1^2 + 2\sqrt{n^2+2^2} + \ldots + n\sqrt{n^2+n^2}\right)$

$$= \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{n}\sqrt{1+\frac{1}{n^2}} + \frac{2}{n}\sqrt{1+\frac{2^2}{n^2}} + \ldots + \frac{n}{n}\sqrt{1+\frac{n^2}{n^2}}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\sqrt{1+\frac{r^2}{n^2}}\right)$$

$$= \int_0^1 x\sqrt{1+x^2}\,dx = \left[\frac{1}{3}(1+x^2)^{\frac{3}{2}}\right]_0^1 = \frac{2\sqrt{2}-1}{3}$$

13. If $(P_n)^{\frac{-1}{n^2}} = \left[\left(1+\frac{1}{n^2}\right)^{1/n}\cdot\left(1+\frac{2}{n^2}\right)^{2/n}\ldots\left(1+\frac{n^2}{n^2}\right)^{n}\right]^{\frac{-1}{n^2}}$

$$\therefore \quad \lim_{n\to\infty}\left(P(n)^{\frac{-1}{n^2}} = e^{\lim\limits_{n\to\infty}\frac{-1}{n}\sum\limits_{r=1}^{n}\frac{r}{n}\log\left(1+\frac{r^2}{n^2}\right)}\right.$$

$$= e^{-\int_0^1 x\ln(1+x^2)dx} = \frac{\sqrt{e}}{2}$$

14. $f(x) = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\frac{n}{n^2+x^2r^2} = \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\frac{1}{1+x^2\frac{r^2}{n^2}} = \int\limits_0^1\frac{dt}{1+x^2t^2}$

$$f(x) = \frac{1}{x^2}\int_0^1\frac{dt}{\frac{1}{x^2}+t^2} = \frac{1}{x}\left(\tan^{-1}\left(\frac{t}{\frac{1}{x}}\right)\right)_0^1 = \frac{\tan^{-1}x}{x}$$

$$\therefore \quad x\,f(x) = \tan^{-1}x$$

$$\sum_{k=1}^{3}kf(k) = \sum_{k=1}^{3}\tan^{-1}k = \pi$$

15. $\lim\limits_{n\to 0}\frac{1}{n}\sum\limits_{r=0}^{3n-1}\sqrt{\frac{n}{n+r}}$

$$= \sum_{r=0}^{3(n-1)}\frac{1}{\sqrt{1+\frac{r}{n}}} = \int_0^3\frac{dx}{\sqrt{1+x}} = \left(2\sqrt{1+x}\right)\Big|_0^3$$

$$= 2(2-1) = 2$$

Test Your Concepts-VII

1. Let $f(x) = x^3 + 10x + 9\sin x + 5$

$$f'(x) = 3x^2 + 10 + 9\cos x > 0 \quad \forall x \in R$$

Hence, $f(x)$ is always increasing

$$\Rightarrow \quad \frac{1}{f(x)} \text{ is decreasing in } (0, \pi)$$

Then absolute maximum of $f(x)$ in $[0, \pi]$ is $f(0) = \frac{1}{5}$ absolute minimum is $f(\pi) = \frac{1}{\pi^3 + 10\pi + 5}$

$$\text{so, } \frac{1}{\pi^3 + 10\pi + 5} < \frac{1}{x^3 + 10x + 9\sin x + 5} < \frac{1}{5}$$

$$\frac{\pi}{\pi^3 + 10\pi + 5} < \int_0^{\pi} \frac{dx}{x^3 + 10x + 9\sin x + 5} < \frac{\pi}{5}$$

2. Since $0 < x < 1$

$$x^2 < x^2 + x^3 < 2x^2$$

$$4 - x^2 > 4 - x^2 - x^3 > 4 - 2x^2$$

$$\text{so } \frac{1}{\sqrt{4 - x^2}} < \frac{1}{\sqrt{4 - x^2 - x^3}} < \frac{1}{\sqrt{4 - 2x^2}}$$

Now using the property;

If $f(x) \le g(x)$ on $[a, b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx$

Integrate the above relation

$$\int_0^1 \frac{dx}{\sqrt{4 - x^2}} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \int_0^1 \frac{dx}{\sqrt{4 - 2x^2}}$$

$$\left(\sin^{-1}\frac{x}{2}\right)_0^1 < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{1}{\sqrt{2}}\left(\sin^{-1}\frac{x}{\sqrt{2}}\right)_0^1$$

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi}{4\sqrt{2}}$$

Hence, proved.

3. $\frac{1}{2} < \frac{1}{\sqrt[3]{\ln x}} < 1,2 \;\; \forall x \in [3, 4]$

$$\Rightarrow \quad \int_3^4 \frac{1}{2}\, dx < \int_3^4 \frac{1}{\sqrt[3]{\log x}}\, dx < \int_3^4 1\, dx \text{ or } \int_3^4 2\, dx$$

$$\Rightarrow \quad \frac{1}{2} < \int_3^4 \frac{1}{\sqrt[3]{\log x}}\, dx < 1 \text{ or } 2$$

4. $f(x) = x \tan x$

$f'(x) = \tan x + x\sec^2 x > 1$

$$x \in \left(0, \frac{\pi}{4}\right)$$

$$0 < x \tan < \frac{\pi}{4} < 1$$

$$(x \tan x)^{2009} > (x \tan x)^{2010} > (x \tan x)^{2011}$$

$$I_1 > I_2 > I_3$$

Single Correct Choice Type Questions

1. Let $\log_e x = t$

$$I = \int_1^{\log e\pi^2} \left[\log_\pi e^t\right] dt$$

$$\Rightarrow \quad I = \int_1^{\log_e \pi^2} \left[t \log_\pi e\right] dt = \int_1^{\log_e \pi} 0\, dt + \int_{\log_e \pi}^{2\log_e \pi} 1\, dt = \log_e \pi$$

Hence, the correct answer is (B).

2. Let $I = \int_0^{\pi/4} 4x\left(\frac{\pi}{4} - x\right)\ln(1 + \tan x)\, dx$

$$= \int_0^{\pi/4} 4\left(\frac{\pi}{4} - x\right)x \ln\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right)$$

$$= 4 \ln 2 \int_0^{\pi/4} x\left(\frac{\pi}{4} - x\right) dx - I$$

$$\Rightarrow \quad I = 2 \ln 2 \int_0^{\pi/4} x\left(\frac{\pi}{4} - x\right) dx$$

$$= \frac{\pi^3}{192} \ln 2$$

Hence, the correct answer is (B).

3. $I = \int_0^{\pi/2} \sin x \sin 2x \sin 3x \sin 4x\; dx$ …(1)

Using $\left(\because \int_0^{\pi/2} f(x)dx = \int_0^{\pi/2} f\left(\frac{\pi}{2} - x\right)dx\right)$

$$I = \int_0^{\pi/2} \cos x \sin 2x \cos 3x \sin 4x\; dx \qquad \text{…(2)}$$

Adding (1) and (2)

$$2I = \int_0^{\pi/2} \sin 2x \sin 4x(\cos 3x \cos x + \sin 3x \sin x)\, dx$$

$$2I = \int_0^{\pi/2} \cos 2x \sin 2x \sin 4x\; dx$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2 4x dx = \frac{1}{2}\int_0^{\pi/2} \frac{1 - \cos 8x}{2} dx = \frac{\pi}{8}$$

$$\Rightarrow \quad I = \frac{\pi}{16}$$

Hence, the correct answer is (C).

4. $I = \int_0^{\pi/2} \frac{\sin x}{1 + \cos x + \sin x} dx$

$$I = \int_0^{\pi/2} \frac{\sin \frac{x}{2}}{\sin \frac{x}{2} + \cos \frac{x}{2}} dx = 2\int_0^{\pi/4} \frac{\sin x}{\sin x + \cos x} dx$$

Let $\sin x = A(\sin x + \cos x) + B(\cos x - \sin x)$

$$A - B = 1 \text{ and } A + B = 0\,,\; A = \frac{1}{2}\,,\; B = -\frac{1}{2}$$

$$I = 2\times\frac{1}{2}\times\frac{\pi}{4} - 2\times\frac{1}{2}\left[\log(\sin x + \cos x)\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} - \log\sqrt{2}$$

Hence, the correct answer is (C).

5. $f(x) = \int_0^1 \frac{dt}{1+|x-t|} = \int_0^x \frac{dt}{1+x-t} + \int_x^1 \frac{dt}{1+t-x}$

$$f(x) = -\log(1+x-t)\Big|_0^x + \log|(1+t-x)|\Big|_x^1$$

$$f(x) = \log(1+x) + \log(2-x)$$

Hence, the correct answer is (A).

6. $x|x| - x^2|x| \ge 0 \Rightarrow |x|(x-x^2) \ge 0$

$$x = 0,1 \text{ or } x - x^2 > 0 \Rightarrow 0 < x < 1$$

$$f(x) = \begin{cases} x|x| & ;\ 0 \le x \le 1 \\ x^2|x| & ;\ x < 0 \text{ or } x > 1 \end{cases}$$

$$g(x) = \begin{cases} x^2|x| & ;\ 0 \le x \le 1 \\ x|x| & ;\ x < 0 \text{ or } x > 1 \end{cases}$$

$$\int_{-1}^{1} [f(x) - g(x)]dx$$

$$= \int_{-1}^{0} \left(x^2|x| - x|x|\right)dx + \int_0^1 \left(x|x| - x^2|x|\right)dx$$

$$= \int_{-1}^{0} \left(x^2 - x^3\right)dx + \int_0^1 \left(x^2 - x^3\right)dx$$

$$= \int_{-1}^{1} (x^2 - x^3)dx = 2\int_0^1 x^2 dx = \frac{2}{3}$$

Hence, the correct answer is (C).

7. $\int_3^{3+3T} f(2x)\,dx = \frac{1}{2}\int_6^{6+6T} f(2x)d2x$

$$= \frac{1}{2}\int_6^{6+6T} f(t)\,dt = \frac{1}{2}\int_0^{6T} f(t)\,dt$$

$$= \frac{6}{2}\int_0^T f(t)\,dt = 3I$$

Hence, the correct answer is (C).

8. Let $y = \lim\limits_{n\to\infty}\left(\cos\frac{\pi}{2n}\cos\frac{2\pi}{2n}\cos\frac{3\pi}{2n}\ldots\cos\frac{(n-1)\pi}{2n}\right)^{1/n}$

$$\ln y = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n-1}\ln\cos\frac{r\pi}{2n} = \int_0^1 \ln\cos\frac{\pi x}{2}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\ln\cos x\,dx = \ln\frac{1}{2} \qquad \left(\because \int_0^{\pi/2}\ln\cos x dx = -\frac{\pi}{2}\ln 2\right)$$

$$\Rightarrow \quad y = \frac{1}{2}$$

Hence, the correct answer is (D).

9. $|\sin x| - |\cos x|$ is periodic with the period π and $\in [-1, 1]$

$$\therefore \quad \int_0^{2\pi}[|\sin x| - |\cos x|]\,dx = 2\int_0^{\pi}[|\sin x| - |\cos x|]\,dx$$

$$I = 2\left(\int_0^{\pi/4}[|\sin x| - |\cos x|]dx + \int_{\pi/4}^{3\pi/4}[|\sin x| - |\cos x|]\,dx + \int_{3\pi/4}^{\pi}[|\sin x| - |\cos x|]\,dx\right)$$

$$I = 2\left(\int_0^{\pi/4}(-1)dx + \int_{\pi/4}^{3\pi/4} 0\,dx + \int_{3\pi/4}^{\pi}(-1)\,dx\right) = -\pi$$

Hence, the correct answer is (C).

10. $\int_0^{n^2}\left[\sqrt{x}\right]dx = \int_0^{1^2}\left[\sqrt{x}\right]dx + \int_{1^2}^{2^2}\left[\sqrt{x}\right]dx + \ldots + \int_{(n-1)^2}^{n^2}\left[\sqrt{x}dx\right]$

$$= 0 + 1\left(2^2 - 1^2\right) + 2\left(3^2 - 2^2\right) + \ldots + (n-1)\left\{n^2 - (n-1)^2\right\}$$

$$= -\left[1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\right] + (n-1)n^2$$

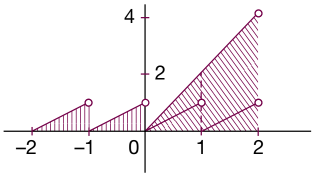
$$= -\frac{n(n-1)(2n-1)}{6} + (n-1)n^2$$

$$= \frac{n}{6}\left\{\left(6n^2 - 6n\right) - \left(2n^2 - 3n + 1\right)\right\}$$

$$= \frac{n}{6}(n-1)(4n+1)$$

Hence, the correct answer is (B).

11. $x + |x| = \begin{cases} 2x & x > 0 \\ 0 & x < 0 \end{cases}$ and $x - [x] = \{x\}$



$$\max\{x + |x|, x - [x]\} = \begin{cases} 2x, x \ge 0 \\ x - [x], x < 0 \end{cases}$$

$$\int_{-2}^{2}\max\{x + |x|, x - [x]\}\ dx = \text{Shaded Area}$$

$$= \frac{1}{2}(1) + \frac{1}{2}(1) + \frac{1}{2}(2)\,4 = 1 + 4 = 5$$

Hence, the correct answer is (B).

12. $I = \int_0^{2\pi}(2\pi - x)\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx \Rightarrow 2I = 2\pi\int_0^{2\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$

$$\Rightarrow \quad I = 2\pi\int_0^{\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx = 2\pi\int_0^{\pi}\ln\left(\frac{3-\cos x}{3+\cos x}\right)dx \text{ (using properly)}$$

$$\Rightarrow \quad I = -2\pi\int_0^{\pi}\ln\left(\frac{3+\cos x}{3-\cos x}\right)dx$$

$$\Rightarrow \quad I = -I \Rightarrow I = 0$$

Hence, the correct answer is (D).

13. $f(x)=\int\limits_{1}^{x}|t|\,dt$

$$f'(x)=|x|>0\ \forall\ x\in\left[\frac{-1}{2}.\frac{1}{2}\right]$$

$\therefore$ Range of $f(x)$ is $\left[f\left(\frac{-1}{2}\right)\cdot f\left(\frac{1}{2}\right)\right]\Rightarrow\left[\frac{-5}{8},\frac{-3}{8}\right]$

Hence, the correct answer is (D).

14. Note that in $\left(-\frac{1}{2},\frac{1}{2}\right)$, $\sin^{-1}(3x-4x^3)$

$$=3\sin^{-1}x \text{ and } \cos^{-1}(4x^3-3x)=2\pi-3\cos^{-1}x$$

Hence, $f(x)=3\sin^{-1}x-2\pi+3\cos^{-1}x=-\frac{\pi}{2}$

$\therefore\quad I=-\frac{\pi}{2}\int\limits_{-1/2}^{1/2}dx=-\frac{\pi}{2}$]

[**Alternate:** $f(x)=\sin^{-1}(3x-4x^3)-[\pi-\cos^{-1}(3x-4x^3)]$

$$=-\pi+(\sin^{-1}(3x-4x^3)+\cos^{-1}(3x-4x^3))=-\frac{\pi}{2}\ ]$$

Hence, the correct answer is (B).

15. Let $f(x)=(\log_e x)\cdot(\log_e x)\ldots\infty$

$$\left.\begin{array}{l} f(x)\to 0 \text{ when } 1<x<e \\ f(x)=1 \quad \text{when } x=e \\ f(x)\to\infty \text{ when } e<x \end{array}\right\}$$

$$g(x)=\begin{cases} x\ ; & 1<x<e \\ \frac{x}{2}\ ; & x=e \\ 0\ ; & e<x<2e \end{cases}$$

$$\int\limits_{1}^{2e}g(x)dx=\int\limits_{1}^{e}x\,dx+\int\limits_{e}^{2e}0\,dx=\left[\frac{x^2}{2}\right]_1^e=\frac{1}{2}(e^2-1)\ .$$

Hence, the correct answer is (C).

16. $f(x)=\int\limits_{0}^{x}\sqrt{1+t^4}\,dt$ on differentiation

We get $f'(x)=\sqrt{1+x^4}$

Let $g(x)=f^{-1}(x)$, we have $g(f(x))=x$

$$g'(f(x))\,f'(x)=1$$

$$g'(f(x))=\frac{1}{f'(x)}=\frac{1}{\sqrt{1+x^4}}$$

Put $x=1$

$$f(1)=C,\ g'(C)=\frac{1}{\sqrt{2}}$$

Hence, the correct answer is (A).

17. $\int\limits_{-3}^{3}\frac{t^2\sin 2t}{t^2+1}dt=0$ as the integrand is an odd function.

Also $\int\limits_{0}^{1}\frac{dt}{t^2+2t\cos\alpha+1}$

$$\int_0^1\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}=\frac{1}{\sin\alpha}\tan^{-1}\frac{t+\cos\alpha}{\sin\alpha}\Bigg|_0^1=\frac{\alpha}{2\sin\alpha}$$

Thus, the given equation reduces to

$$x^2\ \frac{\alpha}{2\sin\alpha}-2=0\Rightarrow x=\pm\ 2\sqrt{\frac{\sin\alpha}{\alpha}}$$

Hence, the correct answer is (D).

18. $f'(x)=A(x+1)\left(x-\frac{1}{3}\right)=\frac{A}{3}\left(3x^2+2x-1\right)$

$$f(x)=\frac{A}{3}\left(x^3+x^2-x\right)+B$$

$$f(-2)=0\Rightarrow B=\frac{2A}{3}$$

$\therefore\quad f(x)=\frac{A}{3}\left(x^3+x^2-x+2\right)$

$$\int\limits_{-1}^{1}f(x)dx=\frac{14}{3}\quad\Rightarrow A=3$$

$\therefore\quad f(x)=x^3+x^2-x+2\qquad f(1)=3$

Hence, the correct answer is (C).

19. $B=\int_1^{\operatorname{cosec}\theta}\frac{1}{t(1+t^2)}$

Let $\frac{1}{t}=u\Rightarrow B=\int_1^{\sin\theta}\frac{-u\,du}{1+u^2}$

$\Rightarrow\quad A+B=0\Rightarrow A=-B$

$$\therefore\quad\begin{vmatrix} A & A^2 & -A \\ e^0 & B^2 & -1 \\ 1 & A^2+B^2 & -1 \end{vmatrix}=0$$

Hence, the correct answer is (C).

20. $\int\limits_{-1}^{1}\left\{x^2+x-3\right\}dx=\int\limits_{-1}^{1}\left\{x^2+x\right\}dx$

$$\int\limits_{-1}^{1}\left(x^2+x-\left[x^2+x\right]\right)dx=\left(\frac{x^3}{3}+\frac{x^2}{2}\right)_{-1}^{1}-\int\limits_{-1}^{0}\left[x^2+x\right]dx-$$

$$\int\limits_{0}^{\frac{\sqrt{5}-1}{2}}\left[x^2+x\right]dx-\int\limits_{\frac{\sqrt{5}-1}{2}}^{1}\left[x^2+x\right]dx$$

$$=\frac{2}{3}+1-0-\left(1-\frac{\sqrt{5}-1}{2}\right)=\frac{1+3\sqrt{5}}{6}$$

Hence, the correct answer is (B).

21. Put $f^{-1}(x)=t\Rightarrow x=f(t)\Rightarrow dx=f'(t)\,dt$

$$I=\int\limits_{a}^{b}(b-t)(2f(t))f'(t)\,dt=\left((b-t)f^2(t)\right)_a^b+\int\limits_{a}^{b}f^2(t)\,dt$$

$$=-(b-a)f^2(a)+\int\limits_{a}^{b}f^2(t)\,dt=\int\limits_{a}^{b}(f^2(x)-f^2(a))\,dx$$

Hence, the correct answer is (B).

22. $\int\limits_{-2n}^{2n+\frac{1}{2}}\left\{\frac{x}{2}\right\}(\sin\pi x)\cdot dx$

It is periodic function with period 2

$$= \int_{-2n}^{2n} \left\{\frac{x}{2}\right\}(\sin \pi x)\cdot dx + \int_{2n}^{2n+1/2} \left\{\frac{x}{2}\right\}(\sin \pi x)\cdot dx$$

$$= 2n\int_{0}^{2} (\sin \pi x)\frac{x}{2}dx + \int_{0}^{1/2} (\sin \pi x)\frac{x}{2}dx$$

$$= 2n.\left[\frac{x}{2}\frac{(-\cos \pi x)}{\pi} + \frac{\sin \pi x}{2\pi^2}\right]_0^2 + \left[\frac{x}{2}\frac{(-\cos \pi x)}{\pi} + \frac{\sin \pi x}{2\pi^2}\right]_0^{1/2}$$

$$= 2n\left[-\frac{1}{\pi} - 0\right] + \left[0 + \frac{1}{2\pi^2}\right] = \frac{-4n\pi + 1}{2\pi^2}$$

Hence, the correct answer is (B).

23. $I_n = \int_0^{\infty} e^{-x}(\sin x)^n \; dx$

$$= \left[\sin^n x(-e^{-x})\right]_0^{\infty} + \int_0^{\infty} n\sin^{n-1} x\cos x e^{-x} \; dx$$

$$= 0 + n\int_0^{\infty} (\sin^{n-1} x\cos x)e^{-x} \; dx$$

$$= n\left[(\sin^{n-1} x\cos x)(-e^{-x})\right]_0^{\infty} -$$

$$n\int_0^{\infty} \{-\sin^n x + (n-1)\sin^{n-2} x\cos^2 x\}(-e^{-x}) \; dx$$

$$= 0 + n\int_0^{\alpha} e^{-x}\{-\sin^n x + (n-1)\sin^{n-2} x(1-\sin^2 x)\} \; dxv$$

$$= n\int_0^{\alpha} e^{-x}\{(n-1)\sin^{n-2} x - n\sin^n x\} \; dx = n(n-1)I_{n-2} - n^2 I_n$$

we have $(1 + n^2)I_n = n(n-1)\, I_{n-2}$ then $\frac{I_n}{I_{n-2}} = \frac{n(n-1)}{n^2+1}$

Hence, the correct answer is (B).

24. $\frac{df(x)}{dx} = \cos\left(\sqrt{x}\right)^2 \frac{d\sqrt{x}}{dx} - \cos\left(\frac{1}{x}\right)^2 . \frac{d}{dx}\left(\frac{1}{x}\right)$

$$= \frac{1}{2\sqrt{x}}\cos x + \frac{\cos x^{-2}}{x^2} = \frac{x\sqrt{x}\cos x + 2\cos(x^{-2})}{2x^2}$$

Hence, the correct answer is (B).

25. $\because \quad A = \int_0^{\pi} \frac{\cos x}{(x+2)^2} dx = \left\{\cos x.\left(-\frac{1}{x+2}\right)\right\}_0^{\pi} - \int_0^{\pi} (-\sin x)\left(-\frac{1}{x+2}\right) dx$

$$A = \frac{1}{\pi+2} + \frac{1}{2} - \int_0^{\pi} \frac{\sin x}{x+2} dx$$

Put $x = 2t$, $dx = 2dt$

$$A = \frac{1}{\pi+2} + \frac{1}{2} - 2\int_0^{\pi/2} \frac{\sin 2t}{2t+2} dt$$

$$\therefore \quad \int_0^{\pi/2} \frac{\sin 2x}{x+1} dx = \frac{1}{\pi+2} + \frac{1}{2} - A$$

Hence, the correct answer is (B).

26. Let $I = \int_0^{n\pi+t} (|\cos x| + |\sin x|) \; dx$

$$I = \int_0^{n\pi} (|\cos x| + |\sin x|) \; dx + \int_{n\pi}^{n\pi+t} (|\cos x| + |\sin x|) \; dx$$

$\because$ Period of $|\cos x| + |\sin x|$ is $\pi/2$,

$$I = \int_0^{2n(\pi/2)} (|\cos x| + |\sin x|) \, dx + \int_{2n(\pi/2)}^{2n(\pi/2)+t} (|\cos x| + |\sin x|) \, dx$$

$$= 2n\int_0^{\pi/2} |\cos x| + |\sin x| dx + \int_0^{t} (|\cos x| + |\sin x|) \, dx \text{ (by property)}$$

$$= 2n\int_0^{\pi/2} (\cos x + \sin x) \; dx + \int_0^{t} (\cos x + \sin x) \; dx$$

$$= 2n[\sin x - \cos x]_0^{\pi/2} + [\sin x - \cos x]_0^{t}$$

$$= 2n((1-0) - (0-1)) + ((\sin t - \cos t) - (0-1))$$

$$= 4n + \sin t - \cos t + 1$$

Hence, the correct answer is (D).

27. $I = 2^{-10}\int_0^{\pi/2} (\sin 2x)^{10} \; dx;$

Put $2x = t \Rightarrow dx = \frac{dt}{2}$

$$I = \frac{2^{-10}}{2}\int_0^{\pi} (\sin t)^{10} \; dt = 2^{-10}\int_0^{\pi/2} (\sin t)^{10} \; dt \qquad \text{(Using Queen rule)}$$

$$I = 2^{-10}\int_0^{\pi/2} (\cos t)^{10} \; dt \qquad \text{(Using King rule)}$$

$$2I = 2^{-10}\int_0^{\pi/2} ((\sin x)^{10} + (\cos x)^{10}) \; dx$$

$$I = 2^{-11}\int_0^{\pi/2} ((\sin x)^{10} + (\cos x)^{10}) \; dx \Rightarrow n = 11.$$

Hence, the correct answer is (D).

28. Let $I = \left|\int_0^{2\pi} [2\sin x]dx\right|$

$$= \left|\int_0^{\pi/6} 0 \, dx + \int_{\pi/6}^{5\pi/6} 1 \, dx + \int_{5\pi/6}^{\pi} 0 \, dx + \int_{\pi}^{7\pi/6} (-1) \, dx + \int_{7\pi/6}^{11\pi/6} (-2) \, dx + \int_{11\pi/6}^{2\pi} (-1)\right| = \pi$$

Hence, the correct answer is (A).

29. We have $\int_0^1 e^{5x^6}(x-\alpha) \; dx = 0$

Algebraic sum of area enclosed by function $f(x) = e^{5x^6}(x-\alpha)$ and x-axis from $x = 0$ to $x = 1$ = zero

It means function $f(x) = e^{5x^6}(x-\alpha)$ changes its sign between 0 to 1.

So, $f(x) = 0$ for some $x \in (0, 1)$

$$e^{5x^6}(x-\alpha) = 0 \Rightarrow 0 < \alpha < 1 \text{ as } e^{5x^6} > 0 \text{ always}$$

Hence, the correct answer is (C).

30. $\int\limits_{\sin x}^{1} t^2.f(t)\,dt = 1 - \sin x$

Differentiating both sides with respect to 'x'

$0 - \sin^2 x \cdot f(\sin x) \cdot \cos x = -\cos x \Rightarrow \cos x[1 - \sin^2 x \cdot f(\sin x)] = 0$

But $\cos x \neq 0$

So, $f(\sin x) = \dfrac{1}{\sin^2 x}$

$$f\left(\frac{1}{\sqrt{3}}\right) = 3$$

Hence, the correct answer is (D).

31. $I = \int\limits_0^{1/\sqrt{3}} \dfrac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $x = \sin\theta$, $dx = \cos\theta\,d\theta$

$$I = \int\limits_0^{\sin^{-1}(1/\sqrt{3})} \frac{\cos\theta\,d\theta}{(1+\sin^2\theta)(\cos\theta)}$$

$$= \int\limits_0^{\sin^{-1}(1/\sqrt{3})} \frac{d\theta}{1+\sin^2\theta} = \int\limits_0^{\sin^{-1}(1/\sqrt{3})} \frac{\operatorname{cosec}^2\theta\,d\theta}{2+\cot^2\theta}$$

Where $t = \cot\theta$

$$= \int\limits_{\sqrt{2}}^{\infty} \frac{dt}{2+t^2} = \frac{1}{\sqrt{2}} \tan^{-1}\frac{t}{\sqrt{2}}\Bigg|_{\sqrt{2}}^{\infty} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$$

Hence, the correct answer is (B).

32. Let $t = \ln x \Rightarrow x = e^t \Rightarrow dx = e^t\,dt \Rightarrow \dfrac{dx}{x} = dt.$

Also $x = 0 \to t \to -\infty$ and $x \to \infty \Rightarrow t \to \infty$

Thus, $\int_0^{\infty} f(x^n + x^{-n}) \ln x \dfrac{dx}{x} = \int_{-\infty}^{\infty} f(e^{nt} + e^{-nt}) \cdot t\,dt = 0$

{$\because$ integrand is on odd function of t}.

Hence, the correct answer is (A).

33. $I = \int\limits_0^{\pi/2} e^{-\alpha\sin x}\,dx > \int\limits_0^{\pi/2} e^{-\alpha}\,dx = \dfrac{\pi}{2} \cdot e^{-\alpha}$

And $I = \int\limits_0^{\pi/2} e^{-\alpha\sin x}\,dx < \int\limits_0^{\pi/2} e^{0}\,dx = \dfrac{\pi}{2}$

Hence, the correct answer is (C).

34. $\int\limits_{-1}^{3} \left(\tan^{-1}\dfrac{x}{x^2+1} + \tan^{-1}\dfrac{x^2+1}{x}\right) dx$

$$= \int\limits_{-1}^{1} \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right) dx + \int\limits_{1}^{3} \left(\tan^{-1}\frac{x}{x^2+1} + \tan^{-1}\frac{x^2+1}{x}\right) dx$$

$$= 0 + \int\limits_1^3 \left(\tan^{-1}\frac{x}{x^2+1} + \cot^{-1}\frac{x}{x^2+1}\right) dx = \left[\frac{\pi}{2}x\right]_1^3 = \pi$$

Hence, the correct answer is (A).

35. $f'(x) = \dfrac{1}{2+x^4}$

By LMVT $f'(C) = \dfrac{f(2) - f(1)}{2-1}$ for some $c \in (1, 2)$

$\Rightarrow \quad f(2) = \dfrac{1}{2+c^4}$ as $f(1) = 0$

$\Rightarrow \quad 1 < c < 2 \Rightarrow 3 < 2 + c^4 < 18 \Rightarrow f(2) < \dfrac{1}{3}$

Hence, the correct answer is (A).

36. $\int\limits_0^{\pi/4} \dfrac{x^2}{(x\sin x + \cos x)^2}\,dx$

$$= \int \frac{x\cos x}{(x\sin x + \cos x)^2} \cdot x\sec x\,dx$$

$$= \left(\frac{-x\sec x}{x\sin x + \cos x}\right)_0^{\pi/4} + \int\limits_0^{\pi/4} \sec^2 x\,dx$$

$$= \frac{-2\pi}{\pi+4} + 1 = \frac{4-\pi}{4+\pi}$$

Hence, the correct answer is (C).

37. Given integral $= 2\int_0^{\pi/4} \dfrac{1}{(\cos x)\sqrt{\cos 2x}}\,dx$

$$= 2\int_0^{\pi/4} \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}\,dx = 2\int_0^1 \frac{dt}{\sqrt{1-t^2}}$$

$$= 2\left[\sin^{-1} t\right]_0^1 = 2 \times \frac{\pi}{2} = \pi$$

Hence, the correct answer is (D).

38. $F'(x) = f(x)$

$$F(x) = x\left(1 + \sqrt{x}\right) = x + x^{3/2}$$

$\therefore \quad F'(x) = f(x) = 1 + \dfrac{3}{2}\sqrt{x}$

$\therefore \quad f(4) = 4$

Hence, the correct answer is (C).

39. $\int\limits_0^1 f(x)\,dx = -\int\limits_1^0 f(x)\,dx = -\left[\int\limits_1^{\frac{1}{2}} 1dx + \int\limits_{\frac{1}{2}}^{\frac{1}{2^2}} \frac{1}{2}\,dx + \ldots\infty\right]$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \left(\frac{1}{2^4} - \frac{1}{2^5}\right) + \ldots = \frac{2}{3}$$

Hence, the correct answer is (C).

40. $f(x) = e^x + e^x\int_0^1 f(t)\,dt = e^x + Ke^x$ where $K = \int\limits_0^1 f(t)\,dt$

$\therefore \quad K = \int\limits_0^1 \left(e^t + Ke^t\right) dt = e + Ke - 1 - K$

$\Rightarrow \quad K = \dfrac{e-1}{2-e} \Rightarrow f(x) = e^x\left(1 + \dfrac{e-1}{2-e}\right) = \dfrac{e^x}{2-e}$

Hence, the correct answer is (A).

41. $I = \int\limits_1^x \dfrac{e^t}{(t+a)^2}\,dt$ put $t + a = z$

$$I = \int\limits_{1+a}^{a+x} \frac{e^{-a}e^z}{z^2}\,dt$$

$$= e^{-a}\left(\int_{1+a}^{1} \frac{e^z}{z^2} dz + \int_{1}^{a+x} \frac{e^z}{z^2} dz\right)$$

$$= e^{-a}\left(f\left(x+a\right) - f\left(1+a\right)\right).$$

Hence, the correct answer is (C).

42. $f(x) = \int_0^x (1+t^3)^{-1/2}\, dt$

i.e., $f[g(x)] = \int_0^{g(x)} (1+t^3)^{-1/2}\, dt$

i.e., $x = \int_0^{g(x)} (1+t^3)^{-1/2}\, dt$ $[\because\ g$ is inverse of $f \Rightarrow f[g(x)] = x]$

Differentiating with respect to x, we have:

$1 = (1+g^3)^{-1/2} \cdot g'$

i.e., $(g')^2 = 1 + g^3$

Differentiating again with respect to x, we have:

$2g'g'' = 3g^2g'$ gives $\frac{g''}{g^2} = \frac{3}{2}$

Hence, the correct answer is (A).

43. $y = f(x) = \int_0^1 \frac{x^2+t^2}{2-t}\, dt$

$$y = \int_0^1 \frac{x^2}{2-t} dt + \int_0^1 \frac{t^2}{2-t} dt$$

$$y = -x^2 \ln\ (2-t)\Big|_0^1\ + C$$

$$y = -x^2\, [0 - \ln 2] + C$$

$$\Rightarrow \quad y = x^2 \ln 2 + c$$

Hence, the correct answer is (C).

44. Put $x - 3 = t \Rightarrow dx = dt$

$$= \int_0^3 \left(\sqrt{t+3+\sqrt{12t}} + \sqrt{t+3-\sqrt{12t}}\right) dt$$

Now replace variable t by x

$$= \int_0^3 \left(\sqrt{(x+3)+2\sqrt{3}\sqrt{x}} + \sqrt{(x+3)-2\sqrt{3}\sqrt{x}}\right) dx$$

$$= \int_0^3 \left(\left(\sqrt{x}+\sqrt{3}\right)+\left(\sqrt{3}-\sqrt{x}\right)\right) dx = \int_0^3 2\sqrt{3}\, dx = 6\sqrt{3}$$

Hence, the correct answer is (A).

45. $\because\ |z| =$ real and positive, imaginary part is zero

$\therefore\quad \arg\ |z|\ = 0$

$\Rightarrow\quad [\arg\ |z|\,] = 0$

$\therefore\quad \int_{x=0}^{100} [\arg|z|\,]\, dx = \int_{x=0}^{100} 0\ \cdot dx = 0$

Hence, the correct answer is (A).

46. Put $x^3 = z, I_1 = \frac{1}{3}\int_0^1 \frac{dz}{(2-z)e^z}$

$1 - z = y \Rightarrow -d_2 = dy$

$$I_1 = -\frac{1}{3}\int_1^0 \frac{dy}{(1+y)e^{1-y}}$$

$$= \frac{1}{3e}\int_0^1 \frac{e^y dy}{1+y}$$

$$I_1 = \frac{1}{3e} I_2$$

Hence, the correct answer is (C).

47. $f\left(x\right) = \int_1^x \frac{\tan^{-1} t}{t} dt$

$$f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\tan^{-1} t}{t} dt$$

$$\Rightarrow \quad f\left(\frac{1}{x}\right) = -\int_1^x \frac{\cot^{-1} t}{t} dt$$

$$\Rightarrow \quad f\left(x\right) - f\left(\frac{1}{x}\right) = \frac{\pi}{2}\int_1^x \frac{dt}{t} = \frac{\pi}{2}\log x \Rightarrow f\left(e^2\right) - f\left(\frac{1}{e^2}\right) = 2\pi$$

Hence, the correct answer is (D).

48. $\frac{f'(x)}{f(x)} = 1$

$\ln f(x) = x + c \Rightarrow f(x) = ke^x\ k = 1,$

We get $f(x) = e^x$ and $g(x) = x^2 - e^x$

$$\int_0^1 e^x(x^2 - e^x)\, dx = \int_0^1 x^2 e^x\, dx - \int_0^1 e^{2x}\, dx$$

$$= x^2 e^x\Big|_0^1 - \int_0^1 2xe^x\, dx - \frac{e^{2x}}{2}\Bigg|_0^1$$

$$= e - \left(e^x 2x - 2e^x\right)_0^1 - \left(\frac{e^2-1}{2}\right) = e - (0+2) - \frac{(e^2-1)}{2}$$

$$= \frac{2e - 4 - e^2 + 1}{2} = e - \frac{3}{2} - \frac{e^2}{2}$$

Hence, the correct answer is (D).

49. $y = \int_1^x x^2\sqrt{\ln t}\ dt = x^2\int_1^x \sqrt{\ln t}\ dt$

$$\frac{dy}{dx} = 2x\int_1^x \sqrt{\ln t}\ dt + x^2\sqrt{\ln x}$$

$$\frac{d^2y}{dx^2} = 2\int_1^x \sqrt{\ln t}\ dt + 2x\sqrt{\ln x}\ + 2x\sqrt{\ln x}\ + \frac{x^2}{2\sqrt{\ln x}} \cdot \frac{1}{x}$$

$$\frac{d^3y}{dx^3} = 2\sqrt{\ln x} + 4\sqrt{\ln x} + \frac{4x}{2\sqrt{\ln x}} \cdot \frac{1}{x} + \frac{1}{2\sqrt{\ln x}} + \frac{x}{2}\left(-\frac{1}{2}(\ln x)^{-\frac{3}{2}}\right)\frac{1}{x}$$

Put at $x = e$

$$\frac{d^3y}{dx^3} = 2 + 4 + 2 + \frac{1}{2} - \frac{1}{4} = 8 + \frac{1}{4} = \frac{33}{4}$$

Hence, the correct answer is (C).

50. Diff. w.r.t. x

$$3x^2x^6 f(x^3) = \frac{3}{13} \cdot 13x^{12}$$

$$f(x^3) = x^4$$

Put $x = \frac{2}{3}$ $\quad f\left(\frac{8}{27}\right) = \frac{16}{81}$

Hence, the correct answer is (C).

51. $$I = \int_{-2\pi}^{2\pi} \frac{\sin^6 x}{(\sin^6 x + \cos^6 x)(1+e^{-x})}\,dx \qquad \ldots(1)$$

$$I = \int_{-2\pi}^{2\pi} \frac{\sin^6 x}{(\sin^6 x + \cos^6 x)(1+e^{x})}\,dx$$

$$I = \int_{-2\pi}^{2\pi} \frac{\sin^6 xe^{-x}\,dx}{(\sin^6 x + \cos^6 x)(1+e^{-x})} \qquad \ldots(2)$$

Adding equation (1) and (2)

$$I = \frac{1}{2}\int_{-2\pi}^{2\pi} \frac{\sin^6 x}{\sin^6 x + \cos^6 x}\,dx$$

$$I = \frac{1}{2}.2\int_{0}^{2\pi} \frac{\sin^6 x}{\sin^6 x + \cos^6 x}\,dx$$

$$= 2\int_{0}^{\pi} \frac{\sin^6 x}{\sin^6 x + \cos^6 x}\,dx$$

$$I = 4\int_{0}^{\pi/2} \frac{\sin^6 x}{\sin^6 x + \cos^6 x}\,dx \qquad \ldots(3)$$

$$I = 4\int_{0}^{\pi/2} \frac{\cos^6 x}{\sin^6 x + \cos^6 x}\,dx \qquad \ldots(4)$$

Adding equation (3) and (4)

$$I = 2\int_{0}^{\pi/2} dx \;= \pi$$

Hence, the correct answer is (B).

52. $$I = \int_{0}^{2a} f(x)g(x)h(x)\,dx$$

$$= \int_{0}^{2a} f(2a-x)g(2a-x)h(2a-x)\,dx$$

$$= \int_{0}^{2a} f(x)g(x)[3-h(x)]\,dx$$

$$I = \frac{3}{2}\int_{0}^{2a} f(x)g(x)\,dx = 3\int_{0}^{a} f(x)g(x)\,dx$$

Hence, the correct answer is (B).

53. $$I_{n+1} = \int_{0}^{\pi/2} \cos^{n+1} x\cos(n+1)x\;dx$$

$$= \int_{0}^{\pi/2} \cos^{n+1} x\,\cos x\cos(n+1)x$$

$$= \int_{0}^{\pi/2} \cos^{n+1} x\;\;(\cos nx\cos x - \sin nx\sin x)\,dx$$

$$= \int_{0}^{\pi/2} \cos^{n} x\cos nx(1-\sin^2 x)\,dx - \int_{0}^{\pi/2} \cos^{n+1} x\cdot\sin nx\sin x\;dx$$

$\therefore \quad I_{n+1} = I_n - I_{n+1} \Rightarrow 2I_{n+1} = I_n$

$\therefore \quad I_n : I_{n+1} = 2$

Hence, the correct answer is (D).

54. $$I_n = \left(\frac{x^{n+1}}{n+1}\tan^{-1} x\right)_0^1 - \int_0^1 \frac{x^{n+1}}{n+1}.\frac{1}{1+x^2}\,dx$$

$$(n+1)I_n = \frac{\pi}{4} - \int_0^1 \frac{x^{n+1}}{1+x^2}\,dx \qquad \ldots(1)$$

$n \to n+2$

$$(n+3)\,I_{n+2} = \frac{\pi}{4} - \int_0^1 \frac{x^{n+3}}{1+x^2}\,dx \qquad \ldots(2)$$

$$\therefore \quad (n+1)\,I_n + (n+3)\,I_{n+2} = \frac{\pi}{2} - \frac{1}{n+2}$$

$\therefore \quad a_n = (n+3) \Rightarrow a_1, a_2, a_3 \ldots$ are in A.P.

$b_n = (n+1) \Rightarrow b_1, b_2 \ldots$ are in A.P.

$c_n = \frac{\pi}{2} - \frac{1}{x+2}$ not in any progression.

Hence, the correct answer is (A).

55. Consider $a_n + a_{n+2} - 2a_{n+1} = A \quad (\text{Say})$

$$\therefore \quad A = \int_0^{\pi/2}\left[\frac{(1-\cos 2nx)+1-\cos(2n+4)x-2(1-\cos(2n+2)x)}{1-\cos 2x}\right]dx$$

$$A = \int_0^{\pi/2}\left[\frac{2\cos(2n+2)x-2\cos(2n+2)x\cos 2x}{1-\cos 2x}\right]dx$$

$$A = \int_0^{\pi/2} 2\cos(2n+2)x\;dx \;= 2\left(\frac{\sin(2n+2)x}{2n+2}\right)_0^{\pi/2} = 0$$

$\therefore \quad a_n, a_{n+1}, a_{n+2} \in AP,$ along with

$$a_1 = \int_0^{\pi/2} \frac{1-\cos 2x}{1-\cos 2x}\,dx \qquad \therefore\; a_1 = \frac{\pi}{2}$$

$$\text{Req.} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$$

$$C_1 \to C_1 + C_3 - 2C_2 = \begin{vmatrix} 0 & a_2 & a_3 \\ 0 & a_5 & a_6 \\ 0 & a_8 & a_9 \end{vmatrix} = 0$$

Hence, the correct answer is (D).

56. $$I_1 = \int_{\sec^2 z}^{2-\tan^2 z} xf(x(3-x))\;dx$$

$$2\,I_2 = \int_{\sec^2 z}^{2-\tan^2 z} (3-x)f((3-x)x)\;dx$$

$$2\,I_1 = \int_{\sec^2 z}^{2-\tan^2 z} 3f(x(3-x))\;dx = 3\int_{\sec^2 z}^{2-\tan^2 z} ((x)(3-x))\;dx$$

$$2\,I_1 = 3\,I_2 \Rightarrow \frac{I_1}{I_2} = \frac{3}{2}$$

Hence, the correct answer is (A).

57. $$\int_{\pi/4}^{\pi/2} \ln\cos x e^{x}\left(\cot x-\operatorname{cosec}^2 x\right)dx=\left[\ln\cos x\cdot e^{x}\cot x+e^{x}\right]_{\pi/4}^{\pi/2}$$

$$=\left[e^{x}\left(\ln\cos x\cot x+1\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}=e^{\frac{\pi}{2}}-e^{\frac{\pi}{4}}\left(\ln\frac{1}{\sqrt{2}}+1\right)$$

$$=e^{\pi/4}\left[e^{\pi/4}+\frac{1}{2}\ln 2-1\right]$$

Hence, the correct answer is (A).

58. $$\int_{-2}^{-1}\frac{x^2dx}{\sqrt{x^4-x^2+1}}=I$$

Let $$I_1=\int_{-2}^{-1}\frac{xdx}{\sqrt{1+x^2+\frac{1}{x^2}-2}}=\int_{-2}^{-1}\frac{x|x|dx}{\sqrt{x^4+1-x^2}}$$

$$=-\int_{-2}^{-1}\frac{x^2dx}{\sqrt{x^4+1-x^2}}=-I$$

and $$I_2=\int_{-1}^{-1/2}\frac{dx}{x^3\sqrt{x^2+\frac{1}{x^2}-1}}=-\int_{-1/2}^{-1}\frac{dx}{x^3\sqrt{x^2+\frac{1}{x^2}-1}}$$

Put $x=\frac{1}{t}\Rightarrow dx=-\frac{1}{t^2}dt$

$$=\int_{-2}^{-1}\frac{\frac{1}{t^2}dt}{\left(\frac{1}{t^3}\right)\sqrt{\frac{1}{t^2}+t^2-1}}$$

$$=\int_{-2}^{-1}\frac{t|t|dt}{\sqrt{t^4-t^2+1}}=\int_{-2}^{-1}\frac{-t^2dt}{\sqrt{t^4-t^2+1}}=-I$$

Required integration $=I_1+I_2=-2I$

Hence, the correct answer is (D).

59. As $\sum_{i=1}^{6}(\sin^{-1}x_i+\cos^{-1}y_i)\;=9\pi$

$\Rightarrow\quad(\sin^{-1}x_1+\sin^{-1}x_2+\ldots\sin^{-1}x_6)+(\cos^{-1}y_1+\cos^{-1}y_2+\ldots\cos^{-1}y_6)=9\pi,$

Which is possible only when $\sin^{-1}x_i=\frac{\pi}{2}$ and $\cos^{-1}y_i=\pi$, $\forall\ i=1, 2, \ldots 6$.

$\therefore\quad x_i=1$ and $y_i=-1$, $\forall\ i=1, 2, \ldots 6$.

$\Rightarrow\quad\sum_{i=1}^{6}x_i\;=6$ and $\sum_{i=1}^{6}y_i\;=-6$.

Now, $$\int_{6}^{-6}\underbrace{x\ln(1+x^2)(e^x+e^{-x})dx=0}_{\text{old function}}$$

(Using the property of definite integral)

Hence, the correct answer is (A).

60. $$f(-x)=\begin{vmatrix}-x & \cos x & e^{x^2}\\ -\sin x & x^2 & \sec x\\ -\tan x & 1 & 2\end{vmatrix}=-f(x)$$

$\therefore\quad\int_{-\pi/2}^{\pi/2}f(x)dx=0\qquad[\because f(x)$ is an odd function].

Hence, the correct answer is (D).

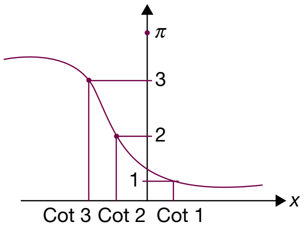
61. $$\lim_{x\to 0}\frac{1}{x}\left(\int_{y}^{c}e^{\sin^2 t}dt-\int_{x+y}^{c}e^{\sin^2 t}dt\right)$$

$$=\lim_{x\to 0}\frac{1}{x}\left(\int_{y}^{c}e^{\sin^2 t}dt+\int_{c}^{x+y}e^{\sin^2 t}dt\right)$$

$$=\lim_{x\to 0}\frac{\int_{y}^{x+y}e^{\sin^2 t}dt}{x}\;=\lim_{x\to 0}\frac{e^{\sin^2(x+y)}-0}{1}\;=e^{\sin^2 y}.$$

Hence, the correct answer is (A).

62. $$\left[\cot^{-1}x\right]=\begin{cases}3; & -\infty<x\le\cot 3\\ 2; & \cot 3<x\le\cot 2\\ 1; & \cot 2<x\le\cot 1\\ 0; & x>\cot 1\end{cases}$$



$\therefore\quad$ The required value

$$=\int_{-2011}^{\cot 3}3\;dx+\int_{\cot 3}^{\cot 2}2\;dx+\int_{\cot 2}^{\cot 1}1dx+\int_{\cot 1}^{\infty}0\;dx$$

$=\;\cot 3+\cot 2+\cot 1+6033$

Hence, the correct answer is (C).

63. $$\int_{1}^{xy}f(t)dt=y\int_{1}^{x}f(t)dt+x\int_{1}^{y}f(t)\;dt\;\forall\,x,y\in R^{+}$$

Diff. w.r.t 'x'

$$y.f(xy)=y\,f(x)+\int_{1}^{y}f(t)dt$$

Put $x=1$, $y.f(y)=3y+\int_{1}^{y}f(t)dt$

Differentiate w.r.t. y'

$f(y)+y\,f'(y)=3+f(y)$

$\Rightarrow\quad f'(y)=\frac{3}{y}$ integrating $f(y)=3\log y+k\qquad\because\ f(1)=3; k=3.$

$\therefore\quad f(x)=3+3\log x$

Hence, the correct answer is (C).

64. $$\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{r}}{\sqrt{n}\cdot n\left(\frac{8}{\sqrt{n}}+10\right)}\;=\int_{0}^{1}\frac{\sqrt{x}}{10}dx$$

$$=\left.\frac{x^{3/2}}{\frac{3}{2}(10)}\right|_{0}^{1}\;=\frac{2}{3(10)}\;=\frac{1}{15}$$

Hence, the correct answer is (B).

65. $\left[x+[x+[x]]\right]=[x+2[x]]=3[x]$

$$\therefore \quad \int_{-2011}^{2012} 3[x]\,dx = 3\left[\int_{-2011}^{0}[x]dx+\int_{0}^{2012}[x]dx\right]=0$$

Hence, the correct answer is (B).

66. $I=\int_0^{\pi}\dfrac{\sec^4\frac{x}{2}dx}{\left(5+5\tan^2\frac{x}{2}+4-4\tan^2\frac{x}{2}\right)^2}$

$$I=\int_0^{\pi}\frac{\sec^4\frac{x}{2}dx}{\left(9+\tan^2\frac{x}{2}\right)^2}$$

$$\tan\frac{x}{2}=t \Rightarrow I=2\int_0^{\infty}\frac{1+t^2}{\left(t^2+9\right)^2}dt$$

But $t=3\tan\theta$

$$I=\frac{2}{27}\int_0^{\pi/2}\left(\cos^2\theta+9\sin^2\theta\right)dx=\frac{5\pi}{27}$$

Hence, the correct answer is (C).

67. $Arg\left|\dfrac{z_1-\left(\dfrac{z_2+z_3}{2}\right)}{z_3-z_2}\right|=\dfrac{\pi}{2}$ = in equilateral triangle, angle between median and opposite side is 90°

Required integration, $I=\int_2^4\frac{\pi}{2}\,dx=2\cdot\frac{\pi}{2}=\pi$

Hence, the correct answer is (A).

68. $I_2=\int_1^{\operatorname{cosec}\theta}\dfrac{dx}{x\left(x^2+1\right)}$

Put $x=\dfrac{1}{t}\Rightarrow dx=-\dfrac{1}{t^2}\,dt$

$I_2=-\int_1^{\sin\theta}\dfrac{t}{1+t^2}dt$, we get $I_2=-I_1$

$$\therefore \quad \begin{vmatrix} I_1 & I_1^2 & I_2 \\ e^{I_1+I_2} & I_2^2 & -1 \\ 1 & I_1^2+I_2^2 & -1 \end{vmatrix}=0$$

Hence, the correct answer is (D).

69. Let $I=\int_{-\pi}^{\pi}\sin^2x\cdot\cos^4x\,dx-\int_{-\pi}^{\pi}\sin^2x\cdot\cos^4x\cdot x^3\,dx$

$$I=4\int_0^{\pi/2}\sin^2x.\cos^4x\,dx-0 \qquad \ldots(1)$$

($\because$ $\sin^2x\cos^4x\cdot x^3$ is an odd function)

$$I=4\int_0^{\pi/2}\cos^2x\sin^4x\,dx=4\left(\frac{1\cdot3\cdot1}{6\cdot4\cdot2}\right)\frac{\pi}{2} \qquad \ldots(2)$$

Using Walli's

$$I=\frac{\pi}{8}$$

Hence, the correct answer is (C).

70. $R_1\to R_1-R_2; R_2\to R_2-R_3$ and Let 'd' be the common difference

$$f(x)=\begin{vmatrix} p-q & q-r & p-r+1 \\ q-r & r-s & -1-s+q \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

$$=\begin{vmatrix} -d & -d & 1-d \\ -d & -d & -1-d \\ r+\sin x & s+\sin x & s-q+\sin x \end{vmatrix}$$

$C_1\to C_1-C_2$

$$=\begin{vmatrix} 0 & -d & 1-d \\ 0 & -d & -1-d \\ r-s & s+\sin x & s-q+\sin x \end{vmatrix}$$

$= (r-s)\,[d+d^2+d-d^2] = (r-s)\,(2d)$

$= -2d^2$

$$\int_0^2 f(x)dx=-4;\ \int_0^2 -2d^2dx=-4;\ d=\pm1$$

Hence, the correct answer is (D).

71. $f(x)=x^2+\int_0^x e^{-t}f(x-t)dt=x^2+\int_0^x e^{-(x-t)}f(t)dt$

$$=x^2+e^{-x}\int_0^x e^{+t}f(t)dt$$

$$f'(x)=2x+e^{-x}\left(e^xf(x)-0\right)-e^{-x}\int_0^x e^tf(t)dt$$

$= 2x + f(x) - (f(x) - x^2) = x^2 + 2x$

$\Rightarrow \quad f(x)=\dfrac{x^3}{3}+x^2+k=\dfrac{x^3}{3}+x^2$ $\qquad (\therefore\ f(0)=0)$

$\therefore \quad f(1)+f(2)+\ldots+f(9)$

$=\dfrac{1}{3}\,(1^3+2^3+\ldots 9^3)+(1^2+2^2+\ldots+9^2)$

$=\dfrac{1}{3}\left(\dfrac{81\times100}{4}\right)+\dfrac{9\times10\times19}{6}=960$

Hence, the correct answer is (A).

72. $(f(x))^{2014}=1+\int_0^x f(t)\,dt$

Put $f(0)=1$

$2014(f(x))^{2013}f'(x)=f(x)$

$2014(f(x))^{2012}f'(x)=1$

Let $y=f(x)$

$2014\int y^{2012}\,dy=\int dx \Rightarrow 2014\,\dfrac{y^{2013}}{2013}=x+c$

$(f(x))^{2013}=\dfrac{2013}{2014}(x+c);\ f(0)=1=\dfrac{2013}{2014}(c)$

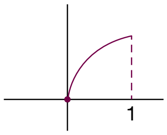
$\Rightarrow \quad x=2014;$

$(f(2014))^{2013}=\dfrac{2013}{2014}(2014+c)=2013+\dfrac{2013}{2014}c=2014$

Hence, the correct answer is (B).

73. $f:[0, 1] \to [0, \infty)$

$$\frac{df'(x)}{dx} \le 0 \Rightarrow f''(x) \le 0$$



$\therefore$ $f(x)$ is convex

$f(0) = 0$

$f'(x) > 0$

$\therefore$ $f(x)$ is increasing function and $f(x) \ge 0$

$$I = \int_0^1 \frac{dx}{f(x)^2+1} = \int_0^1 \frac{1}{f'(x)} \cdot \frac{f'(x)}{f(x)^2+1}\,dx$$

$$= \int_0^1 \frac{1}{f'(x)} (\tan^{-1} f(x))'\,dx$$

$$= \frac{1}{f'(x)} (\tan^{-1} f(x)) - \int_0^1 \frac{-1}{(f'(x))^2} f''(x)\tan^{-1} f(x)\,dx$$

$$I = \frac{\tan^{-1} f(x)}{f'(x)} + \int_0^1 \frac{f''(x)}{(f'(x))^2} \tan^{-1} f(x)\,dx = \frac{\tan^{-1} f(x)}{f(x)} + I_2$$

$f''(x) \le 0, f'(x)^2 > 0, \tan^{-1} f(x) > 0 \Rightarrow I_2 < 0$

$\tan^{-1} x < x$

$$I < \frac{\tan^{-1} f(x)}{f'(x)} = \frac{\tan^{-1} f(1)}{f'(1)} < \frac{f(1)}{f'(1)}$$

$$\therefore \quad I < \frac{f(1)}{f'(1)}$$

Hence, the correct answer is (B).

74. $I(n) = \displaystyle\int_{2016}^{2016+\frac{1}{n}} x\left(\frac{1+\cos(2x-4032)}{2}\right) dx$

$$= \frac{x}{2}\left(x + \frac{\sin(2x-4032)}{2}\right)\Bigg|_{2016}^{2016+\frac{1}{n}} - \int_{2016}^{2016+\frac{1}{n}} \left(\frac{x}{2} + \frac{1}{4}\sin(2x-4032)\right) dx$$

$$= \frac{x^2}{2} + \frac{x}{4}\sin(2x-4032)\Big|_{2016}^{2016+\frac{1}{n}} - \left(\frac{x^2}{4} - \frac{\cos(2x-4032)}{8}\right)_{2016}^{2016+\frac{1}{n}}$$

$$= \frac{1}{2}\left(\frac{1}{n^2} + \frac{4032}{n}\right) + \frac{2016+\frac{1}{n}}{4} \sin\left(\frac{2}{n}\right) - \left[\frac{1}{4}\left(\frac{1}{n^2} + \frac{4032}{n}\right) - \frac{1}{8}\cos\frac{2}{n} + \frac{1}{8}\right]$$

$$I(n) = \frac{1}{4}\left(\frac{1}{n^2} + \frac{4032}{n}\right) + \frac{2016+\frac{1}{n}}{4}\sin\frac{2}{n} + \frac{1}{8}\cos\frac{2}{n} - \frac{1}{8}$$

$$I\left(\frac{1}{\pi}\right) = \frac{1}{4}(\pi^2 + 4032\,\pi) + 0 + \frac{1}{8}\cdot 1 - \frac{1}{8} = \frac{\pi(\pi+4032)}{4}$$

$$\lim_{n\to\infty} nI(n) = \frac{1}{4}\left(\frac{1}{n} + 4032\right) + \frac{2016n+1}{4}\sin\frac{2}{n} + \frac{n}{8}\cos\frac{2}{n} - \frac{n}{8}$$

$$= \frac{1}{4n} + 1008 + 2(504)\frac{\sin\frac{2}{n}}{\frac{2}{n}} + \frac{\sin\frac{2}{n}}{4} + \frac{n}{8}\left(1 - 2\sin^2\frac{1}{n} - 1\right)$$

As $n \to \infty$

$$= 0 + 1008 + 1008(1) + 0 - \left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)^2 \frac{1}{4n} = 2016 + 0 = 2016$$

Hence, the correct answer is (C).

75. $f(x) = \ln\left[\cos|x| + \frac{1}{2}\right]$

$$1 \le \cos|x| + \frac{1}{2} < \frac{3}{2} \Rightarrow \left[\cos|x| + \frac{1}{2}\right] = 1; f(x) = 0$$

$$\int_{x_1}^{x_2}\left(\lim_{n\to\infty}\left(\frac{(f(x))^n}{x^2+\tan^2 x}\right)\right) = \int_{x_1}^{x_2} 0\ dx = 0$$

Hence, the correct answer is (A).

76. Given, that f is a non-negative function defined on [0, 1] and

$$\int_0^x \sqrt{1-(f'(t))^2}\,dt = \int_0^x f(t)dt, \quad 0 \le x \le 1$$

Differentiating both sides with respect to x, we get:

$$\sqrt{1-[f'(x)]^2} = f(x)$$

$$\Rightarrow \quad 1-[f'(x)]^2 = [f(x)]^2 \Rightarrow [f'(x)]^2 = 1-[f(x)]^2$$

$$\Rightarrow \quad \frac{d}{dx} f(x) = \pm\sqrt{1-[f(x)]^2} \Rightarrow \pm\frac{d\,f(x)}{\sqrt{1-f(x)]^2}} = dx$$

Integrating both sides with respect to x, we get:

$$\pm\int \frac{d\,f(x)}{\sqrt{1-[f(x)]^2}} = \int dx \Rightarrow \pm\sin^{-1} f(x) = x + C$$

$\because$ Given that $f(0) = 0 \Rightarrow C = 0$

Hence, $f(x) = \pm\sin x$

But as $f(x)$ is a non-negative function on [0, 1]

$\therefore$ $f(x) = \sin x.$

Now $\sin x < x, \forall x > 0$

$$\therefore \quad f\left(\frac{1}{2}\right) < \frac{1}{2} \text{ and } f\left(\frac{1}{3}\right) < \frac{1}{3}.$$

Hence, the correct answer is (C).

77. $\left.\begin{array}{l}\text{Period of } \tan x \text{ is } \pi \\ \text{Period of } \sin x \text{ is } 2\pi\end{array}\right\}$ L.C.M is 2π.

Required integration simplifies to

$$\int_0^{\pi/4} \sqrt{[\cos x + \{[\tan x] + \{\sin x\}\}]}\,dx$$

$$0 < x < \frac{\pi}{4} \Rightarrow [\tan x] = 0, \{\sin x\} = \sin x$$

$$= \int_0^{\pi/4} \sqrt{[\cos x + \sin x]}dx = \int_0^{\pi/4} \sqrt{1}\ dx = \frac{\pi}{4}$$

Hence, the correct answer is (A).

78. Observe that

$$\sin x + \sin 3x + \sin 5x + \ldots + \sin 17x = \frac{\sin^2 9x}{\sin x}$$

$$\therefore \quad \int_0^{\pi/2} \frac{\sin^2 9x}{\sin x}\,dx$$

$$= -\left[\cos x + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \ldots + \frac{\cos 17x}{17}\right]_0^{\pi/2}$$

$$= 1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{17}$$

Hence, the correct answer is (D).

79. [x], [1 + x] = 1 + [x] are discontinuous at x = 1, 2, 3, 4, 5

$$\therefore \quad I = \int_1^5 \frac{dx}{[x][1+x]} = \int_1^5 \left(\frac{1}{[x]} - \frac{1}{1+[x]}\right) dx$$

$$I = \left(\int_1^2 \left(\frac{1}{1} - \frac{1}{2}\right) dx + \int_2^3 \left(\frac{1}{2} - \frac{1}{3}\right) dx + \int_3^4 \left(\frac{1}{3} - \frac{1}{4}\right) dx + \int_4^5 \left(\frac{1}{4} - \frac{1}{5}\right) dx\right)$$

$$I = \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20}\right) = \frac{4}{5}$$

Hence, the correct answer is (A).

80. In I_1 put $x = \frac{1}{y}$ then $dx = -\frac{1}{y^2}\, dy$

$$I_1 = -\int_\infty^0 \frac{1}{y^2} \frac{1}{y^2} \frac{1}{\sqrt{y}} \frac{y^6\, dy}{(1+y)^6}$$

$$I_1 = \int_0^\infty \frac{y\sqrt{y}\, dy}{(1+y)^6} \Rightarrow I_1 = I_2$$

Hence, the correct answer is (D).

81. In I_1, putting $x = \frac{(4y)}{(3+y)}$

$$dx = \frac{(3+y)4 - 4y}{(3+y)^2} dy = \frac{12}{(3+y)^2} dy$$

$$\text{Now, } I_1 = \int_0^1 \left(\frac{4y}{3+y}\right)^{\frac{5}{2}} \left(\frac{3(1-y)}{3+y}\right)^{\frac{7}{2}} \frac{dy}{(3+y)^2}$$

$$I_1 = \left(864\sqrt{3}\right) \int_0^1 \frac{x^{\frac{5}{2}}(1-x)^{\frac{7}{2}}}{(3+x)^8} dx = 864\sqrt{3}\ I_2$$

Hence, the correct answer is (B).

82. $f'(x) = p\beta - \beta^2 - 2 - \cos^4 x \le 0 \quad \forall x \in R$

$p\beta - \beta^2 - 2 \le 0$

$\beta^2 - p\beta + 2 \ge 0 \quad \forall \beta \in R \Rightarrow p^2 - 8 \le 0$

$p \in \left[-\sqrt{8}, \sqrt{8}\right]$

Hence, the correct answer is (B).

83. $\displaystyle\int_{-y}^{y} f(x)\, dx$

$$= \int_{-y}^{-1} [\sec^{-1} x - (\pi - \sec^{-1} x)] dx + \int_{-1}^{1} 0 dx + \int_{1}^{y} (\sec^{-1} x - \sec^{-1} x) dx$$

$$= 2\int_{-y}^{-1} \sec^{-1} x dx - \int_{-y}^{-1} \pi dx = 2\int_{1}^{y} (\pi - \sec^{-1} x) dx - \pi(y-1)$$

$$= \pi(y-1) - 2\lambda$$

Hence, the correct answer is (B).

84. The given integrals can be equal only if $a > \tan^2 1$.

$$\int_0^a \left[\tan^{-1}\sqrt{x}\right] dx = \int_0^{\tan^2 1} \left[\tan^{-1}\sqrt{x}\right] dx + \int_{\tan^2 1}^{a} \left[\tan^{-1}\sqrt{x}\right] dx$$

$$= 0 + 1. \int_{\tan^2 1}^{a} dx = a - \tan^2 1.$$

$$\int_0^a \left[\cot^{-1}\sqrt{x}\right] dx = \int_0^{\cot^2 1} \left[\cot^{-1}\sqrt{x}\right] dx + \int_{\cot^2 1}^{a} \left[\cot^{-1}\sqrt{x}\right] dx = \int_0^{\cot^2 1} 1 dx + 0$$

$= 1(\cot^2 1 - 0)$

$\Rightarrow a = \tan^2 1 + \cot^2 1$

$$= \frac{\sin^2 1}{\cos^2 1} + \frac{\cos^2 1}{\sin^2 1} = \frac{1-\cos 2}{1+\cos 2} + \frac{1+\cos 2}{1-\cos 2}$$

$$= \frac{(1-\cos 2)^2 + (1+\cos 2)^2}{\sin^2 2}$$

$$= \frac{2\left(1+\cos^2 2\right)}{\sin^2 2} = \frac{2\left(2+2\cos^2 2\right)}{2\sin^2 2} = \frac{2(3+\cos 4)}{(1-\cos 4)}$$

Hence, the correct answer is (A).

85. $I = 2\displaystyle\int_0^{\frac{\pi}{2}} \sqrt{\cos^{2n-1} x} \cdot \sin x\, dx = 2\int_0^{\frac{\pi}{2}} \sqrt{\sin^{2n-1} x} \cdot \cos x\, dx$

Let $\sin x = t \Rightarrow \cos x\, dx = dt$

$$I = 2\int_0^1 t^{\frac{2n-1}{2}} dt = \left.\frac{2.2.t^{\frac{2n-1+2}{2}}}{2n-1+2}\right|_0^1 = \left.\frac{4t^{\frac{2x+1}{2}}}{2n+1}\right|_0^1 = \frac{4}{2n+1}$$

Hence, the correct answer is (C).

86. $I = \displaystyle\int_0^1 e^{2x} e^{e^x} dx$

Let $e^x = t \Rightarrow I = \displaystyle\int_1^e te^t\, dt = \left(te^t - e^t\right)_1^e = e^e(e-1)$

Hence, the correct answer is (B).

87. As $a + b = 16$

$$\int_a^b \frac{x^n}{x^n + (16-x)^n} dx = 6 \qquad \ldots(1)$$

$$\text{Also, } \int_a^b \frac{(16-x)^n}{(16-x)^n + x^n} dx = 6 \qquad \ldots(2)$$

$$\text{Equation } (1)+(2) \Rightarrow \int_a^b dx = 12 \Rightarrow \left.\begin{matrix} b-a=12 \\ a+b=16 \end{matrix}\right\} \Rightarrow \begin{matrix} b=14 \\ a=2 \end{matrix}$$

Hence, the correct answer is (B).

88. $\displaystyle\int_2^x f(t)\, dt = \frac{x^2}{2} - \int_2^x t^2 f(t)\, dt$

Diff. w. r. t. x we get $f(x) = x - x^2 f(x)$

or $f(x) = \dfrac{x}{1+x^2} \Rightarrow k \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right) - \{0\}$.

Hence, the correct answer is (C).

89. Differentiating given equation w.r.t. ‘x’, we get

$$(1 + x^2) f''(x) + 2x f'(x) = 0 \Rightarrow \frac{d}{dx}[(1+x^2) f'(x)] = 0$$

$\Rightarrow (1 + x^2) f'(x) = \lambda$

But $f'(1) = \dfrac{1}{2} \Rightarrow \lambda = 1$

$\therefore \ f'(x) = \dfrac{1}{1+x^2}$

$\Rightarrow f(x) = \tan^{-1} x + c$

But $f(0) = 0 \Rightarrow c = 0$

$\therefore \quad f(x) = \tan^{-1} x$

$\therefore \quad \lim\limits_{x \to \pi/4^+} [f(x)] = 0.$

Hence, the correct answer is (A).

90. Given $\frac{3\pi}{4} < x < \pi \Rightarrow 3\pi < 4x < 4\pi$

$$\Rightarrow \quad 3 < \left(\frac{4x}{\pi}\right) < 4,$$

$$\therefore \quad \left[\frac{4x}{\pi}\right] = 3$$

$$\int_{\frac{3\pi}{4}}^{\pi} \left[\sin x + \left[\frac{4x}{\pi}\right]\right] dx = \int_{\frac{3\pi}{4}}^{\pi} [\sin x + 3]\, dx$$

$$= \int_{\frac{3\pi}{4}}^{\pi} 3 + [\sin x]\, dx = 3\left(\pi - \frac{3\pi}{4}\right) + 0 = \frac{3\pi}{4}$$

Hence, the correct answer is (D).

91. Let $F(k) = \int_0^{\pi/2} \ln(\sin^2\theta + k^2\cos^2\theta)\, d\theta$ where $F(1) = 0$

Use Newton's Leibnitz rule

$$F'(k) = \int_0^{\pi/2} \frac{2k\cos^2\theta}{\sin^2\theta + k^2\cos^2\theta}\, d\theta$$

$$= 2k \int_0^{\pi/2} \frac{\sec^2\theta\, d\theta}{(1+\tan^2\theta)(k^2+\tan^2\theta)}$$

Put $\tan\theta = t \Rightarrow \sec^2 d\theta = dt$

$$F'(k) = 2k \int_0^{\infty} \frac{dt}{(t^2+1)(t^2+k^2)}$$

$$F'(k) = 2k \int_0^{\infty} \frac{dt}{(t^2+1)(t^2+k^2)}$$

$$= \frac{2k}{k^2-1}\left[\int_0^{\infty}\left(\frac{1}{1+t^2} - \frac{1}{k^2+t^2}\right)dt\right]$$

$$F'(k) = \frac{2k}{k^2-1}\left[\tan^{-1} t - \frac{1}{k}\tan^{-1}\frac{t}{k}\right]_0^{\infty}$$

$$= \frac{2k}{k^2-1}\left[\frac{\pi}{2} - \frac{1}{k}\frac{\pi}{2}\right] = \frac{\pi}{k+1}$$

$\Rightarrow \quad F(K) = \pi \ln|k+1| + c$

When $k = 1$, $F(1) = 0$, we get $c = -\pi \ln 2$

$$\Rightarrow \quad F(k) = \pi \ln\left|\frac{k+1}{2}\right|$$

Hence, the correct answer is (D).

92. $$\int_0^{\pi/3} \left[\sqrt{3}\tan x\right] dx = \int_0^{\pi/6} \left[\sqrt{3}\tan x\right] dx +$$

$$\int_{\pi/6}^{\tan^{-1}\frac{2}{\sqrt{3}}} \left[\sqrt{3}\tan x\right] dx + \int_{\tan^{-1}\frac{2}{\sqrt{3}}}^{\pi/3} \left[\sqrt{3}\tan x\right] dx$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{2}{\sqrt{3}} = \sin^{-1}\left(\sqrt{\frac{3}{7}}\right) \Rightarrow k = \frac{3}{7}$$

$$\therefore \quad \int_0^{2k} [\tan x]\, dx = \int_0^{6/7} [\tan x]\, dx$$

$$= \int_0^{\pi/4} [\tan x]\, dx + \int_{\pi/4}^{6/7} [\tan x]\, dx$$

$$= \frac{6}{7} - \frac{\pi}{4}$$

Hence, the correct answer is (C).

93. $$I = \int_{-\pi/4}^{3\pi/4} \frac{e^{\pi/4}\, dx}{(e^x + e^{\pi/4})(\sin x + \cos x)} \qquad \ldots(1)$$

Replacing x by $a + b - x = \frac{\pi}{2} - x$

$$I = \int_{-\pi/4}^{3\pi/4} \frac{e^{\pi/4}\, dx}{(e^{\pi/2-x} + e^{\pi/4})(\sin x + \cos x)}$$

$$= \int_{-\pi/4}^{3\pi/4} \frac{e^{\pi/4}e^x\, dx}{(e^{\pi/2} + e^{\pi/4}e^x)(\sin x + \cos x)} \qquad \ldots(2)$$

Adding (1) + (2)

$$2I = \int_{-\pi/4}^{3\pi/4} \frac{1}{\sin x + \cos x}\, dx$$

$$= \int_{-\pi/2}^{\pi/2} \frac{dx}{\sin\left(x + \frac{\pi}{4}\right) + \cos\left(x + \frac{\pi}{4}\right)}\, dx$$

$$2I = \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{2}} \frac{dx}{\frac{1}{\sqrt{2}}\sin\left(x + \frac{\pi}{4}\right) + \frac{1}{\sqrt{2}}\cos\left(x + \frac{\pi}{4}\right)}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{2}} \frac{dx}{\sin\left(x + \frac{\pi}{2}\right)} = \int_{-\pi/2}^{\pi/2} \frac{1}{\sqrt{2}} \sec x dx$$

$$I = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sec x dx$$

$$\therefore \quad \lambda = \frac{1}{2\sqrt{2}}$$

Hence, the correct answer is (C).

94. Let $I = \int_0^{\pi^2/4} \frac{dx}{1 + \sin\sqrt{x} + \cos\sqrt{x}}$;

Put $x = t^2 \Rightarrow dx = 2t\, dt$

$$\text{Now, } I = \int_0^{\pi/2} \frac{2t}{1 + \sin t + \cos t}\, dt$$

$$2I = \int_0^{\pi/2} \frac{\pi}{1 + \sin t + \cos t}\, dt$$

$$= \int_0^{\pi/2} \frac{\pi\left(1+\tan^2\frac{t}{2}\right)}{\left(1+\tan^2\frac{t}{2}\right)+2\tan\frac{t}{2}+\left(1-\tan^2\frac{t}{2}\right)}\,dt$$

$$\Rightarrow\quad 2I = \pi\int_0^{\pi/2} \frac{\sec^2\frac{t}{2}}{2\left(1+\tan\frac{t}{2}\right)}\,dt;$$

Put $\tan\frac{t}{2} = y \Rightarrow \frac{1}{2}\sec^2\frac{t}{2}\,dt = dy$

So, $I = \frac{\pi}{4}\int_0^1 \frac{2}{1+y}\,dy = \frac{\pi}{2}\ \ln 2$

Hence, the correct answer is (B).

95. $I = \int_0^{2\pi} \frac{dx}{2+\sin 2x}$ …(1)

Using king, we get:

$$I = \int_0^{2\pi} \frac{dx}{2-\sin 2x} \quad \text{…(2)}$$

Adding (1) and (2)

$$2I = \left[\int_0^{2\pi} \frac{4}{4-\sin^2 2x}\,dt\right]$$

Using queen, we get

$$I = 2\int_0^{\pi} \frac{2}{4-\sin^2 2x}\,dx = 2\int_0^{\pi/2} \frac{4}{4-\sin^2 2x}\,dx$$

Put $2x = t \Rightarrow 2dx = dt$

$$I = \int_0^{\pi} \frac{4}{4-\sin^2 t}\,dt = 16\int_0^{\pi/2} \frac{dt}{8-2\sin^2 t}$$

$$= 16\int_0^{\pi/2} \frac{dt}{8-(1-\cos 2t)}$$

$$= 16\int_0^{\pi/2} \frac{dt}{7+2\cos 2t}$$

Hence, $k = 16$.

Hence, the correct answer is (D).

96. Let $I = \int_0^{\pi/2} (\tan^{-1}(\cot x)+\cot^{-1}(\tan x))\,dx$ …(1)

Using King rule

$$I = \int_0^{\pi/2} (\tan^{-1}(\tan x)+\cot^{-1}(\cot x))\,dx \quad \text{…(2)}$$

Adding (1) and (2)

$$\Rightarrow\quad 2I = \int_0^{\pi/2} \pi\,dx$$

$$I = \frac{\pi^2}{4}.$$

Hence, the correct answer is (B).

97. $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{1}{x^3}\int_0^x \sin(t^2)\,dt = \lim\limits_{x\to 0} \frac{\sin x^2}{3x^2} = \frac{1}{3}$

so, $a = \frac{1}{3}$

Hence, the correct answer is (C).

98. Let $I = \int_{\frac{1}{3}}^{3} \frac{1}{x}\ln\left(\left|\frac{x+x^2-1}{x-x^2+1}\right|\right)dx$

Put $x = \frac{1}{t} \Rightarrow dx = \frac{-1}{t^2}\,dt$

So, $I = -\int_3^{\frac{1}{3}} t\ln\left(\left|\frac{\frac{1}{t}+\frac{1}{t^2}-1}{\frac{1}{t}-\frac{1}{t^2}+1}\right|\right)\frac{dt}{t^2} \Rightarrow I = -I$

So, $I = 0$

Hence, the correct answer is (B).

99. $f'(x)+f'(2-x) = 0$

$f(x)-f(2-x) = \lambda$

Put $x = 1$, we get $\lambda = 0$

$\therefore\quad f(x) = f(2-x)$ or $f(1+x) = f(1-x)$,

Therefore, $f(x)$ is symmetrical about line $x = 1$.

$\therefore\quad f(1) = 4$

$\therefore\quad y = f(x) = (x-1)^2+4$

$$\int_1^3 \frac{dx}{(x-1)^2+4} = \left[\frac{1}{2}\tan^{-1}\frac{x-1}{2}\right]_1^3$$

$$= \frac{1}{2}\ [\tan^{-1}1-\tan^{-1}0] = \frac{\pi}{8}$$

Hence, the correct answer is (D).

100. Let, $I = \int_{-\pi}^{\pi} \frac{x^2}{1+\sin x+\sqrt{1+\sin^2 x}}\,dx$

Using King property and Add

$$\therefore\quad 2I = \int_{-\pi}^{\pi} \frac{x^2(2+2\sqrt{1+\sin^2 x})}{\left(1+\sqrt{1+\sin^2 x}\right)^2-\sin^2 x}\,dx$$

$$\text{So, } I = \int_{-\pi}^{\pi} \frac{x^2\left(1+\sqrt{1+\sin^2 x}\right)}{2\left(1+\sqrt{1+\sin^2 x}\right)}\,dx = \frac{1}{2}\int_{-\pi}^{\pi} x^2dx = \frac{\pi^3}{3}.$$

Hence, the correct answer is (B).

101. $\lim\limits_{n\to\infty} \frac{\sin\frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n}}\cdot\sqrt{n}}\left(\frac{1}{\sqrt{n+1}}+\frac{1}{\sqrt{n+2}}+\frac{1}{\sqrt{n+3}}+\ldots+\frac{1}{\sqrt{n+3n}}\right)$

$$= 1\cdot\lim_{n\to\infty} \frac{1}{n}\cdot\frac{1}{\sqrt{1+\frac{1}{n}}}+\frac{1}{\sqrt{1+\frac{2}{n}}}+\frac{1}{\sqrt{1+\frac{3}{n}}}+\ldots+\frac{1}{\sqrt{1+\frac{3n}{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{n}\cdot\sum_{r=1}^{3n} \frac{1}{\sqrt{1+\frac{r}{n}}} = \int_0^3 \frac{dx}{\sqrt{1+x}} = \int_0^3 (1+x)^{\frac{-1}{2}}\,dx$$

$$= 2(1+x)^{\frac{1}{2}}\Big]_0^3 = 2[2-1] = 2$$

Hence, the correct answer is (D).

102. Let $I = \int_{2\pi}^{4\pi} \frac{\sin^{-1}(\sin x)}{\sin^4 x + \cos^2 x}\, dx$

Now, using King property, we get

$$I = \int_{2\pi}^{4\pi} \frac{\sin^{-1}(\sin(6\pi - x)}{\sin^4 x + \cos^2 x}\, dx$$

$$= -\int_{2\pi}^{4\pi} \frac{\sin^{-1}(\sin x)}{\sin^4 x + \cos^2 x}\, dx = -I$$

$\Rightarrow$ $2I = 0 \Rightarrow I = 0$

Hence, the correct answer is (A).

103. Put $-\frac{\theta}{2} = x$, we get

$$I = -2\int_{-\pi/4}^{-\pi/3} \frac{e^x\sqrt{1+\sin 2x}}{1+\cos 2x}\, dx = 2\int_{-\pi/3}^{-\pi/4} \frac{e^x\,|\cos x + \sin x|}{2\cos^2 x}\, dx$$

$$= -\int_{-\pi/3}^{-\pi/4} e^x(\sec x + \sec x \cdot \tan x)\, dx = -\ (e^x \cdot \sec x)_{-\pi/3}^{-\pi/4}$$

$$= -\left(e^{\frac{-\pi}{4}}\sqrt{2} - e^{\frac{-\pi}{3}} \cdot 2\right)$$

$$\therefore\quad I = \left(2 \cdot e^{\frac{-\pi}{3}} - \sqrt{2} \cdot e^{\frac{-\pi}{4}}\right)$$

Hence, the correct answer is (D).

104. Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$$I = \frac{1}{2}\int_0^{\pi/2} \ln(\sin\theta) \cdot 2\cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \ln(\sin\theta) \cdot (1 + \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \ln(\sin\theta)\, d\theta + \frac{1}{2}\int_0^{\pi/2} \ln(\sin\theta)\cos 2\theta\, d\theta$$

$$I = -\frac{\pi}{4}\ln 2 + \frac{1}{2}I_1 \qquad \left(\text{As} \int_0^{\pi/2} \ln(\sin\theta)\, d\theta = \frac{-\pi}{2}\ln 2\right)$$

$$I_1 = \int_0^{\pi/2} \underbrace{\ln(\sin\theta)}_{\text{I}}\underbrace{\cos 2\theta)}_{\text{II}}\, d\theta$$

$$= \underbrace{\frac{1}{2}\ln(\sin\theta)\sin 2\theta\Big]_0^{\pi/2}}_{\text{vanishes}} - \frac{1}{2}\int_0^{\pi/2} \frac{\cos\theta}{\sin\theta}\, 2\sin\theta\cos\theta\, d\theta$$

$$\left(\text{As} \lim_{\theta\to 0^+} \frac{\ln(\sin\theta)}{\text{cosec}\, 2\theta}\left(\frac{\infty}{\infty}\text{ form}\right)\right.$$

$$\left.= \lim_{\theta\to 0^+} \frac{\cot\theta}{-2\,\text{cosec}\, 2\theta\cot 2\theta} = \lim_{\theta\to 0^+} \frac{\tan 2\theta\sin 2\theta}{-2\tan\theta} = 0\right)$$

$$= 0 - \int_0^{\pi/2} \cos^2\theta\, d\theta = -\frac{\pi}{4}$$

$$I = -\frac{\pi}{4}\ln 2 - \frac{\pi}{8} = \frac{-\pi}{4}\left(\ln 2 + \frac{1}{2}\right)$$

Hence, the correct answer is (C).

105. $f(x) = \int_0^x e^t \sin(x - t)\, dt = \int_0^x e^{x-t}\sin(t)\, dt$ (Using King property)

$$f(x) = e^x \int_0^x e^{-t}\sin t\, dt$$

$$f'(x) = e^x \cdot e^{-x}\sin x + \left(\int_0^x e^{-t}\sin t\, dt\right)\mathrm{e}^x$$

$$f'(x) = \sin x + f(x) \qquad \ldots(1)$$

$$f''(x) = \cos x + f'(x) = \cos x + \sin x + f(x) \qquad [\text{Using (1)}]$$

$$f''(x) - f(x) = \sin x + \cos x \qquad \ldots(2)$$

$\Rightarrow$ $g(x) = \sin x + \cos x \Rightarrow g(x) \in \left[-\sqrt{2}, \sqrt{2}\right]$

Hence, the number of integer in the range of $g(x)$ is 3, i.e., – 1, 0, 1

Hence, the correct answer is (D).

106. Given, $\int_2^3 \underbrace{(3-x)}_{\text{I}}\underbrace{f''(x)dx}_{\text{II}} = 7$

$\Rightarrow$ $(3 - x)\ f'(x)\Big]_2^3 + \int_2^3 f'(x)\, dx = 7$

$\Rightarrow$ $0 - (f'(2)) + f(3) - f(2) = 7$

$\Rightarrow$ $f(3) = f'(2) + f(2) + 7 = 4 + (-1) + 7 = 10.$

Hence, the correct answer is (C).

107. $I = \int_0^1 f(x) \cdot x^2\, dx - x(f(x))^2)\, dx$

$$I = \int_0^1 -x(-xf(x) + (f(x))^2)\, dx$$

$$I = -\int_0^1 x\left[f(x)^2 - 2 \cdot \frac{x}{2}f(x) + \frac{x^2}{4} - \frac{x^2}{4}\right]dx$$

$$I = \int_0^1 \left(\frac{x^3}{4} - x\left(f(x) - \frac{x}{2}\right)^2\right)dx,$$

which is maximum when $f(x) = \frac{x}{2}$

i.e., $I = \int_0^1 \frac{x^3}{4}\, dx = \frac{1}{16}$

Hence, the correct answer is (D).

108. $\int_0^{\pi/2} \frac{\sin^3 x}{(\cos^4 x + 3\cos^2 x + 1)\tan^{-1}(\sec x + \cos x)}\, dx$

Put $\sec x + \cos x = t \Rightarrow (\sec x \tan x - \sin x)dx = dt$

$\Rightarrow$ $\sin x\, \frac{\sin^2 x}{\cos^2 x}\, dx = dt \Rightarrow \sin^3 x\, dx = \cos^2 x\, dt$

$\therefore$ $I = \int_2^{\infty} \frac{dt}{(t^2 + 1)\tan^{-1} t}$; Put $\tan^{-1} t = z$

$$I = \ln z\Big]_{\tan^{-1} 2}^{\frac{\pi}{2}} = \ln\frac{\pi}{2} - \ln(\tan^{-1} 2).$$

Hence, the correct answer is (B).

109. As, $\sin(\cos^{-1} x) = \sin\left(\frac{\pi}{2} - \sin^{-1} x\right) = \cos(\sin^{-1} x)$

Hence, $\int_{-1}^{1} (\tan^{-1}\theta + \cot^{-1}\theta)\, d\theta$, where $\theta = \cos(\sin^{-1} x)$

$$= \frac{\pi}{2}(2) = \pi.$$

Hence, the correct answer is (D).

110. $$I = \int_0^{\pi/6} \frac{\sqrt{1+\sin x}}{\cos x}\,dx = \int_0^{\frac{\pi}{6}} \frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\,dx$$

$$= \int_0^{\frac{\pi}{6}} \frac{dx}{\cos\frac{x}{2}-\sin\frac{x}{2}} = \int_0^{\frac{\pi}{6}} \frac{dx}{\sqrt{2}\cos\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{6}} \sec\left(\frac{x}{2}+\frac{\pi}{4}\right)dx$$

$$= \sqrt{2}\left[\ln\left(\left(\sec\frac{\pi}{4}+\frac{x}{2}\right)+\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right)\right]_0^{\frac{\pi}{6}}$$

$$= \sqrt{2}\ln\left(\frac{2+\sqrt{3}}{\sqrt{2}+1}\right) = \sqrt{2}\ln(\sqrt{8}+\sqrt{6}-\sqrt{4}-\sqrt{3}).$$

Hence, the correct answer is (B).

111. Put $\sin^{-1}x = t$ in first integral and $\tan^{-1}x = t$ in second integral, we get,

$$I = 2\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1}x}{x}\,dx - \int_0^1 \frac{\tan^{-1}x}{x}\,dx$$

$$= 2\int_0^{\pi/4} \frac{t\cos t}{\sin t}\,dt - \int_0^{\pi/4} \frac{t\sec^2 t}{\tan t}\,dt$$

$$= \int_0^{\pi/4} \frac{t(1-\tan^2 t)}{\tan t}\,dt = \int_0^{\pi/4} (2t)\cdot(\cot 2t)\,dt$$

Put $2t = y$; we get

$$I = \frac{1}{2}\,\frac{1}{2}\int_0^{\pi/2} \underset{\text{(I)}}{y}\cdot\underset{\text{(II)}}{(\cot y)}\,dy = \frac{1}{2}\left(y\ln\sin y\Big|_0^{\pi/2} - \int_0^{\pi/2}\ln\sin y\,dy\right)$$

$$= \frac{\pi}{4}\ln 2$$

Hence, the correct answer is (B).

112. Consider $(V_{n+1} - V_n) - (V_n - V_{n-1}) = 0$

$\Rightarrow\ 2V_n = V_{n+1} + V_{n-1} \Rightarrow V_1, V_2, V_3, \ldots$ are in A.P.

Now, $V_1 = \frac{\pi}{2}$; $V_2 = \int_0^{\pi/2} \frac{\sin^2 2x}{\sin^2 x}\,dx = \int_0^{\pi/2} 4\cos^2 x\,dx = \pi$

$$\therefore\quad d = \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$

$$S_{100} = \frac{\pi}{2}[1+2+3+\ldots+100] = \frac{(5050)\pi}{2} \equiv \frac{(k)\pi}{2}$$

$\Rightarrow\ k = 5050.$

Hence, the correct answer is (C).

113. $f(x) = \sin^2(2x)$.

So, $$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\sqrt{f\left(\frac{r}{n}\right)} = \int_0^1 \sqrt{f(x)}\,dx = \int_0^1 (\sin 2x)\,dx$$

$$= \frac{1-\cos 2}{2}$$

Hence, the correct answer is (D).

114. $$g(x) = [\cos(\cos x)] + [\sin(\sin x)] = \begin{cases} -1\ ; & -\pi < x < 0 \\ 0\ ; & 0 < x < \pi \end{cases}$$

$$\int_{-\pi}^{\pi} f(g(x))\,dx = \int_{-\pi}^{0} f(g(x))\,dx + \int_0^{\pi} f(g(x))\,dx$$

$$= \int_{-\pi}^{0}\cos^{-1}(-1)\,dx + \int_0^{\pi}\cos^{-1}(0)\,dx = \pi^2 + \frac{\pi^2}{2} = \frac{3\pi^2}{2}.$$

Hence, the correct answer is (A).

115. $$\int (f(x) + xf'(x))\,dx = xf(x) + C$$

So, $$I = x\left(e^{\cos^2 x} + e^{\sin^2 x}\right)\Big]_0^{\pi} = \pi(e+1).$$

Hence, the correct answer is (D).

116. Note that

$$\cos x + \cos 3x + \cos 5x + \cos 7x$$

$$= \frac{\sin 4x}{\sin x}\left[\cos\left(x+(4-1)x\right)\right] = \frac{\sin 8x}{2\sin x}$$

Hence, $$\int_0^{\frac{\pi}{2}} \frac{\sin 8x}{\sin x}\,dx = \int_0^{\frac{\pi}{2}} 2(\cos x + \cos 3x + \cos 5x + \cos 7x)\,dx$$

$$= 2\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \frac{\sin 7x}{7}\right)_0^{\pi/2}$$

$$= 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\right)$$

Hence, the correct answer is (C).

117. $$f(x) = e^x\int_0^x e^{-y}f'(y)dy - (x^2 - x + 1)e^x$$

(Using Leibnitz's rule)

$$f'(x) = e^x\int_0^x e^{-y}f'(y)dy + e^x\cdot e^{-x}f'(x) - (x^2 - x + 1)e^x - (2x-1)e^x$$

$$\int_0^x e^{-y}f'(y)dy = (x^2 + x)$$

$$e^{-x}f'(x) = (2x+1)$$

$$f'(x) = (2x+1)e^x$$

$$f(x) = (2x-1)e^x$$

so, $$f\left(-\frac{1}{2}\right) = -\frac{2}{\sqrt{e}}$$

Hence, the correct answer is (C).

118. Here, $f'(x) = \sin^2 x + 1$

$$f(x) = \int_0^x \frac{3}{2} - \frac{\cos 2t}{2} dt = \frac{3x}{2} - \frac{\sin 2x}{4}$$

$$f\left(\frac{\pi}{2}\right) = \frac{3\pi}{4}$$

$$g(f(x)) = x$$

On differentiation, $g'(f(x))f'(x) = 1$

Put $x = \frac{\pi}{2}$

So, $g'\left(\frac{3\pi}{4}\right) = \frac{1}{f'\left(\frac{\pi}{2}\right)} = \frac{1}{2}$.

Hence, the correct answer is (C).

119. We have $I = \int_2^4 (x(3-x)(4+x)(6-x)(10-x) + \sin x)\, dx$ …(1)

Now, $I = \int_2^4 ((6-x)(3-(6-x)(4+(6-x))(6-(6-x))\ (10-(6-x)) +$

$\sin(6-x))dx$

(Applying King Property)

$$= \int_2^4 ((6-x)(x-3)(10-x)x(4+x) + \sin(6-x))\, dx \quad \text{…(2)}$$

∴ On adding (1) and (2), we get

$$2I = \int_2^4 (\sin x + \sin(6-x)\, dx = (-\cos x + \cos(6-x))_2^4$$

$= -\cos 4 + \cos 2 + \cos 2 - \cos 4$

$= 2(\cos 2 - \cos 4)$

Hence, $I = \cos 2 - \cos 4$

Hence, the correct answer is (B).

120. $f(x) = \cos^4 x + 3\sin^4 x$

$\Rightarrow f'(x) = -4\cos^3 x \sin x + 12 \sin^3 x \cos x$

$= 4 \cos x \sin x\, (3 \sin^2 x - \cos^2 x) = 4 \cos x \sin x\, [4 \sin^2 x - 1]$

Now, for $x \in \left[0, \frac{\pi}{6}\right]$ we have $f'(x) < 0$ and $x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]$

We have $f'(x) \geq 0$

Hence, $I = \int_0^{\pi/2} |f'(x)|\, dx = -\int_0^{\pi/6} (f'(x))\, dx + \int_{\pi/6}^{\pi/2} (f'(x))\, dx$

$$= f\left(\frac{\pi}{2}\right) - f\left(\frac{\pi}{6}\right) - \left(f\left(\frac{\pi}{6}\right) - f(0)\right)$$

$$= 3 - \left(\frac{3}{4}\right) - \left(\frac{3}{4} - 1\right) = \frac{5}{2}$$

Hence, the correct answer is (B).

121. Let $S = \lim\limits_{n\to\infty} \frac{\pi}{n} \left(\frac{1}{\sin\frac{\pi(n+1)}{4n}} + \frac{1}{\sin\frac{\pi(n+2)}{4n}} + \frac{1}{\sin\frac{\pi(n+3)}{4n}} + \ldots + \frac{1}{\sin\frac{\pi(n+n)}{4n}} \right)$

$$\Rightarrow S = \lim_{n\to\infty} \frac{\pi}{n} \sum_{r=1}^{n} \frac{1}{\sin\frac{(n+r)\pi}{4n}}$$

$$= \pi \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{\pi r}{4n}\right)}$$

$$= \pi \int_0^1 \frac{dx}{\sin\left(\frac{(1+x)\pi}{4}\right)}$$

$$S = \pi \int_0^1 \operatorname{cosec}\left(\frac{(x+1)\pi}{4}\right) dx = \pi \times \frac{4}{\pi} \left[\ln\left(\tan\frac{(x+1)}{8}\right)\right]_0^1$$

$= 4[0 - \ln(\sqrt{2} - 1)]$

Hence, $S = 4 \ln(\sqrt{2} + 1)$

Hence, the correct answer is (A).

122.
$$\lim_{n\to\infty} \frac{\left(\sum\limits_{r=1}^{n} \frac{r^2}{n^2}\right)\left(\sum\limits_{i=1}^{n} \frac{i^3}{n^3}\right)\left(\sum\limits_{k=1}^{n} \frac{k^4}{n^4}\right)}{\left(\sum\limits_{r=1}^{n} \frac{r^5}{n^5}\right)^2}$$

$$= \frac{\int_0^1 x^2 dx \cdot \int_0^1 x^3 dx \cdot \int_0^1 x^4 dx}{\left(\int_0^1 x^5 dx\right)^2} = \frac{\frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}}{\frac{1}{6} \times \frac{1}{6}} = \frac{36}{60} = \frac{3}{5}$$

Hence, the correct answer is (C).

123. $I = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx +$

$$\int_{5\pi/4}^{2\pi} (\cos x - \sin x)\, dx = 4\sqrt{2}.$$

Hence, the correct answer is (B).

124. $f(x) = x^5 + 5x - 1$

$f(1) = 5 \Rightarrow f^{-1}(5) = 1$

$f(2) = 41 \Rightarrow f^{-1}(41) = 2$

Put $f^{-1}(x) = t$

$$\therefore\ I = \int_1^2 \frac{f'(t)}{t^5 + 5t} dt = \int_1^2 \frac{5t^4 + 5}{t^5 + 5t} dt$$

$$= \ln(t^5 + 5t)\Big]_1^2 = \ln 42 - \ln 6 = \ln 7$$

Hence, the correct answer is (C).

125.
$$\lim_{h\to 0} \frac{\int_a^{x+h} \frac{(\sin t)^{2014}}{t^{2013} + 1} - \int_a^{x} \frac{(\sin t)^{2014}}{t^{2013} + 1} dt}{h}$$

Differentiate w.r.t. h, we get:

$$\lim_{h\to 0} \frac{\frac{(\sin(s+h))^{2014}}{(x+h)^{2013} + 1} - 0}{1} = \frac{(\sin x)^{2014}}{x^{2013} + 1}$$

Hence, the correct answer is (C).

126. $f(t) = \int_{2t}^{t^2} \tan^{-1}\left|\frac{(1+t)^2 - x}{1+x}\right| dx \qquad \ldots(1)$

Applying King property:

$$f(t) = \int_{2t}^{t^2} \tan^{-1}\left|\frac{1+x}{(1+t)^2 - x}\right| dx \qquad \ldots(2)$$

$\therefore$ (1) + (2)

$$2f(t) = \int_{2t}^{t^2} \frac{\pi}{2} \cdot dx = \frac{\pi}{2} \cdot (t^2 - 2t)$$

$$\Rightarrow\ f(t) = \frac{\pi}{4}((t-1)^2 - 1)$$

$$\therefore\ f(t)|_{\min} = \frac{-\pi}{4}$$

Hence, the correct answer is (A).

127. On expanding the determinant, we get:

$$f(x) = \underbrace{(x^3 - 4x)[\ ]}_{\text{odd}} - \underbrace{(\sin x - \tan x)[\ ]}_{\text{odd}} + \underbrace{\sec^2 x[3a^2 + 1 - 2a(a+k)]}_{\text{even}}$$

$$\therefore\ \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} f(x)\,dx = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \sec^2 x[3a^2 + 1 - 2a)(a+k)]\,dx$$

$\Rightarrow\ 2[3a^2 + 1 - 2a(a+k)] \ge 0\ \forall\ a \in R$

$\Rightarrow\ a^2 - 2ak + 1 \ge 0\ \forall\ a \in R$

$\therefore\ D \le 0 \Rightarrow k \in [-1, 1]$

Hence, the correct answer is (D).

128. $f(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ when $x \in \left[\frac{-\pi}{4}, \frac{3\pi}{4}\right]$

Then $f(x) \in [0, \sqrt{2}\,]$

$\therefore\ f[f(x)] = f[0, \sqrt{2}\,]$, which is always positive.

$$\therefore\ I = \int_{\frac{-\pi}{4}}^{\frac{3\pi}{4}} 1\,dx = \pi$$

Hence, the correct answer is (D).

129. Let locus of the required centre of the variable circle is radical axis*:*

$\therefore$ Locus of the centre $S_1 - S_2 = 0$

$x + 2y - 1 = 0 \Rightarrow a = 1$ and $b = 2$

Now, $I = \int_1^2 \left(\ln\left|\sin\frac{\pi x}{2}\right|\right) dx$

Put $\frac{\pi x}{2} = t$

$$I = \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\ln|\sin t|)\,dt$$

Put $t = \left(\frac{\pi}{2} + \theta\right)$

$$\therefore\ I = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \ln|\cos\theta|\,d\theta = \frac{2}{\pi}\left(\frac{-\pi}{2}\ln 2\right) = -\ln 2$$

Hence, the correct answer is (B).

130. $f'(x) = \frac{xf(x)}{1+x^2}$

$$\therefore\ \int \frac{dy}{y} = \int \frac{x}{1+x^2}\,dx$$

$\Rightarrow\ \ln y = \frac{1}{2}\ln(1 + x^2) + C$ at $x = 0;\ y = 3$

$\therefore\ c = \ln 3$

$\therefore\ y = 3\sqrt{1+x^2} = f(x)$

Hence, $f\left(\frac{4}{3}\right) = 5.$

Hence, the correct answer is (D).

131. f is continuous at $x = \frac{1}{3} \Rightarrow c_1 = 2c_2 \qquad \ldots(1)$

and $c_1 \int_0^{1/3} x\,dx + c_2 \int_{1/3}^{1} (1-x)\,dx = 1$

$\Rightarrow\ c_2 = 3$

Hence, the correct answer is (D).

132. We have $\lim\limits_{x\to 0}\left(\frac{\sqrt[3]{1+\sin^2 x} - \sqrt[4]{1 - 2\tan x}}{\sin x + \tan^2 x}\right) = \frac{1}{2}$

Hence, the correct answer is (C).

133. $I_N = \int_0^{\infty} x^9 \cdot e^{-x^2}\,dx = \int_0^{\infty} \underbrace{x^9}_{\text{I}} \cdot \underbrace{e^{-x^2}}_{\text{II}}\,dx$

Using by parts:

$$I_N = \left.\frac{-x^8 \cdot x^{-x^2}}{2}\right]_0^{\infty} + \frac{8}{2}\int_0^{\infty} x^7 \cdot e^{-x^2}\,dx$$

$$I_N = 9 + 4I_D \Rightarrow I_N = 4I_D$$

Hence, $\frac{I_N}{I_D} = 4.$

Hence, the correct answer is (B).

134. $\lim\limits_{n\to\infty}\left(\frac{1 - \cos\frac{2}{n}}{1 + \cos\frac{2}{n}}\right)\left(\sum_{r=1}^{2n}\frac{n}{\sqrt{8n^2 - r^2}}\right)\left(\sum_{r=1}^{3n}\frac{n}{\sqrt{18n^2 - r^2}}\right)$

$$\lim_{n\to\infty}\frac{1}{2}\cdot\left(\frac{1 - \cos\frac{2}{n}}{\frac{4}{n^2}}\right)\cdot 4 \cdot \left(\sum_{r=1}^{2n}\frac{n}{n \cdot n\sqrt{8 - \frac{r^2}{n^2}}}\right)\left(\sum_{r-1}^{3n}\frac{n}{n \cdot n\sqrt{18 - \frac{r^2}{n^2}}}\right)$$

$$= \frac{1}{2} \times \frac{1}{2} \times 4\left(\int_0^2 \frac{dx}{\sqrt{8 - x^2}}\right)\left(\int_0^3 \frac{dx}{\sqrt{18 - x^2}}\right) = \left(\sin^{-1}\frac{x}{2\sqrt{2}}\right)_0^2$$

$$\left(\sin^{-1}\frac{x}{3\sqrt{2}}\right)_0^3 = \frac{\pi}{4}\cdot\frac{\pi}{4} = \frac{\pi^2}{16}$$

Hence, the correct answer is (C).

135. $I = \int_{-1}^{1} \frac{(x^2 + x + 1) - \sqrt{x^4 + 3x^2 + 1}}{2x(1+x^2)}\,dx = \int_{-1}^{1} \frac{x^2 + x + 1}{2x(1+x^2)}\,dx$

$$= \int_{-1}^{1} \frac{dx}{2(1+x^2)} = \int_{-1}^{1} \frac{dx}{1+x^2} = \frac{\pi}{4}$$

Hence, the correct answer is (B).

136. $I_n = \int_0^1 x^n\sqrt{1-x^2}\; dx = \int_0^I x^{n-1}x\sqrt{1-x^2}\; dx$

$$I_n = -x^{n-1}\frac{(1-x^2)^{3/2}}{3}\Bigg|_0^1 + \int_0^1 (n-1)x^{n-2}\frac{(1-x^2)^{3/2}}{3}\,dx$$

$$I_n = 0 + \frac{n-1}{3}\int_0^1 x^{n-2}(1-x^2)\sqrt{1-x^2}\; dx$$

$$= \frac{n-1}{3}I_{n-2} - \frac{n-1}{3}I_n$$

$$3I_n + (n-1)I_n = (n-1)I_{n-2}$$

$$\frac{I_n}{I_{n-2}} = \frac{n-1}{n+2}$$

$$\lim_{n\to\infty}\frac{I_n}{I_{n-2}} = \lim_{n\to\infty}\frac{n\left(1-\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)} = 1$$

Hence, the correct answer is (D).

137. $L = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\left(\frac{k}{n}\right)^2 + \ln^2\left(1+\frac{k}{n}\right)\right)$

$$= \int_0^1 (x^2 + \ln^2(1+x))\,dx = \frac{1}{3} + 2\ln^2\frac{2}{e}$$

Hence, the correct answer is (B).

138. We have $S = \frac{1}{n}\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)\cos 4\left(\frac{k}{n}\right) = \int_0^1 \underbrace{(1-x)}_{\text{I}}\underbrace{\cos 4x}_{\text{II}}\,dx$

$$= (1-x)\frac{\sin 4x}{4}\Bigg]_0^1 + \frac{1}{4}\int_0^1 \sin 4x\,dx = 0 + \frac{1}{4}\int_0^1 \sin 4x\; dx$$

$$= \frac{-1}{16}\cos 4x\Bigg]_0^1$$

$$= -\left(\frac{1}{16}\cos 4 - \frac{1}{16}\right) = \frac{1}{16}(1-\cos 4)$$

Hence, the correct answer is (D).

139. $\lim_{x\to 0}\dfrac{\int_0^{x^2}(t^2\cos^5 t)dt}{x^3(x-\sin x)}\left(\dfrac{0}{0}\right) = \lim_{x\to 0}\dfrac{\int_0^{x^2}(t^2\cos^5 t)dt}{x^6\left(\dfrac{x-\sin x}{x^3}\right)}$

$$= 6\lim_{x\to 0}\frac{\int_0^{x^2}(t^2\cos^5 t)dt}{x^6}$$

$$\left(\text{As}, \lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots\infty\right)}{x^3} = \frac{1}{6}\right)$$

$$= 6\left(\frac{2x\cdot x^4\cdot\cos^5(x^2)}{6x^5}\right) = \lim_{x\to 0}2\cos^5(x^2) = 2$$

Hence, the correct answer is (C).

140. Given, $I_1 = \int_a^b \frac{f(x)}{c^{g(x)}+1}\,dx$ ...(1)

Using King property, we get:

$$I_1 = \int_a^b \frac{f(a+b-x)}{c^{g(a+b-x)}+1}\,dx = \int_a^b \frac{f(x)}{c^{-g(x)}+1}\,dx$$

(As, $f(a+b-x) = f(x)$ and $g(a+b-x) = -g(x)$)

$$\Rightarrow\quad I_1 = \int_a^b \frac{c^{g(x)}(f(x))}{(c^{g(x)}+1)}\,dx \qquad \ldots(2)$$

Now, adding (1) and (2), we get:

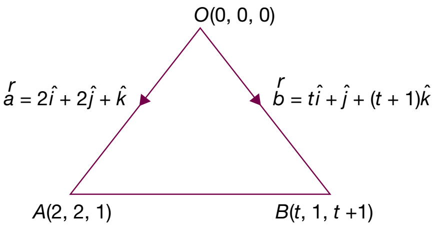
$$2I_1 = \int_a^b f(x)\,dx = I_2 \Rightarrow I_1 = \frac{I_2}{I_1} = 2$$

Hence, the correct answer is (C).

141. We have $S(t) = \frac{1}{2}|\ \vec{a}\times\vec{b}\ |$

Where $\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = t\hat{i} + \hat{j} + (t+1)\hat{k}$

$\Rightarrow\ 4(S(t))^2 = |\ \vec{a}\times\vec{b}\ |^2$



O(0, 0, 0)
$\vec{a} = 2\hat{i} + 2\hat{j} + \hat{k}$
$\vec{b} = t\hat{i} + \hat{j} + (t+1)\hat{k}$
A(2, 2, 1)
B(t, 1, t +1)

But, $4(S(t))^2 = \overrightarrow{a^2}\ \overrightarrow{b^2} - (\vec{a}\cdot\vec{b})^2$

$$= 9(t^2+1+(t+1)^2) - (2t+2+t+1)^2 = 9(t^2+1)$$

$\therefore\quad (S(t))^2 = \frac{9}{4}(t^2+1)$

Now, $I = \int_1^e \frac{9}{4}(t^2+1)\ln t\; dt$

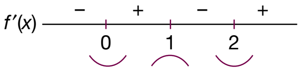
Integration by parts,

We get, $I = \frac{1}{2}(e^3+5)$

Hence, the correct answer is (B).

Multiple Correct Choice Type Questions

1. $f(x) = \int_1^x x(x^2-3x+2)\,dx$



where $1 \le x \le 4$

$f'(x) = x(x-1)(x-2)$

Max. at $x = 1$ and $x = 4$

Local min. at $x = 2$; $f(1) = 0$

$$f(2) = \int_1^2 x^3 - 3x^2 + 2x\; dx$$

$$= \frac{x^4}{4} - x^3 + x^2\Big|_1^2 = 4 - 8 + 4 - \frac{1}{4} + 1 - 1$$

$$= -\frac{1}{4}\ \text{min.}$$

$$f(4) = \int_1^4 x^3 - 3x^2 + 2x\ dx$$

$$= \frac{x^4}{4} - x^3 + x^2\Big|_1^4 = 64 - 64 + 16 - \frac{1}{4} + 1 - 1 = \frac{63}{4}$$

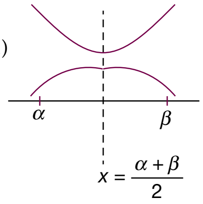
Hence, (B) and (D) are correct.

2. $y = f(x)$ and $y = g(x)$ are symmetric about line $x = \frac{\alpha+\beta}{2}$, α and β both are equidistant from $x = \frac{\alpha+\beta}{2}$ so we have $f(\alpha) = f(\beta), g(\alpha) = g(\beta)$

$$\int_\alpha^\beta f(x)g'(x)\,dx = f(x)\ g(x)\Big|_\alpha^\beta - \int_\alpha^\beta f'(x)g(x)\,dx$$

$$= f(\beta)g(\beta) - f(\alpha)g(\alpha) - \int_\alpha^\beta f'(x)g(x)\,dx$$

$$= 0 - \int_\alpha^\beta f'(x)g(x)\,dx \qquad (B) \qquad \ldots(1)$$



$$\text{Now, } \frac{1}{2}\int_\alpha^\beta \left(f(x)g'(x) - f'(x)g(x)\right)dx$$

$$= \frac{1}{2}\int_\alpha^\beta f(x)g'(x)\,dx - \frac{1}{2}\int_\alpha^\beta f'(x)g(x)\,dx$$

$$= \frac{1}{2}\int_\alpha^\beta f(x)g'(x)\,dx + \frac{1}{2}\int_\alpha^\beta f(x)g'(x)\ dx \qquad \text{using (1)}$$

$$= \int_\alpha^\beta f(x)g'(x)\ dx$$

$$\frac{1}{2}\int_\alpha^\beta (f(x)g'(x) + f'(x)g(x))dx = 0$$

Hence, (B) and (C) are correct.

3. $I_1 = \int_0^{\pi/2} \cos(\pi\sin^2 x)\,dx$

$$I_1 = \int_0^{\pi/2} \cos(\pi\cos^2 x)\,dx = \int_0^{\pi/2} \cos(\pi - \pi\sin^2 x)\,dx$$

$$= \int_0^{\pi/2} -\cos(\pi\sin^2 x)\,dx = -I_1 \Rightarrow 2I_1 = 0 \Rightarrow I_1 = 0$$

$$I_2 = \int_0^{\pi/2} \cos(2\pi\sin^2 x)\,dx = \int_0^{\pi/2} \cos(2\pi - 2\pi\cos^2 x)\,dx$$

$$= \int_0^{\pi/2} \cos(2\pi\cos^2 x)\,dx = \int_0^{\pi/2} \cos\left(2\pi\frac{(1+\cos 2x)}{2}\right)dx$$

$$= \int_0^{\pi/2} \cos(\pi + \pi\cos 2x)\,dx = -\int_0^{\pi/2} \cos(\pi\cos 2x)\,dx$$

(Let $2x = t$; $2dx = dt$)

$$= -\int_0^{\pi} \cos(\pi\cos t)\ \frac{dt}{2} = -\frac{1}{2}\left(2\int_0^{\pi/2} \cos(\pi\cos t)dt\right)$$

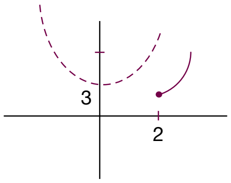
$$= -\int_0^{\pi/2} \cos(\pi\cos t)\,dt$$

$$I_2 = -\int_0^{\pi/2} \cos(\pi\sin t)\,dt;\ I_2 = -I_3 \Rightarrow I_2 + I_3 = 0$$

$$I_1 + I_2 + I_3 = 0$$

Hence, (A), (B) and (C) are correct.

4. $f(x) = \begin{cases} (x-2)^2 + 3 & ;\ x \ge 2 \\ K + x^2 & ;\ x < 2 \end{cases}$



Local minima at $x = 2$ $f(2^-) \ge f(2)$

$\Rightarrow \quad k + 4 \ge 0 + 3 \Rightarrow k \ge -1$

For $k \le -1$, no extrema

If $K \in [3, \infty)$; Range of $f(x) \in [3, \infty)$

$$\left.\begin{aligned} f'(2^-) &= 2\cdot 2 = 4 \\ f'(2^+) &= 2(x-2) = 0 \end{aligned}\right) \text{not diff } \forall\ K \in R$$

$$\int_2^3 (x-2)^2 + 3\,dx = \frac{(x-2)^3}{3} + 3x\Big|_2^3$$

$$= \frac{1}{3} + 9 - 0 - 6 = \frac{10}{3}$$

Hence, (A), (B) and (C) are correct.

5. Let $I = \int_0^\pi \frac{(a-\cos x)dx}{1 - 2a\cos x + a^2}$

$$= \frac{1}{2a}\int_0^\pi \frac{a^2 - 2a\cos x + 1 + a^2 - 1}{1 - 2a\cos x + a^2}dx$$

$$I = \frac{1}{2a}\int_0^\pi \left(1 + \frac{a^2 - 1}{1 - 2a\cos x + a^2}\right)dx \quad \text{where } \tan\frac{x}{2} = t$$

$$I = \frac{\pi}{2a} + \frac{a^2-1}{2a}\int_0^\infty \frac{2\,dt}{(1-a)^2 + (1+a)^2 t^2}$$

$$I = \frac{\pi}{2a} + \frac{1}{a}\frac{(1+a)}{(a-1)}\left|\frac{1-a}{1+a}\right|\left(\tan^{-1}\left|\frac{1+a}{1-a}\right|t\right)_0^\infty$$

$$\Rightarrow \quad I = \frac{\pi}{2a} + \frac{1}{a}\frac{(1+a)}{(a-1)}\left|\frac{1-a}{1+a}\right|\frac{\pi}{2}$$

$$I = \begin{cases} \frac{\pi}{a} & ;\ \text{if } a > 1 \\ 0 & ;\ \text{if } a < 1 \end{cases}$$

Hence, (A) and (C) are correct.

6. $I = \int_0^\alpha \frac{dx}{1 - \cos\alpha\cos x} = \int_0^\alpha \frac{dx}{1 - \cos\alpha\frac{\left(1 - \tan^2\frac{x}{2}\right)}{\left(1 + \tan^2\frac{x}{2}\right)}}$

$$= \int_0^{\alpha} \frac{\sec^2 \frac{x}{2} dx}{1+\tan^2 \frac{x}{2} - \cos\alpha + \cos\alpha \tan^2 \frac{x}{2}}$$

Put $\tan \frac{x}{2} = t \Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} dx = dt$

$$= \int_0^{\tan\alpha/2} \frac{2\, dt}{(1+\cos\alpha)t^2 + (1-\cos\alpha)}$$

$$= \frac{2}{1+\cos\alpha} \int_0^{\tan\alpha/2} \frac{dt}{t^2 + \tan^2 \frac{\alpha}{2}}$$

$$I = \frac{2}{2\cos^2 \frac{\alpha}{2}} \cdot \frac{1}{\tan \frac{\alpha}{2}} \tan^{-1}\left(\frac{\mathrm{t}}{\tan \frac{\alpha}{2}}\right)\Bigg|_0^{\tan\alpha/2}$$

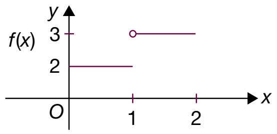
$$I = \frac{2}{\sin\alpha}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{2\sin\alpha} = \frac{A}{\sin\alpha} + B$$

∴ Possible values of $A = \frac{\pi}{2}$, $B = 0$ and $A = \frac{\pi}{4}$, $B = \frac{\pi}{4\sin\alpha}$

Hence, (A) and (B) are correct.

7. $g(x) = \int_0^x f(t)\ dt$ for $0 \le x \le 2$

$$g(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 2+3(x-1) & 1 \le x \le 2 \end{cases} \Rightarrow g(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 3x-1 & 1 \le x \le 2 \end{cases}$$



Continuous at $x = 1$ and not diff. at $x = 1$.

Hence, (A) and (B) are correct.

8. $0 < x < \frac{\pi}{4}$, $0 < \tan x < 1$

For positive integer n,

$\tan^{n+1} x < \tan^n x \Rightarrow \int_0^{\pi/4} \tan^{n+1} x dx < \int_0^{\pi/4} \tan^n x dx$

$I_{n+1} < I_n$ …(1)

$$I_n = \int_0^{\pi/4} \tan^{n-2} x(\sec^2 x - 1)dx$$

$$I_n = \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\ dx - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2} \Rightarrow I_n + I_{n-2} = \frac{1}{n-1}, n > 2$$

From (1) $I_{n+1} < I_n < I_{n-1}$

or $I_{n+2} < I_{n+1} < I_n < I_{n-1} < I_{n-2}$

or $I_{n+2} < I_n < I_{n-2}$ or $I_{n+2} + I_n < 2I_n < I_{n-2} + I_n$

$$\frac{1}{n+1} < 2I_n < \frac{1}{n-1}$$

Hence, (A), (C) and (D) are correct.

9. $\int_0^x f(t)\ dt = x^2 - 2x + \int_0^x tf(t)\ dt$ where $x \in (0, \infty)$

$$f(x) = 2x - 2 + x f(x) \Rightarrow f(x) = -2$$

Constant periodic with no FTP

Hence, (A) and (B) are correct.

10. $f(2 - x) = f(2 + x) \Rightarrow f(4 - x) = f(x + 4)$

Replace $x \to 2 - x$

$$f(2 - (2 - x)) = f(2 + 2 - x) \Rightarrow f(x) = f(4 - x) = f(x + y)$$

$T = 4$, $f(x)$ is periodic function

$$\int_0^4 f(x)\, dx = \int_0^2 f(x)\ + f(4-x)dx = \int_0^2 2f(x)\ dx = 10$$

$$\int_0^{50} f(x)\, dx = \int_0^{48+2} f(x)\, dx = \int_0^{12T} f(x)\, dx + \int_{12T}^{12T+2} f(x)\, dx$$

$$= 12\left(\int_0^2 f(x) + f(4-x)dx\right) + \int_0^2 f(x)\, dx$$

$$= 12 \cdot 2\int_0^2 f(x)\, dx + \int_0^2 f(x)\, dx$$

$$= 25\int_0^2 f(x)\ dx = 25 \times 5 = 125 \quad \text{(A)}$$

$$\int_{-4}^{46} f(x)\, dx = \int_{-T}^{11T+2} f(x)\, dx = \int_{-T}^{11T} f(x)\, dx + \int_{11T}^{11T+2} f(x)\, dx$$

$$= 12\int_0^4 f(x)\, dx + \int_0^2 f(x)\, dx$$

$$= 12 \times 10 + 5 = 125 \quad \text{(B)}$$

$$\int_1^{51} f(x)\, dx = \int_1^{12T+3} f(x)\, dx = \int_1^3 f(x)\, dx + \int_3^{3+12T} f(x)\, dx$$

$$= \int_1^3 f(x)\, dx + 12\int_0^T f(x)\, dx = \int_1^3 f(x)\, dx + 12 \times 10$$

Put $x - 1 = t$

$$\Rightarrow \quad dx = dt = \int_0^2 f(t+1)\ dt + 120 \ne 125$$

$$\int_2^{52} f(x)\, dx = \int_2^{12T+4} f(x)\, dx$$

$$= \int_2^4 f(x)dx + \int_4^{4+12T} f(x)dx = \int_2^4 f(x)\, dx + 12\int_0^T f(x)\, dx$$

$$= 5 + 12 \times 10 = 125 \quad \left(\because \int_2^4 f(x)dx = \int_2^0 f(4-t)(-dt) = \int_0^2 f(t)dt = 5\right)$$

Hence, (A) and (B) are correct.

11. $x^2F(x) = \int_4^x (4t^2 - 2F'(t))\,dt$

$$2x\,F(x) + x^2F'(x) = 4x^2 - 2F'(x) \qquad \ldots(1)$$

Put $x = 4$, $8F(4) + 18F'(4) = 64$

$\Rightarrow$ As $F(4) = 0$ we have $F'(4) = \dfrac{32}{9}$

$$(x^2 + 2)\,F'(x) + 2F(x)\text{x} = 4x^2$$

$$\Rightarrow \int d((x^2+2)F(x)) = \int 4x^2\,dx$$

$$F(x)\,(x^2+2) = \frac{4x^3}{3} + C \Rightarrow F(4)(18) = \frac{4}{3}\cdot 64 + C$$

$$0 = \frac{256}{3} + C \Rightarrow C = -\frac{256}{3}$$

$$F(x) = \left(\frac{4x^3}{3} - \frac{256}{3}\right)\frac{1}{(x^2+2)} = \frac{4}{3}\,\frac{(x^3-64)}{x^2+2}$$

$$F(8) = \frac{4}{3}\left(\frac{64\cdot 8 - 64}{64+2}\right) = \frac{28}{11}\cdot\left(\frac{32}{9}\right) = \frac{28}{11}F'(4)$$

$$F'(4) = \frac{11}{28}F(8)$$

Hence, (A) and (D) are correct.

12. $I_n = \int_0^{\pi/2} \dfrac{\sec^2 x\,dx}{(\sec x + \tan x)^n} = \int_0^{\pi/2} \sec^2 x(\sec x - \tan x)^n\,dx$

$$= \left.\frac{\tan x}{(\sec x + \tan x)^n}\right|_0^{\pi/2} - \int_0^{\pi/2} \frac{\tan x(-n)}{(\sec x + \tan x)^{n+1}}(\sec x \tan x + \sec^2 x)\,dx$$

$$I_n = 0 - 0 + n\int_0^{\pi/2} \frac{\tan x \sec x}{(\sec x + \tan x)^n}\,dx$$

$$= n\left.\frac{\sec x}{(\sec x + \tan x)^n}\right|_0^{\pi/2} - n\int_0^{\pi/2} \frac{\sec x(-n)}{(\sec x + \tan x)^{n+1}}(\sec x \tan x + \sec^2 x)\,dx$$

$$I_n = 0 - \frac{n}{1} + n^2\int_0^{\pi/2} \frac{\sec^2 x}{(\sec x + \tan x)^n}\,dx$$

$$\Rightarrow \quad I_n = -n + n^2 I_n \Rightarrow I_n = \frac{n}{n^2-1};$$

$$I_2 = \frac{2}{3},\ I_3 = \frac{3}{9-1} = \frac{3}{8},\ I_4 = \frac{4}{15},\ I_5 = \frac{5}{24};$$

$$I_2 + I_3 = \frac{2}{3} + \frac{3}{8} = \frac{25}{24};\ I_2 + I_4 = \frac{2}{3} + \frac{4}{15} = \frac{14}{15}$$

$$I_3 + I_5 = \frac{3}{8} + \frac{5}{24} = \frac{14}{24} = \frac{7}{12}$$

Hence, (A), (B) and (D) are correct.

13. $f : (0, 1) \to R$

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{2^n}}{1 - x^{2^{n+1}}} = \sum_{n=0}^{\infty}\left(\frac{x^{2^n} + 1 - 1}{1 - x^{2^{n+1}}}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(x^{2^n} + 1) - 1}{(1 - x^{2^n})(1 + x^{2^n})}$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{1 - x^{2^n}} - \frac{1}{1 - x^{2^{n+1}}}\right)$$

$$= \left(\frac{1}{1-x} - \frac{1}{1-x^2}\right) + \left(\frac{1}{1-x^2} - \frac{1}{1-x^4}\right) + \ldots + \left(\frac{1}{1-x^{2^n}} - \frac{1}{1-x^{2^{n+1}}}\right)$$

as $0 < x < 1$

$$f(x) = \frac{1}{1-x} - \frac{1}{\underset{\overset{\downarrow}{\text{O}}}{1 - x^{2^{n+1}}}} = \frac{1}{1-x} - 1 = \frac{x}{1-x}$$

As $0 < x < 1$; $n \to \infty$, $x^{2^{n+1}} \to 0$

$$y = f(x) = \frac{x}{1-x} \Rightarrow y - xy = x \Rightarrow x = \frac{y}{1+y}$$

$$f^{-1}(x) = \frac{x}{1+x} \Rightarrow \int_0^{1/2} f(x)\,dx$$

$$= \int_0^{1/2} \frac{x}{1-x}\,dx = \int_0^{1/2} \frac{1-(1-x)}{1-x}\,dx = \int_0^{1/2} \frac{1}{1-x} - 1\,dx$$

$$= -\ln|x-1| - x\Big|_0^{1/2} = -\ln\frac{1}{2} - \frac{1}{2} = \ln 2 - \frac{1}{2}$$

$$\int_0^1 f(x)\,dx = -\ln|1-x| - x\Big|_0^1 = -\lim_{x\to 1}\ \ln(1-x) - 1$$

$$= -\lim_{x\to 1}\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots\right) - 1 = 1 + \frac{1}{2} + \frac{1}{3} + \ldots - 1$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots\ \infty$$

Hence, (A), (B) and (D) are correct.

14. $I_1 = \int_0^{\pi} \dfrac{\sin x}{x}\,dx \qquad \ldots(1)$

$$I_1 = \int_0^{\pi} \frac{\sin(\pi - x)}{\pi - x}\,dx = \int_0^{\pi} \frac{\sin x}{\pi - x}\,dx \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2I_1 = \int_0^{\pi} \frac{\sin x}{x} + \frac{\sin x}{\pi - x}\,dx = \int_0^{\pi} \frac{\pi \sin x}{x(\pi - x)}\,dx$$

$$I_1 = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{x(\pi - x)}\,dx$$

$$\Rightarrow \quad I_2 = \int_0^{\pi/2} f(x) f\left(\frac{\pi}{2} - x\right)dx = \int_0^{\pi/2} \frac{\sin x}{x}\,\frac{\sin\left(\frac{\pi}{2} - x\right)}{\frac{\pi}{2} - x}\,dx$$

$$I_2 = \int_0^{\pi/2} \frac{2\sin x \cos x}{x(\pi - 2x)}\,dx = \int_0^{\pi/2} \frac{\sin 2x}{x(\pi - 2x)}\,dx$$

Put $2x = t \Rightarrow 2dx = dt$

$$I_2 = \int_0^{\pi} \frac{\sin t}{\frac{t}{2}(\pi - t)}\,\frac{dt}{2} = \int_0^{\pi} \frac{\sin t}{t(\pi - t)}\,dt$$

$$\Rightarrow \quad I_1 = \int_0^{\pi} \frac{\sin x}{x} < \int_0^{\pi} \frac{x}{x}\,dx \qquad (\because\ \sin x < x) \Rightarrow I_1 < \pi$$

$$I_1 = \frac{\pi}{2}I_2;\ I_1 > I_2;\ \frac{I_1}{I_2} = \frac{\pi}{2} \Rightarrow \text{(D)}$$

Hence, (B) and (D) are correct.

15. $f(x) = \int_{-x}^{x} (t\sin at + bt + c)\, dt = \int_{0}^{x} 2t\sin at + 2c\ dt$

$$= 2t\left(\frac{-\cos at}{a}\right)\Big|_0^x - \int_0^x 2\left(\frac{-\cos at}{a}\right) dt + \ 2ct\Big|_0^x$$

$$= -\frac{2x}{a}\cos ax - 0 + \frac{2}{a}\ \frac{\sin at}{a}\Big|_0^x + 2c(x-0)$$

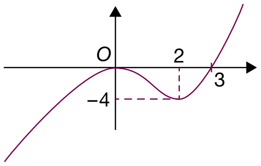
$$f(x) = -\frac{2x}{a}\cos ax + \frac{2}{a^2}(\sin ax - 0) + 2cx$$

$$\lim_{x\to 0}\frac{f(x)}{x} = -\frac{2}{a}\cos ax + \frac{2}{a}\left(\frac{\sin ax}{ax}\right) + 2c$$

$$= -\frac{2}{a} + \frac{2}{a} + + 2c = 2c$$

Hence, (A), (B) and (D) are correct.

16. $\int_0^1 f(xt)\, dt = 0 \quad \forall\ x \in R$

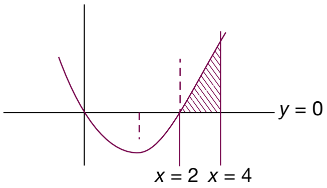


Put $xt = \alpha \Rightarrow x\, dt = d\alpha$

$$\int_0^x f(\alpha)\ \frac{d\alpha}{x} = 0 \Rightarrow \int_0^x f(\alpha)\, d\alpha = 0, \forall\ x \in R$$

$\therefore \quad f(x) = 0$

$f(x)$ is odd and even, periodic



$$y = x^3 - 3x^2 = x^2(x-3)$$

$$\frac{dy}{dx} = 3x^2 - 6x = 3x(x-2)$$

$\Rightarrow \quad x^3 - 3x^2 + P = 0$

For three distinct point of intersection; $P \in (-4, 0)$

Integral P are –3, –2, –1 (3)

$$\text{Area} = \int_2^4 x^2 - 2x\ dx = \frac{x^3}{3} - x^2\Big|_2^4 = \frac{20}{3}$$

Hence, (A), (B) and (C) are correct.

17. $f(x) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k\cos^2 x}} \quad k \in [0, 1)$

$$\frac{1}{4} < \frac{1}{3} < \frac{1}{2} \Rightarrow \frac{1}{4}\cos^2 x < \frac{1}{3}\cos^2 x < \frac{1}{2}\cos^2 x$$

$$\Rightarrow \quad -\frac{1}{4}\cos^2 x > -\frac{1}{3}\cos^2 x > -\frac{1}{2}\cos^2 x$$

$$\Rightarrow \quad \sqrt{1-\frac{1}{4}\cos^2 x} > \sqrt{1-\frac{1}{3}\cos^2 x} > \sqrt{1-\frac{1}{2}\cos^2 x}$$

$$\Rightarrow \quad \int\frac{dx}{\sqrt{1-\frac{1}{4}\cos^2 x}} < \int\frac{dx}{\sqrt{1-\frac{1}{3}\cos^2 x}} < \int\frac{dx}{\sqrt{1-\frac{1}{2}\cos^2 x}}$$

$\therefore \quad f(x)$ is increasing

$$f\left(\frac{1}{4}\right) < f\left(\frac{1}{3}\right) < f\left(\frac{1}{2}\right) \text{ and } f\left(\frac{1}{4}\right) < f\left(\frac{1}{2}\right) < f\left(\frac{3}{4}\right)$$

Hence, (B) and (D) are correct.

18. $A = \int_1^{e^2} \frac{\ln x}{\sqrt{x}}\, dx$

Put $\sqrt{x} = t;\ x = t^2 \Rightarrow dx = 2t\, dt$

$$A = \int_1^e \frac{\ln t^2}{t}\cdot 2t\ dt = 4\int_1^e \ln t\ dt = 4\left(t\ln t\Big|_1^e - \int_1^e t\cdot\frac{1}{t}dt\right)$$

$$A = 4\left(e\cdot\ln e - 1\cdot\ln 1 - (e-1)\right) = 4$$

(A) $2\left(e - \frac{1}{e}\right) = 2\left(2\cdot 71 - \frac{1}{2\cdot 71}\right) = 5{\cdot}42 - \frac{1}{1\cdot 35} = 4.68 > A$

(B) $(e-1)\left(2 + \frac{1}{\sqrt{e}}\right) = (1{\cdot}71)\left(2 + \frac{1}{1\cdot 65}\right) = 3{\cdot}42 + \frac{1\cdot 71}{1\cdot 61} > A$

(C) $\ln x = t \Rightarrow x = e^t\ dx = e^t\, dt$

$$A = \int_0^2 \frac{t}{\sqrt{e^t}}\cdot et\ \ dt = \int_0^2 t\sqrt{e^t}\ dt$$

(D) $3 < A < 5$

Hence, (A), (B), (C) and (D) are correct.

19. $I_{m,n} = \int_0^1 \underset{\overset{\downarrow}{\text{II}}}{x^{n-1}}\underset{\overset{\downarrow}{\text{I}}}{(\ln x)^m}\ dx$

$$= \frac{x^n}{n}\cdot(\ln x)^m\Big|_0^1 - \int_0^1 \frac{x^n}{n}\cdot m(\ln x)^{m-1}\frac{1}{x}dx$$

$$= 0 - \lim_{x\to 0^+}\frac{x^n}{n}(\ln x)^m - \frac{m}{n}\int_0^1 x^{n-1}(\ln x)^{m-1}\, dx$$

$$I_{m,n} = 0 - \lim_{x\to 0^+}\frac{(\ln x)^m}{nx^{-n}} - \frac{m}{n}\cdot I_{m-1,n}$$

$$\Rightarrow \quad I_{m,n} = 0 - \frac{m}{n}I_{m-1,n} \Rightarrow \frac{I_{m,n}}{I_{m-1,n}} = -\frac{m}{n}$$

$$K = \sum_{r=1}^{51}\frac{I_{r,52}}{I_{r-1,52}} = \sum_{r=1}^{51}\frac{-r}{52} = -\frac{1}{52}\ \frac{(51)(52)}{2} = -\frac{51}{2}$$

$[K] = [-25{\cdot}5] = -26$

Hence, (A) and (B) are correct.

20. $I_1 = \int_2^3 \frac{x^2}{x^2 + (5-x)^2}\, dx = \int_2^3 \frac{(5-x)^2}{(5-x)^2 + x^2}\, dx$

$$2I_1 = \int_2^3 dx = 1 \Rightarrow I_1 = \frac{1}{2}$$

$$I_2 = \int_0^1 \ln\left(\cos\frac{\pi}{2}x\right) dx$$

Put $\frac{\pi}{2}x = t \Rightarrow dx = \frac{2}{\pi}dt$

$$= \int_0^{\pi/2}\ln(\cos t)\ \frac{2}{\pi}\, dt = \frac{2}{\pi}\left(-\frac{\pi}{2}\ln 2\right) = -\ln 2$$

$$I_1 - I_2 = \frac{1}{2} + \ln 2 = \ln\sqrt{e} + \ln 2 = \ln 2\sqrt{e} \qquad \text{(A)}$$

$$I_2 = \ln I_1 = \ln\left(\frac{1}{2}\right) = -\ln 2$$

$$I_1 I_2 = \frac{1}{2}(-\ln 2) = -\frac{1}{2}\ln 2 = \frac{1}{\pi}\int_0^{\pi/2} \ln\sin 2x\, dx$$

Hence, (A), (C) and (D) are correct.

21. $f(x+y) = f(x)\cdot f(y) \quad \forall\, x, y \in R$

$f(0) \neq 0; f'(0) = 1$

$f(x) = a^{Kx}$

$f'(x) = a^{Kx}\ln a^K \Rightarrow f'(0) = 1 \Rightarrow \ln a^K = 1 \Rightarrow a^K = e$

$f(x) = e^x$

(A) $$\int_{-1}^{1} \frac{(\ln e^x)^2}{1+(e^x)^2}\, dx = \int_{-1}^{1} \frac{x^2}{1+e^{2x}}\, dx = \int_0^1 \frac{x^2}{1+e^{2x}} + \frac{x^2}{1+\frac{1}{e^{2x}}}\, dx$$

$$= \int_0^1 x^2\, dx = \frac{1}{3}$$

(B) $$\int_{-\pi/2}^{\pi/2} \frac{\cos\ln e^x}{1+e^x}\, dx = \int_{-\pi/2}^{\pi/2} \cos x\left(\frac{1}{e^x+1} + \frac{e^x}{e^x+1}\right) dx$$

$$= \sin x\Big|_0^{\pi/2} = 1 - 0 = 1$$

(C) $$\int_{-2015}^{2015} \ln\left(\ln e^x + \sqrt{1+(\ln e^x)^2}\right) dx;$$

$$= \int_{-2015}^{2015} \ln\left(x + \sqrt{1+x^2}\right) dx = 0 \qquad (\because \text{ odd function})$$

(D) $$\int_0^2 (e^x - x)\, dx = e^x - \frac{x^2}{2}\Bigg|_0^2 = e^2 - 3$$

Hence, (A), (B) and (C) are correct.

22. $$L = \lim_{y\to 0} \frac{\int_0^{\pi} \tan(y\sin x) dx}{y} = \lim_{y\to 0} \frac{\int_0^{\pi} \sec^2(y\sin x)\sin x\, dx}{1}$$

$$= \int_0^{\pi} \sin x\, dx = (-\cos x)_0^{\pi} = 2$$

$$\ell = \lim_{k\to\infty} \int_0^k \left(1 - \frac{x}{k}\right)^k e^{x/3}\, dx = \int_0^{\infty} e^{-x} e^{x/3}\, dx = \int_0^{\infty} e^{-2x/3}\, dx$$

$$= \frac{e^{-2x/3}}{-\frac{2}{3}}\Bigg|_0^{\infty} = \frac{0-1}{-\frac{2}{3}} = \frac{3}{2}$$

$3L = 4\ell$ (B)

$L + 2\ell = 2 + 3 = 5$ (C)

$$2(\ell + L) = 2\left(\frac{3}{2} + 2\right) = 2\left(\frac{7}{2}\right) = 7 \qquad \text{(B, C, D)}$$

Hence, (B), (C) and (D) are correct.

23. $$\int_0^x f(t)\, dt = e^x - ce^{2x}\int_0^1 f(t)e^{-t} dt$$

$f(x) = e^x - 2ce^{2x}\alpha \Rightarrow x = 0; 0 = 1 - c\alpha; c\alpha = 1$

$$\Rightarrow \quad \alpha = \int_0^1 (e^t - 2c\alpha e^{2t})e^{-t}\, dt \Rightarrow \alpha = \int_0^1 1 - 2c\alpha e^t\, dt$$

$$\Rightarrow \quad t - 2\, c\alpha e^t\Big|_0^1$$

$\Rightarrow \quad 1 - 2\, c\alpha e - (0 - 2c\alpha) \Rightarrow 1 + 2\, c\alpha(1-e) - 1 + 2 - 2e$

$$\alpha = 3 - 2e \Rightarrow C = \frac{1}{3-2e}$$

$f(x) = e^x - 2e^{2x}$ (B,D)

Hence, (B) and (D) are correct.

24. $f(f(x)) = 1 - x$

$x \to f(x)$

$f(f(f(x))) = 1 - f(x)$

$f(1-x) = 1 - f(x) \Rightarrow f(x) + f(1-x) = 1$

$$J = \int_0^{1/2} f(x) + f(1-x)\, d\alpha = \frac{1}{2}$$

$f(x) + f(1-x) = 1$

$$f\left(\frac{1}{2}\right) + f\left(\frac{1}{2}\right) = 1 \Rightarrow f\left(\frac{1}{2}\right) = \frac{1}{2}$$

$$\Rightarrow \quad f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) = 1$$

$$f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) = 1$$

$$I = \int_0^{\pi/2} \frac{\sin x}{(\sin x + \cos x)^3}\, dx = \int_0^{\pi/4} \frac{\sin x + \cos x}{(\sin x + \cos x)^3}\, dx$$

$$= \int_0^{\pi/4} \frac{1\, dx}{2\left(\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos\alpha\right)^2}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sin^2 x + \frac{\pi}{4}}\, dx = \frac{1}{2}\int_0^{\pi/4} \operatorname{cosec}^2\left(x + \frac{\pi}{4}\right) dx$$

$$= -\frac{1}{2}\cot\left(x + \frac{\pi}{4}\right)\Bigg|_0^{\pi/4} = -\frac{1}{2}(0-1) = \frac{1}{2} \qquad \text{(A, B, D)}$$

Hence, (A), (B) and (D) are correct.

25. $$f(x) = \frac{\tan x}{1+\tan^2 x} = \frac{\tan x}{\sec^2 x} = \sin x \cos x$$

(A) $$\int_0^{\pi/6} \frac{\sin x}{2}\, dx = -\frac{\cos 2x}{4}\Bigg|_0^{\pi/6} = -\frac{\left(\frac{1}{2}\right)+1}{4} = \frac{1}{8}$$

(B) $$f'(x) = \frac{2\cos 2x}{2}\; f'\left(\frac{\pi}{12}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

(C) $$\lim_{x\to 0^+} \frac{\sin x\cos x}{x} = 1$$

(D) Not odd fn as $x \in \left(0, \frac{\pi}{4}\right)$ only positive side (A, C)

Hence, (A) and (C) are correct.

26. $l_1 = \lim\limits_{x\to 0} \dfrac{\dfrac{x(e^{3x}-1)\ln(1+2x)}{(x^3+3)(1-\cos\sqrt{x})}}{3x^2}$

$$= \lim_{x\to 0} \frac{(e^{3x}-1)}{(1-\cos\sqrt{x})(0+3)}\ \frac{\ln(1+2x)}{2x}\cdot\frac{2}{3}$$

$$= \frac{2}{9}\frac{3e^{3x}}{\sin\sqrt{x}}\cdot 2\sqrt{x} = \frac{4}{3}$$

$$l_2 = \lim_{x\to 0}\frac{1}{x}\left(\int_{\pi/4}^{\pi/4+x} \log_{1/2}\sin^2 t\,dt\right)$$

$$= \lim_{x\to 0}\frac{\log_{1/2}\sin^2\left(\frac{\pi}{4}+x\right)-0}{1} = 1$$

(A) $9l_1^2 + l_2^2 = 16+1 = 17$; (B) $3l_1 + 4l_2 = 4+4 = 8$

(C) $l_1 > 0, l_2 > 0$; (D) $l_1 > l_2 \Rightarrow$ (B,D)

Hence, (B) and (D) are correct.

27. $f(x) = \sin^{-1} t\Big|_0^x = \sin^{-1}x$

$$f'(x) = \frac{1}{\sqrt{1-x^2}}$$

(A) $f'\left(\frac{\sqrt{3}}{2}\right) = 2$

(B) $\sin^{-1}\frac{2}{3} > \frac{2}{3} \Rightarrow \frac{2}{3} > \sin\left(\frac{2}{3}\right)$ $0.66 > \sin 40°$ (True)

(C) $\lim\limits_{x\to 0}\frac{\sin^{-1}x}{x} = 1$

(D) $\sin^{-1}x + \sin^{-1}\sqrt{1-x^2} = \frac{\pi}{2}$

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ True (A, B, C, D)

Hence, (A), (B), (C) and (D) are correct.

28. $I_n = \int_0^1 \frac{dx}{(1+x^2)^n} = \frac{x}{(1+x^2)^n}\Big|_0^1 - \int_0^1 x(-n)(1+x^2)^{-n-1}2x\,dx$

$$I_n = \frac{1}{2^n} + 2n\int_0^1 \frac{x^2+1-1}{(1+x^2)^{n+1}}dx \Rightarrow I_n = \frac{1}{2^n} + 2n(I_n - I_{n+1})$$

$$2nI_{n+1} = 2^{-n} + (2n-1)\,I_n$$

$$I_1 = \int_0^1\frac{dx}{1+x^2} = \tan^{-1}x\Big|_0^1 \Rightarrow I_1 = \frac{\pi}{4}$$

$n = 1$ $2I_2 = \frac{1}{2} + I_1 = \frac{1}{2} + \frac{\pi}{4} \Rightarrow I_2 = \frac{1}{4} + \frac{\pi}{8}$

$n = 2$ $4I_3 = \frac{1}{4} + 3I_2 = \frac{1}{4} + \frac{3}{4} + \frac{3\pi}{8} \Rightarrow I_3 = \frac{1}{4} + \frac{3\pi}{32}$ (A, B)

Hence, (A) and (B) are correct.

29. $f(1^-) = f(1) = 2$

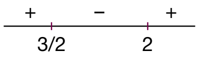
$f(1^+) = 2-6+6 = 2$; continuous

$f'(1^-) = 1$

$f'(1^+) = 4x-6\,|_{x=1} = -2$; not diff.

$g'(t) = f(t) - f(t-1)$; $t \in [1, 2]$

$$g'(t) = \begin{cases} f(1)-f(0) & ;\ t=1 \\ = 2-1 = 1 & \\ (2t^2-6t+6)-(t-1+1)\ ; & 1<t\le 2\end{cases}$$



$$= \begin{cases} 1 & ;\ t=1 \\ 2t^2-7t+6 & ;\ 1<t\le 2\end{cases}$$

$$g'(t) = 2t^2 - 7t + 6 = 2t^2 - 4t - 3t + 6 = (2t-3)\,(t-2)$$

$$g(1) = \int_0^1 f(x)\,dx = \frac{x^2}{2}+x\Big|_0^1 = \frac{3}{2}$$

$$g\left(\frac{3}{2}\right) = \int_{1/2}^{3/2} f(x)\,dx = \int_{1/2}^{1} x+1\,dx + \int_1^{3/2} 2x^2-6x+6\,dx$$

$$= \frac{x^2}{2}+x\Big|_{1/2}^{1} + \frac{2x^3}{3}-3x^2+6x\Big|_1^{3/2}$$

$$= \frac{1}{2}+1-\frac{1}{8}-\frac{1}{2}+\frac{9}{4}-\frac{27}{4}+\frac{18}{2}-\frac{2}{3}+3-6 = \frac{7}{8}+\frac{5}{6}$$

$$g(2) = \int_1^2 2x^2-6x+6\,dx$$

$$= \frac{2x^3}{3} - 3x^2 + 6x\Big|_1^2$$

$$= \frac{16}{3} - 12 + 12 - \left(\frac{2}{3} - 3 + 6\right) = \frac{14}{3} - 3 = \frac{5}{3}$$

Maxima at $t = \frac{3}{2}$, minima at $t = 1$

Hence, (B), (C) and (D) are correct.

30. $f: R \to R \Rightarrow f(x) = \int_1^x 2tf(t)\,dt$

$f(1) = 0$

$f'(x) = 2xf(x) \Rightarrow \int\frac{dy}{y} = \int 2x\,dx$

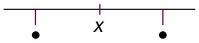
$\ln y = x^2 + c \Rightarrow f(x) = y = e^{x^2}k$

$f(1) = 0 = ek \Rightarrow k = 0$

$f(x) = 0$

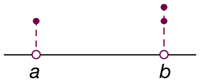
Hence, (A), (B), (C) and (D) are correct.

31. (A) $\int_0^x f(x)\,dx = \int_x^1 f(x)\,dx$ all $x \in [0, 1]$



$f(x) = 0 - f(x) \Rightarrow 2f(x) = 0 \Rightarrow f(x) = 0$

(B) $\int_a^b f(x)\,dx = 0 \Rightarrow f(x) = 0\ \forall\ x \in (a, b)$



$f(x)$ is not given continuous so, we don't know about $\frac{f(a)}{f(b)}$

$f(x)$ also possible

(C) $\int_0^{\pi/2} f(\cos x)\,dx = \int_0^{\pi/2} f(\sin x)\,dx$ (Queen rule)

$$\int_0^{\pi/2} f(\sin 2x)\,dx = \int_0^{\pi} f(\sin t)\,\frac{dt}{2} = \int_0^{\pi/2} f(\sin t)\,dt$$

$2x = t, 2dx = dt$ True

(D) $\int_{-a}^{a} f(x^2)\,dx = 2\int_0^{a} f(x^2)\,dx$

Hence, (A), (C) and (D) are correct.

32. Let $f(x) = \dfrac{x}{1+x^2} \Rightarrow f'(x) = \dfrac{1+x^2-2x^2}{(1+x^2)^2}$

$$f'(x) = \frac{1-x^2}{(1+x^2)^2} \Rightarrow f'(x) < 0, \forall\, x \in (1, \infty)$$

$\Rightarrow$ $f(x)$ is a decreasing function

$$\Rightarrow \quad S_n < \int_2^3 \frac{x}{1+x^2}\,dx < T_n \Rightarrow S_n < \frac{1}{2}\left[\ln(1+x^2)\right]_2^3 < T_n$$

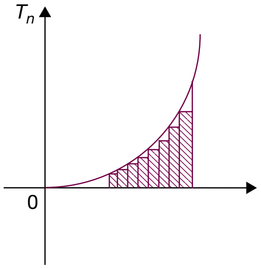
$$\Rightarrow \quad S_n < \frac{1}{2}\ln 2 < T_n$$

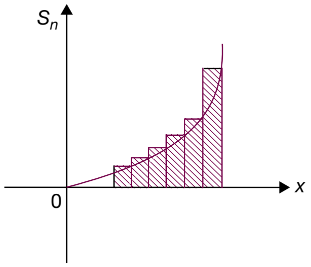
Hence, (A) and (B) are correct.

33. $S_n = \lim\limits_{n\to\infty}\left(\dfrac{1}{n^6} + \dfrac{32}{n^6} + \dfrac{243}{n^6} + \ldots + \dfrac{1}{n}\right)$

$$T_n = \lim_{n\to\infty}\left(\frac{1}{n^6} + \frac{32}{n^6} + \frac{243}{n^6} + \ldots + \frac{(n-1)^5}{n^6}\right)$$

$$S_n = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{r^5}{n^6} = \int_0^1 x^5\,dx = \left[\frac{x^6}{6}\right]_0^1 = \frac{1}{6}$$





$$\Rightarrow \quad S_n = \left(\frac{1}{6}\right)^{+};\ T_n = \left(\frac{1}{6}\right)^{-}$$

But since x^5 is concave upward, the area included in S_n for any two consecutive values of r is more than area excluded in T_n for the same values of r.

$$\Rightarrow \quad S_n - \frac{1}{6} > \frac{1}{6} - T_n \Rightarrow S_n + T_n > \frac{1}{3}$$

Hence, (A), (C) and (D) are correct.

34. $f(x) = x^2 - \int_0^1 f^2(t)\,dt - 2x\int_0^1 f(t)\,dt - x^2\int_0^1 dt$

$$f(x) = x^2 + A + 2Bx - x^2 = A + 2Bx$$

$$A = -\int_0^1 f^2(t)\,dt = -\int_0^1 (A+2Bt)^2\,dt$$

$$B = -\int_0^1 f(t)\,dt = -\int_0^1 (A+2Bt)\,dt$$

$\therefore$ $f(x) = 3x - 3$

Now verify all the option.

Hence, (A), (B) and (C) are correct.

35. $\log_2\left(1+\sqrt{6x-x^2-5}\right) \geq 0 \Rightarrow 1 + \sqrt{6x-x^2-5} \geq 1$

$6x - x^2 - 5 \geq 0$

$\Rightarrow$ $x^2 - 6x + 5 \leq 0$

$\Rightarrow$ $(x-1)(x-5) \leq 0 \Rightarrow x \Rightarrow x \in [1, 5]$

$$f(x) = \int_0^x (3t^2+2t+2)dt \Rightarrow f'(x) = 3x^2 + 2x + 2$$

As, $D < 0$

$\therefore$ $f(x)$ is increasing.

$$f(x)\big|_{\min} = L = f(1) = \int_0^1 (3t^2+2t+2)dt = \left(t^3+t^2+2t\right)_0^1 = 4$$

$$f(x)\big|_{\max} = M = f(5) = \int_0^5 (3t^2+2t+2)dt$$

$$= \left(t^3+t^2+2t\right)_0^5 = 160.$$

Hence, (B) and (C) are correct.

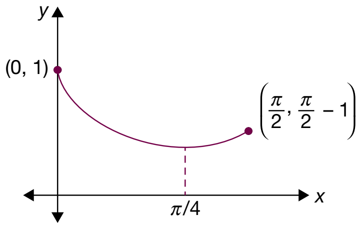
36. Let $A = (x^3+1^3)(x^3+2^3)\ldots(x^3+2014)^3$

$$\therefore \quad \text{L.H.S.} = \frac{1}{3}\int_0^1 dA = \frac{1}{\lambda}\left(\prod_{k=1}^{2014}(1+k^3) - \mu^3\right)$$

$\therefore$ $\lambda = 3, \mu = 2014!$

Hence, (B) and (D) are correct.

37.



Put $x = \cos\phi$

$$\therefore \quad f(\theta) = \int_0^{\theta}(\sin\theta - \sin\phi)d\phi + \int_{\theta}^{\frac{\pi}{2}}(\sin\phi - \sin\theta)d\phi$$

$$= 2(\sin\theta + \cos\theta) - \frac{\pi}{2}\sin\theta - 1$$

$$\Rightarrow \quad f'(\theta) = 2\left(\theta - \frac{\pi}{4}\right)\cdot\cos\theta$$

$\Rightarrow$ $f(\theta)$ decrease in $\left(0, \dfrac{\pi}{4}\right)$ and increase in $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

Hence, (A) and (B) are correct.

38. Given $f^3(x) = \int_0^x t\cdot f^2(t)dt$

Differentiating, $3f^2(x)f'(x) = xf^2(x)$

As $f(x) \# 0, f'(x) = \dfrac{x}{3}; f(x) = \dfrac{x^2}{6} + C$

But $f(0) = 0 \Rightarrow C = 0 \Rightarrow f(x) = \dfrac{x^2}{6}$

$\therefore \quad f(6) = 6.$

Hence, (A) and (C) are correct.

39. We have, $f(x) - 2\underbrace{\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\cos^5 x}\cdot \cos t \cdot f(t)dt}_{\text{A (say)}} = \dfrac{\sin^2 x}{\cos^5 x}$

$$\therefore \quad f(x) - 2A\frac{\sin^2 x}{\cos^5 x} = \frac{\sin^2 x}{\cos^5 x}$$

$$\Rightarrow \quad f(x) = (2A+1)\frac{\sin^2 x}{\cos^5 x} \qquad \ldots(1)$$

$$\text{Now, } A = \int_0^{\frac{\pi}{4}} \cos t \cdot (2A+1)\cdot \frac{\sin^2 t}{\cos^5 t}dt = (2A+1)\int_0^{\frac{\pi}{4}} \frac{\sin^2 t}{\cos^4 t}dt$$

$$= (2A+1)\int_0^{\frac{\pi}{4}} \tan^2 t \cdot \sec^2 t dt$$

Put $\tan t = y \Rightarrow \sec^2 t\, dt = dy$

$$\therefore \quad \text{we get } A = (2A+1)\int_0^1 y^2 dy = (2A+1)\frac{1}{3}$$

$\Rightarrow \quad 3A = 2A + 1$

$\therefore \quad \text{A} = 1$

Hence, from equation (1), we get

$$\therefore \quad f(x) = \frac{3\sin^2 x}{\cos^5 x}$$

(A) Clearly, $\lim\limits_{x\to\frac{\pi}{3}} f(x) = \lim\limits_{x\to\frac{\pi}{3}} \dfrac{3\sin^2 x}{\cos^5 x} = \dfrac{3\left(\frac{\sqrt{3}}{2}\right)^2}{\left(\frac{1}{2}\right)^5} = 72$

(B) As, $f(x) = \dfrac{3\sin^2 x}{\cos^5 x}$ so, $f(x)$ is periodic with period 2π.

(C) $f(x) = \dfrac{3\sin^2 x}{\cos^5 x} \Rightarrow f'(x)$

$$= \left[\frac{\cos^5 x\cdot(2\sin x\cdot\cos x) + \sin^2 x\cdot(5\cos^4 x\cdot\sin x)}{\cos^{10} x}\right]$$

$\therefore f'(\pi) = 0$

$\Rightarrow M(x = \pi, y = 0)$

So, equation of normal to the graph of $f(x)$ at point M whose abscissa is π, is given by $x - \pi = 0$

(D) As, $f(x) = 0 \Rightarrow \dfrac{3\sin^2 x}{\cos^5 x} = 0 \Rightarrow \sin x = 0$

$\therefore \qquad x = n\pi, n \in I$

So, the equation $f(x) = 0$ has no root in (0, 3).

Hence, (A) and (C) are correct.

40. For $x \in [0, 1]$

$\cos^{-1}x + \sin^{-1}x + \tan^{-1}x = \pi - \cot^{-1}x$

i.e., $\cos^{-1}x + \sin^{-1}x + \tan^{-1}x + \cot^{-1}x = \pi$

i.e., identity

For $x \in [-1, 0]$

As, $\cos^{-1}x + \cot^{-1}x - \sin^{-1}x - \tan^{-1}x = \pi$

$\cos^{-1}x + \cos^{-1}x$, $-\sin^{-1}x$ and $-\tan^{-1}x$, all of them are decreasing function and value of LHL $= \pi$ at $x = 0$

$\Rightarrow \quad$ No solution in this interval.

$\therefore \quad$ Solution of above equality is $[0, 1] \Rightarrow \alpha = 0, \beta = 1$

$$\cot^{-1}(1 - x + x^2) = \tan^{-1}\left(\frac{x-(x-1)}{1+x(x-1)}\right) = \tan^{-1}x - \tan^{-1}(x-1)$$

(A) $I = \displaystyle\int_0^1 \frac{\tan^{-1}x}{\cot^{-1}(1-x+x^2)}$

$$= \int_0^1 \frac{\tan^{-1}x}{\tan^{-1}x - \tan^{-1}(x-1)}dx = \int_0^1 \frac{\tan^{-1}(1-x)}{\tan^{-1}(1-x)+\tan^{-1}x}dx$$

Add them, we get $2I = 1 \Rightarrow I = \dfrac{1}{2}$

(B) $I = \displaystyle\int_0^{\pi/2} \sin 2x\cdot\sin(\sin x)dx$

$$= \int_0^{\pi/2} 2\sin x\cos x\sin(\sin x)dx$$

Put $\sin x = t$

$$= \int_0^1 2t\sin t\, dt = 2(\sin 1 - \cos 1)$$

(C) Option C is standard identity.

Hence, (A), (B) and (C) are correct.

41. $I + J = \displaystyle\int_0^1 \frac{(1+\sqrt{x})+(1-\sqrt{x})}{\sqrt{1-x}}dx = \int_0^1 \frac{2}{\sqrt{1-x}}dx$

Put $x = \sin^2\theta$

$$I + J = \int_0^{\pi/2} \frac{2\cdot 2\sin\theta\cos\theta}{\cos\theta}d\theta = 4\int_0^{\pi/2} \sin\theta\, d\theta = 4$$

$$I - J = \int_0^1 \frac{2\sqrt{x}}{\sqrt{1-x}}dx \text{ Put } x = \sin^2\theta$$

$$I - J = \int_0^{\pi/2} \frac{2\sin\theta 2\sin\theta\cos\theta}{\cos\theta}d\theta = 4\int_0^{\pi/2} \sin\theta\, d\theta = \pi,$$

Hence, (B) and (D) are correct.

42. $f(\theta) = |\,|\,|3\cos\theta - \sin\theta| + 2\cos\theta| + \cos\theta|$

$f(\theta) = |\,|\sin\theta - 3\cos\theta + 2\cos\theta| + \cos\theta|$

$= |\,|\sin\theta - \cos\theta| + \cos\theta| = |\sin\theta - \cos\theta + \cos\theta|$

$= |\sin\theta| = \sin\theta.$

(A) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \sin\theta d\theta = -(\cos\theta)_{\frac{\pi}{2}}^{\pi} = -(-1-0) = 1.$

(B) $f(\theta) = \sin\theta \Rightarrow f'(\theta) = \cos$

$$\Rightarrow \quad f'\left(\frac{5\pi}{6}\right) = \frac{-\sqrt{3}}{2}$$

(C) $I = \displaystyle\int_{\frac{\pi}{2}}^{\pi} \sin^2\theta\, d\theta \qquad \ldots(1)$

Applying King property:

$$I=\int_{\frac{\pi}{2}}^{\pi}\cos^2\theta\ d\theta \qquad \ldots(2)$$

Adding (1) and (2), we get:

$$2I=\int_{\frac{\pi}{2}}^{\pi} I\cdot d\theta=\frac{\pi}{2}\Rightarrow I=\frac{\pi}{4}$$

(D) $f'(\theta)=\cos\theta$

$$f'\left(\frac{2\pi}{3}\right)=\frac{-1}{2}.$$

Hence, (A), (B), (C) and (D) are correct.

43. Consider $A_{n+1}-A_n=\int\limits_0^{\pi/2}\dfrac{\sin(2n+1)x-\sin(2n-1)x}{\sin x}dx$

$$=\int_0^{\pi/2}\frac{2\cos 2nxdx}{\sin x}=0$$

$$A_{n+1}-A_n=\int_0^{\pi/2}2\cos 2nx=\frac{2}{2n}\left[\sin 2nx\right]_0^{\pi/2}$$

$$=\frac{1}{n}[0-0]\Rightarrow A_{n+1}=A_n$$

Again consider $B_{n+1}-B_n=\int\limits_0^{\pi/2}\dfrac{\sin^2(n+1)x-\sin^{2n}x}{\sin^2 x}dx$

$$=\int_0^{\pi/2}\frac{\sin(2n+1)x}{\sin x}dx=A_{n+1}]$$

$$B_{n+1}-B_n=\int_0^{\pi/2}\frac{\sin(2n+1)x\sin x}{\sin^2 x}dx\Rightarrow B_{n+1}-B_n=A_{n+1}$$

Hence, (A) and (D) are correct.

44. Given integral differentiating w.r.t x and y = constant

$$f(xy)\,y=yf(x)+\int_1^y f(t)dt\Rightarrow \text{Put } x=1$$

$$\Rightarrow\quad yf(y)=3y+\int_1^y f(t)dt \qquad \ldots(1)$$

Differentiates w.r.t y of (1)

$$\Rightarrow\quad yf'(y)+f(y)=3+f(y)\left(f'(y)=\frac{3}{y}\Rightarrow f'(x)=\frac{3}{x}\right)$$

$f(x)=3\ln x+C$

Put $x=1$; $3=C$

$f(x)=2\ln x+3$; $f(x)=3\ln x+\ln e$

$f(x)=3\ln e\,x$

Hence, (A) and (C) are correct.

45. $f(x)=f(2-x)\Rightarrow f'(x)=-f'(2-x)$,

Put $x=1\Rightarrow f'(1)=-f'(1)\Rightarrow f'(1)=0$

And $f'\left(\frac{1}{2}\right)=0\Rightarrow f'\left(\frac{1}{2}\right)=-f'\left(\frac{3}{2}\right)\Rightarrow f'\left(\frac{3}{2}\right)=0$

$\therefore\quad f''(x)$ vanishes at least twice on [0, 2] and

$f(1+x)\left(\dfrac{e^x-1}{e^x+1}\right)$ is an odd function and $\sin(2\pi-\theta)=-\sin\theta$

Hence, (A), (B) and (C) are correct.

46. $I_1=\int\limits_1^e x\left(\dfrac{1+x}{x}\right)(x+\ln x)^{100}dx$

$$\because\quad \frac{d}{dx}(x+\ln x)=1+\frac{1}{x}=\frac{1+x}{x}$$

$$I_1=\left[\frac{x(x+\ln x)^{101}}{101}\right]_1^e-\int_1^e\frac{(x+\ln x)^{101}}{101}$$

$$\Rightarrow\quad I_1=\frac{e(e+1)^{101}-1}{101}-\frac{1}{101}\int_1^e(x+\ln x)^{101}dx$$

$$I_2=\frac{1}{e}\int_1^e(t+\ln t)^{101}dt$$

Taking $e\sin x=t\Rightarrow I_1+\dfrac{eI_2}{101}=\dfrac{e(e+1)^{101}-1}{101}$

Hence, k is 1.

Hence, (A), (B) and (D) are correct.

47. Let $\int\limits_0^{\pi/4}\cos t\ f(t)dt=A$

Then, $f(x)\ -\mu\sin^2 x\cos^{-5}x\ A=\sin^2 x\ \cos^{-5}x$

$$f(x)=(\mu A+1)\sin^2 x\cos^{-5}x$$

$$A=(\mu A+1)\int_0^{\pi/4}\frac{\sin^2 t}{\cos^4 t}dt$$

$$=(\mu A+1)\left[\frac{\tan^3 t}{3}\right]_0^{\pi/4}=(\mu A+1).\frac{1}{3}$$

$$\therefore\quad f(x)=\frac{3}{3-\mu}\sin^2 x\cos^{-5}x$$

Hence, (A) and (D) are correct.

48. Let e_1, e_2 are eccentricities of hyperbola and it's conjugate

$$\Rightarrow\quad \frac{1}{{e_1}^2}+\frac{1}{{e_2}^2}=1$$

$$\Rightarrow\quad \frac{1}{{e_2}^2}=\frac{{e_1}^2-1}{{e_1}^2}$$

$$\Rightarrow\quad e_2=\frac{e_1}{\sqrt{e_1^2-1}}\,;f(e)=\frac{e}{\sqrt{e^2-1}}$$

$$\Rightarrow\quad f(f(e))=f\left(\frac{e}{\sqrt{e^2-1}}\right)=\frac{\frac{e}{\sqrt{e^2-1}}}{\sqrt{\frac{e^2}{e^2-1}-1}}=e$$

$$\therefore\quad \underbrace{f(f(f\ldots.f(e)))}_{(n\text{ times})}=\begin{cases}\dfrac{e}{\sqrt{e^2-1}} & \text{if } 'n' \text{ is odd}\\ e & \text{if } 'n' \text{ is even}\end{cases}$$

$$\int_1^3\underbrace{f(f(f\ldots..f(e)))}_{n\text{ times}}de=\begin{cases}\dfrac{1}{2}\displaystyle\int_1^3\frac{2e}{\sqrt{e^2-1}}de & ;\quad n\text{ is odd}\\ \displaystyle\int_1^3 e\,de & ;\quad n\text{ is even}\end{cases}$$

$$= \begin{cases} \left[\sqrt{e^2-1}\right]_1^3 = 2\sqrt{2} & ; \quad n \text{ is odd} \\ \left.\dfrac{e^2}{2}\right|_1^3 = 4 & ; \quad n \text{ is even} \end{cases}$$

Hence, (B) and (C) are correct.

49. Let $f^{-1}(x) = t \Rightarrow x = f(t) \Rightarrow dx = f'(t)dt$

$$\therefore \quad 2\int_{-1}^{1} x(f^{-1}(x)-1)\,dx = 2\int_{1}^{0} f(t)(t-1)f'(t)\,dt$$

$$= \int_{0}^{1} f^2(t)\,dt - 1 \qquad \text{(integration by parts)}$$

Also, $f(x)$ is differentiable $\Rightarrow f(x)$ is continuous

$$\therefore \quad \int_{0}^{1} f(t)\,dx \text{ exists}$$

Hence, (A) and (B) are correct.

50. $f'(x) = f(x) + k$

$$\int \frac{f'(x)}{f(x)+k} = \int dx$$

$\log(f(x)+k) = x + C; f(x) = k_1e^x - k$

$$f(0) = k_1 - k = \frac{4-e^2}{3} \qquad \ldots(1)$$

$$k = \int_{0}^{2} f(x)\,dx$$

$$3k = k_1(e^2-1) \qquad \ldots(2)$$

Solving (1) and (2), we get $f(x) = e^x - \left(\dfrac{e^2-1}{3}\right)$

Hence, (A) and (D) are correct.

51. Let $f^{-1}(x) = y \Rightarrow x = f(y) \Rightarrow dx = f'(y)dy$

$$I = \int_{2}^{b} 2f(y)(b-y)f'(y)\,dy$$

$$= b\int_{2}^{b} 2f(y)f'(y)\,dy - \int_{2}^{b} 2yf(y)f'(y)\,dy$$

$$= b(f^2(b) - f^2(a)) - bf^2(b) + af^2(a) + \int_{2}^{b} f^2(y)\,dy$$

$$= \int_{a}^{b} f^2(x)dx + (a-b)f^2(a) = \int_{a}^{b} \left(f^2(x) - f^2(a)\right)dx$$

Hence, (A) and (D) are correct.

52. $f(x+13) + f(x+630) = 0 \qquad \ldots(1)$

Putting $x = x+617 \Rightarrow f(x+630) + f(x+1247) = 0 \qquad \ldots(2)$

$\Rightarrow \quad f(x+12447) - f(x+13) = 0 \Rightarrow f(x+1247) = f(x+13)$

Sub. $x = x-13 \Rightarrow f(x+1234) = f(x)$

$f(x)$ is a periodic with period $T = 1234$

$$\sum_{P=1}^{777}\int_{P}^{P+T} f(x)dx = \sum_{P=1}^{777}\int_{0}^{T} f(x)dx = 777\int_{0}^{T} f(x)dx \Rightarrow \lambda = 777$$

$$\lambda + T = 777 + 1234 = 2011$$

Hence, (A) and (B) are correct.

53. $A(x+y) = A(x)\,A(y) \Rightarrow A(0) = 1$

$\Rightarrow \quad A(x) = e^{kx}$

First note there $B(x)$ is even.

$$\int_{-2010}^{2011} B(x)dx = \int_{-2010}^{2010} B(x)dx + \int_{2010}^{2011} B(x)dx$$

$$= 2\int_{0}^{2010} B(x)dx + \int_{2010}^{2011} B(x)dx$$

$$= \int_{0}^{2010} B(x)dx + \int_{0}^{2011} B(x)dx$$

Hence, (B) and (D) are correct.

54. $I(n) = \displaystyle\int_{0}^{\frac{1}{n}} (2016+x)\cos^2 x dx$

$$\therefore \quad I\left(\frac{1}{\pi}\right) = \int_{0}^{\pi} (2016+x)\cos^2 x dx$$

$$I\left(\frac{1}{\pi}\right) = \int_{0}^{\pi} (2016+\pi-x)\cos^2 x dx$$

$$\therefore \quad 2I\left(\frac{1}{\pi}\right) = (\pi+4032)\int_{0}^{\pi} \cos^2 x dx$$

$$\Rightarrow \quad I\left(\frac{1}{\pi}\right) = \frac{(\pi+4032)\pi}{4}$$

$$\lim_{n\to\infty} nI(n) = \lim_{n\to\infty} \frac{\displaystyle\int_{0}^{1/n} (x+2016)\cos^2 x\,dx}{\dfrac{1}{n}}$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{\left(\dfrac{1}{n}+2016\right)\cos^2\dfrac{1}{n}\times\left(-\dfrac{1}{n^2}\right)}{\left(-\dfrac{1}{n^2}\right)} = 2016$$

Hence, (B) and (C) are correct.

55. $I_n = \displaystyle\int_{0}^{\pi/2} x(\sin x + \cos x)^n dx = 2^{n/2}\int_{0}^{\pi/2} x\cos^n\left(x - \frac{\pi}{4}\right)dx$

Put $x - \dfrac{\pi}{4} = t \Rightarrow I_n = 2^{n/2}\displaystyle\int_{-\pi/4}^{\pi/4} \left(\frac{\pi}{4}+t\right)\cos^n tdt$

$$\Rightarrow \quad I_n = \pi 2^{\frac{n}{2}-1}\int_{0}^{\pi/4} (\cos t)^n dt$$

$$\Rightarrow \quad I_n = \left\{\left((\cos t)^{n-1}\sin t\right)_n^{\pi/4} + (n-1)\int_{0}^{\pi/4} (\cot)^{n-2}(1-\cos^2 t)dt\right\}$$

$$\Rightarrow \quad \frac{nI_n - \dfrac{\pi}{2}}{I_{n-2}} = 2(n-1) \qquad \{\text{After simplification}\}$$

Hence, (A) and (B) are correct.

56. Put $x = 0 \Rightarrow e^0 - a\int_0^1 f(t)e^{-t}dt = 0$

$$\Rightarrow \int_0^1 f(t)e^{-t}\,dt = \frac{1}{a} \Rightarrow \int_0^x f(t)dt = e^x - ae^{2x}\frac{1}{a} = e^x - e^{2x}$$

Differentiating we get:

$$\Rightarrow \quad f(x) = e^x - 2e^{2x}$$

Hence, (A) and (B) are correct.

Linked Comprehension Type Questions

1. $I_n = \int \sin nx \cdot \sec x\,dx$

$$I_{n+2} = \int \sin(n+2)x \sec x\,dx$$

$$I_n + I_{n+2} = \int \sec x(\sin nx + \sin(n+2)x)\,dx$$

$$= \int \sec x \cdot 2\sin(n+1)x\cos x\,dx$$

$$= 2\int \sin(n+1)x\,dx$$

$$I_n + I_{n+2} = -2\frac{\cos(n+1)x}{(n+1)} + C \Rightarrow \text{Replace } n \to n-2$$

$$I_{n-2} + I_n = -\frac{2\cos(n-1)x}{n-1} + C$$

Hence, the correct answer is (C).

2. $S_2 = \int_0^{\pi/2} 2\sin x\cos x\sec x\,dx = 2(-\cos x)_0^{\pi/2} = 2(0+1) = 2$

$$S_n + S_{n+2} = 2\int_0^{\pi/2} \sin(n+1)x\,dx = \left.\frac{-2\cos(n+1)x}{(n+1)}\right|_0^{\pi/2}$$

$$n = 2; \quad S_4 + S_2 = -\frac{2}{3}[0-1] = \frac{2}{3}$$

$$\Rightarrow \quad n = 4; \quad S_6 + S_4 = -\frac{2}{5}[0-1] = \frac{2}{5}$$

$$n = 6; \quad S_8 + S_6 = -\frac{2}{7}[0-1] = \frac{2}{7}$$

Now $S_4 + S_2 - (S_6 + S_4) + S_8 + S_6 = \frac{2}{3} - \frac{2}{5} + \frac{2}{7}$

$$S_8 + S_2 = \frac{2(35-21+15)}{105} \Rightarrow S_8 = \frac{58}{105} - 2$$

Required integration $= \int_0^{\pi/2} \frac{\cos 5\theta \sin 3\theta}{\cos\theta}\,d\theta$

$$= \int_0^{\pi/2} \frac{\sin 8\theta - \sin 2\theta}{2\cos\theta}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin 8\theta \sec\theta\,d\theta - \frac{1}{2}\int_0^{\pi/2} \sin 2\theta \sec\theta\,d\theta$$

$$= \frac{1}{2}(S_8 - S_2) = \frac{1}{2}\left(\frac{58}{105} - 4\right) = \frac{-181}{105}$$

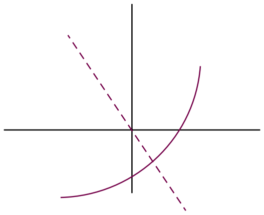
Hence, the correct answer is (B).

3. $f(x) = -x$

Only one solution.

Hence, the correct answer is (B).

4. $\int_0^1 f(x)\,dx = \int_0^1 e^x - \left(\frac{e^2-1}{3}\right)dx$



$$= e^x\Big|_0^1 - \left(\frac{e^2-1}{3}\right)x\Big|_0^1$$

$$= e - 1 - \left(\frac{e^2-1}{3}\right) = \frac{3e - e^2 - 2}{3}$$

Hence, the correct answer is (A).

5. $I_n = \int_0^{\pi/2} \cos^n x\cos nx\,dx$

$$I_1 = \int_0^{\pi/2} \cos^2 x\,dx = \frac{1}{2}\,\frac{\pi}{2} = \frac{\pi}{4} \qquad \text{(Using Walli's formula)}$$

$$I_2 = \int_0^{\pi/2} \cos^2 x\cos 2x\,dx = \int_0^{\pi/2} \cos^4 x - \cos^2 x\sin^2 x\,dx$$

$$= \frac{3\cdot 1}{4\cdot 2} \times \frac{\pi}{2} - \frac{1\cdot 1\cdot \pi}{4\cdot 2\cdot 2} \qquad \text{(Using Walli's formula)}$$

$$= \frac{3\pi}{16} - \frac{\pi}{16} = \frac{\pi}{8}$$

$$I_3 = \int_0^{\pi/2} \cos^3 x\cos 3x\,dx = \int_0^{\pi/2} \cos^3 x(4\cos^3 x - 3\cos x)\,dx$$

$$= 4\int_0^{\pi/2} \cos^6 x\,dx - 3\int_0^{\pi/2} \cos^4 x\,dx$$

$$= 4\left(\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\times\frac{\pi}{2}\right) - 3\left(\frac{3\cdot 1}{4\cdot 2}\times\frac{\pi}{2}\right) = \frac{5\pi}{8} - \frac{9\pi}{16} = \frac{\pi}{16}$$

$$I_1, I_2, I_3, \ldots \longrightarrow \frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{16} \ldots \quad \text{G.P.}$$

Hence, the correct answer is (B).

6. $T_n = (n^2+1)n! = ((n+1)^2 - 2n)n!$

$$= (n+1)^2 n! - 2n\,n!$$

$$= (n+1)\,(n+1)! - 2(n+1-1)n!$$

$$= (n+2-1)\,(n+1)! - 2[(n+1)\,n! - n!]$$

$$T_n = [(n+2)! - (n+1)!] - 2[(n+1)! - n!]$$

$$\sum T_n = (3! - 2!) - 2(2! - 1!) + (4! - 3!) - 2(3! - 2!)$$

$$\vdots$$

$$+ ((n+2)! - (n+1)!) - 2[(n+1)! - n!]$$

$$= (n+2)! - 2! - 2[(n+1)! - 1!]$$

$$T_1 + T_2 + \ldots + T_n = (n+2)! - 2(n+1)!$$

$$a_n = \frac{T_1 + T_2 + \ldots + T_n}{n!} = \frac{(n+2)! - 2(n+1)!}{n!}$$

$$= \frac{(n+2-2)(n+1)!}{n!} = n(n+1)$$

$$\sum_{n=1}^{5}\frac{1}{a_n} = \sum_{n=1}^{5}\frac{1}{n(n+1)} = \sum_{n=1}^{5}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \ldots + \left(\frac{1}{5}-\frac{1}{6}\right) = \frac{5}{6}$$

Hence, the correct answer is (A).

7. $f(f(x)) = (f(x))^2 - 4 \quad \forall\, x \in R$

$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$

$f(f(x)) = a_n(a_nx^n + a_{n-1}\,x^{n-1} + x^{n-1} + \ldots)^n + \ldots = (a_nx^n + \ldots)^2 - 4$

Degree of both, should be the same

$$x^{n^2} = x^{2n}$$

$\Rightarrow \quad n^2 = 2n$

$\Rightarrow \quad n = 0$ or $n = 2$ Quadratic

$f(x) = ax^2 + bx + c$

$a(ax^2 + bx + c)^2 + b(ax^2 + bx + c) + c = (ax^2 + bx + c)^2 - 4$

$a(a^2x^4 + b^2x^2 + c^2 + 2abx^3 + 2bcx + 2a(x^2) + abx^2 + b^2x + bc + c)$

$= a^2x^4 + b^2x^2 + c^2 + 2abx^3 + 2bcx + 2acx^2 - 4$

$a^3x^4 + 2a^2bx^3 + (ab^2 + 2a^2c + ab)x^2 + (2abc + b^2)x + ac^2 + bc + c$

$= a^2x^4 + 2abx^3 + (b^2 + 2ac)x^2 + 2bcx + c^2 - 4$

On comparison, we get, $a = 1$, $b = 0$, $c = -4$

$f(x) = ax^2 + bx + c = x^2 - 4$

$f(2) = 4 - 4 = 0$

$$l = \lim_{t\to\sqrt{2}} \left(\frac{\displaystyle\int_2^{t^2} \frac{f(x)dx}{\ln(5-x^2)}}{\sin(t-\sqrt{2})} \right) = \lim_{t\to\sqrt{2}} \frac{\frac{f(t^2)}{\ln(5-t^4)}\cdot 2t - 0}{\cos(t-\sqrt{2})}$$

$$= \lim_{t\to\sqrt{2}} \frac{2t}{\cos(t-\sqrt{2})}\,\frac{f(t^2)}{\ln(5-t^4)} = \lim_{t\to\sqrt{2}} \frac{2\sqrt{2}}{1}\,\frac{f(t^2)}{\ln(5-t^4)}$$

Now, $l = \lim\limits_{t\to\sqrt{2}} \dfrac{2\sqrt{2}\, f(t^2)}{\dfrac{\ln(5-t^4)(4-t^4)}{(4-t^4)}}$

$$= 2\sqrt{2} \lim_{t\to\sqrt{2}} \frac{(t^4-4)}{1\cdot(4-t^4)} = -\,2\sqrt{2}$$

$\Rightarrow \quad l^2 = 8$

Hence, the correct answer is (C).

8. $\displaystyle\int_0^4 f(f(x))\,d(x^2) = \int_0^4 (x^2-4)^2 - 4\,d(x^2)$

Put $x^2 = t$; $d(x^2) = dt$

$$= \int_0^4 (t-4)^2 - 4\,dt = \frac{(t-4)^3}{3} - 4t\Big|_0^4 = \frac{16}{3}$$

Hence, the correct answer is (C).

9. Writing $\sin^2\theta = x$, we get $2\sin\theta\cos\theta\,d\theta = dx$, and hence, the given integral is equal to

$$\frac{1}{2}\int_0^{\pi/2} \cos^{2m-1}\sin^{2n-1}\theta\,(2\sin\theta\cos\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (\cos^2\theta)^{\frac{2m-1}{2}}(\sin^2\theta)^{\frac{2n-1}{2}}(2\sin\theta\cos\theta)d\theta$$

$$= \frac{1}{2}\int_0^1 (1-x)^{m-1/2}x^{n-1/2}\,dx = \frac{1}{2}\beta\left(m+\frac{1}{2}, n+\frac{1}{2}\right)$$

Hence, the correct answer is (A).

10. Writing $\dfrac{x}{n} = z$, $dx = ndz$ we get:

$$\int_0^n \left(1-\frac{x}{n}\right)^n x^{k-1}dx = \int_0^1 (1-z)^n (nz)^{k-1}\,ndz\,.$$

$$= n^k \int_0^1 (1-z)^n z^{k-1}dz \;= n^k\,\beta\,(k, n+1)$$

Hence, the correct answer is (C).

11. Writing $\dfrac{x}{1+x} = z$, we get $x = \dfrac{z}{1-z}, 1+x = \dfrac{1}{1-z}$ and $dx = \dfrac{dz}{(1-z)^2}$.

$$\text{L.H.S.} = \int_0^1 \frac{z^{m-1}}{(1-z)^{m-1}}(1-z)^{m+n}\frac{dz}{(1-z)^2}$$

$$= \int_0^1 z^{m-1}(1-z)^{n-1}\,dz = \beta(m,n) \;= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Hence, the correct answer is (B).

12. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \left(x+x^2\right)dx$

$$= \frac{1}{\pi}.2\left[\frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\cdot\frac{\pi^2}{3} = \frac{2\pi^2}{3}$$

Hence, the correct answer is (A).

13. $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} \left(x+x^2\right)\cos nx\,dx$

$$a_n = \frac{1}{\pi}\left[0 + 2\int_0^\pi x^2\cos x dx\right];$$

$$a_n = \frac{2}{\pi}\cdot\left[\frac{x^2\sin nx}{n}\Big|_0^\pi - \int_0^\pi \frac{2x\sin x}{n}dx\right]$$

$$= \frac{2}{\pi}\left[0 - 2\left\{\frac{-x\cos nx}{n^2}\Big|_0^\pi + \frac{\sin nx}{n^3}\Big|_0^\pi\right.\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}(-1)^n - 0\right] = \frac{4(-1)^n}{n^2}$$

Hence, the correct answer is (A).

14. $x + x^2 = \dfrac{\pi^2}{3} + 4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}.\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$

Here, $b_n = \dfrac{2(-1)^{n+1}}{n}$ Putting $x = \pi$

$$\pi + \pi^2 = \frac{\pi^2}{3} + 4\left[1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

Putting, $x = -\pi$; $-\pi + \pi^2 = \dfrac{\pi^2}{3} + 4\left[1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots\right]$

Adding, $\pi^2 = \dfrac{\pi^2}{3} + 4\left[1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots\right]$

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

Hence, the correct answer is (A).

15. Here, $f(x) = x^2 + 1$, $\alpha(x) = [x]$

$$\therefore \quad \int_0^3 (x^2+1)\, d[x] + \int_0^3 [x]2x\, dx = [3]\,(9+1) - 0$$

$$\Rightarrow \quad \int_0^3 (x^2+1)\, d[x] = 30 - 2\int_0^3 x[x]\, dx$$

$$= 30 - 2\left\{\int_0^1 0.dx + \int_1^2 xdx + \int_2^3 2xdx\right\} = 30 - 2\left\{0 + \frac{3}{2} + 5\right\}$$

$$= 30 - 3 - 10 = 17$$

Hence, the correct answer is (C).

16. $\int_0^6 (x^2 + [x]d\,|3 - x|)$

$$= \int_0^3 (x^2 + [x])\, d(3-x) + \int_3^6 (x^2 + [x])d(x-3)$$

$$= -\int_0^3 x^2\, dx + \int_3^6 x^2\, dx - \int_0^3 [x]\, dx + \int_3^6 [x]\, dx$$

$$= -\left\{\frac{x^3}{3}\right\}_0^3 + \left\{\frac{x^3}{3}\right\}_3^6 - \int_0^3 [x]\, dx + \int_0^3 [x+3]\, dx$$

$$= -9 + (72 - 9) - \int_0^3 [x]\, dx + \int_0^3 [x]\, dx + 3\int_0^3 1\;dx$$

$$= 72 - 18 + 9 = 72 - 9 = 63$$

Hence, the correct answer is (C).

17. $\int_{-2}^{3} [\,|x|\,]d\,|x| = \int_{-2}^{0} [-x]d(-x) + \int_0^3 [x]\, dx$

$$= -\int_{-2}^{0} [-x]\, dx + \int_0^3 [x]\, dx = \int_2^0 [x]\, dx + \int_0^3 [x]\, dx \text{ (by property)}$$

$$= -\int_0^2 [x]\, dx + \int_0^2 [x]\, dx + \int_2^3 [x]\, dx$$

$$= \int_2^3 [x]\, dx = \int_2^3 2.dx = 2(3-2) = 2$$

Hence, the correct answer is (C).

Combined Solution to 18–19

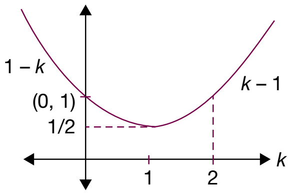
Put $\sin t = x \Rightarrow \cos t\, dt = dx$

$$f(k) = \int_0^1 |k - 2x|\, dx = 2\int_0^1 \left|x - \frac{k}{2}\right| dx$$

If $\frac{k}{2} \le 0$ i.e. $k \le 0$

$$f(k) = 2\int_0^1 \left(x - \frac{k}{2}\right) dx = 2\left(\frac{x^2}{2} - \frac{kx}{2}\right)\Bigg|_0^1 = 2\left(\frac{1}{2} - \frac{k}{2}\right) = 1 - k$$

If $\frac{k}{2} \ge 1 \Rightarrow k \ge 2$



$$f(k) = 2\int_0^1 \left(\frac{k}{2} - x\right) dx \quad \therefore f(k) = (k-1)$$

If $0 < \frac{k}{2} < 1$ or $0 < k < 2$

$$f(k) = 2\left[\int_0^{\frac{k}{2}} \left(\frac{k}{2} - x\right) dx + \int_{\frac{k}{2}}^{1} \left(x - \frac{k}{2}\right) dx\right]$$

$$f(k) = \frac{1}{2}(k^2 - 2k + 2) \;\therefore f(k) = \begin{cases} 1 - k & \text{if } k \le 0 \\ \frac{1}{2}(k^2 - 2k + 2) & \text{if } 0 \le k \le 2 \\ k - 1 & \text{if } k > 2 \end{cases}$$

Now graph of $y = f(k)$

Hence, the minimum value of $f(k)$ is $\frac{1}{2}$ at $k = 1$ and $f(k)$ is continuous and derivable $\forall\, k$.

18. Hence, the correct answer is (B).

19. Hence, the correct answer is (B).

20. Given, $\int_0^x (x - t + 1)g(t)dt = x^4 + x^2 \;\forall\, x \in R$.

$$(x+1)\int_0^x g(t)\, dt - \int_0^x tg(t)\, dt = x^4 + x^2$$

$$\Rightarrow \quad (x+1)\int_0^x g(t)dt \quad \int_0^x tg(t) = x^4 + x^2 \;\forall\, x \in R.$$

$\therefore$ On differentiating both sides with respect to x, we get:

$$(x+1)\, g(x) + \int_0^x g(t)dt - x\, g(x) = 4x^3 + 2x$$

$$\Rightarrow \quad g(x) + \int_0^x g(t)dt = (4x^3 + 2x)$$

So, $g(0) = 0$. Now, again differentiating both sides with respect to x, we get

$g'(x) + g(x) = (12x^2 + 2)$ [Linear differential equation in $g(x)$]

Also, $h(x) = (12x^2 + 2)\;\forall\, x \in R \;\therefore$ Integrating factor $= e^x$

So, general solution is

$$g(x)\cdot e^x = \int (12x^2 + 2)\cdot e^x dx + C$$

$$\Rightarrow \quad g(x)\cdot e^x = 12x^2 - 24x\, e^x + 26e^x + C$$

As $g(0) = 0 \Rightarrow C = -26$

Hence, the function $g(x) = 12x^2 - 24x + 26 - 26^{-x}$.

As, $g(x) = 24x + 26 - 26e^{-x}$

So, $g'(x) = 24x - 24 + 26 - 26e^{-x}$

$\Rightarrow \quad g'''(x) = 26e^{-x}$

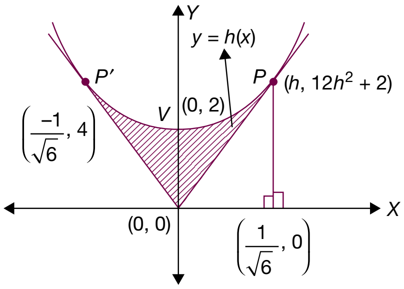
Hence, $\int_0^1 g'''(x)dx = \int_0^1 26\left(e^{-x}\right)_0^1 = -26(e^{-1}-1) = 26(1-e^{-1})$.

Hence, the correct answer is (B).

21. The equation of tangent at $(h, 12h^2+2)$ is

$$(y-(12h^2+2)) = 24h\,(x-h)$$

As (0, 0) satisfy it, so $12h^2+2 = 24h^2$



$\Rightarrow \quad 12h^2 = 2 \Rightarrow h^2 = \frac{1}{6} \Rightarrow h = \pm\frac{1}{\sqrt{6}}$

$\therefore$ Required area $= 2\left[\int_0^{1/\sqrt{6}} (12x^2+2)dx - \frac{1}{2}\times\frac{1}{\sqrt{6}}\times 4\right] = \frac{2\sqrt{6}}{9}$.

Hence, the correct answer is (D).

22. Given, $f(n) = h(n+1) - h(n)$

$= (12(n+1)^2+2) - (12n^2+2) = 24n + 12$

So, $\sum_{r=1}^{10} f(r) = \sum_{r=1}^{10}(24r+12) = 24\sum_{r=1}^{10} r + \sum_{r=1}^{10} 12$

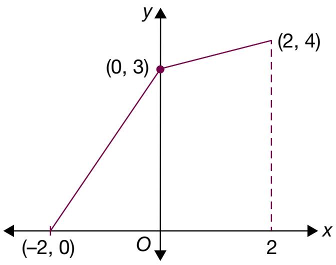
$= 24\left(\frac{10\times 11}{2}\right) + 120 = (120\times 11) = 120 = 1440.$

Hence, the correct answer is (A).

23. $g(x) = \begin{cases} \frac{3}{2}x+3 & ; \quad -2 \le x \le 0 \\ \frac{x}{2}+3 & ; \quad 0 < x \le 2 \end{cases}$

$$f(x) = \int_{-2}^{x} |g(t)-t|\,dt, \quad -2 \le x \le 2$$

$$= \begin{cases} \int_{-2}^{x}\left|\frac{3}{2}t+3-t\right|dt & ; \quad -2 \le x \le 0 \\ \int_{-2}^{0}\left|\frac{3t}{2}+3-t\right|dt + \int_{0}^{x}\left|\frac{t}{2}+3-t\right|dt & ; \quad 0 < x \le 2 \end{cases}$$



$$\Rightarrow \quad f(x) = \begin{cases} \frac{1}{2}\int_{-2}^{x}|t+6|dt & ; \quad -2 \le x \le 0 \\ \frac{1}{2}\int_{-2}^{0}|t+6|dt + \frac{1}{2}\int_{0}^{x}|6-t|dt & ; \quad 0 < x \le 2 \end{cases}$$

$$f(x) = \begin{cases} \frac{x^2}{4}+3x+5 & ; \quad -2 \le x \le 0 \\ -\frac{x^2}{4}+3x+5 & ; \quad 0 < x \le 2 \end{cases}$$

$f'(0^+) = f'(0^-) = 3$

Hence, the correct answer is (A).

24. For $0 \le x \le 2$, $5 \le f(x) \le 10$

$f(x) = \frac{31}{4} \Rightarrow x = 1$

$g\left(\frac{31}{4}\right) = 1$

$g'\left(\frac{31}{4}\right) = \frac{1}{f'(1)} = \frac{2}{5}$

Hence, the correct answer is (B).

25. $f(x) = 2x^2 + 5$

$\Rightarrow \quad \frac{-x^2}{4} + 3x + 5 = 2x^2 + 5 \Rightarrow x = 0, \frac{4}{3}$.

Hence, the correct answer is (C).

26. $\lim\limits_{x\to 0} I_1 = 1 = \dfrac{\lim\limits_{x\to 0}\int_0^x \frac{t^2dt}{\sqrt{\lambda^2+2\lambda t}}}{(x-\sin x)}$

$= \lim\limits_{x\to 0}\dfrac{x^2}{(1-\cos x)\sqrt{\lambda^2+2\lambda x}} = 1 \Rightarrow \frac{2}{|\lambda|} = 1 \Rightarrow |\lambda| = \pm 2$

$\because \quad$ Sum of values of $\lambda = 0$

Hence, the correct answer is (A).

27. $\lim\limits_{x\to 0} I_2 = \lim\limits_{x\to 0}\int_0^{x^2}\frac{e^t-1-t}{(x^6)}dt$

$= \lim\limits_{x\to 0}\dfrac{e^{x^2}-1-x^2}{6x^5}\cdot 2x = \frac{1}{3}\lim\limits_{x\to 0}\dfrac{e^{x^2}-1-x^2}{(x^2)^2} = 1$

$\therefore \quad \lim\limits_{x\to 0} I_1 = 2$ for $\lambda = 1$

$\lim\limits_{x\to 0}(I_1 + 6I_2) = \frac{2}{1} + 1 = 3.$

Hence, the correct answer is (B).

28. We have $f(x) = \lim\limits_{\alpha\to 0}\dfrac{1}{\alpha^4}\int_0^{\alpha}\dfrac{(e^{x+t}-e^x)\ln^2(1+t)}{2t^3+3}dt$

$$e^x\lim_{\alpha\to 0}\frac{(e^2-1)\ln^2(1+\alpha)}{\frac{(22^3+3)}{\alpha^4}} = \frac{e^x\lim\limits_{\alpha\to 0}\frac{(e^{\alpha}-1)\ln^2(1+t)}{(2t^3+3)}}{\alpha^4}$$

$\Rightarrow \quad f(x) = \frac{e^x}{12}$

Clearly $f(\ln 2) = \dfrac{e^{\ln 2}}{12} = \dfrac{2}{12} = \dfrac{1}{6}$

Hence, the correct answer is (B).

29. We have, $\displaystyle\int_0^x g(t)dt = 3x + \int_x^0 \cos^2 t g(t)dt$

Now, on differentiating both the sides with respect to x, we get,

$$g(x) = 3 - \cos^2 x \cdot g(x) \Rightarrow g(x) = \frac{3}{1+\cos^2 x}$$

Clearly, $\dfrac{3}{2} \le \dfrac{3}{1+\cos^2 x} \le 3.$

Hence, range of $g(x) = \left[\dfrac{3}{2}, 3\right]$

Hence, the correct answer is (A).

30. Let $I = \displaystyle\int_0^{\frac{\pi}{2}} g(x)dx = \int_0^{\frac{\pi}{2}} \frac{3}{1+\cos^2 x}dx$

$$= \int_0^{\frac{\pi}{2}} \frac{3\sec^2 x}{\sec^2 x + 1}dx = \int_0^{\frac{\pi}{2}} \frac{3\sec^2 x}{\tan^2 x + 2}dx$$

Put, $\tan x = t \Rightarrow \sec^2 x\, dx = dt,$

So, $I = \displaystyle\int_0^{\infty} \frac{3dt}{t^2+2} = \frac{3}{\sqrt{2}}\left(\tan^{-1}\frac{t}{\sqrt{2}}\right)_0^{\infty} = \frac{3}{\sqrt{2}}\left(\frac{\pi}{2} - 0\right) = \frac{3\pi}{2\sqrt{2}}.$

Hence, the correct answer is (D).

31. $\therefore \quad f(x) = f(2-x) \Rightarrow f'(x) = -f'(2-x)$ …(1)

Putting $x = \dfrac{1}{2}, \dfrac{1}{4}$; we get $f'\left(\dfrac{3}{2}\right) = f'\left(\dfrac{7}{4}\right) = 0$

Putting $x = 1$ in (1)

$f'(1) = -f'(1) \Rightarrow f'(1) = 0$

$\therefore \quad f'(x) = 0$ will have atleast five real roots in [0, 2]

$\therefore \quad f''(x) = 0$ will have atleast four real roots in [0, 2]

Hence, the correct answer is (C).

32. Replacing x by $1 + x$ in (1), we get

$f'(1+x) = -f'(1-x) = 0$

Let $I = \displaystyle\int_{-1}^{1} f'(1+x)x^2 e^{x^2}dx$ …(2)

$I = \displaystyle\int_{-1}^{1} f'(1-x)\cdot x^2 e^{x^2}dx$ (using king)

$I = -\displaystyle\int_{-1}^{1} f'(1+x)\cdot x^2 e^{x^2}dx$ …(3)

$(\because\ f'(1+x) = -f'(1-x))$

From (2) + (3), we get $2I = 0 \Rightarrow I = 0$

Hence, the correct answer is (D).

33. Let $I = \displaystyle\int_1^1 f'(1-t)e^{\cos \pi t}dt - \int_1^2 f'(2-t)e^{\cos \pi t}dt$

$= \displaystyle\int_0^1 f(1-(1-t)e^{-\cos \pi t}dt - \int_1^2 f(2-t)e^{\cos \pi t}dt$ (Using king in I$^{\text{st}}$)

$= \displaystyle\int_0^1 f(t)e^{-\cos \pi t}dt - \int_1^2 f(t)e^{\cos \pi t}dt$ $(\therefore f(2-t) = f(t))$

$\because \quad \displaystyle\int_0^2 f(t)e^{\cos \pi t}dt = 2\int_0^1 f(t)e^{\cos \pi t}dt$

$(\therefore f(2-t)e^{\cos \pi(2-t)} = f(t)e^{\cos \pi t})$

$\Rightarrow \quad \displaystyle\int_1^2 f(t)e^{\cos \pi t}dt = \int_0^1 f(t)e^{\cos \pi t}dt$ $(\therefore I = 0)$

$\displaystyle\int_1^2 f(t)e^{\cos \pi t}dt$ $\{\therefore f'(2-t) = -f'(t)\}$

$$= \int_0^1 f'(t)e^{\cos \pi t} + f'(2-t)e^{\cos(2\pi - \pi t)}dt = 0$$

Hence, the correct answer is (A).

Combined Solution to 34–36

$$I_3 = \int_0^{\pi/2} x \cdot \cot x\, dx = x \cdot \ln(\sin x)_0^{\pi/2} - \int_0^{\pi/2} \ln(\sin x)\, dx$$

$$I_3 = \int_0^{\pi/2} \underbrace{x}_{(\text{I})_{\text{I.B.P}}} \underbrace{\cot x}_{(\text{II})} dx$$

$$= \int_0^{\pi/2} \underbrace{x}_{(\text{I})_{\text{I.B.P}}} \underbrace{\cot x}_{(\text{II})} dx (x \cdot \ln(\sin x)_0^{\pi/2}) - \int_0^{\pi/2} \ln(\sin x)\, dx$$

$$\therefore \quad I_3 = 0 - \left(-\frac{\pi}{2}\ln 2\right) = \frac{\pi}{2}\ln 2$$

Now, $I_2 = \displaystyle\int_0^{\frac{\pi}{2}} \underbrace{x^2}_{(\text{I})} \cdot \underbrace{\text{cosec}^2 x}_{\substack{(\text{II})\\(\text{I.B.P})}} dx = \left(x^2 \cdot (-\cot x)\right)_0^{\pi/2} + \int_0^{\frac{\pi}{2}} 2x \cdot \cot x\, dx$

$$I_2 = 0 + 2\left(\frac{\pi}{2}\ln 2\right) = \pi \ln 2$$

Also, $I_1 = \displaystyle\int_0^{\frac{\pi}{2}} \frac{x^3 \cdot \cos x\, dx}{4\sin^3 x} = \frac{1}{4}\int_0^{\frac{\pi}{2}} \underbrace{x^3}_{(\text{I})} \underbrace{\left(\frac{\cos x}{\sin^3 x}\right)}_{\substack{(\text{II})\\(\text{I.B.P})}} dx$

$$= \frac{1}{4}\left[\left(x^3 \cdot \left(\frac{-1}{2\sin^2 x}\right)\right)_0^{\frac{\pi}{2}} + \frac{3}{2}\int_0^{\frac{\pi}{2}} x^2 \text{cosec}\, x\, dx\right]$$

$$= \frac{1}{4}\left[\left(\frac{\frac{\pi^3}{8}}{-2} - 0\right) + \frac{3}{2}(\pi \ln 2)\right]$$

$$I_3 = \frac{-\pi^3}{64} + \frac{3}{8}\pi \ln 2.$$

Now, verify alternatives.

34. Hence, the correct answer is (A).

35. Hence, the correct answer is (C).

36. Hence, the correct answer is (A).

37. We have $f(x) = x^2 + x \underbrace{\int_{-1}^{1} f(y)dy}_{A} + \underbrace{\int_{-1}^{1} yf(y)dy}_{B}$

$$\therefore \quad f(x) = (x^2 + Ax + B) \qquad \ldots(1)$$

Where $A = \int_{-1}^{1} f(y)dy \; + \int_{-1}^{1} (y^2 + Ay + B)dy$

$$= 2\int_{-1}^{1} (y^2 + B)dy \; = 2\left(\frac{y^3}{3} + By\right)_0^1$$

$$\Rightarrow \quad A = \frac{2}{3} + 2B \Rightarrow (A - 2B) = \frac{2}{3} \qquad \ldots(2)$$

Similarly, $B = \int_{-1}^{1} yf(y)dy \; = \int_{-1}^{1} y(Ay + B + y^2)dy$

$$= 2A\int_{-1}^{1} y^2 dy \; = \frac{2A}{3} \Rightarrow 2A = 3B \qquad \ldots(3)$$

$\therefore$ On solving (2) and (3), we get, $A = -2$, $B = \frac{-4}{3}$

Hence, $f(x) = \left(x^2 - 2x - \frac{4}{3}\right) = (x-1)^2 - \frac{7}{3}$.

$$f(x) > k \; \forall \; x \in R \Rightarrow x^2 - 2x - \left(\frac{4}{3} + k\right) > 0 \; \forall \; x \in R, \text{ hence}$$

$$D < 0 \Rightarrow 4 + 4\left(\frac{4}{3} + k\right) < 0 \Rightarrow \frac{7}{3} + k < 0$$

$$\Rightarrow \quad k < \frac{-7}{3} \Rightarrow k \in \left(-\infty, \frac{-7}{3}\right)$$

$$\therefore \quad k_{\text{largest integer}} = -3 \quad \Rightarrow -3$$

Hence, the correct answer is (B).

38. As, $g(t) = f(-t) - f(t) \; \forall \; x \in R$

$$\Rightarrow \quad g(t) = \left(t^2 + 2t - \frac{4}{3}\right) - \left(t^2 - 2t - \frac{4}{3}\right) \qquad (\therefore g(t) = 4\,t \; \forall \; t \in R)$$

Now, $\int_0^{\frac{\pi}{2}} g(t)\sin t \, dt \; = 4\int_0^{\frac{\pi}{2}} \underset{\text{I}}{t} \cdot \underbrace{\sin t}_{\substack{\text{II} \\ \text{(I.B.P.)}}} \, dt$

$$= 4\left[(-t \cdot \cos t)_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} 1 \cdot \cos t \, dt\right] \qquad \left(\therefore g \int_0^{\frac{\pi}{2}} g(t)\sin t \, dt \; = 4\right)$$

Hence, the correct answer is (D).

39. We have $K(x) = 3f(x)\, b + 4 \; \forall \; x \in R$

$$\therefore \quad k(x) = 3\left(x^2 - 2x - \frac{4}{3}\right) + b + 4$$

$$= (3x^2 - 6x - 4) + b + 4$$

$$\Rightarrow \quad k(x) = (3x^2 - 6x + b) \; \forall \; x \in R$$

So, for range of $k(x)$ to be $[0, \infty)$, we have

$$D = 0 \Rightarrow 36 = 12b \qquad \therefore b = 3.$$

Hence, the correct answer is (C).

40. Let $I = \int_0^1 |x - t| \cos \pi t \, dt$

CASE 1: When $x \geq 1$

So, $I = \int_0^1 \underbrace{(x - t)}_{\text{I}} \cdot \underbrace{\cos \pi t}_{\substack{\text{(II)} \\ \text{(I.B.P.)}}} \, dt$

$$= \left((x - t)\frac{\sin \pi t}{\pi}\right)_{t=0}^{t=1} - \int_0^1 (-1) \cdot \frac{\sin \pi t}{\pi} \, dt$$

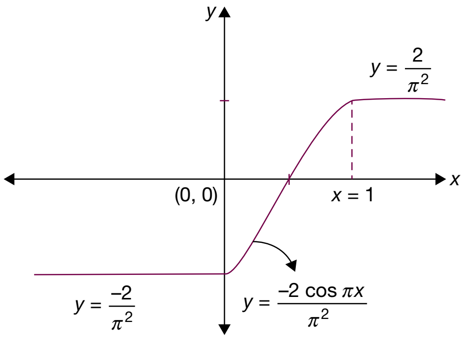
$$= \frac{1}{\pi}\int_0^1 \sin \pi t \, dt \; = \frac{-1}{\pi^2} \; (\cos \pi t)_0^1 \; = \frac{1}{\pi^2} + \frac{1}{\pi^2} = \frac{2}{\pi^2}$$

CASE 2: When $0 < x < 1$,

So, $I = \int_0^1 \underbrace{(x - t)}_{\text{(I)}} \underbrace{\cos \pi t}_{\substack{\text{(II)} \\ \text{(I.B.P.)}}}$

$$= \left[\left((x - t)\frac{\sin \pi t}{\pi}\right)_{t=0}^{t=x} + \frac{1}{\pi}\int_0^x \sin \pi t \, dt\right] +$$

$$\left[\left((x - t)\frac{\sin \pi t}{\pi}\right)_{t=0}^{t=1} - \frac{1}{\pi}\int_0^x 1 \cdot \sin \pi t \, dt\right]$$



$$= \left(\frac{1 - \cos \pi x}{\pi^2}\right) + \left(\frac{-1 - \cos \pi x}{\pi^2}\right) = \frac{-2 \cos \pi x}{\pi^2}$$

CASE 3: When $-\infty < x \leq 0$,

So, $I = \int_0^1 \underbrace{(t - x)}_{\text{(I)}} \underbrace{\cos \pi t}_{\substack{\text{(II)} \\ \text{(I.B.P.)}}} \, dt = \left((t - x)\frac{\sin \pi t}{\pi}\right)_{t=0}^{t=1} - \int_0^1 1 \cdot \frac{\sin \pi t}{\pi} \, dt$

$$= \frac{1}{\pi^2} \; (\cos \pi t)_{t=0}^{t=1} = \frac{-2}{\pi^2} \quad \therefore f(x) = \begin{cases} \frac{-2}{\pi^2} & ; \quad -\infty < x \leq 0 \\ \frac{-2}{\pi^2}\cos \pi x & ; \quad 0 < x < 1 \\ \frac{2}{\pi^2} & ; \quad 1 \leq x < \infty \end{cases}$$

As, range of $f(x) = \left[\frac{-2}{\pi^2}, \frac{2}{\pi^2}\right]$ so there is only one integer in the range of $f(x)$ i.e. 0. Clearly, $f(x)$ is continuous $\forall \; x \in R$.

As, $h(x) = f(x) = \frac{-2 \cos \pi x}{\pi^2}$, $x \in [0, 1] \quad \Rightarrow h\left(\frac{1}{2}\right) = 0$

And $k(x) = h^{-1}(x)$

$$\therefore \quad k'(0) = \frac{1}{h'\left(\frac{1}{2}\right)} = \frac{1}{\left(\frac{-2}{\pi^2}\times(-\pi\sin\pi x)\right)_{x=\frac{1}{2}}} = \frac{1}{\frac{2}{\pi}} = \frac{2}{\pi}.$$

Hence, the correct answer is (B).

41. $\beta = \lim\limits_{x\to 0} \dfrac{x^2\cdot\sin(x-2x)}{\int_0^x t^2\,dt}\left(\dfrac{0}{0}\right)$

$$= \lim_{x\to 0}\left(\frac{x^2\cdot\sin x}{x^3}\right) = 3\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 3.$$

Given, $g(x) = 7 + 2x\ln 25 - 5^{x-1} - 5^{2-x}$

$$= 7 + 4x\ln 5 - \frac{5^x}{5} - \frac{25}{5^x}$$

$$\therefore \quad g'(x) = 0 + 4\ln 5 + 25\cdot 5^{-x}\cdot\ln 5$$

$$= \frac{\ln 5}{5}[20 - 5^x + 125\cdot 5^{-x}]$$

$\therefore$ For maximum or minimum of $g(x)$, we must have $g'(x) = 0$

$$\Rightarrow \quad (5^x - 25)\underbrace{(5^x+5)}_{\text{(Always positive)}} = 0 \quad \therefore 5^x = 25 \Rightarrow x = 2.$$

Also, $g''(x=2) < 0 \Rightarrow g(x)$ is maximum at $x = 2 = \alpha$(Given)

$$\text{So, } \sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{2}+\frac{1}{3}\right)^n = \sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^n$$

$$= 1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \ldots\infty = \frac{1}{1-\frac{5}{6}} = \frac{1}{\frac{1}{6}} = 6.$$

Hence, the correct answer is (C).

42. $2g'(x) = 2g(x) - 3 \Rightarrow \displaystyle\int\frac{2g'(x)dx}{2g(x)-3} = \int 1.dx$

$$\Rightarrow \quad g(x) = \frac{3}{2} - \frac{3}{2}\cdot e^{x-1} \qquad \text{(As, g(1) = 0)}$$

$\therefore$ Equation of normal at (1, 0) is

$$(y-0) = \frac{2}{3}(x-1) \qquad \therefore y \text{ intercept} = \frac{-2}{3}$$

Hence, the correct answer is (B).

43. Range of $g(x) = \left(-\infty, \dfrac{3}{2}\right)\ \forall\ x \in R.$

Hence, the correct answer is (D).

44. Put $x = 1$, in the given functional equation, we get $\displaystyle\int_0^1 g(t)dt = 3 + k$

$$\Rightarrow \quad k = -3 + \int_0^1\left(\frac{3}{2} - \frac{3e^{t-1}}{2}\right)dt \Rightarrow k = \left(\frac{3}{2e} - 3\right)$$

Hence, the correct answer is (B).

45. We have $g(x) = 1 + \displaystyle\int_0^x g(t)dt$ …(1)

Now, on differentiating both the sides of equation (1) with respect to x, we get

$$g'(x) = g(x) \qquad \ldots(2)$$

But $g(x) = 0$ (Not possible as $g(0) = 1$)

$$\text{So, } \int\frac{g'(x)}{g(x)}dx = \int 1\,dx$$

$$\Rightarrow \quad \ln(g(x)) = x + A$$

$$\therefore \quad A = 0 \quad (\text{As } g(0) = 1)$$

Hence, $g(x) = e^x$

Hence, $g(\ln 10) + g'(\ln 10) + g''(\ln 10) = 10 + 10 + 10 = 30$

Hence, the correct answer is (C).

46. We have, $g(x) + g(2x) + \ldots\infty = \dfrac{e^x}{1-e^x}$ If $x < 0$ then $e^x < 1$.

$$\therefore \quad \int_{-3}^{-1}\left(\sum_{r=1}^{\infty} g(rx)\right)dx = \int_{-3}^{-1}\frac{e^x dx}{1-e^x}$$

$$= \left[-\left(\ln\left(1-e^x\right)\right)\right]_{-3}^{-1} = \ln\left(1+\frac{1}{e}+\frac{1}{e^2}\right)$$

Hence, the correct answer is (B).

47. As $g(-x) = g(x)$ gives $e^{-x} = e^x \Rightarrow e^{2x} = 1$

$$\therefore \quad x = 0$$

Hence, the number of solution of given equation is one.

Hence, the correct answer is (B).

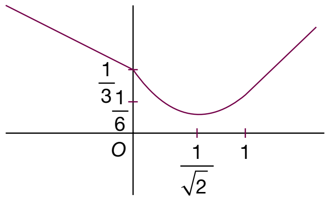
Matrix Match/Column Match Type Questions

1. **[A → t; B → p; C → s; D → p, q, s, t]**

$$f(x) = \int_0^1 t\,|t-x|\,dt = \begin{cases} \displaystyle\int_0^1 t(t-x)dt & ; \quad x \le 0 \\ \displaystyle\int_0^x t(x-t)dt + \int_x^1 t(t-x)dt & ; \quad 0 < x < 1 \\ \displaystyle\int_0^1 t(x-t)dt & ; \quad x \ge 1 \end{cases}$$

$$f(x) = \begin{cases} \left.\dfrac{t^3}{3} - \dfrac{t^2}{2}x\right|_0^1 & ; \quad x \le 0 \\ \left.\dfrac{t^2}{2}x - \dfrac{t^3}{3}\right|_0^x + \left.\dfrac{t^3}{3} - \dfrac{t^2}{2}x\right|_x^1 & ; \quad 0 < x < 1 \\ \left.\dfrac{t^2}{2}x - \dfrac{t^3}{3}\right|_0^1 & ; \quad x \ge 1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{3} - \dfrac{x}{2} & ; \quad x \le 0 \\ \dfrac{x^3}{2} - \dfrac{x^3}{3} + \dfrac{1}{3} - \dfrac{x}{2} - \left(\dfrac{x^3}{3} - \dfrac{x^3}{2}\right) & ; \quad 0 < x < 1 \\ \dfrac{x}{2} - \dfrac{1}{3} & ; \quad x \ge 1 \end{cases}$$



$$f(x)=\begin{cases}\dfrac{1}{3}-\dfrac{x}{2} & ; \quad x\le 0\\[2mm] \dfrac{1}{3}-\dfrac{x}{2}+\dfrac{x^3}{3} & ; \quad 0<x<1\\[2mm] \dfrac{x}{2}-\dfrac{1}{3} & ; \quad x\ge 1\end{cases}$$

$$O<x<1;\ f'(x)=-\frac{1}{2}+x^2=0$$

Hence, $x=\dfrac{1}{\sqrt{2}}$ is point of minima

(A) min $x\in R$

$$f(x)=f\left(\frac{1}{\sqrt{2}}\right)=\frac{1}{3}-\frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}3}=\frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)$$

(B) max $x\in(0,1)\ f(x)=f(0)=\dfrac{1}{3}$ [P]

(C) min $x\ge 1\ f(x)=f(1)=\dfrac{1}{6}$

(D) $f''(x)>0$ for $x\in R^+$ [p, q, s, t]

2. **[A → q; B → s; C → p; D → t]**

(A) $J=\displaystyle\int_0^{\infty}\frac{\ln(1+t^2)}{1+t^2}\,dt$ put $t=\tan\theta\Rightarrow dt=\sec^2\theta\,d\theta$

$$=\int_0^{\pi/2}\frac{\ln(1+\tan^2\theta)}{1+\tan^2\theta}\sec^2\theta\,d\theta=\int_0^{\pi/2}\ln\sec^2\theta\,d\theta$$

$$=2\int_0^{\pi/2}\ln\sec\theta\,d\theta=-2\int_0^{\pi/2}\ln\cos\theta\,d\theta$$

$J=-2K\Rightarrow J+2K=0$

$\alpha=2$ (q)

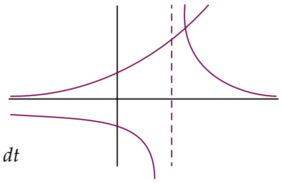
(B) $$4\left(\lim_{t\to 0}\frac{\tan^{-1}t\cdot\tan^2(\tan^{-1}t)\dfrac{1}{1+t^2}}{(\sec^2 t-1)\sec t\tan t}\right)$$

$$=4\left(\lim_{t\to 0}\frac{\tan^{-1}t\cdot t^2}{(1+t^2)\tan^2 t\sec t\tan t}\right)$$

$$=\lim_{t\to 0}4\left(\frac{\tan^{-1}t}{t}\right)\left(\frac{t}{\tan t}\right)^3\left(\frac{1}{1+t^2}\right)$$

$$=\frac{4(1)(1)}{1+0}=4$$

(C) $f(x)=\sin x-\displaystyle\int_0^x(x-t)f(t)\,dt$



$x=0,\ y=0$

$$f(x)=\sin x-x\int_0^x f(t)\,dt+\int_0^x tf(t)\,dt$$

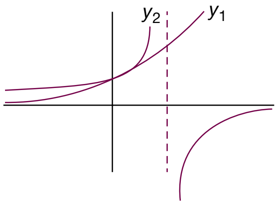
$$f'(x)=\cos x-\int_0^x f(t)\ -xf(x)+xf(x)$$

$f'(0)=1-0=m_T=$ slope of tangent (p)

(D) **CASE 1 :** $k>0$

$e^x=\dfrac{k}{x-3}$ only one solution

CASE 2: $k<0$



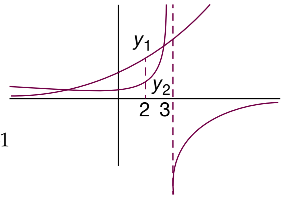
$y=e^x\quad y=\dfrac{k}{x-3}$

let (x_1, y_1) is point of contact of common tangent

$$e^{x_1}=\frac{-k}{(x_1-3)^2}=-\frac{e^{x_1}}{x_1-3}$$

$\Rightarrow\quad x_1-3=-1\Rightarrow x_1=2$

$P(2, e^2)\equiv(2,-k)$ we get $k=-e^2$



For 2 solution $y_2<y_1$

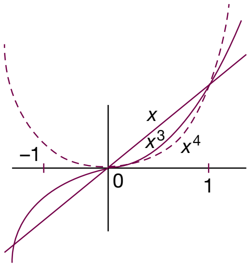
$-K<e^2\Rightarrow K\in(-e^2, 0)$

Acceptable $k=-7,-6,-5,-4,-3,-2,-1$

7 values (t)

3. **[A → p; B → r; C → q; D → p]**

On differently given relations



$u(x)+v(x)=x^3$

$u(x)-v(x)=2x-x^3$

$v(x)=x^3-x$ and $u(x)=x$

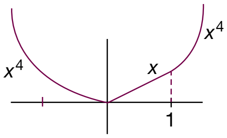
$$\frac{2u(x)}{w(x)}=\frac{2}{x^3}\Rightarrow\frac{2(x)}{w(x)}=\frac{2}{x^3}$$

$\Rightarrow\quad w(x)=x^4$

(A) $$\int_{-\pi/4}^{\pi/4}v(x)=\int_{-\pi/4}^{\pi/4}(x^3-x)\,dx=0\quad\text{(p)}$$

(B) $h(x)=\max(x, x^3, x^4)$

Now diff. at $x=0$, 1 at 2 points (r)



(C) $h(x)$ has 1 extremum point (minima at $x=0$) (q)

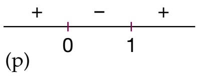
(D) $w(x)<u(x)$

$x^4<x\Rightarrow x(x^3-1)<0$

$\Rightarrow x(x-1)(x^2+x+1)<0$

$\Rightarrow x(x-1)<0$

$\Rightarrow 0<x<1$ ∴ no integral value of x (p)



4. **[A → q; B → p; C → s; D → p]**

(A) $$\int_{6\pi-3}^{6\pi}\frac{\cos\left(\dfrac{x}{3}\right)}{6\pi+3-x}\,dx$$

Let $6\pi-3t=x\ \Rightarrow -dx=3\,dt$

$$\int_0^1\frac{3\cos(2\pi-t)}{3(1+t)}\,dt=\int_0^1\frac{\cos t}{1+t}\,dt=\int_0^1\frac{\cos x}{1+x}\,dx=k$$

(B) $$\int_{-1}^{1}\left(\sin^{-1}\left[x+\frac{3}{4}\right]\right)dx=\int_{-1}^{-3/4}-\frac{\pi}{2}\,dx+\int_{-3/4}^{1/4}0\,dx+\int_{1/4}^{1}\frac{\pi}{2}\,dx$$

$$= -\frac{\pi}{8} + \frac{\pi}{2} \times \frac{3}{4} = \frac{2\pi}{8} = \frac{\pi}{4}$$

(C) $\because \begin{matrix} -|x| \le [x] & \text{for all + ve } x \\ -|x| \ge [x] & \text{for all – ve } x \end{matrix}$

$$\Rightarrow \int_{-3}^{3} f(x)dx = \int_{-3}^{0} (x - |x|)dx + \int_{0}^{3} (x + [x])dx$$

$$= \int_{-3}^{0} 2x\,dx + \int_{0}^{3} x\,dx + \int_{0}^{3} [x]dx = -\frac{3}{2}$$

(D) $\displaystyle\int_{0}^{20} \sqrt{1 - \cos \pi x}\,dx = \int_{0}^{20} \sqrt{2}\left|\sin\frac{\pi x}{2}\right|dx$

$$= 10\sqrt{2}\int_{0}^{2}\left|\sin\frac{\pi x}{2}\right|dx \qquad \left\{\because \left|\sin\frac{\pi x}{2}\right| \text{ is periodic with periods 2}\right\}$$

$$= 10\sqrt{2}\int_{0}^{2}\sin\left(\frac{\pi x}{2}\right)dx = 10\sqrt{2}\left[-\frac{\cos\left(\frac{\pi x}{2}\right)}{\frac{\pi}{2}}\right]_0^2 = \frac{40\sqrt{2}}{\pi}.$$

5. [A → q; B → p; C → p; D → r]

(A) $\displaystyle I = \int_{2}^{3} ((x-1)^3 + (4-x)^3 + x)\cos\pi x\,dx \quad (x \to 5 - x)$ …(1)

$$I = \int_{2}^{3} (4-x)^3 + (x-1)^3 + 5 - x)(-\cos\pi x)\,dx \qquad \text{…(2)}$$

$$2I = \int_{2}^{3} (2x-5)\cos\pi x\,dx$$

$$\Rightarrow \quad 2I = (2x-5)\left.\frac{\sin\pi x}{\pi}\right|_2^3 - 2\int_{2}^{3}\frac{\sin\pi x}{\pi}\,dx$$

$$2I = 0 + 2\left.\frac{\cos\pi x}{\pi^2}\right|_2^3 = 2\frac{(-1-1)}{\pi^2}$$

$$I = \frac{-2}{\pi^2} \Rightarrow |50\pi^2 I| = 100 \quad \text{(q)}$$

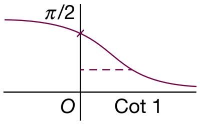
(B) $\displaystyle J = \int_{0}^{10} \text{sgn}(\sin\pi x)\,dx \quad T = 2$

$$= 5\int_{0}^{2}\text{sgn}(\sin\pi x)\,dx = 5\int_{0}^{1} 1\,dx + \int_{1}^{2} -1\,dx$$

$$= 5((1-0) - (x)_1^2)$$

$$= 5(1 - 2 + 1) = 0 \quad \text{(p)}$$

(C) $\displaystyle K = \int_{0}^{102} [\cot^{-1} x]\,dx$



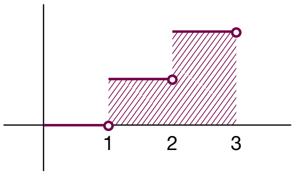
$$= \int_{0}^{\cot 1} 1\,dx + \int_{\cot 1}^{\infty} 0\,dx$$

$$= \cot 1 \approx \frac{1}{\sqrt{3}} \quad 1 = 60^\circ\ [K] = 0 \quad \text{(p)}$$

(D) $\displaystyle L = \frac{\int_{0}^{51} [x] + 25d\alpha}{\int_{0}^{51} \{x\}dx} = \frac{(1+2+\ldots+50) + 25(51-0)}{51\left.\frac{x^2}{2}\right|_0^1}$

$$= \frac{\frac{50.51}{2} + 25.51}{\frac{51}{2}}$$

$$= \frac{50-51}{51} = 100 \quad \text{(q)}$$



6. [A → q ; B → q ; C → r; D → r]

(A) $\displaystyle I = \int_{0}^{\pi} x\left((\sin^2(\sin x) + \cos^2(\cos x)\right)dx$

$$\Rightarrow I = \int_{0}^{\pi} (\pi - x)\left(\sin^2(\sin x) + \cos^2(\cos x)\right)dx$$

$$\text{Adding } 2I = \pi\int_{0}^{\pi}\left(\sin^2(\sin x) + \cos^2(\cos x)\right)dx$$

$$2I = 2\pi\int_{0}^{\pi/2}\left(\sin^2(\sin x) + \cos^2(\cos x)\right)dx\,;$$

$$I = \pi\int_{0}^{\pi/2}\left(\sin^2(\sin x) + \cos^2(\cos x)\right)dx$$

$$\text{Also } I = \pi\int_{0}^{\pi/2}\left[\sin^2(\cos x) + \cos^2(\sin x)\right]dx$$

$$\text{Adding } 2I = \pi\int_{0}^{\pi/2} 2dx = 2\pi\cdot\frac{\pi}{2} = \pi^2 \quad I = \frac{\pi^2}{2}$$

(B) Let $f(x) = 2\sin\sqrt{x}$

$$\text{Then } x\,f'(x) = x\cdot\frac{2\cos\sqrt{x}}{2\sqrt{x}} = \sqrt{x}\cos\sqrt{x}$$

$$I = \int_{0}^{\pi^2/4}\left(f(x) + x\,f'(x)\right)dx = \left[x\,f(x)\right]_0^{\pi^2/4}$$

$$= 2\times\frac{\pi^2}{4}\times\sin\frac{\pi}{2} = \frac{\pi^2}{2}$$

(C) $\displaystyle\int_{-\pi/4}^{\pi/4}\ln\sqrt{1 + \sin 2x}\;dx = \int_{-\pi/4}^{\pi/4}\ln|\sin x + \cos x|dx$

$$= \int_{-\pi/4}^{\pi/4}\ln\left|\sqrt{2}\,\sin\left(x + \frac{\pi}{4}\right)\right|dx$$

$$= \int_{0}^{\pi/2}\ln\left|\sqrt{2}\,\sin t\right|dt = \ln\sqrt{2}\int_{0}^{\pi/2} dt + \int_{0}^{\pi/2}\ln\sin t\,dt$$

$$= \frac{\pi}{4}\ln 2 - \frac{\pi}{2}\ln 2 = -\frac{\pi}{4}\ln 2$$

$$\therefore \frac{\pi}{\ln 2}\int_{-\pi/4}^{\pi/4} \ln\sqrt{1+\sin 2x}\,dx=\left(\frac{\pi}{\ln 2}\right)\left(\frac{-\pi}{4}\ln 2\right)=\frac{-\pi^2}{4}$$

(D) $I=\displaystyle\int_0^{\pi}\frac{x^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}\,dx$

$$I=\int_O^{\pi}\frac{(\pi-x)^3\cos^4 x\sin^2 x}{\pi^2-3\pi x+3x^2}\,dx$$

Adding $2I=\pi\displaystyle\int_0^{\pi}\cos^4 x\sin^2 x\,dx$

$$I=\frac{\pi}{2}\cdot 2\int_0^{\pi/2}\cos^4 x\sin^2 x\,dx=\pi\int_0^{\pi/2}\cos^4 x\sin^2 x\,dx$$

$$=\pi\frac{(3.1)1}{6.4.2}\frac{\pi}{2}=\frac{\pi^2}{32}$$

7. [A → s; B → q; C → q; D → q]

(A) For $O\le x\le\dfrac{\pi}{2}$, $\dfrac{1+2\sin^2 x}{1+\sin^2 x}\in\left[1,\dfrac{3}{2}\right]$

$$\therefore \left[\frac{1+2\sin^2 x}{1+\sin^2 x}\right]=1$$

(B) For $-\dfrac{\pi}{2}\le x\le\dfrac{\pi}{2}$, $\cos x-\cos^2 x\in\left[0,\dfrac{1}{4}\right]$

$\therefore [\cos x-\cos^2 x]=0$

(C) For $\dfrac{\pi}{6}\le x\le\dfrac{\pi}{4}$,

$$\frac{\sec x+\operatorname{cosec} x-\sec x\operatorname{cosec} x}{2}=\frac{\sin x+\cos x-1}{2\sin x\cos x}$$

$$=\frac{1}{\sin x+\cos x+1}$$

$$\therefore x\in\left[\sqrt{2}-1,1-\frac{1}{\sqrt{3}}\right]$$

$$\therefore \left[\frac{\sec x+\operatorname{cosec} x-\sec x\operatorname{cosec} x}{2}\right]=0$$

(D) For $\dfrac{\pi}{6}\le x\le\dfrac{\pi}{4}$, $\dfrac{\cot x-\tan x}{2}=\cot 2x\in\left[0,\dfrac{1}{\sqrt{3}}\right]$

8. [A → r; B → t; C → s; D → q]

(A) $\left(\dfrac{\sin^2 3x}{\sin^2 x}-\dfrac{\cos^2 3x}{\cos^2 x}\right)$

$$=\left(\frac{\sin^2 3x\cos^2 x-\sin^2 x\cos^2 3x}{\sin^2 x\cos^2 x}\right)$$

$$=\left(\frac{(\sin 3x\cos x+\sin x\cos 3x)(\sin 3x\cos x-\sin x\cos 3x)}{\sin^2 x\cos^2 x}\right)$$

$$=\frac{4\sin 4x\sin 2x}{\sin^2 2x}=8\cos 2x \qquad \therefore \int_{\pi/4}^{\pi/3}\left(\frac{\sin^2 3x}{\sin^2 x}-\frac{\cos^2 3x}{\cos^2 x}\right)dx$$

$$=\int_{\pi/4}^{\pi/3}8\cos(2x)dx=4\sin 2x\Big|_{\pi/4}^{\pi/3}=2\sqrt{3}-4$$

(B) $\dfrac{1}{1+\sin 2x}=\dfrac{1}{1+\dfrac{2\tan x}{1+\tan^2 x}}=\dfrac{\sec^2}{(1+\tan x)^2}$

Let $u=\dfrac{\tan x-1}{\tan x+1}$

$$\Rightarrow \quad du=\frac{(1+\tan x)\sec^2 x-(1-\tan x)\sec^2 x}{(1+\tan x)^2}$$

$$=\frac{2\sec^2 x}{(1+\tan x)} \qquad \therefore \int_{\pi/2}^{\pi/4}\frac{1}{1+\sin 2x}\sqrt[2009]{\frac{\tan(x)-1}{\tan(x)+1}}\,dx$$

$$=\frac{-1}{2}\int_2^1 u^{\frac{1}{2009}}du=\frac{-1}{2}\times\frac{2009}{2010}\Rightarrow \text{(t)}$$

(C) Put $x=\tan\theta$

$$\Rightarrow \quad I=\int\frac{x^2}{(x\sin x+\cos x)^2}\,dx=\int\frac{\tan^2\theta}{\cos^2(\tan\theta-\theta)}\,d\theta$$

Put $\tan(\theta)-\theta=y$

$$\Rightarrow \int\sec^2 y\,dy=\tan y=\tan(\tan\theta-\theta)$$

$$=\tan(x-\tan^{-1}x)=\frac{\tan x-x}{1+x\tan x}$$

$$\frac{\tan x-x}{1+x\tan x}\Bigg|_0^{\pi/2}=\frac{1}{\frac{\pi}{2}}-0=\frac{2}{\pi}\Rightarrow \text{(s)}$$

(D) $\displaystyle\int_1^e\frac{\ln x}{x\left(\sqrt{1-\ln x}+\sqrt{\ln x+1}\right)}\,dx$

Rationalising the denominator, we get:

$$\int_1^e\frac{\ln x\left(\sqrt{1-\ln x}-\sqrt{\ln x-1}\right)}{-2x\ln x}\,dx$$

Now put $t=\ln x\Rightarrow dt=\dfrac{1}{x}dx$

At $x=1$, $t=0$ at $x=e$, $t=1$

$$=\frac{1}{2}\int_0^1\left(\sqrt{1+t}-\sqrt{1-t}\right)dt$$

$$=\frac{1}{3}\left((1+t)^{3/2}\Big|_0^1+(1-t)^{3/2}\Big|_0^1\right)=\frac{2\sqrt{2}-2}{3}\Rightarrow \text{(q)}$$

9. [A → q; B → r; C → p; D → s]

(A) $I=\displaystyle\int_0^{\pi/2}\frac{(\sin x)^{\cos x}}{(\sin x)^{\cos x}+(\cos x)^{\sin x}}\,dx$

Using king property and add, we get

$I=\dfrac{\pi}{4}$.

(B) $\displaystyle\int_1^2\frac{(x^2-1)}{x^3\sqrt{2x^4-2x^2+1}}\,dx$

$$=\int_1^2\frac{(x^2-1)}{x^5\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}\,dx=\int_1^2\frac{\left(\frac{1}{x^3}-\frac{1}{x^5}\right)}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}\,dx$$

Put $2-\dfrac{2}{x^2}+\dfrac{1}{x^4}=t^2\Rightarrow\left(\dfrac{4}{x^3}-\dfrac{4}{x^5}\right)dx=2t\,dt$

$$\Rightarrow \int\frac{tdt}{2t}=\frac{1}{2}\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}\Bigg]_1^2=\frac{1}{2}\left(\frac{5}{4}-1\right)=\frac{1}{8}.$$

(C) $\int_{-2}^{2} \frac{x^3 - x + 1}{\sqrt{4-x^2}} dx$

$$\int_{-2}^{2} \underbrace{\frac{x^3}{\sqrt{4-x^2}}}_{\text{odd}} dx - \int_{-2}^{2} \underbrace{\frac{x}{\sqrt{4-x^2}}}_{\text{odd}} dx + \int_{-2}^{2} \underbrace{\frac{dx}{\sqrt{4-x^2}}}_{\text{even}}$$

$$= 0 + 0 + 2\int_{0}^{2} \frac{dx}{\sqrt{4-x^2}} = 2\sin^{-1}\left(\frac{x}{2}\right)\Bigg]_0^2 = \pi.$$

(D) $I_2 = \int_0^1 \frac{x^2}{5x^3(2-x^3)} dx$

$$I_2 = \int_0^1 \frac{x^2}{5^{x^3}(2-t)} dx; \text{ Put } x^3 = t \Rightarrow 3x^2 dx = dt$$

$$I_2 = \frac{1}{3}\int_0^1 \frac{dt}{5^t(2-t)} \Rightarrow 3I_2 = \int_0^1 \frac{5^{-t}}{5(2-t)} dt$$

Using King property

$$3I_2 = \int_0^1 \frac{5^{t-1}}{5(1-t)} dt$$

$$\therefore \quad 3I_2 = \frac{1}{5}\int_0^1 \frac{5^t}{5(1-t)} dt = \frac{I_1}{5} \Rightarrow \frac{I_1}{I_2} = 15$$

Integer/Numerical Answer Type Questions

1. $\frac{1}{\pi} I = \frac{1}{\pi}\left(\int_0^{5\pi/2} f(x)dx - \int_0^{\pi/2} f(x)dx\right) = \frac{1}{\pi}\left(\frac{5\pi}{4} - \frac{\pi}{4}\right) = 1$

Hence, the correct answer is (1).

2. $I_n = \int_0^1 x^{n-1} x\sqrt{1-x^2} dx = x^{n-1}\left(-\frac{(1-x^2)^{3/2}}{3}\right)\Bigg|_0^1 +$

$$\frac{1}{3}\int_0^1 (n-1)x^{n-2}(1-x^2)^{3/2} dx$$

$\therefore \quad 3I_n = (n-1)(I_{n-2} - I_n)$

or $\frac{I_n}{I_{n-2}} = \frac{n-1}{n+2} \to 1 \quad$ as $n \to \infty$

Hence, the correct answer is (1).

3. $f(x) = Kx$ is required function

$$I_n = n\int_0^n Kx\ dx = n\frac{Kx^2}{2}\Bigg|_0^n = \frac{Kn^3}{2}$$

$$\Rightarrow \quad I_1 + I_2 + I_3 + I_4 + I_5 = \frac{K}{2}(1^3 + 2^3 + \ldots + 5^3) = 450$$

$$\frac{K}{2}\left(\frac{5 \cdot 6}{2}\right)^2 = 450; \ \frac{K}{2}(15 \cdot 15) = 450$$

$$\Rightarrow \quad K = 4 = f'(0)$$

Hence, the correct answer is (4).

4. $P = \int_0^1 {}^{207}C_7 x^{200}(1-x)^7 dx = {}^{207}C_7$

$$\left(\frac{x^{201}}{201}(1-x)^7\Bigg|_0^1 - \int_0^1 \frac{x^{201}}{201}(7)(1-x)^6(-1)dx\right)$$

$$= {}^{207}C_7 \times \frac{7}{201}\int_0^1 x^{201}(1-x)^6 dx$$

Similarly ${}^{207}C_7 \times \frac{7}{201}\cdot\frac{6}{202}\cdot\frac{5}{203}\cdot\frac{4}{204}\cdot\frac{3}{205}\cdot\frac{2}{206}\cdot\frac{1}{207}\cdot\frac{1}{208}$

$$\Rightarrow \quad P = \frac{1}{208}$$

Alternate $P = \int_0^1 {}^{207}C_7\, x^{200}(1-x)^7 dx$

Put $x = \sin^2\theta;\ dx = 2\sin\theta\cos\theta\, d\theta$

$$P = {}^{207}C_7 \int_0^{\pi/2} \sin^{400}\theta\left(1-\sin^2\theta\right)^7 2\sin\theta\cos\theta d\theta$$

$$= 2.{}^{207}C_7 \int_0^{\pi/2} \sin^{401}\theta\cos^{15}\theta d\theta$$

$$= 2.{}^{207}C_7 \frac{(400.398\ldots4.2)(14.12\ldots4.2)}{416.414\ldots4.2}$$

$$P = 2.\frac{207.206\ldots201}{7!}\frac{(14.12\ldots4.2)}{416.414\ldots402} \Rightarrow P = \frac{1}{208}$$

Hence, the correct answer is (2).

5. $f(x) + f(1-x) = \frac{\lambda^x}{\lambda^x + \sqrt{\lambda}} + \frac{\lambda^{1-x}}{\lambda^{1-x} + \sqrt{\lambda}}$

$$= \frac{\lambda^x}{\lambda^x + \sqrt{\lambda}} + \frac{\lambda}{\lambda + \sqrt{\lambda}\lambda^x} = 1$$

$$I_1 = \int_{f(1-K)}^{f(K)} xf(x(1-x))\ dx = \int_{f(1-K)}^{f(K)} (1-x)f(1-x)x)\ dx$$

$$\left(\text{using } \int_a^b f(x)\ dx = \int_a^b f(a+b-x)\right)$$

$$I_1 = \int_{f(1-K)}^{f(K)} f((1-x)x)\ dx - \int_{f(1-K)}^{f(K)} xf((1-x)x)\ dx$$

$$I_1 = I_2 - I_1 \Rightarrow \frac{I_1}{I_2} = \frac{1}{2} = \frac{a}{2} \Rightarrow a = 1$$

Hence, the correct answer is (1).

6. $f(0) = f(\pi) = 0$

$$\int_0^{\pi/2} (f(2x) + f''(2x))\sin x\cos x\ dx;$$

$$= \int_0^{\pi} (f(2x) + f''(2x))\frac{\sin 2x}{2}\ dx$$

Put $2x = t \Rightarrow 2dx = dt$

$$= \int_0^{\pi} (f(t) + f''(t)) \sin t \frac{dt}{4}$$

$$= \frac{1}{4}\left[\int_0^{\pi} f(t)\sin t\,dt + \int_0^{\pi} f''(t)\sin t\,dt\right]$$

$$= \frac{1}{4}\left(\int_0^{\pi} f(t)\sin t\,dt + f'(t)\sin\Big|_0^{\pi} - \int_0^{\pi} f'(t)\cos t\,dt\right)$$

$$= \frac{1}{4}\left(\int_0^{\pi} f(t)\sin t\,dt + 0 - 0 - (f(t)\cos t\Big|_0^{\pi} - \int_0^{\pi} f(t)(-\sin t)dt\right)$$

$$= \frac{1}{4}\left(\int_0^{\pi} f(t)\sin t\,dt - 0 - \int_0^{\pi} f(t)\sin t\,dt\right) = 0$$

$\Rightarrow \quad K = 0$

Hence, the correct answer is (0).

7. $$\lim_{n\to\infty} \int_0^{\infty} \frac{\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\cos\frac{x}{2^{n-1}}\left(2\sin\frac{x}{2^{n-1}}\right)}{2\sin\frac{x}{2^{n-1}}}dx = \frac{\pi}{2}$$

$$\Rightarrow \quad \lim_{n\to\infty} \int_0^{\infty} \frac{\sin x}{2^{n-1}\sin\frac{x}{2^{n-1}}}dx = \lim_{n\to\infty} \int_0^{\infty} \frac{\sin x}{x}\frac{\left(\frac{x}{2^{n-1}}\right)}{\left(\sin\frac{x}{2^{n-1}}\right)}dx = \frac{\pi}{2}$$

$$\Rightarrow \quad \int_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$$

$$\int_0^{\infty} \frac{\sin^3 x}{x}dx = \int_0^{\infty} \frac{3\sin x - \sin 3x}{4x}dx \qquad (\because \ \sin 3x = 3\sin x - 4\sin^3 x)$$

$$= \frac{3}{4}\int_0^{\infty} \frac{\sin x}{x}dx - \int_0^{\infty} \frac{\sin 3x}{3x}\left(\frac{1}{4}\right)d(3x)$$

$$= \frac{3}{4}\left(\frac{\pi}{2}\right) - \frac{1}{4}\left(\frac{\pi}{2}\right); = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4} \qquad \therefore \lambda = 4$$

Hence, the correct answer is (4).

8. $f^3(x) = \int_0^x t f^2(t)\,dt \Rightarrow f : [0, \infty) \to R \Rightarrow f(0) = 0$

$3f^2(x) f'(x) = xf^2(x)$

$\Rightarrow \quad f'(x) = \frac{x}{3}$ given slope of normal $= m_N = -\frac{1}{2}$;

Slope of tangent $= m_T = 2$

$\Rightarrow \quad f'(x_0) = \frac{x_0}{3} = 2 \Rightarrow x_0 = 6$

$$\frac{dy}{dx} = \frac{x}{3} \Rightarrow y = \frac{x^2}{6} + c \qquad (\because \ f(0) = 0)$$

We get $c = 0$

$f(6) = \frac{36}{6} = 6$ at point (6, 6), normal $y - 6 = -\frac{1}{2}(x - 6)$

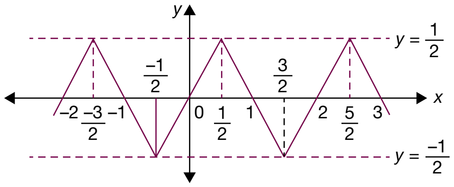
$2y - 12 + x - 6 = 0 \Rightarrow 2y + x - 18 = 0$

$y = -\frac{1}{2}x + 9$, y intercept is $= 9$

Hence, the correct answer is (9).

9. $$f(x) = \begin{cases} x & ; \ \frac{-1}{2} \le x < \frac{1}{2} \\ 1 - x & ; \ \frac{1}{2} \le x < \frac{3}{2} \\ x - 2 & ; \ \frac{3}{2} \le x < \frac{5}{2} \\ 3 - x & ; \ \frac{5}{2} \le x < \frac{7}{2} \end{cases} \quad \text{and so on.}$$

Graph of $f(x)$ will be as shown below.



Clearly, $f(x)$ is an odd function.

$\because$ cos(πx) is an even function, so $f(x)$. cos (πx) will be an odd function.

$$\text{Hence, } \frac{\pi^2}{5}\int_{-20}^{20} f(x)\cos(\pi x)\,dx = 0$$

Hence, the correct answer is (0).

10. Given $\int_{-6}^{12} f(x)\,dx = 9$

$$\Rightarrow \quad \int_{-6}^{0} f(x)\,dx + \int_0^{12} f(x)\,dx = 9 \Rightarrow \int_{-6}^{0} f(x)\,dx = -3$$

$$\int_0^{12} f(x)\,dx = \int_{-6}^{6} f(x)dx = 12$$

$$\Rightarrow \quad \int_{-6}^{0} f(x)\,dx + \int_0^{6} f(x)dx = 12 \Rightarrow \int_0^{6} f(x)dx = 15$$

Hence, the correct answer is (15).

11. We have $\int_{-4}^{\frac{9}{2}} (\sin \pi x)\left\{\frac{x}{2}\right\}dx$

$$= \underbrace{\int_{-4}^{4} (\sin \pi x)\left\{\frac{x}{2}\right\}dx}_{\text{(periodic with } T=2)} + \int_4^{\frac{9}{2}} (\sin \pi x)\left\{\frac{x}{2}\right\}dx$$

$$= 4\int_0^2 \underbrace{(\sin \pi x)}_{\substack{\text{II} \\ \text{(I.B.P.)}}} \underbrace{\frac{x}{2}}_{\text{I}}dx + \int_0^{\frac{1}{2}} \underbrace{(\sin \pi x)}_{\substack{\text{II} \\ \text{(I.B.P.)}}} \underbrace{\frac{x}{2}}_{\text{I}}dx = \frac{-8\pi + 1}{2\pi^2}$$

Hence, the correct answer is (10).

12. $$\int_{\pi/4}^{\pi/2} \frac{dx}{\left(\sqrt{\sin x} + \sqrt{\cos x}\right)^2 \sqrt{\sin x \cos x}}$$

$$= \int_{\pi/4}^{\pi/2} \frac{dx}{\cos x\left(\sqrt{\tan x} + 1\right)^2} \frac{1}{\sqrt{\frac{\sin x}{\cos x}\cos^2 x}}$$

$$= \int_{\pi/4}^{\pi/2} \frac{\sec^2 x}{\left(\sqrt{\tan}+1\right)^2 \sqrt{\tan x}}\, dx$$

Put $\sqrt{\tan x}\ +1 = t \Rightarrow \frac{1}{2\sqrt{\tan x}} \sec^2 x\, dx = dt$

$$= \int_{2}^{\infty} \frac{2dt}{t^2} = -\left.\frac{2}{t}\right|_2^{\infty} = -2\left(0-\frac{1}{2}\right) = 1$$

Hence, the correct answer is (1).

13. $f(x) = \sqrt{\left(\sqrt{x^3+1}-1\right)^2} + \sqrt{\left(\sqrt{x^3+1}-3\right)^2}$

$f(x) = \left|\sqrt{x^3+1}-1\right| + \left|\sqrt{x^3+1}-3\right|$

$(\because\ 0 < x < 2 \Rightarrow 0 < x^3 < 8 \Rightarrow 1 < \sqrt{x^3+1}\ < 3)$

$$\int_0^2 f(x)\, dx = \int_0^2 \left(\sqrt{x^3+1}-1\right)+\left(3-\sqrt{x^3+1}\right) dx$$

$$= \int_0^2 2\, dx = 2(2-0) = 4$$

Hence, the correct answer is (4).

14. $f(x) = x^3 + 3x^2 - 6x + 11,\ g(x) = \ln|x| \Rightarrow g'(x) = \frac{1}{x}$

$f'(x) = 3x^2 + 6x - 6;\ g'(-1) = -1 \Rightarrow f'(-1) = -9$

$A = f(g'(-1)) = f(-1) = -1 + 3 + 6 + 11 = 19$

$B = g(f'(-1)) = g(-9) = \ln 9$

$$C = \int_{-2}^{1} f'(x)g'(f(x))\, dx = \int_{-2}^{1} d(g(f(x)) = g(f(1)) - g(f(-2))$$

$= \ln 9 - g(27) = \ln 9 - \ln 27 = \ln\left(\frac{1}{3}\right);$

$$D = \frac{d}{dt}\int_1^{t^2} g(x)\, dx$$

$D = g(t^2)2t - 0$ at $t = 3 \Rightarrow D = g(9)\cdot 6 = 6\ln 9$

$A(B + C) - D = 19(\ln 9 + \ln(1/3)) - 6\ln 9$

$= 19(\ln 3) - 12\ln 3 = 7\ln 3n = K(\ln 3)$ we get $K = 7$

Hence, the correct answer is (7).

15. $g : R \rightarrow \{4\}$

$\Rightarrow$ Range = 4 only so $g(x)$ must be *a* constant function

$$g(x) = x^3\underset{\overset{||}{O}}{(f'(t)-2)} + x^2\underset{\overset{||}{O}}{f''(t)} + 4\underset{\overset{||}{O}}{(f((0)+6)} + 4$$

$f(0) = -6,\ f''(t) = 0,\ f'(t) = 2$

let $f(t) = at + b,\ f(0) = -6 = b,\ f'(t) = a = 2$ $\quad\therefore f(x) = 2x - 6$

$$h(x) = \begin{cases} \int_0^x |2x-8|\,dt & ;\quad 0 \le x \le 6 \\ (x-6)^2+20 & ;\quad 6 < x \le 12 \end{cases}$$

$$\equiv \begin{cases} \int_0^x 2(4-x)\,dx & ;\quad 0 \le x < 4 \\ (x-6)^2+20 & ;\quad 6 < x \le 12 \\ \int_0^4 2(4-x)\,d\alpha + \int_4^x 2(x-4)\,dx & ;\quad 4 \le x \le 6 \end{cases}$$

$$= \begin{cases} 8x-x^2 & ;\quad 0 \le x < 4 \\ (x-6)^2+20 & ;\quad 6 < x \le 12 \\ \left. 8x-x^2\right|_0^4 + \left. x^2-8x\right|_4^x & ;\quad 4 \le x \le 6 \end{cases}$$

$$= \begin{cases} 8x-x^2 & ;\quad 0 \le x < 4 \\ (x-6)^2+20 & ;\quad 6 < x \le 12 \\ 16+x^2-8x+16 & ;\quad 4 \le x \le 6 \end{cases}$$

$$y = \begin{cases} 16-(x-4)^2 & ;\quad 0 \le x < 4 & ,\quad 0 \le y \le 16 \\ 16+(x-4)^2 & ;\quad 4 \le x \le 6 & ,\quad 16 \le y \le 20 \\ (x-6)^2+20 & ;\quad 6 \le x \le 12 & ,\quad 20 < y \le 56 \end{cases}$$

Range of $h(x) = [0, 56]$; no. of integer $= 57 = N \Rightarrow \frac{N}{19} = 3$

Hence, the correct answer is (3).

16. $$P = \int_{(\ln 4)^{1/3}}^{(\ln 5)^{1/3}} \frac{u^2\cot^{-1}(u^3)}{\cot^{-1}(u^3)+\cot^{-1}(\ln 20-u^3)}\, du$$

Put $u^3 = t \Rightarrow 3u^2\, du = dt$

$$P = \int_{\ln 4}^{\ln 5} \frac{\cot^{-1}(t)}{\cot^{-1}t+\cot^{-1}(\ln 20-t)}\cdot\frac{dt}{3} \qquad \ldots(1)$$

Replace t by $\ln 4 + \ln 5 - t = \ln 20 - t$

$$P = \int_{\ln 4}^{\ln 5} \frac{\cot^{-1}(\ln 20-t)}{\cot^{-1}(\ln 20-t)+\cot^{-1}t}\,\frac{dt}{3} \qquad \ldots(2)$$

Adding (1) + (2), we get

$$2P = \frac{1}{3}\int_{\ln 4}^{\ln 5} dt = \frac{1}{3}\ln\frac{5}{4}$$

$$\Rightarrow \quad P = \ln\left(\frac{5}{4}\right)^{1/6};\ e^P = \left(\frac{5}{4}\right)^{1/6} \Rightarrow e^{6P} = \frac{5}{4} \Rightarrow 4e^{6P} = 5$$

Hence, the correct answer is (5).

17. $f(0) = -4,\quad f'(3) = 0,\quad f'(-3) = 12$

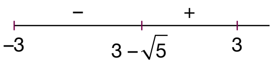
$f''(x) \ge -2 \quad \forall\, x \in [-3, 3]$

$\frac{d}{dx}(f'(x) + 2x) \ge 0 \quad h(x) = f'(x) + 2x \qquad$ (non-decreasing)

$h(3) = f'(3) + 6 = 6$

$h(-3) = f'(-3) - 6 = 6 \qquad \therefore h(x)$ is constant in $x \in [-3, 3]$

$h(x) = f'(x) + 2x = 6$



$f'(x) = 6 - 2x \Rightarrow f(x) = 6x - x^2 + c$

$f(0) = -4 = c \quad f(x) = 6x - x^2 - 4$

$$g(x) = \int_0^x f(t)\, dt \Rightarrow g'(x) = f(x) = -(x^2 - 6x + 4)$$

$$x = \frac{6 \pm \sqrt{20}}{2} = 3 \pm \sqrt{5} \text{ max can be at } x = -3,\ x = 3$$

$$g(3) = \int_0^3 6x - x^2 - 4\, dx = \left. 3x^2 - \frac{x^3}{3} - 4x\right|_0^3 = 27 - 9 - 12 = 6$$

$$g(-3) = \int_0^{-3} 6x - x^2 - 4\,dx = 3x^2 - \frac{x^3}{3} - 4x\Big|_0^{-3}$$

$$= 27 + 9 + 12 = 48$$

$$\frac{g(x)_{\max}}{12} = \frac{48}{12} = 4$$

Hence, the correct answer is (4).

18. $$I = \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})} \quad \ldots(1)$$

$$I = \int_0^1 \frac{dx}{(5+2(1-x)-2(1-x)^2)(1+e^{2-4(1-x)})}\ a$$

$$I = \int_0^1 \frac{e^{2-4x}\,dx}{(5+2x-2x^2)(1+e^{2-4x})} \quad \ldots(2)$$

(1) + (2), we get

$$2I = \int_0^1 \frac{dx}{5+2x-2x^2} = -\frac{1}{2}\int_0^1 \frac{dx}{x^2-x-\frac{5}{2}}$$

$$= -\frac{1}{4}\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)^2 - \frac{11}{4}}$$

$$I = -\frac{1}{4}\left(\frac{1}{2\frac{\sqrt{11}}{2}}\ln\left(\frac{x-\frac{1}{2}-\frac{\sqrt{11}}{2}}{x-\frac{1}{2}+\frac{\sqrt{11}}{2}}\right)\Bigg|_0^1\right)$$

$$= -\frac{1}{4\sqrt{11}}\left(\ln\left|\frac{1-\sqrt{11}}{1+\sqrt{11}}\right| - \ln\left(\left|\frac{1+\sqrt{11}}{\sqrt{11}-1}\right|\right)\right)$$

$$= \frac{1}{2\sqrt{11}}\ln\left|\frac{\sqrt{11}+1}{\sqrt{11}-1}\right| = \frac{1}{2\sqrt{11}}\ln\left|\frac{\left(\sqrt{11}+1\right)^2}{11-1}\right|$$

$$= \frac{1}{\sqrt{11}}\ln\frac{\sqrt{11}+1}{\sqrt{10}} \qquad \therefore a = 1$$

Hence, the correct answer is (1).

19. $f(0) = 1; g(0) = 0$

$$g(x) = \int e^x\left(\int_0^x f(t)dt + f(x)\right)dx$$

$$\Rightarrow \quad g'(x) = e^x\left(\int_0^x f(t)dt + f(x)\right) = e^x\int_0^x f(t)\,dt + e^x f(x)$$

Put $x = 0 \Rightarrow g'(0) = e^0\int_0^0 f(t)dt + e^0 f(0) \Rightarrow g'(0) = 0 + 1$

Hence, the correct answer is (1).

20. $x\int_0^x \sin(f(t))\,dt = (x+2)\int_0^x t\sin(f(t))\,dt$, $x > 0$ on differentiating

$$1 \cdot \int_0^x \sin(f(t))\,dt + x\sin f(x) = \int_0^x t\sin(f(t))\,dt + (x+2)\,x\sin f(x)$$

$$\int_0^x (1-t)\sin f(t)\,dt = (x^2+x)\sin f(x)$$

$$\Rightarrow \quad (1-x)\sin f(x) = (2x+1)\sin f(x) + (x^2+x)\cos f(x)\cdot f'(x)$$

$$\Rightarrow \quad (1-x) = (2x+1) + (x^2+x)\cot f(x)\,f'(x)$$

$$\Rightarrow \quad 0 = (3x+0) + x(x+1)\cot f(x)\,f'(x)$$

$$f'(x)\cot f(x) + \frac{3}{x+1} = 0$$

Hence, the correct answer is (0).

21. $f(x) = 3x^2 + 2x + a$

$\frac{da}{dt} = 3$ let $a = 0$ when $t = 0 \Rightarrow a = 3t$;

we have $f(x) = 3x^2 + 2x + 3t$

$$A(t) = \int_0^t \{f(x)\}\,dx = \int_0^t \{3x^2+2x+3t\}\,dt$$

$$A(2) = \int_0^2 \{3x^2+2x+6\}\,dx = \int_0^t \{3x^2+2x\}\,dx$$

when $x \in [0, 2]$ $3x^2 + 2x \in [0, 16]$

As we know that {.} is discontinuous at every integral so integration will break minimum 16 times, so $n = 16$ min of $n/2 = 8$

Hence, the correct answer is (8).

22. Given, $\int_0^1 f(x)\,dx = \frac{1}{3} + \int_0^1 \left(f(x^2)\right)^2 dx \quad \ldots(1)$

Consider, $I_1 = \int_0^1 \left(f(x^2)\right)^2 dx$

Put $x^2 = t \Rightarrow 2x\,dx = dt \Rightarrow I_1 = \frac{1}{2}\int_0^1 \frac{\left(f(t)\right)^2}{\sqrt{t}}\,dt$

$\therefore$ Equation (1) becomes $\int_0^1 \frac{t}{2\sqrt{t}}\,dt + \int_0^1 \frac{(f(t))^2 - 2\sqrt{t}\,f(t)}{2\sqrt{t}}\,dt$

$$\Rightarrow \quad \int_0^1 \left(f(t) - \sqrt{t}\right)^2 dt = 0 \qquad \therefore f(t) = \sqrt{t}$$

So, reciprocal of $f\left(\frac{1}{4}\right) = 2.$

Hence, the correct answer is (2).

23. $I = \int_{\frac{1}{2}}^1 \frac{dx}{x\sqrt{(3+4x)(3-2x)}}$; put $x = \frac{1}{y} \Rightarrow dx = \frac{-dy}{y^2}$

$$I = \int_1^2 \frac{y^2dy}{y^2\sqrt{(3y+4)(3y-2)}} = \int_1^2 \frac{dy}{9y^2+6y-8}$$

$$= \frac{1}{3}\int_1^2 \frac{dy}{\sqrt{y^2+\frac{2}{3}y-\frac{8}{9}}} = \frac{1}{3}\int_1^2 \frac{dy}{\sqrt{\left(y+\frac{1}{3}\right)^2-1}}$$

Put $y + \frac{1}{3} = t \Rightarrow dy = dt$

$$= \frac{1}{3}\int_{\frac{4}{3}}^{\frac{7}{3}} \frac{dt}{\sqrt{t^2-1}} = \frac{1}{3}\left[\ln\left(t+\sqrt{t^2-1}\right)\right]_{\frac{4}{3}}^{\frac{7}{3}}$$

$$= \frac{1}{3}\left[\ln\left(\frac{7}{3}+\sqrt{\frac{49}{9}-1}\right)-\ln\left(\frac{4}{3}+\sqrt{\frac{6}{9}-1}\right)\right]$$

$$= \frac{1}{3}\left[\ln\frac{7+2\sqrt{10}}{3}-\ln\left(\frac{4+\sqrt{7}}{3}\right)\right] = \frac{1}{3}\left[\ln\frac{7+2\sqrt{10}}{4+\sqrt{7}}\right]$$

$\therefore \quad p = 7, q = 10, r = 4.$

$\therefore \quad (p+q+r) = 7+10+4 = 21.$

Hence, the correct answer is (21).

24. Let $L_1 = \lim\limits_{x\to 0} \frac{1}{x}\int_0^x (1+\cos t)^2\, dt$

$$= \lim_{x\to 0} (1+\cos x)^2 = 4$$

$$L_2 = \lim_{x\to\infty} \frac{1}{x}\int_0^x (1+\cos t)^2\, dt$$

$$= \lim_{x\to\infty} \frac{1}{x}\int_0^x \left(1+\cos t+\frac{1+\cos 2t}{2}\right) dt$$

$$= \lim_{x\to\infty} \frac{1}{x}\int_0^1 \left(\frac{3}{2}+2\cos t+\frac{1}{2}\cos 2t\right) dt = \frac{3}{2}$$

Hence, $2(L_1+L_2) = 2\left(4+\frac{3}{2}\right) = 2\left(\frac{11}{2}\right) = 11.$

Hence, the correct answer is (11).

25. $g(0) = 0.$

Also, $g(x) = 1 - e^{\sin x}$

Clearly, $g(x)$ is periodic function with fundamental period $= 2\pi$ and

Its range $= \left[1-e, 1-\frac{1}{e}\right].$

$\therefore$ Integers in its range $= 0, 1$

Hence, the correct answer is (2).

26. Put $x - 2 = \theta$, we get

$$I = \int_{-2}^{2} \frac{(\theta^2+1)}{(2\theta^2+3)}\sin\theta \; d\theta = 0 \text{ (odd function)}$$

Hence, the correct answer is (0).

27. $y = \lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^3}\right)\left(1+\left(\frac{2}{n}\right)^3\right)\ldots(1+(1)^3)\right]^{1/n}$

$$\log y = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\left(\frac{r}{n}\right)^3\right)$$

$$\log y = \int_0^1 \log(1+x^3)\, dx$$

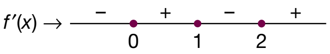
$$= \left(\log(1+x^3)\cdot x\right)_0^1 - \int_0^1 \frac{1}{1+x^3}3x^2\cdot x\, dx$$

$$= \log 2 - 3\int_0^1 \frac{x^3}{1+x^3}dx$$

$$y = 2.e^{-3}e^{\int_0^1 \frac{x^3}{1+x^3}dx} \qquad \alpha-\beta = 2-(-3) = 5$$

Hence, the correct answer is (5).

28. $f(x) = \int_1^x \left(t^3-3t^2+2t\right)dt$



$$f'(x) = x^3-3x^2+2x = x(x-2)(x-1)$$

$$\text{Min } f(x) = f(2) = \left[\frac{t^4}{4}-t^3+t^2\right]_1^2$$

$$= 4-8+4-\frac{1}{4}+1-1 = -\frac{1}{4}$$

$$\text{Max } f(x) = f(3) = 2 \quad \text{Range} = \left[-\frac{1}{4}, 2\right]$$

Hence, the correct answer is (3).

29. LHS $= \frac{1}{a}\int_0^a \left(\sqrt{x+a}-\sqrt{x}\right)dx$

$$= \frac{1}{a}.\frac{2}{3}\left[(x+a)^{3/2}-x^{3/2}\right]_0^a = \frac{2}{3a}\left[(2a)^{3/2}-a^{3/2}-a^{3/2}\right]$$

$$= \frac{2}{3a}.2a^{3/2}\left[\sqrt{2}-1\right]$$

$$\text{RHS} = \int_0^{\pi/8}\frac{2\tan\theta}{2\tan\theta}.\sec^2\theta\, d\theta \qquad \left(\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \tan\theta\Big|_0^{\pi/8} = \sqrt{2}-1$$

Now given equation, $\int_0^a \frac{dx}{\sqrt{x+a}+\sqrt{x}} = \int_0^{\pi/8}\frac{2\tan\theta}{\sin 2\theta}d\theta$

$$\Rightarrow \quad \frac{2}{3a}.2a^{3/2}\left[\sqrt{2}-1\right] = \sqrt{2}-1 \Rightarrow \frac{4}{3}\sqrt{a} = 1 \Rightarrow a = \frac{9}{16}\ ]$$

Hence, the correct answer is (1).

30. $\int_0^{2\pi}\frac{dx}{a+b\cos x+c\sin x} = \int_0^{\pi}\frac{1}{a+b\cos x+c\sin x}+\frac{1}{a+b\cos x-c\sin x}dx$

$$= \int_0^{\pi}\frac{2(a+b\cos x)}{(a+b\cos x)^2-c^2\sin^2 x}dx$$

$$= \int_0^{\pi}\frac{2\left(a+b\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)\right)dx}{\left(a+b\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)\right)^2-c^2\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)^2}$$

$$= \int_0^{\pi}\frac{2\left(a+b+(a-b)\tan^2\frac{x}{2}\right)\sec^2\frac{x}{2}dx}{\left(a+b+(a-b)\tan^2\frac{x}{2}\right)^2-c^2\,4\tan^2\frac{x}{2}}$$

$$= \int_0^{\infty}\frac{4(a+b+(a-b)t^2)dt}{(a+b+(a-b)t^2)^2-4c^2t^2}$$

Put $\tan\frac{x}{2} = t$; $\sec^2\frac{x}{2}dx = 2dt$

$$= \int_0^{\infty}\left(\frac{2dt}{a+b+(a-b)t^2+2ct}+\frac{2dt}{a+b-2ct+(a-b)t^2}\right.$$

$$= \frac{2}{a-b}\int_0^\infty \left(\frac{dt}{t^2+\frac{2c}{a-b}t+\frac{a+b}{a-b}} + \frac{dt}{t^2-\frac{2c}{a-b}t+\frac{a+b}{a-b}} \right)$$

$$= \frac{2}{a-b}\left[\int_0^\infty \frac{dt}{\left(t+\frac{c}{a-b}\right)^2+\frac{a+b}{a-b}-\frac{c^2}{(a-b)^2}} + \int_0^\infty \frac{dt}{\left(t-\frac{c}{a-b}\right)^2+\frac{a+b}{a-b}-\frac{c^2}{(a-b)^2}}\right)$$

$$= \frac{2}{a-b}\int_0^\infty \frac{dt}{\left(t+\frac{c}{a-b}\right)^2+\frac{a^2-b^2-c^2}{(a-b)^2}} + \int_0^\infty \frac{dt}{\left(t-\frac{c}{a-b}\right)^2+\frac{a^2-b^2-c^2}{(a-b)^2}}\Bigg)$$

$$= \frac{2}{a-b}\left[\frac{1}{\frac{\sqrt{a^2-b^2-c^2}}{a-b}}\tan^{-1}\frac{t+\frac{c}{a-b}}{\frac{\sqrt{a^2-b^2-c^2}}{a-b}} + \frac{1}{\frac{\sqrt{a^2-b^2-c^2}}{a-b}}\tan^{-1}\frac{t-\frac{c}{a-b}}{\frac{\sqrt{a^2-b^2-c^2}}{a-b}}\right]_0^\infty$$

$$= \frac{2}{\sqrt{a^2-b^2-c^2}}\left(\frac{\pi}{2}+\frac{\pi}{2}-\tan^{-1}\frac{c}{\sqrt{a^2-b^2-c^2}}-\tan^{-1}\left(\frac{-c}{\sqrt{a^2-b^2-c^2}}\right)\right]$$

$$= \frac{2\pi}{\sqrt{a^2-b^2-c^2}} \quad \text{we get } K=2$$

Hence, the correct answer is (2).

31. $I=\int_0^\pi xf\left(\cos^2 x+\tan^4 x\right)dx$

$$=\int_0^\pi (\pi-x)f\left(\cos^2(\pi-x)+\tan^4(\pi-x)\right)dx$$

$$\Rightarrow \quad I=\frac{\pi}{2}\int_0^\pi f\left(\cos^2 x+\tan^4 x\right)dx$$

$$=\frac{\pi}{2}\cdot 2\int_0^{\pi/2} f\left(\cos^2 x+\tan^4 x\right)dx \text{ . Here } k=1997$$

Hence, the correct answer is (0).

32. Integrating by parts, we have $\int_0^\pi \frac{x^2\cos x}{(1+\sin x)^2}dx$

$$=-\frac{x^2}{1+\sin x}\Bigg|_0^\pi+2\int_0^\pi \frac{x}{1+\sin x}dx=-\pi^2+2I$$

where $I=\int_0^\pi \frac{x}{1+\sin x}dx=\int_0^\pi \frac{\pi-x}{1+\sin x}dx=\pi\int_0^\pi \frac{dx}{1+\sin x}-I$

$$\Rightarrow \quad 2I=\pi\int_0^\pi \frac{dx}{1+\sin x}=2\pi\int_0^{\pi/2}\frac{dx}{1+\sin x}$$

$$\Rightarrow \quad I=\pi\int_0^{\pi/2}\frac{dx}{1+\sin x}=\pi\int_0^{\pi/2}\frac{dx}{1+\sin\left(\frac{\pi}{2}-x\right)}$$

$$=\int_0^{\pi/2}\frac{dx}{1+\cos x}=\frac{\pi}{2}\int_0^{\pi/2}\sec^2\left(\frac{x}{2}\right)dx=\pi\tan\left(\frac{x}{2}\right)\Bigg]_0^{\pi/2}=\pi$$

Hence, $\int_0^\pi \frac{x^2\cos x}{(1+\sin x)^2}dx=-\pi^2+2\pi=\frac{A}{498}\pi-\pi^2$.

We have $A=996$

Hence, the correct answer is (6).

33. $\ln p=\lim\limits_{n\to\infty}\frac{1}{n}\left[\ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{2}{n}\right)+\ldots+\ln\left(1+\frac{1}{n}\right)\right]$

General term of $\ln p=\frac{1}{n}\ln\left(1+\frac{r}{n}\right)$

$$\text{Sum}=\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r}{n}\right)=\int_0^1\ln(1+x)\,dx$$

$$\ln p=\ln\frac{4}{e} \qquad \therefore\ p=4e^{-1}\Rightarrow k=4$$

Hence, the correct answer is (4).

34. $\int_0^{\frac{\pi}{16}}\frac{64x\cot^2(8x)dx}{8\cot 8x}=\int_0^{\frac{\pi}{16}}\underbrace{8x}_{\text{I}}\underbrace{\cot 8x}_{\text{II}}\,dx$

$$=8\left[\underbrace{\left(\frac{x}{8}\ln(\sin 8x)\right)_0^{\frac{\pi}{16}}}_{\text{zero}}-\frac{1}{8}\int_0^{\frac{\pi}{16}}\ln(\sin 8x)dx\right]$$

$$=-\int_0^{\frac{\pi}{16}}\ln(\sin 8x)dx$$

Put $8x=t=-\frac{1}{8}\int_0^{\frac{\pi}{2}}\ln(\sin t)dt=\frac{-1}{8}\left(\frac{-\pi}{2}\ln 2\right)=\frac{\pi}{16}\ln 2$

$\therefore \quad a=16, b=2 \qquad \therefore\ \frac{a}{b}=8$

Hence, the correct answer is (8).

35. Given $P(2+x)+P(2-x)=2P(2)$ …(1)

$x\to x-2$ in (1)

$\Rightarrow \quad P(x)+P(4-x)=2P(2)$ …(2)

$x\to x+1$ in (1)

$\Rightarrow \quad P(x+3)+P(1-x)=2P(2)$ …(3)

$\therefore \quad$ (2) + (3)

$20=4P(2)\Rightarrow P(2)=5$

Now differentiate (1) w.r.t. x, we get, $P'(2+x)-P'(2-x)=0$

$\Rightarrow \quad P'(2+x)=P'(2-x) \qquad \therefore\ P'(x)$ is symmetric about $x=2$

$P'(x)=3x^2+2a+9\Rightarrow-\frac{2a}{6}=2\Rightarrow a=-6$

$P(2)=8+4a+18+b=5$

$\Rightarrow \quad 8-24+18+b=5\Rightarrow b=3$

$\therefore \quad P(x) = x^3 - 6x^2 + 9x + 3$

$$I = \int_{+6}^{-6} (x^3 - 6x^2 + 9x + 3)\, dx$$

$$= \int_{-6}^{6} (6x^2 - 3)\, dx = \int_{0}^{6} (6x^2 - 3)\, dx = 828$$

Hence, the correct answer is (828).

36. $P = \lim\limits_{n\to\infty} \prod\limits_{r=1}^{n} \left(\sin\left(\frac{r\pi}{4n}\right) \cos\left(\frac{r\pi}{4n}\right)\right)^{\frac{1}{n}}$

$$\ln P = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \ln\left(\frac{\sin r\pi}{n} . \frac{\cos r\pi}{n}\right)$$

$$\Rightarrow \quad \ln P = \int_0^1 \ln\left(\sin\frac{\pi x}{4} \cdot \cos\frac{\pi x}{4}\right) dx$$

$$\Rightarrow \quad \ln P = \int_0^1 \ln\left(\sin\frac{\pi x}{2}\right) dx - \ln 2$$

Put $\frac{\pi x}{2} = t$

$$\therefore \quad \ln P = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \ln(\sin t)\, dt - \ln 2$$

$$= -2 \ln 2 = \ln \frac{1}{4}$$

$$\therefore \quad P = \frac{1}{4} \equiv \frac{a}{b} \Rightarrow a + b = 5$$

Hence, the correct answer is (5).

37. $f(x) = 3x^2 \int\limits_0^1 f(t)\, dt + 2x \int\limits_0^1 f(1-t)\, dt + 3$

$$\Rightarrow \quad f(x) = 3Ax^2 + 2Bx + 3$$

where $A = \int\limits_0^1 f(t)\, dt = \int\limits_0^1 (3At^2 + 2Bt + 3)\, dt$

Now, $A = A + B + 3$ $\quad \therefore B = -3$

Similarly $B = \int\limits_0^1 f(1-t)\, dt = \int\limits_0^1 (3A(1-t)^2 + 2B(1-t) + 3)\, dt$ $\quad \therefore A = -3$

Hence, $f(x) = -9x^2 - 6x + 3$

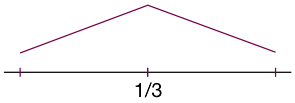
Let $I(a) = \int\limits_{-1}^{a} (-9x^2 - 6x + 3)\, dx \quad (a > -1)$

$$\therefore \quad I(a) = \left.\frac{-9x^3}{3} - \frac{6x^2}{2} + 3x\right]_{-1}^{a} = \left.-3x^3 - 3x^2 + 3x\right]_{-1}^{a}$$

$$\Rightarrow \quad I(a) = -\ 3(x^3 + x^2 - x)\Big|_{-1}^{a}$$

So, $I(a) = -3(a^3 + a^2 - a - 1) \Rightarrow I'(a) = -3(3a^2 + 2a - 1)$

$$= -3(3a^2 + 3a - a - 1) = -3(a + 1)\,(3a - 1)$$



$\therefore$ Maximum value occur at $x = \frac{1}{3}$

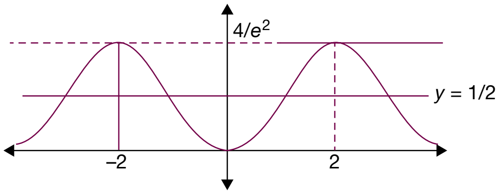
$$\therefore \quad I\left(\frac{1}{3}\right) = -3\left(\frac{1}{27} + \frac{1}{9} - \frac{1}{3} - 1\right) = \frac{32}{9} = S \qquad \therefore [S] = 3$$

Hence, the correct answer is (3).

38. $f(x) = \lim\limits_{t\to 0} \int\limits_0^x \left(\frac{tx^2 + x\left((t+1)^{2t+2} - 1\right)}{e^{-x} \cdot t}\right) dx$

$$f(x) = \int_0^x e^x \left(\lim_{t\to 0} x^2 + x\frac{(t+1)^{2t+2} - 1}{t}\right) dx$$

Now $\lim\limits_{t\to 0} \frac{(t+1)^{2t+2} - 1}{t} = \lim\limits_{t\to 0} \frac{e^{(2t+2)\ln(t+1)} - 1}{t} = 2$



$$\therefore \quad f(x) = \int_0^x e^x (x^2 + 2x)\, dx = e^x x^2$$

Now $2f(-|x|) = 1$ $\qquad \ldots(1)$

$$\Rightarrow \quad \frac{x^2}{e^{|x|}} = \frac{1}{2}$$

Let $h(x) = \frac{x^2}{e^{|x|}}$

Let $h'(x) = \frac{(2x - x^2)}{e^x} = \frac{x(2-x)}{e^x}$

$\therefore$ Number of solution of the equation (1) is 4.

Hence, the correct answer is (4).

39. $I = \int\limits_{-1}^{1} \cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{x}{\sqrt{1-\left(x^2\right)^{|x|}}}\right)\right) dx$

$$= \int_{-1}^{1} \frac{\pi}{2} \cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) dx = \pi \int_0^1 \cot^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right) dx$$

$$= \pi \int_0^1 \tan^{-1} \sqrt{1-x^2}\, dx$$

$$= \pi\left(\left(x \tan^{-1}\sqrt{1-x^2}\right)_0^1 - \int_0^1 x . \frac{1}{1+1-x^2} \frac{2(-2x)}{2\sqrt{1-x^2}} dx\right)$$

$$= 2\pi \int_0^1 \frac{x^2}{\left(2-x^2\right)\sqrt{1-x^2}} dx$$

Put $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta = 2\pi \int\limits_0^{\frac{\pi}{2}} \frac{\sin^2\theta}{\left(2 - \sin^2\theta\right)} d\theta$

$$= 2\pi\left(\int_0^{\frac{\pi}{2}}(-1)\,d\theta + 2\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sin^2\theta + 2\cos^2\theta}\right);$$

$$= 2\pi\left(-\frac{\pi}{2} + 2.\frac{\pi}{2\sqrt{2}}\right)$$

$$= \pi^2\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right) = \pi^2\left(\frac{\sqrt{2}-1}{2}\right) = \frac{\pi^2\left(\sqrt{a}-\sqrt{b}\right)}{\sqrt{c}}$$

$\therefore \quad a + b + c = 2 + 1 + 4 = 7$

Hence, the correct answer is (7).

40. Putting $x = 1 + h$

$$l = \lim_{h\to 0} 2^{-2^{-1/h}} = 2^0 = 1$$

$$m = \lim_{h\to 0}\frac{(1+h)\sin h}{h} = 1 \qquad \therefore \int_1^1 \frac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx = 0$$

Hence, the correct answer is (0).

41. $I = \int_1^e \frac{1+\ln x}{\sqrt{x\ln x}}dx$

$\Rightarrow$ Putting $(x \ln x) = t^2$, $(\ln x + 1)dx = 2tdt$, we get

$$I = \int_0^{\sqrt{e}}\frac{2tdt}{t} = 2\sqrt{e}$$

Hence, the correct answer is (6).

42. $I_n = \int_0^{\pi/2}(\sin x + \cos x)^{n-1}(\sin x - \cos x)'dx$

$$= \left[(\sin x + \cos x)^{n-1}(\sin x - \cos x)\right]_0^{\pi/2} +$$

$$(n-1)\Big|_0^{\pi/2} (\sin x + \cos x)^{n-2}(\cos x - \sin x)^2\,dx$$

$$= 2 + (n-1)\int_0^{\pi/2}(\sin x + \cos x)^{n-2}(\cos x - \sin x)^2\,dx$$

$$= 2 + (n-1)\int_0^{\pi/2}(\sin x + \cos x)^{n-2}\left[2 - (\sin x + \cos x)^2\right]dx$$

$$= 2 + 2(n-1)I_{n-2} - (n-1)I_n$$

$\Rightarrow \quad nI_n - 2(n-1)I_{n-2} = 2$

Hence, the correct answer is (2).

43. As, function is differentiable and continuous at $x = -1$

$\Rightarrow \quad \alpha = 1$ and $\beta = -1$.

$$\therefore \quad f(x) = \begin{cases} \frac{\pi}{4} + \tan^{-1}x & ; \quad x < -1 \\ \frac{3\pi}{4} - \cot^{-1}x & ; \quad x \geq -1 \end{cases}$$

$$\therefore \quad \int_{-\pi/2}^{\pi/2} f(x)d(\tan^{-1}x) = \int_{-\pi/2}^{\pi/2} f(\tan t)\,dt$$

$$= \int_{-\pi/2}^{-1}\left[\frac{\pi}{4} + \tan^{-1}(\tan t)\right]dt + \int_{-1}^{\pi/2}\left[\frac{3\pi}{4} - \cot^{-1}(\tan t)\right]dt$$

$$= \frac{\pi^2}{4}$$

Hence, the correct answer is (3).

44. Put $x = 1$ and $y = 1 \Rightarrow g(1) = 1$ or $g(1) = 2$

But $g(1) \neq 1 \Rightarrow g(1) = 2$

Put $y = \frac{1}{x}$ to get $\Rightarrow g(x)\,g\left(\frac{1}{x}\right) = g(x) + g\left(\frac{1}{x}\right) + g(1) - 2$

$\therefore \quad g(x) = 1 + x^2$

$\therefore \quad I = \int_1^{\infty}\frac{dx}{x(1+x^2)}$; put $x = \tan\theta$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cot\theta\,dt = \ln\sin\theta\Big|_{\pi/4}^{\pi/2}$$

$$= (0) - \ln\frac{1}{\sqrt{2}} = \frac{1}{2}\,n\,2 \equiv \frac{p}{q}\,\ln 2 \Rightarrow p^2 + q^2 = 5$$

Hence, the correct answer is (5).

45. Let $\int_0^1 (1-x^5)^6\,dx = J$

$$I = \int_0^1 (1-x^5)^7\,dx$$

$$\Rightarrow \quad I = \left[(1-x^5)^7\cdot x\right]_0^1 - \int_0^1 7(1-x^5)^6(-5x^4)\cdot x\,dx$$

$$\Rightarrow \quad I = 0 + \int_0^1 35(1-x^5)^6 x^5\,dx$$

$$\Rightarrow \quad I = -35\int_0^1 (1-x^5)^7\,dx + 35\int_0^1 (1-x^5)^6\,dx$$

$$I = -35\,I + 35\,J \Rightarrow \frac{J}{I} = \frac{36}{35} \Rightarrow k = 3$$

Hence, the correct answer is (3).

46. $I_n = \int_{\pi/2}^{\infty} e^{-x}\cos^n x dx$

$$= \left[\frac{\cos^n x.e^{-x}}{-1}\right]_{\pi/2}^{\infty} - \int_0^{\infty} n.\cos^{n-1}x\sin x.e^{-x}dx$$

$$= -n\left\{\left[\frac{\cos^{n-1}x\sin xe^{-x}}{-1}\right]_{\pi/2}^{\infty} + \int_{\pi/2}^{\infty}\left(\cos^{n-1}x\cos x - (n-1)\sin^2 x\cos^{n-2}x\right)e^{-x}dx\right\}$$

$$\Rightarrow \quad I_n = -nI_n + n(n-1)I_{n-2} - n(n-1)I_n$$

$$\Rightarrow \quad I_n = \frac{n(n-1)}{1+n^2}I_{n-2},\ I_{n-2} = \frac{(n-2)(n-3)}{1+(n-2)^2}I_{n-4}$$

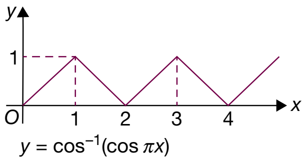
$$\Rightarrow \quad I_n = \frac{n(n-1)(n-2)...2.1}{\left(1+n^2\right)\left(1+(n-2)^2\right)...\left(1+2^2\right)},\ I_0 = \int_{\pi/2}^{\infty} e^{-x}\,dx = e^{-\pi/2}$$

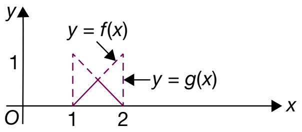
$$I_n = \frac{n!\cdot I_0}{\left(1+n^2\right)\left(1+(n-2)^2\right)...\left(1+2^2\right)} = \frac{720}{3145}e^{-\pi/2}.$$

Comparing we get $n! = 720 \Rightarrow n = 6$.

Hence, the correct answer is (6).

47. $\int_1^2 f(x)dx=\frac{1}{2}\left(1+\frac{1}{2}\right)\times\frac{1}{2}\times 2=\frac{3}{4}$ and $\int_1^2 g(x)dx=\frac{1}{4}$





Hence, the correct answer is (3).

48. $f(t)=\begin{cases}-t & \text{if} \quad t<-1\\ t+2 & \text{if} \quad -1\le t<0\\ 2-t & \text{if} \quad 0\le t<1\\ t & \text{if} \quad t\ge 1\end{cases}$;

$$f(t)=\begin{cases}\frac{3}{2} & \text{if} \quad -\frac{3}{2}\le x<-\frac{1}{2}\\ x+2 & \text{if} \quad -\frac{1}{2}\le x<0\\ 2 & \text{if} \quad 0\le x<2\\ x & \text{if} \quad 2\le x\le 4\end{cases}$$

$$I=\frac{3}{2}+0-\frac{1}{8}+1+4+6=11+\frac{1}{8}\Rightarrow I-\frac{51}{8}=6$$

Hence, the correct answer is (6).

49. $f(x)=\begin{cases}1+x & \text{if} \quad -1\le x<0\\ 1-x & \text{if} \quad 0\le x<1\\ 0 & \text{if} \quad x<-1 \text{ or } x>1\end{cases}$;

$$g(x)=\begin{cases}2x+2 & \text{if} \quad -1\le x<0\\ -2x+2 & \text{if} \quad 0\le x<1\\ 0 & \text{if} \quad x<-1 \text{ or } x>1\end{cases}$$

$$=\int_{-1}^{0}2(x+1)\,dx+\int_0^1(-2x+2)\,dx$$

$$=\left[\frac{2(x+1)^2}{2}\right]_{-1}^{0}-2\left[\frac{(1-x)^2}{2}\right]_0^1$$

$$=1-0-[0-1]=2$$

Hence, the correct answer is (2).

50. Given $f(x)=f(y)\,f(x-y)$

replace x by $(x+y)$

$f(x+y)=f(y).f(x)$

$\Rightarrow \quad f(x)=e^{kx}\Rightarrow f'(x)=ke^{kx}$ But $f'(0)=\int_0^4\{2x\}dx=2$.

$\Rightarrow \quad f'(0)=k=2\Rightarrow f'(x)=2e^{2x}$

$\Rightarrow \quad f'(-3)=2e^{-6}\Rightarrow|\alpha+\beta|=4.$

Hence, the correct answer is (4).

51. $\int_1^c\frac{f(x)}{x}dx=\int_1^{\sqrt{c}}\frac{f(x)}{x}dx+\int_{\sqrt{c}}^{c}\frac{f(x)}{x}dx$

Put $x=\frac{c}{t}\Rightarrow dx=-\frac{c}{t^2}dt$

$$=\int_1^{\sqrt{c}}\frac{f(x)}{x}dx+\int_{\sqrt{c}}^{1}\frac{f\left(\frac{c}{t}\right)}{t}(-dt)=\int_1^{\sqrt{c}}\frac{f(x)}{x}dx+\int_1^{\sqrt{c}}\frac{f(x)}{x}dx=6$$

Hence, the correct answer is (6).

52. For $1\le t\le 3, 1\le x\le 3$

$$f(x)=\int_1^x\frac{dt}{1+|t-x|}+\int_x^3\frac{dt}{1+|t-x|}$$

$$=-\left(\ln(1+x-t)\right)_1^x+\left(\ln(1+t-x)\right)_x^3$$

$$=\ln x+\ln(4-x)$$

$\Rightarrow \quad f(t)=\ln t+\ln(4-t), t\in[1,3]$

$\therefore \quad \int_1^3 f(t)\,dt=\int_1^3\ln t+\ln(4-t)\,dt=6\ln 3-4$

$\Rightarrow \quad k=3$

Hence, the correct answer is (3).

53. $f(x)=x^2+\int_0^x e^{-t}f(x-t)dt=x^2+e^{-x}\int_0^x e^t f(t)dt$

$\Rightarrow \quad f^1(x)=2x-e^{-x}\left(e^x\left(f(x)-x^2\right)\right)+e^{-x}\cdot e^x f(x)$

$\Rightarrow \quad f^{|}(x)=2x+x^2\Rightarrow f(x)=\frac{x^3}{3}+x^2+K$

But, $f(0)=0\Rightarrow k=0 \quad \therefore f(1)=\frac{4}{3}$

Hence, the correct answer is (8).

54. $I=\int_0^\pi(\pi-x)\left(\left(\sin^2(\sin x)\right)+\cos^2(\cos x)\right)dx$

$\Rightarrow \quad 2I=2\pi\int_0^{\frac{\pi}{2}}\left(\sin^2(\sin x)+\cos^2(\cos x)\right)dx$

$\Rightarrow \quad I=\pi\int_0^{\frac{\pi}{2}}\left(\sin^2(\sin x)\right)+\cos^2(\cos x)dx$

$$=\pi\int_0^{\frac{\pi}{2}}\left(\sin^2(\cos x)+\cos^2(\sin x)\right)dx$$

$\Rightarrow \quad 2I=\pi\int_0^{\frac{\pi}{2}}2dx\Rightarrow I=\frac{\pi^2}{2}$

Hence, the correct answer is (4).

55. $I=\int_{\pi/4}^{\pi/2}2\sin t\cos t\,dt+\int_{\pi/2}^{\pi}\{(-\sin t\cos t)+(\sin t\cos t)\}dt+$

$$\int_\pi^{5\pi/4}-2\sin t\cos t\,dt$$

$$I=\int_{\pi/4}^{\pi/2}\sin 2t\,dt-\int_\pi^{5\pi/4}\sin 2t\,dt\Rightarrow I=0$$

Hence, the correct answer is (0).

56. $\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{r}\sum_{r=1}^{n}\frac{1}{\sqrt{r}}}{\frac{n(n+1)}{2}}=\lim_{n\to\infty}\frac{\sum_{r=1}^{n}\sqrt{\frac{r}{n}}\sum_{r=1}^{n}\frac{1}{\sqrt{\frac{r}{n}}}}{\frac{n^2}{2}\left(1+\frac{1}{n}\right)}$

$$= 2\times\int_0^1 \sqrt{x}\,dx\int_0^1 \frac{1}{\sqrt{x}}\,dx \;= 2\times\left[\frac{x^{3/2}}{\frac{3}{2}}\right]_0^1\left[\frac{x^{1/2}}{\frac{1}{2}}\right]_0^1$$

$$= 2\times\frac{2}{3}\times 2 = \frac{8}{3}$$

$\therefore\ k = 8$

Hence, the correct answer is (8).

57. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^{\frac{1}{a}}\left\{n^{a-\frac{1}{a}}+k^{a-\frac{1}{a}}\right\}}{n^{a+1}}$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}.\left\{\left(\frac{k}{n}\right)^{1/a}+\left(\frac{k}{n}\right)^{a}\right\} = \int_0^1\left(x^{1/a}+x^{a}\right)dx$$

$$= \left\{\frac{x^{(1/a)+1}}{\frac{1}{a}+1}+\frac{a^{a+1}}{a+1}\right\}_0^1 = \frac{a}{a+1}+\frac{1}{a+1} = 1$$

Hence, the correct answer is (1).

58. $$\lim_{x\to 0}\frac{\int_0^x \frac{t^2}{\sqrt{a+t}}\,dt}{x-\sin x} = \lim_{x\to 0}\frac{x^2}{\sqrt{a+x}\,(1-\cos x)}$$

$$= \lim_{x\to 0}\frac{x^2}{\sqrt{a+x}\,\left(2\sin^2\frac{x}{2}\right)}$$

$$= \frac{2}{\sqrt{a}} = 1 \Rightarrow a = 4$$

Hence, the correct answer is (4).

59. We have $\underbrace{\left|(|x|-2)^2+1\right|}_{\geq 1} + \underbrace{\left|(|x^3-4x|+3\right|}_{\geq 3} = 4$

Minimum value of L.H.S is 4 when $\left.\begin{array}{ll} |x|-2=0 & \Rightarrow x=\pm 2 \\ x^3-4x=0 & \Rightarrow x=0,\pm 2 \end{array}\right\}$

$\Rightarrow\quad x = 2$ or $x = -2$

Since $x_1 < x_2 \Rightarrow x_1 = -2$ or $x_2 = 2$ which satisfy the given equation.

Also, $\int_0^{3/2} f(x)dx = 6 \Rightarrow 3\int_0^{1/2} f(x)dx = 6 \Rightarrow \int_0^{1/2} f(x)dx = 2$

Now, $\int_{-2}^{2} f(x)dx = \int_{-4\times\frac{1}{2}}^{4\times\frac{1}{2}} f(x)dx = 8\int_0^{\frac{1}{2}} f(x)dx = 16$

Hence, the correct answer is (16).

60. Let $I = 2\int_0^{\ln 2}\frac{(x-\ln 2)e^x}{(1+e^x)^2}\,dx = 2\int_0^{\ln 2}\underbrace{(x-\ln 2)}_{\text{I}}\underbrace{\frac{e^x}{(1+e^x)^2}}_{\substack{\text{II}\\ \text{(I.B.P.)}}}\,dx$

Integrating by parts, we get $I = 2\left[\left.\frac{-(x-\ln 2)}{1+e^x}\right|_0^{\ln} + \int_0^{\ln 2}\frac{dx}{1+e^x}\right]$

$$= 2\left[\left.\frac{x-\ln 2}{1+e^x}\right|_{\ln 2}^{0} + \int_0^{\ln 2}\frac{e^x dx}{e^x(1+e^x)}\right]$$

$$= 2\left[\left(\frac{-\ln 2}{2}\right)+\int_1^2\frac{dt}{t(t+1)}\right]$$

$$= -\ln 2 + 2\left[\ln\frac{t}{t+1}\right]_1^2 = -\ln 2 + 2\left[\ln\frac{2}{3}+\ln 2\right]_1^2$$

$$= -\ln 2 + 2\left[\ln\frac{4}{3}\right] = \ln\frac{16}{9} - \ln 2 = \ln\left(\frac{8}{9}\right) \Rightarrow a+b = 17.$$

Hence, the correct answer is (17).

61. Let $I_1 = \int_0^{\frac{\pi}{2}}(\cos x)^{\sqrt{2}+1}dx$ …(1)

and $I_2 = \int_0^{\frac{\pi}{2}}(\cos x)^{\sqrt{2}-1}dx$ …(2)

Now, $I_1 = \underbrace{\int_0^{\frac{\pi}{2}}\underset{\text{(I)}}{(\cos x)^{\sqrt{2}}}\cdot\underbrace{(\cos x)dx}_{\text{(II)}}}_{\text{(I.B.P.)}}$

$$= \left((\cos x)^{\sqrt{2}}\cdot\sin x\right)_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\sqrt{2}\cdot(\cos x)^{\sqrt{2}-1}\cdot\sin^2 x\,dx$$

$$= \sqrt{2}\int_0^{\frac{\pi}{2}}(\cos x)^{\sqrt{2}-1}\cdot(1-\cos^2 x)\,dx$$

$$\therefore\quad I_1 = \sqrt{2}I_2 - \sqrt{2}I_1 \Rightarrow \frac{\left(\sqrt{2}+1\right)I_1}{\sqrt{2}I_2} = 1$$

$$\therefore\quad \frac{I_1}{I_2} = \left(\frac{\sqrt{2}}{\sqrt{2}+1}\right)\left(\frac{\sqrt{2}-1}{\sqrt{2}-1}\right)$$

$$= \left(2-\sqrt{2}\right) \equiv \left(n-\sqrt{n}\right) \Rightarrow n = 2$$

NOTE: $$\frac{\int_0^{\frac{\pi}{2}}(\cos x)^{\sqrt{2}+1}dx}{\int_0^{\frac{\pi}{2}}(\cos x)^{\sqrt{2}-1}dx} = \frac{\int_0^{\frac{\pi}{2}}(\sin x)^{\sqrt{2}+1}dx}{\int_0^{\frac{\pi}{2}}(\sin x)^{\sqrt{2}-1}dx}$$ [Using King property]

Hence, the correct answer is (2).

62. Put $x = \frac{1}{y}$ $\ln I_1$, we get $I_1 = -I_1 \Rightarrow I_1 = 0$.

Let $I_2 = \int_2^4\frac{\ln 2\cdot(\ln x-1)}{(\ln x)^2}\,dx$; Put $\ln x = t \Rightarrow x = e^t$

So, $I_2 = (\ln 2)\int_{\ln 2}^{\ln 4} e^t\,\frac{(t-1)}{t^2}\,dt = (\ln 2)\left(\frac{e^t}{t}\right)_{\ln 2}^{\ln 4} = 0.$

Hence, the correct answer is (0).

63. $$\int_0^{\pi}\frac{dx}{1+2\sin^2 x} = 2\int_0^{\pi/2}\frac{dx}{1+2\sin^2 x} = 2\int_0^{\pi/2}\frac{\sec^2 x\,dx}{1+3\tan^2 x}$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= 2\int_0^{\infty}\frac{dt}{1+\left(\sqrt{3}t\right)^2} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\sqrt{3}t\right]_0^{\infty} = \frac{\pi}{\sqrt{3}}$$

$$\left[\frac{\pi}{\sqrt{3}}\right] = 1$$

Hence, the correct answer is (1).

64. We have $F(x)+F\left(x+\frac{1}{2}\right)=3$ …(1)

Replace x by $x+\frac{1}{2}$ in (1), we get

$$F\left(x+\frac{1}{2}\right)+F(x+1)=3 \quad \text{…(2)}$$

∴ From (1) and (2), we get $F(x)=F(x+1)$ …(3)

⇒ $F(x)$ is periodic function.

Now consider

$$I=\int_0^{1500}F(x)dx=1500\int_0^1 F(x)dx=1500\left[\int_0^1 F(x)dx+\int_{\frac{1}{2}}^{1}F(x)dx\right]$$

(Using property of periodic function)

Put $x=y+\frac{1}{2}$ in 2nd integral, we get

$$I=1500\left[\int_0^{\frac{1}{2}}F(x)dx+\int_0^{\frac{1}{2}}F\left(y+\frac{1}{2}\right)dy\right]$$

$$=1500\int_0^{\frac{1}{2}}\left(F(x)+F\left(x+\frac{1}{2}\right)\right)dx=1500\int_0^{\frac{1}{2}}3dx \quad |\text{Using } (1)|$$

Hence, $I = 1500(3)\left(\frac{1}{2}\right)=750\times 3=2250$

Hence, the correct answer is (4).

65. Using King property $\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$

$$I=\int_{-1}^{1}(1-x)^{1/2}(1+x)^{3/2}dx$$

$$\Rightarrow \quad 2I=\int_{-1}^{1}(1+x)^{1/2}(1-x)^{1/2}[(1-x)+(1+x)]dx = 2\int_{-1}^{1}\sqrt{1-x^2}dx$$

$$\Rightarrow \quad I=2\int_0^1\sqrt{1-x^2}dx(x=\sin\theta)\Rightarrow dx=\cos\theta d\theta$$

$$\Rightarrow \quad I=2\int_0^{\pi/2}\cos^2\theta\, d\theta=\frac{\pi}{2}$$

Hence, the correct answer is (8).

66. Let $I_n=\int_0^{\pi/2}x^n\sin x\,dx$

Integrating by parts choosing $\sin x$ as the second function, we get

$$I_n=[x^n(-\cos x)]_0^{\pi/2}-\int_0^{\pi/2}nx^{n-1}(-\cos x)dx$$

$$=0+n\int_0^{\pi/2}x^{n-1}\cos x\,dx$$

Again integrating by parts,

$$I_n=n[x^{n-1}\sin x]_0^{\pi/2}-n(n-1)\int_0^{\pi/2}x^{n-2}\sin x\,dx$$

$$\Rightarrow \quad I_n=n\left(\frac{\pi}{2}\right)^{n-1}-n(n-1)I_{n-2}$$

R.H.S. contains π^2. So putting $n=3$, we get

$$I_3=3\left(\frac{\pi}{2}\right)^2-3\times 2I_1 \;=\frac{3\pi^2}{4}-6\int_0^{\pi/2}x\sin x\,dx$$

$$=\frac{3\pi^2}{4}-6[x(-\cos x)+\sin x]_0^{\pi/2}$$

$$=\frac{3\pi^2}{4}-6(1)=\frac{3}{4}(\pi^2-8)\text{, which is true.}$$

Hence, $n=3$.

Hence, the correct answer is (3).

67. $I=\int_1^{\sqrt{3}}\frac{1}{x^2}\ln\left(\sqrt{1+x^2}\right)dx=\frac{1}{2}\int_1^{\sqrt{3}}\frac{1}{x^2}\ln(1+x^2)\,dx$

$$=\frac{1}{2}\left(-\frac{1}{x}\right)\ln(1+x^2)\Big|_1^{\sqrt{3}}-\frac{1}{2}\int_1^{\sqrt{3}}\left(\frac{-1}{x}\right)\frac{2x}{1+x^2}dx$$

$$=-\frac{1}{2\sqrt{3}}(\ln 4)+\frac{1}{2}\ln 2+\int_1^{\sqrt{3}}\frac{dx}{1+x^2}$$

$$=-\frac{1}{\sqrt{3}}\ln 2+\frac{1}{2}\ln 2+\tan^{-1}x\Big|_1^{\sqrt{3}}$$

$$=\left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\ln 2+\frac{\pi}{3}-\frac{\pi}{4}$$

$$=\frac{\left(\sqrt{3}-2\right)}{2\sqrt{3}}\ln 2+\frac{\pi}{12}\Rightarrow k_1=2,\ k_2=12,\ \frac{k_2}{k_1}=6$$

Hence, the correct answer is (6).

68. $3\int_{-3}^{1}(x-k)(x+3)(x-1)\,dx=64$

$$3\int_{-3}^{1}(x^2+2x-3)(x-k)\,dx=64$$

$$\Rightarrow \quad 3\int_{-3}^{1}x^3+2x^2-3x-kx^2-2kx+3k\,dx=64$$

$$3\left(\frac{x^4}{4}+\frac{2x^3}{3}-\frac{3x^2}{2}-\frac{k}{3}x^3-kx^2+3kx\right)\Bigg|_{-3}^{1}=64$$

$$3\left(\frac{1}{4}+\frac{2}{3}-\frac{3}{2}-\frac{k}{3}-k+3k-\left(\frac{81}{4}-18-\frac{27}{2}+9k-9k\right)\right)=64$$

$$3\left(\frac{3+8-18}{12}+\frac{5}{3}k-\frac{81}{4}+18+\frac{27}{2}+9k\right)=64$$

$$3\left(\frac{-7}{12}+\frac{32}{3}k+\frac{(-81+72+54)}{4}\right)=64;$$

$$-\frac{7}{4}+32k+\frac{3}{4}(45)=64$$

$$32k=64+\frac{7-135}{4}=\frac{256+7-135}{4}=\frac{121+7}{4}$$

$$32k=\frac{128}{4}=32 \quad k=1$$

Hence, the correct answer is (1).

69. Let $I=\int_0^\pi\left(\sqrt[2013]{\cos x}+\sqrt[2013]{\sin x}+\sqrt[2013]{\tan x}\right)dx$ …(1)

Using king property

$$I=\int_0^\pi\left(-\sqrt[2013]{\cos x}+\sqrt[2013]{\sin x}-\sqrt[2013]{\tan x}\right)dx \quad …(2)$$

(1) + (2), we get $2I = 2\int_0^\pi\left(\sqrt[2013]{\sin x}\right)dx$

$\Rightarrow \quad I=\int_0^\pi\left(\sqrt[2013]{\sin x}\right)dx \Rightarrow I=2\int_0^{\pi/2}\left(\sqrt[2013]{\sin x}\right)dx$.

Hence, $I=2$.

Hence, the correct answer is (2).

70. Let $I=\int_0^1 f(x)\,dx$ where $f(x)=[(2x^3-3x^2-x+1)]^{1/3}$

Now $I=\int_0^1 f(1-x)\,dx$ …(1)

(Using King)

But $f(1-x)=[2(1-x)^3-3(1-x)^2-(1-x)+1]^{1/3}$ …(2)

$=-f(x)$

Hence, from (1), $2I=0\Rightarrow I=0$

∴ $I=-\int_0^1 f(x)\,dx=-1\Rightarrow I=0$

Hence, the correct answer is (0).

71. Let $f(x)=ax^2+bx+c$

$f'(x)=2ax+b$

$f'(2)=4a+b=1$ …(1)

Now $I=\int_{2-\pi}^{2+\pi} f(x)\cdot\sin\left(\frac{x-2}{2}\right)dx$

Put $x-2=t\Rightarrow dx=dt$

$$I=\int_{-\pi}^{\pi}f(1+t)\cdot\sin\left(\frac{t}{2}\right) \quad …(2)$$

Also, $I=\int_{-\pi}^{\pi}f(2-t)\cdot\sin\left(-\frac{t}{2}\right)dt$ (using King)

$$I=-\int_{-\pi}^{\pi}f(2-t)\cdot\sin\left(-\frac{t}{2}\right)dt \quad …(3)$$

∴ (2) + (3) gives

$$2I=\int_{-\pi}^{\pi}[f(2+t)-f(2-t)]\sin\frac{t}{2}\,dt \quad …(4)$$

∴ $f(x)=a(x-2)^2+b(x-2)+c\Rightarrow f'(x)=2a(x-2)+b$

⇒ $f'(2)=b=1\Rightarrow f(x)=a(x-2)^2+(x-2)+c$

Now from (4), we get $I=\int_0^\pi[f(2+t)-f(2-t)]\sin\frac{t}{2}\,dt$

$$=\int_0^\pi\left[(at^2+t+c)-(at^2-t+c)\right]\sin\frac{t}{2}\,dt$$

$I=2\int_0^\pi t\sin\frac{t}{2}\,dt$; put $\frac{t}{2}=y\Rightarrow I=8\int_0^{\pi/2}y\sin y\,dy$; $I=8$

Hence, the correct answer is (1).

72. Given $f^3(x)=\int_0^x tf^2(t)$ …(1)

Differentiating both the sides of equation (1) by using Leibnitz rule, we get, $3f^2(x)f'(x)=xf^2(x)$

As $f(x)$ is strictly increasing, so $f(x)\neq 0$.

∴ $3f'(x)=x\Rightarrow f'(x)=\frac{x}{3}$

∴ $f(x)=\frac{x^2}{6}+C$ (on integrating both sides with respect to x.)

But $f(0)=0\Rightarrow C=0$

∴ $f(x)=\frac{x^2}{6}$, $x\geq 0$

Slope of the tangent at $P(x_1, y_1)$ to the curve $f(x)=\frac{x_1}{3}$

∴ Slope of normal $=\frac{-3}{x_1}$

i.e., $\frac{-1}{2}=\frac{-3}{x_1}$ (Given) $\Rightarrow x_1=6$

∴ $y_1=6$

So, Equation of normal is $(y-6)=\frac{-1}{2}(x-6)$, we get

∴ For y-intercept, $x=0$, $y=9$

Hence, the correct answer is (9).

73. We have $f(x)=x^3-\frac{3x^2}{2}+x+\frac{1}{4}$;

$$I=\int_{1/4}^{3/4}f(f(x))dx=\int_{1/4}^{3/4}f(f(1-x))dx$$

Now, $f(1-x)=(1-x)^3-\frac{3}{2}(1-x)^2+1-x+\frac{1}{4}$

$=1-x^3-3x+3x^2-\frac{3}{2}(1+x^2-2x)+1-x+\frac{1}{4}$

$f(x)+f(1-x)=1$ …(1)

⇒ $f(f(x))+f(1-f(x))=1$ $(x\to f(x))$ …(2)

Now $I=\int_{1/4}^{3/4}f(f(x))dx$ …(3)

Also $I=\int_{1/4}^{3/4}f(f(1-x))dx=\int_{1/4}^{3/4}f(1-f(x))dx$ …(4)

[using (1)] adding (3) and (4), $2I=\int_{1/4}^{3/4}[f(f(x))+f(x))]dx$

$$= \int_{1/4}^{3/4} dx = 2I = \frac{1}{2} \Rightarrow I = \frac{1}{4}$$

Hence, the correct answer is (4).

74. We have $f(x) = \alpha e^{2x} + \beta e^{x} - \gamma x$

Now $f(0) = -1 \Rightarrow \alpha + \beta = 1$ ...(1)

$f'(\ln 2) = 30 \Rightarrow 8\alpha + 2\beta - \gamma x = 30$ ...(2)

and $\int_{0}^{\ln 4} (f(x) + \gamma x) = 24$

$$\Rightarrow \int_{0}^{\ln 4} \left(\alpha e^{2x} + \beta e^{2x}\right) dx = 24 \Rightarrow \left(\frac{\alpha e^{2x}}{2} + \beta e^{x}\right)_{0}^{\ln 4} = 24$$

$$\Rightarrow (8\alpha + 4\beta) - \left(\frac{\alpha}{2} + \beta\right) = 24 \Rightarrow 15\alpha + 6\beta = 48$$

$\therefore$ On solving (1), (2), (3), we get:

$\alpha = 6, \beta = -7, \gamma = 4.$

Hence, $(\alpha + \beta + \gamma) = 6 - 7 + 4 = 3.$

Hence, the correct answer is (3).

75. Let $J = \int_{0}^{1} \frac{f(x) \cdot g'(x) - f'(x) \cdot g(x)}{f^{2}(x) g(x)} dx$

$$= \int_{0}^{1} \underbrace{\left(\frac{g(x)}{f(x)}\right)}_{\text{II}} \cdot \underbrace{\frac{1}{g(x)}}_{\text{II}} dx \text{ (I.B.P.)}$$

Now, Integrating by parts, $\left.\frac{1}{g(x)} \cdot \frac{g(x)}{f(x)}\right|_{0}^{1} + \int_{0}^{1} \frac{g'(x)}{g^{2}(x)} \cdot \frac{g(x)}{f(x)} dx$

$$J = \left(\frac{1}{f(1)} - \frac{1}{f(0)}\right) + \int_{0}^{1} \frac{g'(x)}{g(x) \cdot f(x)} dx \quad \text{...(1)}$$

Now given, $f^{2}(x) = 1 + g^{2}(x)$

$$\Rightarrow f(x) \cdot f'(x) = g(x) \cdot g'(x) \Rightarrow g'(x) = \frac{f(x) \cdot f'(x)}{g(x)}$$

$$\therefore \quad J = \left(\frac{1}{3} - \frac{1}{2}\right) + \int_{0}^{1} \frac{f(x) \cdot f'(x)}{g^{2}(x) \cdot f(x)} dx$$

$$J = \frac{-1}{6} + \int_{0}^{1} \frac{f'(x)}{f^{2}(x) - 1} dx$$

Now put, $f(x) = t$, we get $J = \frac{-1}{6} + \int_{2}^{3} \frac{dt}{t^{2} - 1} = \left.\frac{-1}{6} + \frac{1}{2} \ln \frac{t-1}{t+1}\right]_{2}^{3}$

$$= \frac{-1}{6} + \frac{1}{2}\left[\ln \frac{1}{2} - \ln \frac{1}{3}\right]$$

$$J = \frac{-1}{6} + \frac{1}{2} \ln \frac{3}{2}$$

$$\Rightarrow J = \frac{1}{2} \ln \frac{3}{2} - \frac{1}{6}.$$

$$\Rightarrow 6J + 1 = \ln \frac{27}{8} \Rightarrow a + b = 27 + 8 = 35.$$

Hence, the correct answer is (35).

76. As, $\frac{x^{3} - \operatorname{cosec}^{3} x + \tan x}{5 - |x|}$ is an odd function.

So, $I = \int_{-4}^{4} \frac{-2x^{2}}{5 - |x|} dx \Rightarrow I = -4 \int_{0}^{4} \frac{x^{2}}{5 - |x|} dx$

$$I = -4\left[\int_{0}^{4} \frac{x^{2} - 25 + 25}{5 - x} dx\right]$$

$$\Rightarrow I = -4\left[\int_{0}^{4} -(x+5) dx + 25 \int_{0}^{4} \frac{dx}{5 - x}\right]$$

$$I = -4\left[\left.\frac{-x^{2}}{2}\right|_{0}^{4} - (5x)\Big|_{0}^{4} - 25(\ln|5 - x|)\Big|_{0}^{4}\right]$$

$$\Rightarrow I = -4\left[\left.\frac{-16}{2}\right| - 20 - 25(0 - \ln 5)\right]$$

$I = -4[-28 + 25 \ln 5] \Rightarrow I = 112 - 100 \ln 5 = (a + b \ln 5)$ (Given)

$\Rightarrow (a + b) = 112 - 100 = 12.$

Hence, the correct answer is (12).

77. As, $\frac{1}{3} < x < \frac{1}{2}$

$$\Rightarrow 2 < \frac{1}{x} < 3 \Rightarrow 4 < \frac{2}{x} < 6$$

$$\Rightarrow \left[\frac{2}{x}\right] = 4 \text{ or } 5$$

CASE 1:

$$4 < \frac{2}{x} < 5 \Rightarrow 2 < \frac{1}{x} < \frac{5}{2}$$

$$\Rightarrow \left[\frac{2}{x}\right] = 4 \Rightarrow \frac{2}{5} < x < \frac{1}{2} \Rightarrow \frac{8}{5} < 4x < 2.$$

CASE 2:

$$5 < \frac{2}{x} < 6 \Rightarrow \frac{5}{2} < \frac{1}{x} < 3$$

$$\Rightarrow \left[\frac{2}{x}\right] = 5 \Rightarrow \frac{1}{3} < x < \frac{2}{5} \Rightarrow \frac{5}{3} < 5x < 2$$

Now, $I = \int_{\frac{1}{3}}^{\frac{2}{5}} \{5x\} dx + \int_{\frac{2}{5}}^{\frac{1}{2}} \{4x\} dx$

$\{5x\} = 5x - [5x] \Rightarrow \{5x\} = 5x - 1 \Rightarrow \{4x\} = 4x - 1$

$$\therefore \quad I = \int_{\frac{1}{3}}^{\frac{2}{5}} (5x - 1) dx + \int_{\frac{2}{5}}^{\frac{1}{2}} (4x - 1) dx$$

$$I = \left(\frac{5x^{2}}{2} - x\right)\Bigg|_{\frac{1}{3}}^{\frac{2}{5}} + (2x^{2} - x)\Bigg|_{\frac{2}{5}}^{\frac{2}{5}}$$

$$I = \frac{5}{2}\left[\frac{4}{25} - \frac{1}{9}\right] - \left(\frac{2}{5} - \frac{1}{3}\right) + 2\left(\frac{1}{4} - \frac{4}{25}\right) - \left(\frac{1}{2} - \frac{2}{5}\right)$$

$$I = \frac{61}{450} = \frac{m}{n} = |m - n| = 389.$$

Hence, the correct answer is (389).

78. $p(x) + 1$ is divisible by $(x - 1)^{3}$

$\Rightarrow \quad p(-x) + 1$ is divisible by $(x + 1)^{3}$ ...(1)

Similarly, $P(x) - 1$ is divisible by $(x+1)^3$

$\Rightarrow$ $p(-x) - 1$ is divisible by $(x-1)^3$ …(2)

$\Rightarrow$ From (1) and (2)

$p(x) + p(-x)$ is divisible by $(x+1)^3$ and $(x-1)^3$ both

$\Rightarrow$ $P(x) + p(-x)$ is divisible by $(x+1)^3 (x-1)^3$

Then $p(x) + p(-x)$ has degree $6 \Rightarrow p(x) + p(-x) = 0$

$\therefore$ $p(-x) = -p(x)$

$\Rightarrow$ $p(x)$ is an odd polynomial function of degree 5.

Since, $p(x)$ is odd function $\Rightarrow \int_{-10}^{10} p(x)dx = 0$.

Hence, the correct answer is (0).

79. Consider $f(a) = \int_0^{\pi/2} (\sin x + a\cos x)^3 \, dx$ …(1)

$$f'(a) = 3\int_0^{\pi/2} (\sin x + a\cos x)^2 \cos x \, dx \quad \text{…(2)}$$

$$f''(a) = 6\int_0^{\pi/2} (\sin x + a\cos x)\cdot\cos^2 x \, dx$$

$$\Rightarrow f''(a) = 6\int_0^{\pi/2} \sin x \cos x^2 dx + 6a\int_0^{\pi/2} \cos^3 x \, dx$$

$$\Rightarrow f''(a) = 6\cdot\left(\frac{-t^3}{3}\right)\Bigg|_{-1}^{0} + 6a\left(\frac{2}{3}\right) = -2(0-1) + 4a$$

$\Rightarrow$ $f''(a) = 4a + 2 \Rightarrow f'(a) = 2a^2 + 2a + c$

$f'(0) = c = 1$ (Using (2))

$f'(a) = 2a^2 + 2a + 1 \Rightarrow f(a) = \dfrac{2a^3}{3} + a^2 + a + c_1$

$f(0) = c_1 = \dfrac{2}{3}$ (Using (1))

Hence, $f(a) = \dfrac{2}{3}a^3 + a^2 + a + \dfrac{2}{3}$

Hence, given equation reduces to,

$$2a^3 + 3a^2 - 3a - 4 = 0$$

$$\sum a_1 = \sum a_1 a_2 = \frac{-3}{2};\ a_1a_2a_3 = 2$$

$$\therefore \quad \left(\sum a_1^2\right) = \left(\sum a_1\right)^2 - 2\sum a_1a_2$$

$$= \frac{9}{4} + 3 = \frac{21}{4}$$

Hence, $[a_1{}^2 + a_2{}^2 + a_3{}^2] = 5$.

Hence, the correct answer is (5).

80. $I_{16} = \int_0^{\frac{\pi}{2}} \underbrace{x\cdot\sin x}_{\text{(II)}}\cdot\underbrace{\sin^{15} x}_{\text{(I)}} \, dx$

$$= \left((\sin x - x\cdot\cos x)\sin^{15} x\right)_{x=0}^{x=\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} 15\cdot\sin^{14} x\cdot\cos x(\sin x - x\cos x)\, dx$$

$$I_{16} = 1 - 15\int_0^{\frac{\pi}{2}} \underset{\text{put } \sin x = t}{\sin^{15} x\cdot\cos x \, dx} + 15\int_0^{\frac{\pi}{2}} x\cdot\sin^{14} x(1-\sin^2 x)dx$$

$$\Rightarrow \quad I_{16} = 1 - \frac{15}{16} + 15I_{14} - 15I_{16}$$

$$\Rightarrow \quad 16I_{16} = \frac{1}{16} + 15I_{14} \Rightarrow \left(\frac{2561_{16} - 1}{I_{14}}\right)$$

$$= 16 \times 15 = 240.$$

Hence, the correct answer is (240).

81. $F(1) = \int xe^{-x}dx = -e^{-x}x - e^{-x}(x+1)$

$$F(2) = \int e^{-x}x^2dx = -e^{-x}(x^2 + 2x + 2)$$

$$\therefore \quad \frac{F(1)}{F(2)} = \frac{(x+1)}{(x+1)^2 + 1}$$

$$\Rightarrow \quad \int \frac{F(1)}{F(2)}dx = \int \frac{(x+1)}{(x+1)^2+1}dx$$

$$(x+1)^2 + 1 = t \Rightarrow (x+1)\,dx = \frac{1}{2}dt$$

$$= \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\ln(t) = \ln\sqrt{t}$$

$$\therefore \quad \left(e^{\int\frac{F(1)}{F(2)}dx}\right)^2 = \left(\sqrt{t}\right)^2 = t = (x+1)^2 + 1$$

$$\text{Hence, } \int_2^5 \left((x+1)^2 + 1\right)dx = \left.\frac{(x+1)^3}{3} + x\right]_2^5$$

$$= (72 + 5) - (9 + 2) = 77 - 11 = 66.$$

Hence, the correct answer is (66).

82. $I = 4\int_0^{\pi/2} \dfrac{dx}{\sin^2 x + k^2\cdot\cos^2 x}$ where $k = 100$

$$= 4\int_0^{\pi/2} \frac{\sin^2 x\,dx}{\tan^2 x + k^2} \quad \text{(put } \tan x = t\text{)}$$

$$= 4\int_0^{\infty} \frac{dt}{t^2 + k^2}\left.\frac{4}{k}\tan^{-1}\frac{t}{k}\right]_0^{\infty}$$

$$= \frac{4}{k}\cdot\frac{\pi}{2} = \frac{2\pi}{k}$$

$$\therefore \quad I = \frac{2\pi}{100} = \frac{\pi}{50}$$

$\therefore$ $a + b = 51$.

Hence, the correct answer is (51).

83. $J = \int_0^1 (1 - x^{50})^{100} x^{100} dx$

$$= \left((1-x^{50})^{100}\frac{x^{101}}{101}\right)_0^1 - \int_0^1 100(1-x^{50})^{99}(-50x^{49})\frac{x^{101}}{101}dx$$

$$= 0 + \frac{5000}{101}\int_0^1 (1-x^{50})^{99}(1-(1-50x^{50})x^{100}\, dx$$

$$= \frac{5000}{101}\left(\int_0^1 (1-x^{50})^{99}x^{100}dx - \int_0^1 (1-x^{50})^{100}x^{100}dx\right)$$

$$\therefore \quad J + \frac{5000}{101} \Rightarrow J = \frac{5000}{101} I$$

$$\Rightarrow \quad \frac{5000}{101} J = \frac{5000}{101} I$$

$$\frac{I}{J} = \frac{5101}{5000} = \frac{m}{n}$$

$$\therefore \quad \frac{m-1-n}{20} = \frac{5101-1-5000}{20} = 5.$$

Hence, the correct answer is (5).

84. $\int_0^{\pi}\left(8g(x)\sin^2 x - (g(x))^2\right)dx = 8\pi.$

$$\Rightarrow \quad \int_0^{\pi} 16\sin^2 x dx - \int_0^{\pi}(g(x) - 4\sin^2 x)^2\, dx = 8\pi$$

$$\Rightarrow \quad 32 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - \int_0^{\pi}\left(g(x) - 4\sin^2 x\right)^2 dx = 8\pi$$

$$\Rightarrow \quad 8\pi - \int_0^{\pi}\left(g(x) - 4\sin^2 x\right)^2 dx = 8\pi$$

$\Rightarrow$ $g(x) = 4\sin^2 x$, max of $g(x) = 4\sin^2 x$ is 4

Hence, the correct answer is (4).

ARCHIVE: JEE MAIN

1. $\lim\limits_{x\to 0^+} \dfrac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x^3}$

$$= \lim_{x\to 0^+} \frac{(\sin x)2x}{3x^2} = \lim_{x\to 0^+}\left(\frac{\sin x}{x}\right) \times \frac{2}{3} = \frac{2}{3}$$

($\because$ using L'Hospital's Rule)

Hence, the correct answer is (D).

2. $\int_1^3 \left([(x-1)^2] - 3\right) dx = \int_0^1 [x^2]\, dx + \int_1^2 [x^2]\, dx - 3\int_1^3 dx$

$$= \int_1^{\sqrt{2}} 1.dx + \int_{\sqrt{2}}^{\sqrt{3}} 2.dx + \int_{\sqrt{3}}^{2} 3.dx - 6$$

$$= \sqrt{2} - 1 + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) - 6$$

$$= -\sqrt{2} - \sqrt{3} - 1$$

Hence, the correct answer is (C).

3. $f'(x) = f'(2-x)$

$f(x) = -f(2-x) + c$

Put $x = 0$; $f(0) = -f(2) + c$

$c = f(0) + f(2) = 1 + e^2$

So, $f(x) + f(2-x) = 1 + e^2$; $I = \int_0^2 f(x)\,dx$

$$I = \int_0^2 f(2-x)\,dx;\ 2I = \int_0^2 (f(x) + f(2-x))dx$$

$2I = 2(1+e^2)$; $I = 1 + e^2$

Hence, the correct answer is (D).

4. $I = \int_{-1}^{1} x^2 e^{[x^3]}\, dx$

$$= \int_{-1}^{0} x^2 e^{[x^3]}\, dx + \int_0^1 x^2 e^{[x^3]}\, dx$$

$$= \int_{-1}^{0} x^2 e^{-1}\, dx + \int_0^1 x^2 e^{0}\, dx$$

$$= \frac{1}{e} \times \frac{x^3}{3}\Bigg|_{-1}^{0} + \frac{x^3}{3}\Bigg|_0^1$$

$$= \frac{1}{e} \times \left(0 - \left(\frac{-1}{3}\right)\right) + \frac{1}{3}$$

$$= \frac{1}{3e} + \frac{1}{3} = \frac{1+e}{3e}$$

Hence, the correct answer is (C).

5. $I_n = \int_{\pi/4}^{\pi/2} \cot^n x\, dx = \int_{\pi/4}^{\pi/2} \cot^{n-2} x(\operatorname{cosec}^2 x - 1)\, dx$

$$= -\frac{\cot^{n-1} x}{n-1}\Bigg|_{\pi/4}^{\pi/2} - I_{n-2} = \frac{1}{n-1} - I_{n-2}$$

$$\Rightarrow \quad I_n + I_{n-2} = \frac{1}{n-1}$$

$$\Rightarrow \quad I_2 + I_4 = \frac{1}{3};\ I_3 + I_5 = \frac{1}{4};\ I_4 + I_6 = \frac{1}{5}$$

$\therefore$ $\dfrac{1}{I_2+I_4}, \dfrac{1}{I_3+I_5}, \dfrac{1}{I_4+I_6}$ are in A.P.

Hence, the correct answer is (A).

6. $\sum\limits_{n=1}^{100} \int_{n-1}^{n} e^{\{x\}}\, dx$, period of $\{x\} = 1$

$$= \sum_{n=1}^{100} \int_0^1 e^{\{x\}}\, dx = \sum_{n=1}^{100} \int_0^1 e^{x}\, dx$$

$$= \sum_{n=1}^{100} (e-1) = 100(e-1)$$

Hence, the correct answer is (A).

7. $I = \int_{-\pi/2}^{\pi/2} \dfrac{\cos^2 x}{1+3^x} dx$ (using x by $-x$)

$$I = \int_{-\pi/2}^{\pi/2} \frac{\cos^2 x}{1+3^{-x}} dx = \int_{-\pi/2}^{\pi/2} \frac{3^x \cos^2 x}{1+3^x} dx;$$

$$2I = \int_{-\pi/2}^{\pi/2} \frac{\left(1+3^x\right)\cos^2 x}{1+3^x}\,dx$$

$$\Rightarrow \quad 2I = \int_{-\pi/2}^{\pi/2} \cos^2 x\,dx = 2\int_0^{\pi/2} \cos^2 x\,dx$$

$$\Rightarrow \quad I = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$$

Hence, the correct answer is (A).

8. $f(x) = \int_1^x \frac{\log_e t}{(1+t)}\,dt$

$$f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\ln t}{1+t}\,dt, \text{ let } t = \frac{1}{y}$$

$$= \int_1^x \frac{\ln y}{1+y}\cdot\frac{y}{y^2}\,dy = \int_1^x \frac{\ln y}{y(1+y)}\,dy$$

$$\text{Now, } f(x) + f\left(\frac{1}{x}\right) = \int_1^x \frac{(1+t)\ln t}{t(1+t)}\,dt$$

$$= \int_1^x \frac{\ln t}{t}\,dt = \frac{1}{2}\ln^2(x)$$

$$\text{So } f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2} \qquad \ldots(3)$$

Hence, the correct answer is (A).

9. $\int_{-a}^{0} (-2x+2)\,dx + \int_0^2 (x+2-x)\,dx + \int_2^a (2x-2)\,dx = 22$

$$\Rightarrow \quad x^2 - 2x\Big|_0^{-a} + 2x\Big|_0^2 + x^2 - 2x\Big|_2^a = 22$$

$$\Rightarrow \quad a^2 + 2a + 4 + a^2 - 2a - (4-4) = 22$$

$$\Rightarrow \quad 2a^2 = 18 \Rightarrow a = 3$$

$$I = \int_3^{-3} (x+[x])\,dx \qquad \ldots(1)$$

$$\Rightarrow \quad I = \int_3^{-3} (-x+[-x])\,dx \qquad \ldots(2)$$

Add (1) and (2), we get

$$2I = \int_3^{-3} (-1)\,dx$$

$$I = -\frac{1}{2}(-1)(-6) \Rightarrow I = 3$$

Hence, the correct answer is (3).

10. $\int_{-2}^{2} 3\,|x^2 - x - 2|\,dx$

$$= 3\int_{-2}^{2} |x^2 - x - 2|\,dx$$

$$= 3\left[\int_{-2}^{-1}(x^2 - x - 2)dx + \int_{-1}^{2}(-x^2 - x - 2)dx\right]$$

$$= 3\left[\left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)\Big|_{-2}^{-1} - \left(\frac{x^3}{3} - \frac{x^2}{2} - 2x\right)\Big|_{-1}^{2}\right]$$

$$= 3\left[7 - \frac{2}{3}\right] = 19$$

Hence, the correct answer is (19).

11. Put $2x = t \Rightarrow 2dx = dt$

$$\Rightarrow \quad I = \frac{1}{2}\int_0^{2\pi} |\sin t|\,dt$$

$$= \int_0^{\pi} |\sin t|\,dt = 2$$

Hence, the correct answer is (2).

12. $I_{m,n} = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = I_{n,m}$

$$\text{Now Let } x = \frac{1}{y+1} \Rightarrow dx = -\frac{1}{(y+1)^2}\,dy$$

$$\text{So } I_{m,n} = -\int_{\infty}^{0} \frac{1}{(y+1)^{m-1}}\,\frac{y^{n-1}}{(y+1)^{n-1}}\,\frac{dy}{(y+1)^2} = \int_0^{\infty} \frac{y^{n-1}}{(y+1)^{m+n}}\,dy$$

$$I_{m,n} = \int_0^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\,dy + \int_1^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy \qquad \ldots(1)$$

$$T = \int_1^{\infty} \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

Put $y = 1/z,\ dy = -1/z^2\,dz$

$$T = \int_1^0 \frac{1}{z^{n-1}(1+1/z)^{m+n}}\cdot\left(-\frac{1}{z^2}\right)dz = \int_0^1 \frac{z^{m+n}}{z^{n+1}(z+1)^{m+n}}\,dz$$

$$\text{Now from (1) } I_{m,n} = \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}} + \frac{t^{m-1}}{(t+1)^{m+n}}\,dt$$

We get, $\alpha = 1$

Hence, the correct answer is (1).

13. $f(x) + f(x+1) = 2$

$\Rightarrow$ $f(x)$ is periodic with period = 2

$$I_1 = \int_0^1 (f(x) + f(1+x))\,dx = 8$$

Similarly, $I_2 = 2 \times 2 = 4$

$$I_1 + 2I_2 = 16$$

Hence, the correct answer is (16).

14. $I = \int_0^{2\pi} \frac{x\sin^8 x}{\sin^8 x + \cos^8 x}\,dx \qquad \ldots(1)$

$$= \left[\int_0^{\pi} \frac{x\sin^8 x}{\sin^8 x + \cos^8 x}\,dx + \int_0^{\pi} \frac{(2\pi - x)\sin^8 x}{\sin^8 x + \cos^8 x}\,dx\right]$$

$$= 2\pi\int_0^{\pi}\frac{\sin^8 x}{\sin^8 x+\cos^8 x}\,dx$$

$$I = 2\pi\left[\int_0^{\pi/2}\frac{\sin^8 x}{\sin^8 x+\cos^8 x}dx+\int_0^{\pi/2}\frac{\cos^8 x dx}{\sin^8 x+\cos^8 x}dx\right]$$

$$= 2\pi\int_0^{\pi/2} 1dx \;= 2\pi\frac{\pi}{2} \;= \pi^2$$

Hence, the correct answer is (A).

15. $\int_0^{1/2}\frac{((x^2-1)+1)}{(1-x^2)^{3/2}}\,dx = \int_0^{1/2}\frac{dx}{(1-x^2)^{3/2}} - \int_0^{1/2}\frac{dx}{\sqrt{1-x^2}}$

$$= \int_0^{1/2}\frac{x^{-3}}{(x^{-2}-1)^{3/2}}\,dx - \left(\sin^{-1}x\right)_0^{1/2}$$

Let $x^{-2}-1 = t^2 \Rightarrow x^{-3}\,dx = -t\,dt$

$$= \int_{\infty}^{\sqrt{3}}\frac{-tdt}{t^3} - \frac{\pi}{6} = \int_{\sqrt{3}}^{\infty}\frac{dt}{t^2} - \frac{\pi}{6}$$

$$= -\frac{1}{t}\Big|_{\sqrt{3}}^{\infty} - \frac{\pi}{6} = \frac{1}{\sqrt{3}} - \frac{\pi}{6} = \frac{2\sqrt{3}-\pi}{6} = \frac{k}{6}$$

$$k = 2\sqrt{3} - \pi$$

Hence, the correct answer is (A).

16. $\int_{-\pi}^{\pi}|\pi-|x||\;dx = 2\int_0^{\pi}|\pi-x|\,dx = 2\cdot\int_0^{\pi}(\pi-x)\,dx$

$$= 2\left[\pi x-\frac{x^2}{2}\right]_0^{\pi} = \pi^2$$

Hence, the correct answer is (A).

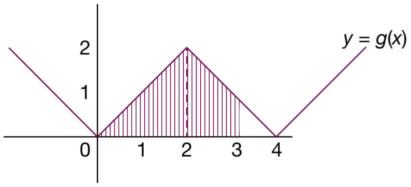
17. $I = \int_{\pi/6}^{\pi/3}((2\tan^3 x\cdot\sec^2 x\cdot\sin^4 3x)+(6\tan^4 x\cdot\sin^3 3x\cdot\cos 3x))\,dx$

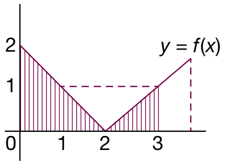
$$\Rightarrow \quad I = \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d((\sin 3x)^4(\tan x)^4) \;=\; \frac{1}{2}\left[(\sin 3x)^4(\tan x)^4\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$\Rightarrow \quad I = -\frac{1}{18}$$

Hence, the correct answer is (C).

18. $\int_0^3\left(g(x)-f(x)\right)dx = \int_0^3||x-2|-2|\,dx - \int_0^3|x-2|\,dx$





$$= \left(\frac{1}{2}\times2\times2+1+\frac{1}{2}\times1\times1\right) - \left(\frac{1}{2}\times2\times2+\frac{1}{2}\times1\times1\right)$$

$$= \left(2+1+\frac{1}{2}\right) - \left(2+\frac{1}{2}\right) = 1$$

Hence, the correct answer is (D).

19. $I = \int_{-\pi/2}^{\pi/2}\frac{dx}{1+e^{\sin x}} = \int_0^{\pi/2}\frac{1}{1+e^{\sin x}}+\frac{1}{1+e^{-\sin x}}\,dx = \int_0^{\pi/2}dx$

$$\Rightarrow \quad I = \frac{\pi}{2}$$

Hence, the correct answer is (D).

20. $\int_1^2 e^x\cdot x^x(2+\log_e x)\,dx = \int_1^2 e^x(2x^x+x^x\log_e x)\,dx$

$$= \int_1^2 e^x\left(\underbrace{x^x}_{f(x)}+\underbrace{x^x(1+\log_e x)}_{f(x)}\right)dx = \left(e^x.x^x\right)_1^2 = 4e^2 - e$$

Hence, the correct answer is (D).

21. $I_2 = \int_0^1\left(1-x^{50}\right)^{101}dx = \int_0^1\left(1-x^{50}\right)\left(1-x^{50}\right)^{100}dx$

$$= \int_0^1\left(1-x^{50}\right)^{100}dx - \int_0^1 x^{50}\cdot\left(1-x^{50}\right)^{100}dx$$

$$I_2 = I_1 - \int_0^1 \underbrace{x}_{I}\cdot\underbrace{x^{49}\cdot(1-x-^{50})^{100}dx}_{II}$$

Now apply by parts, we get

$$I_2 = I_1 - \left[x\int x^{49}\cdot(1-x^{50})^{100}dx - \int x^{49}(1-x^{50})^{100}dx\right]$$

Let $(1-x^{50}) = t \Rightarrow -50x^{49}dx = dt$

$$I_2 = I_1 - \left[x\left(-\frac{1}{50}\right)\frac{(1-x^{50})^{101}}{101}\Bigg|_{x=0}^{x=1} - \int_0^1\left(-\frac{1}{50}\right)\frac{(1-x^{50})^{101}}{101}dx\right]$$

$$I_2 = I_1 - 0 - \frac{1}{50}\cdot\frac{1}{101}\;.\,I_2 = I_1 - \frac{1}{5050}I_2$$

$$I_2 + \frac{1}{5050}I_2 = I_1 \Rightarrow \frac{5051}{5050}I_2 = I_1 \Rightarrow I_2 = \frac{5050}{5051}I_1$$

$\because \quad I_2 = \alpha.I_1$

$\therefore \quad \alpha = \frac{5050}{5051}$

Hence, the correct answer is (A).

22. $F'(x) = x^2g(x) = x^2\int_1^x f(u)du \Rightarrow F'(1) = 0$

$$F''(x) = x^2 f(x) + 2x\int_1^x f(u)du$$

$F''(1) = f(1) + 2\times0$

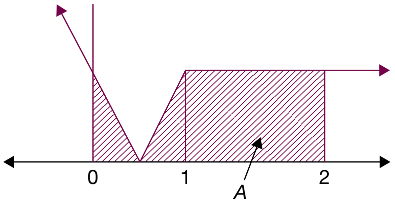
$F''(1) = 3$

$F'(1) = 0$ and $F''(1) = 3 > 0$

So, Minima

Hence, the correct answer is (D).

23.



Let $f(x) = |\,|x-1| - x\,| = \begin{cases} 1 & ; \quad x \geq 1 \\ |1-2x| & ; \quad x \leq 1 \end{cases}$

$$\int_0^2 |\;|x-1|-x|dx = \int_0^{1/2}(1-2x)\,dx + \int_{1/2}^{1}(2x-1) + \int_1^2 1\,dx$$

$$= \left[x-x^2\right]_0^{\frac{1}{2}} + \left[x^2-x\right]_{1/2}^{1} + [x]_1^2 = 3/2$$

Hence, the correct answer is (1.50).

24. $\int_1^2 |2x-[3x]|\,dx$

$$= \int_1^{4/3}(3-2x)\,dx + \int_{4/3}^{5/3}(4-2x)\,dx + \int_{5/3}^{2}(5-2x)\,dx = \frac{2}{9} + \frac{3}{9} + \frac{4}{9} = 1$$

Alternate:

$$\int_1^2 |2x-[3x]|$$

$$= \int_1^2 |3x-[3x]-x|\,dx = \int_1^2 |\{3x\}-x|\,dx = \int_1^2 x-\{3x\}dx$$

$$= \int_1^2 xdx - \int_1^2 \{3x\}dx = \left.\frac{x^2}{2}\right|_1^2 - 3\int_0^{1/3}\{3x\}dx$$

$$= \frac{3}{2} - 3\int_0^{1/3} 3xdx = \frac{3}{2} - 9\left.\frac{x^2}{2}\right|_0^{1/3} = 1$$

Hence, the correct answer is (1.0).

25. $\int_0^n \{x\}\,dx = n\int_0^1 \{x\}\,dx = n\int_0^1 xdx\,dx = \frac{n}{2}$

$$\int_0^n \{x\}\,dx = \int_0^n (x-[x])\,dx = \frac{n^2}{2} - \frac{n}{2}$$

$$\Rightarrow \left(\frac{n^2-n}{2}\right)^2 = \frac{n}{2}\cdot 10(n^2-n) \Rightarrow \frac{n-1}{4} = 5 \Rightarrow n = 21$$

Hence, the correct answer is (21).

26. $I = \int_0^{\pi} |\cos x|^3\,dx = 2\int_0^{\pi/2} |\cos x|^3\,dx$

$$= 2\int_0^{\pi/2} |\cos^3 x \cdot dx = 2\int_0^{\pi/2}\left(\frac{\cos 3x + 3\cos x}{4}\right)\cdot dx$$

$$l = \frac{1}{2}\left[\frac{\sin 3x}{3} + 3\sin x\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left[\frac{1}{3}\cdot\sin 3\cdot\frac{\pi}{2} + 3.1\right]$$

$$= \frac{1}{2}\left[\frac{-1}{3} + 3\right] = \frac{1}{2}\left(\frac{8}{3}\right) = \frac{4}{3}$$

Hence, the correct answer is (C).

27. $|f(x) - f(y)| \leq 2[x-y]^{3/2}$

$$\Rightarrow \left|\frac{f(x)-f(y)}{x-y}\right| \leq 2|x-y|^{1/2}$$

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \leq \lim_{y\to x} 2|x-y|^{1/2}$$

$$\Rightarrow \quad |f'(x)| \leq 0 \Rightarrow f'(x) = 0 \Rightarrow f(x) = \text{constant}$$

as $f(0) = 1 \Rightarrow f(x) = 1$, now $\int_0^1 f^2(x)dx = 1$

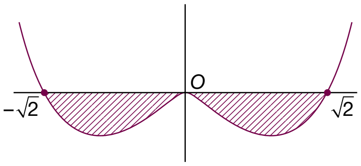
Hence, the correct answer is (D).

28. $\int_0^{\pi/3}\frac{\tan\theta}{\sqrt{2k\sec\theta}}d\theta = \int_0^{\pi/3}\frac{\sin\theta}{\sqrt{2k}\sqrt{\cos\theta}} = \frac{1}{\sqrt{2k}}\left(-2\sqrt{\cos\theta}\right)_0^{\pi/3}$

$$\Rightarrow \quad \frac{\sqrt{2}}{\sqrt{k}}\left[1-\frac{1}{\sqrt{2}}\right] = 1 - \frac{1}{\sqrt{2}} \Rightarrow k = 2$$

Hence, the correct answer is (A).

29. $I = \int_a^b (x^4 - 2x^2)dx$



$f(x) = x^4 - 2x^2$

$= x^2(x - \sqrt{2})(x + \sqrt{2})$

$-\sqrt{2} \leq x \leq \sqrt{2}$ integration will be greatest negative

So for minimum value of I

$$a = -\sqrt{2}, b = \sqrt{2}$$

Hence, the correct answer is (C).

30. $\int_{-\pi/2}^{-1}\frac{dx}{2-1} + \int_{-1}^{0}\frac{dx}{3-1} + \int_0^1\frac{dx}{4} + \int_1^{\pi/2}\frac{dx}{5+0}$

$$= (x)_{-\pi/2}^{-1} + \frac{1}{2}(x)_{-1}^{0} + \frac{1}{4}(x)_0^1 + \frac{1}{5}(x)_1^{\pi/2}$$

$$= \left(-1+\frac{\pi}{2}\right) + \frac{1}{2}(1) + \frac{1}{4} + \frac{1}{5}\left(\frac{\pi}{2}-1\right) = \frac{3}{20}(4\pi - 3)$$

Hence, the correct answer is (D).

31. $f(x) = 2x - x^2 f(x)$

$$f(x) = \frac{2x}{1+x^2}, f\left(\frac{1}{2}\right) = \frac{1}{1+\left(\frac{1}{4}\right)} = \frac{4}{5}$$

$$f'(x) = \frac{(1+x^2)^2 - 2x(2x)}{(1+x^2)} = 2\left[\frac{1-x^2}{(1+x^2)^2}\right]$$

$$f'\left(\frac{1}{2}\right) = \frac{2\left(1-\frac{1}{4}\right)}{\left(1+\frac{1}{4}\right)} = \frac{2\left(\frac{3}{4}\right)}{\frac{25}{16}} = \frac{24}{25}$$

Hence, the correct answer is (C).

32. $I = \int_{-2}^{2}\frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}}dx = \left(\frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} + \frac{\sin^2(-x)}{\left[-\frac{x}{\pi}\right]+\frac{1}{2}}\right)dx$

Now $\left[\frac{x}{\pi}\right] + \left[\frac{-x}{\pi}\right] = -1$ (As $x \neq n\pi$)

$$I = \int_0^2 \left(\frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} - \frac{\sin^2 x}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} \right) dx = 0$$

Hence, the correct answer is (A).

33. $$\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{\sin 2x(\tan^5 x + \cot^5 x)} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{(1+\tan^2 x)}{2\tan x(\tan^5 x + \cot^5 x)} \; dx$$

Let $\tan x = t$, $\sec^2 x dx = dt$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{dt}{2t\left(t^5 + \frac{1}{t^5}\right)} = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{t^5 dt}{2t(t^{10}+1)} = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{t^5 dt}{2(1+t^{10})}$$

Put $t^5 = u \Rightarrow 5t^4\, dt = du$

$$\Rightarrow \quad \int_{\left(\frac{1}{\sqrt{3}}\right)^5}^{1} \frac{du}{5.2(1+u^2)} = \frac{1}{10}\left[\tan^{-1} u\right]_{\left(\frac{1}{\sqrt{3}}\right)^5}^{1}$$

$$= \frac{1}{10}\left[\frac{\pi}{4} - \tan^{-1}\frac{1}{(\sqrt{3})^5}\right]$$

Hence, the correct answer is (C).

34. $I = \int_0^a f(a-x)g(a-x)dx \; ; I = \int_0^a f(x)\{4-g(x)\}dx$

$$I = \int_0^a 4f(x)dx \;-\; \int_0^a f(x)g(x)dx \;= 4\int_0^a f(x)dx \;- I$$

$$\Rightarrow \quad 2I = 4\int_0^a f(x)dx \;\Rightarrow I = 2\int_0^a f(x)dx$$

Hence, the correct answer is (D).

35. $$\int_1^e \left(\left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^{x}\right) \log x \cdot dx$$

$$= \int_1^e \left(\frac{x}{e}\right)^{2x} \log_e x \cdot dx - \int_1^e \left(\frac{e}{x}\right)^{x} \log_e x \cdot dx$$

Put $\left(\frac{x}{e}\right)^{2x} = u \Rightarrow 2x\ln\left(\frac{x}{e}\right) = \ln u \Rightarrow \left(2\ln\left(\frac{x}{e}\right) + \frac{2x}{\frac{x}{e}}\frac{1}{e}\right)dx$

$$= \frac{du}{u} \Rightarrow 2u\ln x dx = du$$

Put $\left(\frac{e}{x}\right)^{x} = v \Rightarrow x\ln\left(\frac{e}{x}\right) = x - x\ln x = \ln v \Rightarrow -\ln x\, dx = \frac{dv}{v}$

$$= \frac{1}{2}\int_{(1/e)^2}^{1} du \;+\; \int_e^1 dv \;=\; \frac{1}{2}\,(u)_{1/e^2}^{1} \;+\; (v)_e^1$$

$$= \frac{1}{2}\left(1 - \frac{1}{e^2}\right) + 1 - e = \frac{3}{2} - \frac{1}{2e^2} - e$$

Hence, the correct answer is (B).

36. $$\lim_{n\to\infty} \sum_{r=1}^{2n} \frac{n}{n^2+r^2} = \lim_{n\to\infty} \sum_{r=1}^{2n} \frac{n}{n^2\left(1+\frac{r^2}{n^2}\right)}$$

$$= \int_0^2 \frac{dx}{(1+x^2)} = \left[\tan^{-1} x\right]_0^2 = \tan^{-1} 2$$

Hence, the correct answer is (A).

37. Let, $1 = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx$ …(1)

Using, $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$, we get

$$1 = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}} dx \qquad \text{…(2)}$$

Adding (1) and (2), we get

$$2I = \int_{-\pi/2}^{\pi/2} \sin^2 x\, dx \;\Rightarrow\; 2I = 2.\int_0^{\pi/2} \sin^2 x\; dx$$

$$\Rightarrow \quad 2I = 2\times\frac{\pi}{4} \Rightarrow I = \frac{\pi}{4}$$

Hence, the correct answer is (C).

38. $1 = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos x} = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1-\cos x}$ …(1)

Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Adding (1) and (2), we get

$$2I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{2}{\sin^2 x} dx \;\Rightarrow\; I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \operatorname{cosec}^2 x\; dx$$

$$\Rightarrow \quad I = -(\cot x)_{\pi/4}^{3\pi/4} = -\left[\cos\frac{3\pi}{4} - \cot\frac{\pi}{4}\right] = 2$$

Hence, the correct answer is (C).

39. $$L = \lim_{n\to\infty}\left(\frac{(n+1)(n+2)\dots(n+2n)}{n^{2n}}\right)^{1/n}$$

$$\therefore \quad \log_e L = \frac{1}{n}\left(\lim_{n\to\infty}\sum_{r=1}^{2n}\log\left(1+\frac{r}{n}\right)\right)$$

$$\Rightarrow \quad \log_e L = \int_0^2 \log(1+x)\, dx = \left(x\log(1+x)\right)_0^2 \;-\; \int_0^2 \frac{x}{1+x} dx$$

$$\Rightarrow \quad \log_e L = 2\log_e 3 - \int_0^2\left(1 - \frac{1}{1+x}\right)dx = 2\log 3 - \left(x - \log(1+x)\right)_0^2$$

$$\Rightarrow \quad \log_e L = 2\log 3 - (2 - \log 3) = 3\log 3 - 2 = \log\frac{27}{e^2}$$

$$\Rightarrow \quad L = \frac{27}{e^2}$$

Hence, the correct answer is (C).

40. $$I=\int_2^4 \frac{\log x^2}{\log x^2+\log(36-12x+x^2)}\,dx$$

$$I=\frac{2}{2}\int_2^4 \frac{\log|x|}{\log|x|+\log|6-x|}\,dx \qquad \ldots(1)$$

$$I=\int_2^4 \frac{\log|6-x|}{\log|6-x|+\log|x|}\,dx$$

$$\left\{\int_a^b f(x)dx=\int_a^b f(a+b-x)dx\right\} \qquad \ldots(2)$$

Adding (1) and (2), we get

$$2I=\int_2^4 \frac{\log|x|+\log|6-x|}{\log|x|+\log|6-x|}\,dx=\int_2^4 dx\ =2$$

Hence, $I=1$

Hence, the correct answer is (A).

41. $$I=\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$$

$$=\int_0^{\pi}\left|1-2\sin\frac{x}{2}\right|dx=\int_0^{\pi/3}\left(1-2\sin\frac{x}{2}\right)dx+\int_{\pi/3}^{\pi}\left(2\sin\frac{x}{2}-1\right)dx$$

$$=\left(x+4\cos\frac{x}{2}\right)\Bigg|_0^{\pi/3}+\left(-4\cos\frac{x}{2}-x\right)\Bigg|_{\pi/3}^{\pi}\ =\ 4\sqrt{3}\ -4-\frac{\pi}{3}$$

Hence, the correct answer is (D).

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. $$g(x)=\int_{\sin x}^{\sin(2x)}\sin^{-1}(t)dt\ ;$$

$$g'(x)=2\cos 2x\sin^{-1}(\sin 2x)-\cos x\sin^{-1}(\sin x)$$

$$g'\left(\frac{\pi}{2}\right)=2(-1)(0)-\cos\left(\frac{\pi}{2}\right)(1)=0$$

$$g'\left(-\frac{\pi}{2}\right)=0$$ No options is/are matching.

Hence, the correct answer is (Bonus).

2. $$I=\int_{-\pi/2}^{\pi/2}\frac{x^2\cos x}{(1+e^x)}dx \Rightarrow I=\int_0^{\pi/2}\left(\frac{x^2\cos x}{1+e^x}+\frac{x^2\cos x}{1+e^{-x}}\right)dx$$

$$I=\int_0^{\pi/2}x^2\cos x dx=x^2\sin x-\int_0^{\pi/2}2x\sin x dx$$

$$I=(x^2\sin x-2x(-\cos x)+(2)(-\sin x))_0^{\pi/2}$$

$$=\left(\frac{\pi^2}{4}-2\right)-(0)=\frac{\pi^2}{4}-2$$

Hence, the correct answer is (A).

3. $$\frac{192x^3}{3}\le\frac{192x^3}{2+\sin^4\pi x}\le\frac{192x^3}{2}$$

$$\Rightarrow \int_{1/2}^{x}\frac{192x^3}{3}dx\le\int_{1/2}^{x}f'(x)dx\le\int_{1/2}^{x}\frac{192x^3}{2}dx$$

$$\Rightarrow 16x^4-1\le f(x)\le 24x^4-\frac{3}{2}$$

$$\Rightarrow \int_{1/2}^{1}(16x^4-1)dx\le\int_{1/2}^{1}f(x)dx\le\int_{1/2}^{1}\left(24x^4-\frac{3}{2}\right)dx$$

$$1<\frac{26}{10}\le\int_{1/2}^{1}f(x)dx\le\frac{39}{10}<12$$

Hence, the correct answer is (D).

4. Put $\ln\tan x/2=-t$

$$\Rightarrow \quad \tan\frac{x}{2}=e^{-t}\Rightarrow\frac{1}{2}\sec^2\frac{x}{2}dx=e^{-t}dt$$

$$\Rightarrow \quad dx=\frac{-2e^{-t}dt}{1+e^{-2t}}=\frac{-2dt}{e^{-t}+e^{t}}$$

and $\sin x=\dfrac{2e^t}{1+e^{2t}}\ \Rightarrow 2\operatorname{cosec} x=e^t+e^{-t}$

$$I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\operatorname{cosec} x)^{17}dx$$

$$=-2\int_{\ln\left(\sqrt{2}+1\right)}^{0}(e^t+e^{-t})^{16}\cdot dt\ =\ \int_0^{\ln(\sqrt{2}+1)}2(e^t+e^{-t})^{16}dt$$

Hence, the correct answer is (A).

5. $$F(x)=\int_0^{x^2}f(\sqrt{t})dt\ \Rightarrow F'(x)=f'(x)=2xf(x)$$

$$\Rightarrow \quad \frac{f'(x)}{f(x)}=2x\ \Rightarrow\ln(f(x))=x^2+c$$

Using $f(0)=1$ we get $c=0$

$\therefore \quad \ln(f(x))=x^2\Rightarrow f(x)=e^{x^2}$

$\therefore \quad F(x)=f(x)+c\ \Rightarrow F(x)=e^{x^2}+c$

Given $F(0)=0$

$\therefore \quad c=-1$

$\therefore \quad f(x)=e^{x^2}-1\Rightarrow f(2)=e^4-1$

Hence, the correct answer is (B).

Multiple Correct Choice Type Questions

6. (A) Let g(x) = $f(x)$ – 3 cos 3x

$$\text{Now } \int_0^{\pi/3} g(x)\,dx = \int_0^{\pi/3} f(x)\,dx - 3\int_0^{\pi/3} \cos 3x\,dx = 0$$

Hence, $g(x) = 0$ has *a* root in $\left(0, \frac{\pi}{3}\right)$

(B) Let $h(x) = f(x)$ –3sin3x + $\frac{\pi}{6}$

$$\text{Now } \int_0^{\pi/3} h(x)\,dx = \int_0^{\pi/3} f(x)\,dx - 3\int_0^{\pi/3} \sin 3x\,dx + \int_0^{\pi/3} \frac{6}{\pi}\,dx$$

= 0 – 2 +2 = 0.

Hence, $h(x) = 0$ has *a* root in $\left(0, \frac{\pi}{3}\right)$,

(C) $$\lim_{x\to 0} \frac{x\int_0^x f(t)dt}{1-e^{x^2}} = \lim_{x\to 0} \underbrace{\left(\frac{x^2}{1-e^{x^2}}\right)}_{-1} \underbrace{\frac{\int_0^x f(t)dt}{x}}_{\text{Apply L'Hopital's Rule}}$$

$$= -1 \lim_{x\to 0} \frac{f(x)}{1} = -1$$

(D) $$\lim_{x\to 0} \frac{(\sin x)\int_0 f(t)dt}{x^2}$$

$$= \lim_{x\to 0} \underbrace{\left(\frac{\sin x}{x}\right)}_{1} \underbrace{\frac{\int_0^x f(t)dt}{x}}_{\text{Apply L'Hopital's Rule}} = 1 \lim_{x\to 0} \frac{f(x)}{1} = 1$$

Hence, (A), (B) and (C) are correct.

7. (A) $$\int_0^1 x\cos x\,dx = \int_0^1 x\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ldots\right)dx \ge \int_0^1 x\left(1 - \frac{x^2}{2}\right)dx$$

$$= \int_0^1 x - \frac{x^3}{2}\,dx = \frac{x^2}{2} - \frac{x^4}{8}\Big|_0^1 = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

(B) $$\int_0^1 x\sin x = \int_0^1 x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots\right)dx\,dx \ge \int_0^1 x\left(x - \frac{x^3}{3!}\right)dx$$

$$= \int_0^1 x^2 - \frac{x^4}{6}\,dx = \frac{x^3}{3} - \frac{x^5}{30}\Big|_0^1 = \frac{1}{3} - \frac{1}{30} = \frac{9}{30} = \frac{3}{10}$$

(C) $$\int_0^1 x^2\cos x\,dx = \int_0^1 x^2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ldots\right)dx \ge \int_0^1 x^2\left(1 - \frac{x^2}{2!}\right)dx$$

$$= \int_0^1 x^2 - \frac{x^4}{2!}\,dx = \frac{x^3}{3} - \frac{x^5}{10}\Big|_0^1 = \frac{1}{3} - \frac{1}{10} = \frac{7}{30}$$

(D) $$\int_0^1 x^2\sin x\,dx = \int_0^1 x^2\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\ldots\right)dx \ge \int_0^1 x^2\left(x - \frac{x^3}{3!}\right)dx$$

$$= \int_0^1 x^3 - \frac{x^5}{3!}\,dx = \frac{x^4}{4} - \frac{x^6}{36}\Big|_0^1 = \frac{1}{4} - \frac{1}{36} = \frac{8}{36} = \frac{2}{9}$$

Hence, (A), (B) and (D) are correct.

8. $$\lim_{n\to\infty} \frac{\sqrt[3]{1} + \sqrt[3]{2} + \ldots + \sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(na+1)^2} + \frac{1}{(na+2)^2} + \ldots + \frac{1}{(na+n)^2}\right)}$$

$$= \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^{1/3}}{n^{7/3}\left(\sum_{r=1}^{n} \frac{1}{(na+r)^2}\right)} = \lim_{n\to\infty} \frac{\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}}{\sum_{r=1}^{n}\left(\frac{1}{\left(a+\frac{r}{n}\right)^2}\right)}$$

$$= \frac{\int_0^1 x^{1/3}dx}{\int_0^1 \frac{dx}{(a+x)^2}} = \frac{\frac{3}{4}x^{4/3}\Big|_0^1}{-\left(\frac{1}{a+x}\right)\Big|_0^1} = \frac{\frac{3}{4}}{-\left(\frac{1}{a+1} - \frac{1}{a}\right)} = 54$$

$$\Rightarrow \quad \frac{3}{4\left(\frac{1}{a(a+1)}\right)} = 54 \Rightarrow a = 8 \text{ or } a = -9$$

Hence, (A) and (D) are correct.

9. $$F(x) = \int_0^x f(t)dt$$

$F'(x) = f(x) = (x - 1)\,(x - 2)\,(x - 5)$

$\Rightarrow$ $x = 1, 5$ is point of local minima and $x = 2$ is point of local maxima

$F(2) = \int_0^2 f(t)dt < 0$ at point of maxima, function is negative,

then $F(x) < 0\ \forall\ x \in (0, 5)$

Hence, (A), (B) and (D) are correct.

10. $$\frac{k+1}{(k+1)(x+1)} < \frac{k+1}{x(x+1)} < \frac{k+1}{x(k+1)} \text{ where } x \in (k, k+1)$$

$$\sum_{k=1}^{98}\int_k^{k+1} \frac{1}{x+1}dx < \sum_{k=1}^{98}\int_k^{k+1} \frac{k+1}{x(x+1)}dx < \sum_{k=1}^{98}\int_k^{k+1} \frac{dx}{x}$$

$$\sum_{k=1}^{98}\left(\ln(k+2) - \ln(k+1)\right) < I < \sum_{k=1}^{98}\left(\ln(k+1) - \ln k\right)$$

$$\ln 50 < I < \ln 99 \quad \Rightarrow \frac{49}{50} < \ln 50 < I < \ln 99.$$

Hence, (B) and (D) are correct.

11. Function $f : R \to (0, 1)$

(A) Let $g(x) = x^9 - f(x)$ is a continuous function

$g(0) = -f(0) < 0$

$g(1) = 1 - f(1) > 0$

By intermediate mean value theorem, $g(x) = 0$ will be for some $x \in (0, 1)$

(B) Let $M(x) = x - \int\limits_0^{\frac{\pi}{2}-x} f(t)\cos t\, dt$

$M(0) < 0$ and $M(1) > 0$

As $M(x)$ is continuous in $x \in [0, 1]$

From Intermediate mean value theorem

$M(x) = 0$ for some $x \in (0, 1)$

(C) Let $h(x) = e^x - \int\limits_0^x f(t) \sin t\, dt$ is a continuous function

$h'(x) = e^x - f(x)$, $\sin x > 0$ in $x \in (0, 1)$

then $h(x)$ is an increasing function

$h(0) = 1$; $h(x)$ cannot be zero in $x \in (0, 1)$

(D) Let $k(x) = f(x) + \int\limits_0^{\frac{\pi}{2}} f(t)\sin t\, dt > 0$ for $x \in (0, 1)$

So $k(x)$ cannot be zero

Hence, (A) and (B) are correct.

12. $$f(x) = \lim_{n\to\infty}\left\{\frac{n.n.n\ldots n}{1.2.3\ldots n}\frac{\left(x+\frac{n}{1}\right)\left(x+\frac{n}{2}\right)\ldots\left(x+\frac{n}{n}\right)}{\left(x^2+\frac{n^2}{1}\right)\left(x^2+\frac{n^2}{2^2}\right)\ldots\left(x^2+\frac{n^2}{n^2}\right)}\right\}^{x/n}$$

$$\ln f(x) = \lim_{n\to\infty}\frac{x}{n}\left\{\sum_{r=1}^{n}\ln\frac{n}{r} + \sum_{r=1}^{n}\ln\left(x+\frac{n}{r}\right) - \sum_{r=1}^{n}\ln\left(x^2+\frac{n^2}{r^2}\right)\right\}$$

Replace $r/n = t$

$$\ln(f(x)) = x\left(\int\limits_0^1 \ln\frac{1}{t}dt + \int\limits_0^1 \ln\left(x+\frac{1}{t}\right) - \int\limits_0^1 \ln\left(x^2+\frac{1}{t^2}\right)dt\right)$$

$$\ln f(x) = x\int\limits_0^1 \ln\frac{\frac{1}{t}.\left(x+\frac{1}{t}\right)}{\left(x^2+\frac{1}{t^2}\right)}dt = x\int\limits_0^1 \ln\frac{(tx+1)}{(t^2x^2+1)}dt$$

Put $tx = p$ then $x\, dt = dp$

$$\ln(f(x)) = \int\limits_0^x \ln\frac{(p+1)}{(p^2+1)}dp\,;\quad \frac{f'(x)}{f(x)} = \ln\left(\frac{1+x}{1+x^2}\right);$$

$$\frac{f'(2)}{f(2)} = \ln\left(\frac{3}{5}\right) < 0 \Rightarrow f'(2) < 0$$

$$\frac{f'(3)}{f(3)} = \ln\left(\frac{4}{10}\right) = \ln\left(\frac{2}{5}\right) \Rightarrow \frac{f'(2)}{f(2)} \ge \frac{f'(3)}{f(3)}$$

Now $\frac{f'(x)}{f(x)} > 0$ in $(0, 1)$ and $\frac{f'(x)}{f(x)} < 0$ in $(1, \infty)$

$f(x)$ is increasing in $(0, 1)$ and decreasing in $[1, \infty)$ (as $f(x)$ is positive)

Hence, $f(1) \ge f\left(\frac{1}{2}\right)$ and $f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$

Hence, (B) and (C) are correct.

13. Let $\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt = A$

$$I = \int_{\pi}^{2\pi} e^t(\sin^6 at + \cos^4 at)dt$$

Put $t = \pi + x$ then $dt = dx$ for $a = 2$ as well as $a = 4$

$$I = e^{\pi}\int_{\pi}^{\pi} e^x(\sin^6 ax + \cos^4 ax)dx = e^{\pi}A$$

Similarly, $\int_{2\pi}^{3\pi} e^t(\sin^6 at + \cos^4 at)dt = e^{2\pi}A$

Using Put $t = 2\pi + x$ then $dt = dx$, For both $a = 2, 4$

$$\int_{3\pi}^{4\pi} e^t(\sin^6 at + \cos^4 at)dt = e^{3\pi}A$$

Using Put $t = 3\pi + x$ then $dt = dx$, For both $a = 2, 4$

So, $L = \frac{A + e^{\pi}A + e^{2\pi}A + e^{3\pi}A}{A} = \frac{e^{4\pi}-1}{e^{\pi}-1}$. For both $a = 2, 4$

Hence, (A) and (C) are correct.

14. $f(x) = (7\tan^6 x - 3\tan^2 x)(\tan^2 x + 1)$

$$\int\limits_0^{\pi/4} f(x) = \int\limits_0^{\pi/4} (7\tan^6 x - 3\tan^2 x)\sec^2 x dx = \int\limits_0^1 (7t^6 - 3t^2)dt = 0$$

Using $\tan x = t \Rightarrow \int\limits_0^{\pi/4} f(x)dx = 0$

$$\int\limits_0^{\pi/4} xf(x)dx = \left[x\int f(x)dx\right]_0^{\pi/4} - \int\limits_0^{\pi/4}\left[\int f(x)dx\right]dx$$

$$= 0 - \int\limits_0^{\pi/4} \tan^7 x - \tan^3 x dx$$

$$\Rightarrow \quad \int\limits_0^{\pi/4} xf(x)dx = \int\limits_0^{\pi/4} -\tan^7 x + \tan^3 x dx$$

$$= \int\limits_0^{\pi/4} \tan^3 x(1-\tan^2 x)\sec^2 x dx = \int\limits_0^1 t^3 - t^5 dt = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

Hence, (A) and (B) are correct.

15. $$f(x) = \int\limits_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t} \Rightarrow f'(x) = \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{x}{x^2}e^{-\left(x+\frac{1}{x}\right)}$$

$$\Rightarrow \quad f'(x) = \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x} > 0$$

$f(x)$ is always increasing function for $x \in (0, \infty)$.

$$f(x) + f(1/x) = \int\limits_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t} + \int\limits_{x}^{1/x}\frac{e^{-\left(t+\frac{1}{t}\right)}}{t}dt$$

Put $t = \frac{1}{p} \Rightarrow dt = -\frac{1}{p^2}dp$ in second integral

$$f(x) + f(1/x) = \int\limits_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)}\frac{dt}{t} - \int\limits_{1/x}^{x} e^{-\left(p+\frac{1}{p}\right)}\frac{dp}{p} = 0$$

Since, $f(x) = -f\left(\frac{1}{x}\right)$

$$\Rightarrow \quad f(2^x) = -f\left(\frac{1}{2^x}\right) \Rightarrow f(2^x) = -f(2^{-x})$$

$f(2^x)$ is an odd function

Hence, (A), (C) and (D) are correct.

16. $$g(x)=\begin{cases}0 & ; \quad x<a\\ \displaystyle\int_a^x f(t)dt & ; \quad a\le x\le b\\ \displaystyle\int_a^b f(t)dt & ; \quad x>b\end{cases}$$

$$\lim_{x\to a^-} g(x)=0\,,\ g(a)=\int_a^a f(t)dt=0$$

$$\lim_{x\to a^+} g(x)=\lim_{x\to a^+}\int_a^x f(t)dt=\int_a^a f(t)dt=0\,,$$

$g(x)$ is continuous at $x=a$

Similarly, at $x=b$ we will get $g(x)$ as continuous

So, $g(x)$ is continuous $\forall\ x\in R$

$$g'(x)=\left[\begin{array}{ccc}0 & ; & x<a\\ f(x) & ; & a\le x\le b\\ 0 & ; & x>b\end{array}\right.$$

$g'(a^-)=0,\ '(a^+)=f(a)$ and $g'(b^-)=f(b),\ g'(b^+)=0$

Since $f(x)$ co-domain is $[1,\infty)$ then $f(a)$ and $f(b)$ can never be zero.

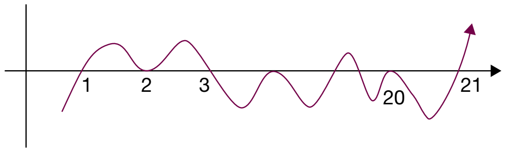
Hence, it is non-derivable at $x=a$ and $x=b$.

Hence, (A) and (C) are correct.

Linked Comprehension Type Questions

17. $f_1(x)=\displaystyle\int_0^x \prod_{j=1}^{21}(t-j)^j\ dt$

$$f_1'(x)=\prod_{j=1}^{21}(t-j)^j\ =(x-1)\,(x-2)^2\,(x-3)^3\ldots(x-21)^{21}$$



So points of minima one $4m+1$ where $m=0,1,\ldots 5\Rightarrow m_1=6$

Points of maxima are $4m-1$ where $m=1,2,\ldots 5\Rightarrow n_1=5$

$\Rightarrow\quad 2m_1+3n_1+m_1n_1=57$

Hence, the correct answer is (57).

18. $f'_2(x)=98(x-1)^{50}-600(x-1)^{49}+2450$

$\Rightarrow\quad 2f'(x)=2\times 49\times 50(x-1)^{49}-50\times 12\times 49(x-1)^{48}$

$=50\times 49\times 2(x-1)^{48}(x-1-6)$

$=50\times 49\times 2\,(x-1)^{48}\,(x-7)$

Point of minima $=7\Rightarrow m_2=1$

No point of maxima $\Rightarrow n_2=0$

$6m_2+4n_2+8m_2n_2=6$

Hence, the correct answer is (6).

19. $S_1=\displaystyle\int_{\pi/8}^{3\pi} f(x)\,dx=\int_{\pi/8}^{3\pi/8}\sin^2 x\ dx$

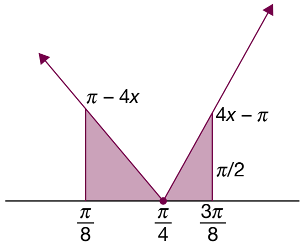
$$=\int_{\pi/8}^{3\pi/8}\sin^2\left(\frac{\pi}{8}+\frac{3\pi}{8}-x\right)dx=\int_{\pi/8}^{3\pi/8}\cos^2 x\,dx$$

$$2S_1=\int_{\pi/8}^{3\pi/8}(\sin^2 x+\cos^2 x)\,dx=\frac{3\pi}{8}-\frac{\pi}{8}=\frac{\pi}{4}$$

$\Rightarrow\quad \dfrac{16S_1}{\pi}=2$

Hence, the correct answer is (2).

20.



$$S_2=\int_{\pi/8}^{3\pi/8} f(x)g_2(x)\,dx=\int_{\pi/8}^{3\pi/8}\sin^2 x\,|4x-\pi|\,dx$$

$$=\int_{\pi/8}^{3\pi/8}\sin^2\left(\frac{\pi}{2}-x\right)\left|4\left(\frac{\pi}{2}-x\right)-\pi\right|dx$$

$$=\int_{\pi/8}^{3\pi/8}(\cos^2 x)\,|\pi-4x|\,dx$$

$$\Rightarrow\quad 2S_2=\int_{\pi/8}^{3\pi/8}|4x-\pi|\,(\sin^2 x+\cos^2 x)\,dx$$

$$=\int_{\pi/8}^{3\pi/8}|4x-\pi|\,dx=2\times\frac{1}{2}\times\frac{\pi}{8}\times\frac{\pi}{2}$$

$$=\frac{\pi^2}{16}\ \Rightarrow\ \frac{48S_2}{\pi^2}=\frac{3}{2}=1.5$$

Hence, the correct answer is (1.5).

21. $f'(x)=(|\,x\,|-x^2)\ e^{-x^2}\ +(|\,x\,|-x^2)\,e^{-x^2}\ ,x\ge 0$

$f'=2(x-x^2)\ e^{-x^2}$

Hence, option (D) is wrong.

$g'(x)=\ xe^{-x^2}\ 2x\Rightarrow f'(x)+g'(x)=2\times\ e^{-x^2}$

$f(x)+g(x)=-e^{-x^2}\ +c\Rightarrow f(x)+g(x)=-e^{-x^2}\ +1$

$F(\ln 3)+g(\sqrt{\ln 3}\,)=1-\dfrac{1}{3}=\dfrac{2}{3}$ (option (A) is wrong)

$H(x)=\psi_1(x)-1-\alpha x=e^{-x}+x-1-\alpha x\,,x\ge 1$ and $\alpha\in(1,x)$

$H(1)=e^{-1}+1-1-\alpha<0$

$H'(x)=-e^{-x}+1-\alpha>0\Rightarrow H(x)$ is $\downarrow$

$\Rightarrow\quad$ option (B) is wrong

(C) $\quad\psi_2(x)=2x\,(\psi_1(\beta)-1)$

Applying L.M.V.T to $\psi_2(x)$ in $[0,x]$

$$\psi_2'(\beta)=\frac{\psi_2(x)-\psi_2(0)}{x}\Rightarrow 2\beta-2+23^{-\beta}=\frac{\psi_2(x)-0}{x}$$

$\Rightarrow\psi_2(x)=2x(\psi_1(\beta-1))$ has one solution

Option (C) is correct.

Hence, the correct answer is (C).

22. (A) $\quad\psi_1(x)=e^{-x}+x\,,x\ge,0$

$\psi_1'(x)=1-e^{-x}>0\Rightarrow\psi_1(x)$ is $\uparrow$

$\psi_1(x)\ge\psi_1(0)\ \forall\ x\ge 0\Rightarrow\psi_1(0)\ge 1$

(B) $\psi_2(x) = x^2 - 2x + 2 - 2e^{-x}$

$x \geq 0$

$\psi_2'(x) = 2x - 2 + 2e^{-x} = 2\psi_1(x) - 2 \geq 0 \ \forall \ x \geq 0$

$\Rightarrow \psi_2(x) \geq 1$ is $\uparrow \Rightarrow \psi_2(x) \geq \psi_2(x) \geq 0$

(C) $f(x) = 2\int_0^x (t - t^2)\ e^{-t^2}\ dt$ and $x \in \left(0, \frac{1}{2}\right)$

$$= \int_0^x 2te^{-t^2}\ dt - \int_0^x 2t^2 e^{-t^2}\ dt = -\ e^{-x^2}\Big|_0^x -$$

Let $H(x) = f(x) - 1 + e^{-t^2} + \frac{2}{3}x^3 - \frac{2}{5}x^5, x \in \left(0, \frac{1}{2}\right)$

$H(0) = 0$

$H'(x) = 2(x \,.\, x^2)\ e^{-x^2} - 2xe^{-x^2} + 2x^2 - 2x^4$

$= -2x^2 e^{-x^2} + 2x^2 - 2x^4$

$= 2x^2(1 - x^2 - e^{-x^2})$ $\quad \because\ e^{-x} \geq 1 - x \ \forall \ x \geq 0$

$\Rightarrow \quad H'(x) \leq 0 \Rightarrow H(x)$ is $\downarrow \Rightarrow 1.1(x) < 0 \ \forall \ x \in \left(0, \frac{1}{2}\right)$

Let $P(x) = g(x) - \frac{2}{3}x^3 + \frac{2}{5}x^5 - \frac{1}{7}x^7\ x \in \left(0, \frac{1}{2}\right)$

$P'(x) = 2x^2\ e^{-x^2} - 2x^2 + 2x^4 - x^6$

$= -\frac{x^8}{3} + \frac{x^{10}}{12} \ \dots$

$\Rightarrow \quad P'(x) \leq 0 \Rightarrow P(x)$ is $\downarrow \Rightarrow P(x) \leq 0$

Option (D) is correct

Hence, the correct answer is (D).

23. Let $f(x) = \left(\frac{10x}{x+1}\right)$

So, $f'(x) = 10\left(\frac{(x+1) - x}{(x+1)^2}\right) = \frac{10}{(x+1)^2} > 0 \ \forall \ x \in [0, 10]$,

So, (x) is an increasing function

So, range of $f(x)$ is $\left[0, \sqrt{\frac{100}{11}}\right]$

$$I = \int_0^{1/9}\left[\sqrt{\frac{100x}{x+1}}\right]dx + \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx +$$

$$\int_{1/9}^{2/3}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{2/3}^{9}\left[\sqrt{\frac{10x}{x+1}}\right]dx + \int_{9}^{10}\left[\sqrt{\frac{10x}{x+1}}\right]dx$$

$$= 0 + \int_{1/9}^{2/3} dx + 2\int_{2/3}^{9} dx + 3\int_{9}^{10} dx$$

$$= \frac{2}{3} - \frac{1}{9} + 2\left(9 - \frac{2}{3}\right) + 3(10 - 9)$$

$$= \frac{6-1}{9} + 2 \times \frac{25}{3} + 3 = \frac{5}{9} + \frac{50}{3} + 3$$

$$= \frac{5 + 150 + 27}{9} = \frac{182}{9} = 182$$

Hence, the correct answer is (182).

24. $f'(x) = F(x) + xF'(x)$

$\Rightarrow \quad f'(1) = F(1) + F'(1) = F'(1) < 0$

$\Rightarrow \quad f'(1) < 0$

$f(2) = 2F(2)$

$F(x)$ is decreasing and $F(1) = 0$

Hence, $F(2) < 0 \Rightarrow f(2) < 0$

$f'(x) = F(x) + xF'(x)$

$F(x) < 0 \, \forall x \in (1, 3)$ and $F'(x) < 0 \, \forall x \in (1, 3)$

Hence, $f'(x) < 0 \ \forall \ x \in (1, 3)$

Hence, (A), (B) and (C) are correct.

25. $\int_1^3 f(x)dx = \int_1^3 xF(x)dx$

$$= \left[\frac{x^2}{2}F(x)\right]_1^3 - \frac{1}{2}\int_1^3 x^2F'(x)dx = \frac{9}{2}F(3) - \frac{1}{2}F(1) + 6 = -12$$

$$40 = \int_1^3 x^3 F'''(x)dx = \left[x^3F'(x)\right]_1^3 - 3\int_1^3 x^2 F'(x)\,dx$$

$40 = 27F'(3) - F'(1) + 36 \Rightarrow 27F'(3) - F'(1) = 4$ …(1)

$f'(x) = F(x) + xF'(x)$

$f'(3) = F(3) + 3F'(3) = 3F'(3) - 4$

and $f'(1) = F(1) + F'(1) = F'(1)$

$9f'(3) - f'(1) = 27F'(3) - F'(1) - 36 = 4 - 36 = -32$

using result (1)

Hence, (C) and (D) are correct.

26. $g'(1/2) = \lim\limits_{h \to 0^+} \int_h^{1-h} t^{-1/2}(1-t)^{-1/2}dt$

$$= \int_0^1 \frac{1}{\sqrt{t - t^2}}dt = \int_0^1 \frac{1}{\sqrt{(1/2)^2 - (t - 1/2)^2}}dt$$

$$= \left(\sin^{-1}\frac{t - \frac{1}{2}}{\frac{1}{2}}\right)_0^1 = \sin^{-1}(2t - 1)\Big|_0^1 = \sin^{-1}(1) - \sin^{-1}(-1) = \pi$$

Hence, the correct answer is (A).

27. $g(a) = \lim\limits_{h \to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}dt$

$g(1-a) = \lim\limits_{h \to 0^+} \int_h^{1-h} t^{-(1-a)}(1-t)^{(1-a)-1}dt$

$= \lim\limits_{h \to 0^+} \int_h^{1-h} t^{a-1}(1-t)^{-a}dt$

Using $\left\{\text{by} \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right\}$

$$= \lim_{h \to 0^+} \int_h^{1-h} (1-t)^{a-1}(1-(1-t))^{-a}dt = \lim_{h \to 0^+} \int_h^{1-h} (1-t)^{a-1}t^{-a}dt$$

$g(1-a) = g(a)$ on differentiation

$\Rightarrow \quad -g'(1-a) = g'(a)$

Now put $a = 1/2$, we obtain

$$-g'\left(\frac{1}{2}\right) = g'\left(\frac{1}{2}\right) \Rightarrow g'\left(\frac{1}{2}\right) = 0$$

Hence, the correct answer is (D).

Matrix Match/Column Match Type Questions

28. (P) Let $f(x) = ax^2 + bx$, $a, b \in W$ (as $f(0) = 0$)

$$\int_0^1 f(x)dx = \int_0^1 ax^2 + bxdx = \frac{a}{3} + \frac{b}{2} = 1 \Rightarrow 2a + 3b = 6$$

Possible (a, b) are $(3, 0), (0, 2)$, hence, number of such polynomials $= 2$

(Q) $f(x) = \sin x^2 + \cos x^2 = \sqrt{2}\sin\left(x^2 + \frac{\pi}{4}\right)$

$$f(x)_{\max} = \sqrt{2}\sin\left(x^2 + \frac{\pi}{4}\right) = \sqrt{2}$$

$$\Rightarrow x^2 + \frac{\pi}{4} = 2n\pi + \frac{\pi}{2} \Rightarrow x^2 = 2n\pi + \frac{\pi}{4} \quad \text{for } n = 0, 1$$

$x^2 \in [0, 13]$

We get 4 values of x

(R) $$\int_{-2}^{2} \frac{3x^2}{1+e^x}dx = \int_0^2 3x^2\left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right)dx$$

using $\left\{\int_{-a}^{a} f(x)dx = \int_0^a (f(x) + f(-x))dx\right\}$

$$= \int_0^2 3x^2\left(\frac{1}{1+e^x} + \frac{e^x}{1+e^x}\right)dx = \int_0^2 3x^2dx = x^3\Big|_0^2 = 8$$

(S) Numerator $= \displaystyle\int_{-1/2}^{1/2} \cos 2x \ln\left(\frac{1+x}{1-x}\right)dx = 0$

(as it is an odd function)

Required term $= 0$

Hence, A → q, B → r, C → p, D → s

Hence, the correct answer is (D).

Integer/Numerical Answer Type Questions

29. $$I = \int_{-1}^{0} \frac{x \cdot 0}{2+0}dx + \int_0^1 \frac{x \cdot 0}{2+1}dx + \int_1^{\sqrt{2}} \frac{x \cdot 1}{2+0}dx + 0 = \frac{1}{4}$$

$\Rightarrow \quad 4I - 1 = 0$

Hence, the correct answer is (0).

30. $$\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$$

Put $9x + 3\tan^{-1}x = t \Rightarrow \left(9 + \frac{3}{1+x^2}\right)dx = dt$

$$\Rightarrow \quad \alpha = \int_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$$

$$\Rightarrow \left(\log_e|1+\alpha| - \frac{3\pi}{4}\right) = 9$$

Hence, the correct answer is (9).

31. $f(-x) = -f(x)$ hence, $F(1) = 0$

$$G(1) = \int_{-1}^{1} t\,|f(f(t))|\,dt = 0$$

($t\,|f(f(t))|$ is an odd function)

Given $f(1) = \frac{1}{2}$; $$\lim_{x\to1}\frac{F(x)}{G(x)} = \lim_{x\to1}\frac{\frac{F(x)-F(1)}{x-1}}{\frac{G(x)-G(1)}{x-1}} = \frac{f(1)}{|f(f(1))|} = \frac{1}{14}$$

$$\Rightarrow \quad \frac{\frac{1}{2}}{\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14} \quad \Rightarrow f\left(\frac{1}{2}\right) = 7.$$

Hence, the correct answer is (7).

32. $F(x) = \displaystyle\int_0^x f(t)\,dt \quad \Rightarrow F'(x) = f(x)$

$$\int_0^{\pi} (f'(x) + F(x))\cos x dx$$

$$= \int_0^{\pi} f'(x)\cos x\,dx + \left[F(x)\sin x\Big|_0^{\pi} - \int_0^{\pi} F'(x)\sin xdx\right] = 2$$

$$\Rightarrow \quad \int_0^{\pi} f'(x)\cos x\,dx + 0 - 0 - \int_0^{\pi} \underset{\substack{\downarrow\\ \text{I}}}{f(x)}\underset{\substack{\downarrow\\ \text{II}}}{\sin x}\,dx = 2$$

$$\Rightarrow \quad \int_0^{\pi} f'(x)\cos x\,dx - \left(f(x)(-\cos x)\Big|_0^{\pi} - \int_0^{\pi} f'(x)(-\cos x)dx\right) = 2$$

$$\Rightarrow \quad \int_0^{\pi} f'(x)\cos x\,dx - \left[+f(\pi) + f(0) + \int_0^{\pi} f'(x)\cos xdx\right] = 2$$

$\Rightarrow \quad -f(\pi) - f(0) = 2 \Rightarrow +6 - f(0) = 2 \Rightarrow f(0) = 4$

Hence, the correct answer is (4).

33. $$I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{\left(1+e^{\sin x}\right)(2-\cos 2x)}$$

$$= \frac{2}{\pi}\int_0^{\pi/4} \frac{1}{\left(1+e^{\sin x}\right)(2-\cos 2x)} + \frac{1}{\left(1+e^{-\sin x}\right)(2-\cos 2x)}dx$$

$$I = \frac{2}{\pi}\int_0^{\pi/4} \frac{1+e^{\sin x}}{\left(1+e^{\sin x}\right)(2-\cos 2x)}dt$$

$$I = \frac{2}{\pi}\int_0^{\pi/4} \frac{dt}{2-\cos 2t} = \frac{2}{\pi}\int_0^{\pi/4} \frac{\sec^2 t}{1+3\tan^2 t}dt = \frac{2}{\pi}\cdot\frac{\pi}{3\sqrt{3}}$$

$3\sqrt{3}I = 2 \Rightarrow 27I^2 = 4$

Hence, the correct answer is (4).

34. $$I = \int_0^{\pi/2} \frac{3\sqrt{\cos\theta}d\theta}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^5} \qquad \ldots(1)$$

Using property replace θ by $\pi/2 - \theta$

$$I = \int_0^{\pi/2} \frac{3\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^5}d\theta \qquad \ldots(2)$$

$$\Rightarrow \quad 2I = 3\int_0^{\pi/2} \frac{d\theta}{\left(\sqrt{\cos\theta} + \sqrt{\sin\theta}\right)^4} = \int_0^{\pi/2} \frac{3d\theta}{\cos^2\theta\left(1+\sqrt{\tan\theta}\right)^4}$$

$$\Rightarrow \quad I = \frac{3}{2}\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{\left(1+\sqrt{\tan\theta}\right)^4}$$

Let $1+\sqrt{\tan\theta}=t \Rightarrow \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta d\theta = dt$

$$I=\frac{3}{2}\int_1^{\infty}\frac{2(t-1)}{t^4}dt=\frac{3}{2}\left|\frac{2}{-2t^2}+\frac{3}{3t^3}\right|_1^{\infty}$$

$$=\frac{3}{2}\left(\frac{2}{2}-\frac{2}{3}\right)=\frac{1}{2}=0.5$$

Hence, the correct answer is (0.5).

35. $I_2=\int_0^{1/2}\frac{1+\sqrt{3}}{\sqrt{1-x^2}(1-x)}dx$

Let $x=\sin\theta \Rightarrow dx=\cos\theta d\theta$

$$I_2=\int_0^{\pi/6}\frac{1+\sqrt{3}}{(1-\sin\theta)}d\theta$$

$$=\int_0^{\pi/6}\frac{1+\sqrt{3}}{\left(\sin\frac{\theta}{2}-\cos\frac{\theta}{2}\right)^2}d\theta=\int_0^{\pi/6}\frac{(1+\sqrt{3})\sec^2\frac{\theta}{2}}{\left(\tan\frac{\theta}{2}-1\right)^2}d\theta$$

Let $\tan\frac{\theta}{2}=y$

$$\Rightarrow \quad I=\int_0^{2-\sqrt{3}}\frac{2(1+\sqrt{3})dy}{(y-1)^2}=2(1+\sqrt{3})\left|\frac{-1}{y-1}\right|_0^{2-\sqrt{3}}$$

$$=2(1+\sqrt{3})\left|\frac{-1}{1-\sqrt{3}}-1\right|=2(1+\sqrt{3})\left(\frac{\sqrt{3}-1}{2}\right)=2$$

Hence, the correct answer is (2).

36. $\ln(L)=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{r}{n}\right)$

$$=\int_0^1\ln(1+x)\,dx=|x\ln(1+x)|_0^1-|x-\ln(1+x)|_0^1$$

$$=\ln 2-(1-\ln 2)$$

$$\Rightarrow \quad L=\frac{4}{e}\Rightarrow[L]=1$$

Hence, the correct answer is (1).

37. $g(x)=\int_x^{\pi/2}\left[f'(t)\operatorname{cosec}t-\cot t\operatorname{cosec}t\cdot f(t)\right]dt$

$$\Rightarrow \quad g(x)=f(t)\operatorname{cosec}(t)\Big|_x^{\pi/2}-\int_x^{\pi/2}f(t)(-\operatorname{cosec}t\cdot\cot t)dt-\int_x^{\pi/2}\cot t\operatorname{cosec}t\,f(t)dt$$

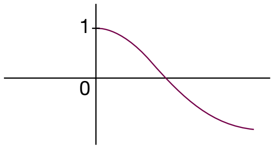
$$=3-\frac{f(x)}{\sin x}$$

$$\Rightarrow \quad \lim_{x\to 0}g(x)=\lim_{x\to 0}\left(3-\frac{f(x)}{\sin x}\right)=3-f'(0)=2$$

Hence, the correct answer is (2).

38. Let $f(x)=\int_0^x\frac{t^2}{1+t^4}dt+1-2x$

$f'(x)=\frac{x^2}{1+x^4}-2<0$; $f(x)$ is a decreasing function



∵ $\quad \because \frac{1}{t^2}+t^2>2\Rightarrow\frac{1}{\frac{1}{t^2}+t^2}<\frac{1}{2}$

$f(1)=\int_0^1\frac{t^2}{1+t^4}dt-1=\int_0^1\frac{1}{\frac{1}{t^2}+t^2}dt-1<0$ and $f(0)=1$

It means $f(x)=0$ for exactly once

Hence, the correct answer is (1).

39. $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\}dx$

$$=4x^3\cdot\frac{d}{dx}\left(1-x^2\right)^5\Big|_0^1-12\int_0^1x^2\cdot\frac{d}{dx}\left(1-x^2\right)^5dx$$

$$=-12\left[\left(x^2\cdot\left(1-x^2\right)^5\right)_0^1-2\int_0^1x\cdot\left(1-x^2\right)^5dx\right]$$

$$=12\int_0^1 2x\left(1-x^2\right)^5dx.$$

Put $1-x^2=t$, $2x\,dx=-dt$

$$=-12\int_1^0t^5dt=\frac{12}{6}\left(t^6\right)_0^1=2$$

Alternate:

$$\frac{d}{dx}\left(\frac{d\left(1-x^2\right)^5}{dx}\right)=\frac{d}{dx}\left(5\left(1-x^2\right)^4(-2x)\right)$$

$$=-10\left[\left(1-x^2\right)^4+4x\left(1-x^2\right)^3(-2x)\right]$$

$$=-10\left(1-x^2\right)^3(1-9x^2)$$

Hence, Integral $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\}dx$

$$=-40\int_0^1x^3\left(1-x^2\right)^3\left(1-9x^2\right)dx$$

Put $x=\sin\theta$

$$=-40\int_0^{\pi/2}\sin^3\theta\cos^7\theta d\theta+360\int_0^1\sin^5\theta\cos^7\theta d\theta$$

Using Wallis formula

$$=-40.1.\frac{2.6.4.2}{10.8.6.4.2}+360.\frac{4.2.6.4.2}{12.10.8.6.4.2}=-1+3=2$$

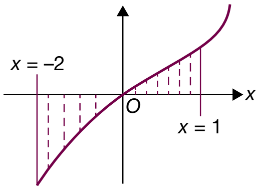
Hence, the correct answer is (2).

CHAPTER 10: AREA UNDER THE CURVE

Test Your Concepts-I

1. $A = \int_{-2}^{0} \left(-x^3\right) dx + \int_{0}^{1} x^3 dx$

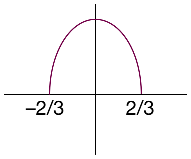
$$= \left[\frac{-x^4}{4}\right]_{-2}^{0} \left[\frac{x^4}{4}\right]_{0}^{1} = \frac{17}{4}$$



2. Required area $= 2 \int_{0}^{2/3} (-3x^2 - x + 2)\, dx$

$$= -2\left[x^3\right]_0^{2/3} - \left[x^2\right]_0^{2/3} + 4\,[x]_0^{2/3}$$

$$= \frac{-16}{27} - \frac{4}{9} + \frac{8}{3} = \frac{44}{27}$$



3. $y = xe^{-x}$

$$dy/dx = e^{-x} - xe^{-x} = e^{-x}(1 - x)$$

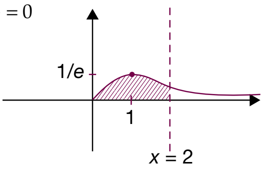
$$d^2y/dx^2 = \frac{e^x(-1) - (1 - x)e^x}{e^{2x}} = \frac{x - 2}{e^x} = 0$$

Point of inflection $x = 2$

Required area $= \int_{0}^{2} xe^{-x}\, dx$

$$= \left[[-xe^{-x}]_0^2 + \int_0^2 e^{-x}\, dx\right] = \left[(-2e^{-2}) - [e^{-x}]_0^2\right]$$

$$= -2e^{-2} - (e^{-2} - 1) = 1 - 3/e^2$$



4. $\int_{0}^{x_1} f(x)\,dx = x_1 \cdot e^{x_1}$

On differentiation

$$f(x_1) = x_1 e^{x_1} + e^{x_1}$$

$$f(x) = xe^x + e^x$$

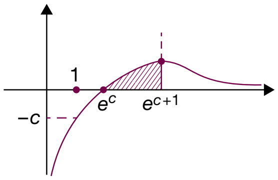
5. $y = \frac{\ell nx - c}{x} \Rightarrow dy/dx = 0; \quad x > 0$

$\Rightarrow$ $(x)\,[1/x] - (\ln x - c)\,(1) = 0$

$\Rightarrow$ $1 - \ln x + c = 0 \Rightarrow \ln x = 1 + c$

$\Rightarrow$ $x = e^{1+c}$ vertical line

$$\lim_{x\to 0} y = \lim_{x\to\infty} \frac{\ln x - c}{x} = -\infty$$



$$A = \int_{e^c}^{e^{c+1}} \left(\frac{\ln x - c}{x}\right) dx \text{ put } \ln x - c = t;\ 1/x\, dx = dt$$

$$A = \int_0^1 t.dt = \left(\frac{t^2}{2}\right)_0^1 = \frac{1}{2}$$ (independent of c). Hence, Proved.

6. $ay^2 = x^2(a - x)$

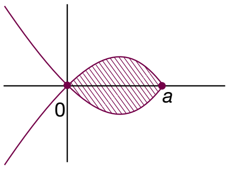
$\Rightarrow$ $y^2 = x^2(a - x)/a \Rightarrow x \le a$

$\Rightarrow$ Area $= 2\frac{1}{\sqrt{a}} \int_0^a \sqrt{x^2(a - x)}\, dx = 2\frac{1}{\sqrt{a}} \int_0^a x\sqrt{a - x}\,dx$

Put $a - x = t^2$; $x = a - t^2$; $dx = -2t\, dt$

Area $= 2\frac{1}{\sqrt{a}} \int_{\sqrt{a}}^{0} (a - t^2)(t)(-2t)\, dt$

Area $= 2\, \frac{1}{\sqrt{a}} \int_{0}^{\sqrt{a}} (2at^2 - 2t^4)\, dt$

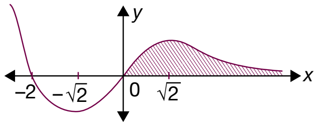


$$= 4\, \frac{1}{\sqrt{a}} \int_0^{\sqrt{a}} (at^2 - t^4)\, dt = 4a^2[1/3 - 1/5] = 8a^2/15$$

$\Rightarrow$ $k = 8$

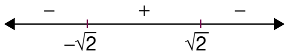
7. Curve $y = (x^2 + 2x)e^{-x}$

$$\frac{dy}{dx} = (2x + 2)e^{-x} + (x^2 + 2x)\,(-e^{-x})$$



$$= e^{-x}[2x + 2 - x^2 - 2x] = e^{-x}[2 - x^2]$$

$$f(\sqrt{2}) = \frac{(2 + 2\sqrt{2})}{e^{\sqrt{2}}};$$



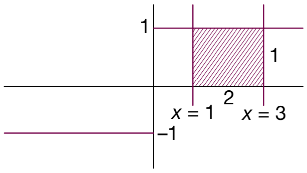
Required area $A = \int_0^\infty (x^2 + 2x)e^{-x}\, dx$

$$= (-e^{-x})\,(x^2 + 2x)\Big|_0^\infty - \int_0^\infty (-e^{-x})(2x + 2)\, dx$$

$$= 0 - 0 + (-e^{-x})(2x + 2)\Big|_0^\infty - \int_0^\infty (-e^{-x})2\, dx$$

$$= -\left(0 - \frac{2}{e^0}\right) + 2\left(-e^{-x}\right)_0^\infty = 2 + 2(0 + 1) = 4$$

8. Curve $y = \frac{|x|}{x};\ \ x \ne 0,\ \ y(x-1)(x-3) = 0$ is

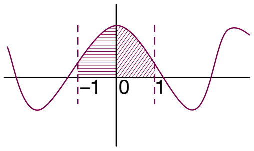


Area $= 2 \times 1 = 2$ square units

9. $|x| = \cos^{-1} y$ or $y = \cos|x|$ and line $|x| = 1$ and x-axis

Required area $= 2\int_0^1 \cos x\, dx$

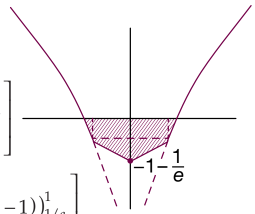
$$= 2[\sin x]_0^1 = 2\sin 1$$



10. $f(x) = \begin{cases} \log_e|x| & ; \quad |x| \ge \frac{1}{e} \\ |x| - 1 - \frac{1}{e}; & |x| < \frac{1}{e} \end{cases}$

$$= 2\left[\int_0^{1/e} -\left(x - 1 - \frac{1}{e}\right)dx + \int_{1/e}^{1} -\ln x\, dx\right]$$

$$= 2\left[-\left\{\frac{x^2}{2} - \left(1 + \frac{1}{e}\right)x\right\}_0^{1/e} - \left(x(\ln x - 1)\right)_{1/e}^{1}\right]$$



$$= -2\left[\frac{1}{2e^2} - \frac{1}{e} - \frac{1}{e^2}\right] - 2\left[(-1) + \left(\frac{2}{e}\right)\right]$$

$$= -\frac{1}{e^2} + \frac{2}{e} + \frac{2}{e^2} + 2 - \frac{4}{e} = \frac{1}{e^2} + 2 - \frac{2}{e} \text{ sq. units}$$

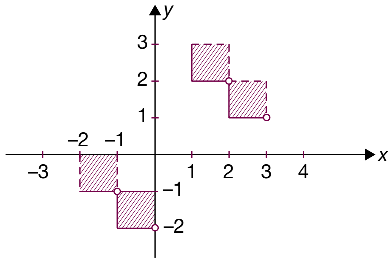
11. $\int_1^b f(x)\,dx = (b-1)\sin(3b+4)$

$\Rightarrow \int_1^x f(x)\,dx (x-1)\sin(3x+4)$ [replacing b by x]

$\Rightarrow f(x) = 3(x-1)\cos(3x+4) + \sin(3x+4)$

Test Your Concepts-II

1.



$[x].[y] = 2$

$[x] = 2, [y] = 1; 2 \le x < 3, 1 \le y < 2$

$[x] = 1, [y] = 2; 1 \le x < 2, 2 \le y < 348.$

$[x] = -1, [y] = -2; -1 \le x < 0, -2 \le y < -1$

$[x] = -2, [y] = -1; -2 \le x < -1, -1 \le y < 0$

∴ Required Area = (1 × 1) + (1 × 1) + (1 × 1) + (1 × 1) = 4

2. $x^2y - y^3 - x^2 + 5y^2 - 8y + 4 = 0$

$\Rightarrow x^2(y-1) - y^3 + y^2 + 4y^2 - 4y - 4y + 4 = 0$

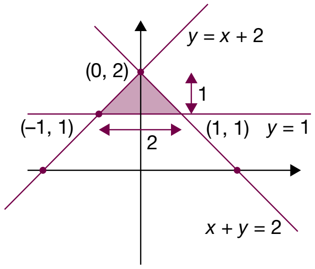
$\Rightarrow x^2(y-1) - y^2(y-1) + 4y(y-1) - 4(y-1) = 0$

$\Rightarrow (y-1)[x^2 - y^2 + 4y - 4] = 0$

$\Rightarrow (y-1)[x^2 - (y-2)^2] = 0$

$y = 1, x = y - 2, x = -y + 2$

Required area = (1/2) (1) (2) = 1



3. Circle $x^2 + y^2 = 1$; curve $y^2 = 1 - |x|$

Both curves are symmetric about x-axis and y-axis

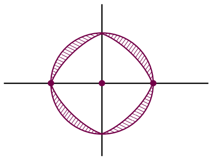
In first quadrant they intersect at point (1,0) and (0,1)

$$\text{Area} = \pi - 4\int_0^1 (1 - y^2)\,dy = -\pi + 4\left[\frac{y^3}{3} - y\right]_0^1$$

$= \pi + 4[1/3 - 1] = \pi - 8/3$

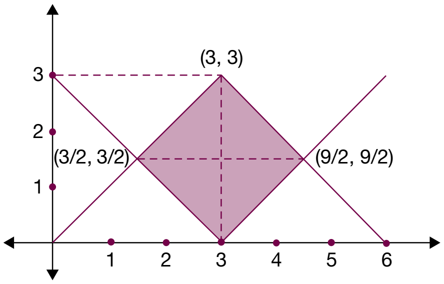
We get $k = 8/3$

$\Rightarrow [k] = 2$



4. $y \le 3 - |3 - x|$ and $y \ge |x - 3|$

Both curves are shown in figure

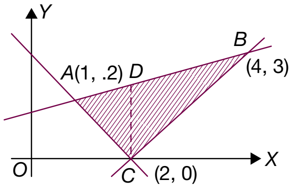


Required Area = shaded area of square

$A = (1/2)(9/2 - 3/2)(3) = 9/2 \Rightarrow 2A = 9$

5. The curves are three straight lines. These lines are drawn in fig. The required region is ABC.

Area of region ABC = Area of region ACD + Area of region BCD. Hence,



$$\text{Required Area} = \int_1^2 \left[\frac{x+5}{3} - (4 - 2x)\right]dx + \int_2^4 \left[\frac{x+5}{3} - \frac{3x-6}{2}\right]dx$$

$$= \frac{7}{3}\int_1^2 (x-1)\,dx + \frac{7}{6}\int_2^4 (-x+4)\,dx$$

$$= \frac{7}{3}\left[\frac{1}{2}x^2 - x\right]_1^2 + \frac{7}{6}\left[\frac{1}{2}x^2 + 4x\right]_2^4$$

$$= \frac{7}{3}\left[(2-2) - \left(\frac{1}{2} - 1\right)\right] + \frac{7}{6}[(-8+16) - (-2+8)]$$

$$= \frac{7}{6} + \frac{7}{3} = \frac{7}{2} \text{ square units}$$

6. $(x^2 - y^2)(x^2 - y^2 - 8x + 16) = 0$

$(x^2 - y^2)[(x-4)^2 - y^2] = 0$

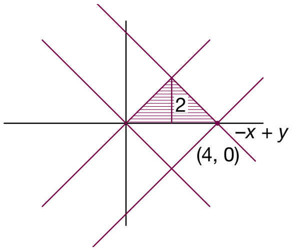
$y = \pm x$

$y = \pm(x - 4)$

$y = x - 4$ or $y = -x + 4$

Area $\frac{1}{2} \times 2 \times 4$

Total area = 2 × 4 = 8 square units.

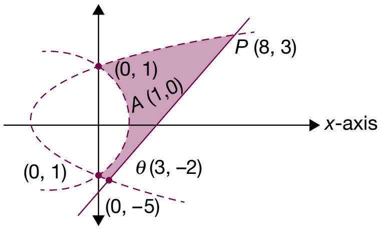


7. Given equation of curve is $x = |y - 1|$

$\Rightarrow x = (y^2 - 1)$ for $y \le 1$ and $y \ge 1$ ...(i)

$= -(y^2 - 1)$ for $-1 \le y \le 1$ ...(ii)

and $y = x - 5$ ...(iii)



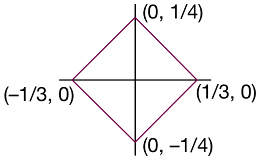
$y = x - 5$ intersect x-axis of (5, 0) and y-axis of (0, 5). Now, parabola (i) and line (iii) intersect of (8, 3) and (3, –2).

$$\text{Required area bounded} = \int_{-2}^{3}(y-5)\,dy - \int_{-2}^{3}|y^2-1|\,dy = \frac{109}{6}\text{ sq. units}$$

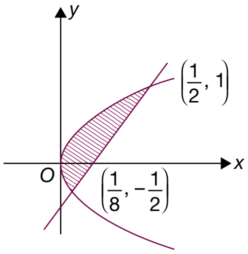
8. $3|x| + 4|y| \le 1$

$3x + 4y = 1$

$$\text{Required area } = 4\left(\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\right) = \frac{1}{6}$$

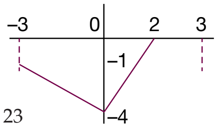


9. $$A = \int_{-1/2}^{1}\left(\left(\frac{y+1}{4}\right) - \frac{y^2}{2}\right)dy = \frac{9}{32}$$


y
x
O
$\left(\frac{1}{2}, 1\right)$
$\left(\frac{1}{8}, -\frac{1}{2}\right)$

10. $$f|x| = \begin{cases} -2 & ; \ -3 \le x \le 0 \\ x-2; & 0 < x \le 3 \end{cases}; \ |f(x)| = \begin{cases} 2 & ; \ -3 \le x \le 0 \\ -x+2; & 0 < x \le 2 \\ x-2 & ; \ 0 < x \le 3 \end{cases}$$

$$f|x| - |f(x)| = \begin{cases} -x-4 & -3 \le x \le 0 \\ 2x-4 & 0 < x \le 2 \\ 0 & 2 < x \le 3 \end{cases}$$



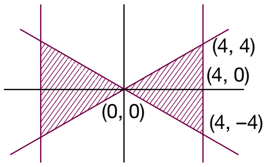
$$\text{Required area} = \frac{1}{2}(1+4)\times 3 + \frac{1}{2}\times 2\times 4 + 0 = \frac{23}{2}$$

11. $$\text{Required area} = -\int_{0}^{\beta} f(x)dx = \beta\sin\beta + 2$$

$\Rightarrow$ $f(\beta) = -\beta\cos\beta - \sin\beta$

Hence, $f\left(\frac{\pi}{2}\right) = -1$

12.



Required area = 32 sq. units

13. $$\int_{0}^{a} f(x)dx = \frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$$

(Differentiation w.r.t a)

$$\therefore f(a) = a + \frac{a}{2}\cos a + \frac{1}{2}\sin a - \frac{\pi}{2}\sin a$$

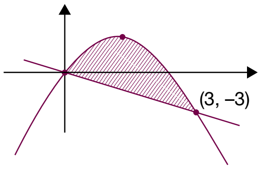
$$f(\pi) = \frac{\pi}{2};\ f(-\pi) = \frac{-\pi}{2};\ f\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

$$\Rightarrow\ f(\pi) + f(-\pi) + f\left(\frac{\pi}{2}\right) = \frac{1}{2}$$

Test Your Concepts-III

1. Given curves $y = 2x - x^2$ and $y = -x$

$\Rightarrow$ $-x = 2x - x^2 \Rightarrow x = 0, x = 3$

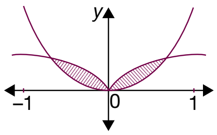


$$A = \int_{0}^{3}(2x - x^2 + x)dx$$

$$A = \int_{0}^{3}(3x - x^2)dx = \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)_0^3 = 27/2 - 27/3 - 0 = 9/2$$

2. Curves $y = x^2$ and $y =$ meet at point (0, 0), (1, 1), (–1, 1)

$$\text{Required area} = 2\int_{0}^{1}\sqrt{x} - x^2\,dx$$

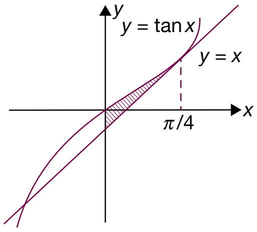


$$= 2\left(\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right)_0^1 = 2\left(\frac{2}{3} - \frac{1}{3}\right) = \frac{2}{3}$$

3. Curve $y = \tan x$

At $x = \pi/4 \Rightarrow y = 1$

$$\frac{dy}{dx} = \sec^2 x \Rightarrow \left(\frac{dy}{dx}\right)_{(\pi/4-1)} = 2$$



Hence, tangent at $\left(\frac{\pi}{4}, 1\right)$ is

$$(y-1) = 2\left(x - \frac{\pi}{4}\right) \Rightarrow 2x - y = \pi/2 - 1$$

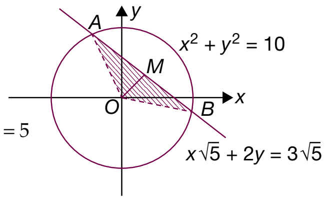
$$\text{Required area} = \int_{0}^{\pi/4}\tan x\,dx - \int_{0}^{\pi/4}(2x + 1 - \pi/2)dx$$

$$= \left(\log|\sec x|\right)_0^{\pi/4} - \left(x^2 + x - \pi x/2\right)_0^{\pi/4} = \log\sqrt{2} - \left(\frac{\pi}{4} - \frac{\pi^2}{16}\right)$$

$$= \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\log 2 \text{ sq units}$$

4. We want to find the area of DOAB for this, we draw a perpendicular from (0, 0) the given line

$$\Rightarrow\ OM = \left|\frac{-3\sqrt{5}}{\sqrt{5+4}}\right| = \sqrt{5}$$



Given that $OB = \sqrt{10}$

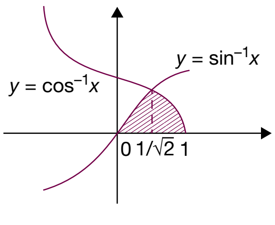
$(MB)^2 = (OB)^2 - (OM)^2 = 10 - 5 = 5$

$\Rightarrow$ $MB = \sqrt{5}$

$\Rightarrow$ $AB = 2MB = 2\sqrt{5}$

$$\text{Area of } \Delta OAB = \frac{1}{2}.AB.OM = \frac{1}{2}.2\sqrt{5}\,.\,\sqrt{5} = 5$$

5. Given curves $y = \cos^{-1}x$, $y = \sin^{-1}x$, x-axis, $0 \le x \le 1$



$$\text{Required area} = \int_{0}^{1/\sqrt{2}}\sin^{-1}x\,dx + \int_{1/\sqrt{2}}^{1}\cos^{-1}x\,dx$$

$$= \left[x\sin^{-1}x\right]_0^{1/\sqrt{2}} - \int_{0}^{1/\sqrt{2}}\frac{x}{\sqrt{1-x^2}}dx + \left[x\cos^{-1}x\right]_{1/\sqrt{2}}^{1}$$

$$+ \int_{1/\sqrt{2}}^{1}\frac{x}{\sqrt{1-x^2}}dx$$

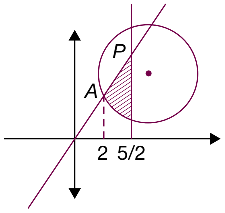
$$= \left(\frac{\pi}{4\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right) - \int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^2}}dx + \int_{1/\sqrt{2}}^{1} \frac{x}{\sqrt{1-x^2}}dx$$

$$= -\int_0^{1/\sqrt{2}} \frac{x}{\sqrt{1-x^2}}dx + \int_{1/\sqrt{2}}^{1} \frac{x}{\sqrt{1-x^2}}dx$$

$$= \sqrt{1-x^2}\Big|_0^{1/\sqrt{2}} - \sqrt{1-x^2}\Big|_{1/\sqrt{2}}^{1} = (\sqrt{2} - 1) \text{ sq. unit.}$$

6. Given region $x^2 + y^2 - 6x - 4y + 12 \le 0$,

$y \le x$, and $x \le \dfrac{5}{2}$



Curves $x^2 + y^2 - 6x - 4y + 12 = 0$ and $y = x$ meet at $x = 2, 3$

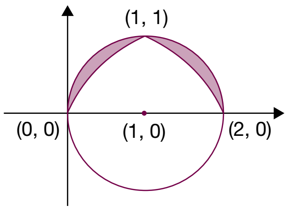
Circle $(x-3)^2 + (y-2)^2 = 1$

$$\text{Required area} = \int_2^{5/2} \left[x - \left(2 - \sqrt{1-(x-3)^2}\right)\right]dx$$

$$= \left[\frac{x^2}{2} - 2x + \frac{x-3}{2}\sqrt{1-(x-3)^2} + \frac{1}{2}\sin^{-1}(x-3)\right]_2^{5/2}$$

$$= \frac{25}{8} - 5 - \frac{\sqrt{3}}{8} - \frac{\pi}{12} - 2 + 4 - \frac{\pi}{4} = \frac{1}{8} - \frac{\sqrt{3}}{8} + \frac{\pi}{6} \text{ sq units}$$

7.



$x^2 + y^2 - 2x = 0 \Rightarrow (x-1)^2 + y^2 = 1$ and $y = \sin\left(\dfrac{\pi x}{2}\right)$

$$\text{Required Area} = \frac{\pi}{2} - \int_0^2 \sin\frac{\pi x}{2}dx$$

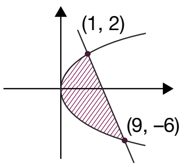
$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\cos\frac{\pi x}{2}\right]_0^2 = \frac{\pi}{2} + \frac{2}{\pi}[-1-1] = \left(\frac{\pi}{2} - \frac{4}{\pi}\right)\text{sq. unit}$$

8. End point of latus rectum P (1, 2)

$2yy' = 4$; Slope of tangent at $P = 1$

Slope of normal $= -1$

Normal $y - 2 = -1(x-1) \quad \Rightarrow \quad x = 3 - y$

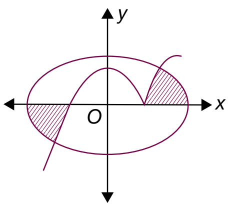


$$A = \int_{-6}^{2} (3-y)dy - \int_{-6}^{2} \frac{y^2}{4}dy$$

$$A = 24 + 16 - \frac{56}{3} = \frac{64}{3}$$

9. C_1: $\dfrac{x^2}{\pi^2} + \dfrac{y^2}{4} = 1$

$$\text{And curve } C_2{}^{\circ}\ y = \begin{cases} -\cos x; & x > \dfrac{\pi}{2} \\ \cos x; & x < \dfrac{\pi}{2} \end{cases}$$



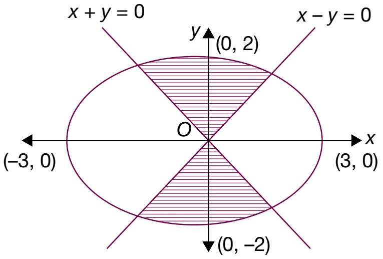
Shaded Areas are equal

$\therefore$ Required ratio $= \dfrac{\pi^2 - 2}{\pi^2 + 2}$

10. $\dfrac{x^2}{9} + \dfrac{y^2}{4} \le 1$ and $x^2 - y^2 < 0$

$$\text{Required area} = 4\int_0^{\frac{6}{\sqrt{3}}} \left(\frac{2}{3}\sqrt{9-x^2} - x\right)dx = 12\sin^{-1}\frac{2}{\sqrt{13}}$$

$a + b + c = 27$



11. Curve $y = ax^2 + bx + c$

$dy/dx = 2ax + b$

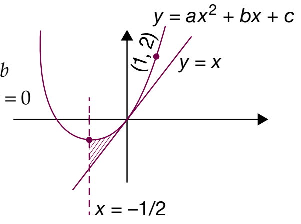
Slope of tangent at origin, $m = 1 = b$

As origin lies on curve, we have $c = 0$ and it passes through point (1, 2)

$2 = a + b$

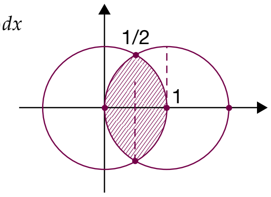
We get curve as $y = x^2 + x$

$x = -b/2a = -1/2$



$$\text{Required area} = \int_{-1/2}^{0} (x^2 + x) - x dx = \left[\frac{x^3}{3}\right]_{-1/2}^{0} = 1/24 \text{ sq. units}$$

12. Required shaded area $= 4\displaystyle\int_{1/2}^{1} (\sqrt{1-x^2})dx$



For $x = \sin\theta$; $dx = \cos\theta\, d\theta$

$$\text{Area} = 4\int_{\pi/6}^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2\int_{\pi/6}^{\pi/2} 1 + \cos 2\theta\, d\theta = 2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/6}^{\pi/2}$$

$$= (2\theta + \sin 2\theta)_{\pi/6}^{\pi/2} = \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \text{ sq. units}$$

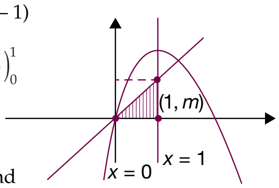
13. $y = 2x - x^2 - 1 + 1$

$-(y-1) = -(x-1)^2 \Rightarrow (x-1)^2 = -(y-1)$

$$\text{Area } A = \int_0^1 (2x - x^2)dx = \left(x^2 - x^3/3\right)_0^1$$

$\Rightarrow \quad A = 1 - 1/3 = 2/3$

$\Rightarrow \quad A/2 = 1/3$

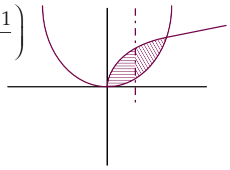


Area enclosed by line, $x = 0$, $x = 1$ and x-axis $= (1/2)(1)(m) = 1/3 \quad \Rightarrow \quad m = 2/3$

Required line is $y = (2/3)\,x$

14. $A_1 = \displaystyle\int_0^{1/2} \sqrt{x} - x^2 = \frac{2x^{3/2}}{3} - \frac{x^3}{3}\Big|_0^{1/2}$

$$= \frac{1}{3}\left[\frac{2}{2\sqrt{2}} - \frac{1}{8}\right] = \frac{1\left(4\sqrt{2}-1\right)}{3\times 8} = \left(\frac{4\sqrt{2}-1}{24}\right)$$

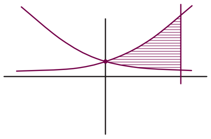


$$A_2 = \int_{1/2}^{1} \sqrt{x} - x^2 = \frac{2x^{3/2}}{3} - \frac{x^3}{3}\Big|_0^{1/2}$$

$$= \frac{2x^{3/2}}{3} - \frac{x^3}{3}\Bigg|_{1/2}^{1}$$

$$= \left[\frac{2}{3} - \frac{1}{3}\right] - \left[\frac{4\sqrt{2}-1}{24}\right] = \frac{1}{3} - \frac{\left(4\sqrt{2}-1\right)}{24} = \frac{8-4\sqrt{2}+1}{24}$$

15. $\displaystyle\int_0^5 \left(e^x - e^{-x}\right)dx$



$$= e^x + e^{-x}\Big|_0^5 = e^5 + e^{-5} - 2 \text{ sq. units}$$

16. The curve consist of a loop lying between the line $x = 0$ and $x = 2a$ and is symmetrical about the x-axis.

The required area $= 2\displaystyle\int_0^{2a} y\,dx = \frac{2}{a^2}\int_0^{2a} x^{5/2}\sqrt{(2a-x)}\,dx$

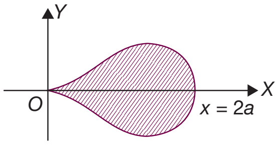
To evaluate this integral, we put $x = 2a\sin^2\theta$

When $x = 0$, $\theta = 0$ and when $x = 2a$, $\theta = \frac{1}{2}\pi$

$\therefore$ The required area $= \displaystyle\frac{2}{a^2}\int_0^{\pi/2} (2a)^{5/2}\sin^5\theta.\sqrt{2a}\,.\cos\theta.4a\sin\theta\cos\theta\,d\theta$

$$= 64a^2\int_0^{\pi/2} \sin^6\theta\cos^2\theta\,d\theta$$

$$= 64a^2\frac{5.3.1.1}{8.6.4.2}.\frac{\pi}{2} = \frac{5a^2\pi}{4}$$



$= \frac{5}{4} \times$ area of the circle whose radius is a.

17. The curve will intersect the x-axis when $y = 0$

$\Rightarrow$ $2 - 3x - 2x^2 = 0$ or $(2 + x)(1 - 2x) = 0$

or $x = -2$, $x = \frac{1}{2}$

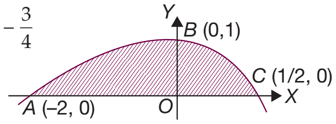
Thus, the curve passes through the points $(-2, 0)$ and $(\frac{1}{2}, 0)$ on the x-axis.

It will have a turning point where $\frac{dy}{dx} = 0$

$\therefore$ $\frac{dy}{dx} = \frac{1}{2}(-3 - 4x) = 0 \Rightarrow x = -\frac{3}{4}$

Also, $\frac{d^2y}{dx^2} = -4$.

That is, it is a max. at $x = -\frac{3}{4}$



Also, it cuts y-axis where $x = 0$. Then $y = 1$

Thus, the shape of the curve is as shown in the figure.

The required area is ABC. It is given by

$$\int_{-2}^{1/2} y\,dx = \int_{-2}^{1/2} \frac{1}{2}(2 - 3x - 2x^2)dx$$

$$= \frac{1}{2}\left[2x - \frac{3}{2}x^2 - \frac{2x^3}{3}\right]_{-2}^{1/2}$$

$$= \frac{1}{2}\left[2\left(\frac{1}{2}\right) - \frac{3}{2}\left(\frac{1}{2}\right)^2 - \frac{2}{3}\left(\frac{1}{2}\right)^3\right] - \frac{1}{2}\left[2(-2) - \frac{3}{2}(-2)^2 - \frac{2}{3}(-2)^3\right]$$

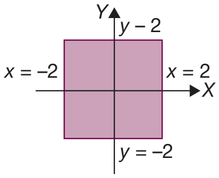
$$= \frac{1}{2}\left(\frac{13}{24}\right) - \frac{1}{2}\left(-\frac{14}{3}\right) = \frac{125}{48} \text{ sq. units.}$$

18. $\displaystyle\frac{|x|+|y|}{2} + \frac{|x|-|y|}{2} \le 2$

$\Rightarrow \big||x| - |y|\big| \le 4 - (|x| + |y|)$ (Here $|x| + |y| \le 4$)

$\Rightarrow |x| + |y| - 4 \le |x| - |y| \le 4 - (|x| + |y|)$

$\Rightarrow |x| \le 2$ and $|x| + |y| \le 2$



So, that the region satisfying $|x| \le 2$, $|y| \le 2$ and $|x| + |y| \le 4$ is shown in diagram

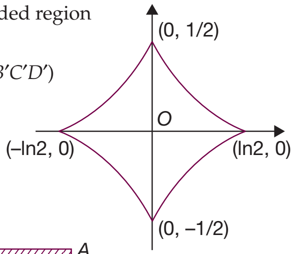
Also, curve $|y| = e^{-|x|} - \frac{1}{2}$ is symmetric about both x and y-axis. So, that the curve is as shown

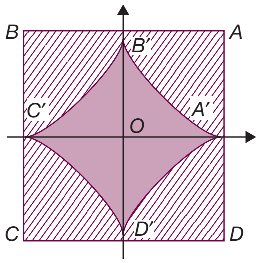
So, that the required region is shaded region

= (Area of square $ABCD$) –
(Area of curve $A'B'C'D'$)

$$= 16 - 4\int_0^{\ln^2}\left(e^{-x} - \frac{1}{2}\right)dx$$

$= 14 + \ln 4$ sq. units





Test Your Concepts-IV

1. $x = a\cos^3 t$ $y = a\sin^3 t$ where $t \in [0, 2\pi]$ as both are periodic with 2π

Curve $x^{2/3} + y^{2/3} = a^{2/3}(\cos^2 t + \sin^2 t) = a^{2/3}$

$y = (a^{2/3} - x^{2/3})^{3/2}$ symmetric about x and y-axis

Required area is 4 times the area enclosed in first quadrant

x is decreasing when $0 < t < \pi/2$

Required area $= -4\displaystyle\int_0^{\pi/2} y\,dx = -4\int_0^{\pi/2} y\,dx/dt.dt$

$$= -4\int_0^{\pi/2} a\sin^3 t\; 3a\cos^2 t\,(-\sin t)\,dt$$

$= 12a^2\displaystyle\int_0^{\pi/2} \sin^4 t\cos^2 t\,dt$ [Using Walli’s formula]

$$= 12a^2\left(\frac{3.1.1}{6.4.2}\frac{\pi}{2}\right) = \frac{3\pi a^2}{8}$$

2. Required area $= \displaystyle\int_0^{2\pi} y\left(\frac{dx}{d\theta}\right)d\theta = \int_0^{2\pi} a(1-\cos\theta)a(1-\cos\theta)\,d\theta$

$$= a^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = a^2\int_0^{2\pi} 1 - 2\cos\theta + \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{\sin 2\theta}{4}\right]_0^{2\pi} = 3\pi a^2$$

3. Clearly t can be any real number

Let $t = \tan\theta \Rightarrow x = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \sin 2\theta \Rightarrow x^2 + y^2 = 1$

Thus, required area is π sq. units.

4. Required area $A = \int_0^1 y dx = \int_0^1 (a^2x^2 + ax + 1)dx = \frac{a^2}{3} + \frac{a}{2} + 1$

$$\frac{dA}{da} = \frac{2a}{3} + \frac{1}{2} = 0; \text{ We get } a = -\frac{3}{4}$$

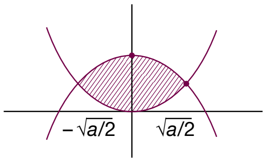
$$\frac{d^2A}{da^2} = \frac{2}{3} > 0 \quad \Rightarrow \quad a = -\frac{3}{4} \text{ Point of minima}$$

5. Solving both given curves $y = a - x^2$ and $y = x^2$

$$x^2 = a - x^2 \Rightarrow 2x^2 = a$$

$$x = \pm\sqrt{a/2}$$

Total area enclosed between both curves is $18\sqrt{2}$



$$\therefore \quad 9\sqrt{2} = \int_0^{\sqrt{a/2}} (a - x^2) - (x^2)dx$$

$$\Rightarrow \quad 9\sqrt{2} = 2/3\sqrt{a/2} \Rightarrow 27 = a\sqrt{a} \Rightarrow a^{3/2} = 27 \Rightarrow a = 9$$

6. $y = x^n \Rightarrow dy/dx = nx^{n-1}dx$

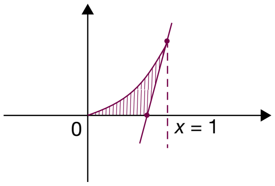
At point (1, 1)

Slope of tangent= $dy/dx = n$

$\Rightarrow \quad y - 1 = n(x - 1)$

$\Rightarrow \quad y = 1 + n(x - 1)$

For $y = 0, x = 1 - 1/n$



Required Area $A = \int_0^1 x^n \, dx - \frac{1}{2}.1.\left(\frac{1}{n}\right) = \frac{1}{n+1} - \frac{1}{2n}$

For max area $\frac{dA}{dn} = -\frac{1}{(n+1)^2} + \frac{1}{2n^2} = 0$

$$\Rightarrow \quad 2n^2 = (n + 1)^2 \Rightarrow 2n^2 = n^2 + 1 + 2n$$

$$\Rightarrow \quad n^2 - 2n - 1 = 0 \Rightarrow n = \frac{2 + \sqrt{8}}{2} = 1 + \sqrt{2}$$

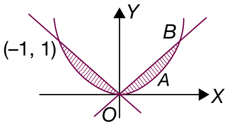
$\frac{d^2A}{dn^2} = \frac{2}{(n+1)^3} - \frac{1}{n^3} < 0$ at $n = 1 + \sqrt{2}$ it means it a point of maxima

7. The curves intersect each other at $x = 0$ and $x = \pm 1$ as shown in figure. The points of intersection are (–1, 1), (0, 0) and (1, 1).

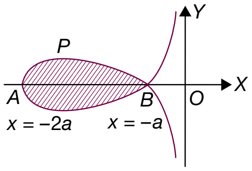
Since, the region is symmetric about y-axis, the required area is 2[area of region $OABO$] Hence,

$$\text{Area} = 2\int_0^1 (x - x^2)dx = 2\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$

$$= \frac{1}{3} \text{sq. units}$$



8. Here let us first of all trace the curve. The equation can be put in the form $y^2 = -\{(x + a)^2 (x + 2a)\}/x$. We notice that $y = 0$ at the points B and A where $x = -a$ and $x = -2a$, and $y \to \pm \infty$ when $x \to 0$. For positive values of x, and also for x less than $-2a$, y^2 is negative and so y is imaginary. Therefore graph will not exit for $x > 0$ and $x < -2a$. From A to B, for each value of x, y has two equal and opposite finite values and a loop is thus formed within this range, symmetrical about the x-axis. From B to O, each value of x gives two equal and opposite values of y which gradually increase in magnitude to ∞ as x approaches 0. The curve, therefore, is a shown in the figure



The required area of the loop = 2. area APB

$$= 2\int_{-2a}^{-a} y\,dx = 2\int_{-2a}^{-a} \sqrt{-\frac{(x+a)^2(x+2a)}{x}}\, dxv$$

Put $z = x + 2a$

$$\text{Area} = 2\int_0^a (a - z)\sqrt{\frac{z}{2a - z}}\, dz$$

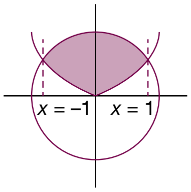
$$= 2\int_0^{\pi/2} \text{a cos } \theta. \frac{\sin\frac{1}{2}\theta}{\cos\frac{1}{2}\theta} 2a \sin\frac{\theta}{2}\cos\frac{\theta}{2} d\theta \qquad \left[\text{putting } z = 2a\sin^2\frac{\theta}{2}\right]$$

$$= 2a^2 \int_0^{\pi/2} \cos\theta\,(1 - \cos\theta) d\theta = 2a^2\left(1 - \frac{\pi}{4}\right)$$

$$= \frac{1}{2}a^2(4 - \pi).$$

9. $\because \quad \lim_{x\to 0}\left\{1 + \frac{f(x)}{x^3}\right\}^{1/x} = e$ exists

$$\Rightarrow \quad \underset{(x\to 0)}{\frac{f(x)}{x}} = 0$$



Thus, $f(x) = a_4x^4 + a_5x^5 ..... + a_nx^n \;\; n \geq 4$

$$\because \quad \lim_{x\to 0}\left\{1 + \frac{f(x)}{x^3}\right\}^{1/x} = e \quad \Rightarrow e^{a_4} = e \Rightarrow a_4 = 1 \Rightarrow f(x) = x^4$$

Now, required area $= 2\int_0^1 \left(\sqrt{x - x^2} - x^4\right) dx$

$$= 2\left[\frac{x}{2}\sqrt{2 - x^2} + \sin^{-1}\frac{x}{\sqrt{2}} - \frac{x^5}{5}\right]_0^1 = \left(\frac{\pi}{2} + \frac{3}{5}\right)\text{sq. unit}$$

10. The equation can be put in the form

$$by^2 + 2hxy + (ax^2 - 1) = 0$$

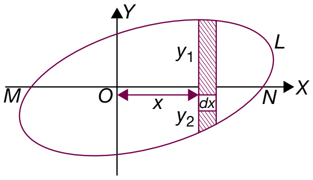
Cut an elementary strip.

Let the thickness of strip = dx.

If y_1, y_2 be the values of y corresponding to any value of x,

Length of strip = $y_1 - y_2$

$$= \frac{2}{b}\sqrt{h^2x^2 - b(ax^2 - 1)} = \frac{2}{b}\sqrt{b - (ab - h^2)x^2}$$



$ab - h^2$ being positive here, since the conic is an ellipse.

The extreme values of x, where the ordinates touch the ellipse, are given by

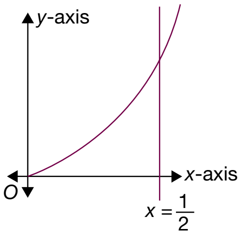
$$y_1 - y_2 = 0, \text{ i.e., } x = \pm\sqrt{\frac{b}{ab - h^2}}$$

Hence, the area required $= \int_{-\sqrt{b/(ab-h^2)}}^{\sqrt{b/(ab-h^2)}} (y_1 - y_2)dx$

$$= \int_{-\sqrt{b/(ab-h^2)}}^{\sqrt{b/(ab-h^2)}} \frac{2}{b}\sqrt{b - (ab - h^2)x^2}\, dx \qquad \text{put } \sqrt{(ab - h^2)}\text{x} = \sqrt{b}\sin\theta$$

$$= \frac{2}{\sqrt{(ab - h^2)}}\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = \frac{\pi}{\sqrt{(ab - h^2)}}$$

11. $A_n = \int_0^{1/2} x^n dx \; A_n = \frac{1}{2^{n+1}(n+1)}$



$\therefore \quad \frac{2^n A_n}{n} = \frac{1}{2(n+1)n}$

Hence, $\sum_{n=1}^{n} \frac{2^n A_n}{n}$

$$= \frac{1}{2}\sum_{n=1}^{n}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{1}{2}\sum_{n=1}^{n}\left(1 - \frac{1}{n+1}\right)$$

Given $\frac{1}{2}\left(1 - \frac{1}{n+1}\right) = \frac{1}{3}$

$$1 - \frac{2}{3} = \frac{1}{n+1};\; \frac{1}{3} = \frac{1}{n+1} \Rightarrow n = 2$$

12. Required Area $= \int_{\pi/6a}^{\pi/2a} \cos(ax)dx - \int_{\pi/2a}^{5\pi/6a} \cos(ax)dx$

$$= \left(\frac{\sin ax}{a}\right)_{\pi/6a}^{\pi/2a} - \left(\frac{\sin ax}{a}\right)_{\frac{\pi}{2a}}^{\frac{5\pi}{6a}}$$

$$= \left(1 - \frac{1}{2}\right)\frac{1}{a} - \left(\frac{1}{2} - 1\right)\frac{1}{a} = \frac{1}{a} > 3 \quad \Rightarrow \quad 0 < a < \frac{1}{3}$$

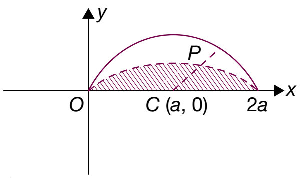
Single Correct Choice Type Questions

1. $y = \sqrt{2ax - x^2} = \sqrt{a^2 - (x-a)^2}$

$(x-a)^2 + y^2 = a^2$ Circle

Let, $P(x_1, y_1)$ lying inside the curve

$r - PC > y_1$



According to given condition,

$\Rightarrow \quad a - \sqrt{(x-a)^2 + y^2} > y_1$

$\Rightarrow \quad a - y_1 > \sqrt{(x-a)^2 + y^2}$

$\Rightarrow \quad a^2 + y_1^2 - 2ay > x^2 + a^2 - 2ax + y^2$

$\Rightarrow \quad -2ay > x^2 - 2ax$

$y < -\frac{1}{2a}(x^2 - 2ax)$ it is a Parabola.

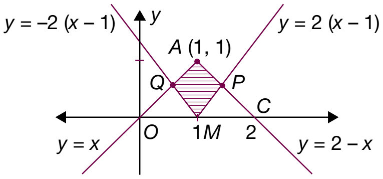
Required area $= -\int_0^{2a} \frac{1}{2a}(x^2 - 2ax)dx = \int_0^{2a} -\frac{x^2}{2a} + x\,dx$

$$= -\frac{x^3}{6a} + \frac{x^2}{2}\Big|_0^{2a} = -\frac{8a^3}{6a} + \frac{4a^2}{2} = -\frac{4}{3}a^2 + 2a^2 = \frac{2a^2}{3}$$

Hence, the correct answer is (C).

2. $0 \le a \le 4$

For maximum area bounded by given curves $y = 1 - |x - 1|$ and $y = |2x - a|$



Then $\frac{a}{2} = 1$ or $a = 2$

Now, second curve is $y = 2|x - 1|$

Curve $y = x$ and $y = -2x + 2$ meeting at Q (2/3,2/3)

Curve $y = 2x - 2$ and $y = 2 - x$ meeting at point P (4/3,2/3)

Shaded area = area of ΔOAC – area of ΔOAM – Area of ΔMPC

$$= \frac{1}{2}(2) \times 1 - \left(\frac{1}{2}(1)\frac{2}{3} + \left(\frac{1}{2}\cdot 1 \times \frac{2}{3}\right)\right)$$

$$= 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$$

Hence, the correct answer is (A).

3. $f(x) = ||\tan x + \cot x| - |\tan x - \cot x||$

$f(x) = |\tan x + \cot x - |\tan x - \cot x||$ where $x \in \left(0, \frac{\pi}{2}\right)$

If $x \in \left(0, \frac{\pi}{4}\right]$ then $\cot x > \tan x$

If $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ then $\cot x < \tan x$

$$f(x) = \begin{cases} 2|\tan x| = 2\tan x; & x \in \left(0, \frac{\pi}{4}\right] \\ 2|\cot x| = 2\cot x; & x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \end{cases}$$

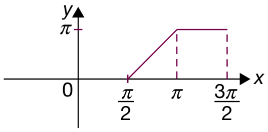
Required Area $= \int_0^{\pi/4} 2\tan x\,dx + \int_{\pi/4}^{\pi/2} 2\cot x\,dx$

$$= 2\ln \sec x\Big|_0^{\pi/4} + 2\ln \sin x\Big|_{\pi/4}^{\pi/2}$$

$$= 2(\ln\sqrt{2} - \ln 1) + 2\left(\ln 1 - \ln\frac{1}{\sqrt{2}}\right) = 4\ln\sqrt{2} = 2\ln 2 = \ln 4$$

Hence, the correct answer is (A).

4. $f(x) = \tan^{-1}(\tan x) + \cos^{-1}(\cos x)$



$$f(x) = \begin{cases} x - \pi + x & ; \; x \in \left(\frac{\pi}{2}, \pi\right) \\ x - \pi + 2\pi - x; & x \in \left(\pi, \frac{3\pi}{2}\right) \end{cases} = \begin{cases} 2x - \pi; & x \in \left(\frac{\pi}{2}, \pi\right) \\ \pi & ; \; x \in \left(\pi, \frac{3\pi}{2}\right) \end{cases}$$

Required area $= \frac{1}{2}\left(\frac{\pi}{2} + \pi\right)\pi = \frac{3\pi^2}{4}$

Hence, the correct answer is (B).

5. $f : [0, \infty) \longrightarrow [0, \infty)$

According to question $\int_0^t f(x)dx = \frac{1}{2}\cdot\left(\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ t & 0 & 1 \\ t & f(t) & 1 \end{vmatrix}\right)$

$$\int_0^t f(x)dx = \frac{1}{4}(tf(t))$$

$\Rightarrow \quad f(t) = \frac{1}{4}(f(t) + tf'(t))$

$\Rightarrow \quad 3f(t) = tf'(t)$

Let $y = f(t) \Rightarrow 3y = t\frac{dy}{dt} \Rightarrow \int 3\frac{dt}{t} = \int \frac{dy}{y}$

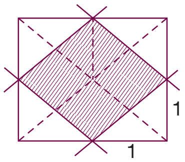
$\Rightarrow \quad 3\ln t = \ln y + \ln k \Rightarrow t^3 = ky$

$f(t) = \frac{1}{k}t^3$

$\Rightarrow \quad f(x) = x^3$ if $k = 1$

Hence, the correct answer is (C).

6. Side of square is 2
Shaded region the area where point P is nearer to point of intersection of diagonals than any of vertex = it is a square of side $\sqrt{2}$



Required area $= 4 - \left(\sqrt{2}\right)^2 = 2$

Hence, the correct answer is (B).

7. $(a, 0) : a > 0$

$$y = \sin 2x - \sqrt{3}\sin x$$

$$y = \sin x\left(2\cos x - \sqrt{3}\right) = 0 \text{ at } x = \frac{\pi}{6} = a$$

$$A = \int_0^a \sin 2x - \sqrt{3}\sin x\, dx$$

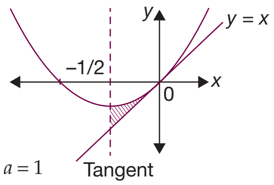
$$A = -\frac{\cos 2x}{2} + \sqrt{3}\cos x\Big|_0^a \text{ when } a = \frac{\pi}{6}$$

$$A = -\frac{1}{4} + \sqrt{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2} - \sqrt{3} = \frac{7}{4} - \sqrt{3}$$

$$4A = 7 - 4\sqrt{3} \quad\Rightarrow\quad 4A + 4\sqrt{3} = 7 \quad\Rightarrow\quad 4A + 8\cos a = 7$$

Hence, the correct answer is (A).

8. $y = ax^2 + bx + c$, origin lies on it. $c = 0$



Tangent at origin $\frac{1}{2}(y + 0)$

$= ax(0) + \frac{b}{2}(x + 0) \Rightarrow y = bx$, then $b = 1$

Point (1, 2) lies on it $2 = a + b$. we get $a = 1$

Curve $y = x^2 + x$

$$\text{Required area} = \int_{-1/2}^{0} (x^2 + x) - x\ \ dx = \frac{x^3}{3}\Big|_{-1/2}^{0} = \frac{1}{24}$$

Hence, the correct answer is (A).

9. $\int_0^{x_1} f(x)dx = x_1 e^{x_1}$

$$f(x_1) - 0 = x_1 \cdot e^{x_1} + e^{x_1}$$

$$f(x) = xe^x + e^x$$

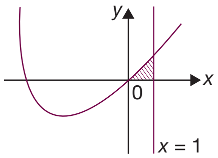
Hence, the correct answer is (D).

10. $\frac{dy}{dx} = 2x + 1 \Rightarrow dy = 2x + 1\, dx \quad\Rightarrow\quad y = x^2 + x + c$

Cure passes through $(1, 2) \Rightarrow 2 = 1 + 1 + c$

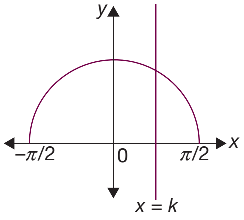
$c = 0$

$$\int_0^1 x^2 + x\, dx = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$



Hence, the correct answer is (A).

11. $\int_{-\pi/2}^{k} \cos x\ \ dx = 3\int_{k}^{\pi/2} \cos x\, dx$



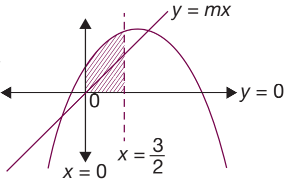
$$\sin x \int_{-\pi/2}^{k} = 3\sin x\Big|_k^{\pi/2}$$

$\Rightarrow$ $\sin k - (-1) = 3(1 - \sin k)$
$\Rightarrow$ $\sin k + 1 = 3 - 3\sin k$
$\Rightarrow$ $4\sin k = 2$
$\Rightarrow$ $\sin k = \frac{1}{2} \Rightarrow k = \frac{\pi}{6}$

Hence, the correct answer is (C).

12. $y = 1 + 4x - x^2$ has vertex (2, 5)

According to given condition



$$\frac{1}{2}\int_0^{3/2} (1 + 4x - x^2)\, dx = \int_0^{3/2} mx\ dx$$

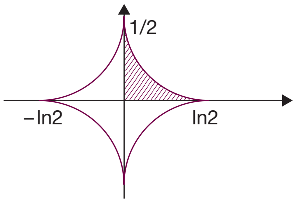
$$x + 2x^2 - \frac{x^3}{3}\Big|_0^{3/2} = 2m\frac{x^2}{2}\Big|_0^{3/2}$$

$$\frac{3}{2} + \frac{9}{2} - \frac{27}{24} = m\,\frac{9}{4} \quad\Rightarrow\quad \frac{144 - 27}{24} = \frac{9m}{4}$$

$$\Rightarrow\quad 6 \times 9\, m = 117 \quad\Rightarrow\quad m = \frac{13}{6}$$

Hence, the correct answer is (A).

13. $|y| \le e^{-|x|} - \frac{1}{2}$;



$$\Rightarrow\quad y = e^{-|x|} - \frac{1}{2}$$

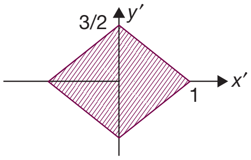
$$\Rightarrow\quad |y| = e^{-|x| - \frac{1}{2}}$$

$$\text{Required area} = 4\int_0^{\ln 2}\left(e^{-x} - \frac{1}{2}\right)dx$$

$$= 4\left(-e^{-x} - \frac{x}{2}\right)_0^{\ln 2} = 4\left(\frac{1}{2} - \frac{\ln 2}{2}\right) = 2(1 - \ln 2)$$

Hence, the correct answer is (D).

14. $|3x - 18| + |2y + 7| \le 3$



$$\Rightarrow\quad \frac{3|x - 6|}{3} + \frac{2\left|y + \frac{7}{2}\right|}{3} \le 1$$

$$\Rightarrow\quad |x - 6| + \frac{\left|y + \frac{7}{2}\right|}{3/2} \le 1$$

Area remains invariant under translation of co-ordinate axes

Let $x - 6 \equiv x'$ and let $y + \frac{7}{2} \equiv y'$ $\quad\Rightarrow\quad |x'| + \frac{|y'|}{3/2} \le 1$

Required shaded Area $= 4\left(\frac{1}{2}(1)\left(\frac{3}{2}\right)\right) = 3$

Hence, the correct answer is (A).

15. Required area is equal to $\int_0^{\pi/2} (\sin^2 x\cos^3 x - \sin^4 x\cos^3 x)dx$

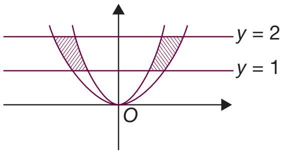
Hence, the correct answer is (B).

16. The curves $y = [a]x^2$ and $y = \frac{1}{2}[a]x^2$ represent parabolas which are symmetric about y-axis. The equation $y^2 - 3y + 2 = 0$ gives a pair of straight lines $y = 1$, $y = 2$ which are parallel to x-axis. The shaded region in figure determines the area bounded by the two parabolas and two lines. Let us slice this region into horizontal strips. For the approximating rectangle shown in figure. We have

Length $= x_2 - x_1$, width $= \Delta y$,

Area $= (x_2 - x_1)dy$

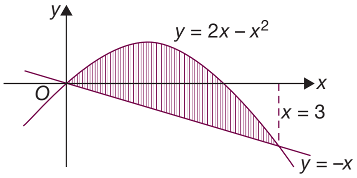
AT it can move vertically $y = 1$ and $y = 2$. So, Required area



$$= 2\int_1^2 (x_2 - x_1)dy$$

Hence, the correct answer is (B).

17. Curve $y = 2x - x^2$ and the straight line $y = -x$ meet at point (0,0) and (3,–3)



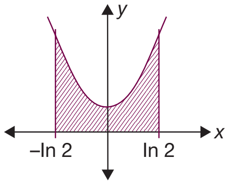
$$\text{Required Area} = \int_0^3 \left((2x - x^2) - (-x)\right) dx$$

$$= \int_0^3 (3x - x^2) dx = \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$$

Hence, the correct answer is (A).

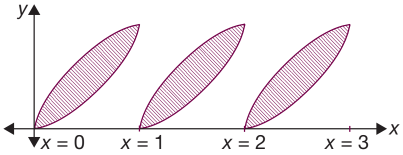
18. $A = 2 \cdot \frac{1}{2} \int_0^{\ln 2} (e^x + e^{-x}) dx = e^x - e^{-x} \Big|_0^{\ln 2}$

$$= e^{\ln 2} - e^{-\ln 2} = 2 - \frac{1}{2} = \frac{3}{2}.$$



Hence, the correct answer is (C).

19. Both functions are periodic with period 1.



$$\text{Hence, area} = 10\int_0^1 \left(\sqrt{x} - x^2\right) dx$$

$$= 10\left(\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3}\right)_0^1 = 10\left(\frac{2}{3} + \frac{-1}{3}\right) = \frac{10}{3}$$

Hence, the correct answer is (C).

20. We have $f^3(x) = \int_0^x t f^2(t)\, dt$

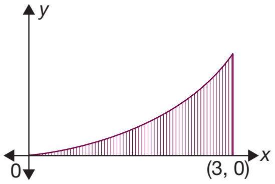
On differentiate with respect to x, we get

$\Rightarrow \ 3f^2(x) f'(x) = x f^2(x)$

$\Rightarrow \ f^2(x) \, [3f'(x) - x] = 0$

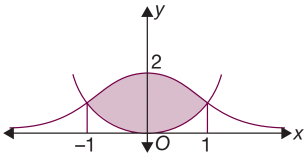
As, $f(x) \neq 0 \Rightarrow f'(x) = \frac{x}{3}; f(x) = \frac{x^2}{6} + C$

$$A = \frac{1}{6}\int_0^3 x^2 dx = \frac{x^3}{18}\bigg]_0^3 = \frac{3}{2}.$$



Hence, the correct answer is (B).

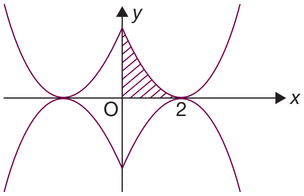
21. $\text{Required area} = \int_{-1}^{1} \left(\frac{2}{1+x^2} - x^2\right) dx = 2\int_0^1 \left(\frac{2}{1+x^2} - x^2\right) dx$



$$= \pi - \frac{2}{3} \ \Rightarrow \ \left[\pi - \frac{2}{3}\right] = 2$$

Hence, the correct answer is (A).

22.



$$\text{Required area} = 4\int_0^2 (x-2)^2 \, dx = \frac{32}{3}$$

Hence, the correct answer is (B).

23. $(y - \text{arc}\sin x)^2 = x - x^2$

$\Rightarrow \ y = \text{arc}\sin x \pm \sqrt{x - x^2}$

$$\text{Required area} = 2\int_0^1 \sqrt{x - x^2} dx = \frac{\pi}{4}$$

Hence, the correct answer is (A).

24. Since, $\int_0^a f(x) dx = \frac{a^2}{2} + \frac{a}{2}\sin a + \frac{\pi}{2}\cos a$

Differentiating both sides w.r.t. a, we get

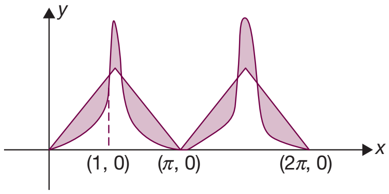
$$\Rightarrow \ f(a) = a + \frac{a}{2}\cos a + \frac{1}{2}\sin a - \frac{\pi}{2}\sin a$$

$$\therefore \ f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 0 + \frac{1}{2}.1 - \frac{\pi}{2}.1 = \frac{1}{2}$$

Hence, the correct answer is (A).

25. $y = \sin^{-1} |\sin x| = \begin{cases} x & ; \ 0 \leq x < \frac{\pi}{2} \\ \pi - x & ; \ \frac{\pi}{2} \leq x < \pi \\ x - \pi & ; \ \pi \leq x < \frac{3\pi}{2} \\ 2\pi - x & ; \ \frac{3\pi}{2} \leq x < 2\pi \end{cases}$

The required area A is shown shaded in the figure.



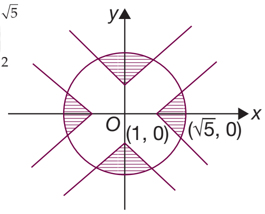
$$\Rightarrow \ 4\int_0^1 \left(x - x^2\right) dx + 4\int_1^{\pi/2} \left(x^2 - x\right) dx$$

$$= \frac{4}{3} + 4\left[\frac{\pi^3}{24} - \frac{\pi^2}{8}\right] = \frac{4}{3} + \left[\frac{\pi^3}{6} - \frac{\pi^2}{2}\right] = \frac{4}{3} + \pi^2\left(\frac{\pi - 3}{6}\right)$$

Hence, the correct answer is (D).

26. If $y = x - 1 \ \Rightarrow \ x^2 + (x-1)^2 = 5 \ \Rightarrow \ x = 2 \ (x \neq -1)$

$$\text{Required area} = 8\left(\frac{1}{2}.1.1 + \int_2^{\sqrt{5}} \sqrt{5 - x^2} dx\right)$$



$$= 4 + 8\left(\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right)_2^{\sqrt{5}}$$

$$= 10\pi - 20\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - 4$$

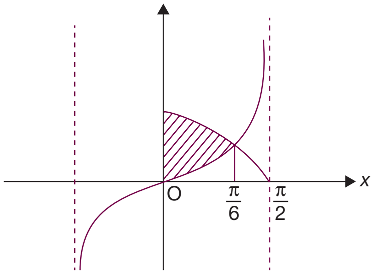
$$= 20\cos^{-1}\left(\frac{2}{\sqrt{5}}\right) - 4$$

$$= 20\tan^{-1}\left(\frac{1}{2}\right) - 4 = 5\left(\pi - \tan^{-1}\left(\frac{24}{7}\right)\right) - 4$$

$$\left(\because 4\tan^{-1}\left(\frac{1}{2}\right) = \pi - \tan^{-1}\left(\frac{24}{7}\right)\right)$$

Hence, the correct answer is (B).

27.



Solving 2 cos x = 3 tan x we get, $2 - 2\sin^2 x = 3\sin x$

$$\Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}.$$

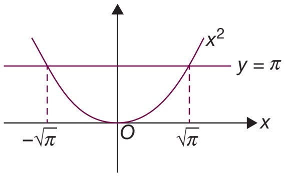
$$\text{Required area} = \int_0^{\pi/6} (2\cos x - 3\tan x)dx$$

$$= 2\sin x - 3\ln\sec x\Big|_0^{\pi/6} = 1 - 3\ln 2 + \frac{3}{2}\ln 3.$$

Hence, the correct answer is (B).

28. $[-\sin^2 x] = 0$ or -1 but $\sec^{-1}(0)$ is not defined.

$$\Rightarrow \sec^{-1}[-\sin^2 x] = \sec^{-1}(-1) = p.$$



$$\text{The required area} = \int_{-\sqrt{\pi}}^{\sqrt{\pi}} (\pi - x^2)\,dx$$

$$= \frac{4}{3}\pi\sqrt{\pi}.$$

Hence, the correct answer is (B).

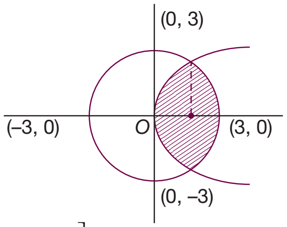
29. $$\text{Area} = 2\int_{-2}^{0} y\,dx = 2\int_{-2}^{0} x^2\sqrt{x+2}\,dx = 4\sqrt{2}\int_{-2}^{0}(z^2-2)^2 z^2\,dz$$

(where $\sqrt{x+2} = z$)

$$= 4\left[\frac{z^7}{7} - \frac{4z^5}{5} - \frac{4z^3}{3}\right]_0^{\sqrt{2}} = \frac{256\sqrt{2}}{105}$$

Hence, the correct answer is (D).

30. $x^2 + y^2 = 9,$



$$\Rightarrow x^2 + 8x - 9 = 0$$

$$\Rightarrow x = \frac{-8 \pm \sqrt{64+36}}{2}$$

$$x = \frac{-8 \pm 10}{2} = -9, 1$$

$$\Rightarrow x = 1$$

$$\text{Area enclosed} = 2\left[\int_0^1 2\sqrt{2x}dx + \int_1^3 \sqrt{9-x^2}dx\right]$$

$$= 2\left[2\sqrt{2x}\int_0^1 \sqrt{x}dx + \int_1^3 \sqrt{9-x^2}dx\right]$$

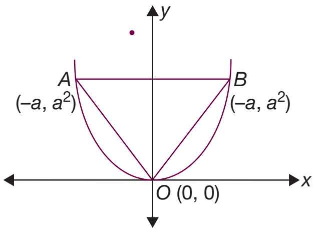
$$= 2\left[\frac{\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{9}{2}\sin^{-1}\left(\frac{1}{3}\right)\right]$$

$$= 2 + \left[\left[\frac{4\sqrt{2}}{3}x^{3/2}\Big|_0^1 + \left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right]_1^3\right]\right]$$

$$= 2\left[\frac{4\sqrt{2}}{3} + \left[0 + \frac{9\pi}{4} - \sqrt{2} - \frac{9}{2}\sin^{-1}\frac{1}{3}\right]\right] = -9, 1 \Rightarrow x = 1$$

Hence, the correct answer is (D).

31. $$\text{Area of curve } OAB = 2\int_0^{a^2} x\,dy = 2\int_0^{a^2}\sqrt{y}\,dy = \frac{4}{3}(a^3)$$



Also, area of $\Delta AOB = \frac{1}{2}(2a)(a^2) = a^3.$

$$\therefore \frac{\text{Area of } \Delta OAB}{\text{Area of curve } OAB} = \frac{a^3}{\frac{4}{3}a^3} = \frac{3}{4}.$$

Hence, the correct answer is (A).

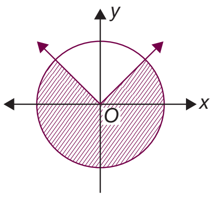
32. From the first radical sign

$x^2 + y^2 \le 16$

i.e. interior of a circle with circle (0, 0) and radius 4.

From the 2nd radical sign, $y \le |x|$

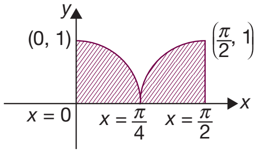
i.e. Required area = 12π.



Hence, the correct answer is (D).

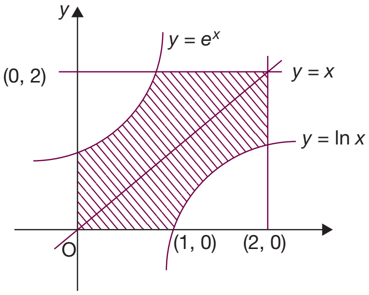
33. So, required area

$$= \int_0^{\pi/4}(\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} -(\cos x - \sin x)\,dx = 2\left(\sqrt{2}-1\right)$$



Hence, the correct answer is (B).

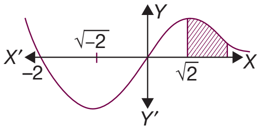
34.



Required area $= 2\left[2 - \int_1^2 \ln x\, dx\right] = 6 - 4\ln 2$

Hence, the correct answer is (A).

35. Required area $A = \int_0^{\infty} \underset{\text{(II)}}{e^{-x}} \underset{\text{(I)}}{(x^2 + 2x)} dx = 4$
(I.B.P.)



Hence, the correct answer is (C).

36. Let $A_n = \int_0^1 x^3(1 - x^2)^n dx$

Put $x^2 = t \Rightarrow x\, dx = \frac{1}{2}dt$

$\therefore\ A_n = \frac{1}{2}\int_0^1 t^n(1-t)^n dt = \frac{1}{2}\int_0^1 t^n(1-t)dt$ (Using King property)

$\Rightarrow A_n = \frac{1}{2}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) \Rightarrow \sum_{n=1}^{\infty} A_n = \frac{1}{4}.$

Hence, the correct answer is (C).

37. $A = \int_0^{\frac{\pi}{4}} (\sec x - \cos x)dx + \int_0^{\frac{\pi}{2}} (\cos ec x - \sin x)dx$

$= 2\ln(\sqrt{2} + 1) - \sqrt{2}$

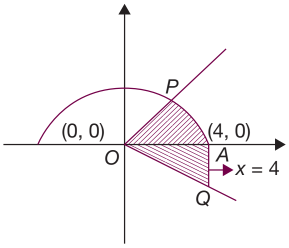
Hence, the correct answer is (C).

38. Area $= 2\int_0^{1/2} \sqrt{\frac{x}{2}}dx + \frac{1}{4} = \frac{7}{12}$

Hence, the correct answer is (D).

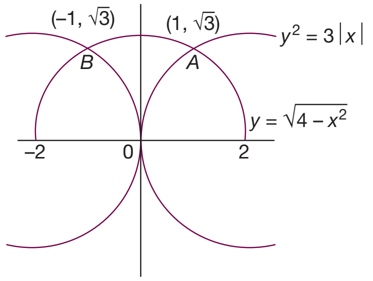
39. Required area is the area of shaded region (*APOQ*)
= Area of ΔOAQ + Area of sector (*OAP*)

$= \frac{1}{2} \times 4 \times 4\sqrt{3} + \frac{\pi(4 \times 4)}{6} = \left(\frac{8\pi}{3} + 8\sqrt{3}\right)$



Hence, the correct answer is (B).

40. Required area $= 2\int_0^1 \left(\sqrt{4 - x^2} - \sqrt{3x}\right)dx$



$= 2\left(\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right) - \frac{\sqrt{3}.2x^{3/2}}{3}\right)_0^1 = \frac{2\pi - \sqrt{3}}{3}$

Hence, the correct answer is (C).

41. $f^3(x) = \int_0^x t\, f^2(t)dt;\quad f(0) = 0$

Differentiation w.r.t *x*, we get

$3f^2(x)\, f'(x) = xf^2(x)$

$\Rightarrow\ f'(x) = \frac{x}{3} \ \Rightarrow\ f(x) = \frac{x^2}{6} + C$

$\Rightarrow\ A = \frac{1}{6}\int_0^3 x^2 dx = 3/2$

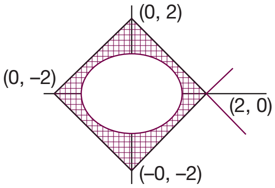
Hence, the correct answer is (B).

42. $2x^2 + 6y^2 \geq 3$ …(1)

Area of ellipse $= \pi \times \frac{\sqrt{3}}{\sqrt{2}} \times \frac{1}{\sqrt{2}} = \frac{\pi\sqrt{3}}{2}$

$|x + y| \leq 2 \ \Rightarrow\ -2 \leq (x + y) \leq 2$ …(2)

$|x - y| \leq 2 \ \Rightarrow\ -2 \leq (x - y) \leq 2$ …(3)



Required area $= \left(8 - \frac{\pi\sqrt{3}}{2}\right)$ sq. units

Hence, the correct answer is (B).

43. $F(x) = \sin x\int_0^x \cos t\, dt + 2\int_0^x t\, dt - x^2 + \cos^2 x$

$= \sin x(\sin t)_0^x + 2\left(\frac{t^2}{2}\right)_0^x - x^2 + \cos^2 x$

$= \sin^2 x + x^2 - x^2 + \cos^2 x = 1$

$A = \int_0^5 xF(x)dx = \int_0^5 (x)(1)dx = \left[\frac{x^2}{2}\right]_0^5 = \frac{25}{2}$

Hence, the correct answer is (B).

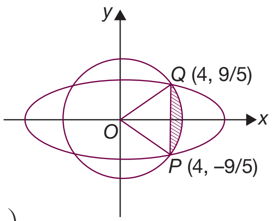
44. $A_1 = \frac{10}{3} - 1,\ A_2 = \frac{10}{3} + 1 \ \Rightarrow\ \frac{A_1}{A_2} = \frac{7}{13}$

Hence, the correct answer is (A).

45. As eccentricity of ellipse $= \frac{4}{5}$

Co-ordinate of foci = (4, 0), (–4, 0)

$\Rightarrow\ \left(4, -\frac{9}{5}\right)$ is one of the end point of latus – rectum



$\Rightarrow$ Required area is

$= \frac{1}{2\pi} \times \pi \times \left(4^2 + \frac{9^2}{5^2}\right) \times 2\tan^{-1}\left(\frac{9}{20}\right) -$ area of ΔPOQ

$= \frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right) - \frac{1}{2} \times 4 \times \left(\frac{18}{5}\right) = \frac{481}{25}\tan^{-1}\left(\frac{9}{20}\right) - \frac{36}{5}$

Hence, the correct answer is (A).

46. Clearly, we have $\frac{k\sin k}{2}+\int_k^\pi \sin x dx=1\left(u\sin g\int_0^\pi \sin x\,dx=2\right)$

$\Rightarrow\ \frac{k\sin k}{2}-\cos x|_k^\pi=1$

$\Rightarrow\ k\sin k+2[1+\cos k]=2$

So, $k\sin k+2\cos k=0$.

Hence, the correct answer is (B).

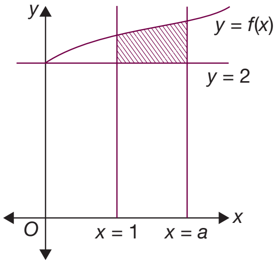
47. Required area $=\int_1^a (f(x)-2)dx$

$=\frac{2}{3}\left((2a)^{\frac{3}{2}}-3a+3-2\sqrt{2}\right)$

Differentiate with respect to '*a*'

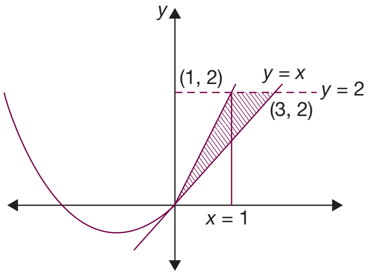
$\therefore\ f(a)-2=\frac{2}{3}\left[\frac{3}{2}\sqrt{2a}\times 2-3\right]$

$\Rightarrow\ f(a)=2\sqrt{2a};\ a\le 1$ or $f(x)=2\sqrt{2};\ x\ge 1$



Hence, the correct answer is (A).

48. $A_1=\int_0^1\left(\frac{(2x+1)^2-1}{4}-x\right)dx=\frac{1}{3}$



$A_2=\frac{1}{2}$

Hence, the correct answer is (B).

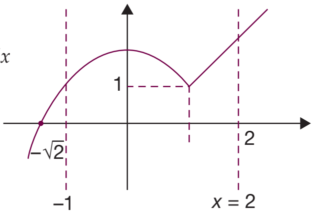
49. By the graph,

$A=\int_{-1}^{1}(-x^2+2)\ +\int_1^2(2x-1)dx$

$=\left[2x-\frac{x^3}{3}\right]_{-1}^{1}+\left[x^2-x\right]_1^2$

$=2-\frac{1}{3}-\left(-2+\frac{1}{3}\right)+2-0$

$=2-\frac{1}{3}+2-\frac{1}{3}+2=6-\frac{2}{3}=\frac{16}{3}$ sq. unit



Hence, the correct answer is (A).

50. $A=\int_0^1(a^2x^2+ax+1)\,dx$

$=\frac{a^2}{3}+\frac{a}{2}+1$ which is minimum for $a=-3/4$

Hence, the correct answer is (C).

51. Circle $x^2+y^2-4x+2y+1=0$

$(x-2)^2+(y+1)^2=4$

Clearly $y=\left[\frac{x^3}{100}+\frac{x}{35}\right]=0,\ \forall\ x\in[0,4]$

Hence, the correct answer is (A).

52. **CASE 1:** If $[[x[]=1$ and $[\,|y|\,]=0$

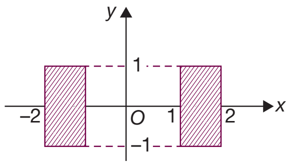
Then, $1\le|x|<2, 0\le|y|<1\Rightarrow x\in(-2,-1]\cup[1,2), y\in(-1,1)$

CASE 2 : If $\left[|x|\right]=0,\left[|y|\right]=1,$

Then, $x\in(-1,1), y\in(-2,-1]\cup[1,2)$

Similarly, for case 2

$\therefore$ Required area $=4(2-1)\left[1-(-1)\right]=8$



Hence, the correct answer is (D).

53. $\int_{\frac{\pi}{4}}^{\beta} f(x)dx=\beta\sin\beta+\frac{\pi}{4}\cos\beta+\sqrt{2}\beta$, differentiate both sides w.r. to 'β'

$f(\beta)=b\cos\beta+\sin\beta+\frac{\pi}{4}(-\sin\beta)+\sqrt{2}$

$f'(\beta)=\beta(-\sin\beta)+\cos\beta+\cos\beta-\frac{\pi}{4}\cos\beta$

$f'\left(\frac{\pi}{2}\right)=-\frac{\pi}{2}+0+0-0=-\frac{\pi}{2}$

Hence, the correct answer is (C).

54. $x_{\text{intercept}}=r$

$y_{\text{intercept}}=r+1$

Area $=\frac{1}{2}r(r+1)$

$$\sum_{r=1}^{n}A_r=\sum_{r=1}^{n}\frac{1}{2}\left[r^2+r\right]=\frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$$

$$=\frac{1}{2}\frac{n(n+1)}{6}\left[2n+1+3\right]=\frac{n(n+1)}{2\times 6}(2n+4)=\frac{n(n+1)(n+2)}{6}$$

Hence, the correct answer is (D).

55. $A=\sqrt{2}\int_2^4\sin\frac{\pi x}{4}dx=\frac{4\sqrt{2}}{\pi}$ square units

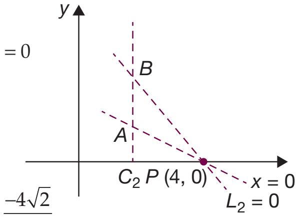
Let the line $L_1=0$ is $y-m_1(x-4)=0$

$L_2=0\ is\ y-m_2(x-4)=0$

$A(2,-2m_1), B(2,-2m_2);$

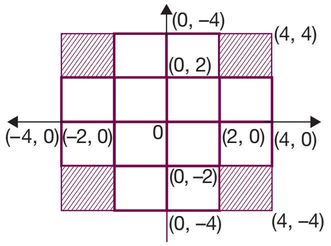
$A(2,-2m_1), B(2,-2m_2)$

$\Delta_{ACD}=\frac{\Delta}{3}\Rightarrow m_1=\frac{-2\sqrt{2}}{3\pi}; m_2=\frac{-4\sqrt{2}}{3\pi}$



Hence, the correct answer is (A).

56. Required area $=4[2\times 2]=16$



Hence, the correct answer is (C).

57. $f(x)$ and $g(x)$ are periodic with period 1.

$$\therefore \int_0^6 [f(x)-g(x)]\,dx = 6\int_0^1 \left(\sqrt{x}-x^2\right)dx$$

$$= 6\left[\frac{2}{3}x^{3/2}-\frac{x^3}{3}\right]_0^1 = 6\left[\frac{2}{3}-\frac{1}{3}\right] = 2$$

Hence, the correct answer is (B).

58. $A_2 = \int_0^{x_1}(bx^2)\,dx + \frac{1}{2}(1-x_1)\cdot bx_1^2$

$$= \frac{bx_1^2}{6}\cdot(3-x_1) = \frac{(1-x_1)(3-x_1)}{6} \quad (\text{As, } bx_1{}^2 = 1-x_1)$$

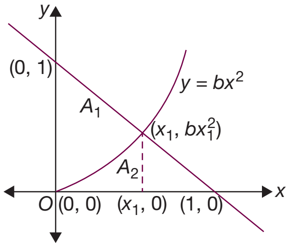
Now, $\frac{A_1}{A_2} = \frac{11}{16} \Rightarrow \frac{A_1+A_2}{A_2} = \frac{27}{16}$

$$\left(\text{As, } A_1 + A_2 = \frac{1}{2}(1)(1) = \frac{1}{2}\right)$$

$\Rightarrow A_2 = \frac{8}{27}$

$\therefore \frac{(1-x_1)(3-x_1)}{6} = \frac{8}{27}$

$\Rightarrow x_1 = \frac{1}{3} \Rightarrow b = 6.$



Hence, the correct answer is (D).

59. $f(a) = \frac{1}{\ln\sqrt{1+a^2}}\int_0^a \frac{\ln(1+ax)}{1+x^2}\,dx$

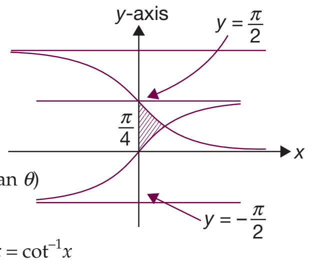
$$\int_0^a \frac{\ln(1+ax)}{1+x^2}\,dx$$

$= \tan^{-1}a \cdot \ln\sqrt{1+a^2}$ (Put $x = \tan\theta$)

$\therefore f(a) = \tan^{-1}a$

Curves : $y = \tan^{-1}x$ $y = \frac{\pi}{2} - \tan^{-1}x = \cot^{-1}x$

$$A = \int_0^{\frac{\pi}{4}} \tan y\,dy + \int_0^{\frac{\pi}{2}} \cot y\,dy = \ln 2$$



Hence, the correct answer is (A).

60. $\int_0^\alpha P(x)\,dx = \frac{\alpha^2}{3}(\alpha+3)$

Differentiating with respect to α, we get

$P(\alpha) = \alpha^2 + 2\alpha$

$\therefore P(1) = 3.$

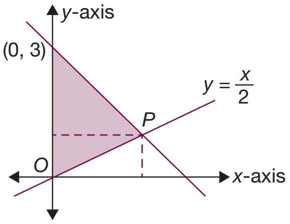
Hence, the correct answer is (C).

61. At P, $\frac{x}{2} = 3 - ax \Rightarrow x = \frac{6}{2a+1}$

Now, Area $= \frac{1}{2}\times 3\times x = \frac{3}{2}\times\frac{6}{2a+1}$

$= \frac{9}{2a+1} = 3$. ($A = 3$ given)

Hence, $a = 1$ which lies in $\left(\frac{3}{4},\frac{3}{2}\right)$.



Hence, the correct answer is (B).

62. $y = fog(x) = \cos^2 x, \alpha = \frac{\pi}{6}$ and $\beta = \frac{\pi}{3}$

$$A = \int_{\pi/6}^{\pi/3} \cos^2 x\,dx = \frac{\pi}{12}.$$

Hence, the correct answer is (C).

63. Area $= -\int_{-1}^{1} y\times\frac{dx}{dt}\,dt = -\int_{-1}^{1} at\left(1-t^2\right)a(-2t)\,dt$

$$= +2a^2\int_{-1}^{1} t^2\left(1-t^2\right)dt = \frac{8a^2}{15}$$

Hence, the correct answer is (C).

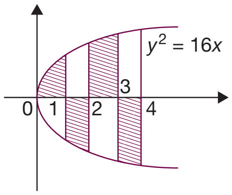
64. S_1: $4 - x > 0 \Rightarrow x < 4;\ x \neq 3$

$16x - y^2 > 0 \Rightarrow y^2 < 16x$

S_2: $\frac{xy(x-1)(x-2)}{(x-3)(x-4)} > 0$

$y > 0 \quad x \in (0, 1) \cup (2, 3)$

$y < 0 \quad x \in (1, 2) \cup (3, 4)$



Required area $= 4\int_0^4 \sqrt{x}\,dx = \frac{64}{3}$

Hence, the correct answer is (B).

65. $$\frac{2\sqrt{2}}{2\pi} < \int_\pi^{2\pi} \frac{|\sin x + \cos x|}{x}\,dx < \frac{2\sqrt{2}}{\pi} \quad \ldots(1)$$

$\because \pi < x < 2\pi \Rightarrow \frac{1}{2\pi} < \frac{1}{x} < \frac{1}{\pi}$

$$\frac{2\sqrt{2}}{3\pi} < \int_{2\pi}^{3\pi} \frac{|\sin x + \cos x|}{x}\,dx < \frac{2\sqrt{2}}{2\pi} \quad \ldots(2)$$

Adding (i) and (ii)

$\frac{5\sqrt{2}}{3\pi} < A < \frac{3\sqrt{2}}{\pi}$; $0.75 < A < 1.3$.

Hence, the correct answer is (B).

66. $P(|x|)$ has two roots only when $P(|b|) = 0$

$\Rightarrow b^2 = 4 \Rightarrow |b| = 2$

$P(x) = x^3 - 12x + 16$ $\therefore$ the roots of $P(x)$ are $-4, 2$

$$\int_{-4}^{2} P(x)\,dx = \int_{-4}^{2}\left(x^3 - 12x + 16\right)dx$$

$$= \frac{1}{4}\left(2^4 - 4^4\right) - \frac{12}{2}\left(2^2 - 4^2\right) + 16(6)$$

$= -60 + 72 + 96 = 108.$

Hence, the correct answer is (B).

67. $f(x) = \tan^{-1}(\tan x) + \cos^{-1}(\cos x)$ for $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

When $\frac{\pi}{2} < x < \pi \Rightarrow \tan^{-1}(\tan x) = x - \pi$

$\cos^{-1}(\cos x) = x$

$\Rightarrow f(x) = \tan^{-1}(\tan x) + \cos^{-1}(\cos x) = -\pi + x + x = (2x - \pi)$

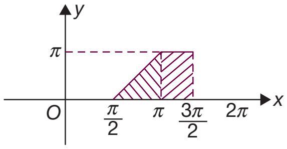
When $\pi \leq x < \frac{3\pi}{2}$

$\tan^{-1}(\tan x) = x - \pi$

$\cos^{-1}(\cos x) = 2\pi - x$

$\Rightarrow \quad f(x) = -\pi + x + 2\pi - x = \pi$

So, graph of $f(x)$ and required area is shown in figure.



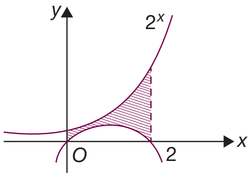
$\Rightarrow \quad \text{Area} = \pi^2 - \frac{\pi^2}{4} = \frac{3\pi^2}{4}.$

Hence, the correct answer is (B).

68. $m = \int_0^2 \left|2^x - 2x + x^2\right| dx$

$= \left[\frac{2^x}{\ln 2} - x^2 + \frac{x^3}{3}\right]_0^2$

$= \left[\frac{4}{\ln 2} - 4 + \frac{8}{3} - \frac{1}{\ln 2}\right] = \left[\frac{3}{\ln 2} - \frac{4}{3}\right]$



Hence, the correct answer is (D).

69. Curve passes through (1, 2) $\Rightarrow a + b = 2$

Given $\int_1^b \left(a\sqrt{x} + bx\right) dx = 8 \Rightarrow \frac{2a}{3} + b = 1$

Hence, the correct answer is (A).

70. $\int_1^b f(x)\,dx = (b-1)\cos(3b+4)$

Diff. both sides w.r.to b

$f(b) = -(b-1)\sin(3b+4)(3) + \cos(3b+4)$

$f(x) = \cos(3x+4) - 3(x-1)\sin(3x+4)$

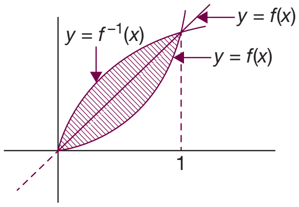
Hence, the correct answer is (C).

71. Required area $= \int_{-1}^{1} y \frac{dx}{dt} dt\, dt = \int_{-1}^{1} (1-t^4)(1-3t^2)\ dt$

Hence, the correct answer is (C).

72. Clearly required area

$= 2\int_0^1 (f^{-1}(x) - x)\, dx = 2k - 1$ sq. units



Hence, the correct answer is (B).

73. By the graph
Area of the left of y-axis is π, Area of the right of y-axis

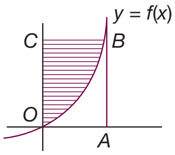
$= \int_0^{\sqrt{2}} \left(\sqrt{4-x^2} - \sqrt{2}\sin\left(\frac{x\pi}{2\sqrt{2}}\right)\right) dx$

Hence, the correct answer is (D).

74. Interchange the axis required area is

$=$ Area of rectangle $OABC$ $-$

$\int_0^1 f(x)dx = 5 - \int_0^1 (3x^3 + 2x)dx = \frac{13}{4}$



Hence, the correct answer is (D).

75. Solving we get $x^4 + 4x^2 - 32 = 0 \Rightarrow x = -2, 2$

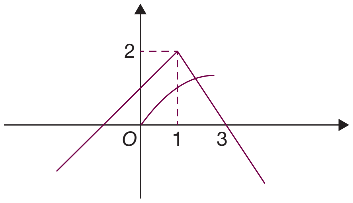
$\therefore \quad$ The area is $S = \int_{-2}^{2} \left(\frac{8}{x^2+4} - \frac{x^2}{4}\right) dx$

$= \left(4tan^{-1}\left(\frac{x}{2}\right) - \frac{x^3}{12}\right)_{-2}^{2} = 2\pi - \frac{4}{3}$

Hence, the correct answer is (B).

76. Area $= \int_0^1 (1 + x - \sin x)dx + \int_1^2 (3 - x - \sin x)dx$

$= \frac{3}{2} + (\cos 1 - 1) + 3 - \frac{3}{2} + (\cos 2 - \cos 1) = 2 + \cos 2$



Hence, the correct answer is (A).

77. We have, $y = \frac{4-x^2}{4+x^2}$

We observe that all powers of x in the above equation are even, so it is symmetric about y-axis. The curve intersects y-axis at (0, 1). Also,

$\lim_{x\to\infty} y = \lim_{x\to\infty} \frac{4-x^2}{4+x^2} = -1$

Therefore, as $x \to \infty$, $y \to -1$ i.e. $y = -1$ is an asymptote to the curve.

Now, $y = \frac{4-x^2}{4+x^2} = x = \pm 2\sqrt{\frac{1-y}{y+1}}$

This means that x is imaginary for $y > 1$ or $y < -1$ i.e. the curve lies between $y = -1$ and $y = 1$. At (0, 1) the tangent to the curve $y = \frac{4-x^2}{4+x^2}$ is $y = 1$.

Hence, the correct answer is (B).

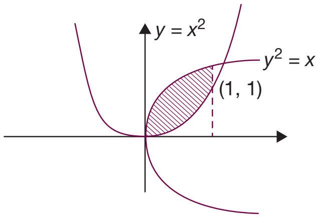
78. Let $X = x - 1$, $Y = y - 1$ (translating origin to (1, 1)), we get $\max\{|x|, |y|\} \le 1$ and $xy \le \frac{1}{2}$ area required is as shown (shaded)

Area $= 2 + 2\int_{1/2}^{1} \frac{1}{2x}dx + 2\left(\frac{1}{2}\right) = 3 + \ln 2$

Hence, the correct answer is (C).

Multiple Correct Choice Type Questions

1. $x^2 = y \quad y^2 = x$

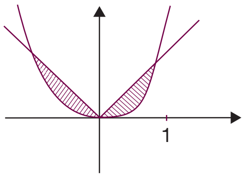


(A) Required area $= \int_0^1 \sqrt{x} - x^2 = \left.\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right|_0^1$

$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

(B) $2\int_0^1 x - x^2\,dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1$

$= 2\left(\frac{1}{6}\right) = \frac{1}{3}$



(C) $x^2 \le y \le |x|$

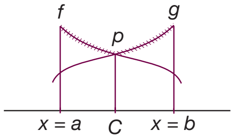
Required $= 2\int_0^1 (x - x^2)dx = \frac{1}{3}$

Hence, (A), (B) and (C) are correct.

2. $p(x) = \max\{f(x), g(x)\} \quad q(x) = \min\{f(x), g(x)\}$

Area $= \int_a^b |p(x) - q(x)|\,dx = \int_a^b p(x) - q(x)\,dx$

$= \int_a^b |f(x) - g(x)|\,dx$



Hence, (B), (C) and (D) are correct.

3. $f(x) = \frac{\sin x}{x}$

$f'(x) = \frac{x\cos x - \sin x}{x^2}$

$= \frac{\cos x(x - \tan x)}{x^2} < 0$

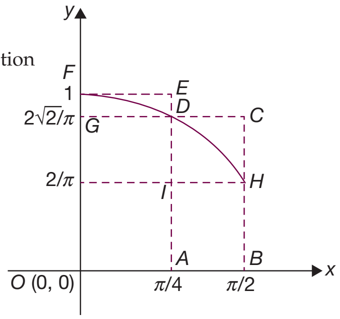
$\Rightarrow$ $f(x)$ is an decreasing function

$f\left(\frac{\pi}{2}\right) < f(x) < f(0^+)$;

$\frac{1}{\pi/2} < \frac{\sin x}{x} < 1$

$\Rightarrow \int_0^{\pi/2} \frac{2}{\pi}dx < A < \int_0^{\pi/2} dx$

$\Rightarrow 1 < A < \frac{\pi}{2}$



Sum of area of smaller rectangles $OADG + ABHI < A <$ sum of area of larger rectangles $OAEF + ABCD$

$$\frac{\pi}{4}\left(\frac{2\sqrt{2}}{\pi}\right) + \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\frac{2}{\pi} < A < 1.\frac{\pi}{4} + \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\frac{2\sqrt{2}}{\pi}$$

$$\frac{1+\sqrt{2}}{2} < A < \frac{\pi + 2\sqrt{2}}{4}$$

Hence, (A) and (B) are correct.

4. $f(x) = 2x - x^2 \quad g(x) = x^n \quad n \in N$

According to given condition $\int_0^1 |f(x) - g(x)|\,dx = \frac{1}{2}$

Point of intersection $\Rightarrow 2x - x^2 = x^n \Rightarrow x^{n-1} + x - 2 = 0 \Rightarrow x = 1$; $x \in (0, 1)$;

Required area $= \int_0^1 2x - x^2 - x^n\,dx = 1/2$

$\Rightarrow x^2 - \frac{x^3}{3} - \frac{x^{n+1}}{n+1}\Big|_0^1 = 1/2$

$\Rightarrow 1 - \frac{1}{3} - \frac{1}{n+1} = \frac{1}{2} \Rightarrow \frac{1}{n+1} = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

Thus, $n + 1 = 6 \quad \Rightarrow \quad n = 5$

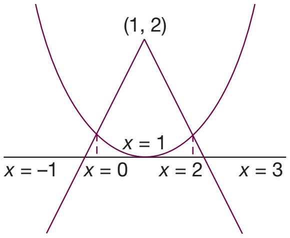
Hence, (B), (C) and (D) are correct.

5. Area bounded by f and

$g = 2\int_0^1 ((x+1) - (x-1)^2)dx$

$= 2\left(\frac{(x+1)^2}{2} - \frac{(x-1)^3}{3}\right)\Big|_0^1 = \frac{7}{3}$

Area bounded by $f(x)$, $g(x)$ and x-axis $= 4 - \frac{7}{3} = \frac{5}{3}$

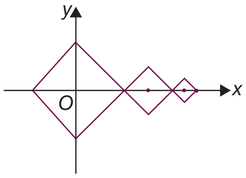


Hence, (B) and (C) are correct.

6. $a_1 = 0, b_1 = 32, a_2 = a_1 + \frac{3}{2}b_1 = 48,$

$b_2 = \frac{b_1}{2} = 16.$

$a_3 = 48 + \frac{3}{2}16 = 72, b_3 = \frac{16}{2} = 8$



So, the three loops from $i = 1$ to $i = 3$ are alike.

Now, area of i^{th} loop (square) $= \frac{1}{2}(\text{diagonal})^2 = \frac{1}{2}(2b_i)^2 = 2(b_i)^2$

So, $\frac{2(b_{i+1})^2}{2(b_i)^2} = \frac{1}{4}$

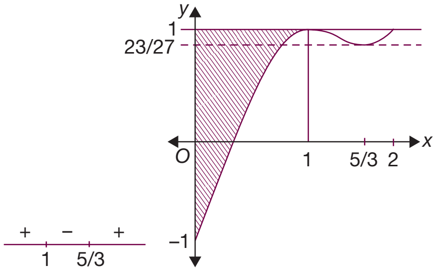
So, the areas form a *G.P.* series

So, the sum of the *G.P.* upto infinite terms.

$$= A_1\frac{1}{1-r} = 2(32)^2 \times \frac{1}{1-\frac{1}{4}} = 2 \times (32)^2 \times \frac{4}{3} = \frac{8}{3}(32)^2 \text{ square units.}$$

Hence, (A) and (C) are correct.

7. $f(x) = (x-1)^2(x-2) + 1$



$f'(x) = (x-1)^2 + 2(x-2)(x-1)$

$= x^2 - 2x + 1 + 2x^2 - 6x + 4$

$= 3x^2 - 8x + 5 = (3x-5)(x-1)$

$\Rightarrow x = 5/3$

$y = (5/3 - 1)^2(5/3 - 2) + 1$

$\Rightarrow y = -4/27 + 1 = 23/27$

Point (1,1) maxima and point $\left(\frac{5}{3}, \frac{23}{27}\right)$ is minima

The area enclosed by $y = f(x)$, the lines $x = 0$ and $y = 1$ as x varies from 0 to 1

Required area $= \int_0^1 1 - ((x-1)^2(x-2) + 1)dx$

$= \int_0^1 -(x-1)^3 + (x-1)^2\,dx = \frac{7}{12}$

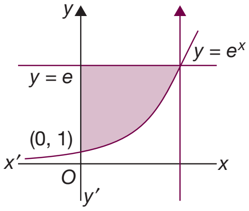
Hence, (B), (C) and (D) are correct.

8. Required area $= \int_1^e \ln y\, dy$

$= (y\ln y - y)_1^e = (e - e) - (-1) = 1$

Also, $\int_1^e \ln y\, dy = \int_1^e \ln(e+1-y)\, dy$

Further Required area $= e \times 1 - \int_0^1 e^x dx$



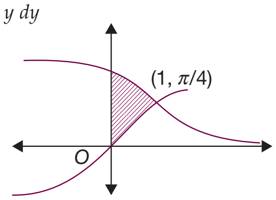
Hence, (B), (C) and (D) are correct.

9. Required area $= \int_0^{\pi/4} \tan y\; dy + \int_{\pi/4}^{\pi/2} \cot y\; dy$

$= \int_0^{\pi/4} \tan x dx + \int_{\pi/4}^{\pi/2} \cot x dx$

$= [\ln(\sec x)]_0^{\pi/4} + [\ln(\sec x)]_{\pi/4}^{\pi/2}$

$= \ln\sqrt{2} + \ln\sqrt{2} = \ln 2 = \dfrac{-2}{\pi}\int_0^{\pi/2} \log(\sin x)\; dx$



Hence, (A) and (C) are correct.

10. Equation of tangent at $(\lambda, \lambda^2 + 1)$ is $y - (\lambda^2 + 1) = 2\lambda(x - 1)$

$\Rightarrow \; y = 2\lambda x - \lambda^2 + 1$

$\therefore$ Required area

$= \int_0^{\lambda}(x^2+1)dx - ar(\Delta PMN)$

$\Rightarrow \; \dfrac{\lambda^3}{3} + \lambda - \dfrac{1}{2}\dfrac{(\lambda^2+1)^2}{2\lambda} = \dfrac{11}{3}$

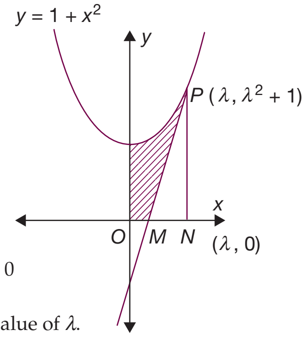
$\Rightarrow \; 3\lambda^4 + 18\lambda^2 - 132\lambda - 9 = 0$

$\therefore \; (\lambda - 3)\,(3\lambda^3 + 9\lambda^2 + 45\lambda + 3) = 0$

↓

Not satisfy by any positive value of λ.

$\therefore \; \lambda = 3$



Hence, (A), (B) and (C) are correct.

11. Area of $(T) = \dfrac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 0 & c^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = \dfrac{1}{2}|-c^3| = \dfrac{c^3}{2}$

$\Rightarrow \; y = cx, y = x^2 \;\; \Rightarrow \;\; x^2 = cx$

Point of intersection $x = 0, x = c$

Required area$(R) = \int_0^c x^2 - cx\, dx = \dfrac{x^3}{3} - \dfrac{cx^2}{2}\Big|_0^c = \dfrac{c^3}{3} - \dfrac{c^3}{2} = \dfrac{c^3}{6}$

$\lim\limits_{c\to 0^+} \dfrac{\text{Area}(T)}{\text{Area}(R)} = \lim\limits_{c\to 0^+} \dfrac{c^3/2}{c^3/6} = 3$

Hence, (A) and (C) are correct.

12. $f : R \longrightarrow [-1, 1]\; f(x) = \dfrac{x^2-1}{x^2+1} = y$

$\Rightarrow \; x^2 - 1 = yx^2 + y \;\; \Rightarrow \;\; x^2(1 - y) = y + 1$

Even function and many one $x^2 = \dfrac{y+1}{1-y} \geq 0$

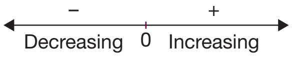
$y \in [-1, 1)$

Range $\subset$ Co domain so it is into function

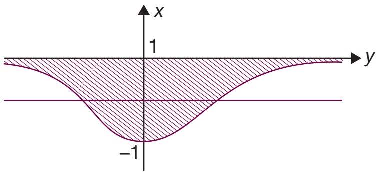
$f(x) = \dfrac{x^2-1}{x^2+1} = 1 - \dfrac{2}{x^2+1}$

$f'(x) = \dfrac{2}{(x^2+1)^2}\cdot 2x = \dfrac{4x}{(x^2+1)^2}$

$f(x)_{\min} = \dfrac{0-1}{0+1} = -1$



At $x = 0$



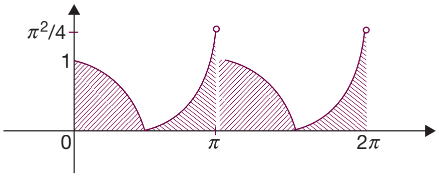
Area included by the curve $y = f(x)$ and the line $y = 1$ is

$= \int_{-\infty}^{\infty} 1 - \dfrac{(x^2-1)}{x^2+1}dx$

$= \int_{-\infty}^{\infty} \dfrac{2}{x^2+1}dx = 4\int_0^{\infty}\dfrac{dx}{x^2+1} = 4\tan^{-1}x\Big|_0^{\infty} = 4\left(\dfrac{\pi}{2}\right) = 2\pi$

Hence, (A), (C) and (D) are correct.

13. $f(x) = \begin{cases} \cos x & 0 \le x < \dfrac{\pi}{2} \\ \left(\dfrac{\pi}{2} - x\right)^2 & \dfrac{\pi}{2} \le x < \pi \end{cases}$ period $T = \pi$



Range $\in \left[0, \dfrac{\pi^2}{4}\right)$ discontinuous at $x = n\pi$

Required area $= \int_{-n\pi}^{n\pi} f(x)dx = 2n\int_0^{\pi} f(x)\, dx$

$= 2n\left(\int_0^{\pi} \cos x dx + \int_{\pi/2}^{\pi} (x - \pi/2)^2\, dx\right)$

$= 2n\left[\sin x\Big|_0^{\pi/2} + \dfrac{(x-\pi/2)^3}{3}\Bigg|_{\pi/2}^{\pi}\right) = 2n\left(1 - 0 + \dfrac{\pi^3}{24}\right) = 2n\left(\dfrac{\pi^3}{24} + 1\right)$

Hence, (A) and (D) are correct.

14. Let line $y = c$ cuts curve at point $x = a, b$

Also, $c = 8b - 27b^3$ and $c = 8a - 27a^3$ …(1)

Shaded area$_1$ = Shaded area$_2$

$\because \;\; \int_0^a c - (8x - 27x^3)\, dx = \int_a^b (8x - 27x^3) - c\, dx$

$\Rightarrow \; \left[cx - 4x^2 + \dfrac{27x^4}{4}\right]_0^a = \left[4x^2 - \dfrac{27x^4}{4} - cx\right]_a^b$

$ac - 4a^2 + 27a^4/4 = (4b^2 - 27b^4/4 - bc) - (4a^2 - 27a^4/4 - ac)$

$bc + 27b^4/4 = 4b^2$

Using from (1), we have

$$c + 27b^3/4 = 4b$$

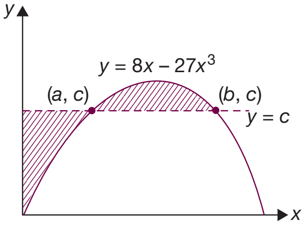
$$8b - 27b^3 + 27b^3/4 = 4b$$

$$\Rightarrow \quad b^2 = 16/81$$

$$c = 8(4/9) - 27(84/81 \times 9)$$

$$= 32/9 - 64/27$$

$$b = 4/9 \quad c = 32/27 \quad \therefore \text{B, C}$$



Hence, (B) and (C) are correct.

15. The area could be of the rectangle defined by $y = 10$ from $x = 0$ to $x = 10$, which is 100. The integral could range from 0 (example : $f(x) = 2x - 10$) to 100. So, A is always true and all other options are not always true.

Hence, (B), (C) and (D) are correct.

16. $\sec^{-1}x = \cot^{-1}x$

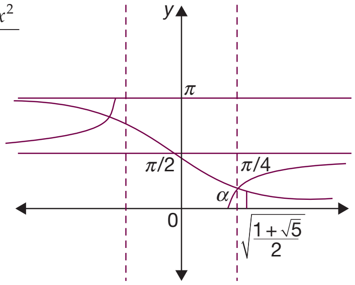
$$\Rightarrow \quad x = \sec(\cot^{-1}x) = \frac{\sqrt{1+\sqrt{x^2}}}{x}$$

$$\Rightarrow \quad x^4 - x^2 - 1 = 0$$

$$\Rightarrow \quad x^2 = \frac{1+\sqrt{5}}{2}$$

Required area

$$= \int_1^{\sqrt{\frac{1+\sqrt{5}}{2}}} (\cot^{-1}x - \sec^{-1}x)\, dx$$



$$\text{or} \int_0^{\alpha} (\sec y - 1)dy + \int_{\alpha}^{\pi/4} \cot y\, dy - \frac{\pi}{4} + \alpha$$

where $\sec^{-1}x = \alpha = \cot^{-1} x$

$$\Rightarrow \quad x = \sec\alpha = \cot\alpha \quad \Rightarrow \quad \cos^2\alpha = \sin\alpha.$$

Hence, (A) and (B) are correct.

17. $y = (x-1)e^x; \dfrac{dy}{dx} = xe^x$ and $y(0) = -1$

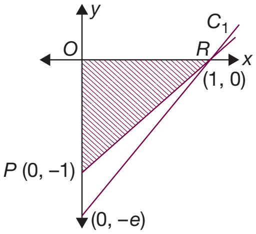
$$T : y - 0 = e(x-1) \quad \Rightarrow \quad y = e(x-1)$$

$$A_2 = -\int_0^1 (x-1)e^x dx$$

$$= -\left[(x-1)e^x - e^x\right]_0^1 = e - 2$$

$$A_1 = \frac{1}{2} \times 1 \times e - (e-2) = 2 - \frac{e}{2}$$

$$\therefore \quad 2A_1 + A_2 = 4 - e + e - 2 = 2 \text{ and } A_1 < A_2$$

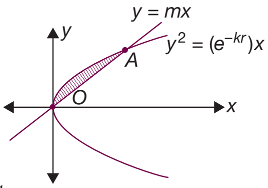


Hence, (B) and (C) are correct.

18. Solving the two equations,

$$m^2x^2 = (e^{-kr})x; x_1 = 0, x_2 = \frac{e^{-lr}}{m^2}.$$

$$\text{So, A}_\text{r} = \int_{x_1}^{x_2} \left(e^{\frac{-kr}{2}}\sqrt{x} - mx \right) dx = \frac{e^{-2kr}}{6m^3}$$



$$\text{Now, } \frac{A_{r+1}}{A_r} = \frac{e^{-2k(r+1)}}{e^{-2kr}} = e^{-2k} = \text{constant.}$$

So, the sequence $A_1, A_2, A_3, \ldots$ is in *G.P.*

$$\lim_{n\to\infty} \sum_{i=1}^{n} A_i = \frac{e^{-2k}}{6m^3} \times \frac{e^{2k}}{e^{2k}-1} = \frac{1}{6m^3(e^{2k}-1)}$$

Hence, (A) and (C) are correct.

19. Since, $-1 \le \sin x \le 1$, the curves $y = e^{-x}\sin x$ is bounded by the curves $y = e^{-x}$ and $y = -e^{-x}$

Also, the curve $y = e^{-x} \sin x$ intersects the positive semi axis OX at the points where $\sin x = 0$, where $x_n = n\pi$, $n \in$ Z. Also, $|y_n| =$ $|y$ coordinate in the half wave $S_n|$

$= (-1)^n e^{-x} \sin x$, and in S_n, $n\pi \le x \le (n+1)\pi$.

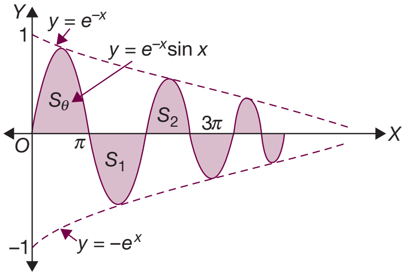
$$\therefore \quad S_n = (-1)^n \int_{n\pi}^{(n+1)\pi} e^{-x}\sin x\, dx$$

$$= \frac{(-1)^{n+1}}{2}\left[e^{-x}(-\sin x + \cos x)\right]_{n\pi}^{(n+1)\pi}$$

$$= \frac{(-1)^{n+1}}{2}\left[e^{-(n+1)\pi}(-\sin x + \cos x)\right]_{n\pi}^{(n+1)\pi}$$

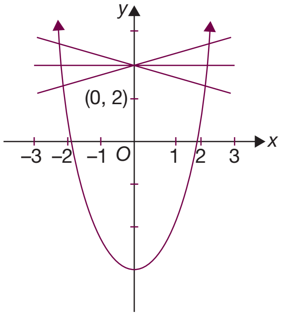
$$= \frac{e^{-n\pi}}{2} = e^{-}\pi \text{ and } S_0 = \frac{1}{2}(1 + e\pi)$$

$\therefore$ The sequence $S_0, S_1, S_2, \ldots$ forms an infinite *G.P.* with common ratio $e^{-}\pi$



Hence, (A) and (C) are correct.

20. Line $y = kx + 2$ passes through fixed point (0, 2) for different value of i.



Also, it is obvious that minimum $A(k)$ occurs when $k = 0$, as When line is rotated from this position about point (0, 2) the increased part of area is more than the decreased part of area.

$$\text{Area} = 2\int_0^{\sqrt{5}} (2 - (x^2 - 3))dx = 2\int_0^{\sqrt{5}} (5 - x^2)\, dx$$

$$= \left[5x - \frac{x^3}{3}\right]_0^{\sqrt{5}} = 2\left[5\sqrt{5} - \frac{5\sqrt{5}}{3}\right] = \frac{20\sqrt{5}}{3} \text{ sq. units}$$

Hence, (B) and (C) are correct.

21. Since, the curve $y = a\sqrt{x} + bx$ passes through the point (1, 2)

$$\therefore \quad 2 = a + b \qquad \ldots(1)$$

By observation the curve also passes through (0, 0). Therefore, the enclosed by the curve, x-axis and $x = 4$ is given by

$$A = \int_0^4 (ax^{1/2} + bx)dx = 18$$

$$\Rightarrow \frac{2a}{3}\times 8+\frac{b}{2}\times 16=8 \Rightarrow \frac{2a}{3}+b=1 \quad \ldots(2)$$

∴ Solving (1) and (2), we get $a = 3$, $b = -1$.

Hence, (C) and (D) are correct.

22. Clearly $\int_1^b f(x)dx=(b-1)\sin(3b+4)$

Now, apply Leibnitz theorem

$f(b) = 3(b-1)\cos(3b+4)+\sin(3b+4)$

$\Rightarrow f(x) = \sin(3x+4)+3(x-1)\cos(3x+4)$

$f(1) = \sin 7 > 0$

$f(2) = 3\cos 10+\sin 10 < 0$

From IMT $f(x) = 0$ has atleast one real root in (1,2)

Hence, (B), (C) and (D) are correct.

23. (A, B, C) for $x\in\left(0,\frac{\pi}{2}\right)$

$0<\sin x<1<\operatorname{cosec} x<\infty$

So, $\sin x^{\operatorname{cosec}x} < \sin x^{\sin x} < \operatorname{cosec}x^{\sin x} < \operatorname{cosec}x^{\operatorname{cosec}x}$

$A_1<A_3<A_2<A_4$

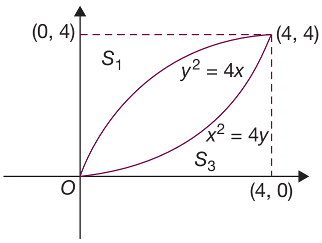
Hence, (A), (B) and (C) are correct.

24. $S_1=\int_0^4 \frac{y^2}{4}dy = \frac{1}{4}\left(\frac{y^3}{3}\right)_0^4 = \frac{16}{3}$

$S_3 = \int_0^4 \frac{x^2}{4}dx = \frac{16}{3}$

$S_2 = 16-(S_1+S_3)=\frac{16}{3}$

$\therefore S_1;S_2;S_3 = 1:1:1$



Hence, (A), (C) and (D) are correct.

Linked Comprehension Type Questions

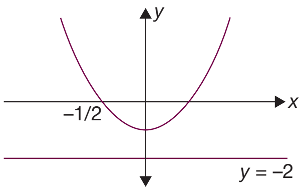
1. Equation of normal $y-y_1=-\frac{1}{m}(x-x_1)$

At $x=0$

$y = y_1+\frac{x_1}{m}=\sqrt{x_1^2+y_1^2}$

$\Rightarrow \frac{x_1}{m}=\sqrt{x_1^2+y_1^2}-y$

$\Rightarrow m=\frac{x}{\sqrt{x^2+y^2}-y}$



$\Rightarrow \frac{dy}{dx}=\frac{x}{(x^2+y^2-y)}\left(\sqrt{x^2+y^2}+y\right)$

$\Rightarrow \frac{dy}{dx}=\frac{1}{x}\left(\sqrt{x^2+y^2}+y\right)=\sqrt{1+\left(\frac{y}{x}\right)^2}+\frac{y}{x}$

Put $\frac{y}{x}=t; y=tx \Rightarrow \frac{dy}{dx}=t+x\frac{dt}{dx}$

$t+x\frac{dt}{dx}=\sqrt{1+t^2}+t$

$\Rightarrow \int\frac{dt}{\sqrt{1+t^2}}=\int\frac{dx}{x}\Rightarrow \ln\left(t+\sqrt{1+t^2}\right)=\ln x+\ln k$

$kx=t+\sqrt{1+t^2}$

$\Rightarrow kx^2=y+\sqrt{x^2+y^2}$

At point $M\left(-\sqrt{3},1\right)\Rightarrow 3k=1+\sqrt{3+1};\ 3k=3\Rightarrow k=1$

Put $k=1\Rightarrow x^2=y+\sqrt{x^2+y^2}\Rightarrow x^2-y=\sqrt{x^2+y^2}$

$\Rightarrow x^4+y^2-2x^2y=x^2+y^2\Rightarrow x^2-2y=1\Rightarrow y=\frac{x^2-1}{2}$

$f(x)=\frac{x^2-1}{2}$

$f'(x)=x$

$f''(x)=1$ no. point of intersection

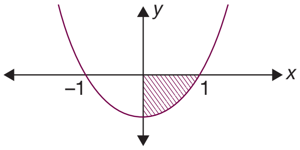
minima at $x=0$

$f'(x)=x|_{x=1}\Rightarrow \tan\theta=1\Rightarrow\theta=\tan^{-1}1$

Hence, the correct answer is (D).

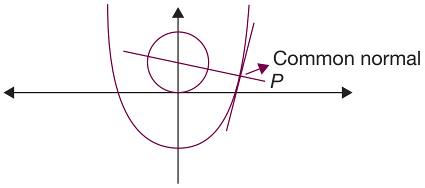
2. Required area $=\int_0^1\frac{1-x^2}{2}dx$

$=\frac{1}{2}\left(\frac{x-x^3}{3}\right)\Big|_0^1=\frac{1}{3}$



Hence, the correct answer is (D).

3. $f(x)=\frac{x^2-1}{2}$ equation of circle $x^2+y^2-2y=0$



$\Rightarrow f'(x)=x$

Equation of normal at $(x_1\, y_1)$ is $y-y_1=-\frac{1}{x_1}(x-x_1)$

Centre of circle $c(0, 1): 1-y_1=-\frac{1}{x_1}(0-x_1)$

$1-y_1=1\Rightarrow y_1=0$

$x_1^2-1=0\Rightarrow x_1=1$

$y_1=0$

$CP=\sqrt{1+1}=\sqrt{2} \quad P(1, 0)$

Minimum distance $=CP-r=\sqrt{2}-1$

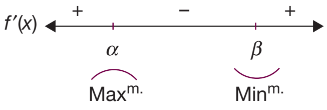
Hence, the correct answer is (B).

4. $f(x)=2x^3+3\left(1-\frac{3a}{2}\right)x^2+3(a^2-a)x+b$ where $a, b\in R$

$\Rightarrow f'(x)=6x^2+3\left(1-\frac{3a}{2}\right)2x+3(a^2-a)$

$\Rightarrow f''(x)=12x+3\left(1-\frac{3a}{2}\right)2=12x+3(2-3a)$

$=3[4x+2-3a]$



$f(x)$ has negative point of local minimum, then $\beta<0$

Both roots of $f'(x) = 0$ are negative

$f'(x)=6x^2+3(2-3a)x+3(a^2-a)$

$=3[2x^2+(2-3a)x+a^2-a]=6[(x-a/2)(x-a+1)]=0$

$x=\frac{a}{2}, a-1$

CASE 1: Let $\frac{a}{2} < a - 1 \Rightarrow a < 2a - 2 \Rightarrow a > 2$

For negative point of minima, greater root $a - 1 < 0 \Rightarrow a < 1$ not true as $a > 2$.
This case is not possible.

CASE 2: Let $a - 1 < \frac{a}{2} \Rightarrow a < 2$

For negative point of minima, greater root $\frac{a}{2} < 0 \Rightarrow a < 0$.

Final set of a, we get $a < 0$

$a \in (-\infty, 0)$

Hence, the correct answer is (A).

5. $g(x) = \frac{f'(x)}{6} = (x - \frac{a}{2})(x - a + 1)$

$\Rightarrow\ g(x) = x^2 - \frac{a}{2}x + (1 - a)x + \frac{a}{2}(a - 1);$

$\Rightarrow\ g'(x) = 2x - \frac{a}{2} + (1 - a) = 0$

$2x + 1 - 3\frac{a}{2} = 0 \quad \Rightarrow x_0 = 3\frac{a}{4} - \frac{1}{2}$

As given $g(x_0) < 0 \quad \Rightarrow \quad \left(\left(3\frac{a}{4} - \frac{1}{2}\right) - \frac{a}{2}\right)\left(\left(3\frac{a}{4} - \frac{1}{2}\right) - a + 1\right) < 0$

$\Rightarrow\ \left(\frac{a}{4} - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{a}{4}\right) < 0 - (a - 2)\,(a - 2) < 0;\ (a - 2)\,(a - 2) > 0$

$(a - 2)^2 > 0$

$a \neq 2$

Hence, the correct answer is (C).

6. $h''(x) = 6x - 4$

$h'(x) = 3x^2 - 4x + c \Rightarrow h'(1) = 0 \Rightarrow 3 - 4 + c = 0;\ c = 1$

$h(x) = x^3 - 2x^2 + cx + d$

$\Rightarrow\ h(1) = 5 = 1 - 2 + c + d$

$\Rightarrow\ c + d = 6 \Rightarrow d = 5$

$\Rightarrow\ h(x) = x^3 - 2x^2 + x + 5$

Required area $= \int_0^2 x^3 - x^2 + x + 5\ dx \qquad h\left(\frac{1}{3}\right) = \frac{1}{27} - \frac{2}{9} + \frac{1}{3} + 5$

$= \frac{x^4}{4} - \frac{2x^3}{3} + \frac{x^2}{2} + 5x\Big|_0^2$

$= 4 - \frac{16}{3} + 2 + 10 = 16 - \frac{16}{3} = \frac{32}{3}$

Hence, the correct answer is (D).

Combined solution to 7–9

Given, $A(c) = \int_0^c f(x)dx$ and $B(c) = \int_c^5 f(x)dx$

$\therefore\ A'(c) = f(c)$ …(1)

And $B'(c) = -f(c)$ …(2)

$\Rightarrow\ A'(c) + B'(c) = 0$

Now, $A(3) + B(3) = \int_0^3 f(x)dx + \int_3^5 f(x)\,dx$

$\Rightarrow\ A(3) + B(3) = \int_0^5 f(x)dx = 8$

Also, given $R(3) = 1 \quad \Rightarrow A(3) = B(3)$

Hence, $A(3) = B(3) = 4$ [Using equation (3)]

Now, $R(c) = \frac{A(c)}{B(c)}$

$\Rightarrow\ R'(3) = \frac{B(3)A'(3) - A(3)B'(3)}{(B(3))^2} = \frac{4 \cdot A'(3) - 4 \cdot B'(3)}{4^2}$

$\Rightarrow\ 7 = \frac{A'(3) - B'(3)}{4} = \frac{2f(3)}{4}$

[From equation (1) and (2), $A'(3) = f(3)$ and $B'(3) = -f(3)$]

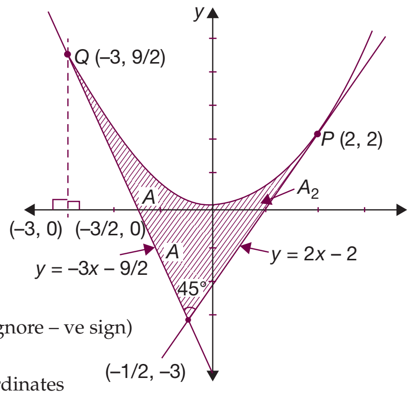
$\therefore\ f(3) = 14.$

7. **Hence, the correct answer is (B).**

8. **Hence, the correct answer is (B).**

9. **Hence, the correct answer is (B).**

Combined Solution 10–11



$C : 2y = x^2$

$\left.\frac{dy}{dx}\right]_P = x_1 = 2$

$y_1 = \frac{x_1^2}{2} = 2$

$\Rightarrow\ P \equiv (2, 2)$

Now, let m be the slope of l_2

$\therefore\ \frac{m - 2}{1 + 2m} = \pm 1$

$m - 2 = 1 + 2m$ (ignore – ve sign)

$m = -3$

If (x_2, y_2) are the coordinates of Q then

$x_2 = -3$ and $y_2 = \frac{x_2^2}{2} = \frac{9}{2}$

Hence, tangent at P is $y = 2x - 2$

Tangent at Q is $y = -3x - \frac{9}{2}$

On solving, we get $R = \left(\frac{-1}{2}, -3\right)$

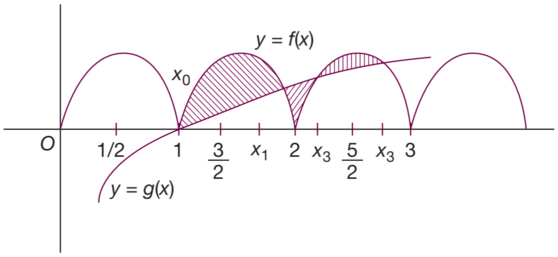
Now, $A = A_1 + A_2 + A_3 = \frac{125}{24}$,

Where $A_1 = \frac{90}{24}, A_2 = \frac{8}{24}, A_3 = \frac{27}{24}$

10. **Hence, the correct answer is (A).**

11. **Hence, the correct answer is (C).**

12.



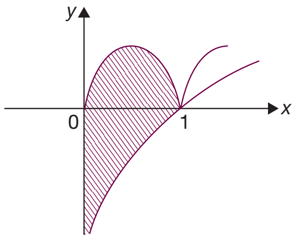
Hence, the correct answer is (A).

13. $x_{n+1} = x_3 \Rightarrow n = 2$

Hence, the correct answer is (B).

14. Required area $= \int_0^1 \sin^4 \pi x\ dx - \int_0^1 \ln x\ dx$

$= \frac{1}{4}\int_0^1 (1 - \cos 2\pi x)^2 dx - [\ln x.x - x]_0^1$



$$=\frac{1}{4}\int_0^1(1+\cos^2 2\pi x-2\cos 2\pi x)dx-[0-1]$$

$$=\frac{1}{8}\left[3x+\frac{\sin 4\pi x}{4\pi}-\frac{4\sin 2\pi x}{2\pi}\right]_0^1+1$$

$$=\frac{1}{8}[3]+1=\frac{11}{8}$$

Hence, the correct answer is (A).

Combined Solution 15–17

$$f(x)=3+\int_0^x\frac{1+\sin t}{t^2+2}dt\ \forall x\in R\Rightarrow f(0)=3$$

$$f'(x)=\frac{1+\sin x}{x^2+2}\Rightarrow f'(0)=\frac{1}{2}$$

$$f''(x)=\frac{(x^2+2)\cos x-(1+\sin x)\times 2x}{(x^2+2)^2};f''(0)=\frac{2-0}{2^2}=\frac{1}{2}$$

Let $P(x)=ax^2+bx+3$. Hence,

$$P'(x)=2ax+b\Rightarrow b=\frac{1}{2}\text{ and }P''(x)=2a=\frac{1}{2}\Rightarrow a=\frac{1}{4}$$

$$\therefore\quad P(x)=\frac{1}{4}x^2+\frac{x}{2}+3=\frac{1}{4}\left(x^2+2x+12\right)=\frac{1}{4}\left[(x+1)^2+11\right]$$

$$P'(x)=\frac{x+1}{2}\Rightarrow P'(-2)=-\frac{1}{2}$$

Equation of tangent at (–2, 3) is (–2, 3) is $y-3=-\frac{1}{2}(x+2)$

Put $x=0, y=3-1=2;\ y=-\frac{x}{2}+2$

$$\text{Area }=\int_{-2}^{0}\left(\frac{x^2+2x+12}{4}-\left(2-\frac{x}{2}\right)\right)dx$$

$$=\int_{-2}^{0}\frac{x^2+4x+4}{4}dx=\frac{1}{4}\times\int_{-2}^{0}(x+2)^2dx$$

$$=\frac{1}{12}\times\left[(x+2)^3\right]_{-2}^{0}=\frac{8}{12}=\frac{2}{3}$$

15. **Hence, the correct answer is (D).**

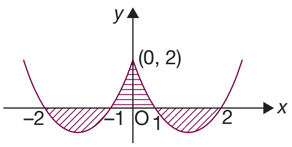
16. **Hence, the correct answer is (A).**

17. **Hence, the correct answer is (D).**

18. The graph of $f(|x|)$ is

$\therefore$ Required area

$$=2\int_0^1 f(x)\,dx+2\left|\int_1^2 f(x)dx\right|$$



$$=2\int_0^1(x^2-3x+2)dx+2\left|\int_1^2(x^2-3x+2)\right|dx$$

$$=2\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_0^1+2\left[\frac{x^3}{3}-\frac{3x^2}{2}+2x\right]_1^2$$

$$=2\left[\frac{1}{3}-\frac{3}{2}+2-0\right]+2\left[\frac{8}{3}-6+4-\frac{1}{3}+\frac{3}{2}-2\right]$$

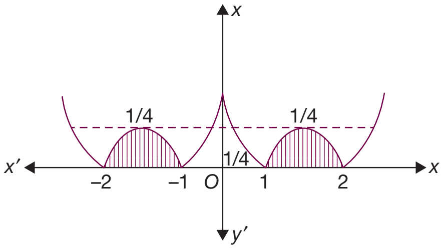
$$=2\times\frac{5}{6}+2\times\frac{1}{6}\quad=\frac{5}{3}+\frac{1}{3}=2$$

Hence, the correct answer is (A).

19. The graph of $|f(|x|)|$ is

$\because 1\le|x|\le 2\quad\because x\in[-2,-1]\cup[1,2]$

$$\because\text{ Required area}=2\left|\int_1^2 f(x)dx\right|=2\left|\int_1^2(x^2-3x+2)dx\right|$$



$$=2\left|\left\{\frac{x^3}{3}-\frac{3x^2}{2}+2x\right\}_1^2\right|=2\left|\left\{\left(\frac{8}{3}-\frac{12}{2}+4\right)-\left(\frac{1}{3}-\frac{3}{2}+2\right)\right\}\right|$$

$$=2\left|\frac{2}{3}-\frac{5}{6}\right|=\frac{1}{3}\text{sq. units}$$

Hence, the correct answer is (C).

20. For six real roots of $|f(|x|)|=a$

$$\Rightarrow\quad a=\frac{1}{4}.$$

Hence, the correct answer is (D).

21. $A=\int_0^{\frac{6e^4}{\pi^3 r}}\left(\frac{1}{\sqrt{6\pi^3 r}}\sqrt{x}-\frac{x}{6e^2}\right)dx=\left[\frac{1}{\sqrt{6\pi^3 r}}\frac{2}{3}x^{3/2}-\frac{x}{12e^2}\right]_0^{\frac{6e^4}{\pi^3 r}}=\frac{e^6}{\pi^6 r^2}$

$$\sqrt{A_r}=\left(\frac{e}{\pi}\right)^3\frac{1}{r}\quad\Rightarrow\quad\sqrt{A_1}\,\sqrt{A_2}\,\sqrt{A_3}\text{ .... are in H.P.}$$

Hence, the correct answer is (C).

22. $\sqrt{A_r\,A_{r+1}\,A_{r+2}}=\left(\frac{e}{\pi}\right)^9\frac{1}{r(r+1)(r+2)}=\frac{1}{2}\left(\frac{e}{\pi}\right)^9\left[\frac{1}{r(r+1)}-\frac{1}{(r+1)(r+2)}\right]$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\sqrt{A_r\,A_{r+1}\,A_{r+2}}=\frac{1}{2}\left(\frac{e}{\pi}\right)^9\frac{1}{1.2}=\frac{1}{4}\left(\frac{e}{\pi}\right)^9$$

Hence, the correct answer is (D).

23. $e^{\frac{-1}{\sqrt{A_r}}}=e^{-\left(\frac{\pi}{e}\right)^3\gamma}=\left(e^{\frac{\pi^3}{e^3}}\right)^{-\gamma}=k^{-}\gamma$

$$\lim_{n\to\infty}\left[e^{\frac{-1}{\sqrt{A_1}}}+e^{\frac{-1}{\sqrt{A_2}}}+e^{\frac{-1}{\sqrt{A_3}}}+\cdots n\text{ terms}\right]$$

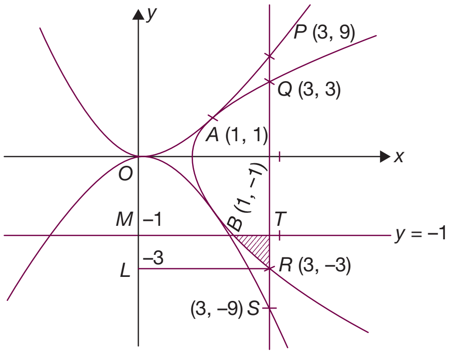
$$=\lim_{n\to\infty}\sum_{r=1}^{n}k^{-\gamma}=\frac{1}{k-1}=(k-1)^{-1}=\left(e^{\frac{\pi^3}{e^3}}-1\right)^{-1}$$

Hence, the correct answer is (B).

24. Area OAB

$$=2\int_0^1\left(\frac{y^2+3}{4}-\sqrt{y}\right)dy=2\left[\frac{1}{4}\left(\frac{y^3}{3}+3y\right)-\frac{2}{3}y^{3/2}\right]_0^1$$

$$=2\left[\frac{1}{4}\left(\frac{1}{3}+3\right)-\frac{2}{3}\right]=\frac{1}{3}$$



Hence, the correct answer is (D).

25. Area $APQ = \int_{1}^{3}\left(x^2 - \sqrt{(4x-3)}\right)dx$

$$= \left[\frac{x^3}{3} - \frac{2}{3}\frac{(4x-3)^{3/2}}{4}\right]_1^3 = \left[9 - \frac{9}{2} - \frac{1}{3} + \frac{1}{6}\right] = \frac{13}{3}$$

Hence, the correct answer is (A).

26. Area BTR = Area of rectangle LMRT – Area LMRB

$$= 6 - \int_{-3}^{-1}\frac{y^2+3}{4}dy = 6 - \left[\frac{y^2}{12} + \frac{3}{4}y\right]_{-3}^{-1}$$

$$= 6 - \left[-\frac{1}{12} - \frac{3}{4} + \frac{27}{12} + \frac{9}{4}\right] = \frac{7}{3} \text{ sq. unit}$$

Hence, the correct answer is (A).

Combined Solution to 27–28

$\therefore\ P''(x) = 0$ has only two roots so $P''(x)$ should be of 2nd degree only.

$\therefore\quad P(x)$ is of least degree polynomial.

$\therefore\quad P''(x) = P''(1) = 0$

So, $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$; $P''(x) = 2a_2 + 6a_3x + 12a_4x^2$

$a_2 = 0$ and $6a_3 + 12a_4 = 0 \Rightarrow a_3 = -2a_4$; $P(x) = a_0 + a_1x - 2a_4x^3 + a_4x^4$

$\therefore\quad P(-1) = 1$ and $P(1) = 2$ $\therefore\ P'(0) = 1$ so $a_1 = 1 \Rightarrow a_4 = \frac{1}{4}$

$\Rightarrow\quad a_0 = \frac{3}{2} - a_4$ and $a_1 = \frac{1+4a_4}{2}$;

So, $P(x) = \frac{1}{4}x^4 - \frac{x^3}{2} + x + \frac{5}{4}$;   So, $\int_{-2}^{2} f(x)dx = \frac{41}{5}$

$\because\quad A \equiv (-1,1) B \equiv (1,2), C = (0, 5/4)$; So, $\Delta ABC = \frac{1}{4}$

27. **Hence, (A), (B) and (C) are correct.**

28. **Hence, (C) and (D) are correct.**

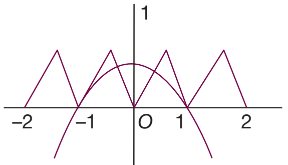
29. From graph, there are 4 solutions

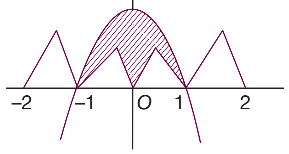
$2f(x) = 1 - x^2$

Hence, the correct answer is (B).

30. Area $= 2 \times \int_{0}^{1}(1-x^2)dx - 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{2}\right)$

$$= 2\left(1 - \frac{1}{3}\right) - \frac{1}{2} = \frac{4}{3} - \frac{1}{2} = \frac{5}{6}$$





Hence, the correct answer is (A).

Matrix Match/Column Match Type Questions

1. **A → r ; B → q ; C → t ; D → t**

(A) $[x]^2 = [y]^2 \quad 0 \le x \le 4$

For $0 \le x < 1$; $[x] = 0$; $[y]^2 = 0 \Rightarrow 0 \le y < 1$

For $1 \le x < 2$; $[x] = 1$; $[y]^2 = 1 \Rightarrow 1 \le y < 2$ and $-1 \le y < 0$

For $2 \le x \le 3$; $[x] = 2$; $[y]^2 = 4$; $2 \le y < 3$ or $-2 \le y < -1$

For $3 \le x < 4$; $[x] = 3$; $[y]^2 = 9$; $3 \le y < 4$ or $-3 \le y < -2$

Required area = 1 + (1 + 1) + (1 + 1) + (1 + 1) + 0 = 7

(B) $x + y \le 6$

Circle $x^2 + y^2 - 6y \le 0$ and Parabola $y^2 \le 8x$

$x^2 + (y-3)^2 \le 9$

$x = 6 - y$

At point A : $y^2 = 8x$; $y^2 = 8(y-6)$

$y^2 + 8y - 48 = 0$

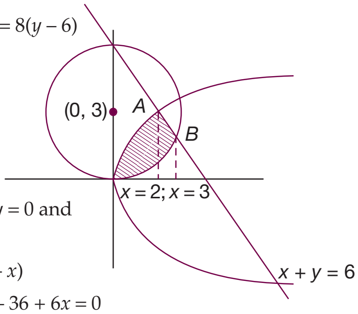
$\Rightarrow\quad (y + 12)(y - 4) = 0$

$\Rightarrow\quad y = -12, y = 4$

$x = 2$

$A(2, 4)$

At point B : $x^2 + y^2 - 6y = 0$ and $y = 6 - x$



$x^2 + (6-x)^2 - 6(6-x)$

$x^2 + x^2 - 12x + 36 - 36 + 6x = 0$

$2x^2 - 6x = 0$; $x = 3$

$B(3, 3)$

Required area $= \int_{0}^{2}\sqrt{8x} - (3 - \sqrt{9-x^2})dx + \int_{2}^{3} 6 - x - \left(3 - \sqrt{9-x^2}\right)dx$

$$= \frac{2\sqrt{2}x^{3/2}}{3/2} - 3x + \left(\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\right)\Bigg|_0^2 + 6x - \frac{x^2}{2} - 3x + \frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\frac{x}{3}\Bigg|_2^3$$

$$= \frac{4\sqrt{2}}{3}\left(2\sqrt{2}\right) - 6 + \sqrt{5} + \frac{9}{2}\sin^{-1}\frac{2}{3} + 18 - \frac{9}{2}\cdot\frac{\pi}{2} - 12 + 2 + 6 - \sqrt{5} - \frac{9}{2}\sin^{-1}\frac{2}{3}$$

$$= \frac{16}{3} - 6 + \frac{9\pi}{4} + \frac{9}{2} - 4 = \frac{9\pi}{4} + \frac{16}{3} + \frac{9}{2} - 10 = \frac{27\pi - 2}{12} \quad K = 27$$

(C) $y = x^2 + bx - b$

$\frac{dy}{dx} = 2x + b \Rightarrow m = 2 + b$ at point (1,1)

$$y-1=(2+b)(x-1)$$

$$A\left(1-\frac{1}{2+b},0\right)=\left(\frac{b+1}{2+b},0\right)$$

$$B(0,1-(2+b))=(0,-1-b)$$

$$\text{Area}=\frac{1}{2}\left(\frac{b+1}{2+b}\right)(-1-b)=\pm 2$$

We get $b=-3$

2. **A → r; B → s; C → q; D → p**

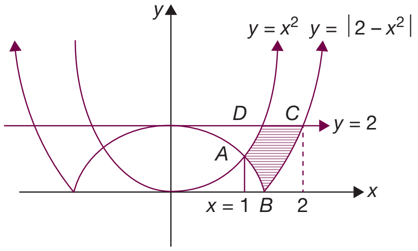
(A) The points in the graph are :

$$A(1,1),B(\sqrt{2},0),C(2,2),D(\sqrt{2},2)$$

Required area

$$=\int_{1}^{\sqrt{2}}\left(x^2-\left(2-x^2\right)\right)dx+\int_{\sqrt{2}}^{2}\left(2-\left(x^2-2\right)\right)dx$$

$$=\int_{1}^{\sqrt{2}}\left(2x^2-2\right)dx+\int_{\sqrt{2}}^{2}\left(4-x^2\right)dx$$

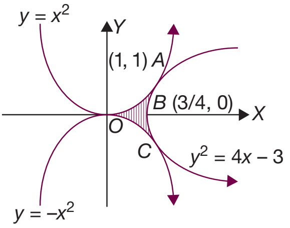


$$=\left(\frac{2x^3}{3}-2x\right)_{1}^{\sqrt{2}}+\left(4x-\frac{x^3}{3}\right)_{\sqrt{2}}^{2}=\left(\frac{20-12\sqrt{2}}{3}\right)\text{ sq.units.}$$

(B) The region bounded by the curves $y=x^2, y=-x^2$ and $y^2=4x-3$ is symmerical about x-axis.

Where $y^2=4x-3$ meets at (1, 1)

Hence, are ($OABCO$)



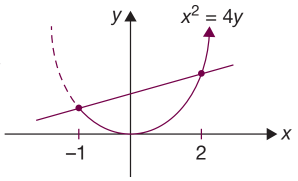
$$=2\left\{\int_{0}^{1}x^2dx-\int_{3/4}^{1}\left(\sqrt{4x-3}\right)dx\right\}$$

$$=2\left\{\left(\frac{x^3}{3}\right)_{0}^{1}-\left(\frac{(4x-3)^{3/2}}{(3/2)}\right)_{3/4}^{1}\right\}$$

$$=2\left\{\frac{1}{3}-\frac{1}{6}\right\}=2.\frac{1}{6}=\frac{1}{3}\text{ square units.}$$

(C) The curves $x^2=4y$ and $x=4y-2$ could be sketched as, whose point of intersection are $x=-1$, and

Thus,

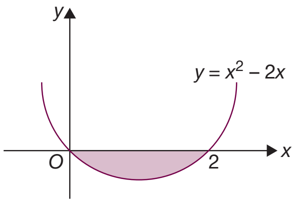


$$\text{Area}=\int_{-1}^{2}\left\{\left(\frac{x+2}{4}\right)-\left(\frac{x^2}{4}\right)\right\}dx$$

$$=\frac{1}{4}\left\{\frac{x^2}{2}+2x-\frac{x^3}{3}\right\}_{-1}^{2}$$

$$=\frac{1}{4}\left\{\left(2+4-\frac{8}{3}\right)-\left(\frac{1}{2}-2+\frac{1}{3}\right)\right\}=\frac{1}{4}\cdot\frac{9}{2}=\frac{9}{8}\text{ sq.units.}$$

(D) Here, slope of tangent



$$\Rightarrow\frac{dy}{dx}=\frac{(x+1)^2+y-3}{(x+1)},$$

$$\Rightarrow\frac{dy}{dx}=(x+1)+\frac{(y-3)}{(x+1)}$$

Put $x+1=X$ and $y-3=Y$

$$\Rightarrow\frac{dy}{dx}=\frac{dY}{dX},$$

$$\therefore\frac{dY}{dX}=X+\frac{Y}{X}\text{ or }\therefore\frac{dY}{dX}-\frac{1}{X}Y=X$$

Where integrating factor

$$=e^{\int\frac{1}{X}dX}=e^{-\log X}=\frac{1}{X}$$

$$\therefore\quad\text{Solution is, }Y\cdot\frac{1}{X}=\int X\cdot\frac{1}{X}dX+c\text{ or }\frac{Y}{X}=X+c$$

$$y-3=(x+1)^2+c(x+1),$$

which passes through (2, 0) $-3=1+c\Rightarrow c=-4$

∴ Required curve,

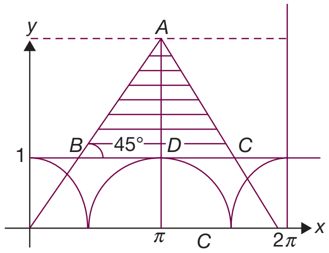
$$y=(x+1)^2-4(x+1)+3\text{ or }y=x^2-2x$$

Shows as, thus,

$$\text{Required area}=\left|\int_{0}^{2}\left(x^2-2x\right)dx\right|=\left|\left(\frac{x^3}{3}-x^2\right)_{0}^{2}\right|=\frac{4}{3}\text{ sq.units}$$

3. **A → q ; B → s ; C → p ; D → p**

(A) Tangent at $x=\pi$ to the curve

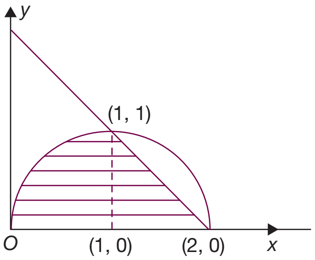


$y=|\cos x|$ parallel to x-axis and cuts the curve $y=\cos^{-1}(\cos x)$ at B and C

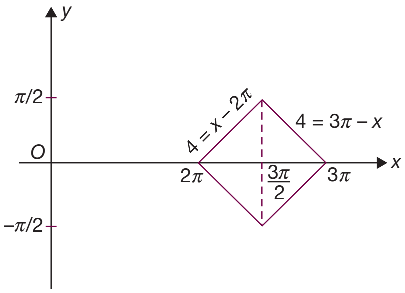
$$\therefore\quad AD=\pi-1$$

Area of $\Delta ABC=(\pi-1)^2$

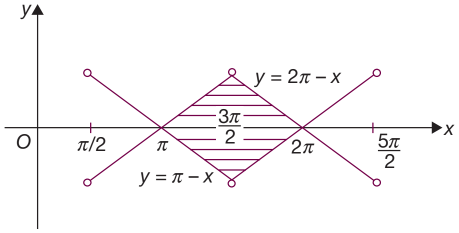
(B) $\text{Req. area}=\frac{\pi}{4}(1)^2+\frac{1}{2}\cdot 1\cdot 1=\frac{\pi}{4}+\frac{1}{2}$



(C) $\text{Req. area }2\left(\frac{1}{2}\times\pi\times\frac{\pi}{2}\right)=\frac{\pi^2}{2}$



(D) Req. area $= 2\left(\frac{1}{2}\times\pi\times\frac{\pi}{2}\right)=\frac{\pi^2}{2}$



4. A → r; B → p; C → s; D → q

(A) $\int_{-1}^{1}|x|x||dx=\int_{-1}^{1}|x^2|dx=2\int_{0}^{1}x^2dx=2\left(\frac{x^3}{3}\right)_0^1=\frac{2}{3}$

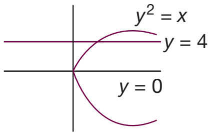
(B) $y-2=\sqrt{y}\Rightarrow t^2-t-2=0$ let $\sqrt{y}=t\Rightarrow(t-2)(t+1)=0$

$\Rightarrow y=4, x=2$

$A=\int_0^2\left(x+2-x^2\right)dx=\left(\frac{x^2}{2}+2x-\frac{x^3}{3}\right)_0^2=2+4-\frac{8}{3}=\frac{10}{3}$ sq.w

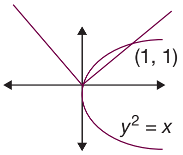
(C) $A=\int_0^1\left(\sqrt{x}-x\right)dx=\left(\frac{2}{3}x^{3/2}-\frac{x^2}{2}\right)_0^1$

$=\frac{2}{3}-\frac{1}{2}=\frac{1}{6}$



(D) $\int_0^4 y^2dy=\left(\frac{y^3}{3}\right)_0^4$

$=\frac{64}{3}$

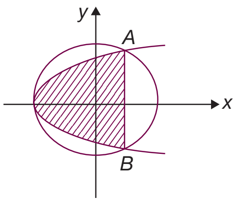


5. A → r, B → r, C → p, D → s

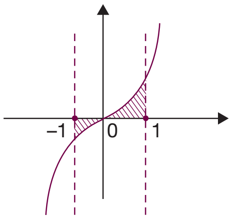
(A) Let $f(a)=A^2=\frac{64}{9}a(1-a)^3$

$f'(a)=0\ \Rightarrow a=\frac{1}{4}=\lambda$

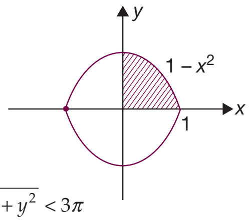
$8\lambda=2$



(B) $A=2\int_0^1 xe^x\ dx=2$



(C) $A=4\int_0^1(1-x^2)\ dx=\frac{8}{3}$



(D) Let $z=x+iy;\ |z|=\sqrt{x^2+y^2}$

$\sin|2|=\sin\sqrt{x^2+y^2}>0$

$\Rightarrow\quad 0<\sqrt{x^2+y^2}<\pi$ and $2\pi<\sqrt{x^2+y^2}<3\pi$

$\Rightarrow\quad 0<x^2+y^2<\pi^2$ and $4\pi^2<x^2+y^2<9\pi^2$

$\Rightarrow$ Required Area $=\pi.\pi^2+\pi(9\pi^2-4\pi^2)=6\pi^3$

Integer/Numerical Answer Type Questions

1. $y=(2011)^x\log^{(2011)}=k^x\log k$ (let $k=2011$)

$y=(2011)^{-x}\log^{(2011)}=\log k/k^x$

Area $=\int_0^1 k^x\log k-\int_0^1\log k.k^{-x}\,dx$

$=\log k\frac{\left\{k^x\right\}_0^1}{\log k}+\log k\frac{\left\{k^{-x}\right\}_0^1}{\log k}$

$=(k-1)+\left(\frac{1}{k-1}\right)=k+\frac{1}{k}-2$

$\Rightarrow\quad A=2009+\frac{1}{2011}$

$\Rightarrow\quad A+2=2011+\frac{1}{2011}$

$\Rightarrow\quad [A+2]=2011;\ \{A\}=\frac{1}{2011}$

$\therefore\quad [A+2]\ \{A\}=1$

Hence, the correct answer is (1).

2. $f(x,y)=\cos^{-1}(3|x|+4|y|)$

$-1\le 3|x|+4|y|\le 1$

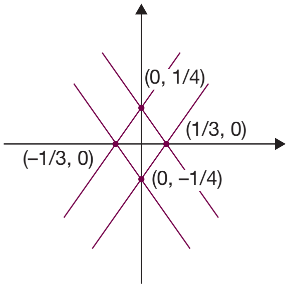
Area enclosed by lines

$3|x|+4|y|=1$

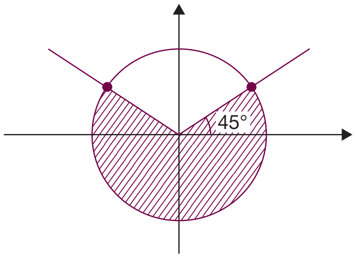
Area $=\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\left(\frac{2}{3}\right)=\frac{1}{12}=\frac{1}{6}$

$\Rightarrow\quad \mu=1/6\Rightarrow 6\mu=1$

Hence, the correct answer is (1).



3. $f(x,y)=\sqrt{16-x^2-y^2}-\sqrt{|x|-y}$



$\Rightarrow\quad 16-(x^2+y^2)\ge 0\Rightarrow x^2+y^2\le 16$

Also, $|x|-y\ge 0\Rightarrow y\le|x|$

Area $=16\pi/2+(2)\,\frac{3}{4}(\pi 16)=12\pi=k\pi$

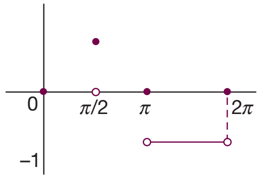
$\Rightarrow\quad k/2=6$

Hence, the correct answer is (6).

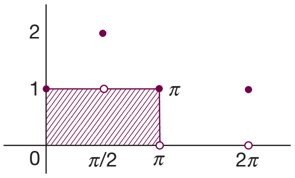
4. $f(x) = \cos^{-1}\left(\sin\left(\frac{\pi}{2} - 1 - [\sin x]\right)\right) = \cos^{-1}(\cos(1 + [\sin x]))$

∴ $f(x) = 1 + [\sin x] \ \forall\ x \in R$

Graph of [sin *x*]



f(*x*) = 1 + [sin *x*]



∴ Required area = π sq. unit.

Hence, the correct answer is (1).

5. $S_k = \int_0^1 x^2(1-x)^k\,dx = \int_0^1 (1-x)^2 x^k\,dx$

$$= \int_0^1 (x^k - 2x^{k+1} + x^{k+2})\,dx = \frac{1}{k+1} - \frac{2}{k+2} + \frac{1}{k+3}$$

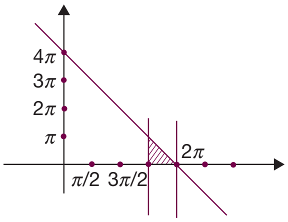
Now, $\lim_{n\to\infty}\sum_{k=1}^{n} S_k = \sum_{k=1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) - \sum_{k=1}^{\infty}\left(\frac{1}{k+2} - \frac{1}{k+3}\right)$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \frac{p}{q} \Rightarrow (p+q) = 7.$$

Hence, the correct answer is (7).

6. For $x \in [3\pi/2, 2\pi]$

$\sin^{-1}(\sin x) = x - 2\pi$ and $\cos^{-1}(\cos x) = 2\pi - x$



$y = |\cos^{-1}(\sin x)| - |\sin^{-1}(\cos x)|$

$y = |\pi/2 - \sin^{-1}(\sin x)| - |\pi/2 - \cos^{-1}(\cos x)|$ …(1)

∴ $y = |\pi/2 - (x - 2\pi)| - |\pi/2 - (2\pi - x)|$

$y = |5\pi/2 - x| - |x - 3\pi/2|$

$y = 5\pi/2 - x - x + 3\pi/2 = 4\pi - 2x$

∴ Required area $= (1/2)(\pi)(\pi/2) = \pi^2/4 = \pi^2/k$

⇒ $k = 4$

Hence, the correct answer is (4).

7. $f(x) = \frac{\pi}{2} - \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \frac{\pi}{2} - 2\tan^{-1}x$

$g(x) = \frac{\pi}{2} - \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{\pi}{2} + 2\tan^{-1}x$

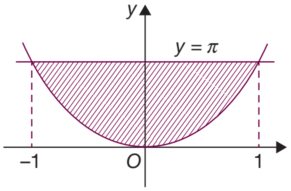
$y = f(x) + g(x) = \pi$

$y = \pi x^2$

Required area $= 2\pi - 2\int_0^1 \pi x^2\,dx$

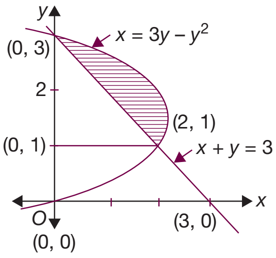
$A = 2\pi - 2\pi\cdot\frac{1}{3} = \frac{4\pi}{3} = 4.19$

∴ $[A] = 4$



Hence, the correct answer is (4).

8. Area between $x = 3y - y^2$ and $x + y = 3$



$A = \int_1^3 (3y - y^2) - (3 - y)\,dy$

$= \left[4\cdot\frac{y^2}{2} - \frac{y^3}{3} - 3y\right]_1^3$

$= (2\cdot 9 - 9 - 9) - \left(2 - \frac{1}{3} - 3\right)$

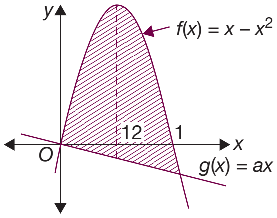
$A = \frac{4}{3}$

Curves $y = x - x^2 = ax$

$x^2 + (a-1)x = 0$

$x_1 + x_2 = 1 - a$

But $x_1 = 0,\ x_2 = 1 - a$



Now, area bounded by *f*(*x*) above the *x*-axis is

$$= \int_0^{1-a}\left((x - x^2) - (ax)\right)dx = \frac{4}{3}$$

Solving $\frac{1}{6}(1-a)^3 = \frac{4}{3}$

$(1-a)^3 = 8 \Rightarrow 1 - a = 2 \quad \Rightarrow a = -1$

⇒ $|[a]| = 1.$

Hence, the correct answer is (1).

9. $y^2 + y = x$

⇒ $\left(y + \frac{1}{2}\right)^2 = x + \frac{1}{4}$

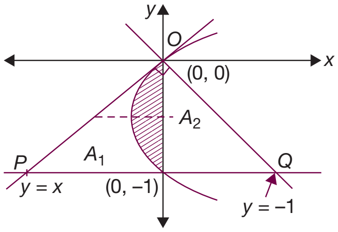
$(2y + 1)y' = 1$

$y' = \frac{1}{2y+1}$

$y'|_{(0,0)} = 1$

∴ Tangent at origin is $y = x$

ΔOPQ formed by $y = x$, $y = -1$ and $x + y = 0$ is right angled.



Area of ΔOPQ $= \frac{1}{2}\times 2\times 1 = 1$ sq. units.

Area of shaded region

$$= -\int_{-1}^{0}(y^2 + y)dy = -\left(\frac{y^3}{3} + \frac{y^2}{2}\right)_{-1}^{0} = -\left(0 - \left(\frac{1}{3} + \frac{1}{2}\right)\right)$$

$$= -\left(\frac{1}{3} + \frac{1}{2}\right) = \frac{1}{6}$$

∴ $\frac{A_1}{A_2} = \frac{\frac{1}{2} - \frac{1}{6}}{\frac{1}{2} + \frac{1}{6}} = \frac{4}{8} = \frac{1}{2}$

∴ $p + q = 3.$

Hence, the correct answer is (3).

10. $f(x) = \cos x$

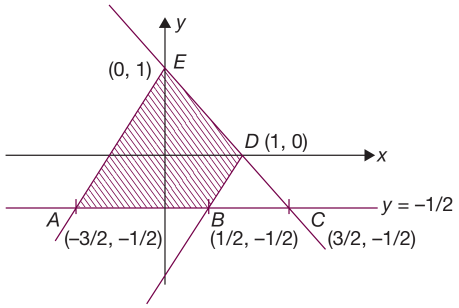
⇒ $f'(x) = -\sin x \quad \Rightarrow \quad f''(x) = -\cos x = 0$

⇒ $x = \pi/2,\ x = 3\pi/2$

$x = \pi/2$, point $P\ (\pi/2, 0)\ f'(x) = -\sin x = -1$

Tangent at $P\ x + y = \pi/2$ …(1)

$x = 3\pi/2$, point $Q(3\pi/2, 0)\ f'(x) = 1$



Tangent at $Q\ x - y = 3\pi/2$ ...(2)

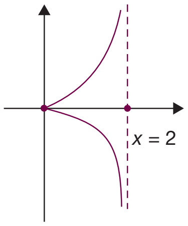
Point of intersection of above tangents $R(\pi, -\pi/2)$

Required area = $1/2PQ$.ordinate of $R = (1/2)\ (\pi)\ (\pi/2) = \pi^2/4 = k\pi^2$

$\Rightarrow k = 1/4$

Hence, the correct answer is (9).

11. $y^2 = \dfrac{-x^3}{(x-2)}$ i.e., $x \in [0, 2)$ for which curve exists



$$2yy' = \frac{(2-x)(3x^2)+x^3}{(2-x)^2}$$

$$= \frac{6x^2-2x^3}{(2-x)^2} = \frac{2x^2(3-x)}{(2-x)^2} \ge 0$$

Asymptote, $x = 2$

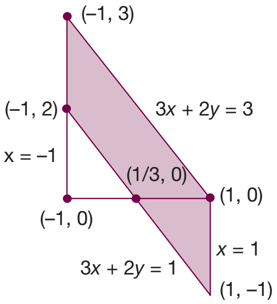
$ax - by + c = 0$

$a = 1, c = -2, b = 0$

$|a + b + c| = |-1| = 1$

Hence, the correct answer is (1).

12. $|x + y - 1| + |2x + y - 1| = 1$



CASE 1: $x + y - 1 > 0; 2x + y - 1 > 0$

$\Rightarrow\ 3x + 2y - 2 = 1$

$\Rightarrow\ 3x + 2y = 3$ ...(1)

CASE 2: $x + y - 1 > 0; 2x + y - 1 < 0$

$\Rightarrow\ x + y - 1 + 1 - y - 2x = 1$

$\Rightarrow\ -x - 1 = 0 \Rightarrow x = -1$ ...(2)

CASE 3: $x + y - 1 < 0; 2x + y - 1 > 0$

$\Rightarrow\ 1 - x - y + 2x + y - 1 = 1$

$\Rightarrow\ x = 1$ ...(3)

CASE 4: $x + y - 1 < 0; 2x + y - 1 < 0$

$\Rightarrow\ -(x + y - 1) - (2x + y - 1) = 1$

$\Rightarrow\ -x - y + 1 = 2x - y + 1 = 1$

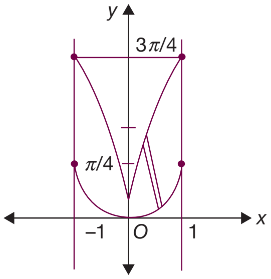
$\Rightarrow\ -3x - 2y + 1 = 0 \Rightarrow 3x + 2y = 1$ ...(4)

Now, area enclosed by lines $3x + 2y = 1; x = -1; x = 1; 3x + 2y = 3$

$$\text{Required area} = \frac{1}{2}\begin{vmatrix} 1 & -1 \\ 1 & 0 \\ -1 & 3 \\ -1 & 2 \\ 1 & -1 \end{vmatrix} = (1/2)\ (0 + 1 + 3 + 0 - 2 + 3 + 1 - 2) = 2$$

Hence, the correct answer is (2).

13. $|x| = \tan\theta$



$\therefore\ \ \theta = \tan^{-1}|x| \quad \Rightarrow \theta \in \left[0, \dfrac{\pi}{2}\right)$

$$f(\theta) = \sin^{-1}(\sin 2\theta) + \sin^{-1}(\sin\theta)$$

$$f(\theta) = \begin{bmatrix} 2\theta + \theta, & if\ \theta \in \left[0, \dfrac{\pi}{4}\right] \\ \pi - 2\theta + \theta, & if\ \theta \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right) \end{bmatrix}$$

$$f(\theta) = \begin{bmatrix} 3\theta = 3\tan^{-1}|x|, & if\ x \in [-1, 1] \\ \pi - \tan^{-1}|x|, & if\ |x| > 1 \end{bmatrix}$$

$$A = 2\int_0^1 (3\tan^{-1}x - \tan^{-1}x)\,dx$$

$$= 2\int_0^1 2\tan^{-1}x\,dx = 4\int_0^1 \tan^{-1}x\,dx$$

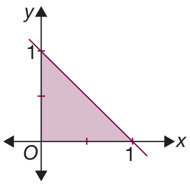
$$= 4\left[\left[\tan^{-1}x \cdot x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx\right]$$

$$= 4\left[\frac{\pi}{4} - \frac{\ln 2}{2}\right] = \pi - \ln 4 = -\ln p$$

$\therefore\ \ p = 4$

Hence, the correct answer is (4).

14. **CASE 1:**



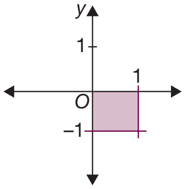
$x \ge 0$ and $y \ge 0$

$S_1 : 2x + 2y \le 2 \Rightarrow x + y \le 1$ and

S_2 : True

$A = \dfrac{1}{2} \times 1 \times 1 = \dfrac{1}{2}$

CASE 2:



$x \ge 0$ and $y \le 0$

$S_1 : 2x \le 2 \Rightarrow x \le 1$

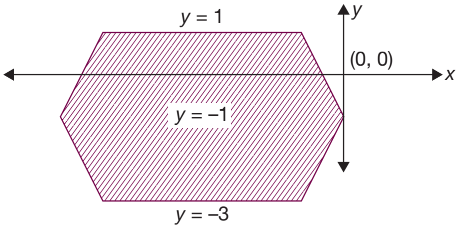
$S_2 : -2y \le 2 \Rightarrow y \le -1$

Area = 1.

$\therefore\ \ \text{Area} = \dfrac{1}{2} + 1 = \dfrac{3}{2} = A \Rightarrow 2A = 3.$

Hence, the correct answer is (3).

15. $y = f(x + 4) = \begin{cases} 2x + 8, & -4 \le x < -3 \\ 2, & -3 \le x < -1 \\ 2x, & -1 < x \le 0 \end{cases}$.



Curve of $y = -1 \pm f(x + 4)$ are shown in in the figure, and area of the required region $S = 12$

Hence, the correct answer is (4).

16. $$\frac{\left(\dfrac{3x - 4y + 5}{5}\right)^2}{4} + \frac{\left(\dfrac{4x + 3y - 10}{5}\right)^2}{2} = 1$$

$\therefore\ \ M = 4, m = 2\sqrt{2}$

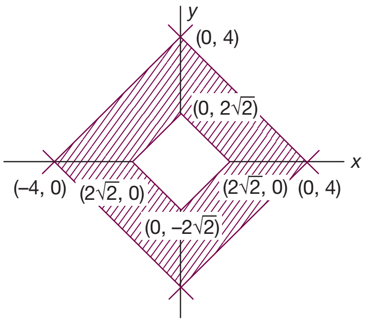
Major axis : $4x + 3y - 10 = 0$

Minor axis : $3x - 4y + 5 = 0$

$m \le p_1 + p_2 \le M$

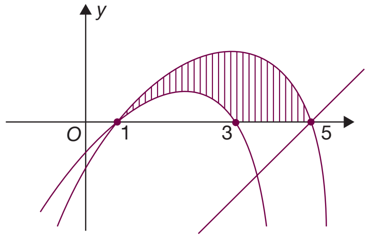
$\Rightarrow\ 2\sqrt{2} \le |x| + |y| \le 4$

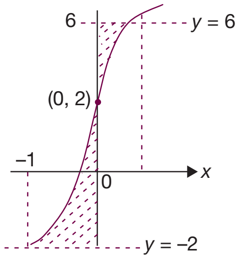
Required area = 32 – 16 = 16



Hence, the correct answer is (7).

17.





The required area will be equal to area enclosed by

$y = f(x)$, the y-axis between the abscissa at $y = -2$ and $y = 6$

Required area

$$= \int_0^1 \{6 - f(x)\}dx + \int_{-1}^0 [f(x) - (-2)]dx$$

$$\frac{9}{2} = \frac{m}{n} \Rightarrow m - n = 7$$

Hence, the correct answer is (7).

18. Area $= \left|\int_1^5 (6x - x^2 - 5)\,dx - \int_1^3 (4x - x^2 - 3)\,dx\right|$

$$+ \left|\int_3^4 (4x - x^2 - 3)dx\right| + \left|\int_4^5 (3x - 15)dx\right| = \frac{73}{6}$$

Hence, the correct answer is (6).

19. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x+0)}{h}$

$$\Rightarrow \lim_{h\to 0} \frac{f(x) + f(h) + h\sqrt{f(x)} - f(x) - f(0) - 0\sqrt{f(x)}}{h}$$

$$\Rightarrow \lim_{h\to 0}\left(\frac{f(h) - f(0)}{h - 0}\right) + \sqrt{f(x)}$$

$$f'(x) = \sqrt{f(x)}$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}dx = \int dx$$

$$2\sqrt{f(x)} = x + c$$

$$f(x) = \frac{x^2}{4}$$

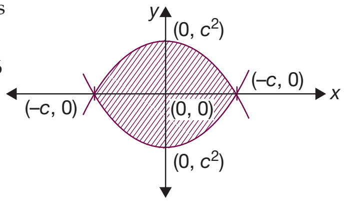
When $a = 0$ area is minimum

Required minimum area $= 2\int_0^9 2\sqrt{y}\,dy$

$$\Rightarrow 4\left(\frac{y^{3/2}}{3/2}\right)_0^9 = 72 \text{ sq. unit.}$$

Hence, the correct answer is (1).

20. (i) Clearly, required area is



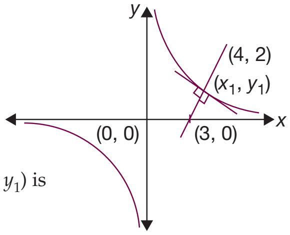
$$4\int_0^c (c^2 - x^2)\,dx = 576$$

$$\Rightarrow \left(c^2x - \frac{x^3}{3}\right)_{x=0}^{x=c} = 144$$

$$\Rightarrow c^3 - \frac{c^3}{3} = 144 \Rightarrow \frac{2}{3}c^3 = 144$$

$$\Rightarrow c^3 = 216 \quad \Rightarrow c = 6$$

$L = 6$



(ii) $xy = 8 \quad \Rightarrow xy' + y = 0$

$$\Rightarrow y' = \frac{-y_1}{x_1} = \frac{-8}{x_1^2}$$

∴ Equation of normal at (x_1, y_1) is

$$\left(y - \frac{-8}{x_1}\right) = \frac{x_1^2}{8}(x - x_1)$$

As (3, 0) lies on it, so

$$\frac{-8}{x_1} = \frac{x_1^2}{8}(3 - x_1) \Rightarrow -64 = x_1^3(3 - x_1)$$

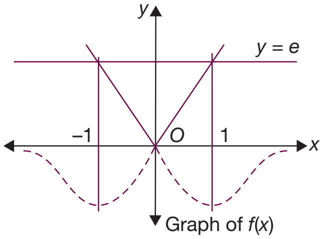
$$\Rightarrow x_1^4 - 3x_1^3 - 64 = 0 \Rightarrow x_1 = 4, y_1 = 2.$$

$M = 6$

$L + M = 12$

Hence, the correct answer is (12).

21. $f(x) = \begin{cases} xe^x, & x \ge 0 \\ -xe^{-x}, & x \le 0 \end{cases}$



Graph of $f(x)$

For maximum area, $a = e$

$$A_{\max} = 2\int_0^1 (e - ex^x)dx$$

$$= 2\left[ex - \left(xe^x - e^x\right)\right]_0^1$$

$$= 2(e - 1)$$

Hence, $[A] = 3$.

Hence, the correct answer is (3).

22. We have $f(x) = \begin{bmatrix} x^2; & x \in \left[0, \frac{\pi}{2}\right] \\ (\pi - x)^2; & x \in \left[\frac{\pi}{2}, \pi\right] \end{bmatrix}$,

$$g(x) = \begin{bmatrix} x; & x \in \left[0, \frac{\pi}{2}\right] \\ (\pi - x); & x \in \left[\frac{\pi}{2}, \pi\right] \end{bmatrix}$$

Both f and g are periodic with period $= \pi$ and symmetric about $x = \pi/2$.

$$\therefore \text{ Required area} = 2\left(\int_0^1 (x - x^2)dx + \int_1^{\pi/2} (x^2 - x)dx\right)$$

$$= 2\left(\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{\pi^3}{24}-\frac{1}{3}\right)-\left(\frac{\pi^3}{8}-\frac{1}{2}\right)\right)$$

$$= 2\left(\frac{\pi^3}{24}-\frac{\pi^2}{8}+\frac{1}{3}\right)=\frac{\pi^3}{12}-\frac{\pi^2}{4}+\frac{2}{3}\equiv a\pi^3+b\pi^2+c \text{ (Given)}$$

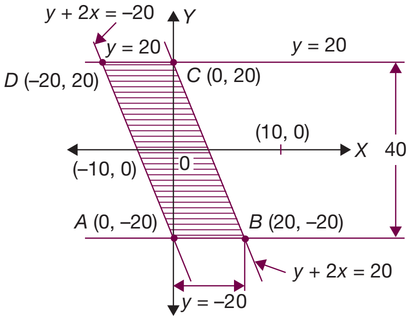
So, $a=\frac{1}{12}, b=\frac{-1}{4}, c=\frac{2}{3}$

Hence, $(|a| + |b| + |c|)$

$$= \left|\frac{1}{12}\right|+\left|\frac{-1}{4}\right|+\left|\frac{2}{3}\right|=\frac{1}{12}+\frac{1}{4}+\frac{2}{3}=\frac{1+3+8}{12}=\frac{12}{12}=1.$$

Hence, the correct answer is (1).

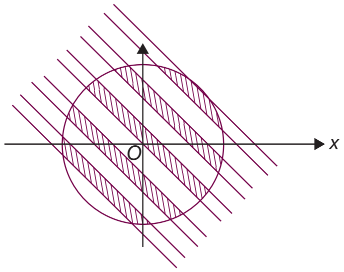
23. For $x \geq 0$, the equation is $(y + 20)\,(y + 2x - 20) = 0$ and for $x \leq 0$, the equation is $(y - 20)\,(y + 2x + 20) = 0$.



Clearly, S = area of parallelogram $ABCD = 20 \times 40 = 800$.

Hence, the correct answer is (800).

24. $x^2 + y^2 \leq 144$ and $\sin(x + y) \geq 0 \Rightarrow 2np \leq x + y \leq (2n + 1)\pi;\ n \in I$

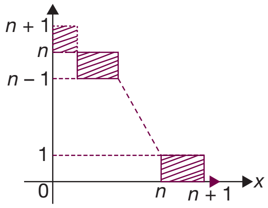


Hence, we get the area

$$S=\frac{\pi.144}{2} \Rightarrow \frac{S}{9\pi}=8$$

Hence, the correct answer is (8).

25. Area $= n + 1$



Hence, the correct answer is (1).

26. Let α, β be roots of the equation. $18x^2 - 9\pi x + \pi^2 = 0$

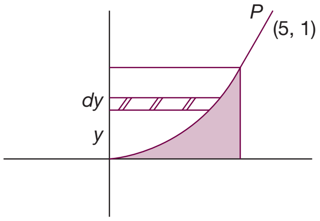
$$\alpha,\beta : 18x^2-9\pi x+\pi^2=0 \Rightarrow \alpha=\frac{\pi}{6},\beta=\frac{\pi}{3}$$

$$(fog)(x)=f(\cos x)=\cos^2 x$$

$$\text{Area} = \int_{\pi/6}^{\pi/3} \cos^2 x\, dx = \frac{\pi}{12}$$

Hence, the correct answer is (3).

27. Inverse of $y = 3x^3 + 2x$ is $x = 3y^3 + 2y$



Required area

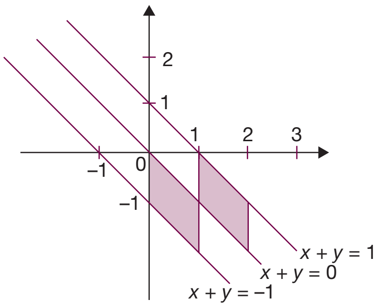
$$A=\int_0^1\left[5-(3y^3+2y)\right]dy=\left[5y-\left(\frac{3y^4}{4}+y^2\right)\right]_0^1$$

$$=5-\left(\frac{3}{4}+1\right)=\frac{20-7}{4}=\frac{13}{4}$$

$$4A = 13 \qquad \therefore \quad 4A - 7 = 6$$

Hence, the correct answer is (6).

28. If $x \in (0, 1)$



Then $-1 \leq x + y < 0$

And if $x \in [1, 2)$

$0 \leq x + y < 1$

Required area $= 2.2.1.\frac{1}{2}$

$= 2$ sq. unit

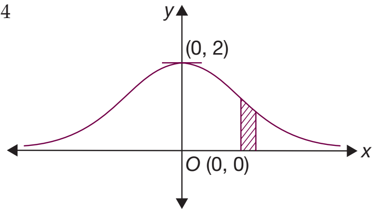
Hence, the correct answer is (2).

29. $f(0) = g(0) \quad \Rightarrow \quad 2 = a$

$f'(0) = g'(0) \quad \Rightarrow \quad 0 = -a \cdot c \quad \therefore \quad c = 0$

$f''(0) = g''(0) \quad \Rightarrow b = \frac{1}{4}$

$$\therefore \quad g(x)=\frac{2}{\frac{x^2}{4}+1}=\frac{8}{x^2+4}$$



$$A=2\int_0^{\infty}\frac{8}{x^2+4}dx$$

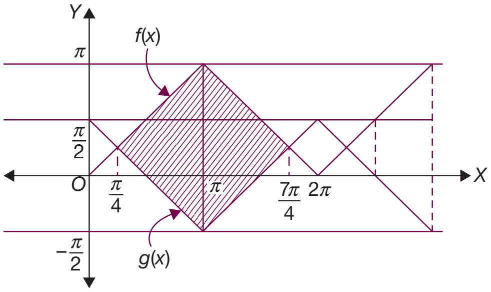
$$=16\cdot\frac{1}{2}\tan^{-1}\frac{x}{2}\Big]_0^{\infty}$$

$$=8\cdot\frac{\pi}{2}=4\pi \quad \therefore k=4$$

Hence, the correct answer is (4).

30. We have $g(x) = \sin^{-1}(\cos x) = \frac{\pi}{2} - \cos^{-1}(\cos x)$

Both the curves bound the regions of same area in



$\left[\frac{\pi}{4}, \frac{7\pi}{4}\right], \left[\frac{9\pi}{4}, \frac{15\pi}{4}\right]$ and so on

$\therefore$ Required area = area of shaded square = $\frac{9\pi^2}{8} \equiv \frac{a\pi^2}{b}$

$\therefore$ $a = 9$ and $b = 8$

Hence, a + b = 17.

Hence, the correct answer is (17).

31. Let the line PQ be $y = l(x)$

Obviously, $y = l(x)$ is above $y = x^2$

P and Q are the intersection point of line and parabola.

Now, $l(x) - x^2 = 0 \begin{cases} \alpha \\ \beta \end{cases}$, $\alpha < \beta$ where $l(x)$ is a linear function.

Also, $l(x) - x^2 \equiv \underbrace{-(x-\alpha)(x-\beta)}_{+ve}$

$\therefore$ $-\int_{\alpha}^{\beta} \underbrace{\underbrace{(x-\alpha)}_{I}\underbrace{(x-\beta)}_{II}}_{I.B.P.} dx = \frac{4}{3}$

$\Rightarrow$ $(\beta - \alpha)^3 = 8$ $\Rightarrow (\beta - \alpha) = 2$

Now, $2h = (\beta + \alpha)$

And $2k = (\alpha^2 + \beta^2)$ and $2 = (\beta - \alpha)$

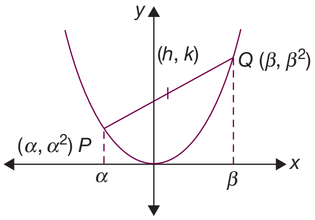
Again, $(\alpha + \beta)^2 + (\beta - \alpha)^2 = 2(\alpha^2 + \beta^2)$

$\Rightarrow$ $4h^2 + 4 = 2(2k)$

$\Rightarrow$ $h^2 + 1 = k \Rightarrow y = x^2 + 1$

$\therefore$ $a = 1, b = 0$ and $c = 1$ $\Rightarrow a + b + c = 2.$

Hence, the correct answer is (2).



32. $P = (2, 3)$

$$2(y-2)\frac{dy}{dx} = 1 \quad \Rightarrow \frac{dy}{dx} = \frac{1}{2(y-2)}$$

$\frac{dy}{dx} = \frac{1}{2}$. Equation of tangent, $(y-3) = \frac{1}{2}(x-2) \Rightarrow x - 2y + 4 = 0$

$$\text{Area} = \int_0^3 x\,dy = \int_0^3 (1 + (y-2)^2)\,dy = \left[y + \frac{(y-2)^3}{3}\right]_0^3$$

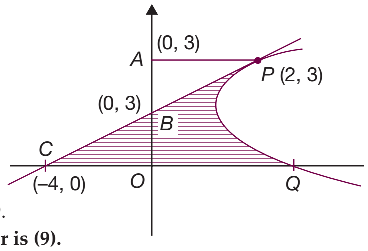
$$= 3 + \frac{1}{3} + \frac{8}{3} = 6$$

Area $ABP = \frac{1}{2}(1)(2) = 1$

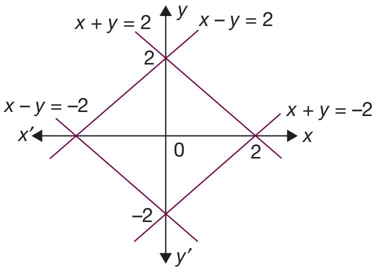
Area $BOC = \frac{1}{2}(4)(2) = 4$

Area $CPQC = 6 + 4 - 1 = 9.$

Hence, the correct answer is (9).



33. Sides of a square is $x = \pm 1, y = \pm 1$



Required area = 2×2 – Area of circle = $4 - \pi \times \frac{3}{\pi} = 1$ Sq. unit.

Hence, the correct answer is (1).

34. $2 \geq \max.\{|x-y|, |x+y|\}$

$\Rightarrow$ $|x-y| \leq 2$ and $|x+y| \leq 2,$

Which forms a square of diagonal length 4 units.

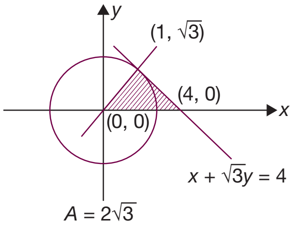
$\Rightarrow$ The area of the region is $\frac{1}{2} \times 4 \times 4 = 8$ sq.units

Hence, the correct answer is (8).

35. $A = \frac{1}{2} \times 1 \times \sqrt{3} + \frac{1}{2} \times (4-1) \times \sqrt{3}$

$$= \frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{2} = 2\sqrt{3}$$

Hence, the correct answer is (2).



36. Here $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}\frac{1}{x} + \tan^{-1}x$

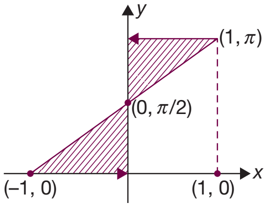
Domain $x \in [-1, 0) \cup (0, 1]$

For $x \in [-1, 0), f(x) = \frac{\pi}{2} - \pi + \frac{\pi}{2} = 0$

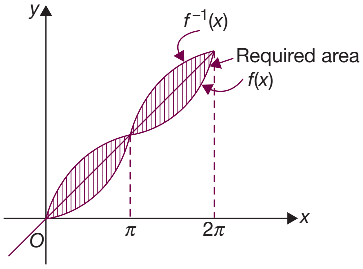
$x \in (0, 1], f(x) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

So, required area $= 2 \times \frac{1}{2} \times \frac{\pi}{2} \times 1 = \frac{\pi}{2}$

Hence, the correct answer is (2).



37. $f(x)$ is symmetric about $y = x$



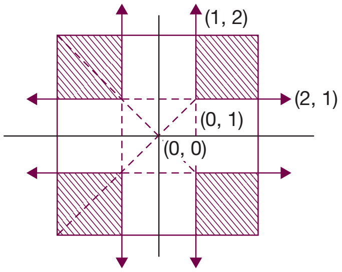
$$\text{Required area} = 2 \times \int_0^{2\pi} |f(x) - x|\,dx$$

$$= 2 \times \left[\int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} (-\sin x)\,dx\right] = 2\,(2 + 2) = 8$$

$\Rightarrow 2^k = 8 \Rightarrow k = 3$

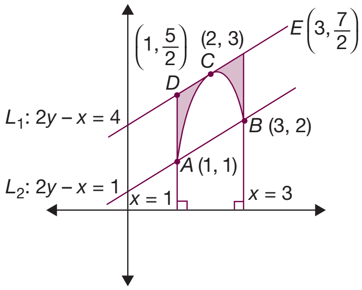
Hence, the correct answer is (3).

38. Area $= 4\times 1\times 1 = 4$.



Hence, the correct answer is (4).

39. Area of shaded region



= (area of trapezium) – (area under $f(x)$)

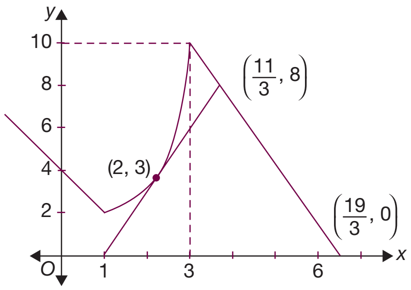
$$= \frac{1}{2}\left(\frac{5}{2}+\frac{7}{2}\right)\times 2-4$$

$$= 6-4 = 2.$$

Hence, the correct answer is (2).

40. $y = \begin{cases} 4-2x, & x<1 \\ (x-1)^3+2, & x\in[1,3] \\ 19-3x, & x\in(3,\infty) \end{cases}$

Tangent at $x = 2$, is $\Rightarrow (y-3) = 3(x-2) \Rightarrow 3x-y = 3$



$$\Rightarrow \Delta = \frac{1}{2}\begin{vmatrix} \frac{11}{8} & 8 & 1 \\ 1 & 0 & 1 \\ \frac{19}{3} & 0 & 0 \end{vmatrix} \Rightarrow \Delta = \frac{64}{3} \equiv \frac{p}{q}$$

$\therefore \quad p-19q = 64-57 = 7.$

Hence, the correct answer is (7).

41. $\displaystyle\int_{\frac{n\pi}{3}}^{\frac{(n+1)\pi}{3}} \left|e^{-2x}\sin 3x\right| dx$

$$= \left[\frac{e^{-2x}}{4+9}(2\sin 3x+3\cos 3x)\right]_{\frac{n\pi}{3}}^{(n+1)\frac{\pi}{3}} = \frac{3}{13}e^{-\frac{2n\pi}{3}}\left(1+e^{-\frac{2\pi}{3}}\right)$$

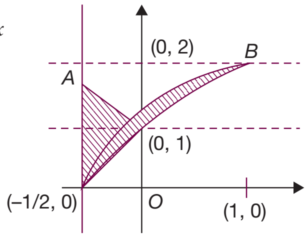
Here, $\dfrac{S_{n+1}}{S_n} = e^{-\frac{2\pi}{3}} \Rightarrow S_0, S_1, S_2 \ldots$ are in $G.P$ with common ratio $e^{-\frac{2\pi}{3}} < 1$. The sum exists. Also, $S_0 = \dfrac{3}{13}\left(1+e^{-\frac{2\pi}{3}}\right)$

Hence, the correct answer is (3).

42. $\displaystyle\int_{-1/2}^{0}\left[(1-x)-\left(\frac{-2x^2+5x+3}{3}\right)\right]dx$

$$+\int_{0}^{1}\left[\left(\frac{-2x^2+5x+3}{3}\right)-(1+x)\right]dx$$

$$= \frac{17}{36}\text{sq.units}.$$



Hence, the correct answer is (2).

43. Clearly, the given curves intersect at $x = 0, \dfrac{k}{k^2+1}$

$\therefore$ Required area

$$= \int_0^{\frac{k}{k^2+1}}\left((x-kx^2)-\frac{x^2}{k}\right)dx = \frac{1}{6\left(k+\frac{1}{k}\right)^2}$$

Now, for maximum area, $\left(k+\dfrac{1}{k}\right)^2$ will be minimum

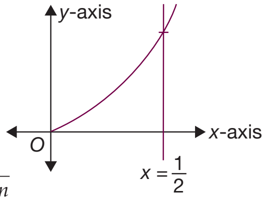
$\Rightarrow \; k+\dfrac{1}{k} = 2 \;\Rightarrow k = 1.$

Hence, the correct answer is (1).

44. $A_n = \displaystyle\int_0^{1/2} x^n\,dx$

$$A_n = \frac{1}{2^{n+1}(n+1)}$$

$\therefore \; 2^n A_n = \dfrac{1}{2(n+1)} \Rightarrow \dfrac{2^n A_n}{n} = \dfrac{1}{2(n+1)n}$



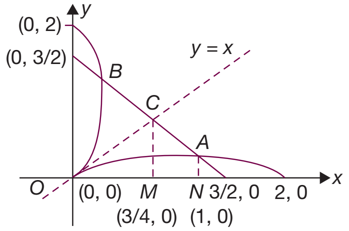
Hence, $\displaystyle\sum_{n=1}^{n}\frac{2^n A_n}{n} = \frac{1}{2}\sum_{n=1}^{n}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \frac{1}{2}\left(1-\frac{1}{n+1}\right)$

Given $\dfrac{1}{2}\left(1-\dfrac{1}{n+1}\right) = \dfrac{1}{3} \Rightarrow 1-\dfrac{2}{3} = \dfrac{1}{n+1}$

$\dfrac{1}{3} = \dfrac{1}{n+1} \Rightarrow n = 2.$

Hence, the correct answer is (2).

45. Clearly $f(x) = \dfrac{2x-x^2}{2}$



Since, $2x+2y = 3$; Passes through $A\left(1,\dfrac{1}{2}\right)$ and $B\left(\dfrac{1}{2},1\right)$ so bounded area

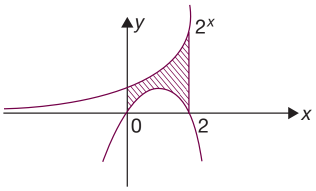
A = area OAB = 2[area OCM + area $CMNA$ – area ONA]

$$= 2\left[\frac{1}{2}\times\frac{3}{4}\times\frac{3}{4}+\frac{1}{2}\left(\frac{3}{4}+\frac{1}{2}\right)\times\frac{1}{4}-\frac{1}{2}\int_0^1\left(2x-x^2\right)dx\right]$$

$$= \frac{5}{24} \Rightarrow 24A = 5$$

Hence, the correct answer is (5).

46. The two curves do not intersect between $x = 0$ and $x = 2$. The curve $y = 2x - x^2 = -(x^2 - 2x) = -(x - 1)^2 + 1$ is a parabola with vertex (1, 1). Moreover, $2^x \geq 2x - x^2$ for $n \in [0, 2]$. Hence, the required area is



$$\int_0^2\left[2^x-\left(2x-x^2\right)\right]dx = \left.\frac{2^x}{\log 2}\right|_0^2 - \left.\left(x^2-\frac{x^3}{3}\right)\right|_0^2$$

$$= 4\left(\log 2\right)^{-1} - \left(\log 2\right)^{-1} - \left(4-\frac{8}{3}\right) = 3\left(\log 2\right)^{-1} - \frac{4}{3}$$

Hence, the correct answer is (4).

47. Here, $f(x)$ is an odd function

$\Rightarrow$ $g(x)$ is $f^{-1}(x)$

$$\therefore \quad \text{Area} = \int_1^{2\pi} f^{-1}(y)\,dy = \int_0^{2\pi} f'(x)\,dx$$

$$= \left[x\left(f(x)\right)\Big|_0^{2\pi} - \int_0^{2\pi}(x+\sin x)\,dx\right] = 2\pi^2$$

Hence, the correct answer is (2).

48. Required Answer $= \int_1^3 |y_1 - y_2|\,dx$

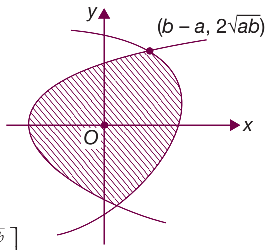
(where y_1, y_2 are roots of $2y^2 + 6(1+x)y + 5x^2 + 7x + 6 = 0$)

and use $\int_\alpha^\beta \sqrt{(\beta-x)(x-\alpha)}\,dx = \frac{\pi}{8}(\beta-\alpha)^2$

$$K = \frac{\pi}{8}(4) = \frac{\pi}{2}$$

Hence, the correct answer is (1).

49. $\Rightarrow \frac{8}{3}\sqrt{ab}(a+b) = \frac{16}{3}$



$$\Rightarrow \quad \sqrt{ab}\,\frac{(a+b)}{2} = 1$$

$$A = 2\int_0^{2\sqrt{ab}}\left\{\frac{4b^2-y^2}{4b} - \left(\frac{y^2-4a}{4a}\right)^2\right\}dx$$

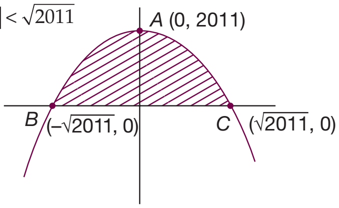
$$= 2\left[\left[\frac{4b^2y-\frac{y^3}{3}}{4b}\right]_0^{2\sqrt{ab}} - \left[\frac{\frac{y^3}{3}-4ay}{4a}\right]_0^{2\sqrt{ab}}\right]$$

$$= \frac{8}{3}\sqrt{ab}\,(a+b)$$

Hence, the correct answer is (1).

50. $x^2 + y = 2011 \Rightarrow x^2 = -(y - 2011)$

Parabola vertex A (0, 2011)



$$-\sqrt{2011} < x < \sqrt{2011} \Rightarrow |x| < \sqrt{2011}$$

$$\Rightarrow \quad 0 \leq \frac{|x|}{\sqrt{2011}} < 1$$

$$\therefore \quad \left[\frac{|x|}{\sqrt{2011}}\right] = 0$$

First curve will be $y = 0$

Required area $= \int_{-\sqrt{2011}}^{\sqrt{2011}} (2011 - x^2)dx = 2\left(\frac{2}{3}\right)(2011)^{3/2}$

$\lambda = 2/3$

Hence, the correct answer is (2).

51. Equation of line : $x + y = 3 \Rightarrow x + \frac{c}{x+1} = 3$

$$\Rightarrow x^2 + x + c = 3x + 3 \Rightarrow x^2 + 2x + (c-3) = 0$$

For tangency $D = 0 \Rightarrow 4 = 4(c-3) \Rightarrow c - 3 = 1 \Rightarrow \boxed{c = 4}$

$$y = \frac{4}{x+1}$$

Required Area $= \int_1^2 \frac{4}{x+1}dx = 4\ln\left(\frac{3}{2}\right)$

$$= 4\left\{\frac{\frac{1}{2}}{1} - \frac{\left(\frac{1}{2}\right)^2}{2} + \frac{\left(\frac{1}{2}\right)^3}{3}\ldots\right\} = 2\left\{1 - \frac{1}{2.2} + \frac{1}{2^2 3} - \ldots\infty\right\}$$

Clearly, $\lambda = 2$

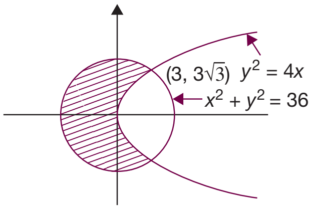
Hence, the correct answer is (2).

ARCHIVE: JEE MAIN

1. Required area $= \pi \times (6)^2 - 2\int_0^3 \sqrt{9x}\,dx - 2\int_3^6 \sqrt{36 - x^2}\,dx$

$$= 36\pi - 12\sqrt{3} - 2\left(\frac{x}{2}\sqrt{36-x^2} + 18\sin^{-1}\frac{x}{6}\right)_3^6$$

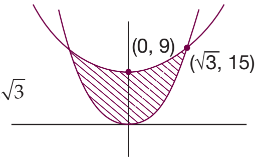
$$= 36\pi - 12\sqrt{3} - 2\left(9\pi - 3\pi - \frac{9\sqrt{3}}{2}\right) = 24\pi - 3\sqrt{3}$$



Hence, the correct answer is (C).

2. Required area $= 2\int_0^{\sqrt{3}} (2x^2 + 9 - 5x^2)dx$

$$= 2\int_0^{\sqrt{3}} (9 - 3x^2)dx = 2\left|9x - x^3\right|_0^{\sqrt{3}} = 12\sqrt{3}$$



Hence, the correct answer is (C).

3. $A_1 = \int_0^{\pi/4} (\cos x - \sin x\, dx)$

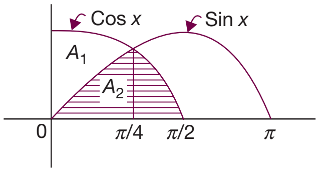
$A_1 = (\sin x + \cos x)_0^{\pi/4} = \sqrt{2} - 1$

$A_2 = \int_0^{\pi/4} \sin x\, dx + \int_{\pi/4}^{\pi/2} \cos x\, dx$

$$= (-\cos x)_0^{\pi/4} + (\sin x)_{\pi/4}^{\pi/2}$$

$A_2 = \sqrt{2}(\sqrt{2} - 1);$

$A_1 : A_2 = 1 : \sqrt{2}, A_1 + A_2 - 1$



Hence, the correct answer is (B).

4. $4y^2 = x^2(4 - x)\ (x - 2)$

$|y| = \frac{|x|}{2}\sqrt{(4-x)(x-2)} \quad \Rightarrow y_1 = \frac{x}{2}\sqrt{(4-x)(x-2)}$

And $y_2 = \frac{-x}{2}\sqrt{(4-x)(x-2)}$

$D : x \in [2, 4]$

Required Area $= \int_2^4 (y_1 - y_2)dx = \int_2^4 x\sqrt{(4-x)(x-2)}\ dx$ ...(1)

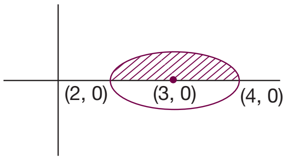
Applying $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Area $= \int_2^4 (6-x)\sqrt{(4-x)(x-2)}\ dx$ ...(2)

(1) + (2)

$2A = 6\int_2^4 \sqrt{1-(x-3)^2}\, dx$

$A = 3.\frac{\pi}{2}.1^2 = \frac{3\pi}{2}$



Hence, the correct answer is (C).

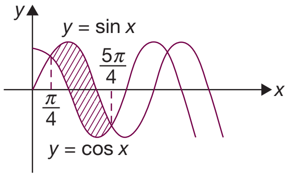
5. $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)dx$

$$= (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4}$$

$$= \left(-\left(\frac{-1}{\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}}\right)\right) - \left(-\left(\frac{1}{\sqrt{2}}\right) - \left(\frac{1}{\sqrt{2}}\right)\right)$$

$\Rightarrow \quad A = \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} = 2\sqrt{2}$

$\Rightarrow \quad A^4 = (2\sqrt{2}^4) = 16 \times 4 = 64$

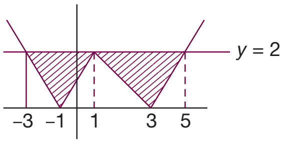


Hence, the correct answer is (64).

6. Remark: Given question was incomplete in exam it should be area bounded by

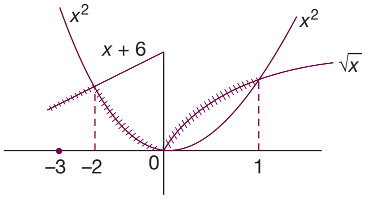
$y = ||x\text{-}1| - 2|$ and $y = 2$

Area $= 2\left(\frac{1}{2}.4.2\right)$



Hence, the correct answer is (8).

7. $f : [-3, 1] \to R$



$$f(x) = \begin{cases} \min\{(x+6), x^2\}, & -3 \le x \le 0 \\ \max\{\sqrt{x}, x^2\}, & 0 \le x \le 1 \end{cases}$$

Area bounded by $y = f(x)$ and x-axis

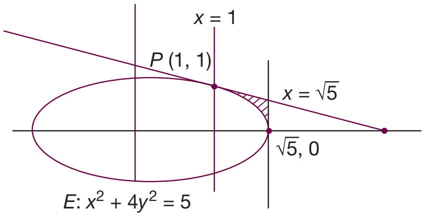
$$= \int_{-3}^{-2} (x+6)dx + \int_{-2}^{0} x^2\, dx + \int_0^1 \sqrt{x}\, dx$$

$$= \left[\frac{x^2}{2} + 6x\right]_{-3}^{-2} + \left[\frac{x^3}{3}\right]_{-2}^{0} + \left[\frac{2x^{3/2}}{3}\right]_0^1$$

$$= 2 - 12 - \frac{9}{2} + 18 + 0 + \frac{8}{3} + \frac{2}{3} \quad A = \frac{41}{6}; \quad 6A = 41$$

Hence, the correct answer is (41).

8. Tangent at $P : x + 4y = 5$



Required Area $= \int_1^{\sqrt{5}} \left(\frac{5-x}{4} - \frac{\sqrt{5-x^2}}{2}\right) dx$

$$= \left[\frac{5x}{4} - \frac{x^2}{8} - \frac{x}{4}\sqrt{5-x^2} - \frac{5}{4}\sin^{-1}\frac{x}{\sqrt{5}}\right]_1^{\sqrt{5}}$$

$$= \frac{5}{4}\sqrt{5} - \frac{5}{4} - \frac{5}{4}\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$$

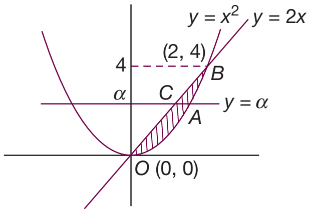
It we assume $\alpha, \beta, \gamma, \in Q$ (Not given in question)

Then $\alpha = \frac{5}{4}, \beta = -\frac{5}{4}$ and $\gamma = -\frac{5}{4}$

$|\alpha + \beta + \gamma| = 1.25$

Hence, the correct answer is (1.25).

9. $y \ge x^2$ upper region of $y = x^2$; $y \le 2x$ lower region of $y = 2x$



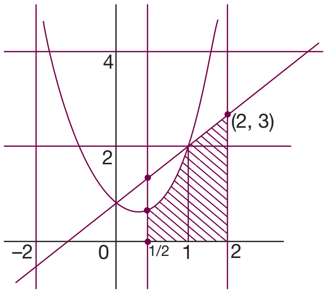
According to ques, area of $OABC$ = 2 area of OAC

$$\Rightarrow \int_0^4 \left(\sqrt{y}-\frac{y}{2}\right)dy = 2\int_0^{\alpha}\left(\sqrt{y}-\frac{y}{2}\right).dy$$

$$\Rightarrow \frac{4}{3} = 2\left[\frac{2}{3}\alpha^{3/2}-\frac{1}{4}.\alpha^2\right] \Rightarrow 3\alpha^2-8\alpha^{3/2}+8=0$$

Hence, the correct answer is (D).

10.

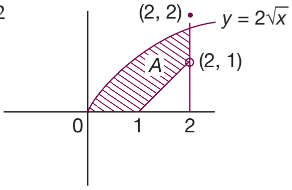


$$\text{Required area} = \int_{1/2}^{1}(x^2+1)\,dx + \frac{1}{2}(2+3)\times 1$$

$$= \frac{19}{24}+\frac{5}{2}=\frac{79}{24}$$

Hence, the correct answer is (C).

11. $(x-1)\,[x] \le y \le 2\sqrt{x}$ where $0 \le x \le 2$



$y = 2\sqrt{x} \Rightarrow y^2 = 4x \quad x \ge 0$

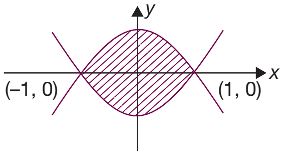
$$y = (x-1)\,[x] = \begin{cases} 0 & , \quad 0 \le x < 1 \\ x-1 & , \quad 1 \le x < 2 \\ 2 & , \quad x = 2 \end{cases}$$

$$A = \int_0^2 2\sqrt{x}\,dx - \frac{1}{2}\,1\cdot 1$$

$$A = 2\cdot\left[\frac{x^{3/2}}{(3/2)}\right]_0^2 - \frac{1}{2} = \frac{8\sqrt{2}}{3}-\frac{1}{2}$$

Hence, the correct answer is (A).

12. $y = x^2 - 1$ and $y = 1 - x^2$



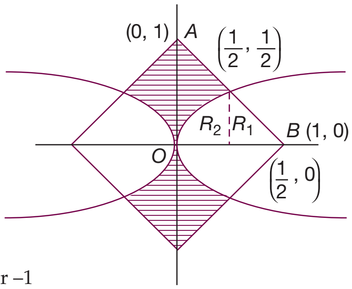
$$A = \int_{-1}^{1}((1-x^2)-(x^2-1))\,dx$$

$$A = \int_{-1}^{1}(2-2x^2)dx = 4\int_0^1(1-x^2)dx$$

$$A = 4\left(x-\frac{x^3}{3}\right)_0^1 = 4\left(\frac{2}{3}\right)=\frac{8}{3}$$

Hence, the correct answer is (B).

13. $|x| + |y| \le 1$ and $2y^2 \ge |x|$



For point of intersection

$x + y = 1$

$\Rightarrow \quad x = 1 - y$

$y^2 = \dfrac{x}{2}$

$\Rightarrow \quad 2y^2 = x$

$2y^2 = 1 - y \Rightarrow 2y^2 + y - 1 = 0$

$(2y-1)(y+1) = 0 \Rightarrow y = \dfrac{1}{2}$ or -1

Now, Area of $\Delta OAB = \dfrac{1}{2}\times 1\times 1 = \dfrac{1}{2}$

Area of region $R_1 = \dfrac{1}{2}\,\dfrac{1}{2}\,\dfrac{1}{2} = \dfrac{1}{8}$

Area of Region $R_2 = \dfrac{1}{\sqrt{2}}\displaystyle\int_0^{\frac{1}{2}}\sqrt{x}\,dx = \dfrac{1}{6}$

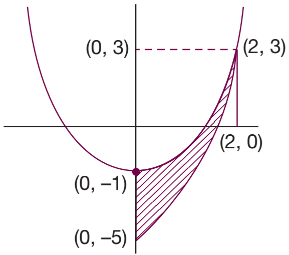
Now, area of shaded region in first quadrant

$= \text{Area of } \Delta OAB - R_1 - R_2 = \dfrac{1}{2}-\left(\dfrac{1}{6}\right)-\left(\dfrac{1}{8}\right)=\dfrac{5}{24}$

So, required area $= 4\left(\dfrac{5}{24}\right)=\dfrac{5}{6}$

Hence, the correct answer is (D).

14. $y = x^2 - 1 \Rightarrow \dfrac{dy}{dx} = 2x$



$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)_{(2,3)} = 4$

$(y-3) = 4(x-2) \Rightarrow y = 4x - 5$

$$\int_{-5}^{3}\left(\frac{y+5}{4}\right)dy - \int_{1}^{3}\sqrt{y+1}\,dy$$

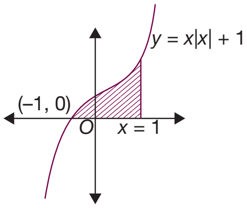
$$= \frac{1}{4}\left[\frac{y^2}{2}+5y\right]_{-5}^{3} - \left[2\frac{(y+1)^{3/2}}{3}\right]_{-1}^{3}$$

$$= \frac{1}{4}\left[\left(\frac{9}{2}+15\right)-\left(\frac{25}{2}-25\right)\right]-\frac{2}{3}[8]$$

$$= \frac{1}{4}\times\frac{64}{2}-\frac{64}{3}=8-\frac{16}{3}=\frac{8}{3}$$

Hence, the correct answer is (C).

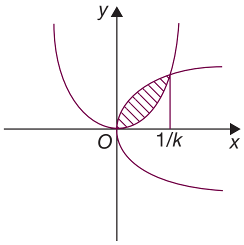
15. Required area $= \displaystyle\int_{-1}^{1}(x|x|+1)dx$



$= 0 + (x)_{-1}^{1} = 2$

Hence, the correct answer is (D).

16. $y = kx^2$



$x = ky^2 \Rightarrow x = k(kx^2)^2$

$\Rightarrow \quad x = k^3x^4 \Rightarrow x = 0$

$x = \dfrac{1}{k}$

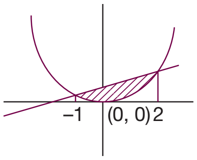
$$\text{Area} = \int_0^{1/k}\left(\sqrt{\frac{x}{k}}-kx^2\right)dx$$

$$= \frac{1}{\sqrt{k}}\frac{x^{3/2}}{3/2} - k\,\frac{x^3}{3}\Bigg|_0^{1/k} = 1 \Rightarrow \frac{2}{3k^2} - \frac{1}{3k^2} = 1$$

$$k^2 = \frac{1}{3} \Rightarrow k = \frac{1}{\sqrt{3}}$$

Hence, the correct answer is (D).

17. $x^2 = 4y$ and $x = 4y - 2$



Solving given curves $x + 2 = 4y \Rightarrow x = 4y - 2$

$$x^2 = 4y \Rightarrow (4y - 2)^2 = 4y$$

$$16y^2 + 4 - 16y = 4y$$

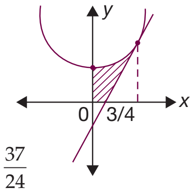
$$4y^2 - 5y + 1 = 0 \Rightarrow (4y - 1)(y - 1) = 0$$

$$y = 1 \text{ or } y = \frac{1}{4} \text{ and } x = 2 \text{ or } x = -1$$

$$A = \int_{-1}^{2}\left(\frac{x+2}{4} - \frac{x^2}{4}\right)dx = \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{8}$$

Hence, the correct answer is (D).

18. Tangent $y - 5 = 4(x - 2)$



$\Rightarrow\ y = 4x - 3$

$$\text{Area} = \int_0^2 (x^2 + 1)dx - \frac{1}{2} \times 5 \times \left(2 - \frac{3}{4}\right)$$

$$= \left(\frac{x^3}{3} + x\right)_0^2 - \frac{25}{8} = \frac{8}{3} + 2 - \frac{25}{8} = \frac{14}{3} - \frac{25}{8} = \frac{37}{24}$$

Hence, the correct answer is (D).

19. Here, $18x^2 - 9\pi x + \pi^2 = 0$

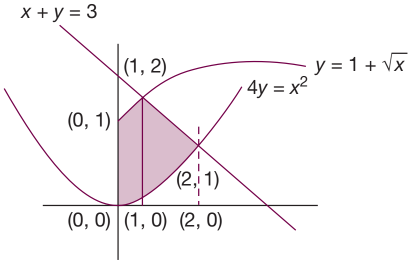
$\Rightarrow\ (3x - \pi)(6x - \pi) = 0$

$\Rightarrow\ \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{3}$

Also, $gof(x) = \cos x$ $\therefore$ Req. area $= \int_{\pi/6}^{\pi/3} \cos x dx = \frac{\sqrt{3} - 1}{2}$

Hence, the correct answer is (D).

20. Area of shaded region

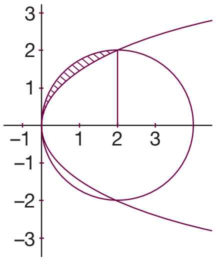


$$= \int_0^1 (1 + \sqrt{x})dx + \int_1^2 (3 - x)dx - \int_0^2 \frac{x^2}{4}dx$$

$$= x\Big]_0^1 + \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_0^1 + 3x\Big]_1^2 - \frac{x^2}{2}\Bigg]_1^2 - \frac{x^3}{12}\Bigg]_0^2 = \frac{5}{2}\text{sq.units}$$

Hence, the correct answer is (A).

21. $y^2 = 2x$ and $x^2 + y^2 = 4x$ meet at $O(0, 0)$ and $B(2, 2)$ {(2, –2) is not considered as $x, y \geq 0$}



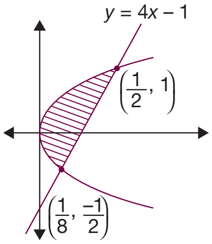
Now, required area $= \frac{1}{4}\pi \times 4 - \int_0^2 \sqrt{2x}dx$

$$= \pi - \sqrt{2}\frac{2}{3}\left[x^{3/2}\right]_0^2$$

$$= \pi - \frac{8}{3}$$

Hence, the correct answer is (C).

22. $y^2 \leq 2x$ represents the interior of parabola $y^2 = 2x$ (towards focus) and $y \geq 4x - 1$ represents the half region of plane to the left of the line $y = 4x - 1$.



$$\int_{-1/2}^{1}\left(\frac{y+1}{4} - \frac{y^2}{2}\right)dy$$

$$= \frac{1}{4}\left\{\frac{y^2}{2} + y\right\}_{-1/2}^{1} - \frac{1}{6}\left\{y^3\right\}_{-1/2}^{1}$$

$$= \frac{1}{4}\left\{\left(\frac{1}{2} + 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right) - \frac{1}{6}\left\{1 + \frac{1}{8}\right\}\right\}$$

$$= \frac{1}{4}\left\{\frac{3}{2} + \frac{3}{8}\right\} - \frac{1}{6}\left\{\frac{9}{8}\right\} = \frac{15}{32} - \frac{6}{32} = \frac{9}{32}$$

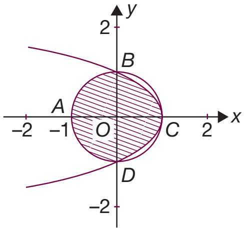
Hence, the correct answer is (D).

23. $x^2 + y^2 \leq 1$, represents the region inside the circle $x^2 + y^2 = 1$.

$y^2 \leq 1 - x$, represents the region inside parabola $y^2 = 1 - x$ towards focus.

The common region is as shown in the figure. The area of shaded region

= Area of semicircle *ADOBA* + Area of region *ODCBO*



$$= \frac{\pi}{2} + 2\int_0^1 (1 - y^2)dy$$

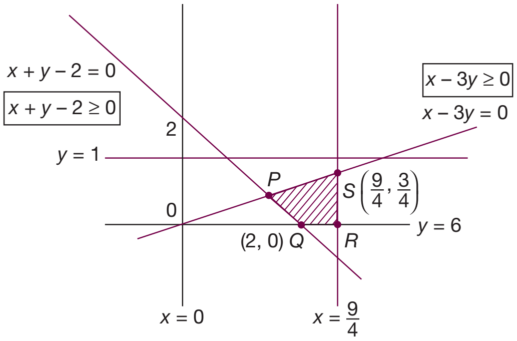
$$= \frac{\pi}{2} + 2\left[y - \frac{y^3}{3}\right]_0^1$$

$$= \frac{\pi}{2} + 2\left[1 - \frac{1}{3}\right] = \frac{\pi}{2} + \frac{2}{3}$$

Hence, the correct answer is (D).

ARCHIVE: JEE ADVANCED

1. $x + y - 2 = 0$



$$P\left(\frac{3}{2},\frac{1}{2}\right);\ Q(2,0);\ R\left(\frac{9}{4},0\right);\ S\left(\frac{9}{4},\frac{3}{4}\right)$$

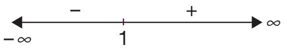
$$\text{Area} = \frac{1}{2}\left|\begin{vmatrix}\frac{3}{2} & \frac{1}{2}\\ 2 & 0\end{vmatrix} + \begin{vmatrix}2 & 0\\ \frac{9}{4} & 0\end{vmatrix} + \begin{vmatrix}\frac{9}{4} & \frac{3}{4}\\ \frac{3}{2} & \frac{1}{2}\end{vmatrix} + \begin{vmatrix}\frac{9}{4} & \frac{3}{4}\\ \frac{3}{2} & \frac{1}{2}\end{vmatrix}\right|$$

$$= \frac{1}{2}\left|(0-1)+(0-0)+\left(\frac{27}{16}-0\right)+\left(\frac{9}{8}-\frac{9}{8}\right)\right| = \frac{11}{32}$$

Hence, the correct answer is (A).

2. $g'(x) = \frac{1}{2}(e^{x-1} - e^{1-x}) = 0 \Rightarrow e^{x-1} = e^{1-x} \Rightarrow x = 1$

$g(1) = 1$ is minimum value of $g(x)$

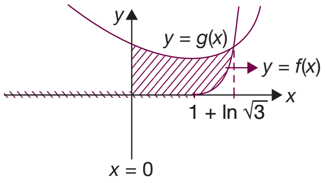


$$f(x) = \begin{cases} e^{x-1} - e^{-(x-1)} & x \ge 1 \\ e^{x-1} - e^{(x-1)} & x < 1 \end{cases} = \begin{cases} e^{x-1} - e^{-(x-1)} & x \ge 1 \\ 0 & x < 1 \end{cases}$$

For $x \ge 1$ we have $f'(x) = e^{x-1} + e^{-x+1} > 0$; $f(x)$ is an increasing function for $x \ge 1$

Point of intersection of both curves $f(x) = g(x)$

$$e^{x-1} - e^{1-x} = \frac{1}{2}(e^{x-1} + e^{1-x}) \Rightarrow e^{x-1} = 3e^{1-x} \Rightarrow x - 1 = \ln 3 + 1 - x$$



$$\Rightarrow 2x = 2 + \ln 3 \Rightarrow x = 1 + \ln\sqrt{3}$$

$$\text{Area} = \int_0^{1+\ln\sqrt{3}} \frac{1}{2}(e^{x-1} + e^{1-x})dx - \int_1^{1+\ell n\sqrt{3}} e^{x-1} - e^{1-x}\,dx$$

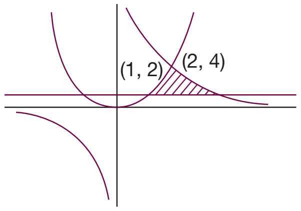
$$= \frac{1}{2}(e^{x-1} - e^{1-x})\Big|_0^{1+\ln\sqrt{3}} - \left(e^{x-1} + e^{1-x}\right)_1^{1+\ell n\sqrt{3}}$$

$$= \frac{1}{2}\left[\sqrt{3} - \frac{1}{\sqrt{3}} - (e^{-1} - e)\right] - \left(\sqrt{3} + \frac{1}{\sqrt{3}} - (1+1)\right)$$

$$= 2 - \sqrt{3} + \frac{1}{2}(e - e^{-1})$$

Hence, the correct answer is (A).

3. Given curves intersect at point (1, 1), (2, 4) and (8, 1)



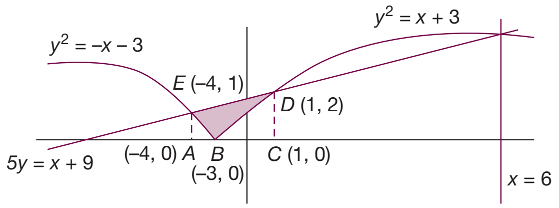
Lets calculate the shaded area with respect to y-axis

$$\text{Required area} = \int_1^4 \left(\frac{8}{y} - \sqrt{y}\right)dy = \left(8\ln y - \frac{2}{3}y^{3/2}\right)_1^4$$

$$= 16\ln 2 - \frac{14}{3}$$

Hence, the correct answer is (A).

4. Let draw given regions $y \ge \sqrt{|x+3|}$ and $5y \le x + 9 \le 15$



$$\text{Area } ABE \text{ (under parabola)} = \int_{-4}^{-3} \sqrt{-x-3}\,dx = \frac{2}{3}$$

$$\text{Area } BCD \text{ (under parabola)} = \int_{-3}^{1} \sqrt{x+3}\,dx = \frac{16}{3}$$

$$\text{Area of trapezium } ACDE = \frac{1}{2}(1+2)5 = \frac{15}{2}$$

$$\text{Required shaded area} = \frac{15}{2} - \frac{16}{3} - \frac{2}{3} = \frac{3}{2}$$

Hence, the correct answer is (C).

5. $F'(a) + 2 = \int_0^a f(x)dx$

Differentiating w.r.t. a

$F''(a) = f(a)$

$$F'(x) = 2\cos^2\left(x^2 + \frac{\pi}{6}\right)\cdot 2x - 2\cos^2 x$$

$$F''(x) = 4\cos^2\left(x^2 + \frac{\pi}{6}\right) - 16x^2\cos\left(x^2 + \frac{\pi}{6}\right)\sin\left(x^2 + \frac{\pi}{6}\right) + 4\cos x\sin x$$

$$F''(0) = f(0) = 4\cos^2\frac{\pi}{6} = 3.$$

Hence, the correct answer is (3).

6. $y_1 = \sin x + \cos x$

$y_2 = |\sin x - \cos x|$

$$\Rightarrow \text{Area} = \int_0^{\frac{\pi}{4}} ((\sin x + \cos x) - (\cos x - \sin x))dx$$

$$+ \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} ((\sin x + \cos x) - (\sin x - \cos x))dx$$

$$\Rightarrow \text{Area} = \int_0^{\frac{\pi}{4}} 2\sin x dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos x dx = -2\cos x |_0^{\pi/4} + 2\sin x |_{\pi/4}^{\pi/2} = 4 - 2\sqrt{2}$$

Hence, the correct answer is (B).

7. Let the equation of common tangent is : $y = mx + \frac{1}{m}$

$$\therefore \left| \frac{0+1+\frac{1}{m}}{\sqrt{1+m^2}} \right| = \frac{1}{\sqrt{2}}$$

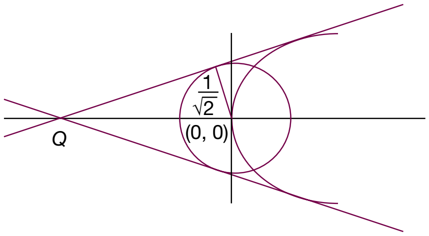
$\Rightarrow \quad m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$

$\therefore$ Equation of common tangents are

$y = x + 1$ and $y = -x - 1 \quad \therefore Q \equiv (-1, 0)$

$\therefore$ Equation of ellipse is : $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$

(A) $e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$



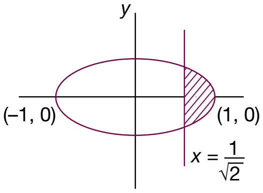
And latus rectum = $= \frac{2b^2}{a} = \frac{2\left(\frac{1}{\sqrt{2}}\right)^2}{1} = 1$

(C) $\therefore$ Required area $= 2 \cdot \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{\sqrt{2}} . \sqrt{1-x^2} dx$

$$= 2\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_{1/\sqrt{2}}^{1}$$

$$= \sqrt{2}\frac{\pi}{4} - \left(\frac{1}{4} + \frac{\pi}{8}\right)$$

$$= \sqrt{2}\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi - 2}{4\sqrt{2}}$$



Hence, (A) and (D) are correct.

8. $f(x) = 1 - 2x + e^x \int_0^x e^{-t} f(t)\, dt$

$\Rightarrow \quad f'(x) = -2 + e^x \int_0^x e^{-t} f(t)\, dt + f(x) \quad \Rightarrow \quad f'(x) - 2f(x) = 2x - 3$

$\Rightarrow \quad f'(x) = -2 + f(x) - 1 + 2x + f(x)$

Integrating factor $= e^{\int -2dx} = e^{-2x}$

$$f(x)e^{-2x} = \int e^{-2x}(2x-3)dx = e^{-2x}(1-x) + C$$

Put $x = 0,\ f(0) = 1 \Rightarrow C = 0$

$f(x)e^{-2x} = e^{-2x}(1-x) \Rightarrow f(x) = 1 - x$ Which passes through (2, –1)

Now region $1 - x \le y \le \sqrt{1-x^2}$

Area $= \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}$.

Hence, (B) and (C) are correct.

9. $\frac{1}{2}\int_0^1 \left(x - x^3\right)dx = \int_0^{\alpha} \left(x - x^3\right)dx$

$\Rightarrow \frac{1}{8} = \left|\frac{x^2}{2} - \frac{x^4}{4}\right|_0^{\alpha}$

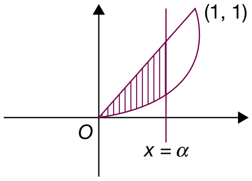
$\Rightarrow \frac{1}{2} = 2\alpha^2 - \alpha^4 \Rightarrow 2\alpha^4 - 4\alpha^2 + 1 = 0$

$f(x) = 2x^4 - 4x^2 + 1$

$f(1/2)f(1) < 0$

It means α lies between 1/2 to 1

Hence, (B) and (C) are correct.

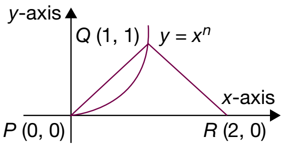


10. Area (ΔPQR) = 1 sq. unit

Required area $= \frac{1}{2} - \int_0^1 x^n dx = \frac{3}{10}$

$\Rightarrow \quad \frac{1}{n+1} = \frac{1}{5} \Rightarrow \quad n = 4.$

Hence, the correct answer is (4).



CHAPTER 11: DIFFERENTIAL EQUATION

Test Your Concepts-I

1. $xy\left(\frac{d^2y}{dx^2}\right)^{1/2} + \left(x^2\frac{dy}{dx}-1\right)^{3/4} = 0$

$\Rightarrow \quad xy\left(\frac{d^2y}{dx^2}\right)^{1/2} = -\left(x^2\cdot\frac{dy}{dx}-1\right)^{3/4}$

$\Rightarrow \quad (xy)^4\left(\frac{d^2y}{dx^2}\right)^{2} = \left(x^2\frac{dy}{dx}-1\right)^{3}$

$\therefore$ Order is 2, Degree is 2

2. $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/3} = \left(\frac{d^2y}{dx^2}\right)^{3}$

$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)^{3} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{5}$

$\therefore$ Order is 2 and degree is 3.

3. $y = (c_1 + c_2)\cos(x + c_3) - c_4e^x \,.\, e^{c_5}$

$\Rightarrow \quad y = A\cos(x + c_3) - Be^x$

Hence three effective arbitrary constants A, c_3 and B

Order of differential equation = 3

4. $y^2 = 2c\,(x + \sqrt{c})$

$\Rightarrow \quad 2y\frac{dy}{dx} = 2c \Rightarrow c = y\frac{dy}{dx}$

$\Rightarrow \quad y^2 = 2y\frac{dy}{dx}\left(x + \sqrt{\frac{ydy}{dx}}\right)$

$\Rightarrow \quad y - 2x\frac{dy}{dx} = 2y^{1/2}\left(\frac{dy}{dx}\right)^{3/2}$

$\Rightarrow \quad \left(y - 2x\frac{dy}{dx}\right)^2 = 4y\left(\frac{dy}{dx}\right)^3$ is differential equation of order 1 and degree 3.

5. The equation of family of hyperbola whose transverse and conjugate axis be coordinate axes is given by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. It has two arbitrary constant

$\Rightarrow \quad \frac{2x}{a^2} = \frac{2y_1y}{b^2} \Rightarrow \frac{b^2}{a^2} = \frac{yy_1}{x}$

Differentiating it

$$\frac{x(yy_2+(y_1)^2)-yy_1}{x^2} = 0$$

$\Rightarrow \quad \left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2} = \frac{y}{x}\frac{dy}{dx}$

6. (i) The given equation contains one constant. Differentiating the equation once, we get

$$2x - 2yy' = 2c\,(x^2 + y^2)(2x + 2y')$$

But $c = \frac{x^2-y^2}{(x^2+y^2)^2}$

Substituting for c, we get

$$(x - yy') = \frac{(x^2+y^2)(x^2-y^2)}{(x^2+y^2)^2}.2\,(x + yy')$$

$\Rightarrow \quad (x^2 + y^2)(x - yy') = 2\,(x^2 - y^2)(x + yy')$

$\Rightarrow \quad yy'\,[(x^2 + y^2) + 2\,(x^2 - y^2)] = x\,(x^2 + y^2) - 2x\,(x^2 - y^2)$

$yy'\,(3x^2 - y^2) = x\,(3y^2 - x^2)$

Hence, $\frac{dy}{dx} = \frac{x(3y^2-x^2)}{y(3x^2-y^2)}$

(ii) The given equation contains only one constant. Differentiating once, we get

$2a\,(y + a)\,y' = 3x^2$ …(1)

Multiplying by $y + a$, we get

$$2a\,(y + a)^2y' = 3x^2\,(y + a)$$

Using the given equation, we obtain

$$2x^3y' = 3x^2\,(y + a) \text{ or } 2xy' = 3y + 3a$$

or $\quad a = \frac{1}{3}(2xy' - 3y)$.

Substituting the value of a in (1) we obtain

$$\frac{2}{3}(2xy' - 3y)\left[y + \frac{1}{3}(2xy\cdot - 3y)\right]y' = 3x^2.$$

$\Rightarrow \quad \frac{2}{9}(2xy' - 3y)(2xy')y' = 3x^2$

Cancelling x, we obtain $8x\,(y')^3 - 12y\,(y')^2 - 27x = 0$

7. Given curve has two arbitrary constant so order of differential equation is 2

$y = A\cdot e^{2x} + B\cdot e^{-2x}$

$y' = 2A\,e^{2x} - 2B\cdot e^{-2x}$

$y'' = 4A\cdot e^{2x} + 4\,Be^{-2x} = 4\,(y)$

$\Rightarrow \quad y'' - 4\,y = 0$

8. The given equation is: $y = A\cos 2x + B\sin 2x$ …(1)

Diff. w.r.t. x, $\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$

Again diff. w.r.t. x, $\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x$

$= -4\,(A\cos 2x + B\sin 2x) = -4y$ [Using (1)]

Hence $\frac{d^2y}{dx^2} + 4y = 0$, which is the required differential equation.

9. Clearly A and B are arbitrary constant, order of differential equation is 2

$\frac{dy}{dt} = \omega A\cos(\omega t + \omega_0) - B\omega\sin(\omega t + \omega_0)$

$\Rightarrow \quad \frac{d^2y}{dt^2} = -\omega^2A\sin(\omega t + \omega_0) - B\omega^2\cos(\omega t + \omega_0)$

$\Rightarrow \quad -\omega^2(A\sin(\omega t + \omega_0) + B\cos(\omega t + \omega_0)) = -\omega^2y$

$\Rightarrow \quad \frac{d^2y}{dt^2} + \omega^2y = 0.$

10. $c\,(y + c)^2 = x^3$

$\Rightarrow \quad 2c\,(y + c)\cdot y' = 3x^2$

$\Rightarrow \quad 3x^2/2y' \times (y + c) = x^3$

$\Rightarrow \quad y + c = \frac{2y'\cdot x^3}{3x^2} \Rightarrow \frac{2xy'}{3}$

$\Rightarrow \quad c = 2xy'/3 - y \quad \Rightarrow \quad 2\,(2xy'/3 - y)\,(2xy'/3) - y' = 3x^2$

$\Rightarrow \quad 2\,(2xy' - 3y)\,(2xy')\cdot y' = 27x^x$

$\Rightarrow \quad 12y(y')^2 = x[3(y')^3 - 27]$

11. Differentiating the given equation $(y+c)^2 = cx$ ...(1)

$$2(y+c)\cdot\frac{dy}{dx} = c \Rightarrow 2y\cdot\frac{dy}{dx} = c - 2c\cdot\frac{dy}{dx} = c\left(1-2\cdot\frac{dy}{dx}\right)$$

$$\Rightarrow c = \frac{2y\cdot\frac{dy}{dx}}{1-2\cdot\frac{dy}{dx}}$$

$$\Rightarrow \left(y + \frac{2y\cdot\frac{dy}{dx}}{1-2\cdot\frac{dy}{dx}}\right)^2 = \frac{2y\cdot\frac{dy}{dx}}{1-2\cdot\frac{dy}{dx}}x$$

$$\Rightarrow \left(\frac{y}{1-2\frac{dy}{dx}}\right)^2 = x\frac{2y\frac{dy}{dx}}{1-2\frac{dy}{dx}} \Rightarrow y = 2x\frac{dy}{dx}\left(1-2\frac{dy}{dx}\right)$$

12. $y^2 = 4a(x+a)$ ...(1)

$$\Rightarrow 2y\cdot\frac{dy}{dx} = 4a \quad \Rightarrow y\cdot\frac{dy}{dx} = 2a \qquad \text{...(2)}$$

Eliminating a from (1) and (2) we get

$$y^2\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} - y^2 = 0$$

13. Differentiating the given differential equation $cy^2 = 2x + c$ ...(1)

$$c(2y)\frac{dy}{dx} = 2 \Rightarrow c = \frac{1}{y}\cdot\frac{dx}{dy}$$

Putting this value of *c* in equation (1) we get

$$\frac{1}{y}\cdot\frac{dx}{dy}\cdot y^2 = 2x + \frac{1}{y}\cdot\frac{dx}{dy} \Rightarrow y^2 = 2xy\cdot\frac{dy}{dx} + 1$$

14. $x^2 + y^2 + 2gx + 2fy + c = 0$

$\Rightarrow \quad 2x + 2yy' + 2g + 2fy' = 0$

$\Rightarrow \quad 2 + 2yy' + 2(y')^2 + 0 + 2fy'' = 0$

$\Rightarrow \quad yy' + 2y''\cdot 2y'\cdot y'' + 2fy''' = 0$

$$\Rightarrow \quad 2f = \frac{-yy'' - (y')^2}{y''}$$

$$\Rightarrow \quad yy''' + y''\cdot y' + 2y'\cdot y'' - fy'''\left(\frac{1+yy''+(y')^2}{y''}\right) = 0$$

$\Rightarrow \quad -y'''(1+(y')^2) + 3y'(y'')^2 = 0$

$\Rightarrow \quad y'''(1+(y')^2) - 3y'(y'')^2 = 0$

15. Equation of parabola whose axes is parallel to *x*-axis and having latus rectum '*a*' is

$(y-\beta)^2 = a(x-\alpha)$ has two arbitrary constant α, b

Differentiating both sides, we get

$$2(y-\beta)\frac{dy}{dx} = a$$

$$\Rightarrow 2(y-\beta)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow a\cdot\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^3 = 0.$$

Test Your Concepts-II

1. Let $x + y = v \quad \Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{dv}{dx} - 1 = \sec v \Rightarrow \frac{dv}{dx} = 1 + \sec v = \frac{\cos v + 1}{\cos v}$$

$$\Rightarrow \int\frac{\cos v\, dv}{1+\cos v} = \int dx \Rightarrow \int\frac{1+\cos v - 1}{1+\cos v}dv = \int dx$$

$$\Rightarrow v - \int\frac{dv}{1+\cos v} = \int dx \Rightarrow v - \int\frac{dv}{1+2\cos^2 v/2 - 1} = x + c$$

$$\Rightarrow v - \int\frac{1}{2}\sec^2\frac{v}{2}dv = x + c \Rightarrow v - \tan\frac{v}{2} = x + C$$

$\Rightarrow \quad (x+y) - \tan\dfrac{(x+y)}{2} = x + C$ is solution of given differential equation

2. Here $\dfrac{dy}{dx} = \{f(x) - y\}\, f'(x)$ ...(1)

Put $f(x) - y = z$; then $f'(x) - \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore \quad$ (1) becomes, $f'(x) - \dfrac{dz}{dx} = zf'(x) \quad \therefore \dfrac{dz}{dx} = (1-z)f'(x)$

or $\quad \dfrac{dz}{1-z} = f'(x)\,dx \quad \therefore \dfrac{dz}{1-z} = \displaystyle\int f'(x)dx$

or $\quad -\log(1-z) = f(x) - c$

$\therefore \quad f(x) + \log(1-z) = c$

$\therefore \quad f(x) + \log\{1 + y - f(x)\} = c$

3. $\dfrac{dy}{dx} = e^{ax+by} \Rightarrow \dfrac{dy}{dx} = e^{ax}\cdot e^{by}$

$\Rightarrow \quad \displaystyle\int e^{-by}dy = \int e^{ax}dx + c$ (c; constant)

$$\Rightarrow \quad -\frac{e^{-by}}{b} = \frac{e^{ax}}{a} + c$$

4. $x\dfrac{d^2y}{dx^2} = 1 \Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{1}{x}$

$$\Rightarrow \quad \frac{dy}{dx} = \log x + c_1$$

Again integrating, we get

$y = x\log x - x + c_1 x + c_2$

Given $y = 1$ and $\dfrac{dy}{dx} = 0$ at $x = 1$

$\Rightarrow \quad c_1 = 0$ and $c_2 = 2$

Therefore, the required solution is $y = x\log x - x + 2$

5. We have $\dfrac{dx}{x} = \dfrac{y\,dy}{1+y^2}$

Integrating, we get

$$\log|x| = \frac{1}{2}\log(1+y^2) + \log c$$

or $\quad |x| = c\sqrt{(1+y^2)}$

But it passes through (1, 0), so we get $c = 1$

$\therefore \quad$ Solution is $x^2 = y^2 + 1$

or $\quad x^2 - y^2 = 1$

6. $\sin\left(\dfrac{dy}{dx}\right) = a \Rightarrow \dfrac{dy}{dx} = \sin^{-1} a$

$\Rightarrow \quad dy = \sin^{-1} a.\, dx$

On integration, we get $y = x\sin^{-1} a + c$

But it passes through (0, 1), so $1 = 0 + c \Rightarrow c = 1$

Hence $y = x\sin^{-1} a + 1$

$$\Rightarrow \quad \frac{y-1}{x} = \sin^{-1} a \Rightarrow \sin\left(\frac{y-1}{x}\right) = a$$

7. $\dfrac{\ln(\sec x + \tan x)}{\cos x}.dx = \dfrac{\ln(\sec y + \tan yz)}{\cos y}.dy$

$\ln(\sec x + \tan x) = t$

$1/(\sec x + \tan x)\,(\sec^2 x + \sec x \tan x) = dt$

$= \int t.dt = \int z.dz \Rightarrow t^2/2 = z^2/2 + c$

$\Rightarrow \quad [\ln(\sec x + \tan x)]^2 - [\ln(\sec y + \tan y)]^2 = 2c$

8. $dy/dx + \dfrac{\sqrt{(x^2-1)}\cdot\sqrt{(y^2-1)}}{x\cdot y} = 0$

$\Rightarrow \quad dy \times y/\sqrt{(y^2-1)} = -\sqrt{x^2-1}\,/\,x.dx$

$y^2 - 1 = t^2 \qquad\qquad x^2 - 1 = z^2$

$y\,dy = dt/z \cdot 2t \qquad\qquad x\,dx = 2\,dz$

$\Rightarrow \quad \int \dfrac{t.dt}{t} = -\int \dfrac{(z)(zdz)}{(z^2+1)}$

$\Rightarrow \quad t = -\int \dfrac{(z^2+1-1)dx}{(z^2+1)}$

$\Rightarrow \quad t = -z + \tan^{-1}(z) + c$

$\Rightarrow \quad \sqrt{y^2-1} = -\sqrt{x^2-1} + \tan^{-1}(\sqrt{x^2-1})$

$\Rightarrow \quad \sqrt{y^2-1} + \sqrt{x^2-1} = \tan^{-1}(\sqrt{x^2-1}) + c$

$\Rightarrow \quad \sqrt{x^2-1} - \sec^{-1} x + \sqrt{y^2-1} = c$

9. $\dfrac{dy}{dx} = -\dfrac{x\left(1+y^2\right)}{y\left(1+x^2\right)}$

Put $y = vx \Rightarrow \dfrac{y}{1+y^2}dy = -\dfrac{x}{1+x^2}\dfrac{dv}{dx}$

$\Rightarrow \quad \ln(1+y^2) + \ln(1+x^2) = \ln k$

$\Rightarrow \quad \left(1+x^2\right)\left(1+y^2\right) = c$

10. $e^{dy/dx} = x + 1 \Rightarrow dy/dx = \ln(x+1)$

$\Rightarrow \quad \int dy = \int \ln(x+1)\,dx$

$\Rightarrow \quad y = \ln(x+1)(x+1) - (x+1) + c$

$(x, y) = (0, 3) \Rightarrow 3 = 0 - 1 + c \Rightarrow c = 4$

$\Rightarrow \quad y = (x+1)[\ln(x+1) - 1] + 4$

$\Rightarrow \quad y = (x+1)\ln(x+1) - x + 3$

11. The given equation $(x^3 - y^2x^3)\dfrac{dy}{dx} + y^3 + x^2y^3 = 0$

$\Rightarrow \quad \dfrac{1-y^2}{y^3}dy + \dfrac{1-x^2}{x^3}dx = 0$

$\Rightarrow \quad \int\left(\dfrac{1}{y^3} - \dfrac{1}{y}\right)dy + \int\left(\dfrac{1}{x^3} + \dfrac{1}{x}\right)dx = 0$

$\Rightarrow \quad \log\left(\dfrac{x}{y}\right) = \dfrac{1}{2}\left(\dfrac{1}{y^2} - \dfrac{1}{x^2}\right) + C$

12. $\dfrac{dy}{dx} = \cos(x+y) - \sin(x+y)$

Putting $x + y = t$, we get,

$\dfrac{dy}{dx} = \dfrac{dt}{dx} - 1$

Therefore, $\dfrac{dt}{dx} - 1 = \cos t - \sin t$

$\Rightarrow \quad \dfrac{dt}{1+\cos t - \sin t} = dx \Rightarrow \dfrac{\sec^2 t/2\,dt}{2(1-\tan t/2)} = dx$

Integrating, we get,

$-\ln\left|1 - \tan\dfrac{x+y}{2}\right| = x + C.$

Test Your Concepts-III

1. Given D.E. is $\dfrac{dy}{dx} = \dfrac{2x-y+1}{6x-5y+4}$. [Remark : This equation is in $\dfrac{dy}{dx} = \dfrac{ax+by+c}{a'x+b'y+c'}$ form in which $\dfrac{a}{a'} \neq \dfrac{b}{b'}$]

Put $x = X + h,\ y = Y + k \Rightarrow dx = dX,\ dy = dY$

$\therefore$ The equation becomes

$\dfrac{dY}{dX} = \dfrac{2X - Y + (2h - k + 1)}{6X - 5Y + (6h - 5k + 4)}$

Now, we choose h, k such that $\left.\begin{aligned} 2h - k + 1 = 0 \\ 6h + 5k + 4 = 0 \end{aligned}\right\}$

$\Rightarrow \quad \dfrac{h}{-4+5} = \dfrac{k}{6-8} = \dfrac{1}{-10+6}$

i.e., $h = \dfrac{1}{4},\ k = \dfrac{1}{2}$

$\therefore$ Actual substitution are

$x = X - \dfrac{1}{4},\ y = Y + \dfrac{1}{2},$

Which reduce the differential equation to $\dfrac{dY}{dX} = \dfrac{2X - Y}{6X - 5Y}$

This is a homogeneous D.E., and so we put

$Y = vX$ i.e., $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ The D.E. becomes

$v + X\dfrac{dv}{dX} = \dfrac{2-v}{6-5v}$ or $X\dfrac{dv}{dX} = \left(\dfrac{2-v}{6-5v} - v\right) = \dfrac{2 - 7v + 5v^2}{6-5v}$

or $\quad \dfrac{dX}{X} = \dfrac{6-5v}{2-7v+5v^2}dv$

Thus the variables are separated, and so the solution is

$\int\dfrac{dX}{X} = \int\dfrac{6-5v}{5v^2-5v-2v+2}dv = \int\dfrac{6-5v}{(5v-2)(v-1)}dv$

or $\quad \int\dfrac{1}{X}dX = \left(\dfrac{A}{5v-2} + \dfrac{B}{v-1}\right)$, let

$\Rightarrow \quad A(v-1) + B(5v-2) = 6 - 5v \Rightarrow A + 5B = -5,\ A + 2B = -6$

$\Rightarrow \quad A = -\dfrac{20}{3},\ B = \dfrac{1}{3}.$

$\therefore \quad \int\dfrac{1}{X}dX = \dfrac{1}{3}\left(\dfrac{-20}{5v-2} + \dfrac{1}{v-1}\right)dv$

$\log X + \log k = -\dfrac{4}{3}\log(5v-2) + \dfrac{1}{3}\log(v-1)$

or $\quad 3\log kX = \log\dfrac{v-1}{(5v-2)^2}$ or $k^3X^3 = \dfrac{v-1}{(5v-2)^4}$

or $\quad \dfrac{Y-X}{(5Y-2X)^4} = k^3$ or $\dfrac{\left(y-\frac{1}{2}\right) - \left(x+\frac{1}{4}\right)}{\left[5\left(y-\frac{1}{2}\right) - 2\left(x+\frac{1}{4}\right)\right]^4} = k^3.$

or $\quad \left(y - x - \dfrac{3}{4}\right) = k^3(5y - 2x - 3)^4.$

2. Write the differential equation as $y' = \dfrac{y^2+x^2}{2xy} = \dfrac{(y/x)^2+1}{2(y/x)}$.

The equation is homogeneous. Substituting, $y = vx$, we get

$$v + xv' = \frac{v^2+1}{2v} \text{ or } xv' = \frac{v^2+1}{2v} - v = \frac{1-v^2}{2v}$$

Separating the variables and integrating, we get

$$-\int \frac{2v\,dv}{v^2-1} = \int \frac{dx}{x} + \log c$$

or $\quad -\log(v^2-1) = \log x + \log c = \log(cx)$ or $\dfrac{1}{v^2-1} = c.$

Substituting $v = y/x$, we obtain

$$\frac{x^2}{y^2-x^2} = cx, \text{ or } x = c\,(y^2 - x^2)$$

Using the initial condition $y(1) = 2$, we find $1 = c(4-1)$ or $c = 1/3$. The solution is $y^2 - x^2 = 3x$.

3. Rewriting given differential equation $\dfrac{dy}{dx} = \dfrac{x^3+y^3}{3xy^2}$ (it is a homogenous diff. eq.)

Put $y = vx$ and $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{1+v^3}{3v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^3}{3v^2} - v = \frac{1-2v^3}{3v^2} \quad \Rightarrow \quad \frac{3v^2}{1-2v^3}\,dv = \frac{dx}{x}$$

Integrating both side we get

$$\int \frac{3v^2}{1-2v^3}\,dv = \int \frac{dx}{x} + \log c$$

$$\Rightarrow \quad -\frac{1}{2}\log(1-2v^3) = \log x + \log c$$

$$\Rightarrow \quad \frac{1}{(1-2v^3)^{1/2}} = c\,x$$

$$\Rightarrow \quad c^2x^2\left(1 - \frac{2y^3}{x^3}\right) = 1 \Rightarrow (x^3 - 2y^3) = kx$$

(remember we have replaced $\dfrac{1}{c^2}$ by k)

4. We have $\dfrac{dy}{dx} = \dfrac{xy}{x^2+y^2}$.

Solving the homogenous differential equation by writing $y = vx$, we get

$$-\frac{x^2}{2y^2} + \ln y = -\frac{1}{2}.$$

For $y = e$, $\dfrac{-x_0^2}{2e^2} + \ln e = -\dfrac{1}{2} \Rightarrow x_0{}^2 = 3e^2 \Rightarrow x_0 = \sqrt{3}\,e.$

5. The equation is homogeneous of degree 0. Put $x = vy$,

$$dx = v\,dy + y\,dv,$$

Then $(1 + 2e^v)(v\,dy + y\,dv) + 2e^v(1 - v)\,dy = 0$

$$\Rightarrow \quad (v + 2e^v)\,dy + y\,(1 + 2e^v)\,dv = 0$$

$$\Rightarrow \quad \frac{dy}{y} + \frac{1+2e^v}{v+2e^v}\,dv = 0$$

Integrating and replacing v by x/y

$$\ln y + \ln(v + 2e^v) = \ln C$$

or $\quad x + 2ye^{x/y} = C.$

6. Put $x = X$, $y^2 = Y$. Then $dx = dX$, $2y\,dy = dY$

$\therefore \quad 2y\dfrac{dy}{dx} = \dfrac{dY}{dX}$. Using this in the given equation, we get

$$y^3\frac{dy}{dx} + x + y^2 = 0; \quad \text{or} \quad \frac{1}{2}Y\frac{dY}{dX} + X + Y = 0$$

Which is a homogenous equation. So in order to solve the equation

Put $y^2 = vx$; then $2y\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

$\therefore \quad$ The equation becomes $y^3\dfrac{dy}{dx} + x + y^2 = 0$,

$$\text{or} \quad y^2 \cdot \frac{1}{2}\left(v + x\frac{dv}{dx}\right) + (x + y^2) = 0$$

$$\text{or} \quad vx \cdot \frac{1}{2}\left(v + x\frac{dv}{dx}\right) + x + vx = 0$$

$$\text{or} \quad \frac{v}{2}\left(v + x\frac{dv}{dx}\right) + 1 + v = 0$$

$$\therefore \quad 1 + v + \frac{v^2}{2} + \frac{vx}{2}\frac{dv}{dx} = 0$$

$$\text{or} \quad \frac{dx}{x} + \frac{v\,dv}{2\left(1 + v + \dfrac{v^2}{2}\right)} = 0$$

$$\therefore \quad \frac{dx}{x} + \frac{v\,dv}{v^2 + 2v + 2} = 0$$

$$\text{or} \quad \int \frac{dx}{x} + \frac{1}{2}\int \frac{(2v+2) - 2}{v^2+2v+2}\,dv = 0$$

$$\text{or} \quad \log x + \frac{1}{2}\int \frac{2v+2}{v^2+2v+2}\,dv - \int \frac{dv}{(v+1)+1} = 0$$

$$\text{or} \quad \log x + \frac{1}{2}\log(v^2 + 2v + 2) - \tan^{-1}(v+1) = c$$

$$\text{or} \quad \log x + \frac{1}{2}\log\left(\frac{y^4}{x^2} + 2\frac{y^2}{x} + 2\right) - \tan^{-1}\left(\frac{y^2}{x} + 1\right) = c$$

$$\text{or} \quad \frac{1}{2}\log(y^4 + 2y^2x + 2x^2) - \tan^{-1}\frac{y^2+x}{x} = c$$

7. $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$

Put $y = vx$

$$\Rightarrow \quad (1 - 3v^2) = (v^3 - 3v)\left(v + x\frac{dv}{dx}\right)$$

$$\Rightarrow \quad \frac{1 - 3v^2}{v^3 - 3v} - v = x\frac{dv}{dx}$$

$$\Rightarrow \quad \frac{1 - v^4}{v^3 - 3v} = x\frac{dv}{dx} \Rightarrow \frac{(v^3 - v - 2v)dv}{(1-v^2)(1+v^2)} = dx/x$$

$$\Rightarrow \quad -v\,dv/(1 + v^2) - 2v\,dv/(1 - (v^2)^2) = dx/x$$

$$\Rightarrow \quad -1/2\ln(1 + v^2) + \int \frac{dt}{t^2-1} = \ln(x) + \ln c$$

Put $v^2 = t$; $2v\,dv = dt$

$\Rightarrow \quad -1/2\ln(1 + v^2) + 1/2\ln|t - 1/t + 1| = \ln x + \ln c$

$\Rightarrow \quad -1/2\ln(1 + v^2) + 1/2\ln(v^2 - 1)/(v^2 + 1) = \ln(xc)$

$\Rightarrow \quad (v^2 - 1)^{1/2} = (xc)\cdot(1 + v^2) \Rightarrow (y^2 - x^2)^{1/2}/x = (xc)\dfrac{(x^2+y^2)}{x^2}$

$\Rightarrow \quad (y^2 - x^2)^{1/2} = c\,(x^2 + y^2) \Rightarrow (y^2 - x^2) = c^2(x^2 + y^2)^2$

$\Rightarrow \quad (y^2 - x^2) = k\,(x^2 + y^2)^2$

8. $y^2 = a - x \Rightarrow 2y\,dy/dx = da/dx - 1$

$$y^2 \cdot y\,dy/dx + x + y^2 = 0 \Rightarrow (a - x)\frac{(da/dx - 1)}{2} + a = 0$$

$\Rightarrow \quad (a - x)\,da/dx - a + x + 2a = 0$

$\Rightarrow \quad da/dx = a + x/x - a$ put $a = vx$

$\Rightarrow \quad v + x\,dv/dx = 1 + v/1 - v \Rightarrow x\,dv/dx = 1 + v/1 - v - v$

$\Rightarrow \quad x\,dv/dx = 1 + v^2/1 - v \Rightarrow (1 - v)dv/(1 + v^2) = dx/x$

$$\Rightarrow \quad \int \frac{dv}{1+v^2} - 1/2\int \frac{2v\,dv}{1+v^2} = \int \frac{dx}{x}$$

$\Rightarrow \quad \tan^{-1}(v) - 1/2\ln|1 + v^2| = \ln x + \ln c$

$$\Rightarrow \quad \tan^{-1}(a/x) - 1/2 \ln|1 + a^2/x^2| = \ln x + c$$

$$\Rightarrow \quad \frac{1}{2}\ln|x^2 + a^2| - \tan^{-1}\left(\frac{a}{x}\right) = c, \text{ where } a = x + y^2$$

9. $(x - y)\,dy = (x + y + 1)dx$

Reducing to Homogenous type $x = X + h$; $y = Y + k$

$$\Rightarrow \quad dy/dx \Rightarrow dY/dX = \frac{(X+Y)+(h+k+1)}{(X-Y)+(h-k)}$$

Where $h + k + 1 = 0$; $h - k = 0 \Rightarrow h = -1/2$; $k = -1/2$

$$\Rightarrow \quad dY/dX = X + Y/X - Y$$

Put $Y = vX \Rightarrow dY/dX = v + xdv/dX$

$$\Rightarrow \quad v + Xdv/dX = 1 + v/1 - v \Rightarrow Xdv/dX = 1 + v^2/1 - v$$

$$\Rightarrow \quad (1 - v)\,dv/(1 + v^2) = dX/X \Rightarrow \int\frac{dv}{1+v^2} - \int\frac{vdv}{1+v^2} = dX/X$$

$$\Rightarrow \quad \tan^{-1}(v) - 1/2\ln(1 + v^2) = \ln X + c$$

$$\Rightarrow \quad \tan^{-1}(Y/X) - 1/2\ln(1 + Y^2/X^2) = \ln X + c$$

We have $X = x + 1/2$; $Y = y + 1/2$

$$\Rightarrow \quad \tan^{-1}\left(\frac{y+1/2}{x+1/2}\right) - 1/2\ln\left[1 + \frac{(y+1/2)^2}{(x+1/2)^2}\right] = \ln(x + 1/2) + c$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2y+1}{2x+1}\right) = 1/2\ln(x^2 + y^2 + x + y + 1/4 + 1/4) + \ln c$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2y+1}{2x+1}\right) = \ln(c\cdot\sqrt{x^2+y^2+x+y+1/2})$$

10. $dy/dx = \dfrac{x + 2y - 3}{2x + y - 3}$

Put $x = X + h$; $y = Y + k$

$$\Rightarrow \quad dY/dX = \frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}$$

Put $h + 2k = 3$; $2h + k = 3$, we get $h = 1$; $k = 1$

$$\Rightarrow \quad dY/dX = X + 2Y/2X + Y$$

Put $Y = vX$

$$\Rightarrow \quad v + X\frac{dv}{dX} = 1 + 2v/(2 + v) \Rightarrow Xdv/dX = (1 + 2v - 2v - v^2)/(2 + v)$$

$$\Rightarrow \quad (2 + v)/(1 - v^2)dv = dX/X \Rightarrow \int\frac{2dv}{1-v^2} + \int\frac{vdv}{1-v^2} = \int\frac{dx}{x}$$

$$\Rightarrow \quad -\ln|v - 1/v + 1| - 1/2\ln(|1 - v^2|) = \ln X + \ln c$$

$$\Rightarrow \quad -\ln|v-1| + \ln|v + 1| - 1/2\ln(v - 1|) - 1/2\ln|v + 1|$$
$$= \ln X + \ln c$$

$$\Rightarrow \quad -3/2\ln|v - 1| + 1/2\ln(v + 1) = \ln X + \ln c$$

$$\Rightarrow \quad \ln(v + 1) - 3\ln(v - 1) = 2\ln X + 2\ln cn$$

$$\Rightarrow \quad \ln((v + 1)/(v - 1)^3) = \ln(X^2c^2)$$

$$\Rightarrow \quad \frac{(Y+X)X^2}{(Y-X)^3} = X^2c^2 \Rightarrow (Y + X)/(Y - X)^3 = c^2$$

where $X = x - 1$; $Y = y - 1$

$$\Rightarrow \quad \frac{(y-1)+(x-1)}{(y-1-x+1)^3} = c^2 \Rightarrow (x + y - 2) = c'(y - x)^3$$

11. $dy/dx = \dfrac{x+y+1}{2x+2y+3}$

Put $x + y = t$; $1 + dy/dx = dt/dx$

$$\Rightarrow \quad dt/dx - 1 = (t + 1)/(2t + 3)$$

$$\Rightarrow \quad dt/dx = (3t + 4)/(2t + 3)$$

$$\Rightarrow \quad \int\frac{2t+3}{3t+4}dt = dx \Rightarrow \int\frac{2}{3} + \frac{1}{3(3t+4)}dt = dx$$

$$\Rightarrow \quad dt(2t + 3)/(3t + 4) = dx$$

$$\Rightarrow \quad 2t/3 + 1/9\ln(3t + 4) = x + c$$

$$\Rightarrow \quad 2(x + y)/3 + 1/9\ln(3x + 3y + 4) = x + c$$

$$\Rightarrow \quad 6x + 6y + \ln(3x + 3y + 4) = 9x + 9c$$

$$\Rightarrow \quad 3x - 6y - \ln(x + y + 4/3) - \ln 3 = 9c$$

$$\Rightarrow \quad x + y + \frac{4}{3} = ce^{3(x-2y)}$$

12. Putting $y = vx$ in the given equation, we have

$$v + x\frac{dv}{dx} = v(\log v + 1)$$

$$\Rightarrow \quad x\frac{dv}{dx} = v\log v$$

$$\Rightarrow \quad \frac{dv}{v\log v} = \frac{dx}{x}$$

$$\Rightarrow \quad \log|\log v| = \log|x| + \log C$$

$$\Rightarrow \quad \log v = Cx \Rightarrow y = xe^{Cx}.$$

13. Put $x = X + h$, $y = Y + k$.

We have $\dfrac{dy}{dx} = \dfrac{X+2Y+(h+2k+3)}{2X+3Y+(2h+3k+4)}$

To determine h and k, we write

$h + 2k + 3 = 0$, $2h + 3k + 4 = 0 \Rightarrow h = 1$, $k = -2$ so that

$$\frac{dY}{dX} = \frac{X+2Y}{2X+3Y}$$

Putting $Y = VX$, we get $V + X\dfrac{dV}{dX} = \dfrac{1+2V}{2+3V}$

$$\Rightarrow \quad \frac{2+3V}{3V^2-1}dV = -\frac{dX}{X} \Rightarrow \left[\frac{2+\sqrt{3}}{2(\sqrt{3}V-1)} - \frac{2-\sqrt{3}}{2(\sqrt{3}V+1)}\right]dV = -\frac{dX}{X}$$

$$\Rightarrow \quad \frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V - 1) - \frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}V + 1) = (-\log X + A)$$

$$\Rightarrow \quad \frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y - X) - \frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}Y + X) = A$$

Where A is another constant and $X = x - 1$, $Y = y + 2$.

$$\Rightarrow \quad \frac{2+\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}y - x + 1 + 2\sqrt{3}) -$$
$$\frac{2-\sqrt{3}}{2\sqrt{3}}\log(\sqrt{3}y + x + 2\sqrt{3} - 1) = A$$

Test Your Concepts-IV

1. Dividing each term of the given equation by x^2, we get

$$\frac{ydx - xdy}{x^2} + \frac{1+x^2}{x^2}dx + \sin y\,dy = 0$$

or $\quad -\dfrac{xdy - ydx}{x^2} + \left(\dfrac{1}{x^2} + 1\right)dx + \sin y\,dy = 0$

or $\quad -d(y/x) + (1 + 1/x^2)\,dx + \sin y\,dy = 0$

Integrating, $-\dfrac{y}{x} + x - \dfrac{1}{x} - \cos y = c.$

or $\quad -y + x^2 - 1 - x\cos y = cx,$

where c is an arbitrary constant.

2. Re-writing, $2xy^2\,dx + ye^x\,dx - e^x dy = 0$

or $\quad 2x\,dx + \dfrac{ye^x dx - e^x dy}{y^2} = 0$

or $\quad 2x\,dx + d\left(\dfrac{e^x}{y}\right) = 0$

Integrating, $x^2 + \dfrac{e^x}{y} = c$ or $yx^2 + e^x = cy.$

3. Separating the variables and integrating, we get

$$\int\frac{3\sec^2(3y)dy}{\tan 3y} = \int\frac{2(e^{2x}-1)}{e^{2x}}dx + c$$

Put $\tan 3y = t$, $3\sec^2 3y\, dy = dt$ in left integral

$$\int \frac{dt}{t} = 2\int(1 - e^{-2x})\, dx + c$$

$$\Rightarrow \quad \log|t| = 2x + e^{-2x} + c$$

$$\text{or} \quad \log|\tan 3y| = 2x + e^{-2x} + c$$

4. $ydx + y^2 dy = x\, dy$

$$\Rightarrow \quad \frac{ydx - xdy}{y^2} = -dy$$

$$\Rightarrow \quad \frac{x}{y} = -y + c$$

$$y(1) = 1 \Rightarrow c = 2$$

$$y^2 - 2y + x = 0$$

For $y(-3)$, put $x = -3$

$$y^2 - 2y - 3 = 0 \Rightarrow (y-3)(y+1) = 0$$

$$y = 3, -1 \text{ but } y > 0$$

So, $y(-3) = 3$

5. It is the homogenous equation. Put $y = vx$

Then $\dfrac{dy}{dx} = v + x + \dfrac{dv}{dx}$ …(1)

We have the equation $x \times \dfrac{dy}{dx} - y = \sqrt{x^2 + y^2}$.

Putting in the equation from (1), we get

$$x\left(v + x\frac{dv}{dx}\right) - vx = \sqrt{x^2 + v^2x^2} \text{ or } x^2\frac{dv}{dx} = x\sqrt{1+v^2}$$

$$\therefore \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

$$\text{or} \quad \int \frac{dv}{\sqrt{1+x^2}} = \int \frac{dx}{x}$$

$$\therefore \quad \log\{v + \sqrt{v^2+1}\} = \log x + \log c$$

$$\text{or} \quad v + \sqrt{v^2+1} = cx \quad \therefore \frac{y}{x} + \sqrt{\frac{y^2}{x^2}+1} = cx$$

$$\therefore \quad y + \sqrt{x^2+y^2} = cx^2$$

6. $dy/dx = \dfrac{x(2\ln x + 1)}{(\sin y + y\cos y)} \Rightarrow \sin y.dy + y \cdot \cos y\, dy = x(\ln x^2 + 1)\, dx$

$$\Rightarrow \quad \int d(y\sin y) = \int [\ln(t) + 1]\frac{dt}{2}$$

Put $x^2 = t$; $xdx = dt/2$

$$\Rightarrow \quad y\sin y = (t\ln t - t/2) + t/2 \Rightarrow y\sin y = x^2\ln x + c$$

7. $xdy + ydx + \dfrac{xdy - ydx}{(x^2+y^2)} = 0 = xdy + ydx = \left(\dfrac{ydx - xdy}{y^2} \times \dfrac{y^2}{(x^2+y^2)}\right)$

$$\Rightarrow \quad \int d(xy) = \int \frac{d(x/y)}{1 + (x/y)^2}$$

$$\Rightarrow \quad xy = \tan^{-1}(x/y) + c \quad \Rightarrow \quad xy + \tan^{-1}(y/x) = k$$

Test Your Concepts-V

1. It is a LDE, Here,
$P = \cos x$ and $Q = \sin x$.

Then, I.F. $= e^{\int \cot x\, dx} = e^{\int \log \sin x} = \sin x$

$$\Rightarrow \quad y\sin x = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + c$$

2. Re-writing given equation, we have

$$\frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right) = \frac{\sec x}{x}$$

$$\text{I.F.} = e^{\int(\tan x + 1/x)dx} = e^{\log\sec x + \log x}$$

$$= e^{\log x\sec x} = x\sec x$$

Hence the required solution is

$$yx\sec x = \int \sec^2 x\, dx + c$$

$$\text{or} \quad yx\sec x = \tan x + c.$$

3. Dividing by $\cos^2 y$, $\sec^2 y\dfrac{dy}{dx} + 2x(\tan y) = x^3$.

Put $\tan y = v$ so that $\sec^2 y\dfrac{dy}{dx} = \dfrac{dv}{dx}$.

Hence the above equation. becomes $\dfrac{dv}{dx} + 2xv = x^3$,

Which is linear in v and x. Hence its I.F. $= e^{\int 2xdx} = e^{x^2}$

and its solution is given by $v.\, e^{x^2} = \int x^3 e^{x^2}\, dx = c$

$$= \frac{1}{2}\int te^t\, dt + c \qquad [\text{Put } x^2 = t \;\therefore\; 2xdx = dt]$$

$$= \frac{1}{2}\int [t.e^t - \int 1.\, e^t\, dt] + c \qquad [\text{Integrating by parts}]$$

$$= \frac{1}{2}(te^t - e^t) + c \text{ or } \tan y\,.\, e^{x^2} = \frac{1}{2}e^{x^2}(x^2 - 1) + c$$

$$[\therefore\; v = \tan y \text{ and } t = x^2]$$

$$\text{or} \quad \tan y = \frac{1}{2}(x^2 - 1) + ce^{-x^2} \qquad [\text{Dividing by } e^{x^2}]$$

4. The given differential equation is

$$(x^2 - 1)\frac{dy}{dx} + 2xy = \frac{1}{x^2 - 1}$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{1}{(x^2-1)^2} \qquad \text{…(1)}$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q, \text{ where } P = \frac{2x}{x^2-1} \text{ and } Q = \frac{1}{(x^2-1)^2}$$

$$\therefore \quad \text{I.F.} = e^{\int P\, dx}$$

$$= e^{\int 2x/(x^2-1)\,dx} = e^{\log(x^2-1)} = (x^2 - 1)$$

$$\therefore \quad \text{Solution is } y\,(\text{I.F.}) = \int Q.\,(\text{I.F.})dx + C$$

$$y(x^2 - 1) = \int \frac{1}{x^2-1}\, dx + C$$

$$\Rightarrow \quad y(x^2 - 1) = \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C.$$

This is the required solution.

5. $(1 + y + x^2y)\, dx + (x + x^3)\, dy = 0$

$$\Rightarrow \quad (x + x^3)\frac{dy}{dx} + y(x^2 + 1) + 1 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + y/x = -\frac{1}{x(x^2+1)}$$

$$\text{If} = e^{\int 1/x\, dx} = e^{\ln x} = x$$

$$\Rightarrow \quad y(x) = -\int \frac{1}{(x^2+1)}dx + c = -\tan^{-1}(x) + c$$

$$\Rightarrow \quad xy = c - \tan^{-1}(x)$$

6. $x\dfrac{dy}{dx} - y = 2x^2 \cdot \operatorname{cosec} 2x$

$$\Rightarrow \quad dy/dx - y/x = 2x \cdot \operatorname{cosec} 2x$$

$$\text{I.F.} = e^{\int -1/x\, dx} = 1/x$$

$$\Rightarrow \quad y/x = \int 2\operatorname{cosec}2x\, dx + c \Rightarrow y/x = \frac{-2\ln|\operatorname{cosec}2x + \cot 2x|}{2} + c$$

$\Rightarrow \quad y/x = -\ln\left|\dfrac{1+\cos 2x}{\sin 2x}\right| + c = \ln \tan x + c$

$\Rightarrow \quad y = cx + x \cdot \ln(\tan x)$

7. $dy/dx + xy = y^2 \cdot e^{x^2/2}\sin x$

$\Rightarrow \quad 1/y^2 \times dy/dx + x/y = e^{x^2/2}\sin x$

Put $1/y = t;\ 1/y^2\, dy/dx = -\,dt/dx$

$\Rightarrow \quad dy/dx - xt = +\,e^{x^2/2}\sin x$

I.F. $= e^{\int -x\,dx} \Rightarrow e^{-x^2/2}$

$\Rightarrow \quad t(e^{-x^2/2}) = \displaystyle\int -\sin x\, dx = \cos x + c$

$\Rightarrow \quad e^{-x^2/2} = y(\cos x + c)$

8. $x^2y - x^3\, dy/dx = y^4 \cos x$

$\Rightarrow \quad x^3/y^4\, dy/dx - 1/y^3\, x^2 = -\cos x$

Put $-1/y^3 = t \Rightarrow 3/y^4\, dy = dt$

$\Rightarrow \quad dt/dx \times (x^3/3) + x^2 t = -\cos x$

$\Rightarrow \quad dt/dx + 3t/x = -\,3\cos x/x^3 \Rightarrow$ I.F. $e^{\int 3/x\,dx} = x^3$

$\Rightarrow \quad t(x^3) = \displaystyle\int -3\cos x\, dx = -3\sin x + c$

$\Rightarrow \quad -x^3/y^3 = -3\sin x + c \Rightarrow x^3 y^{-3} = 3\sin x + k$

9. $2xy\dfrac{dy}{dx} = x^2 + y^2 + 1$ or $2xy\dfrac{dy}{dx} - y^2 = 1 + x^2$.

Dividing by x, $2y\dfrac{dy}{dx} - \dfrac{1}{x}y^2 = \left(\dfrac{1}{x} + x\right)$. …(1)

Put $y^2 = v$; so that $2y\,(dy/dx) = dv/dx$

So (1) becomes $\dfrac{dv}{dx} - \dfrac{1}{x}v = \left(\dfrac{1}{x} + x\right)$,

Which is linear.

Whose I.F. $= e^{-\int (1/x)dx} = e^{-\log x} = x^{-1}$.

$\therefore$ Solution is $v.\ x^{-2} = \int x^{-1}\,(1/x + x)\,dx + c$

$= \int (x^{-2} - 1)dx + c = -\,x^{-1} + x + c$

or $\quad y^2x^{-1} = -x^{-1} + x - c$ or $y^2 = x\,(c + x) - 1$.

10. Write the equation as $x + y\dfrac{dx}{dy} = y^3$

Consider now, x as the dependent variable and y as independent variable. Write the equation as

$$\frac{dx}{dy} + \frac{x}{y} = y^2$$

The integrating factor is $e^{\int (1/y)dy} = e^{\log y} = y$

Multiplying the differential equation by the integrating factor, we get

$$y\frac{dx}{dy} + x = y^3$$

or $\quad \dfrac{d}{dy}(xy) = y^3$

Integrating, we obtain $xy = (1/4)\,y^4 + c$,

Which is the required solution.

11. Here, $\therefore\ \dfrac{dy}{dx} - \dfrac{1}{x}\cdot y = -\dfrac{1}{x}\cos\left(\dfrac{1}{x}\right)$; this is in the linear form.

Integrating factor $= v$

Multiplying by the integrating factor,

$$\frac{1}{x}\cdot\frac{dy}{dx} - \frac{1}{x^2}\,y = \frac{-1}{x^2}\cos\left(\frac{1}{x}\right) \text{ or } \frac{d}{dx}\left\{\frac{1}{x}\cdot y\right\} = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$$

or $\quad d\left(\dfrac{y}{x}\right) = -\dfrac{1}{x^2}\cos\left(\dfrac{1}{x}\right)dx$ or $\displaystyle\int d\left(\frac{y}{x}\right) = \int -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)dx$

$\therefore \quad \dfrac{y}{x} = \displaystyle\int \cos\left(\frac{1}{x}\right) d\left(\frac{1}{x}\right)$ or $\dfrac{y}{x} = \sin\left(\dfrac{1}{x}\right) + c$

$\therefore \quad y - x\sin\left(\dfrac{1}{x}\right) = cx$

12. The given differential equation can be written as $\dfrac{1}{y^6}\dfrac{dy}{dx} + \dfrac{1}{xy^5} = x^2$

Putting $y^{-5} = v$ so that $-5y^{-6}\dfrac{dy}{dx} = \dfrac{dv}{dx}$ or $y^{-6}\dfrac{dy}{dx} = -\dfrac{1}{5}\dfrac{dv}{dx}$

We get $-\dfrac{1}{5}\dfrac{dv}{dx} + \dfrac{1}{x}v = x^2$

$\Rightarrow \quad \dfrac{dv}{dx} - \dfrac{5}{x}v = -5x^2$ …(1)

This is the standard form of the linear differential equation having integrating factor

We get $v \cdot \dfrac{1}{x^5} = \displaystyle\int -5x^2 \cdot \frac{1}{x^5}dx$

$\Rightarrow \quad \dfrac{v}{x^5} = \dfrac{5}{2}x^{-2} + c \Rightarrow y^{-5}x^{-5} = \dfrac{5}{2}x^{-2} + c$

Which is the required solution.

13. The given differential equation can be written as

$x\dfrac{dy}{dx} + y = y^2 \log x$ …(1)

Divide by xy^2.

Hence, $\dfrac{1}{y^2}\dfrac{dy}{dx} + \dfrac{1}{xy} = \dfrac{1}{x}\log x$

Let $\dfrac{1}{y} = v \Rightarrow -\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

So that $\dfrac{dv}{dx} - \dfrac{1}{x}v = -\dfrac{1}{x}\log x$ …(2)

Equation (2) is the standard linear differential equation with

$P = -\dfrac{1}{x},\ Q = -\dfrac{1}{x}\log x$

I.F. $= e^{\int p dx} = e^{\int -1/x\, dx} = 1/x$

The solution is given by $v \cdot \dfrac{1}{x}\displaystyle\int \frac{1}{x}\left(-\frac{1}{x}\log x\right)dx = -\int \frac{\log x}{x^2}\,dx$

$= \dfrac{\log x}{x} - \displaystyle\int \frac{1}{x}\cdot\frac{1}{x}\,dx = \frac{\log x}{x} + \frac{1}{x} + c$

$\Rightarrow \quad v = 1 + \log x + cx$

$= \log ex + cx$ or $\dfrac{1}{y} = \log ex + cx$ or $y\,(\log ex + cx) = 1$.

14. The given equation can be written as:

$\dfrac{dx}{dy} + \dfrac{2}{y}x = 10y^2$ [Linear Equation in x] …(1)

Here 'P' $= \dfrac{2}{y}$ and 'Q' $= 10y^2$.

I.F. $= e^{\int P\,dy} = e^{\int \frac{2}{y}dy} = e^{2\log|y|} = e^{\log y^2} = y^2$

Multiplying (1) by y^2, we get : $y^2\dfrac{dx}{dy} + 2yx = 10y^4$

$\Rightarrow \quad \dfrac{d}{dy}(x.\ y^2) = 10y^4$

Integrating, $xy^2 = 10\displaystyle\int y^4\, dy + c$

$\Rightarrow \quad xy^2 = 2y^5 + c$ which is required solution.

Test Your Concepts-VI

1. Straight line $y = mx + c$

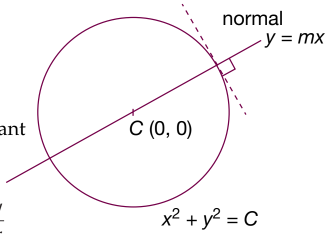
Origin (0, 0) lies on it

$$0 = 0 + c \Rightarrow c = 0$$

$y = mx$ has one arbitrary constant

$$\frac{dy}{dx} = m$$

Differential equation is $\frac{dy}{dx} = \frac{y}{x}$



Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in above differential equation

New differential equation, $-\frac{dx}{dy} = \frac{y}{x}$

$$\Rightarrow \int -x\,dx = \int y\,dy \Rightarrow -\frac{x^2}{2} = \frac{y^2}{2} + k$$

$$\Rightarrow x^2 + y^2 = -2k = c$$

which is a circle having centre at origin

2. $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$

Let $\frac{dy}{dx} = p$ then $p^2 - xp + y = 0$

$\Rightarrow y = xp - p^2$ …(1)

By differentiation $\frac{dy}{dx} = p + x\frac{dp}{dx} - 2p\frac{dp}{dx}$

$$\frac{dp}{dx}(x - 2p) = 0$$

$\frac{dp}{dx} = 0 \Rightarrow p = c$ from (1), general solution is $y = cx - c^2$

Put $c = 2$, we get $y = 2x - 4$ or $x - 2p = 0$ from (1)

We have $y = x(x/2) - (x/2)^2$ or $x^2 = 4y$ is singular solution

3. Differentiate both sides of $x^2 + 2y^2 - y = c$

w.r.t. x, we get $2x + 4y\frac{dy}{dx} - \frac{dy}{dx} = 0$ or $2x + (4y - 1)\frac{dy}{dx} = 0,$

is the differential equation of the given family of curves.

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ to obtain the differential equation of the orthogonal trajectories, we get

$$2x + \frac{(1-4y)}{dy/dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{4y-1}{2x}$$

$$\Rightarrow \int\frac{dy}{4y-1} = \int\frac{dx}{2x} \Rightarrow \frac{1}{4}\ln(4y-1) = \frac{1}{2}\ln x + \frac{1}{2}\ln a,$$

Where a is any constant.

$\Rightarrow \ln(4y - 1) = 2\ln x + 2\ln a$

or $4y - 1 = a^2x^2$ or, $y = \frac{1}{4}a^2x^2 + \frac{1}{4}$

is the required orthogonal trajectory, which is of the form $y = cx^2 + k$ where

$$c = \frac{a^2}{4}, k = \frac{1}{4}$$

4. (i) $x^2p^2 - 2xyp + 2y^2 - x^2 = 0$

$(xp - y)^2 = x^2 - y^2;\ xp - y = \pm\sqrt{x^2 - y^2}$

$p = \frac{y \pm \sqrt{x^2-y^2}}{x}\ \frac{dy}{dx} = \left(\frac{y}{x}\right) \pm \sqrt{1 - \left(\frac{y}{x}\right)^2}$ …(1)

Put $\frac{y}{x} = t \Rightarrow y = xt$

$$\frac{dy}{dx} = t + x\frac{dt}{dx};$$

Now from (1), we have $t + x\frac{dt}{dx} = t \pm \sqrt{1-t^2}$

$$\int\frac{dt}{\sqrt{1-t^2}} = \int\pm\frac{dx}{x}$$

$\Rightarrow \sin^{-1}t = \pm\ln x + c$

$\Rightarrow \frac{y}{x} = \sin(\pm\ln x + c)$

(ii) $y = px + p - p^2$ …(1)

On differentiation, we get

$$\frac{dy}{dx} = p = p + \frac{dp}{dx}.x + \frac{dp}{dx} - 2p\frac{dp}{dx}$$

$$\frac{dp}{dx}(x + 1 - 2p) = 0;$$

$\frac{dp}{dx} = 0 \Rightarrow p = k$ (constant) $\Rightarrow y = kx + k - k^2$

or $x + 1 = 2p$

From (1), we have $y = p(x + 1 - p)$

$\Rightarrow 2y = (x + 1)(x + 1 - (x + 1)/2) \Rightarrow 4y = (x + 1)^2$

Test Your Concepts-VII

1. $dy/dx = y + 2x \Rightarrow dy/dx - y = 2xv$

I.F. $= e^{\int -1\,dx} = e^{-x}$

$\Rightarrow y(e^{-x}) = \int(2x)\cdot e^{-x}dx$

$\Rightarrow ye^{-x} = 2[-xe^{-x} - e^{-x}] + c$

$\Rightarrow 0 = 2[0 - 1] + c \Rightarrow c = 2$

$\Rightarrow y = 2[e^x - x - 1]$

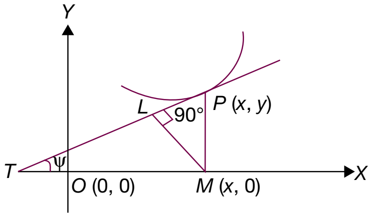
2. The differential equation is given by $dy/dt = ky$ where k is the constant of proportionality. Assuming that at time $t = 0$, the compound present is y_0, the solution is $y = y_0e^{kt}$. Since, it is given that $y(1) = 3y_0$, we get $k = \log 3$. At time $t = 2.5$ hours,

$y = y_0e^{2.5\log 3}$.

Hence, $y/y_0 = e^{2.5\log 3} \approx 15.6$. At the end of 2.5 hours, the compound grows 15.6 times the original amount.

3. The equation of the tangent line at point $P(x, y)$ is-

$$Y - y = \frac{dy}{dx}(X - x) \text{ or } \frac{dy}{dx}X - Y + \left(y - x\frac{dy}{dx}\right) = 0$$



∴ The length of perpendicular from foot of ordinate $M(x, 0)$ to the tangent line is

$$= \frac{\left|x\frac{dy}{dx} - 0 + \left(y - x\frac{dy}{dx}\right)\right|}{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}} = \frac{y}{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}} = k,$$

Let or $\frac{y^2}{k^2} = \left(\frac{dy}{dx}\right)^2 + 1$ or $\frac{dy}{dx} = \frac{\sqrt{y^2 - k^2}}{k}$

or $\frac{1}{k}dx = \frac{1}{\sqrt{y^2 - k^2}}dy$

Thus the variables are separated, and so integrating, we get solution

as $\frac{1}{k}x = \log\,(y + \sqrt{y^2 - k^2}) - \log c$

$$= \log \frac{y + \sqrt{y^2 - k^2}}{c} \text{ or } e^{x/k} = \frac{y + \sqrt{y^2 - k^2}}{c}$$

or $\sqrt{y^2 - k^2} = ce^{x/k} - y$ or $y^2 - k^2 = (ce^{x/k} - y)^2$.

4. Let $y(t)$ be the amount of temperature that is reduced at time t. $y(t)$ reduces with time t. The rate of cooling is proportional to $27 - y$. The differential equation is

$$\frac{dy}{dt} = k\,(27 - y)$$

Where k is the constant of proportionality and t is in minutes.

Solving, we get $y - 27 = ce^{-kt}$. Using the initial condition $y(0) = 19°C$, we get the solution as

$y - 27 = -8e^{-kt}$ or $y = 27 - 8e^{-kt}$

Since, it is given that $y(1) = 23°\,C$, we find $k = \log 2$.

Hence, $y(t) = 27 - 8e^{-t\log 2}$.

The time taken when temperature is 26°C, is $t = \log 8/\log 2 = 3$ minutes.

5. Let population $= x$

Time $= t$ (in years)

$\therefore$ Given condition is $\frac{dx}{dt} = ax$ where 'a' is a constant. …(1)

The D.E. can be expressed as $\frac{1}{x}\,dx = a\,dt$

$\therefore$ Its solution is $\int \frac{1}{x}\,dx = a\int dt$

or $\log x = at + \log k$ or $\log \frac{x}{k} = at$

$\Rightarrow$ $x = ke^{at}$ …(2)

If initially i.e., when time $t = 0$, $x = x_0$

From (2), $x_0 = k$, and hence

$x = x_0e^{at}$ …(3)

Now, as given: $t = 30$, $x = 2x_0 \Rightarrow 2x_0 = x_0e^{30a}$ or $2 = e^{30a}$

or $\log 2 = 30\,a$ …(4)

Next suppose, $t = t_1$, $x = 3x_0 \Rightarrow 3x_0 = x_0\,e^{at_1}$

or $\log 3 = at_1$ …(5)

Dividing (5) by (4), $\frac{at_1}{30a} = \frac{\log 3}{\log 2}$

or $t_1 = 30\frac{\log 3}{\log 2} = 30 \times \frac{48}{30}$, (approximately) $\Rightarrow t_1 = 48$ years.

6. $dc/dt = -k(\mathrm{c}) \Rightarrow dm/dt = k.(m)$

$\Rightarrow$ $dm/m = -k \times dt \Rightarrow (\ln m)_{m_0}^{m} = -k.t$

$\Rightarrow$ $(\ln m/m_0) = -kt \Rightarrow m = m_0.e^{-kt}$

$$\alpha/100 = \frac{-m_0 \cdot e^{-kt} + m_0}{m_0} \Rightarrow \alpha/100 - 1 = -e^{-kt}$$

$\Rightarrow$ $e^{-kt} = 1 - \alpha/100 \Rightarrow k = -1/t_0.\ln(1 - \alpha/100)$

$\therefore$ $m = m_0.e^{+1/t_0 \ell n(1-\alpha/100)}$

7. $dP/dt = -k(P - 1000)$

$\Rightarrow$ $[\ln(P - 1000)]_{2500}^{P} = -kt \Rightarrow \ln\left(\frac{P - 1000}{1500}\right) = -kt$

$\Rightarrow$ $\frac{P - 1000}{1500} = e^{-kt}$

(a) population at time 't' $P = 1000 + 1500.e^{-kt}$

(b) $P = 1000 + 1500.e^{-kt}$

Given $t = 10$ yrs, $P = 1900$

$\Rightarrow$ $1900 = 1000 + 1500.e^{-k\times 10}$

$\Rightarrow$ $900/1500 = e^{-k10} \Rightarrow e^{10k} = 5/3$

$\Rightarrow$ $k = 1/10\ln(5/3)$

For, $P = 1500$

$1500 = 1000 + 1500.e^{-1/10\ln(5/3)t}$

$\Rightarrow$ $1/3 = e^{-\ln(5/3)t/10} \Rightarrow e^{\ln(5/3)t/10} = 3$

$\Rightarrow$ $e^{(5/3)t/10} = \ln 3 \Rightarrow t = \frac{(10\ln 3)}{\ln(5/3)} = 10\log_{5/3}(3)$

(c) $P = 1000 + 1500.e^{-kt}$

For long run, population will exponentially decrease

$t \to \infty.\ P \to 1000$

8. Length of tangent $= \left| y\sqrt{1 + \left(\frac{dx}{dy}\right)^2} \right| \Rightarrow 1 = y^2\left[1 + \left(\frac{dx}{dy}\right)^2\right]$

$$\Rightarrow \frac{dy}{dx} = \pm\frac{y}{\sqrt{1 - y^2}} \Rightarrow \int \frac{\sqrt{1 - y^2}}{y}\,dy = \pm x + c.$$

Writing $y = \sin\theta$, $dy = \cos\theta\,d\theta$ and integrating, we get the equation of the curve as

$$\sqrt{1 - y^2} + \ln\left|\frac{1 - \sqrt{1 - y^2}}{y}\right| = \pm x + c.$$

9. Let the curve C be $y = f(x)$

If $P\,(x, y)$ be a point on $y = f(x)$, then the equation of tangent at P is

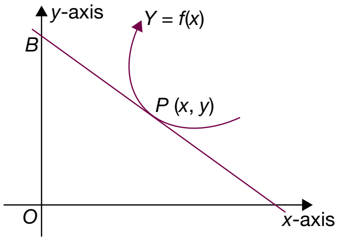
$Y - y = \frac{dy}{dx}(X - x)$

Putting $Y = 0$, x – intercept

$= x - y\frac{dx}{dy}$

Putting $X = 0$, y – intercept

$= y - x\frac{dy}{dx}$



Tangent cuts the co-ordinate axes at A and B and P Is midpoint of AB

$$\therefore \quad x = \frac{0 + x - y\frac{dx}{dy}}{2},\ y = \frac{0 + y - x\frac{dy}{dx}}{2}$$

In each case we get $\frac{xdy}{dx} = -y \Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0$

Integrating we have $\ln|(xy)| = k \Rightarrow xy = \pm e^k = c$

Since curve passes through (1, 1)

$\therefore$ $C = 1$

Required curve is $xy = 1$.

10. If V is the volume of the rain drop at any time t, r the radius

$\frac{dV}{dt} = -k \cdot 4\pi r^2$ where k is constant of proportionality …(1)

$$V = \frac{4}{3}\pi r^3$$

$\frac{dV}{dt} = \frac{4}{3} \times 3\pi r^2\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}$ …(2)

From (1) and (2)

$$4\pi r^2\frac{dr}{dt} = -k4\pi r^2$$

$\therefore$ $\frac{dr}{dt} = -k$

11. The equation of the normal at any point (x, y) is $Y - y = -\frac{dx}{dy}(X - x)$

This passes through (h, k) If $k - y = -\frac{dx}{dy}(h - x)$

$\Rightarrow \quad (k - y)dy + (h - x)\, dx = 0$

Integrating, we have $(k - y)^2 + (h - x)^2 = \text{constant}$

i.e. $(x - h)^2 + (y - k)^2 = a^2$

Single Correct Choice Type Questions

1. $y = c_1 \cos 2x + \frac{c_2}{2}(1 + \cos 2x) + \frac{c_3}{2}(1 - \cos 2x) + c_4$

$$= \left(\frac{c_2}{2} + \frac{c_3}{2} + c_4\right) + \left(c_1 + \frac{c_2}{2} + \frac{c_3}{2}\right)\cos 2x$$

$$= A + B \cos 2x$$

$\therefore$ Equation has only two independent parameters

Hence, order is 2.

Hence, the correct answer is (A).

2. On Putting $x = \tan A$, $y = \tan B$ we get

$$\sec A + \sec B = \lambda(\tan A \sec B - \tan B \sec A)$$

$$\cos A + \cos B = \lambda(\sin A - \sin B)$$

$$\tan\left(\frac{A - B}{2}\right) = \frac{1}{\lambda}$$

$$\tan^{-1}x - \tan^{-1}y = 2\tan^{-1}\frac{1}{\lambda}$$

On differentiating $\frac{1}{1+x^2} - \frac{1}{1+y^2}\frac{dy}{dx} = 0$, which is a differential equation of degree 1.

Hence, the correct answer is (A).

3. $y^2\left(\frac{d^2y}{dx^2}\right)^2 + x^2y^2 - \sin x = -3x\left(\frac{dy}{dx}\right)^{\frac{1}{3}}$

$$\left(y^2\left(\frac{d^2y}{dx^2}\right)^2 + x^2y^2 - \sin x\right)^3 = -9x^3\left(\frac{dy}{dx}\right)$$

Here order $= 2 = p$

Degree $= 6 = q$

$\therefore \quad p < q$

Hence, the correct answer is (D).

4. Given $\frac{d^2y}{dx^2} \propto \log x$ (according to condition)

$$\frac{d^2y}{dx^2} = K \ln x \qquad \text{let } y = f(x)$$

$$\int f''(x)dx = \int k \ln x\, dx$$

$$\Rightarrow \quad f'(x) = K\left(x \ln x - \int x \cdot \frac{1}{x}dx\right)$$

$$\Rightarrow \quad f'(x) = K(x \ln x - x) + c \Rightarrow f'(x) = Kx(\ln x - 1) + c$$

$$f(x) = K\int x(\ln x - 1)dx + cx + d$$

$$= K\left(\frac{x^2}{2}(\ln x - 1) - \int \frac{x^2}{2}\left(\frac{1}{x} - 0\right)dx\right) + cx + d$$

$$= K\left(\frac{x^2}{2}\ln x - \frac{x^2}{2}\right) - K\left(\frac{x^2}{4}\right)cx + d$$

$$f(x) = \frac{Kx^2}{2}\ln x - \frac{3K}{4}x^2 + cx + d$$

$$= \frac{K}{4}x^2(2\ln x - 3) + cx + d \Rightarrow f(x) = c_1x^2(2\ln x - 3) + c_2x + c_3$$

Hence, the correct answer is (A).

5. If (h, k) is the vertex of the parabola, its equation will be $(x - h)^2 = \pm 4a(y - k)$

or this reduces to general form $x^2 + ax + by + c = 0$

Differentiating $2x + a + by_1 = 0$

Again $2 + by_2 = 0$ or $y_2 = \frac{-2}{b}$

Again $y_3 = 0$; $\frac{d^3y}{dx^3} = 0$ is the required equation

Hence, the correct answer is (D).

6. $(x^2 \sin^3 y - y^2 \cos x)\, dx + (x^3 \cos y \sin^2 y - 2y \sin x)\, dy = 0$

$$\frac{dy}{dx} = \frac{y^2 \cos x - x^2 \sin^3 y}{x^3 \cos y \sin^2 y - 2y \sin x};$$

$$(x^3 \cos y \sin^2 y - 2y \sin x)\, dy = (y^2 \cos x - x^2 \sin^3 y)\, dx$$

$$\Rightarrow \quad \left(\frac{x^3}{3}d\sin^3 y - \sin x\, dy^2\right) - \sin^3 y\, d\left(\frac{x^3}{3}\right) + y^2 d \sin x = 0$$

$$\Rightarrow \quad \frac{x^3}{3}d\sin^3 y + \sin^3 y\, d\left(\frac{x^3}{3}\right) - (\sin x\, dy^2 + y^2 d \sin x) = 0$$

$$\Rightarrow \quad d\left(\frac{x^3}{3}\sin^3 y\right) - (y^2 \sin x) = 0$$

$$\frac{x^3}{3}\sin^3 y - y^2 \sin x = c$$

Hence, the correct answer is (A).

7. $\frac{dx}{dy} - \frac{x \ln x}{1 + \ln x} = \frac{e^y}{1 + \ln x}$

$$(1 + \ln x)\frac{dx}{dy} - x \ln x = e^y \Rightarrow (1 + \ln x)e^{-y}\frac{dx}{dy} - e^{-y}x \ln x = 1$$

$$\int d(x \ln x \cdot e^{-y}) = \int dy \Rightarrow \frac{x \ln x}{e^y} = y + c$$

given $y(1) = 0$

$$0 = (0 + 1)e^{\circ} \Rightarrow c = 0$$

We have curve as $x \ln x = ye^y \Rightarrow \ln x^x = ye^y \Rightarrow x^x = e^{ye^y}$

Hence, the correct answer is (A).

8. $y''x^2 + x(y')^2 = 2x\, y'$

$y''x^2 - 2xy' = -x(y')^2$

$$\frac{y''x^2 - 2xy'}{(y')^2} = -x \Rightarrow \frac{d}{dx}\left(\frac{x^2}{y'}\right) = +xv$$

$$\Rightarrow \quad \frac{x^2}{y'} = \frac{x^2}{2} + c$$

$$2x^2 = x^2y' + 2cy' \Rightarrow 2x^2 = (x^2 + 2c)y'$$

$$y' = \frac{2x^2}{x^2 + 2c} \qquad \text{given } f'(1) = 1$$

$$f'(1) = 1 = \frac{2}{1 + 2c}, \text{ we get } c = \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{2x^2}{x^2 + 1} \Rightarrow y = 2\int \frac{(x^2 + 1 - 1)}{x^2 + 1}dx$$

$y = 2(x - \tan^{-1}x) + c'$ given $f(0) = 0$

$$c' = 0$$

$$f(x) = 2(x - \tan^{-1}x)$$

Hence, the correct answer is (A).

9. We have, $\frac{dy}{dx} + \left(\frac{-1}{x}\right)y = x(xe^x + e^x - 1)$ [Linear differential equation]

$\therefore \quad \text{I.F.} = e^{-\ln x} = \frac{1}{x}$

Now, general solution is

$$y\left(\frac{1}{x}\right) = \int (e^x(x + 1) - 1)\, dx + C \Rightarrow \frac{y}{x} = x\, e^x - x + C$$

As, $y\,(x = 1) = e - 1 \Rightarrow \frac{e - 1}{1} = e - 1 + C$

$\Rightarrow \quad C = 0$

$\therefore \quad x + \frac{y}{x} = xe^x$

Now, $2 + \frac{y(2)}{2} = 2e^2 \Rightarrow y(2) = 4e^2 - 4$

and $y(1) = (e - 1)$ (Given)

Hence $y(2) = 4\,(e + 1)\,(e - 1) \Rightarrow y(2) = 4y(1)\,[y(1) + 2]$

$\therefore \quad k = 4.$

Hence, the correct answer is (D).

10. $\frac{f(x)}{1+x^2} = 1 + \int_0^x \frac{f^2(t)dt}{1+t^2}$ [**NOTE:** $f(0) = 1$]

Differentiate both sides with respect to x,

$$\frac{(1+x^2)f'(x) - f(x)(2x)}{(1+x^2)^2} = \frac{f^2(x)}{(1+x^2)}$$

$$\frac{dy}{dx} - \left(\frac{2x}{1-x^2}\right) y = y^2$$

$$\frac{1}{y^2}\frac{dy}{dx} - \left(\frac{2x}{1-x^2}\right)\frac{1}{y} = 1$$

Let $\frac{-1}{y} = t$

$$\frac{dt}{dx} + \left(\frac{2x}{1-x^2}\right) t = 1$$

$\therefore$ Solving the above equation we get,

$$f(x) = \frac{-3(1+x^2)}{x^3 + 3x - 3}$$

$$f(-2) = \frac{-3(5)}{-8-6-3} = \frac{15}{17}$$

Hence, the correct answer is (D).

11. Given equation

$$\frac{1}{2}d(x^2 + y^2) + d\left(\tan^{-1}\frac{y}{x}\right) = 0$$

$\Rightarrow \quad \frac{1}{2}(x^2 + y^2) + \tan^{-1}\frac{y}{x} = C_1$

$\Rightarrow \quad y = x\tan\left(\frac{c - x^2 - y^2}{2}\right)$

Hence, the correct answer is (B).

12. $\left(y\left(1 + \frac{1}{x}\right) + \sin y\right)dx + (x + \ln x + x\cos y)\,dy = 0$

$\Rightarrow \quad y\,dx + \frac{y}{x}\,dx + \sin y\,dx + x\,dy + \ln x\,dy + x\cos y\,dy = 0$

$\Rightarrow \quad \int d(xy) + \int d(y\ln x) + \int d(x\sin y) = 0$

$\Rightarrow \quad xy + y\ln x + x\sin y = c$

Hence, the correct answer is (B).

13. Rewriting $2xy\frac{dy}{dx} - y^2 = x^2 + 1$

or $\quad x\frac{d\left(y^2\right)}{dx} - y^2 = x^2 + 1$ or $x\frac{du}{dx} - u = x^2 + 1$ where $u = y^2$

or $\quad \frac{du}{dx} - \frac{1}{x}u = x + \frac{1}{x}$

The integrating factor is $\frac{1}{x}$

Solution $\frac{1}{x}u = \int\left(1 + \frac{1}{x^2}\right)dx$ or $\frac{y^2}{x} = x - \frac{1}{x} + c$

Since $y(1) = 1$, $C = 1$ $y^2 = x^2 - x + 1$ or $y^2 - \left(x - \frac{1}{2}\right)^2 = \frac{3}{4}$

Which represents a hyperbola

Hence, the correct answer is (C).

14. $\sin 2y \cdot dy - x\cdot\cos y dy = \sin y \cdot dx + x \cdot dx$

$\Rightarrow \quad d\left(x\cdot\sin y\right) + x\cdot dx = \sin 2y \cdot dy \;\Rightarrow -\frac{\cos 2y}{2} = \frac{x^2}{2} + x\cdot\sin y + c$

$\Rightarrow \quad \frac{-1 + 2\cdot\sin^2 y}{2} = \frac{x^2}{2} + x\cdot\sin y + c \;\Rightarrow \sin^2 y = \frac{x^2}{2} + x\cdot\sin y + c$

Hence, the correct answer is (A).

15. $y^4\left(2xe^y dx + x^2e^y dy\right) + y^2\left(2xydx - x^2dy\right) + \left(ydx - 3xdy\right) = 0$

$\Rightarrow \quad d\left(x^2e^y\right) + \frac{2xydy - x^2y}{y^2} + \frac{ydx - 3xdy}{y^4} = 0$

$\Rightarrow \quad d\left(x^2e^y\right) + d\left(\frac{x^2}{y}\right) + d\left(\frac{x}{y^3}\right) = 0$

$\Rightarrow \quad x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = k$

Hence, the correct answer is (C).

16. $e^y dx + e^y.x\,dy + 2y\,dy = 0$

$d\,(x.e^y) + 2y\,dy = 0$

$xe^y + y^2 = C$

$\because \quad y(0) = 1 \Rightarrow C = 1$

Particular solution is $xe^y + y^2 = 1$

$y = 0 \Rightarrow x + 0 = 1$

$\Rightarrow \quad x = 1$

Hence, the correct answer is (A).

17. $f'(x)\sin x + f(x)\cos x = 1$

$\Rightarrow \quad d(f(x)\sin x) = 1 \Rightarrow f(x)\sin x = x + c$

$\Rightarrow \quad f(x) = \frac{x}{\sin x} + \frac{c}{\sin x}$

When $x \to 0$ then $f(x)$ is bounded $c = 0$

$$f(x) = \frac{x}{\sin x}$$

We know $1 < \frac{x}{\sin x} < \frac{\pi}{2}$

$$\left(\frac{\pi}{2} - 0\right) < \int_0^{\pi/2}\left(\frac{x}{\sin x}\right)dx < \frac{\pi}{2}\left(\frac{\pi}{2} - 0\right)$$

$$\frac{\pi}{2} < I < \frac{\pi^2}{4}$$

Hence, the correct answer is (A).

18. Tangent at $P\,(x, y)$ is $Y - y = m\,(X - x)$

Point $Q\left(x - \frac{y}{m}, 0\right)$

Let circumcentre is $C\,(t/2, t)$, it is point which is equidistant from all three vertices

$CO = CP = CQ$

$$\frac{t^2}{4} + t^2 = (t/2 - x)^2 + (t - y)^2 = (t/2 - x - \frac{y}{m})^2 + t^2$$

$\Rightarrow \quad y^2 + \frac{xy}{m} + \frac{2y^2}{m} - 2xy = 0$

$\Rightarrow \quad -y^2 + 2xy = \frac{y}{m}\left(x + 2y\right) \quad \Rightarrow \quad m = \frac{dy}{dx} = \frac{x + 2y}{2x - y}.$

Hence, the correct answer is (A).

19. $\frac{ydx + xdy}{xy} = \left(\cot\frac{y}{x}\right)\left(\frac{xdy - ydx}{x^2}\right) \quad \Rightarrow \quad \frac{1}{xy}d\left(xy\right) = \cot\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right)$

$\Rightarrow \quad \log xy = \log\sin\left(\frac{y}{x}\right) + \log c \quad \Rightarrow \quad xy = c\sin\frac{y}{x}$

Hence, the correct answer is (C).

20. $\frac{dy}{dx} = \frac{1}{dx/dy}, \frac{d^2y}{dx^2} = -\frac{-1}{(dx/dy)^3}\frac{d^2x}{dy^2}$

then $x\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \left(\frac{dy}{dx}\right) = 0$ changes to

$$x\frac{d^2x}{dy^2} + \left(\frac{dx}{dy}\right)^2 - 1 = 0$$

Hence, the correct answer is (B).

21. $f''(x) = -f(x)$

$$\int 2f'(x)f''(x)\,dx = \int -2f(x)f'(x)\,dx$$

$(f'(x))^2 = -(f(x))^2 + c$

$(f(x))^2 + (f'(x))^2 = c$

$f^2(2) + (f'(2))^2 = (f(4))^2 + (f'(4))^2$

$$\frac{[f(2)]^2 + [f'(2])^2}{[f(4)]^2 + [f'(4)]^2} = 1$$

Hence, the correct answer is (C).

22. $f(-1) = a - b + c; f(1) = a + b + c; f(0) = c$

$$A = \int_{-1}^{1} ax^2 + bx + c\,dx = \left.\frac{ax^3}{3} + \frac{bx^2}{2} + cx\right|_{-1}^{1}$$

$$A = 2\left(\frac{a}{3} + c\right) = \frac{1}{3}(2a + 6c) = \frac{1}{3}(f(-1) + 4f(0) + f(1))$$

Hence, the correct answer is (D).

23. $x^2dy - y^2dx + xy^2(x - y)\,dy = 0$

$$\frac{dy}{y^2} - \frac{dx}{x^2} + \left(1 - \frac{y}{x}\right)dy = 0 \Rightarrow d\left(\frac{1}{x} - \frac{1}{y}\right) + \left(\frac{1}{y} - \frac{1}{x}\right)y\,dy = 0$$

$$\Rightarrow \int \frac{d\left(\frac{1}{x} - \frac{1}{y}\right)}{\left(\frac{1}{x} - \frac{1}{y}\right)} = \int y\,dy \Rightarrow \ln\left(\frac{1}{x} - \frac{1}{y}\right) = \frac{y^2}{2} + c$$

$$\Rightarrow \frac{1}{x} - \frac{1}{y} = e^{y^2/2} \cdot e^c$$

$$\Rightarrow y - x = xy \cdot Ke^{y^2/2} \Rightarrow y = x + Kxye^{y^2/2}$$

Hence, the correct answer is (A).

24. $\frac{dx}{dy} + \frac{x^2}{y^2} - \frac{x}{y} + 1 = 0$

Put $x = vy$ where v is a function of 'y'

$$\Rightarrow \frac{dx}{dy} = v + y\frac{dv}{dy}$$

$$\therefore v + y\frac{dv}{dy} + v^2 - v + 1 = 0 \Rightarrow -y\frac{dv}{dy} = (1 + v^2) \Rightarrow \frac{dv}{v^2 + 1} = -\frac{dy}{y}$$

$$\tan^{-1} v + C = -\ln y \Rightarrow \tan^{-1}\left(\frac{x}{y}\right) + \ln y + C = 0$$

Where C is arbitrary constant

Hence, the correct answer is (A).

25. $\frac{dy}{dx} + \int_0^5 y\,dx = 27$

$$K = \int_0^5 y\,dx \Rightarrow \frac{dy}{dx} = 27 - K$$

$f(0) = 0$

$y = 27x - Kx + \lambda$; origin (0, 0) lies on it so we get $\lambda = 0$

$y = (27 - K)x$

$$\Rightarrow K = \int_0^5 y\,dx = (27 - K)\left.\frac{x^2}{2}\right|_0^5$$

$$\Rightarrow K = (27 - K)\frac{25}{2} \Rightarrow 2K = (27 - K)\,25 \Rightarrow 27K = 27 \cdot 25$$

$K = 25$; curve $y = 2x$

$a, b \in \{1, 2, 3, 4\}$

a	b
1	2
2	4

Required probability $P = \frac{2}{4 \times 4} = \frac{1}{8}$

Hence, the correct answer is (C).

26. Tangent $y - y_1 = m(x - x_1)$

Tangent cuts y-axis at Point $P(0, y_1 - mx_1)$

Where slope of tangent $m = \frac{df}{dx}(x_1, y_1)$

Line perpendicular to tangent and passing through P, we have

$$y - (y_1 - mx_1) = -\frac{1}{m}(x - 0)$$

It passes through point (1, 0)

$$0 - y_1 + mx_1 = -\frac{1}{m} \Rightarrow (y_1 - mx)\,m = 1$$

$$\Rightarrow my - m^2x = 1$$

$$\Rightarrow y\left(\frac{dy}{dx}\right) - x\left(\frac{dy}{dx}\right)^2 = 1$$

Hence, the correct answer is (A).

27. $\frac{dy}{dx} = y + \int_0^2 y\,dx$

$$\frac{dy}{dx} = y + K \Rightarrow \int \frac{dy}{y + K} = \int dx$$

$\ln(y + K) = x + \lambda$; given $y(0) = 1$

$\ln(1 + K) = 0 + \lambda$;

Curve $\ln(y + K) = x + \ln(1 + K)$

$$\Rightarrow \ln\left(\frac{y + K}{1 + K}\right) = x$$

$$\Rightarrow y + K = e^x(1 + K) \Rightarrow y = e^x(1 + K) - K \quad \ldots(1)$$

$$\Rightarrow K = \int_0^2 y\,dx = \int_0^2 e^x(1 + K) - K\,dx$$

$$\Rightarrow K = e^x(1 + K) - Kx\Big|_0^2 = e^2(1 + K) - 2K - (1 + K) + 0$$

$$\Rightarrow K = e^2 + e^2K - 3K - 1 \Rightarrow 4K - e^2K = e^2 - 1$$

$$\Rightarrow K = \frac{e^2 - 1}{4 - e^2}$$

From (1)

$$y(2) = e^2(1 + K) - K = e^2\left[\frac{4 - e^2 + e^2 - 1}{4 - e^2}\right] - \left(\frac{e^2 - 1}{4 - e^2}\right)$$

$$= \frac{3e^2 - e^2 + 1}{4 - e^2} = \frac{2e^2 + 1}{4 - e^2}$$

$$\Rightarrow |y(x)| = \frac{2e^2 + 1}{e^2 - 4} = \frac{2(7 \cdot 3) - 1)}{7 \cdot 3 - 4} = \frac{15 \cdot 6}{3 \cdot 3} \simeq 4 \cdot 9;$$

$[|y(2)|] = 4$

Hence, the correct answer is (C).

28. Dividing by x^3y^3

$$\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0$$

$$\Rightarrow \quad \frac{ydx+xdy}{x^2y^2}+\frac{2}{x}dx-\frac{1}{y}dy=0 \Rightarrow -\frac{1}{xy}+2\log|x|-\log|y|=c$$

Hence, the correct answer is (A).

29. $\frac{dy}{dx}+\frac{1}{x}y=y^2$(Bernoule formula)

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{xy}=1$$

Let $\frac{1}{y}=t \Rightarrow -\frac{1}{y^2}\frac{dy}{dx}=\frac{dt}{dx}$

$$-\frac{dt}{dx}+\frac{1}{x}t=1;\ \frac{dt}{dx}-\frac{1}{x}t=-1$$

I.F. $= e^{-\ln x}=\frac{1}{x}$;

Solution of D.E. $\frac{t}{x}=\int\left(-\frac{1}{x}\right)dx+C$

$$\frac{1}{xy}-\ln x+C;\ \frac{1}{xy}=-\ln x+1$$

$$\frac{e}{y(1/e)}=+1+1$$

$\Rightarrow \quad [y(1/e)]=1$

Hence, the correct answer is (C).

30. $\frac{dy}{dx}=\frac{-\cos x.(y+1)}{(2+\sin x)}$

$\Rightarrow \quad \int\frac{dy}{y+1}=\int\frac{-\cos x}{2+\sin x}dx$

$\Rightarrow \quad \ln(y+1)=-\ln(2+\sin x)+\ln c$

$\Rightarrow \quad (y+1)=\left(\frac{c}{2+\sin x}\right)$

As $y(0)=0 \Rightarrow 1=\frac{c}{2} \Rightarrow c=2$

$\Rightarrow \quad y+1=\frac{2}{2+\sin x}$

So, $y\left(\frac{5\pi}{6}\right)=\frac{2}{2+\frac{1}{2}}-1=\frac{4}{5}-1=\frac{-1}{5}$.

Hence, the correct answer is (B).

31. I.F. $= e^{\int\frac{g'(x)}{g(x)}}=e^{\ln g(x)}=g(x)$

$\Rightarrow \quad y\,g(x)=\frac{1}{2}\int\frac{2g(x)g'(x)}{1+g^2(x)}dx=\frac{1}{2}\ln(1+g^2(x))+C.$

Here $C=1-\frac{\ln 2}{2}$

$\therefore \quad y(e)g(e)=\frac{1}{2}\ln(1+2e-1)+1-\frac{\ln 2}{2}=\frac{3}{2}.$

Hence, the correct answer is (A).

32. $x=\tan^{-1}t \Rightarrow \frac{dx}{dt}=\frac{1}{1+t^2}$

$$\frac{dy}{dx}=\frac{dx}{dt}\cdot\frac{dt}{dx}=\frac{dy}{dt}(1+t^2) \qquad \ldots(1)$$

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left[\frac{dy}{dt}(1+t^2)\right]\cdot\frac{dt}{dx}=\left[\frac{dy}{dt}2t+(1+t^2)\frac{d^2y}{dt^2}\right](1+t^2) \qquad \ldots(2)$$

Hence the given differential equation

$$\frac{d^2y}{dx^2}+xy\frac{dy}{dx}+\sec^2 x=0 \text{ becomes}$$

$$(1+t^2)\left[2t\frac{dy}{dt}+(1+t^2)\frac{d^2y}{dt^2}\right]+y\tan^{-1}t\left[\frac{dy}{dt}(1+t^2)\right]+(1+t^2)=0,$$

Cancelling $(1+t^2)$ throughout, we get

$$(1+t^2)\frac{d^2t}{dt^2}+(2t+y\tan^{-1}t)\frac{dy}{dt}=-1$$

$\Rightarrow \quad k=-1.$

Hence, the correct answer is (B).

33. $x=\ln\left(1+\frac{dy}{dx}\right) \Rightarrow 1+\frac{dy}{dx}=e^x \Rightarrow \frac{dy}{dx}=e^x-1$

$\Rightarrow \quad y=e^x-x+c$

At $x=0$, $y=1$

$\therefore \quad 1=1-0+c \Rightarrow c=0$

At $x=0$, $y=1$

$\therefore \quad 1=1-0+c \Rightarrow c=0$

$y=e^x-x$

$\therefore \quad f'\left(\ln\frac{1}{2}\right)+f\left(\ln\frac{1}{2}\right)=\frac{1}{2}-1+\frac{1}{2}+\ln 2 \Rightarrow \ln 2$

Hence, the correct answer is (B).

34. Differentiate with respect to x.

$y'(x)+y(x)=2x$

$\frac{dy}{dt}+y=2x$

I.F. $= e^x$

$ye^x=2\int xe^x dx$

$ye^x=2[xe^x-e^x]+c$

$y=2(x-1)+ce^{-x}$

At $x=1$; $y(1)=1 \Rightarrow c=e$

$y=2(x-1)+e^{1-x}$.

Hence, the correct answer is (A).

35. $\frac{dy}{dx}-f(x).y=0$

$\frac{dy}{y}=f(x)\,dx$

$\ln y=\int f(x)\,dx$

$y_1(x)=e^{\int f(x)dx}$ Then for given equation I.F $= e^{\int f(x)dx}$

Hence Solution $y.y_1(x)=\int r(x).y_1(x)\,dx$

$$y=\frac{1}{y_1(x)}\int r(x).y_1(x)dx$$

Hence, the correct answer is (A).

36. $y=-1 \rightarrow x=\frac{1}{4}, \frac{dy}{dx}=-\frac{16}{5}$

$$\frac{dx}{dy}-xy^3=x^2y^7;\quad \Rightarrow \frac{dx}{x^2dy}-y^3\left(\frac{1}{x}\right)=y^7$$

Let $\frac{1}{x}=t \quad \Rightarrow -\frac{1}{x^2}\frac{dx}{dy}=\frac{dt}{dy}$

$\Rightarrow \quad -\frac{dt}{dy}-y^3t=y^7 \Rightarrow \frac{dt}{dy}+y^3t=-y^7$

$\Rightarrow \quad t\,e^{y^4/4}=-\int y^7e^{y^4/4}\,dy+C \Rightarrow y^4=v$

$$4y^3\,dy=dv;\ t.e^{y^4/4}=-\frac{1}{4}\int ve^{v/4}\,dv+C$$

$$t.e^{y^4/4} = -\frac{1}{4}\left[\frac{ve^{v/4}}{4} - \int \frac{e^{v/4}}{4}\right] + C;$$

$$t.e^{y^4/4} = -\frac{1}{4}\left[\frac{ve^{v/4}}{4} - \frac{e^{v/4}}{16}\right] + C$$

$$\frac{e^{y^4/4}}{x} = -\frac{1}{16}\, y^4\, e^{y^4/4} + \frac{1}{48}\, e^{y^4/4} + C;$$

$$4e^{1/4} = -\frac{1}{16}\, e^{1/4} + \frac{1}{48} e^{1/4} + C$$

$$\Rightarrow \quad C = \frac{195}{48}\, e^{1/4}$$

Hence, the correct answer is (D).

37. $\therefore$ Equation of normal at (x, y) is $Y - y = -\frac{dx}{dy}(X - x)$ put $Y = 0$

Then

$$X = x + y\frac{dy}{dx} \quad \text{Given, } y^2 = 2x\ X$$

$$\Rightarrow \quad y^2 = 2x\ (x + y\frac{dy}{dx})$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - 2x^2}{2xy} = \frac{\left(\frac{y}{x}\right)^2 - 2}{2\left(\frac{y}{x}\right)}$$

Put $y = vx \quad \therefore\ v + x\frac{dv}{dx} = \frac{v^2 - 2}{2v}$

$$\text{or } x\frac{dv}{dx} = -\frac{(2 + v^2)}{2v} \quad \text{or} \quad \frac{2vdv}{(2 + v^2)} + \frac{dx}{x} = 0$$

Integration we get

$$\ln(2 + v^2) + \ln x = \ln c \Rightarrow \ln\ (x(2 + v^2)) = \ln c \ \text{ or } \ x\left(2 + \frac{y^2}{x^2}\right) = c$$

$\because$ It passes through $(2, 1)$ then $c = \frac{9}{2}$ or $4x^2 + 2y^2 = 9x$

Hence, the correct answer is (D).

38. We have $y_1 = 1 + y^2$.

$$\Rightarrow \quad \frac{dy}{dx} = 1 + y^2 \Rightarrow \frac{dy}{1 + y^2} = dx$$

Integrating, we get $\tan^{-1} y = x + c$

$\Rightarrow \quad \tan^{-1}(h(x)) = x + c$

Given $h(0) = 0$

$\therefore \quad 0 = 0 + c$

$\therefore \quad c = 0$

Then $\tan^{-1} h(x) = x$

Since, the range of $\tan^{-1} h(x)$ is $(-\pi/2, \pi/2)$

$\therefore \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

But given $-c < x < c$

Hence the largest value of c such that $f(x)$ is differentiable is $\pi/2$

Hence, the correct answer is (C).

39. $\int \frac{dy}{(y - 2)(y + 1)} = \int \frac{dx}{(x + 3)(x - 1)}$

$$\Rightarrow \quad 4\log\left|\frac{y - 2}{y + 1}\right| = 3\log x\left|\frac{x - 1}{x + 3}\right| + c$$

Hence, the correct answer is (C).

40. $(x - 2y + 3)dx = (2x - y + 5)dy$

$\Rightarrow \quad 2(xdy + ydx) - ydy - xdx + 5dy - 3dx = 0$

$\Rightarrow \quad x^2 - 4xy + y^2 + 6x - 10y + c = 0$

Whose point of intersection is $\left(-\frac{7}{3}, \frac{1}{3}\right)$

Hence, the correct answer is (C).

41. $(\cos y - \sin y)\frac{dy}{dx} + (\cos y + \sin y)e^{-x} = e^{e^{-x}}$

$\cos y + \sin y = u$

$$\Rightarrow \quad \frac{du}{dx} + e^{-x}u = e^{e^{-x}} \Rightarrow ue^{-e^{-x}} = x + C$$

$\Rightarrow \quad (\cos y + \sin y)\ e^{-e^{-x}} = x + C$

$\left(0, -\frac{\pi}{4}\right) \Rightarrow C = 0$

$\Rightarrow \quad e^{-e^{-x}} = t \quad$ (Putting $(t, 0)$) $\Rightarrow t \cdot e^{e^{-t}} = 1$

Hence, the correct answer is (B).

42. Dividing by e^{2x}

$$\frac{dy}{dx} = \frac{y^3 e^{-2x}}{1 + y^2 e^{-2x}}$$

$\Rightarrow \quad dy + y^2 e^{-2x} dy = y^3 e^{-2x} dx$

$$\Rightarrow \quad \int 2\frac{dy}{y} + \int 2\left(ye^{-2x}dy - y^2 e^{-2x}dx\right) = 0$$

$$\Rightarrow \quad \int 2\frac{dy}{y} + \int d\left(e^{-2x}y^2\right) + c = 0$$

$\Rightarrow \quad 2\ln|y| + e^{-2x}y^2 = k.$

Hence, the correct answer is (A).

43. The equation of the normal at any point to the curve $y = f(x)$ is

$$Y - y = \frac{-1}{\frac{dy}{dx}}(X - x) \Rightarrow \frac{X}{x + y\frac{dy}{dx}} + \frac{Y}{y + \frac{x}{\frac{dy}{dx}}} = 1$$

Given that

$$\frac{1}{2}\left(x + y\frac{dy}{dx}\right)\left(y + \frac{x}{\frac{dy}{dx}}\right) = 1$$

$\Rightarrow \quad x^2 + 2xy\frac{dy}{dx} + y^2\left(\frac{dy}{dx}\right)^2 = 2\frac{dy}{dx}$ is differentiable equation.

Hence, the correct answer is (D).

44. $\frac{dy}{dx} + \frac{y}{x}(\log y) = \frac{y}{x^2}(\log y)^2$

$$\Rightarrow \quad \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{\log y} = \frac{1}{x^2}, \text{ put } \frac{1}{\log y} = v$$

$$\Rightarrow \quad -\frac{1}{y(\log y)^2}\frac{dy}{dx} = \frac{dV}{dx}$$

$$\Rightarrow \quad -\frac{dV}{dx} + \frac{1}{x}V = \frac{1}{x^2}$$

$$\Rightarrow \quad \frac{dV}{dx} - \frac{1}{x}V = -\frac{1}{x^2}$$

$$\text{I.F.} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}; \ \frac{V}{x} = -\int \frac{1}{x^3} + C; \quad \frac{V}{x} = \frac{1}{2x^2} + C$$

$$\frac{1}{x\log y} = \frac{1}{2x^2} + C$$

Hence, the correct answer is (B).

45. $\dfrac{dy}{dx} + \dfrac{\tan y}{x} = \dfrac{\tan y \sin y}{x^2}$

$\Rightarrow \quad \cot y \operatorname{cosec} y \dfrac{dy}{dx} + \dfrac{\cos ec\, y}{x} = \dfrac{1}{x^2}$

$\Rightarrow \quad \dfrac{1}{x} \cot y \operatorname{cosec} y \dfrac{dy}{dx} + \dfrac{1}{x^2} \operatorname{cosec} y = \dfrac{1}{x^3}$

$\Rightarrow \quad \displaystyle\int d\left(\frac{\cos ec\, y}{x}\right) = \int -\frac{1}{x^3}\, dx$

$\Rightarrow \quad \dfrac{\cos ec\, y}{x} = \dfrac{1}{2x^2} + c \Rightarrow \dfrac{1}{x \sin y} = \dfrac{1+2cx^2}{2x^2}$

$\Rightarrow \quad 2x = \sin y(1 + 2c'x^2) \Rightarrow 2x = \sin y(1 + cx^2)$

Hence, the correct answer is (B).

46. $y\, dx - x\, dy + (1 + x^2)\, dx + x^2 \sin y\, dy = 0$

$\Rightarrow \quad \dfrac{ydx - xdy}{x^2} + \left(\dfrac{1}{x^2} + 1\right) dx + \sin y\, dy = 0$

$\Rightarrow \quad -\displaystyle\int d\left(\frac{y}{x}\right) + \left(-\frac{1}{x} + x\right) + (-\cos y) + K = 0$

$\Rightarrow \quad -\dfrac{y}{x} - \dfrac{1}{x} + x - \cos y + K = 0$

$\Rightarrow \quad \dfrac{y}{x} + \dfrac{1}{x} - x + \cos y = K$

Hence, the correct answer is (A).

47. $x\dfrac{dy}{dx} - 4y = x^2\sqrt{y}$

Put $y = t^2$

$$\frac{dy}{dx} = 2t\frac{dt}{dx} \Rightarrow x\left(2t\frac{dt}{dx}\right) = 4t^2 + x^2 t$$

$\Rightarrow \quad 2x\dfrac{dt}{dx} = x^2 + 4t \Rightarrow \dfrac{dt}{dx} - \dfrac{2}{x}t = \dfrac{x}{2}$

Integrating factor $= e^{-\int \frac{2}{x}dx}\, dx = e^{-2\ln x} = \dfrac{1}{x^2}$

$$t \cdot \frac{1}{x^2} = \int \frac{x}{2} \cdot \frac{1}{x^2} dx \Rightarrow \frac{\sqrt{y}}{x^2} = \frac{1}{2}\ln x + c$$

Point $(1, (\ln 2)^2)$ lies on it

$$\frac{\ln 2}{1} = 0 + c;\ \sqrt{y} = x^2\left(\frac{1}{2}\ln x + \ln 2\right)$$

$$\sqrt{y(2)} = 4\left(\frac{1}{2}\ln 2 + \ln 2\right) = 4\left(\frac{3}{2}\ln 2\right);\ y(2) = (6\ln 2)^2$$

Hence, the correct answer is (C).

48. I.F. $= e^{-x}$

$\therefore \quad ye^{-x} = \displaystyle\int e^{-x}(\cos x - \sin x)\, dx$ put $-x = t$

$$= -\int e^{t}(\cos t + \sin t)\, dt$$

$$= -e^{t}\sin t + c$$

$$y\, e^{-x} = e^{-x}\sin x + c$$

Since y is bounded when $x \to \infty \Rightarrow c = 0$

$\therefore \quad y = \sin x$

$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)\, dx = \sqrt{2} - 1$$

Hence, the correct answer is (A).

49. $c(y+c)^2 = x^3 \Rightarrow c[2(y+c)]\dfrac{dy}{dx} = 3x^2$

$\Rightarrow \quad \dfrac{2x^3}{y+c}\dfrac{dy}{dx} = 3x^2 \Rightarrow y + c = \dfrac{2x}{3}\dfrac{dy}{dx}.$

$\Rightarrow \quad \left(\dfrac{2x}{3}\dfrac{dy}{dx} - y\right)\left(\dfrac{2x}{3}\dfrac{dy}{dx}\right)^2 = x^3$

$$\left(2x\frac{dy}{dx} - 3y\right)4x^2\left(\frac{dy}{dx}\right)^2 = 27x^3$$

$\Rightarrow \quad 8x\left(\dfrac{dy}{dx}\right)^3 - 12y\left(\dfrac{dy}{dx}\right)^2 - 27x = 0$

Hence, the correct answer is (B).

50. $\dfrac{ydx - xdy}{y^2} + 3x^2e^{x^3}dx = 0 \Rightarrow d\left(\dfrac{x}{y}\right) + d\left(e^{x^3}\right) = 0$

$\Rightarrow \quad \dfrac{x}{y} + e^{x^3} = C$

Hence, the correct answer is (C).

51. $\cos y = ae^{-x}$

$\Rightarrow \quad -\sin y\dfrac{dy}{dx} = -ae^{-x}$

$\Rightarrow \quad -\sin y\dfrac{dy}{dx} = -\cos y \Rightarrow \tan y\dfrac{dy}{dx} = 1$

For orthogonal trajectory, replace $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$

$$\tan y\left(-\frac{dx}{dy}\right) = 1 \Rightarrow \int -dx = \int \cot y\, dy$$

$\Rightarrow \quad -x + c = \ln \sin y$

$\Rightarrow \quad \sin y = e^{-x+c} \quad \Rightarrow \quad \sin y = Ke^{-x}$

Hence, the correct answer is (C).

52. $\displaystyle\lim_{t\to x}\frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$

Using L'Hopital Rule, differentiate with respect to t.

$\Rightarrow \quad \displaystyle\lim_{t\to x}\frac{2t\, f(x) - x^2 f'(t)}{1} = 1$

$\Rightarrow \quad 2x\, f(x) - x^2 f'(x) = 1$

Let $f(x) = y$

$$x^2\frac{dy}{dx} - 2xy = -1 \Rightarrow \frac{dy}{dx} - \frac{2}{x}y = -\frac{1}{x^2},$$

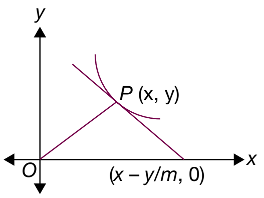
$\therefore \quad$ I.F. $= e^{-\int \frac{2}{x}dx} = e^{-2\ln x} \Rightarrow y\dfrac{1}{x^2} = -\displaystyle\int \frac{1}{x^4}\, dx;$

As $y(1) = 1$, $c = \dfrac{2}{3}$

$\therefore \quad \dfrac{y}{x^2} = \dfrac{1}{3x^3} + \dfrac{2}{3} \Rightarrow y = \dfrac{1}{3x} + \dfrac{2x^2}{3}$

Hence, the correct answer is (A).

53. $\dfrac{1}{2}\left(x - \dfrac{y}{m}\right)y = 2$



$\Rightarrow \quad xy - \dfrac{y^2}{m} = 4$

$\Rightarrow \quad xy - \dfrac{y^2 dx}{dy} = 4$

$\Rightarrow \quad \dfrac{dx}{dy} - \dfrac{1}{y}x = \dfrac{-4}{y^2}$

I.F. $= e^{-\int \frac{1}{y}dy} = \dfrac{1}{y}$

$$x\frac{1}{y} = -\int \frac{4}{y^3}\, dy = \frac{4}{2y^2} + C = \frac{2}{y^2} + C$$

$$x = 2;\ y = 1 \Rightarrow C = 0$$

$\Rightarrow \quad xy = 2$

Hence, the correct answer is (B).

54. $f(x) = e^x + \sin x + x = f'(x) = e^x + \cos x + 1$

$\Rightarrow \quad f'(x) > 0 \; x \in R \Rightarrow f(x)$ is increasing

$f(x + y) = f(x) - f(y)$

Put $y = -x$;

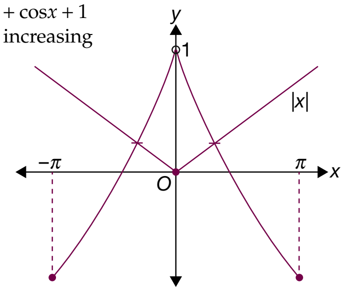
$\Rightarrow \quad f(0) = f(x) - f(-x)$

$\Rightarrow \quad f(-x) = f(x)$

$\Rightarrow \quad f(-x) = f(x)$

$\Rightarrow \quad f(x)$ is even.

$\therefore$ From the graph it is clear that $f(x) = |x|$ has 3 solutions.



Hence, the correct answer is (D).

55. $x + y\dfrac{dy}{dx} = 3x \Rightarrow \displaystyle\int y\,dy = \int 2x\,dx$

$\therefore \quad \dfrac{y^2}{2} = x^2 + c$

Now, (1, 2) satisfy it, $2 - 1 = c$,

$\Rightarrow \quad c = 1$, so $y^2 = 2x^2 + 2$ or $2x^2 - y^2 + 2 = 0$.

Hence, the correct answer is (C).

56. $y = e^{mx}$

$$\frac{dy}{dx} = my, \frac{d^2y}{dx^2} = m\frac{dy}{dx} = m^2y$$

$$\frac{d^3y}{dx^3} = m^3y$$

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 12y = 0$$

$$(m^3 - 3m^2 - 4m + 12)\; y = 0$$

$$(m - 3)\;(m^2 - 4) = 0$$

$$m = 3, m = 2, m = -2$$

As given $m \in N$;

$\therefore \quad$ 2 values $m = 2, 3$

Hence, the correct answer is (C).

57. $f(x) \sin 2x + (1 + \sin^2 x) f'(x) - \cos x = 0$

$\Rightarrow \quad \dfrac{d}{dx}(f(x)\;(1 + \sin^2 x)) = \cos x$

$\Rightarrow \quad f(x)\;(1 + \sin^2 x) = \sin x + c$

$\Rightarrow \quad x = 0, f(0) = 0 \Rightarrow c = 0$

$\Rightarrow \quad f(x) = \dfrac{\sin x}{1 + \sin^2 x}$

$$f\left(\frac{\pi}{6}\right) = \frac{1/2}{1 + 1/4} = \frac{1/2}{5/4} = \frac{2}{5}$$

Hence, the correct answer is (D).

58. $e^x(x + 1)\,dx + (ye^y - xe^x)\,dy = 0$

$\Rightarrow \quad e^x x dx + e^x dx + ye^y dy - xe^x dy = 0$

$\Rightarrow \quad xe^x(dx - dy) + e^x dx + ye^y dy = 0$

$\Rightarrow \quad e^x(x + 1)\dfrac{dx}{dy} + ye^y - xe^x = 0$

Put $xe^x = t$, $(xe^x + e^x)\dfrac{dx}{dy} = \dfrac{dt}{dy}$

$$\frac{dt}{dy} + ye^y - t = 0$$

$\Rightarrow \quad \dfrac{dt}{dy} - t = -ye^y$

Integrating Factor $= e^{\int -dy} = e^{-y}$

$$te^{-y} = \int -ye^y \cdot e^{-y}\,dy \Rightarrow te^{-y} = -\int y\,dy = -\frac{y^2}{2} + c$$

$$xe^x e^{-y} = -\frac{y^2}{2} + c$$

Given $f(0) = 0$; $0 = 0 + c \Rightarrow c = 0$

$$xe^x = -\frac{y^2}{2}e^y \Rightarrow 2xe^x + y^2e^y = 0$$

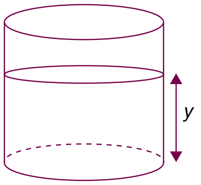
Hence, the correct answer is (B).

59. $\dfrac{dy}{dt} = -K\sqrt{y}$

$\Rightarrow \quad \displaystyle\int_4^0 \frac{dy}{\sqrt{y}} = -\int_0^t K\,dt$

Given $K = \dfrac{1}{15} \Rightarrow 2\sqrt{y}\Big|_0^4 = Kt = 4$

$$\frac{t}{15} = 4 \Rightarrow t = 60 \text{ min}$$



Hence, the correct answer is (C).

60. $dy + xy\,dx = x\,dx$

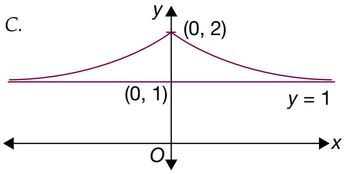
$$\frac{dy}{dx} + xy = x \Rightarrow \text{Integrating factor} = e^{\frac{x^2}{2}}$$

$\therefore \quad ye^{\frac{x^2}{2}} = \displaystyle\int xe^{\frac{x^2}{2}}\,dx = e^{\frac{x^2}{2}} + C.$

Hence, $y = 1 + C\,e^{\frac{-x^2}{2}}$

$$y(0) = 2 \Rightarrow C = 1$$

Hence, $y(x) = 1 + e^{\frac{-x^2}{2}}$



$$A = \int_0^\infty \left(e^{\frac{-x^2}{2}} + 1\right) - 1\,dx = \int_0^\infty e^{\frac{-x^2}{2}}\,dx.$$

Hence, the correct answer is (A).

61. $\left(\dfrac{y+1}{y}\right)dy + e^x(\cos^2 x - \sin 2x)\,dx = 0$

$\therefore \quad$ On integrating, we get

$$y + \ln y + e^x \cos^2 x = c.$$

As, $y(0) = 1 \Rightarrow c = 2$.

Hence, the correct answer is (D).

62. Given $\dfrac{d}{dx}\left(\dfrac{f'(x)}{x}\right) = 1$

$\therefore \quad \dfrac{f'(x)}{e^x} = x + C$

Put $x = 0$

$$\frac{1}{1} = 0 + C$$

$\therefore \quad C = 1$

$\therefore \quad f'(x) = e^x(x + 1) \Rightarrow f(x) = xe^x + k$

Put $x = 0 \Rightarrow k = 0 \Rightarrow f(x) = xe^x$

$\therefore \quad \dfrac{f(x)}{x} = e^x$

$\therefore \quad \displaystyle\int_0^1 e^x\,dx = e - 1$

Hence, the correct answer is (B).

63. $x^4dy + x^3y\,dx + \text{cosec}\,(xy)\,dx = 0$

$\Rightarrow \quad x^3(x\,dy + y\,dx) + \text{cosec}\,(xy)\,dx = 0$

$\therefore \quad \displaystyle\int \frac{d(xy)}{\text{cos}\,ec(xy)} + \int \frac{dx}{x^3} = 0$

$$-\cos xy + \frac{x^{-3+1}}{-3+1} = C$$

$\therefore \quad 2\cos(xy) + x^{-2} = C$

Hence, the correct answer is (A).

64. $y = Ax + Be^{2x}$

$$\frac{dy}{dx} = A + 2Be^{2x} \Rightarrow x\frac{dy}{dx} = Ax + 2Bxe^{2x}$$

$$\left(x\frac{dy}{dx} - y\right) = Be^{2x}(2x-1);\ \frac{d^2y}{dx^2} = 4Be^{2x}$$

$$\frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y\right)\frac{4}{(2x-1)}$$

$$\Rightarrow \quad (1-2x)\frac{d^2y}{dx^2} + 4\left(x\frac{dy}{dx} - y\right) = 0 \qquad \ldots(1)$$

Given $(1-2x)\dfrac{d}{dx}\left(\dfrac{dy}{dx} + ly\right) + K\left(\dfrac{dy}{dx} + ly\right) = 0$

$$\Rightarrow \quad (1-2x)\left(\frac{d^2y}{dx^2} + l\frac{dy}{dx}\right) + K\frac{dy}{dx} + Kly = 0$$

$$\Rightarrow \quad (1-2x)\frac{d^2y}{dx^2} + (\underset{\substack{||\\4x}}{K + l(1-2x)})\frac{dy}{dx} + \underset{\substack{\downarrow\\-4}}{Kl}\,y = 0 \qquad \ldots(2)$$

We have $K + l - 2lx = 4x$ and $Kl = -4$

$K + l = 0;\ 2l = -4$

We get $K = 2$, $l = -2$

$(K, l) \equiv (2, -2)$

Hence, the correct answer is (A).

65. $\dfrac{xdy - ydx}{x^2} = e^x\left(\ln x + \dfrac{1}{x}\right)dx$

$$\int d\left(\frac{y}{x}\right) = \int e^x\left(\ln x + \frac{1}{x}\right)dx$$

$\dfrac{y}{x} = e^x \ln x + C$

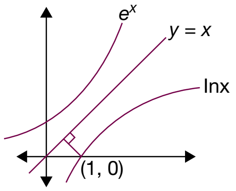
$y(1) = 0 \Rightarrow C = 0$

$\therefore \quad y = xe^x \ln x$

$h(x) = \ln x$

$g(x) = x$

Hence, least distance $= 2\cdot\left(\dfrac{1}{\sqrt{2}}\right) = \sqrt{2}$



Hence, the correct answer is (A).

66. The given equation is linear in y and can be written as

$$\frac{dy}{dx} + \frac{x}{1-x^2}y = \frac{ax}{1-x^2}$$

Its integrating factor is $e^{\int \frac{x}{1-x^2}dx} = e^{-(1/2)\log(1-x^2)} = \dfrac{1}{\sqrt{1-x^2}}$

If $-1 < x < 1$ and if $x^2 > 1$ then I.F. $= \dfrac{1}{\sqrt{x^2-1}}$

$$\frac{d}{dx}\left(y\frac{1}{\sqrt{1-x^2}}\right) = \frac{ax}{(1-x^2)^{3/2}} = -\frac{1}{2}a\frac{-2x}{(1-x^2)^{3/2}}$$

$$\Rightarrow \quad y\frac{1}{\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + C \Rightarrow y = a + C\sqrt{1-x^2}$$

$$\Rightarrow \quad (y-a)^2 = C^2(1-x^2) \Rightarrow (y-a)^2 + C^2x^2 = C^2$$

Thus if $-1 < x < 1$ the given equation represents an ellipse. If $x^2 > 1$ then the solution is of the form $-(y-a)^2 + C^2x^2 = C^2$ which represents a hyperbola.

Hence, the correct answer is (A).

67. $x^2\,dy - 2x^3y^3\,dy = 3x^2y^4\,dx + 2xy\,dx$

$$\Rightarrow \quad x^2\,dy - 2xy\,dy = 3x^2y^4\,dx + 2x^3y^3\,dx$$

$$\Rightarrow \quad \frac{2xydx - x^2dy}{y^2} + 3x^2y^2\,dx + 2x^3y\,dy = 0$$

$$\Rightarrow \quad d\left(\frac{x^2}{y}\right) + d(x^3y^2) = 0 \Rightarrow \frac{x^2}{y} + x^3y^2 = C$$

Hence, the correct answer is (B).

68. Applying C and D, we get

$$\frac{dy}{dx} = \frac{e^{-x}}{e^x} = e^{-2x} \Rightarrow 2y = -e^{-2x} + C$$

or $\quad 2y\,e^{2x} = C.e^{2x} - 1.$

Hence, the correct answer is (B).

69. $x - y\dfrac{dx}{dy} = 3x \Rightarrow \dfrac{dx}{x} + 2\dfrac{dy}{y} = 0$

$\Rightarrow \quad \ln(xy^2) = k \Rightarrow xy^2 = c.$

Hence, the correct answer is (B).

70. The point on y-axis is $\left(0, y - x\dfrac{dy}{dx}\right)$.

According to given condition,

$$\frac{x}{2} = y - \frac{x}{2}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = 2\frac{y}{x} - 1$$

Putting $\dfrac{y}{x} = v$ we get

$$\frac{dv}{dx} = v - 1 \Rightarrow \ln\left|\frac{y}{x} - 1\right| = \ln|x| + c$$

$\Rightarrow \quad l - \dfrac{y}{x} = x \quad$ (as $f(1) = 0$).

Hence, the correct answer is (C).

71. We have $\dfrac{\sqrt{x}dx + \sqrt{y}dy}{\sqrt{x}dx - \sqrt{y}dy} = \sqrt{\dfrac{y^3}{x^3}}$

$$\Rightarrow \quad \frac{d(x^{3/2}) + d(y^{3/2})}{d(x^{3/2}) - d(y^{3/2})} = \frac{y^{3/2}}{x^{3/2}}$$

$\Rightarrow \quad \dfrac{du + dv}{du - dv} = \dfrac{v}{u}$, where $u = x^{3/2}$ and $v = y^{3/2}$

$$\Rightarrow \quad u\,du + u\,dv = v\,du - v\,dv \Rightarrow u\,du + v\,dv = v\,du - u\,dv$$

$$\Rightarrow \quad \frac{udu + vdv}{u^2+v^2} = \frac{vdu - udv}{u^2+v^2} \Rightarrow \frac{d(u^2+v^2)}{u^2+v^2} = -2d\tan^{-1}\left(\frac{v}{u}\right) + c$$

On integrating, we get $\log(u^2+v^2) = -2\tan^{-1}\left(\dfrac{v}{u}\right) + c$

$$\Rightarrow \quad \frac{1}{2}\log(x^3+y^3) + \tan^{-1}\left(\frac{y}{x}\right)^{3/2} = \frac{c}{2}$$

Hence, the correct answer is (D).

72. Since $f''(x) = 6(x-1)$

$\Rightarrow \quad f'(x) = 3(x-1)^2 + c \qquad$ (integrating) $\qquad \ldots(1)$

Also, at the point (2, 1), the tangent to the graph is $y = 3x - 5$ and slope of the tangent $= 3$

$\Rightarrow \quad f'(2) = 3$

$3(2-1)^2 + c = 3 \qquad$ [from Eq (1)]

$\Rightarrow \quad 3 + c = 3 \quad \Rightarrow c = 0$

From Eq (1) we have $f'(x) = 3(x-1)^2$

$\Rightarrow \quad f(x) = (x-1)^3 + k \qquad$ (Integrating) $\qquad \ldots(2)$

$\therefore \quad 1 = (2-1)^3 + k \Rightarrow k = 0$

Hence the equation of the function is $f(x) = (x-1)^3$.

Hence, the correct answer is (B).

73. $\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$ It is homogenous differential equation

Put $y/x = t \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$

Given diff equation $t + \frac{xdt}{dx} = t - \cos^2 t$

$\Rightarrow \quad \int \frac{dt}{\cos^2 t} = \int -\frac{dx}{x} \Rightarrow \tan t = -\ln x + c$

$\Rightarrow \quad \tan\frac{y}{x} = -\ln x + c$

Curve passes through point $\left(1, \frac{\pi}{4}\right)$, then $1 = 0 + c \Rightarrow c = 1$

$\tan\frac{y}{x} = -\ln x + 1 \Rightarrow \tan\frac{y}{x} = \ln\frac{e}{x} \Rightarrow y = x\tan^{-1}\left(\ln\frac{e}{x}\right)$

Hence, the correct answer is (A).

74. Tangent $y - y_0 = m\,(x - x_0)$

$mx - y = mx_0 - y_0$

$$\frac{x}{\left(\frac{mx_0 - y_0}{m}\right)} + \frac{y}{\left(\frac{y_0 - mx_0}{1}\right)} = 1$$

Given condition $\frac{mx_0 - y_0}{m} = y_0 \Rightarrow$ slope of tangent $m = \frac{y_0}{x_0 - y_0}$

$\frac{dy}{dx} = \frac{y}{x-y} \Rightarrow \frac{dx}{dy} = \frac{x-y}{y};\ \frac{dx}{dy} - \frac{x}{y} = -1$

Integrating Factor $= e^{-\int\frac{1}{y}dy} = e^{-\ln y} = \frac{1}{y}$

$\frac{x}{y} = \int -\frac{1}{y}dy \Rightarrow \frac{x}{y} = -\ln y + c$

Point (1, 1) lies on curve, $\frac{1}{1} = -0 + c \Rightarrow \frac{x}{y} = -\ln y + 1 = \ln\left(\frac{e}{y}\right)$

$e^{x/y} = \frac{e}{y} \quad \Rightarrow ye^{x/y} = e$

Hence, the correct answer is (A).

75. $\int_0^1 f(tx)\,dt = n\,f(x) \qquad x > 0$

Put $tx = p \Rightarrow x\,dt = dp$

$\Rightarrow \quad \int_0^x f(P)\frac{dP}{x} = nf(x) \Rightarrow \int_0^x f(P)dP = nxf(x)$

$\Rightarrow \quad f(x) - 0 = nf(x) + nxf'(x) \Rightarrow (1 - n)\,f(x) = n\,x\,f'(x)$

$\Rightarrow \quad (1 - n)\,y = nx\frac{dy}{dx} \Rightarrow \int(1-n)\frac{dx}{x} = \int\frac{ndy}{y}$

$(1 - n)\ln x = n\ln y + \ln K$

$\ln x^{1-n} = \ln y^n + \ln K$

$\frac{x^{1-n}}{y^n} = K \Rightarrow y^n = \frac{1}{K}x^{1-n}$

$y = c\,x^{\frac{1-n}{n}}$

Hence, the correct answer is (A).

76. $\frac{dy}{dx} = -xe^{-x^2/2}\ \int dy = \int -xe^{-x^2/2}dx$

$\frac{d^2y}{dx^2} = -e^{-x^2/2} - x\left(e^{-x^2/2}(-x_1)\right) = e^{-x^2/2}\ [x^2 - 1] = 0$

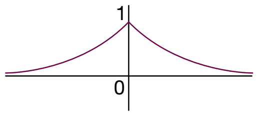
Put $-\frac{x^2}{2} = t - xdx = dt$

$x = 1, -1$

2 point infection $-xdx = dt$

$y = \int e^t dt \Rightarrow y = e^t + c \Rightarrow y = e^{-x^2/2} + c$

At point $\left(1, \frac{1}{\sqrt{e}}\right)$



$\frac{1}{\sqrt{e}} = \frac{1}{\sqrt{e}} + c$

$\Rightarrow \quad c = 0 \quad \Rightarrow \quad y = e^{-x^2/2}$

Differentiable not symmetric about origin but it is symmetric about y-axis

Hence, the correct answer is (B).

77. $x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - y = 0$

$y = mx + c$ where $\frac{dy}{dx} = m \Rightarrow xm^2 + m - y = 0$

$\Rightarrow \quad y = xm^2 + m$ and $y = mx + c$

$\frac{1}{1} = \frac{m^2}{m} = \frac{m}{c}$

$m = 1 \qquad c = m = 1 \qquad m = 0, c = 0$

$y = x + 1 \qquad y = 0$

Two straight lines

Hence, the correct answer is (B).

78. $x^2\frac{dy}{dx}\cdot\cos 1/x - y\sin 1/x = -1$

$\frac{dy}{dx} - \frac{y\tan 1/x}{x^2} = -1/x^2\sec 1/x$

Integrating factor $= e^{\int\frac{-\tan 1/x}{x^2}dx} = e^{\ln\sec 1/x} = \sec 1/x$

$\Rightarrow \quad y\,(\sec 1/x) = \int -\frac{\sec^2 1/x}{x^2}\,dx$

$\Rightarrow \quad y\left(\sec 1/x\right) = \tan 1/x + c;$

$\Rightarrow \quad y = \sin 1/x + c\cos 1/x$

$x \rightarrow \infty, -1 = 0 + c; c = -1$

$y = \sin 1/x - \cos 1/x$

Hence, the correct answer is (A).

79. Normal at point (x, y) is given by $Y - y = -\frac{1}{m}(X - x)$

It cuts x-axis, then $Y = 0$; $X = my + x$

According to given condition, we have $x\,(my + x) = 2\left(x^2 + y^2\right)$

$\Rightarrow \quad mxy = x^2 + 2y^2$

Put $\frac{y}{x} = t \Rightarrow y = tx \Rightarrow \frac{dy}{dx} = t + \frac{xdt}{dx}$

$\frac{dy}{dx} = \frac{x}{y} + \frac{2y}{x}$

$\Rightarrow \quad t + \frac{xdt}{dx} = \frac{1}{t} + 2t \Rightarrow x\frac{dt}{dx} = \frac{1}{t} + t = \frac{1+t^2}{t}$

$\Rightarrow \quad \int\frac{t}{1+t^2}dt = \int\frac{dx}{x} \Rightarrow \frac{1}{2}\ln(1 + t^2) = \ln x + \ln k$

$\Rightarrow \quad \sqrt{1+\frac{y^2}{x^2}} = Kx$

$x^2 + y^2 = K^2x^4$

Point (1, 0) lies on it then $1 + 0 = K^2 \Rightarrow K^2 = 1$

Required curve is $x^2 + y^2 = x^4$

Hence, the correct answer is (A).

80. Equation of normal $y - y_1 = -\frac{1}{m}(x - x_1)$

At x-axis $(x_1 + my_1, 0) \equiv (x_1 + 1, 0)$

$my = 1$

$\int ydy = \int dx$

$$\Rightarrow \quad \frac{y^2}{2} = x + c$$

Origin (0, 0) lies on it $c = 0$

$$y^2 = 2x$$

Latus rectum = 2

Hence, the correct answer is (B).

81. $y = c\,(x - c)^2$

$$\Rightarrow \quad \frac{dy}{dx} = 2c\,(x - c) \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4c^2(x - c)^2$$

$$\left(\frac{dy}{dx}\right)^2 = 4cy \Rightarrow c = \frac{1}{4y}\left(\frac{dy}{dx}\right)^2$$

$$y = \frac{1}{4y}\left(\frac{dy}{dx}\right)^2\left[x - \frac{1}{4y}\left(\frac{dy}{dx}\right)^2\right]^2$$

$$4y^2 = \left(\frac{dy}{dx}\right)^2\left(x - \frac{1}{4y}\left(\frac{dy}{dx}\right)^2\right)^2$$

$$\pm 2y = \left(\frac{dy}{dx}\right)\left(x - \frac{1}{4y}\left(\frac{dy}{dx}\right)^2\right)$$

order = 1 degree = 3

Hence, the correct answer is (C).

82. $\frac{dx}{dt} = -Kx \Rightarrow \int_{x_0}^{x}\frac{dx}{x} = \int_{0}^{t} -K\,dt$

$$\Rightarrow \quad \ln x\Big|_{x_0}^{x} = -Kt \Rightarrow \ln\frac{x}{x_0} = -Kt \Rightarrow x = x_0e^{-Kt}$$

At time, $t = 1$ minutes; $\frac{x_0}{2} = x_0e^{-K}$

$e^{-K} = \frac{1}{2} \Rightarrow x = x_0\left(\frac{1}{2}\right)^t$ is required relation between substance present at time t

$$x = \frac{x_0}{2^t} \Rightarrow \left(1 - \frac{99}{100}\right)x_0 = \frac{x_0}{2^t} \Rightarrow 2^t = 100$$

$$\Rightarrow \quad t = \frac{\log_{10}100}{\log_{10}2}$$

$$t = \frac{2}{0\cdot 3010} = \frac{20}{3\cdot 01} = 6.64$$

Hence, the correct answer is (B).

83. $y = ax^2 \Rightarrow y' = 2ax$

$$\Rightarrow \quad y' = 2x\left(\frac{y}{x^2}\right) = \frac{2y}{x}$$

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$$\Rightarrow \quad -\frac{dx}{dy} = \frac{2y}{x} \Rightarrow \int -x\,dx = \int 2y\,dy$$

$$\Rightarrow \quad -\frac{x^2}{2} = y^2 + c \Rightarrow \frac{x^2}{2} + y^2 = K$$

Hence, the correct answer is (A).

84. Put $y = u^m$

$$\frac{dy}{dx} = mu^{m-1}\frac{du}{dx}$$

$$\Rightarrow \quad 2x^4u^m\left(mu^{m-1}\frac{du}{dx}\right) + u^{4m} = 4x^6$$

$$\Rightarrow \quad 2mx^4\,u^{2m-1}\frac{du}{dx} = 4x^6 - u^{4m}$$

$$\Rightarrow \quad \frac{du}{dx} = \frac{4x^6 - u^{4m}}{2mx^4u^{2m-1}}$$

$$m = \frac{3}{2}$$

Hence, the correct answer is (C).

85. $(x + 1)\,f'(x) - 2(x^2 + x)\,f(x) = \frac{e^{x^2}}{(x+1)}$

$$\Rightarrow \quad f'(x) - 2x\,f(x) = \frac{e^{x^2}}{(x+1)^2}$$

Integrating Factor $= e^{\int -2x\,dx} = e^{-x^2}$

$$\Rightarrow \quad f(x)\,e^{-x^2} = \int \frac{e^{x^2}}{(x+1)^2}e^{-x^2}\,dx \Rightarrow f(x)\,e^{-x^2} = -\frac{1}{(x+1)} + c$$

$$\Rightarrow \quad f(0) = 5 = \frac{-1}{1} + c\text{, we get } c = 6$$

$$f(x)\,e^{-x^2} = -\frac{1}{(x+1)} + 6 = \frac{6x+5}{x+1}$$

$$f(x) = \left(\frac{6x+5}{x+1}\right)e^{x^2}$$

Hence, the correct answer is (B).

86. $\frac{dy}{dx} - y = y^2(\sin x + \cos x)$

$$\Rightarrow \quad \frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = \sin x + \cos x$$

Put $\frac{1}{y} = t \Rightarrow -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \quad -\frac{dt}{dx} - t = \sin x + \cos x \Rightarrow \frac{dt}{dx} + t = -\sin x - \cos x$$

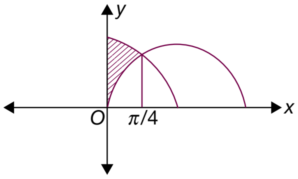
$$\Rightarrow \quad e^xt = -\int e^x(\sin x + \cos x)dx \Rightarrow \frac{e^x}{y} = -e^x\sin x + c$$

Given $y(0) = 1 \Rightarrow \frac{1}{1} = 0 + c \Rightarrow e^x = (1 - e^x\sin x)\,y$

At $x = \pi \Rightarrow e^{\pi} = (1 - 0)\,y(\pi) \Rightarrow y(\pi) = e^{\pi}$

Hence, the correct answer is (A).

87. $\frac{dy}{dx} - y = \cos x - \sin x$



Integrating Factor $= e^{-x}$

$$ye^{-x} = \int e^{-x}(\cos x - \sin x)\,dx$$

Put $x = -t$; $dx = -dt$

$$ye^{-x} = \int e^{t}(\cos t + \sin t)\,(-dt)$$

$$ye^{-x} = e^t\sin t\,(-1) + c \Rightarrow ye^{-x} = -e^{-x}\sin(-x) + c$$

$x \to \infty$; y is bounded $\Rightarrow 0 = 0 + c$, we get $c = 0$

$$y = \sin x$$

Required area $= \int_{0}^{\pi/4} \cos x - \sin x\,dx = \sin x + \cos x\Big|_0^{\pi/4}$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 0 - 1$$

$$= \sqrt{2} - 1$$

Hence, the correct answer is (A).

88. $\int_{0}^{x} t(y(t))\,dt = x^2y(x)$ where $x > 0$

$$\Rightarrow \quad xy(x) - 0 = 2x\,y(x) + x^2\frac{dy(x)}{dx}$$

$$\Rightarrow \quad xy(x) = -x^2\frac{dy(x)}{dx} \Rightarrow \int\frac{dx}{x} = -\int\frac{dy(x)}{y(x)}$$

$\Rightarrow$ $\ln x = -\ln y(x) + c$ curve passes through (2, 3)

$\Rightarrow$ $\ln 2 = -\ln 3 + c \Rightarrow c = \ln 6 \Rightarrow \ln xy(x) = \ln 6 \Rightarrow xy = 6$

Hence, the correct answer is (D).

89. $(y\,dx - x\,dy)\cot\left(\frac{x}{y}\right) = ny^2\,dx$

$$\Rightarrow \quad \left(\frac{ydx - xdy}{y^2}\right)\cot\left(\frac{x}{y}\right) = n\,dx$$

$$\Rightarrow \quad d\left(\frac{x}{y}\right)\cot\left(\frac{x}{y}\right) = n\,dx \Rightarrow \int \cot\left(\frac{x}{y}\right) d\left(\frac{x}{y}\right) = \int n\,dx$$

$$\Rightarrow \quad \ln\sin\left(\frac{x}{y}\right) = nx + c \Rightarrow \sin\left(\frac{x}{y}\right) = e^{nx+c} = ke^{nx}$$

Hence, the correct answer is (C).

90. P – Population, y – population after 't' years

$$\frac{dy}{dt}\,ay \Rightarrow \frac{dy}{dt} = ky \Rightarrow \int \frac{dy}{y} = \int k\,dt$$

$$\Rightarrow \quad \log y = kt + c$$

$$t = 0 \text{ and } y = p \Rightarrow \log p = 0 + C$$

$$\Rightarrow \quad C = \log p$$

$$t = 50 \text{ and } y = 2p \Rightarrow \log 2p = 50k + \log p$$

$$\Rightarrow \quad \log\left(\frac{2p}{p}\right) = 50k \Rightarrow k = \frac{\log(2)}{50}$$

$$t = ? \text{ and } y = 3p \Rightarrow kt = \log y - c$$

$$\left(\frac{\log 2}{50}\right)t = \log 3p - \log p$$

$$t = \frac{\log 3}{(\log 2/50)} = 50\log_2 3$$

Hence, the correct answer is (B).

91. Putting $\sqrt{1+x+y} = v$, we have,

$$x + y - 1 = v^2 - 2 \Rightarrow 1 + \frac{dy}{dx} = 2v\frac{dv}{dx}$$

Then the given equation transforms to

$$\left(2v\frac{dv}{dx} - 1\right)v = v^2 - 2 \Rightarrow \frac{dv}{dx} = \frac{v^2 + v - 2}{2v^2}$$

$$\Rightarrow \quad \int \frac{2v^2}{v^2 + v - 2}dv = \int dx \Rightarrow 2\int\left[1 + \frac{1}{3(v-1)} - \frac{4}{3(v+2)}\right]dv = \int dx$$

$$\Rightarrow \quad 2\left[v + \frac{1}{3}\log|v-1| - \frac{4}{3}\log|v+2|\right] = x + c$$

Where $v = \sqrt{1+x+y}$

Hence, the correct answer is (A).

92. $(x+1)\frac{dy}{dx} + 1 = \frac{e^x}{e^y} \Rightarrow e^y(x+1)\frac{dy}{dx} + e^y = e^x$

$$\int \frac{d}{dx}(e^y(x+1))dx = \int e^x\,dx \Rightarrow e^y(x+1) = e^x + c$$

Hence, the correct answer is (B).

93. Let $y = f(x)$ be the curve and let $P(x, y)$ be any point on the curve. The equation of the normal at $P(x, y)$ to the given curve is

$$Y - y = -\frac{1}{\frac{dy}{dx}}(X - x)$$

The distance of (1) from the origin is

$$d_1 = \left|\frac{y + \frac{x}{\frac{dy}{dx}}}{\sqrt{1 + \frac{1}{(dy/dx)^2}}}\right| = \left|\frac{x + y\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right| \quad d_2 = \left|\frac{(a-x) + (b-y)\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right|$$

Now $d_1 = kd_2$

$$\left|\frac{x + y\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}\right| = k\left|\frac{(a-x) + (b-y)\frac{dy}{dx}}{1 + \left(\frac{dy}{dx}\right)^2}\right|$$

$$x\,dx + y\,dy = \pm k((a-x)\,dx + (b-y)\,dy)$$

On integrating, we get

$$\frac{x^2}{2} + \frac{y^2}{2} = \pm k\left[-\frac{(a-x)^2}{2} - \frac{(b-y)^2}{2}\right] + C$$

Hence, the correct answer is (A).

94. S_1 - Equation of parabola is $y^2 = \pm 4ax$

$$2y\frac{dy}{dx} = \pm 4a$$

D.E of parabola $\Rightarrow y^2 = 2yx\frac{dy}{dx}$

$$2\frac{dy}{dx} = \frac{dx}{x}$$

Which is variable seperable

S_2 - Equation of line which is fixed distance. P from origin can be equation of tangent to circle $x^2 + y^2 = p^2$

Line is $y = mx + p\sqrt{1+m^2}$ $\qquad \left(m = \frac{dy}{dx}\right)$

$$\left(y - x\frac{dy}{dx}\right)^2 = P\left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

So, degree is 2

S_3 - Equation of conic whose both axis co-incide with co-ordinate axis is $ax^2 + by^2 = 1$

As there are two constants, so order of D.E is 2

Hence, the correct answer is (A).

95. $\because \quad \left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx} = \left(\frac{x+y+1}{x+y+2}\right)$

Put $x + y = v$

$$\therefore \quad \left(\frac{v-1}{v-2}\right)\left(\frac{dv}{dx} - 1\right) = \left(\frac{v+1}{v+2}\right)$$

$$\Rightarrow \quad \frac{dv}{dx} - 1 = \frac{(v+1)(v-2)}{(v-1)(v+2)} = \frac{v^2 - v - 2}{v^2 + v - 2}$$

$$\text{or} \quad \frac{dv}{dx} = \frac{2v^2 - 4}{(v^2 + v - 2)}$$

$$\Rightarrow \quad \int \frac{v^2 - 2 + v}{2(v^2 - 2)}\,dv = \int dx$$

$$\frac{1}{2}\int\left[1 + \frac{2v}{2(v^2 - 2)}\right]dv = \int dx;\ \frac{1}{2}\int\left[v + \frac{1}{2}\ln(v^2 - 2)\right] = x + C_1$$

$$y - x + \frac{1}{2}\ln\{(x+y)^2 - 2\} = C$$

Given that $y = 1$, when $x = 1$

$$\therefore \quad 0 + \frac{1}{2}\ln 2 = c$$

$$(y - x) + \frac{1}{2}\left|\frac{\ln(x+y)^2 - 2}{2}\right| = 0 \ \text{ or } \ln\left|\frac{(x+y)^2 - 2}{2}\right| = 2(x - y)$$

Hence, the correct answer is (B).

96. The given equation can be written as

$$\left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{(x^2dy - y^2dx)}{(x-y)^2} = 0$$

$$\Rightarrow \quad \left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{\left(\frac{dy}{y^2} - \frac{dx}{x^2}\right)}{\left(\frac{1}{y} - \frac{1}{x}\right)^2} = 0 \Rightarrow \left(\frac{dx}{x} - \frac{dy}{y}\right) + \frac{\frac{dy}{y^2} - \frac{dx}{x^2}}{\left(\frac{1}{x} - \frac{1}{y}\right)^2} = 0$$

Integrating, we get

$$\ln|x|-\ln|y|-\frac{1}{\left(\frac{1}{x}-\frac{1}{y}\right)}=c$$

or $$\ln\left|\frac{x}{y}\right|-\frac{xy}{(y-x)}=c \;=\ln\left|\frac{x}{y}\right|+\frac{xy}{x-y}=c$$

Hence, the correct answer is (A).

97. $\int\frac{dx}{x}=\int\frac{ydx}{1+y^2}\Rightarrow \log x=\frac{1}{2}\log(1+y^2)$

$$0=\frac{1}{2}\log 1+C\Rightarrow C=0\Rightarrow x^2-y^2=1$$

for C $\frac{dy}{dx}=\frac{y}{x}$ and for E $\frac{dy}{dx}=-\frac{x}{3y}$

for point of intersection

$$x^2=\frac{3}{2}\text{ and }y^2=\frac{1}{2}$$

$$m_1m_2=-1$$

The conic C is $x^2-y^2=1$ and E is $x^2+3y^2=3$.

Angle of intersection is $\pi/2$

Hence, the correct answer is (D).

98. Divide both sides by x^2

$$x^3\,dy+y.3x^2\,dx+\frac{x\,dy-y\,dx}{x^2}+2y\,dy=0$$

$$\Rightarrow\quad d(x^3y)+d\left(\frac{y}{x}\right)+d(y^2)=0$$

$$\Rightarrow\quad x^4y+xy^2+y=cx$$

Hence, the correct answer is (A).

99. Equation of tangent at any point (x, y) is $Y-y=\frac{dy}{dx}(X-x)$

$$\Rightarrow\quad x-y\frac{dx}{dy}=ky;\quad\Rightarrow\quad x\,dy-y\,dx=ky\,dy$$

$$\Rightarrow\quad \frac{-xdy+ydx}{y^2}=-\frac{k\,dy}{y};$$

$$\Rightarrow\quad d\left(\frac{x}{y}\right)=-k\ln y+C\Rightarrow\frac{x}{y}=-k\ln y+C$$

$$\Rightarrow\quad x=y\,(C-k\ln y)$$

Hence, the correct answer is (A).

100. $f''(x)-f'(x)=e^x$

put $f'(x)=v$

$$\frac{dv}{dx}+v(-1)=e^x\Rightarrow ve^{-x}=\int e^x.e^{-x}\,dx$$

$$ve^{-x}=x+C_1, f'(0)=1\Rightarrow C_1=1$$

$$f'(x)=xe^x+e^x$$

$$f(x)=xe^x+C_2\Rightarrow f(0)=0\Rightarrow C_2=0$$

$$\Rightarrow\quad f(x)=xe^x\Rightarrow f(2)=2e^2$$

$$\ln\left(\frac{(f(2))^2}{4}\right)=4.$$

Hence, the correct answer is (D).

101. Put $x+y=Y$ then equation given becomes $\frac{dy}{dx}+Y=x^3Y^3$.

$$\Rightarrow\quad \frac{1}{Y^3}\frac{dY}{dx}+\frac{x}{Y^2}=x^3$$

Putting $z=\frac{1}{Y^2}$ makes it $\frac{dz}{dx}-2xz=-2x^3$

$$\Rightarrow\quad z=2+2x^2+ce^{x^2}$$

$$\Rightarrow\quad \frac{1}{(x+y)^2}=2+2x^2+ce^{x^2}$$

putting $x=0$, $y=1$ gives $c=-1$

Hence, the correct answer is (D).

102. $\frac{dy}{dx}+\frac{\log x}{x}\,y=x^{-\frac{1}{2}\log x}$

$$\text{I.F}=e^{\int\frac{\log x}{x}dx}=e^{\frac{(\log x)^2}{2}}=\left(e^{\log x}\right)^{\frac{\log x}{2}}=x^{\frac{1}{2}\log x}$$

General Solution is $x^{\frac{1}{2}\log x}\,.\,y=\int dx$

$$yx^{\frac{1}{2}\log x}=x+c$$

Hence, the correct answer is (A).

103. Rearranging the terms

$$x^2(x\,dy+y\,dx)+((xy)^2+1)(y\,dx-x\,dy)=0$$

$$\Rightarrow\quad \frac{d(xy)}{((xy)^2+1)}-d\left(\frac{y}{x}\right)=0\,\frac{d(xy)}{((xy)^2+1)}-d\left(\frac{y}{x}\right)=0$$

$$\Rightarrow\quad \tan^{-1}(xy)-\frac{y}{x}=C$$

$$\Rightarrow\quad C=\frac{\pi}{4}-1$$

Hence, the correct answer is (D).

104. Given differential equation can be written as

$$\sec^2(xy)\,(y\,dx+x\,dy)+\sin x\,dx+\sin y\,dy=0$$

$$d\,(\tan xy)+d(-\cos x)+d(-\cos y)=0$$

$$\tan(xy)-\cos x-\cos y=C$$

Hence, the correct answer is (D).

105. $\frac{x(xdy+ydx)}{dx}+\sqrt{1-(xy)^2}=0\Rightarrow\frac{d(xy)}{\sqrt{1-(xy)^2}}=-\frac{dx}{x}$

$$\Rightarrow\quad \sin^{-1}(xy)=-\ln x+c\Rightarrow xy=-\sin(\ln x)$$

Hence, the correct answer is (A).

Multiple Correct Choice Type Questions

1. $\left(\frac{dy}{dx}\right)^2-\frac{dy}{dx}(e^x+e^{-x})+1=0$

$$\frac{dy}{dx}=\frac{(e^x+e^{-x})\pm\sqrt{(e^x+e^{-x})^2-4}}{2}$$

$$\frac{dy}{dx}=\frac{(e^x+e^{-x})\pm(e^x-e^{-x})}{2}$$

On taking positive $\frac{dy}{dx}=e^x\Rightarrow\int dy=\int e^x dx\Rightarrow y=e^x+c$

On taking negative sign

$$\frac{dy}{dx}=e^{-x}\Rightarrow\int dy=\int e^{-x}dx$$

$$\Rightarrow\quad y=-e^{-x}+c$$

Hence, (A) and (D) are correct.

2. $y=(A+Bx)\,e^{3x}$

$$\frac{dy}{dx}=(A+Bx)\,3e^{3x}+Be^{3x}=e^{3x}(3A+B+3\,Bx)$$

$$\frac{d^2y}{dx^2}=3e^{3x}\,(3A+B+3Bx)+e^{3x}(3B)$$

$$\frac{d^2y}{dx^2}=3e^{3x}\,(3A+2B+3Bx)$$

$$\frac{d^2y}{dx^2} + m\frac{dy}{dx} + ny = e^{3x}(9A + 6B + 9Bx) + e^{3x}(3Am + Bm + 3Bxm) + e^{3x}(nA + nBx)$$

$$= e^{3x}\left((9+3m+n)A + B(6+m) + Bx(9+3m+n)\right)$$

This is the solution of diff. equation

$\therefore$ $9 + 3m + n = 0; 6 + m = 0$

$9 + 3m + n = 0; m = -6$

$9 - 18 + n = 0 \Rightarrow n = 9$

$m + n = 3$ and $n^2 - m^2 = 81 - 36 = 45$

Hence, (A), (C) and (D) are correct.

3. $x = 1 + xy\frac{dy}{dx} + \frac{x^2y^2}{2!}\left(\frac{dy}{dx}\right)^2 + \ldots$

$$x = e^{xy\frac{dy}{dx}} \Rightarrow \ln x = xy\frac{dy}{dx}$$

$$\int \frac{\ln x}{x}\,dx = \int y\,dy \Rightarrow \frac{\ln^2 x}{2} + c = \frac{y^2}{2}$$

At point (1, 0) $\Rightarrow 0 + c = 0 \Rightarrow c = 0; y = \pm \ln x$

Hence, (A) and (B) are correct.

4. $\frac{d^2y}{dx^2} + y + \cot^2 x = 0$

We can not solve it

$$y = \cos x \ln\left(\tan\frac{x}{2}\right) + 2$$

$$\frac{dy}{dx} = -\sin x \ln(\tan x/2) + \cos x\,\frac{\sec^2 x/2}{\tan x/2}\,1/2$$

$$= -\sin x \ln(\tan x/2) + \frac{\cos x}{\sin x}$$

$$\frac{dy}{dx} = -\sin x \ln(\tan x/2) + \cot x$$

$$\frac{d^2y}{dx^2} = -\cos x \ln(\tan x/2) - \sin x\left(\frac{\sec^2 x/2}{\tan x/2}\cdot 1/2\right) - \text{cosec}^2 x$$

$$= -\cos x \ln\tan x/2 - \sin x\left(\frac{1}{\sin x}\right) - \text{cosec}^2 x$$

$$= -\cos x \ln\tan x/2 - 1 - 1 - \cot^2 x$$

$$\frac{d^2y}{dx^2} = -y - \cot^2 x \Rightarrow \frac{d^2y}{dx^2} + y + \cot^2 x = 0$$

$$y = 2 + c_1\cos x + \sqrt{c_2}\sin x$$

$$\frac{dy}{dx} = -c_1\sin x + \sqrt{c_2}\cos x \Rightarrow \frac{d^2y}{dx^2} = -c_1\cos x - \sqrt{c_2}\sin x$$

$\Rightarrow$ $\frac{d^2y}{dx^2} + y = 0$

$y = 2 + c_1\cos x + c_2\sin x + \cos x \ln\tan\frac{x}{2}$ is also solution of given diff equation

Hence, (B) and (C) are correct.

5. Equation of tangent $y - y_1 = m(x - x_1) \Rightarrow mx - y + y_1 - mx_1 = 0$

Equation of normal $y - y_1 = -\frac{1}{m}(x - x_1)$

$$my - my_1 + x - x_1 = 0 \Rightarrow x + my - my_1 - x_1 = 0$$

$$\left|\frac{0 - 0 + y_1 - mx_1}{\sqrt{m^2+1}}\right| = \left|\frac{0 + 0 - my_1 - x_1}{\sqrt{1+m^2}}\right|$$

$$y_1 - mx_1 = \pm(my_1 + x_1)$$

Taking positive sign $y_1 - mx_1 = my_1 + x_1$

$$y_1 - x_1 = m(y_1 + x_1) \Rightarrow \frac{dy}{dx} = \frac{y/x - 1}{y/x + 1}$$

Put $\frac{y}{x} = t \Rightarrow y = tx$

$\Rightarrow$ $\frac{dy}{dx} = t + \frac{xdt}{dx}$

$\Rightarrow$ $\frac{xdt}{dx} = \frac{t-1}{t+1} - t = \frac{t - 1 - t^2 - t}{t+1}$

$\Rightarrow$ $\frac{t-1}{t^2+1}dt = -\frac{dx}{x} \Rightarrow \frac{1}{2}\ln(t^2+1) + \tan^{-1}t = -\ln x + \ln K$

$\Rightarrow$ $\tan^{-1}t = \ln\frac{K}{x} - \ln\sqrt{t^2+1} \Rightarrow \tan^{-1}t = \ln\frac{K}{x\sqrt{\frac{y^2+x^2}{x}}}$

$\Rightarrow$ $\tan^{-1}\frac{y}{x} = \ln\frac{K}{\sqrt{x^2+y^2}} \Rightarrow K = \sqrt{x^2+y^2}\,e^{\tan^{-1}\frac{y}{x}}$

$$y_1 - mx_1 = -my_1 - x_1 \Rightarrow y_1 + x_1 = m(x_1 - y_1)$$

$\Rightarrow$ $\frac{dy}{dx} = \frac{y/x + 1}{1 - y/x} \Rightarrow t + \frac{xdt}{dx} = \frac{t+1}{1-t}$

$\Rightarrow$ $\frac{xdt}{dx} = \frac{t+1}{1-t} - t = \frac{t + 1 - t + t^2}{1-t}$

$\Rightarrow$ $\frac{1-t}{1+t^2}dt = \frac{dx}{x} \Rightarrow \tan^{-1}t - \frac{1}{2}\ln(1+t^2) = \ln x + \ln k$

$\Rightarrow$ $\tan^{-1}t = \ln K\sqrt{1+t^2} \Rightarrow \tan^{-1}\frac{y}{x} = \ln K\sqrt{x^2+y^2}$

$\Rightarrow$ $K\sqrt{x^2+y^2} = e^{\tan^{-1}\frac{y}{x}}$

Hence, (B) and (C) are correct.

6. (A) $f(x, y) = e^{y/x} + \tan^{y/x}$ (T)

(B) $\ln\frac{y}{x} + \left(\frac{y}{x}\right)^2 \sin^{-1}\frac{y}{x}\frac{dy}{dx} = 0$ is differential equation of degree 1 → True

(C) $f(x, y) = x^2 + \sin x \cos y \rightarrow$ not homogenous function

(D) $(x^2 + y^2)\,dx - (xy^2 - y^3)\,dy = 0$

$\frac{dy}{dx} = \frac{x^2+y^2}{xy^2 - y^3}$ It is not homogenous

Hence, (A), (B) and (C) are correct.

7. $\frac{dy}{dx} - y\cot x = -\frac{\sin x}{x^2}$

Integrating Factor $= e^{-\int\cot x\,dx} = e^{-\ln\sin x} = \text{cosec}\,x$

$\Rightarrow$ $y\,\text{cosec}\,x = \int -\frac{\sin x\,\text{cosec}\,x}{x^2}\,dx$

$\Rightarrow$ $\frac{y}{\sin x} = \frac{1}{x} + c$

As $x \rightarrow \infty; y \rightarrow 0$

We have $0 = 0 + c; c = 0$

$y = \frac{\sin x}{x}$ even function

(A) $\lim_{x\to 0}\frac{\sin x}{x} = 1$

$\Rightarrow$ $f(0^+) > f(x) > f\left(\frac{\pi}{2}\right)$ $\Rightarrow$ $1 > \frac{\sin x}{x} > \frac{2}{\pi}$

$\Rightarrow$ $\int_0^{\pi/2} 1\,dx > \int_0^{\pi/2}\frac{\sin x}{x}\,dx > \int_0^{\pi/2}\frac{\pi}{2}\,dx$

$\Rightarrow$ $\frac{\pi}{2} > I > 1$

Hence, (A), (B) and (C) are correct.

8. $x^2\dfrac{dy}{dx} = y^2e^{1/x}\ (x \neq 0)$

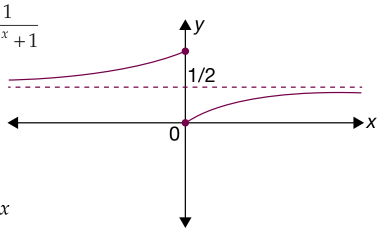
$\Rightarrow \quad \displaystyle\int \frac{dy}{y^2} = \int \frac{1}{x^2}e^{1/x}\,dx \,\Big|\, \frac{1}{x} = t \Rightarrow -\frac{1}{x^2}\,dx = dt$

$-\dfrac{1}{y} = \displaystyle\int e^t(-dt) = \frac{1}{y} = e^t + c$

$\Rightarrow \quad \dfrac{1}{y} = e^{1/x} + c$

$\displaystyle\lim_{x\to 0^-} f(x) = 1 \Rightarrow \frac{1}{1} = e^{-\infty} + c \Rightarrow 1 = 0 + c$

$\dfrac{1}{y} = e^{1/x} + 1;\ y = \dfrac{1}{e^{1/x}+1}$



Range $\in (0, 1) - \left\{\dfrac{1}{2}\right\}$

Bounded $\displaystyle\lim_{x\to 0^+} f(x) = 0$

$\displaystyle\int_0^e f(x)\,dx > \int_0^1 f(x)\,dx$

True as RHS covers more area

Hence, (A), (B) and (D) are correct.

9. $\dfrac{dy}{dx} + y = f(x)$ $\qquad y(0) = 1$

Integrating Factor $= e^{\int dx} = e^x$ $\qquad f(x) = \begin{cases} e^{-x}; & 0 \le x \le 2 \\ e^{-2}; & x > 2 \end{cases}$

$\displaystyle ye^x\Big|_0^1 = \int_0^1 e^x f(x)\,dx \Rightarrow y' + y = f(x)$

$\displaystyle y(1)e - y(0) = \int_0^1 e^x \cdot e^{-x}\,dx \Rightarrow y(1)\,e - 1 = 1 - 0$

$y(1) = 2/e$ option (A) is correct

Now $y'(1) + y(1) = f(1)$

$y'(1) + \dfrac{2}{e} = \dfrac{1}{e} \Rightarrow y'(1) = -\dfrac{1}{e}$ option (B) is correct

$\displaystyle ye^x\Big|_0^3 = \int_0^3 e^x f(x) \Rightarrow y'(3) + y(3) = f(3)$

$\displaystyle y(3)e^3 - y(0) = \int_0^2 e^{+x}e^{-x}\,dx + \int_2^3 e^xe^{-2}\,dx \Rightarrow y'(3) + (2+e)e^{-3} = e^{-2}$

$y(3)e^3 - 1 = (2-0) + e^{x-2}\Big|_2^3 \Rightarrow y'(3) + 2e^{-3} + e^{-2} = e^{-2}$

$y(3)e^3 = 3 + (e-1) \Rightarrow y'(3) = 2e^{-3}$ option (D) is correct

$y(3)e^3 = 2 + e \Rightarrow y(3) = (2+e)e^{-3}$

Hence, (A), (B) and (D) are correct.

10. $l_{\text{subnormal}} = 4\,l_{\text{subtangent}}$

$ym = 4\,\dfrac{y}{m} \Rightarrow m^2 = 4 \Rightarrow m = \pm 2$

$\dfrac{dy}{dx} = \pm 2\,dx \Rightarrow y = \pm 2x + c$

(A) $m = \pm\dfrac{1}{2} \Rightarrow y - 2 = \pm\dfrac{1}{2}(x-4)$

$2y - 4 = \pm(x-4) \Rightarrow 2y = \pm(x-4) + 4$

Taking ⊕ sign $2y = x - 4 + 4 \Rightarrow x = 2y$

Taking negative sign $2y = -x + 4 + 4 \Rightarrow x + 2y = 8$

(B) Orthogonal $m = \pm\dfrac{1}{2}$

(C) $m = \pm 2 \Rightarrow m^2 = 4$

order = 1 and degree = 2

(D) $\left(\dfrac{dy}{dx}\right)^2 = 4$

Hence, (A) and (C) are correct.

11. $\displaystyle\int (by + K)dy = \int (ax + h)dx \Rightarrow \frac{by^2}{2} + Ky = \frac{ax^2}{2} + hx + c$

Clearly for $a = -2$, $b = 0$ and for $a = 0$, $b = 2$. It represents a parabola

$(\because\ y = ax^2 + bx + c\ \text{ and }\ x = ay^2 + by + c)$

Hence, (A) and (C) are correct.

12. Dividing throughout by $\cos^2 x \sin^2 y$, the given differential equation becomes

$(3\tan x + 4\cot y - 7)\sec^2 x\,dx - (4\tan x + 7\cot y - 5)\,\mathrm{cosec}^2 y\,dy = 0$

$\Rightarrow \quad 3\tan x\sec^2 x\,dx - 7\sec^2 x\,dx - 7\cot y\,\mathrm{cosec}^2 y\,dy + 5\,\mathrm{cosec}^2 y dy + 4\cot y\sec^2 x\,dx - 4\tan x\,\mathrm{cosec}^2 y\,dy = 0$

$\Rightarrow \quad 3\tan x\,d(\tan x) - 7\,d(\tan x) + 7\cot y\,d(\cot y) - 5\,d(\cot y) + 4\,d(\tan x\cot y) = 0$

On integrating, we obtain

$$\frac{3}{2}\tan^2 x - 7\tan x + \frac{7}{2}\cot^2 y - 5\cot y + 4\tan x\cot y = C$$

Hence, (B) and (C) are correct.

13. $\dfrac{dy}{dt} = \dfrac{1}{3}(t - t^2)$

$\Rightarrow \quad y = \dfrac{1}{3}\left(\dfrac{t^2}{2} - \dfrac{t^3}{3}\right)$

$\dfrac{dx}{dt} = \dfrac{1}{3}(t + t^2)$

$\Rightarrow \quad x = \dfrac{1}{3}\left(\dfrac{t^2}{2} + \dfrac{t^3}{3}\right)$

From the given equations we get $x = \dfrac{t^2}{3} + \dfrac{t^3}{9}$ and $y = \dfrac{t^2}{6} - \dfrac{t^3}{9}$.

Eliminating t gives $8(x+y)^3 = 9(x-2y)^2$

Hence, (B) and (D) are correct.

14. $\dfrac{dy}{dx} = \dfrac{(x+1)^2 + y - 3}{x+1} = x + 1 + \dfrac{y-3}{x+3}$

Putting $x + 1 = X$ and $y - 3 = Y$

$\dfrac{dY}{dX} = X + \dfrac{4}{X}$

$\Rightarrow \quad \dfrac{dY}{dX} - \dfrac{Y}{X} = X$

Solution is $\dfrac{Y}{X} = X + c \Rightarrow y - 3 = (x+1)^2 + (2+1)$

It passes Through $(2, 0) \Rightarrow c = -4$

Equation of curve is $y = x^2 - 2x$

Area bounded $= \displaystyle\int_0^2 \left(2x - x^2\right)dx = \frac{4}{3}$ s.u.

Hence, (A) and (D) are correct.

15. $p^2 + 2py\cot x - y^2 = 0$

$p = \dfrac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2} \Rightarrow \dfrac{dy}{dx} = (-\cot x \pm \mathrm{cosec}x)y$

$\Rightarrow \quad \dfrac{dy}{dx} + y\,\dfrac{(\cos x \pm 1)}{\sin x} = 0$

On integration

$\log y + 2\log\sin\dfrac{x}{2} = \log c \Rightarrow y(1 - \cos x) = 2c$ and $y(1 + \cos x) = c$

$\Rightarrow \quad y\left(2\sin^2 {}^{x}\!/\!_{2}\right) = C \Rightarrow \sin^2 {}^{x}\!/\!_{2} = \dfrac{C}{2y} \Rightarrow x = 2\sin^{-1}\sqrt{\dfrac{c}{2y}}$

Hence, (A), (B) and (C) are correct.

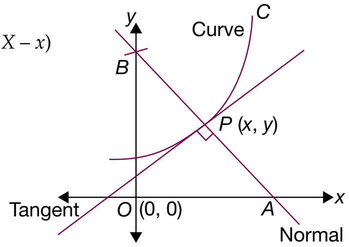
16. The equation of normal of $P(x, y)$ is $(Y - y)$

$$= \frac{-1}{\frac{dy}{dx}}(X - x)$$

$P(x, y)$ is $(Y - y) = \dfrac{-1}{\frac{dy}{dx}}(X - x)$

$\therefore \quad A\left(x + y\dfrac{dy}{dx}, 0\right)$ and

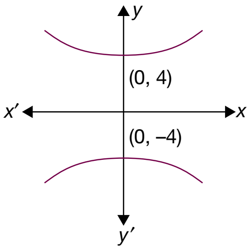
$$B\left(0, y + \frac{x}{\frac{dy}{dx}}\right)$$



Now $\dfrac{1\left(x + y\frac{dy}{dx}\right) + 2(0)}{1+2} = x$

$\Rightarrow \quad x + y\dfrac{dy}{dx} = 3x; y\dfrac{dy}{dx} = 2x \qquad \ldots(1)$

$\Rightarrow \quad \displaystyle\int y\, dy = \int 2x\, dx \Rightarrow \frac{y^2}{2} = x^2 + C$



Also (0, 4) satisfy it, so $C = 8$

$\therefore \quad y^2 = 2x^2 + 16$ (equation of curve)

Which represent a hyperbola

Also $\left.\dfrac{dy}{dx}\right]_{(4,4\sqrt{3})} = \dfrac{2(4)}{4\sqrt{3}} = \dfrac{2}{\sqrt{3}}$

Hence, (A) and (D) are correct.

17. $\dfrac{1}{y(\ln y)^2}\dfrac{dy}{dx} + \dfrac{1}{x\ln y} = \dfrac{1}{x^2}$

Let $\dfrac{1}{\ln y} = t \Rightarrow -\dfrac{1}{y(\ln y)^2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$-\frac{dt}{dx} + \frac{t}{x} = \frac{1}{x^2}$$

$\Rightarrow \quad \dfrac{dt}{dx} - \dfrac{t}{x} = -\dfrac{1}{x^2}$

I.F. $= e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \dfrac{1}{x}$

$$\frac{t}{x} = \int -\frac{1}{x^3}\, dx + C \Rightarrow \frac{t}{x} = \frac{1}{2x^2} + C$$

$\Rightarrow \quad \dfrac{1}{x\ln y} = \dfrac{1 + 2cx^2}{2x^2} \Rightarrow 2x = \ln y\,(1 + 2cx^2)$

Hence, (A) and (C) are correct.

18. $\dfrac{xdx + ydy}{ydx - xdy} = \dfrac{x\sin^2(x^2 + y^2)}{y^3}$

$$\frac{xdx + ydy}{\sin^2(x^2 + y^2)} = \frac{x}{y}\,\frac{ydx - xdy}{y^2}$$

$\Rightarrow \quad \dfrac{1}{2}\operatorname{cosec}^2(x^2 + y^2)\, d\,(x^2 + y^2) = \dfrac{x}{y}\, d\left(\dfrac{x}{y}\right)$

$\Rightarrow \quad -\dfrac{1}{2}\cot(x^2 + y^2) = \dfrac{1}{2}\left(\dfrac{x}{y}\right)^2 + C \Rightarrow -\cot(x^2 + y^2) = \dfrac{x^2}{y^2} + K$

$\Rightarrow \quad \dfrac{y^2}{x^2 + ky^2} = -\tan(x^2 + y^2)$

Hence, (A) and (B) are correct.

19. Slope of normal $\tan\theta = -\dfrac{dx}{dy}$

$\therefore$ The given equation becomes at a general point (x, y) is

$$y\left(x^2\left(-\frac{dx}{dy}\right) + \frac{dy}{dx}\right) = x\left(y^2 + 1\right); -yx^2 + y\left(\frac{dy}{dx}\right)^2 = \frac{dy}{dx}.x\left(y^2 + 1\right)$$

$$y\left(\frac{dy}{dx}\right)^2 - x\left(y^2 + 1\right)\frac{dy}{dx} - yx^2 = 0;\ yy'^2 - xy^2y' - xy' - yx^2 = 0$$

$$yy'\left(y' - xy\right) - x\left(y' - xy\right) = 0$$

$\therefore \quad \dfrac{dy}{dx} = xy$ or $\dfrac{dy}{dx} = \dfrac{x}{y}$

$$\log y = \frac{x^2}{2} + c\,(or)\, x^2 - y^2 = c;\ y = ke^{\frac{x^2}{2}}\,(or)\,\log y^2 = x^2 - \log k$$

$$\log ky^2 = x^2$$

Hence, (B), (C) and (D) are correct.

20. $\dfrac{dx}{dy} = \dfrac{3y}{2x} \Rightarrow \displaystyle\int 2xdx = \int 3ydy \Rightarrow x^2 = \frac{3y^2}{2} + c$

or $\quad \dfrac{x^2}{3} - \dfrac{y^2}{2} = \dfrac{c}{3}$

If C is positive then $e = \sqrt{1 + \dfrac{2}{3}} = \sqrt{\dfrac{5}{3}}$

If C is negative then

$$e = \sqrt{1 + \frac{3}{2}} = \sqrt{\frac{5}{2}}$$

Hence, (B) and (D) are correct.

21. $\dfrac{dy}{dx} = \dfrac{\left(x^2 - y^2\right) \pm \sqrt{\left(x^2 - y^2\right)^2 + 4x^2y^2}}{2xy} = \dfrac{\left(x^2 - y^2\right) \pm \left(x^2 + y^2\right)}{2xy}$

$$= \frac{x}{y}\ \text{(or)}\ \frac{-y}{x}$$

$$x\ dx = y\ dy,\ x\,dy + y\,dx = 0$$

$$x^2 - y^2 = c\,d(x) = 0 \Rightarrow xy - c = 0$$

Hence solutions $= (xy - c) = 0$ *or* $y^2 - x^2 - c = 0$

Hence, (A), (B) and (C) are correct.

22. Replacing $\dfrac{dy}{dx}$ by $-\,dx/dy$, we get

$$\left(\frac{dy}{dx}\right)^2 = x/a \Rightarrow dy = \pm(x^{1/2}/a^{1/2})dx$$

Integrating we get $y + c = \pm\dfrac{2}{3a^{1/2}}x^{3/2}$

$\Rightarrow \quad 9a\,(y + c)^2 = 4x^3$

Hence, (A), (B) and (C) are correct.

23. I.F. $= e^{\int \cos x\ dx} = e^{\sin x}$ or $y\,.\,e^{\sin x} = \displaystyle\int e^{+\sin x}\cos x$

$ye^{\sin x} = e^{\sin x} + C$

Passing through (0, 1) $\quad \Rightarrow C = 0 \quad \Rightarrow y = 1$

Hence, (A), (B) and (D) are correct.

24. $\dfrac{dy}{dx} = \dfrac{y - x \pm \sqrt{x^2 + y^2 - 2xy + 4xy}}{2y}$

$$\frac{dy}{dx} = \frac{y - x \pm (x + y)}{2y}$$

$$\frac{dy}{dx} = 1 \text{ and } \frac{dy}{dx} = \frac{-x}{y}$$

$$y = x + 1 \text{ and } y^2 + x^2 = 25$$

Hence, (A) and (B) are correct.

25. $y = \ln\left\{1 + y' + \frac{(y')^2}{2!} + \frac{(y')^3}{3!} + \ldots\infty\right\} = \ln e^{y'} = y'$

$\Rightarrow \quad y' = y$

Hence, (A) and (B) are correct.

26. $\frac{dy}{dx} = 2cx$

$\therefore y = cx^2 \Rightarrow y = \left(\frac{1}{2x}\frac{dy}{dx}\right)x^2 \Rightarrow y = \frac{x}{2}\frac{dy}{dx}$ …(1)

For orthogonal trajectory Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

Then eq.(1) becomes $y = \frac{x}{2}\left(\frac{-dx}{dy}\right) \Rightarrow 2y\,dy + x\,dx = 0$

By solving $x^2 + 2y^2 = c$

Hence, (A), (B) and (C) are correct.

27. We have, $2xydy = (x^2 + 1)\,dx + y^2dx$

$\Rightarrow \quad \frac{2xydy - y^2dx}{x^2} = \left(1 + \frac{1}{x^2}\right)dx \quad \Rightarrow \quad d\left(\frac{y^2}{x}\right) = d\left(x - \frac{1}{x}\right)$

$\therefore \quad$ We get, $\frac{y^2}{x} = x - \frac{1}{x} + c$

As, $y(1) = 0 \Rightarrow c = 0$

$\therefore \quad y^2 = x^2 - 1$

Now, $y\,(x_0) = \sqrt{3} \Rightarrow 3 = x_0^2 - 1 \Rightarrow x_0^2 - 4$

$\therefore \quad x_0 = \pm 2$

Hence, (A) and (B) are correct.

28. Given, $\frac{dy}{dx} - 3y\cot x = \sin 2x,\ y\left(\frac{\pi}{2}\right) = 2$

$\therefore \quad$ I.F. $= e^{-\int 3\cot x dx} = e^{-3\ln(\sin x)} = \frac{1}{\sin^3 x}$

$\therefore \quad$ General solution is

$$y\left(\frac{1}{\sin^3 x}\right) = \int \frac{2\sin x\cos x}{\sin^3 x}\,dx + C = 2\int \operatorname{cosec} x \cot x\,dx + C$$

$\Rightarrow \quad \frac{y}{\sin^3 x} = -2\operatorname{cosec} x + C$

$\Rightarrow \quad \frac{2}{(1)^3} = -2 + c \quad \Rightarrow \quad c = 4 \qquad \left(\text{As, } y\left(\frac{\pi}{2}\right) = 2\right)$

$\therefore \quad y = (4\sin^3 x - 2\sin^2 x)$

(A) $y\left(\frac{\pi}{6}\right) = 4\left(\frac{1}{2}\right)^3 - 2\left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{1}{2} = 0.$

(B) $y'(x) = (12\sin^2 x\cos x - 4\sin x\cos x)$

(C) As, $y'(x) = (4\sin x\cos x)(3\sin x - 1)$
$= 2\sin 2x(3\sin x - 1)$

So, $y'\left(\frac{\pi}{3}\right) = \left(\frac{12\times 3}{4}\times\frac{1}{2} - 4\times\frac{\sqrt{3}}{2}\times\frac{1}{2}\right)$

$= \frac{9}{2} - \frac{2\sqrt{3}}{2} = \left(\frac{9 - 2\sqrt{3}}{2}\right).$

$\therefore \quad y(x)$ increases in $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$

(D) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\underbrace{4\sin^3 x}_{\text{odd function}} - \underbrace{2\sin^3 x}_{\text{even function}}\right)dx = 0 - 4\int_0^{\frac{\pi}{2}} \sin^2 x dx$

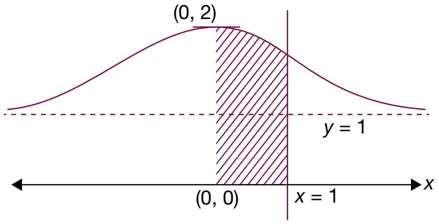
$= -4\left(\frac{\pi}{4}\right) = -\pi.$

Hence, (A) and (C) are correct.

29. We have, $\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \left(\frac{2x}{1+x^2}\right)$

So, I.F. $= e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = (1 + x^2)$

Now, general solution is



$$y\,(1 + x^2) = \int\left(\frac{2x}{1+x^2}\right)\cdot(1 + x^2)\,dx + C$$

$\therefore \quad y\,(1 + x^2) = x^2 + C$ …(1)

As $y(0) = 2 \quad \Rightarrow \quad 2 = C$

$\therefore \quad f(x) = \left(\frac{x^2+2}{x^2+1}\right) = 1 + \frac{1}{x^2+1}$ (even function)

Range of $f(x) = (1, 2]$

Also, Area $= \int_0^1 \left(1 + \frac{1}{1+x^2}\right)dx = (x + \tan^{-1} x)_0^1$

$= 1 + \frac{\pi}{4} = \frac{\pi + 4}{4}.$

Now, verify alternatives.

Hence, (B), (C) and (D) are correct.

30. We have $\lim\limits_{a\to x} \frac{a\,f(x) - x\,f(a)}{a - x}\left(\frac{0}{0}\right) = 2$

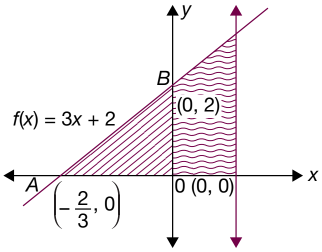
$\Rightarrow \quad \lim\limits_{a\to x} \frac{f(x) - x\,f'(a)}{1 - 0} = 2$

$\Rightarrow \quad f(x) - x\,f'(x) = 2$

$\Rightarrow \quad \frac{f'(x)}{f(x) - 2} = \frac{1}{x}$

$\therefore \quad$ On integrating with respect to x, we get

$\int \frac{f'(x)}{f(x) - 2}\,dx = \int \frac{1}{x}\,dx$



$\Rightarrow \quad \ln(f(x) - 2) = \ln x + \ln c$

$\Rightarrow \quad f(x) = cx + 2$

As $f(1) = 5$, so

$5 = c + 2$

$\Rightarrow \quad c = 3$

Hence, $f(x) = 3x + 2$

Clearly ar. $(\Delta OAB) = \frac{1}{2}\left(\frac{2}{3}\right)(2) = \frac{2}{3}$ (square units)

Also $\int_0^2 (3x + 2)\,dx = \left(\frac{3x^2}{2} + 2x\right)_0^2 = 6 + 4 = 10$

Hence, (A), (B), (C) and (D) are correct.

31. We have

$\frac{dy}{dx} + (2\tan x)y = \sin x$ (Linear differential equation)

$\therefore \quad \text{I.F.} = e^{2\int \tan x dx} = e^{2\ln(\sec x)} = \sec^2 x$

So, general solution is

$$y \cdot \sec^2 x = \int \frac{\sin x}{\cos^2 x} dx + C \quad \Rightarrow \quad y \cdot \sec^2 x = \sec x + C$$

Now, $y\left(\frac{\pi}{3}\right) = 0 \quad \Rightarrow \quad C = -2.$

$\Rightarrow \quad y(x) = \cos x - 2\cos^2 x$

(A) $y'(x) = -\sin x + 4\cos x \cdot \sin x$

$$\therefore \quad y'\left(\frac{\pi}{6}\right) = \frac{-1}{2} + 4\frac{\sqrt{3}}{2} \times \frac{1}{2} = \frac{-1}{2} + \sqrt{3} = \frac{2\sqrt{3}-1}{2}$$

(B) $y(\pi) = -3.$

(C) Consider the equation,

$y(x) = -1 \quad \Rightarrow \quad \cos x - 2\cos^2 x = -1$

$\Rightarrow \quad 2\cos^2 x - \cos x - 1 = 0$

$\Rightarrow \quad (2\cos x + 1)(\cos x - 1) = 0$

Either $\cos x = \frac{-1}{2}$ or $\cos x = 1$

But $x \in (0, 2\pi)$, so $x = \frac{2\pi}{3}, \frac{4\pi}{3}.$

(D) $\int_0^{\frac{\pi}{2}} y(x)dx = \int_0^{\frac{\pi}{2}} (\cos x - 2\cos^2 x\, dx) = 1 - 2\left(\frac{\pi}{4}\right) = 1 - \frac{\pi}{2}.$

Hence, (A) and (D) are correct.

32. Given, $f'(t) = e^t(\cos^2 t - \sin 2t)$

$\therefore$ Integrate both sides with respect to t, we get $f(t) = e^t \cos^2 t + C$

But, $f(0) = 1 \Rightarrow C = 0$

So, $f(t) = e^t \cos^2 t.$

Clearly, f is bounded in $(-\infty, 0)$.

Note that f is neither odd nor even function.

Also, $f(t) = e^t \Rightarrow e^t \cos^2 t = e^t \Rightarrow \cos^2 t = 1$

$\therefore \quad t = 0, \pi, 2\pi \in [0, 2\pi].$

Also $\lim_{t\to 0} \left(f(t)\right)^{\frac{1}{t}} = (1^\infty \text{ form}) = e^L$, where L

$$= \lim_{t\to 0}\left(\frac{e^t \cos^2 t - 1}{t}\right)$$

$$= \lim_{t\to 0}\frac{e^t(1-\sin^2 t) - 1}{t} = 1$$

Hence, (A), (B), (C) and (D) are correct.

33. $f(x) \cdot f''(x) - f(x) \cdot f'(x) = (f'(x))^2$

$f(x) f''(x) - f'(x) = (f'(x))^2$

$$\therefore \quad \frac{f''(x) - f'(x)}{f'(x)} = \frac{f'(x)}{f(x)}$$

$$\frac{f''(x)}{f'(x)} - 1 = \frac{f'(x)}{f(x)}$$

On integration w.r.t. x, we get

$\ln(f'(x)) - x = \ln(f(x)) + C$

At $x = 0$; $f(0) = 1$, $f'(0) = 1$

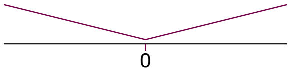
$\therefore \quad C = 0$

$$\therefore \quad \frac{f'(x)}{f(x)} = e^x$$

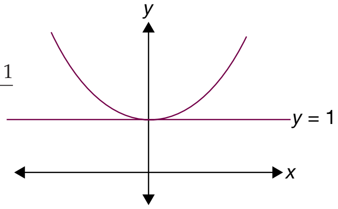
On integrating w.r.t. x, we get

$\therefore \quad \ln(f(x)) = e^x + C$

$x = 0$; $f(0) = 1 \quad \therefore C = -1$



$f(x) = e^{e^x - 1}$



Hence, $\lim_{x\to -\infty} f(x) = e^{-1} = \frac{1}{e}$

Also $\lim_{x\to 0}\frac{f(x)-1}{x} = \lim_{x\to 0}\frac{e^{e^x-1}-1}{x}$

$$= \lim_{x\to 0}\frac{e^x - 1}{x} = 1$$

Now, $f(x) = e^x$

$\Rightarrow \quad e^x - 1 = x \Rightarrow e^x - x = 1$

$f(x) = e^x - x$

$f'(x) = e^x - 1$

Hence, $f(x) = e^x$ has exactly one solution.

Hence, (B), (C) and (D) are correct.

34. $(y - y_0) = \frac{-1}{m}(x - x_0)$

$\therefore \quad my - my_0 = -x + x_0 \Rightarrow x + my - my_0 = -x + x_0$

$$OQ = \frac{|-my_0 - x_0|}{\sqrt{(1+m^2)}} = y_0$$

$\Rightarrow \quad m^2y_0^2 + x_0^2 + 2my_0x_0 = y_0^2 + y_0^2m^2$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

$2xy\frac{dy}{dx} - y^2 = -x^2$; put $t = y^2$

$\frac{dt}{dx} - \frac{t}{x} = -x$ so Integrating Factor $= e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$

$$t.\frac{1}{x} = -\int x \cdot \frac{1}{x}dx$$

$\Rightarrow \quad \frac{y^2}{x} = -x + c$ it passes through $(1, 1)$ so $c = 2$

$y^2 = -x^2 + 2x$ so $(x-1)^2 + y^2 = 1.$

Hence, (A), (B) and (C) are correct.

35. Let $y^3 = t \Rightarrow 3y^2\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} + (\sin x)t = e^{\cos x} \quad \Rightarrow \quad te^{-\cos x} = \int dx$$

$\Rightarrow \quad te^{-\cos x} = x + C \quad \Rightarrow \quad y^3 = xe^{-\cos x}$

$\Rightarrow \quad y = x^{\frac{1}{3}}e^{\frac{\cos x}{3}}$

$\Rightarrow \quad f(x) + f(-x) = 0$ and range is R.

Hence, (B) and (D) are correct.

36. Differential equation is of variable separable form on solving

$y = x\cos x - \sin x = \cos x\,[x - \tan x]$

$= (+\text{ ve})\,(+\text{ve}) = +\text{ve}$

$y'' = -[x\cos x + \sin x] < 0$

$\therefore$ No inflection point

Hence, (A), (C) and (D) are correct.

37. $x\,dy - y^2\,dy = y\,dx \Rightarrow -y^2dy = y\,dx - x\,dy$

$$\Rightarrow \quad -dy = d\left(\frac{x}{y}\right)$$

$$\therefore \quad -y = \frac{x}{y} + c$$

$$-2 = \frac{4}{2} + c \quad \Rightarrow \quad c = -4$$

$\therefore \quad -y^2 = x - 4y$

$x = 4y - y^2 = 4 - (y-2)^2$

$(y-2)^2 = -(x-4)$

$\therefore$ Focus is $\equiv \left(\frac{15}{4}.2\right)$

Equation of directrix $x = \frac{17}{4} \Rightarrow 4x - 17 = 0$

$e = 1.$

Hence, (A), (B) and (C) are correct.

38. $2x^3y\,dy + (1-y^2)(x^2y^2 + y^2 - 1)\,dx = 0$

$$\Rightarrow \frac{2y}{(1-y^2)^2}\frac{dy}{dx} + \frac{y^2}{1-y^2}\frac{1}{x} = \frac{1}{x^3} \qquad \ldots(1)$$

Put $\frac{y^2}{1-y^2} = u \Rightarrow \frac{1}{1-y^2} - 1 = u \Rightarrow \frac{2y}{(1-y^2)^2}\frac{dy}{dx} = \frac{du}{dx}$

From (1), we have $\frac{du}{dx} + \frac{u}{x} = \frac{1}{x^3}$

Integrating factor $= e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$$u.x = \int \frac{1}{x^2}dx = c - \frac{1}{x}$$

$$\Rightarrow x^2y^2 = (Cx-1)(1-y^2)$$

We have $m = 2, y = 2, k = -1$

Hence, (A), (C) and (D) are correct.

Linked Comprehension Type Questions

1. $P \rightarrow$ rumbour is spread among p persons

According to given information $\frac{dP}{dt} = \mathrm{KP}(5000 - P)$

$$\Rightarrow \int \frac{dP}{P(5000-P)} = \int dt \Rightarrow \frac{1}{5000}\int \frac{1}{P} + \frac{1}{5000-P}\,dP = Kt + c$$

$$\Rightarrow \frac{1}{5000}(\ln P - \ln(5000-P)) = Kt + c \Rightarrow \ln\frac{P}{5000-P} = 5000\,(Kt + c)$$

At $t = 0$; $P = 100 \Rightarrow \ln\frac{100}{4900} = 5000\,(c)$

At $t = 2$ day; $P = 500 \Rightarrow \ln\frac{500}{4900} = 5000\,(2K + c)$

$\ln\left(\frac{1}{9}\right) = 10000\,K + \ln\frac{1}{49}$; $10000\,K = \ln\left(\frac{49}{9}\right) = \frac{229}{129}$

$K = \frac{229}{129 \times 10^4}$

According to question, half population $P = 2500$

$\ln\frac{2500}{2500} = 0 = 5000\,(Kt + c)$

$$\Rightarrow t = \frac{-c}{K} = -\frac{1/5000\left(\ln\frac{1}{49}\right)}{\frac{229}{129\times 10^4}} = 129/229 \cdot 2\ln 49 = 4 \cdot 58 \text{ days}$$

Hence, the correct answer is (C).

2. $\ln\frac{P}{5000-P} = 5000\,(Kt + c)$

$$\Rightarrow \frac{P}{5000-P} = e^{5000\left(Kt - \frac{\ell n 49}{5000}\right)} \Rightarrow \frac{P}{5000-P} = e^{5000Kt - \ln 49} = \frac{1}{49}e^{5000kt}$$

$P_{49} = (5000 - P)\,e^{5000Kt}$

$P\,(49 + e^{5000Kt}) = 5000\,e^{5000Kt}$

$P = \frac{5000}{1 + 49e^{-5000Kt}}$

Hence, the correct answer is (B).

3. $f(x+y) = f(x) + f(y) - 8xy \xrightarrow{\quad} f(0) = 0 \quad \forall\, x, y \in R$

$$\Rightarrow \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{f(h) - 8xh}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{f(h) - f(0)}{h} - 8x$$

$$\Rightarrow \frac{dy}{dx} = f'(x) = f'(0) - 8x \Rightarrow \int dy = \int (f'(0) - 8x)\,dx$$

$$y = xf'(0) - 4x^2 + c$$

When $x = 0, y = 0$, we get $c = 0$

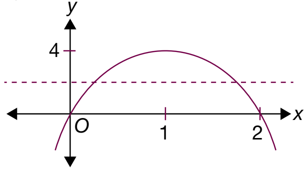
$f(x) = xf'(0) - 4x^2$

$f(x) = 8x - 4x^2$

$g\,(x+y) = g(x) + g(y) + 3xy\,(x+y) \xrightarrow{\quad} g(0) = 0 \quad \forall\, x, y \in R$

$$\lim_{h\to 0}\frac{g(x+h) - g(x)}{h} = \lim_{h\to 0}\frac{g(h) + 3xh(x+h)}{h}$$

$$g'(x) = \lim_{h\to 0}\frac{g(0+h) - g(0)}{h} + 3x(x+0)$$



$g'(x) = g'(0) + 3x^2 = 3x^2 - 4$

$g(x) = x^3 - 4x + c$

$g(x) = x^3 - 4x$

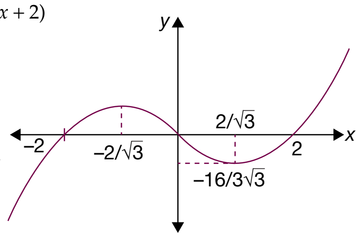
$g(0) = 0$

$f(x) = 8x - 4x^2 = 4x\,(2 - x) = K$

$K \le 4 \Rightarrow$ real roots

Hence, the correct answer is (B).

4. $g(x) = x\,(x^2 - 4) = x\,(x-2)\,(x+2)$



$g'(x) = 3x^2 - 4 = 0$

$x = +\frac{2}{\sqrt{3}}, x = -\frac{2}{\sqrt{3}}$

$g\left(\frac{2}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}} - \frac{4\cdot 2}{\sqrt{3}}$

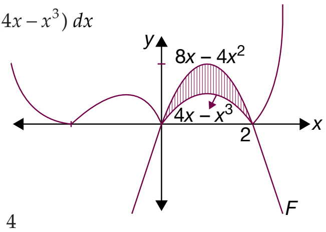
$= \frac{8}{\sqrt{3}}\left(\frac{-2}{3}\right) = -\frac{16}{3\sqrt{3}}$

$K \in [-3, 3] \Rightarrow 3$ real roots

Hence, the correct answer is (B).

5. $y = f(x) \quad y = |g(x)|$

Required area $= \int_0^2 (8x - 4x^2) - (4x - x^3)\,dx$



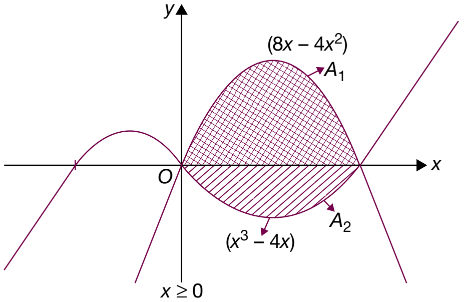
$= \int_0^2 (4x - 4x^2 + x^3)\,dx$

$= 2x^2 - 4/3x^3 + x^4/4\Big|_0^2$

$= 8 - \frac{32}{3} + \frac{16}{4} = 12 - \frac{32}{3} = \frac{4}{3}$

Hence, the correct answer is (A).

6.



$$\frac{A_1}{A_2} = \frac{\int_0^2 8x - 4x^2 dx}{-\int_0^2 x^3 - 4x dx} = \frac{4x^2 - 4/3x^3\Big|_0^2}{-x^4/4 + 2x^2\Big|_0^2}$$

$$= \frac{16 - 32/3}{-4+8} = \frac{16/3}{4} = \frac{4}{3}$$

Hence, the correct answer is (A).

7. $(1 + x^2)\dfrac{dy}{dx} + 2xy = 4x^2$

$$\int d(y(1+x^2)) = \int 4x^2\, dx$$

$$y\,(1 + x^2) = \frac{4}{3}x^3 + c$$

(0, 0) lies on it; we get $c = 0$

$f(x) = y = \dfrac{4x^3}{3(1+x^2)} \rightarrow$ rational function Bijective domain/range R

$$\frac{dy}{dx} = \frac{4}{3}\left[\frac{(1+x^2)3x^2 - x^3(2x)}{(1+x^2)^2}\right] = \frac{4}{3}x^2\left[\frac{3+3x^2-2x^2}{(1+x^2)^2}\right] = \frac{4}{3}\frac{x^2(3+x^2)}{(1+x^2)^2} \geq 0$$

∴ Strictly increasing

Hence, the correct answer is (C).

8. At $x = \dfrac{2}{3} \Rightarrow y = \dfrac{4x^3}{3(1+x^2)} = \dfrac{4\left(\frac{8}{27}\right)}{3\left(1+\frac{4}{9}\right)} = \dfrac{32}{117}$

Area surrounded by $y = f^{-1}(x)$, x-axis and $x = \dfrac{2}{3}$

$$= \int_0^{2/3} f^{-1}(x)dx \text{ put } x = f(t);\ dx = f'(t)dt$$

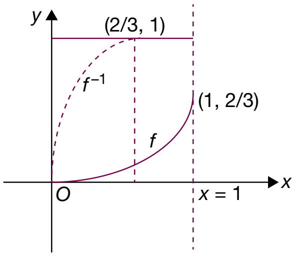
$$= \int_0^1 tf'(t)dt = tf(t)\Big|_0^1 - \int_0^1 f(t)dt$$

$$= f(1) - \int_0^1 \frac{4t^3}{3(1+t^2)}dt$$

$$= \frac{2}{3} - \frac{4}{3}\left(\int_0^1 tdt - \frac{1}{2}\int_0^1 \frac{2t}{1+t^2}dt\right)$$

$$= \frac{2}{3} - \frac{4}{3}\left(t^2/2\Big|_0^1 - 1/2\ln(1+t^2)\Big|_0^1\right)$$

$$= \frac{2}{3} - \frac{4}{3}\left(\frac{1}{2} - \frac{1}{2}\ln 2\right) = \frac{2}{3}\ln 2 \text{ sq units}$$



Hence, the correct answer is (C).

9. $\dfrac{ydx + dy}{e^x y^2} = -\,dy \Rightarrow \dfrac{e^{-x}}{y}dx + e^{-x}\dfrac{dy}{y^2} = -\,dy$

$$\Rightarrow \quad -d\left(\frac{e^{-x}}{y}\right) = -\,dy$$

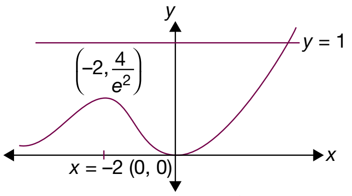
$$\therefore \quad \frac{e^{-x}}{y} = y + \lambda. \text{ So, } = f_1(x)$$

$$= e^{\frac{-x}{2}}, f_2(x) = -\,e^{\frac{-x}{2}}$$

$$f_1(x) \cdot f_2(x) + x^2 = 0$$

$$e^{\frac{-x}{2}} \cdot \left(-e^{\frac{-x}{2}}\right) + x^2 = 0$$

$$\Rightarrow \quad -e^{-x} + x^2 = 0 \quad \Rightarrow x^2 = e^{-x}$$



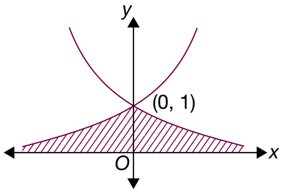
Graph of $f(x) = x^2e^x$

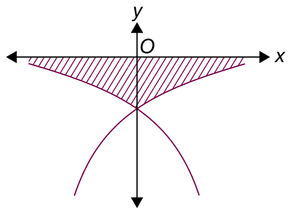
Hence, the correct answer is (B).

10. Area enclosed by $y = f_1(x)$, $y = e^x$ and the x-axis

$$A_1 = \int_{-\infty}^0 e^x dx + \int_0^\infty e^{\frac{-x}{2}}\, dx = \left(e^x\right)_{-\infty}^0 + \left(-2\left(e^{\frac{-x}{2}}\right)_0^\infty\right)$$

$$= 1 - 0 - 2(0 - 1) = 3$$





Area enclosed by $y = f_2(x)$; $y = -e^x$ and the x-axis.

$$A_2 = \left|\int_{-\infty}^0 (-e^x)dx + \int_0^\infty\left(-e^{\frac{-x}{2}}\right)dx \Rightarrow A_2 = 3\right|$$

$$\therefore \quad A_1 = A_2 = 6.$$

Hence, the correct answer is (C).

11. Let $P\ (x, y)$ be any point on curve.

∴ Equation of tangent at P is

$$(Y - y) = f'(x)\,(X - x)$$

$$\Rightarrow \quad \left|x - \frac{y}{f'(x)}\right| = x^2 \quad \Rightarrow x - \frac{f(x)}{f'(x)} = \pm\, x^2$$

$$\Rightarrow \quad \frac{xf'(x) - f(x)}{x^2} = \pm f'(x) \Rightarrow \frac{d}{dx}\left(\frac{f(x)}{x}\right) = \pm f(x) + C$$

$$\Rightarrow \quad \frac{f(x)}{x} = \pm f(x) + C$$

As curve passes through M (2, 1)

$$\Rightarrow \quad \frac{1}{2} = \pm 1 + C \quad \Rightarrow \quad C = \frac{-1}{2}, \frac{3}{2}$$

$$\therefore \quad f(x) = \frac{x}{2(x-1)} \text{ and } g(x) = \frac{3x}{2(1+x)}$$

Hence, the correct answer is (C).

12. $\displaystyle\int_0^1 y(x)dx = \int_0^1 \frac{3x}{2(1+x)}dx$

$$= \frac{3}{2} - \frac{3}{2}\ln 2$$

Hence, the correct answer is (C).

13. $g(x) = 8x^3 - 3x$

$$g(x) = 8x^3 - 3x$$

$$\Rightarrow \quad g(2) = 58$$

Area enclosed between ordinates $x = -\,\alpha$ to $x = \alpha$, is $2\displaystyle\int_0^\alpha dx = 2\alpha$.

Hence, the correct answer is (A).

14. Also, $\dfrac{dy}{dx} + g'(x)y = (g(x) + 1)\,g'(x)$

∴ Integrating factor $= e^{g(x)}$

Also, general solution is $y\ e^{g(x)} = \displaystyle\int e^{g(x)}\ (g(x) + 1)\,g'(x)dx$

Put $g(x) = t \quad \Rightarrow g'(x)\ dx = dt$

$$= \int e^t(t+1)dt = t\,e^t + c = g(x)e^{g(x)} + C$$

$$h(x) = y = g(x) + Ce^{-g(x)}$$

As, $h(0) = 1 \quad \Rightarrow 1 = 0 + C \quad \Rightarrow C = 1$

$\therefore \quad h(x) = g(x) + e^{-g(x)}$

Hence, $h(1) = 5 + e^{-5}$

Hence, the correct answer is (A).

15. Equation of tangent at $P\,(x, y)$ is $(Y - y) = \frac{dy}{dx}(X - x)$

As, M is mid point at PT, so $\dfrac{\left(x - \dfrac{y}{\frac{dy}{dx}}\right) + x}{2} = 0$

$\Rightarrow \quad 2x = \dfrac{y}{\frac{dy}{dx}} \Rightarrow \dfrac{dy}{dx} = \dfrac{y}{2x}$

$\Rightarrow \quad \dfrac{2dy}{y} = \dfrac{dx}{x}$

$\therefore \quad \Rightarrow 2\ln y = \ln x + \ln C'$

$\Rightarrow \quad C'x$

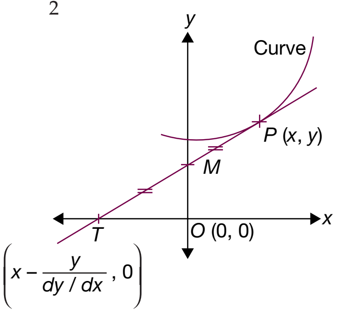
As, above curve passes through $(9, -6)$ so $36 = 9C'$

$\Rightarrow \quad C' = 4 \quad \therefore \; C : y^2 = 4x$

Clearly, focus of C is $(1, 0)$

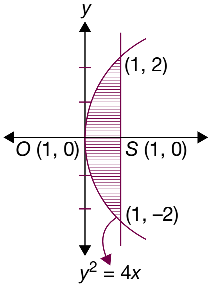
Director circle of C is $x + 1 = 0$

Also, $l\,(L, R) = 4$ and equation of tangent at vertex is y-axis, so $x = 0$.



Hence, the correct answer is (C).

16. So, required area $= 2\int_0^1 2\sqrt{x}\,dx = \frac{8}{3}$.



Hence, the correct answer is (D).

17. We have

$2h = t_1^2 + t_2^2 \qquad \ldots(1)$

$k = (t_1 + t_2) \qquad \ldots(2)$

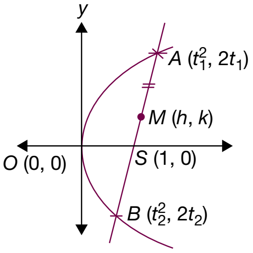
As, AB is focal chord, so

$t_1 \cdot t_2 = -1 \qquad \ldots(3)$

$\therefore$ From (2), we get

$k^2 = 2h - 2$ (Using (1) and (3))

$\Rightarrow$ Locus is $y^2 = 2(x - 1)$.



Hence, the correct answer is (B).

18. Given, $f'(x) + f(x) = x^2 + 3x + 2$

$\Rightarrow \quad \frac{dy}{dx} + y = x^2 + 3x + 2$ (Linear differential equation)

Now, I.F. $= e^x$

$\therefore$ General solution is

$$y\,.\,e^x = \int \underbrace{e^x}_{(II)} \underbrace{(x^2 + 3x + 2)}_{\substack{(I)\\ (I.B.P.)}}\,dx$$

$\Rightarrow \quad y = (x^2 + x + 1) + C\,e^{-x}$

As, $f(q) \in Q \;\forall\; q \in Q$,

So, $C = 0$

Hence, $y = f(x) = x^2 + x + 1$

Now, verify alternatives.

Hence, the correct answer is (D).

19. Area $= \int_0^1 x^2 + x + 1\,dx = \frac{1}{3} + \frac{1}{2} + 1 = \frac{11}{6}$

Hence, the correct answer is (B).

20. $f(f(x)) = 0 \Rightarrow f(x^2) + f(x) + 1 = 0$

No real solution.

Hence, the correct answer is (D).

21. Tangent to $C_1 : Y - y = m\,(X - x)$ at y-axis; $X = 0$

$\therefore \quad Y = y - mx = f(x) - xf'(x)$

Given, $f(x) - xf'(x) = g(x) - xg'(x)$

$\therefore \quad f(x) - g(x) = x(f'(x) - g'(x))$

$\Rightarrow \quad \dfrac{f'(x) - g'(x)}{f(x) - g(x)} = \dfrac{1}{x}$

$\therefore \quad$ Integrate $\ln(f(x) - g(x)) = \ln x + C$

$\therefore \quad f(x) - g(x) = C_1 x \qquad \ldots(1)$

Again Normal to $C_1 : Y - y = \frac{-1}{m}(X - x)$

At x-axis; $Y = 0$

$x + my = x + f'(x)\,f(x)$

$\therefore \quad x + f(x)\,f'(x) = x + g(x)\,g'(x)$ [given]

Integrate

$\Rightarrow \quad f^2(x) - g^2(x) = C_2 \qquad \ldots(2)$

(1) and (2) $\quad \Rightarrow (f(x) + g(x)) = \dfrac{C_2}{C_1 x} \qquad \ldots(3)$

$\therefore \quad$ (1) + (3) $\quad \Rightarrow 2f(x) = C_1 x + \dfrac{C_2}{C_1 x}$

And (3) – (1) $\quad \Rightarrow 2g(x) = \dfrac{C_2}{C_1 x} - C_1 x$

Now, $f(1) = 1 \quad \Rightarrow 2 = C_1 + \dfrac{C_2}{C_1}$ and

$g(2) = 3 \quad \Rightarrow 6 = \dfrac{C_2}{2C_1} - 2C_1$

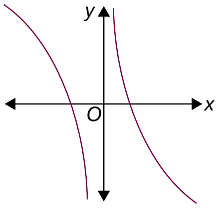
$\Rightarrow \quad C_1 = -2$ and $C_2 = -8$

$\therefore \quad f(x) = -x + \dfrac{2}{x} = \dfrac{2}{x} - x$ and $g(x) = \dfrac{2}{x} + x$

$$\frac{\int_1^2 \left(\frac{2}{x} - x\right)dx}{\int_1^2 \left(\frac{2}{x} + x\right)dx} = \frac{\left.2\ln x - \frac{x^2}{2}\right]_1^2}{\left.2\ln x + \frac{x^2}{2}\right]_1^2} = \frac{(2\ln 2 - 2) - \left(\frac{-1}{2}\right)}{(2\ln 2 + 2) - \left(\frac{1}{2}\right)} = \frac{\ln 16 - 3}{\ln 16 + 3}$$

Hence, the correct answer is (D).

22. Graph of $f(x)$

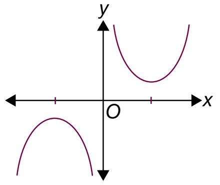


$R_f = (-\infty, \infty)$

Hence, the correct answer is (C).

23. Graph of $g(x)$

$y = \dfrac{2}{x} + x; \; \dfrac{dy}{dx} = 1 - \dfrac{2}{x^2}$



Hence, the correct answer is (D).

24. $x^2+y^2-ax=0$

$$a=\frac{x^2+y^2}{x}$$

$$\therefore\quad 0=\frac{x(2x+2yy')-(x^2+y^2)}{x^2}\Rightarrow x^2+2xy\,.\,y'-y^2=0$$

For Orthogonal curve $x^2-\dfrac{2xy}{y'}-y^2=0$

$$y'=\frac{2xy}{x^2-y^2}\qquad\ldots(1)$$

Put $y=vx$

$$v+x\frac{dv}{dx}=\frac{2v}{1-v^2}\Rightarrow x\frac{dv}{dx}=\frac{2v}{1-v^2}-v=\frac{v+v^3}{1-v^2}$$

$$\Rightarrow\quad \frac{1-v^2}{v+v^3}dv=\frac{dx}{x}\Rightarrow\frac{(1+v^2)-2v^2}{v(1+v^2)}dv=\frac{dx}{x}$$

$$\left(\frac{1}{v}-\frac{2v}{1+v^2}\right)dv=\frac{dx}{x}\Rightarrow \ln v-\ln(1+v^2)=\ln x+\ln c$$

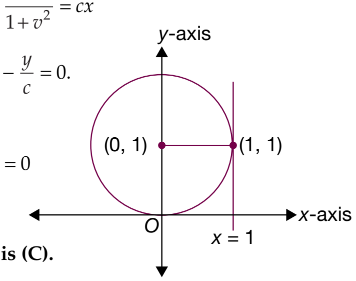
$$\Rightarrow\quad \ln\left(\frac{v}{1+v^2}\right)=\ln(cx)\Rightarrow\frac{v}{1+v^2}=cx$$

$$\Rightarrow\quad \frac{xy}{x^2+y^2}=cx\Rightarrow x^2+y^2-\frac{y}{c}=0.$$

This passes through (1, 1)

$$\therefore\quad c=\frac{1}{2}\therefore\ C: x^2+y^2-2y=0$$

Required area $=1-\dfrac{\pi}{4}$

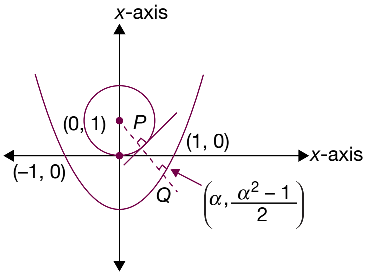


Hence, the correct answer is (C).

25. Let $Q\left(\alpha,\dfrac{\alpha^2-1}{2}\right)$

Slope of tangent at P and Q are same m_{PQ} = slope of normal at Q

$$\frac{-1}{\alpha}=\frac{\frac{\alpha^2-1}{2}}{\alpha-1}\quad\Rightarrow a^3-\alpha=0\quad\Rightarrow\alpha=0,\pm1$$



$\therefore\quad Q(1, 0)$

$PQ=\sqrt{2}-1.$

Hence, the correct answer is (A).

26. $\dfrac{dy}{dx}=e^{x-y}\left(e^x-e^y\right)$

$$e^y\frac{dy}{dx}=e^{2x}-e^{x+y}\Rightarrow e^y\frac{dy}{dx}+e^x.e^y=e^{2x}$$

Let $e^y=u,$ so that $e^y\dfrac{dy}{dx}=\dfrac{du}{dx}$

We have $\dfrac{du}{dx}+e^x.u=e^{2x}$ I.F is $e^{\int e^x dx}=e^{e^x}$

The solution is $u.e^{e^x}=\displaystyle\int e^{e^x}.e^{2x}dx$

$$=\int e^t.t\,dt\ \text{ where } t=e^x=e^t(t-1)+k$$

$$=e^{e^x}\left(e^x-1\right)+k$$

Then $u=e^x-1+ke^{-e^x}\quad\Rightarrow e^y=e^x-1+k\,e^{-e^x}$

Hence, the correct answer is (A).

27. $\dfrac{dy}{dx}+y\dfrac{\ln y}{x}=\dfrac{y}{x^2}\left(\ln y\right)^2$

We have $\dfrac{1}{y(\ln y)^2}\dfrac{dy}{dx}+\dfrac{1}{x}.\dfrac{1}{\ln y}=\dfrac{1}{x^2}$

Let $\dfrac{-1}{\ln y}=u,$ so that $\dfrac{du}{dx}\ -\dfrac{u}{x}=\dfrac{1}{x^2}$

$$\text{I.F}=e^{-\int\frac{dx}{x}}=e^{-In\ x}=1/x$$

Then the solution is $u.\dfrac{1}{x}=\displaystyle\int\frac{1}{x^3}dx+k$

Which on simplification gives $2x=\left[1+2cx^2\ln y\right]$

Hence, the correct answer is (C).

28. $\dfrac{dx}{dy}=\dfrac{y^2+e^{2x}}{y^3}$

$$\frac{dx}{dy}=\frac{1}{y}+\frac{e^{2x}}{y^3}\Rightarrow e^{-2x}\frac{dx}{dy}-e^{-2x}\frac{1}{y}=\frac{1}{y^3}$$

$\dfrac{-e^{-2x}}{2}=u$, so that $e^{2x}\dfrac{dx}{dy}=\dfrac{du}{dy}$

$$\frac{du}{dy}+\frac{2}{y}u=\frac{1}{y^3}\ I.F.=e^{\int\frac{2}{y}dy}=e^{2lny}=y^2$$

Solution is $u.y^2=\displaystyle\int\frac{1}{y}dy+k\Rightarrow uy^2=\ln y+k\Rightarrow-\frac{e^{-2x}}{2}y^2=\ln y+k$

$$\Rightarrow\quad -e^{-2x}y^2=2\ln y+2k$$

$$\therefore\quad e^{-2x}y^2+2\ln y=\text{ constant}$$

Hence, the correct answer is (A).

Matrix Match/Column Match Type Questions

1. **A → s; B → q; C → r; D → s**

(A) $\dfrac{ydx-xdy}{y^2}+3x^2e^{x^3}dx=0;\ d\left(\dfrac{x}{y}\right)+d\left(e^{x^3}\right)=0$

(B) $y\dfrac{dx}{dy}-x=2y^3$

$$\frac{dx}{dy}-\frac{x}{y}=2y^2\ ;\ \text{I.F.}=e^{-\int\frac{1}{y}dy}=\frac{1}{y}$$

(C) $x^2dx+y\,dx-x\,dy=0$

$x^2dx=x\,dy-y\,dx$

$$dx=\frac{xdy-ydx}{x^2};\ \text{I.F.}=\frac{1}{x^2}$$

(D) $y\,dx+xy^2\,dx-x\,dy=0$

$$\frac{ydx-xdy}{y^2}=-x\,dx\quad\Rightarrow\text{I.F.}=\frac{1}{y^2}$$

2. **A → q; B → r; C → s; D → p**

(A) $\dfrac{d^2y}{dx^2}-(5+3)\dfrac{dy}{dx}+5\times3y=0$

$$\frac{d^2y}{dx^2}-8\frac{dy}{dx}+15y=0$$

(B) $xy_1 + y = ae^{x} - be^{-x} + 2x$

$xy_2 + 2y_1 = ae^{x} + be^{-x} + 2 = xy - x^2 + 2$

(C) $\frac{dy}{dx} = 2ax \Rightarrow \frac{d^2y}{dx^2} = 2a; \quad \Rightarrow \frac{dy}{dx} = x\frac{d^2y}{dx^2}$

$x^2y_2 - (2+1-1)xy_1 + 2y = 0$

(D) $y = a\sin(4x+b)$

$\frac{dy}{dx} = 4 - a\cos(4x+b); \frac{d^2y}{dx^2} = -16a\sin(4x+b)$

$\frac{d^2y}{dx^2} = -16y$

$y_2 = -16y;$

3. A → q; B → r; C → p; D → s

(A) $(3x^2y + 2xy - e^{x}(1+x))\,dx + (x^3+x^2)\,dy = 0$

$d((x^3+x^2)y) - e^{x}(1+x)\,dx = 0$

$\int d((x^3+x^2)y) - \int d(xe^{x}) = 0$

$(x^3+x^2)y - xe^{x} = c$

(B) $y\,dx - x\,dy = 3xy^2e^{x^2}\,dx$

$\int \frac{ydx - xdy}{y^2} = \int 3xe^{x^2}\,dx$

Put $x^2 = t$; $2x\,dx = dt$

$\int d\left(\frac{x}{y}\right) = 3\int e^{t}\frac{dt}{2}$

$\frac{x}{y} = \frac{3}{2}e^{t} + c \Rightarrow \frac{x}{y} - \frac{3}{2}e^{x^2} = c$

(C) $\frac{dy}{dx} = xy(x^2y^2 - 1) = x^3y^3 - xy$

$\frac{dy}{dx} + xy = x^3y^3 \Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \frac{x}{y^2} = x^3$

Put $\frac{1}{y^2} = t; -\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \quad -\frac{1}{2}\frac{dt}{dx} + xt = x^3 \Rightarrow \frac{dt}{dx} - 2tx = -2x^3$

I.F. $= e^{\int -2xdx} = e^{-x^2}$

$\Rightarrow \quad te^{-x^2} = \int -2x^3e^{-x^2}\,dx$ Put $-x^2 = P; -2x\,dx = dP$

$\frac{e^{-x^2}}{y^2} = \int (-P)e^{P}dP = (-P)e^{P} - \int (-1)e^{P}dP$

$\frac{e^{-x^2}}{y^2} = -Pe^{P} + e^{P} + c = x^2e^{-x^2} + e^{-x^2} + c$

$\frac{1}{y^2} = (x^2+1) + ce^{x^2}$

(D) $\frac{dy}{dx}(x^2y^3 + xy) = 1 \Rightarrow \frac{dx}{dy} - xy = x^2y^3$

$\frac{1}{x^2}\frac{dx}{dy} - \frac{y}{x} = y^3$

Put $-\frac{1}{x} = t; \Rightarrow \frac{1}{x^2}\frac{dx}{dy} = \frac{dt}{dy}$

$\frac{dt}{dy} + ty = y^3$

I.F. $= e^{\int ydy} = e^{y^2/2}$

$te^{y^2/2} = \int y^3e^{y^2/2}\,dy$

Put $y^2/2\ P; y\,dy = dp$

$-\frac{1}{x}e^{y^2/2} = \int e^{P}\cdot 2pdp = e^{P}2p - \int e^{P}\cdot 2dp$

$\Rightarrow \quad -\frac{e^{y^2/2}}{x} = e^{P}\cdot 2p - 2e^{P} + c$

$-\frac{e^{y^2/2}}{x} = y^2e^{y^2/2} - 2e^{y^2/2} + c;$

$-\frac{1}{x} = y^2 - 2 + ce^{-y^2/2}$

$\frac{1}{x} = 2 - y^2 - ce^{-y^2/2}$

4. A → p, r, s; B → q; C → p, q, r; D → p

(A) $x\,dy = y(dx + y\,dy) \quad y > 0 \quad y(1) = 1$

$y(x_0) = -3$

$x\,dy - y\,dx = y^2dy \Rightarrow -\left(\frac{ydx - xdy}{y^2}\right) = dy$

$\int -d\left(\frac{x}{y}\right) = \int dy \Rightarrow -\frac{x}{y} = y + c$

$-1 = 1 + c \Rightarrow c = -2 \Rightarrow -\frac{x}{y} = y - 2 \Rightarrow x \equiv x_0$

$y(x_0) = -3 \Rightarrow \frac{-x_0}{-3} = -3 - 2$

$x_0 = -15$

(B) $\frac{dy}{dt} - \frac{t}{t+1}y = \frac{t}{t+1}$

$\text{IF} = e^{-\int\frac{t}{t+1}dt}\ e^{-\int\left(1-\frac{t}{t+1}\right)dt} = e^{-(t-\ln(t+1))} = e^{-t}\cdot(t+1)$

$\Rightarrow \quad ye^{-t}(t+1) = \int e^{-t}\,dt = -e^{-t} + c \Rightarrow y(t+1) = -1 + ce^{t}$

$\Rightarrow \quad y(t+1) = -1 + ce^{t} \quad \Rightarrow$ Given $y(0) = -1$

$-1(0+1) = -1 + ce^{\circ} \quad \Rightarrow c = 0$

$t = 1 \Rightarrow y(2) = -1 + 0$

$y(1) = -\frac{1}{2} = \lambda$

(C) $(x^2+y^2)\frac{dy}{dx} = xy \Rightarrow \frac{dx}{dy} = \frac{x}{y} + \frac{y}{x}$

Put $x/y = t \Rightarrow x = ty \Rightarrow dx = t + y\frac{dt}{dy}$

$\Rightarrow \quad t + \frac{ydt}{dy} = t + \frac{1}{t}$

$\int tdt = \int \frac{dy}{y} \Rightarrow \frac{t^2}{2} = \ln y + c$

$\frac{x^2}{2y^2} = \ln y + c \Rightarrow y(1) = 1$

$\frac{1}{2} = 0 + c \Rightarrow y(x_0) = \sqrt{e} \quad \frac{x_0^2}{2e} = \ln\sqrt{e} + \frac{1}{2}$

$x_0^2 = 2e \Rightarrow x_0 = \sqrt{2e}$

(D) $\frac{dy}{dx} = -\frac{2y}{x} \Rightarrow \int\frac{dy}{y} = \int -\frac{2}{x}\,dx$

$\ln y = -2\ln x + \ln K = \ln\frac{K}{x^2} \Rightarrow y = \frac{K}{x^2}$

$\Rightarrow \ y = \frac{K}{x^2}\ (y(1) = 1)$

$K = 1 \quad y(2) = \frac{1}{4}$

5. A → r; B → p; C → s; D → q

(A) The given equation can be written

$\left(\frac{dy}{dx} - e^{-x}\right)\left(\frac{dy}{dx} - e^{x}\right) = 0$

$$\Rightarrow \quad \frac{dy}{dx} - e^{-x} = 0 \text{ or } \frac{dy}{dx} - e^{x} = 0$$

$$\Rightarrow \quad dy - e^{-x}dx = 0 \text{ or } dy - e^{x}dx = 0$$

$$\Rightarrow \quad y + e^{-x} = c \text{ or } y - e^{x} = c$$

(B) $\because (x + y + 1)dy = dx$

Put $x + y + 1 = v$

Then $dx + dy = dv$

And the given equation reduces to $v(dv - dx) = dx$

$\Rightarrow x + c = v - \ln(v + 1)$

$\Rightarrow \ln(v + 1) = v - x - c$

or $\ln(x + y + 2) = y + 1 - c = y + c\,(Q)$

Also, $x + y + 2 = e^{y+c} = e^{y}.e^{c} = ce^{y}\,(P)$

(C) $(x + y + 2)dx + (2x + 2y - 1)dy = 0$

$\frac{dy}{dx} = -\frac{x + y + 2}{2x + 2y - 1}$ put $x + y + 2 = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

Now, differential equation $\frac{dt}{dx} - 1 = -\frac{t}{2t - 5} \Rightarrow \frac{dt}{dx} = \frac{t - 5}{2t - 5}$

$\left(2 + \frac{5}{t - 5}\right)dt = dx$ we have $2t + 5\ln(t - 5) = x + c$ or $2(x + y + 2) + 5\ln(x + y - 3) = x + c,$

(D) Solution of $x\frac{dy}{dx} - y = (x - 1)e^{x} \Rightarrow \frac{dy}{dx} - \frac{y}{x} = \left(\frac{x - 1}{x}\right)e^{x}$ I.F. $= 1/x$

$$\frac{y}{x} = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)e^{x} = \frac{e^{x}}{x} + c \Rightarrow y = e^{x} + cx$$

6. A → r; B → s; C → p; D → q

(A) $\frac{dy}{dx} = 3$ and $\frac{dy}{dx} = 4$

(B) Tangent $y - y_1 = m(x - x_1)$

$x = 0 \quad y = y_1 - mx_1$

Given, $x_1 - (y_1 - mx_1) = 2$

$x - y + mx = 2$

$$x - \frac{ydx + xdy}{dx} = 2$$

$$\int\frac{(x - 2)dx}{x^2} = \int\frac{ydx - xdy}{x^2}$$

$$K + \ln x + \frac{2}{x} - \int d\left(y/x\right)$$

$y = -x\ln x - 2 + Kx$

(C) $\frac{xdy + ydx}{(xy)^2} = \frac{xdy - ydx}{x^2} \quad \Rightarrow -\frac{1}{xy} + \frac{y}{x} + C = 0$

$\Rightarrow \quad y^2 - 1 + C\,xy = 0$

(D) $2y\frac{dy}{dx} = 4a$

$$\frac{dy}{dx} = \frac{2a}{y} \quad \left\{\frac{dy}{dx} \to -\frac{dx}{dy}\right.$$

$$\Rightarrow -\frac{dx}{dy} = \frac{2}{y}\frac{y^2}{4x} \Rightarrow \int y\,dy = -2\int x\,dx$$

$$\Rightarrow \frac{y^2}{2} = -x^2 + C \Rightarrow y^2 = -2x^2 = 2C$$

$\Rightarrow y^2 + 2x^2 = 2C.$

Integer/Numerical Answer Type Questions

1. $\sqrt{x^2 + y^2}\,(y\,dx + x\,dy) = x\,dx + y\,dy$

$$d(xy) = \frac{1}{2}\frac{d(x^2 + y^2)}{\sqrt{x^2 + y^2}}$$

$$\int d(xy) = \int d\left(\sqrt{x^2 + y^2}\right) \Rightarrow xy = \sqrt{x^2 + y^2} + c$$

At point (0, 2) $0 = \sqrt{4} + c \Rightarrow c = -2$

$$xy = \sqrt{x^2 + y^2} - 2$$

$$\Rightarrow \quad y = 1 \quad x = \sqrt{x^2 + 1} - 2 \Rightarrow x + 2 = \sqrt{x^2 + 1}$$

$$\Rightarrow \quad x^2 + 4 + 4x = x^2 + 1 \Rightarrow x = -\frac{3}{4} = h;\ 4|h| = 3$$

Hence, the correct answer is (3).

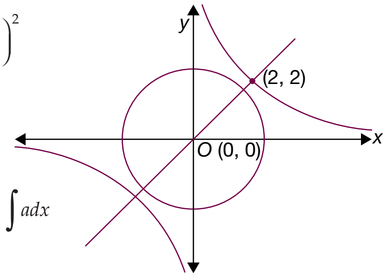
2. $f'(x) + \frac{x}{1 + x^2}\cdot f(x) = \frac{x}{1 + x^2}$

I.F. $= \sqrt{1 + x^2}$

$$\therefore \quad f(x) = 1 + \frac{1}{3\sqrt{1 + x^2}}$$

$$\therefore \quad f(\sqrt{8}) + \frac{8}{9} = 2$$

Hence, the correct answer is (2).



3. Given $y\left(\frac{d^2y}{dx^2}\right) = 2\left(\frac{dy}{dx}\right)^2$

$$\Rightarrow \quad \frac{y''}{y'} = \frac{2y'}{y}$$

$$\Rightarrow \quad \ln y' = 2\ln y + \ln a$$

$$\Rightarrow \quad \frac{y'}{y^2} = a \quad \Rightarrow \int\frac{dy}{y^2} = \int a dx$$

$$\Rightarrow \quad \frac{-1}{y} = ax + b$$

Since curve is passing through (2, 2) and $\left(8, \frac{1}{2}\right)$, so

$2a + b = \frac{-1}{2}$ …(1)

$8a + b = -2$ …(2)

$\therefore$ On solving (1) and (2), we get $a = \frac{-1}{4}$, $b = 0$

Hence, $C_1 : xy = 4$ and curve C_2 is $x^2 + y^2 = 4$.

$\therefore$ Shortest distance between C_1 and C_2 is $= 2\sqrt{2} - 2 = \sqrt{8} - 2$

Hence, $(p^2 - q) = 64 - 2 = 62$

Hence, the correct answer is (62).

4. $f(x) + \int_0^x f(t)\tan t\,dt - \int_0^x \underbrace{\tan(t - x)}_{\text{Using king}}\,dt = 0;$

$$f(x) + \int_0^x f(t)\tan t\,dt + \int_0^x \tan t\,dt = 0$$

Differentiate w.r.t x, we get $f'(x) + (f(x) + 1)\tan x = 0$

$$\Rightarrow \quad \frac{f'(x)}{f(x) + 1} + \tan x = 0$$

Integrate w.r.t. x, we get

$\ln(f(x) + 1) + \ln(\sec x) = C$

$f(0) = 0 \Rightarrow C = 0$

Hence, $\frac{(f(x) + 1)}{\cos x} = 1 \Rightarrow f(x) = \cos x - 1$

$\therefore \quad f(x) = 0 \Rightarrow \cos x = 1$

$\Rightarrow \quad x = 2n\pi\ n \in I \Rightarrow$ Number of solution is 1.

Hence, the correct answer is (1).

5. Given, $\frac{xdy}{dx} - 2y = x^4y^2$ or $\frac{dy}{dx} - \frac{2}{x}y = x^3y^2$.

Now, dividing by y^2, we get $\frac{1}{y^2}\frac{dy}{dx} - \frac{2}{y}\frac{1}{x} = x^3$ …(1)

This is a Bernoulli's differential equation, substituting $\frac{-2}{y} = t$, we get

$$\Rightarrow \quad \frac{2}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

So, equation (1) becomes

$$\Rightarrow \quad \frac{1}{2}\frac{dt}{dx} + \frac{t}{x} = x^3 \Rightarrow \frac{dt}{dx} + \frac{2}{x}t = 2x^3$$

$\text{I.F} = e^{2\ln x} = e^{\ln x^2} = x^2$

So, general solution is given by

$$x^2t = \frac{2x^6}{6} + C \Rightarrow \frac{-2}{y} = \frac{x^4}{3} + \frac{C}{x^2}$$

If $x = 1, y = -6 \Rightarrow C = 0$

$$\therefore \quad \frac{-2}{y} = \frac{x^4}{3} \Rightarrow y = \frac{-6}{x^4} \text{ i.e., } f(x) = \frac{-6}{x^4}$$

Now, $\frac{dy}{dx} = 24x^{-5} = \frac{24}{x^5}$.

Hence, $\left.\frac{dy}{dx}\right]_{x=3^{\frac{1}{5}}} = \frac{24}{3} = 8.$

Hence, the correct answer is (8).

6. We have, $\left|x - y\frac{dx}{dy}\right| = 2\left|\frac{dy}{dx}\right| \Rightarrow x - y\frac{dx}{dy} = \pm 2\frac{dy}{dx}$

Taking positive sign,

$$2\left(\frac{dy}{dx}\right)^2 - x\left(\frac{dy}{dx}\right) + y = 0 \Rightarrow \frac{dy}{dx} = \frac{x \pm \sqrt{x^2 - 8y}}{4}$$

But no solution exist, as it does not pass through (1, 1)

Now, taking negative sign,

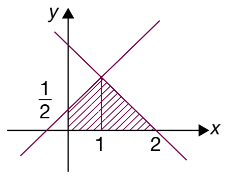
$$2\left(\frac{dy}{dx}\right)^2 + x\left(\frac{dy}{dx}\right) - y = 0 \Rightarrow \frac{dy}{dx} = \frac{-x \pm \sqrt{x^2 + 8y}}{4}$$

Let $x^2 + 8y = v^2 \Rightarrow 2x + 8\frac{dy}{dx} = 2v\frac{dv}{dx}$

So, $v\frac{dv}{dx} - x = -x \pm v \Rightarrow \frac{dv}{dx} = \pm 1$

$\Rightarrow \quad v = \pm x + c$

$\Rightarrow \quad \sqrt{x^2 + 8y} = \pm x + C$



So, curves are $\sqrt{x^2 + 8y} = x + 2$ and $\sqrt{x^2 + 8y} = (4 - x)$

$\therefore \quad$ On squaring, we get $2y = x + 1$ and $x + y = 2$

$\therefore \quad$ Clearly, required area $= \frac{3}{4} + \frac{1}{2} = \frac{5}{4} = \frac{m}{n}$ (given)

So, $(m + n)$Least $= 5 + 4 = 9$.

Hence, the correct answer is (9).

7. $y^3y'' = -1$

$$y'y'' = \frac{-y'}{y^3}$$

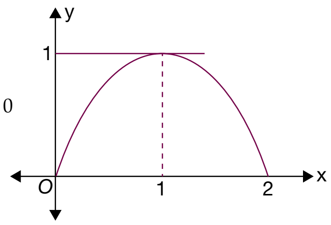
Integrating both sides

$$\frac{(y')^2}{2} = \frac{1}{2y^2} + c$$

Putting $x = 1, y = 1$ and $y'(1) = 0$

$$0 = \frac{1}{2} + c \Rightarrow c = \frac{-1}{2}$$

$$\frac{(y')^2}{2} = \frac{1}{2y^2} - \frac{1}{2}$$



$$(y')^2 = \frac{1}{y^2} - 1 \quad \Rightarrow (y')^2 = \frac{1-y^2}{y^2} \Rightarrow y' = \pm\frac{\sqrt{1-y^2}}{y}$$

$$\Rightarrow \quad \frac{y}{\sqrt{1-y^2}}y' = \pm 1$$

Taking as +ve sign,

$$\int \frac{y}{\sqrt{1-y^2}}y'dx = \int 1dx, \text{ put } 1 - y^2 = t$$

$\therefore \quad y = \sqrt{2x - x^2} = f(x) \quad \{\because y \geq 0 \ \forall \ x \in D_f\}$

Taking as negative sign we will obtain the same function.

Now, $f'(x) = \frac{2-2x}{2\sqrt{2x-x^2}} = 0 \quad \Rightarrow x = 1$, which is point of maxima.

$\therefore \quad$ Maximum value of $f(x)$ is $= f(1) = \sqrt{2-1} = 1$.

Hence, the correct answer is (1).

8. $f'(x) = \frac{1}{x} + \pi\cos(\pi x) + C$

$$f'(2) = \frac{1}{2} + \pi + C = \frac{1}{2} + \pi \Rightarrow C = 0$$

$\therefore \quad f(x) = \ln|x| + \sin(\pi x)$

Now, $J = \int_0^1 (\ell x|x| + \sin\pi x)\,dx$

$$= [x\ln x - x]_0^1 - \left[\frac{1}{\pi}\cos\pi x\right]_0^1$$

$$J = -1 + \frac{2}{\pi}.$$

$\therefore \quad \pi(J + 1) = 2.$

Hence, the correct answer is (2).

9. At $P(x, y)$, for minima $\Rightarrow f'(x) = 0 \qquad f''(x) > 0$

$\Rightarrow \quad x^2 + f^2(x) - 6 > 0$

$\Rightarrow \quad x^2 + y^2 > 6$ (i.e., P lies outside $x^2 + y^2 = 6$)

$\therefore \quad$ No. of tangents to $x^2 + y^2 = 4$ is 2.

Hence, the correct answer is (2).

10. $\frac{1}{y^3}\frac{dy}{dx} - \frac{x}{y^2} = x^3$

$$z = \frac{1}{y^2} \Rightarrow \frac{dz}{dx} = -\frac{2}{y^3}\frac{dy}{dx}$$

$\therefore \quad \frac{dz}{dx} + 2x\,z = -2x^3 \quad$ integrating factor $= e^{x^2}$

$$ze^{x^2} = -\int x^2e^{x^2}2xdx = \left(1 - x^2\right)e^{x^2} + c$$

$$\frac{e^{x^2}}{y^2} = \left(1 - x^2\right)e^{x^2} + c \quad \text{given } y(0) = 1, c = 0$$

We have curve $y^2 = \frac{1}{1-x^2}$ now $3\left(y\left(\frac{1}{2}\right)\right)^2 = 4$

Hence, the correct answer is (4).

11. Since, $\sin y\left(3\sin x - 7\cos y + 7\right)dy + \cos x\left(3\cos y - 7\sin x - 3\right)dx = 0$

$$\Rightarrow \quad \sin y\left((5-2)\sin x - (5+2)\cos y - (5+2)\right)dy +$$
$$\cos x\left((5-2)\sin x - (5+2)\cos y + (5+2)\right)dx = 0$$

$$\Rightarrow \quad 5\sin y\left(\sin x - \cos y + 1\right)dy \ +5\cos x \times \left(\cos y - \sin x - 1\right)dx -$$
$$2\sin y\left(\sin x + \cos y - 1\right)dy \ -2\cos x\left(\cos y + \sin x - 1\right)dx = 0$$

$$\Rightarrow \quad 5\left(\sin x - \cos y + 1\right)\left(\sin ydy - \cos xdx\right) -$$
$$2\left(\sin x + \cos y - 1\right)\left(\sin ydy + \cos xdx\right) = 0$$

$$\Rightarrow \quad \frac{5(\cos x dx - \sin y dy)}{(\sin x + \cos y - 1)} + \frac{(\cos x dx + \sin y dy)}{(\sin x - \cos y + 1)} = 0$$

On integrating, we have

$$5\ln(\sin x + \cos y - 1) + 2\ln(\sin x - \cos y + 1) = \ln c$$

$$\Rightarrow \quad (\sin x + \cos y - 1)^5 (\sin x - \cos y + 1)^2 = c$$

$$\therefore \quad \lambda = 5, \mu = 2$$

Hence, the correct answer is (7).

12. Put $x^2 = \sin\alpha; y^2 = \sin\beta$

$$\cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$$

$$\cot\left(\frac{\alpha - \beta}{2}\right) = a;\ \alpha - \beta = 2\cot^{-1}(a)$$

$$\sin^{-1}(x^2) - \sin^{-1}(y^2) = 2\cot^{-1}(a)$$

$$\frac{1}{\sqrt{1-x^4}}(2x) - \frac{1}{\sqrt{1-y^4}} 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}\sqrt{\frac{1-y^4}{1-x^4}}$$

Order = 1, degree = 1

Hence, the correct answer is (2).

13. Given $y = C_1(\cos x \cos C_2 - \sin x \sin C_2) - C_3 e^{C_4}.e^{-x} + C_5 \sin x$

$$= (C_1 \cos C_2)\cos x - (C_1 \sin C_2 - C_5)\sin x - (C_3 e^{C_4})e^{-x}$$

$\Rightarrow \quad y = l\cos x + m\sin x - ne^{-x}$.where l, m, n are arbitrary constant

$\therefore \quad$ order = 3

Hence, the correct answer is (3).

14. $a = -(2 + 1 - 1) = -2$

$b = 2.1 + 1.(-1) + (-1)(-2) = -1$

$\therefore \quad a^2 + b^2 + c^2 = 9$

$c = -(2)(1)(-2) = 2$

Hence, the correct answer is (9).

15. We have $f(x) = e + (1 - x)\ln\left(\frac{x}{e}\right) + \int_1^x f(t)\,dt$ …(1)

Differentiate both sides with respect to x, we get

$$f'(x) = (1 - x)\frac{1}{x} + \ln\left(\frac{x}{e}\right)(-1) + f(x)$$

$$\Rightarrow \quad f'(x) - f(x) = \left(\frac{1}{x} - 1 - \ln\left(\frac{x}{e}\right)\right)$$

$\therefore \quad$ Multiplying both sides with e^{-x}, we get

$$e^{-x}f'(x) - e^{-x}f(x) = e^{-x}\left(\frac{1}{x} - 1 - \ln\left(\frac{x}{e}\right)\right)$$

$$\Rightarrow \quad \frac{d}{dx}(e^{-x} \cdot f(x)) = e^{-x}\left(\frac{1}{x} - 1 - \ln\left(\frac{x}{e}\right)\right)$$

$\therefore \quad$ On integrating both sides with respect to x, we get

$$e^{-x}f(x) = e^{-x} - \int e^{-x}\left(\ell n\left(\frac{x}{e}\right) - \frac{1}{x}\right)dx$$

$$= e^{-x} + e^{-x}\ln\left(\frac{x}{e}\right) + \lambda$$

$$\Rightarrow \quad f(x) = 1 + \ln\left(\frac{x}{e}\right) + \lambda e^x \qquad \text{…(2)}$$

Now, putting $x = 1$ in equation (1), we get $f(1) = e$ …(3)

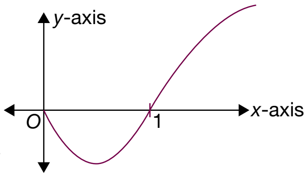
$\therefore \quad$ Using (3) and (2), we get $\lambda = 1$.

Thus, $f(x) = 1 + e^x + \ln\left(\frac{x}{e}\right)$

or $\quad f(x) = e^x + \ln x$.

Hence, $g(x) = x\ln x$;

Now, $A = \int_0^1 x\ln x\,dx = \frac{1}{4}$



Hence, the correct answer is (4).

16. Given, $f(x) = e^x \cdot \int_0^x e^{-y} f'(y)\,dy - (x^2 - x + 1)e^x$ …(1)

Differentiating both the sides

$$f'(x) = e^x \cdot e^{-x} \cdot f'(x) + e^x \int_0^x e^{-y} \cdot f'(y)\,dy - [(x^2 - x + 1)e^x + e^x(2x - 1)]$$

$$f'(x) = f'(x) + \int_0^x e^{x-y} \cdot f'(y)\,dy - e^x(x^2 + x)$$

$$\Rightarrow \quad 0 = f(x) + (x^2 - x + 1)e^x - e^x(x^2 + x)$$

[Substituting $\int_0^x e^{x-y} f'(y)\,dy = f(x) + (x^2 - x + 1)e^x$ from (1)]

$$f(x) = e^x(x^2 + x - x^2 + x - 1)$$

$$\therefore \quad f(x) = e^x \cdot (2x - 1) \quad \Rightarrow f(1) = e.$$

Also, $f'(x) = e^x \cdot 2 + (2x - 1)e^x$

$$\Rightarrow \quad f'(1) = 2e + e = 3e$$

And $f''(x) = 2e^x + (2x - 1)e^x + 2e^x$

$$\Rightarrow \quad f''(1) = 2e + e + 2e = 5e.$$

Hence, $f(1) + f'(1) + f'(1) = 9e \quad \Rightarrow k = 9$.

Hence, the correct answer is (9).

17. $f(x) = x\ln x + x - \int_1^x \frac{f(t)\,dt}{t\ln t}$

$$\therefore \quad f'(x) = \frac{x}{x} + \ln x + 1 - \frac{f(x)}{x\ln x}$$

$$\therefore \quad f'(x) + \frac{f(x)}{x\ln x} = 2 + \ln x \Rightarrow \frac{dy}{dx} + \frac{y}{x\ln x} = 2 + \ln x$$

This is linear differentiable equation.

$$\text{I.F.} = e^{\int \frac{dx}{x\,\ell n x}} = e\ln^{(\ln x)} = \ln x$$

Solution is $y\ln x = \int (2 + \ln x)\ln x\,dx$

$$\therefore \quad y\ln x = \int (\ln^2 x + 2\ln x)\,dx = x\ln^2 x + C.$$

Put $x = 1$ get $C = 0$

$y\ln x = x\ln^2 x \quad \therefore y = f(x) = x\ln x$.

$$f'(x) = 1 + \ln x \quad \therefore f''(x) = \frac{1}{x}$$

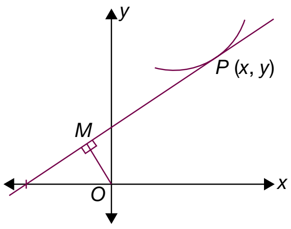
$$\Rightarrow \quad f'(x) + f''\left(\frac{1}{4}\right) = (1 + \ln e) + 4 = 2 + 4 = 6.$$

Hence, the correct answer is (6).

18. We have $Y - y = m(X - x)$

$$mX - Y + (y - mx) = 0$$

Now, $p = \left|\frac{y - mx}{\sqrt{1 + m^2}}\right| = x$ (given)



$$\Rightarrow \quad (y - mx)^2 = x^2(1 + m^2)$$

(on squaring both sides)

$$\Rightarrow \quad y^2 - 2xym = x^2$$

$$\text{or} \quad = \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \qquad \text{…(1)}$$

Put $y^2 = t$

$\Rightarrow \quad 2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$; substituting it in (1), we get

$x\dfrac{dt}{dx} = t - x^2 \quad \Rightarrow \dfrac{dt}{dx} - \dfrac{1}{x}t = -x$ (Linear differential equation)

Now, I.F. $= \dfrac{1}{x} =$ integrating factor

So, general solution is $t.\dfrac{1}{x} = \int -dx$

$\Rightarrow \quad \dfrac{t}{x} = -x + C$

$\Rightarrow \quad y^2 = -x^2 + Cx$

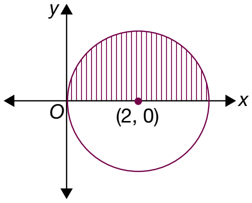
$\Rightarrow \quad x^2 + y^2 = Cx$

If $x = 2, y = -2 \quad \Rightarrow C = 4$

So, $x^2 + y^2 - 4x = 0$, which is circle of centre (2, 0) and radius 2

$\therefore \quad$ Area above the x-axis $= 2\pi$

$\Rightarrow \quad k = 2$



Hence, the correct answer is (2).

19. The equation of tangent at any point $P(x, y)$ is $Y - y = \dfrac{dy}{dx}(X - x)$

As per given, $x - y\left(\dfrac{dy}{dx}\right) = 2x \Rightarrow -\dfrac{dy}{y} = \dfrac{dx}{x}$

By integrating, $xy = c$

By sub (1, 2) , $C = 2$

$\therefore \quad$ Equation of curve is $xy = 2$

If $y = 1/3$ then $x = 6$

Hence, the correct answer is (6).

20. We have $\dfrac{dy}{dx} = 2x + 1 \quad \Rightarrow y = x^2 + x + C$, is passes through (1, 2)

$\therefore \quad y = x^2 + x$

Required area $= \int_0^1 (x^2 + x)\, dx = \dfrac{5}{6}$ sq unit

$A = \dfrac{5}{6} \quad \Rightarrow \quad 5A = 5$

Hence, the correct answer is (5).

21. (h, k) centre

$(x-h)^2 + (y-k)^2 = r^2$

$2h + 3k = 5$

$(x-h)^2 + (y - \dfrac{5-2h}{3})^2 = r^2$

Only two parameters $\Rightarrow$ order = 2

Hence, the correct answer is (2).

22. Given equation can be written as

$2x^4ydy + y^2dy + 4x^3y^2\,dx - x^2dx = 0$

or $\quad 2x^2y(x^2dy + 2xy\,dx) + y^2dy - x^2dx = 0$

or $\quad 2x^2\,yd(x^2y) + y^2\,dy - x^2\,dx = 0$

After integration $x^4y^2 - \dfrac{x^3}{3} + \dfrac{y^3}{3} = c$

$3x^4y^2 - x^3 + y^3 = c$

$\lambda_1 x^4y^2 + \lambda_2 x^3 + \lambda_3 y^3 = c$

$\lambda_1 + \lambda_2 + \lambda_3 = 3$

Hence, the correct answer is (3).

23. $y = e^{4x} + 2e^{-x}$; $y_1 = 4e^{4x} - 2e^{-x}$; $y_2 = 16e^{4x} + 2e^{-x}$; $y_3 = 64e^{4x} - 2e^{-x}$

Now, $y_3 - 13y_1 = (64e^{4x} - 2e^{-x}) - 13\,(4e^{4x} - 2e^{-x}) = 12e^{4x} + 24e^{-x}$

$= 12\,(e^{4x} + 2e^{-x}) = 12y$

$\therefore \quad \dfrac{y_3 - 13y_1}{y} = 12 = 4k;\ k = 3$

Hence, the correct answer is (3).

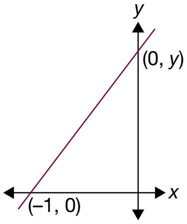
24. Slope of the normal at $P(x, y)$

$\dfrac{-1}{dy/dx} = x^2y \Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{x^2y} \Rightarrow y\,dy = \dfrac{-dx}{x^2}$

$\Rightarrow \quad \dfrac{y^2}{2} = \dfrac{1}{x} + c$ at $x = -2, y = 1 \Rightarrow c = 1$

$\Rightarrow \quad \dfrac{y^2}{2} = \dfrac{1}{x} + 1 \Rightarrow y^2 = \dfrac{2}{x} + 2 = f^2(x)$

$y = xf^2(x) = x\left(\dfrac{2}{x} + 2\right) = 2 + 2x$



Area bounded by $y = 2x + 2$ and coordinate axes

$A = \dfrac{1}{x} \times 1 \times 2 = 1.$

Hence, the correct answer is (1).

25. We have $2\int_1^x f(t)dt + x^k = xf(x)$ $\qquad$ …(1)

Differentiate both sides of equation (1) with respect to x, we get

$2f(x) + kx^{k-1} = xf'(x) + f(x)$

$\Rightarrow \quad xf'(x) - f(x) = k\,x^{k-1}$

$\Rightarrow \quad f'(x) - \dfrac{1}{x}f(x) = k\,x^{k-2} \quad \Rightarrow \dfrac{dy}{dx} - \dfrac{1}{x}y = k\,x^{k-2}$

Integrating factor = I.F. $= \dfrac{1}{x}$

So, $\dfrac{f(x)}{x} = k\int x^{k-3}\,dx \Rightarrow \dfrac{f(x)}{x} = \dfrac{k\cdot x^{k-2}}{k-2} + C$

Put $x = 1$ in (1), we get $f(1) = 1$

Now, $C = 1 - \dfrac{k}{(k-2)} \Rightarrow C = \dfrac{2}{(2-k)} \Rightarrow f(x) = \dfrac{k\,x^{k-1}}{k-2} + \dfrac{2x}{2-k}$

Since coefficients of $f(x)$ are integer, so $\dfrac{k}{k-2}$ and $\dfrac{2}{2-k}$ are integer

$\Rightarrow \quad 2 - k = \pm 1, \pm 2 \quad \Rightarrow k = 0, 1, 3, 4$

But $k \neq 0$ and at $k = 1, 3, 4$, $\dfrac{k}{k-2}$ is also integer hence possible values of k are 1, 3, 4 Hence, sum of possible values of $k = 1 + 3 + 4$.

Hence, sum of possible values of $k = 1 + 3 + 4 = 8$.

Hence, the correct answer is (8).

26. $\dfrac{dy}{dx} = 3x \Rightarrow y = \dfrac{3x^2}{2} + C$

$-3 = \dfrac{3}{2}(1)^2 + C \Rightarrow C = \dfrac{-9}{2}$

$\therefore \quad y = \dfrac{3}{2}(x^2 - 3) \Rightarrow 2y = 3x^2 + 0x - 9$

$\Rightarrow \quad c + b - a = 9 + 0 - 3 = 6.$

Hence, the correct answer is (6).

27. $f'(x) = f^2(x) \Rightarrow \dfrac{f'(x)}{f^2(x)} = 1 \Rightarrow \int \dfrac{f'(x)}{f^2(x)}\,dx = \int 1.dx + C$

$\Rightarrow \quad -\dfrac{1}{f(x)} = x + C$

At $x = 0, f(0) = -1 \Rightarrow C = 1 \quad \therefore f(x) = \dfrac{-1}{x+1}$

$$I=\frac{\sqrt{2}}{\pi}\int_0^1\left(\frac{1-x}{-\frac{1}{x+1}(x^4+1)}+\frac{2}{x^4+1}\right)dx$$

$$=\frac{\sqrt{2}}{\pi}\int_0^1\left(\frac{x^2-1}{x^4-1}+\frac{2}{x^4+1}\right)dx=\frac{\sqrt{2}}{\pi}\int_0^1\frac{x^2+1}{x^4+1}dx$$

$$=\frac{\sqrt{2}}{\pi}\int_0^1\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx=\frac{\sqrt{2}}{\pi}\int_0^1\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+2},\ x-\frac{1}{x}=t$$

$$\Rightarrow\quad\left(1+\frac{1}{x^2}\right)dx=dt$$

$$=\frac{\sqrt{2}}{\pi}\frac{at}{t^2+2}=\frac{\sqrt{2}}{\pi}\frac{1}{\sqrt{2}}\left(\tan^{-1}\left(\frac{t}{\sqrt{2}}\right)\right)_{-\infty}^{0}$$

$$=\frac{1}{\pi}\left(0-\left(\frac{-\pi}{2}\right)\right)=\frac{1}{2}=\frac{p}{q}\quad\therefore (p+q)=3.$$

Hence, the correct answer is (3).

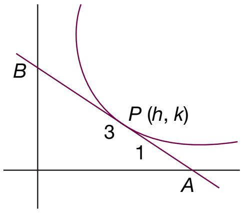
28. $(x-a)^2+(y-a)^2=a^2$ …(1)

$2(x-a)+2(y-a)y_1=0\Rightarrow a=\dfrac{x+yy_1}{1+y_1}$ …(2)

From (1) and (2) $\left(x-\dfrac{x+yy_1}{1+y_1}\right)^2+y-\left(\dfrac{x+yy_1}{1+y_1}\right)^2=\left(\dfrac{x+yy_1}{1+y_1}\right)^2$

Hence, the correct answer is (2).

29. $\dfrac{3x}{h}+\dfrac{y}{k}=1$ is the equation of the tangent whose slope is $-\dfrac{3k}{h}$



$\dfrac{dy}{dx}=-\dfrac{3y}{x}\Rightarrow x\dfrac{dy}{dx}+3y=0$

Solving $\dfrac{dy}{y}=-3\dfrac{dx}{x}$

$\Rightarrow\quad \log y=-3\log x+\log c$

$\Rightarrow\quad y=cx^{-3}\Rightarrow y=\dfrac{c}{x^3}$ which passes through (1, 1)

$\Rightarrow\quad c=1$

Equation of the curve is $y=\dfrac{1}{x^3}$

Hence, the correct answer is (2).

30. $\dfrac{dy}{dx}=\dfrac{y}{x^2}\Rightarrow\dfrac{dy}{y}=\dfrac{dx}{x^2}\Rightarrow\log y=-\dfrac{1}{x}+K$

$\Rightarrow\quad y=c\,e^{-\frac{1}{x}}$ which passes through $(1,3)$

$\Rightarrow\quad ce^{-1}=3\Rightarrow c=3e\quad\therefore y=e^{1-\frac{1}{x}}$ is the curve

Here $f(x)=3\,e^{1-\frac{1}{x}}$

$\Rightarrow\quad f'(x)=3e^{1-\frac{1}{x}}\left(\dfrac{1}{x^2}\right)>0\,\forall x\in R-\{0\}$

Also $\lim\limits_{x\to-\infty}f(x)=\lim\limits_{x\to\infty}f(x)=3e$

Hence, the correct answer is (3).

31. $\dfrac{dy}{dx}+\dfrac{y}{x}=x^2y^6\Rightarrow\dfrac{1}{y^6}\dfrac{dy}{dx}+\dfrac{1}{xy^5}=x^2$

Put $\dfrac{1}{y^5}=z\Rightarrow\dfrac{-5}{y^6}\dfrac{dy}{dx}=\dfrac{dz}{dx}\qquad\therefore-\dfrac{1}{5}\left(\dfrac{dz}{dx}\right)+\dfrac{z}{x}=x^2$

$\Rightarrow\quad\dfrac{dz}{dx}-\dfrac{5}{x}z=-5x^2$

$\text{I.F.}=e^{-\int\frac{5}{x}dx}=e^{-5\ln x}=\dfrac{1}{x^5}$

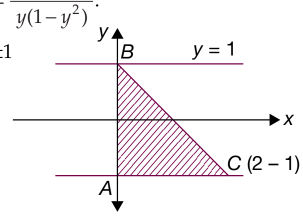
$$\frac{z}{x^5}=\int(-5x^2)\left(\frac{1}{x^5}\right)dx=-5\int\frac{1}{x^3}dx=\frac{5}{2}\frac{1}{x^2}+c$$

$\Rightarrow\quad\dfrac{1}{x^5y^5}=\dfrac{5}{2x^2}+c$

Hence, solution is $\dfrac{1}{x^5y^5}=\dfrac{5}{2x^2}+c$

Hence, the correct answer is (7).

32. $y(1-y^2)\,dx=x(y^2+1)\,dy\Rightarrow\dfrac{dx}{x}=\dfrac{(1+y^2)dy}{y(1-y^2)}.$



$\Rightarrow\quad x(y^2-1)=0\quad\Rightarrow x=0,\ y=\pm1$

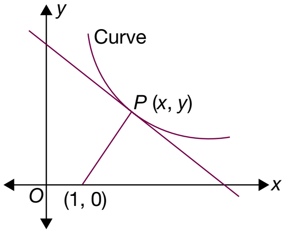
$f_3: x=0$ (y-axis)

$f_1: y=1;\quad f_2: y=-1$

Area of $\Delta ABC=\dfrac{1}{2}\times2\times2=2$

Hence, the correct answer is (2).

33. $Y-y=m(X-x)$



Now, $\left(\dfrac{y-0}{x-1}\right)m=-1$

$y\dfrac{dy}{dx}=1-x$

Integrating $\dfrac{y^2}{2}=x-\dfrac{x^2}{2}+C$

$x^2+y^2-2x=C$

This is the equation of a circle with centre (1, 0).

$\therefore\quad 2x+3y-15=0$ is tangent at (1)

$\therefore\quad$ Perpendicular from (1, 0) on the line $=r$

$\Rightarrow\quad\left|\dfrac{2-15}{\sqrt{13}}\right|=\sqrt{1+C}$

$C+1=13\quad\Rightarrow C=12$

Hence, the curve is $x^2+y^2-2x=12$

Length of tangent is $\sqrt{S_1}=\sqrt{3}=\sqrt{n}$

$\therefore\quad n=3.$

Hence, the correct answer is (3).

34. We have, $f''(x)=12x-4\quad\Rightarrow f'(x)=6x^2-4x+c$

As, $f'(1)=0\Rightarrow c=-2$

$\therefore\quad f'(x)=6x^2-4x-2\Rightarrow f(x)=2x^3-2x^2-2x+\lambda$

As, $f(1)=0\quad\Rightarrow\lambda=2$

Hence, $f(x)=2(x^3-x^2-x+1)$.

or $\quad f(x)=2(x-1)^2(x+1)$.

Now, $M(x=2,\ y=6)$ and $f'(2)=14$.

So, the equation of normal at M is $(y-6)=\dfrac{-1}{14}(x-2)$

For x-intercept, put $y=0$. $\therefore$ We get $x=86$.

Hence, the correct answer is (86).

35. $xy=c^2\Rightarrow$ Diff Eq. is $x\dfrac{dy}{dx}+y=0$

Replace $\dfrac{dy}{dx}$ with; $x\left(-\dfrac{dx}{dy}\right)+y=0$

$y=\dfrac{xdx}{dy}\Rightarrow\dfrac{y^2}{2}=\dfrac{x^2}{2}+c$

$$y^2 = x^2 + c_1 \text{ or } x^2 - y^2 = c$$

Hence, the correct answer is (0).

36. The given equation can be written as

$$\frac{dy}{dx} - y\frac{\phi'(x)}{\phi(x)} = \frac{-y^2}{\phi(x)}$$

$$\Rightarrow \quad -\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\frac{\phi'(x)}{\phi(x)} = \frac{1}{\phi(x)} \qquad \ldots(1)$$

Put $\frac{1}{y} = u$ so that $-\frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}$. Now, (1) becomes $\frac{du}{dx} + u\frac{\phi^1(x)}{\phi(x)} = \frac{1}{\phi(x)}$

This is a linear equation in u so its integrating factor is

$$e^{\int \frac{\phi^1(x)}{\phi(x)}} = e^{\log\phi(x)} = \phi(x)$$

Multiplying both sides by the integrating factor, we have

$$\frac{d}{dx}\left[u\phi(x)\right] = 1 \Rightarrow u\phi(x) = x + cons\tan t \Rightarrow y = \frac{\phi(x)}{x + C}$$

$$\Rightarrow \quad y(1) = \frac{1}{1 + C} \Rightarrow C = 0. \text{ Thus } y = \frac{\phi(x)}{x}$$

Hence, $y(4) = \frac{1296}{4} = 324$

Hence, the correct answer is (6).

37. $dy = x\tan y\,(\sin y - 1)dx$

$$\frac{dy}{dx} + x\tan y = x\tan y\sin y\,;\ \cos y\frac{dy}{dx} + x\sin y = x\sin^2 y$$

Let $\sin y = t$; $\frac{dt}{dx} + xt = xt^2$

$$\frac{1}{t^2}\frac{dt}{dx} + x\left(\frac{1}{t}\right) = x\,;\ \frac{1}{t} = u \Rightarrow -\frac{du}{dx} + xu = x$$

$$\frac{du}{dx} + (-x)u = -x\,;\ ue^{-\int xdx} = \int -xe^{-\int xdx}dx$$

$$ue^{-\frac{x^2}{2}} = -\int xe^{-\frac{x^2}{2}}dx = e^{-\frac{x^2}{2}} + c$$

$$u = 1 + ce^{\frac{x^2}{2}}\,;\ t = \frac{1}{1 + ce^{\frac{x^2}{2}}}$$

$$\sin y = \frac{1}{1 + ce^{\frac{x^2}{2}}}$$

Hence, the correct answer is (1).

38. $x^2dy + xydx = y^3(xdy - ydx) \quad \Rightarrow xd(xy) = y^3x^2d(y/x)$

$$\Rightarrow \quad \frac{d(xy)}{x^2y^2} = \frac{yd\left(\frac{y}{x}\right)}{x} \quad \Rightarrow \frac{y^2}{2x^2} + \frac{1}{xy} = C \text{ (}C\text{ is an integration constant)}$$

Put $x = 1 \Rightarrow C = \frac{y^3 + 2}{2y} = 0 \quad \Rightarrow y^3 = -2x$ is the function.

$$\Rightarrow \quad -8 = -2x \Rightarrow x = 4$$

Hence, the correct answer is (4).

39. $\tan y = t \quad \Rightarrow \sec^2 y\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} + 2xt = x^3;\ e^{\int 2xdx} = e^{x^2}$$

$$t.e^{x^2} = \frac{1}{2}\int 2x^2 : xe^{x^2};\quad \tan y\,e^{x^2} = \frac{1}{2}\int t.e^t\,dt$$

$$\tan y\,.\,e^{x^2} = \frac{1}{2}.\,t\,e^t - \frac{1}{2}e^t + c$$

$$\tan y\,e^{x^2} = \frac{1}{2}e^{x^2}(x^2 - 1) + c$$

Hence, the correct answer is (1).

40. $2xy\,dy - y^2dx = x^2dx \ \Rightarrow \frac{xd(y^2) - y^2dx}{x^2} = dx \Rightarrow d\left(\frac{y^2}{x}\right) = dx$

Solution: $\frac{y^2}{x} = x + k$ passing $(2,1) \Rightarrow k = -\frac{3}{2}$

Hence: $y^2 = x^2 - \frac{3}{2}x \Rightarrow \left(x - \frac{3}{4}\right)^2 - y^2 = \frac{9}{16} \Rightarrow e = \sqrt{2}, 2e^2 = 4$

Hence, the correct answer is (4).

41. Put $e^x = X$ and $e^y = Y$

So that $e^x dx = dX$ and $e^y dy = dY$

$$\Rightarrow \quad \frac{e^y}{e^x}\cdot\frac{dy}{dx} = \frac{dY}{dX} \quad \text{or } \frac{Y}{X}p = P \text{ (say)}$$

or $\quad p = \frac{X}{Y}\cdot P$

∴ The given equation becomes

$$X^3\left(\frac{X}{Y}P - 1\right) + \frac{X^3}{Y^3}.Y^2P^3 = 0$$

or $\quad XP - Y + P^3 = 0 \quad$ or $Y = PX + P^3$

Which is of Clairaut's form.∴ the solution is $Y = cX + c^3$

or $\quad e^y = ce^x + c^3$

Hence, the correct answer is (3).

42. Put $x = \frac{1}{X}$ and $y = \frac{1}{Y}$

$$dx = -\frac{1}{X^2}dX \text{ and } dy = -\frac{1}{Y^2}dY$$

$$\Rightarrow \quad p = \frac{dy}{dx} = \frac{X^2}{Y^2}\frac{dY}{dX} = \frac{X^2}{Y^2}P$$

The given equation becomes $\frac{1}{Y^2}\left(\frac{1}{Y} - \frac{1}{X}\cdot\frac{X^2}{Y^2}P\right) = \frac{1}{X^4}\cdot\frac{X^4}{Y^4}P^2$

$$\Rightarrow \quad Y - XP = P^2 \text{ or } Y = PX + P^2$$

Which is the Claraut's form

∴ The solution is $Y = cX + c^2$ or $\frac{1}{y} = \frac{c}{x} + c^2$

Hence, the correct answer is (2).

43. $\cos y\frac{dy}{dx} + \frac{4}{x}\sin y = \frac{e^x}{x^3}$

Put $\sin y = t \Rightarrow \cos y\frac{dy}{dx} = \frac{dt}{dx} \quad ∴ \quad \frac{dt}{dx} + \frac{4}{x}t = \frac{e^x}{x^3}$

$$\text{I.F.} = e^{\int 4/xdx} = e^{4\ln x} = x^4$$

$$\sin y\,x^4 = \int\frac{x^4e^x}{x^3}dx + c;\ \sin y\,x^4 = x\,.\,e^x - e^x + c$$

$$0 = e - e + c \Rightarrow c = 0$$

$\sin y = e^x(x - 1)x^{-4}$

Hence, the correct answer is (4).

44. $\frac{xdy - ydx}{y^2} = dy$

$$-\int d\left(\frac{x}{y}\right) = \int dy \Rightarrow -\frac{x}{y} = y + c$$

At $x = 1, y = 1, c = -2$

At $x = -3, \frac{3}{y} = y - 2 \Rightarrow y^2 - 2y - 3 = 0$

$\therefore \quad y = 3$

Hence, the correct answer is (3).

45. $\sin y.\dfrac{dy}{dx} - \cos x \cos y = -\cos x - \cos y = t \Rightarrow \sin y \dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$\frac{dt}{dx} + \cos x\, t = -\cos x$$

$$t\,.\,e^{\sin x} = -\int \cos x e^{\sin x}\, dx$$

$$t\,.\,e^{\sin x} = -1 + C\, e^{\sin x} \quad \Rightarrow C = 1$$

$$e^{\sin x} + \cos y = 1$$

Hence, the correct answer is (1).

46. $\dfrac{xdx}{dy} - \dfrac{x^2}{y} = -y^3$

Put $x^2 = t$

$$\frac{dt}{dy} - \frac{2}{y}t = -2y^3$$

$$\frac{dt}{dy} - \frac{2}{y}\, t = -2y^3$$

$$t.e^{-2\ln y} = -\int 2y^3 e^{-2\ln y}\, dy + C$$

$$\frac{t}{y^2} = -2\int y\, dy + C$$

$$\frac{t}{y^2} = -y^2 + C \quad \Rightarrow x^2 + y^4 = Cy^2 \quad \Rightarrow C = 4$$

$$x^2 + y^4 = 4y^2$$

Hence, the correct answer is (4).

47. $y = ax^2$ …(1)

$$\frac{dy}{dx} = 2ax \qquad \text{…(2)}$$

$$\frac{(2)}{(1)} \Rightarrow \frac{dy/dx}{y} = \frac{2}{x} \Rightarrow -\frac{1}{y\dfrac{dy}{dx}} = \frac{2}{x}$$

$$\Rightarrow \quad 2ydy = -xdx \Rightarrow y^2 = -\frac{x^2}{2} + C$$

Hence, the correct answer is (4).

48. $(f'(x) + g'(x)) + (f(x) + g(x)) = e^x + x$ Let $f(x) + g(x) = t$

$$\frac{dt}{dx} + t = (e^x + x)$$

$$(f(x) + g(x)e^x = \frac{e^{2x}}{2} + (xe^x - e^x) + C$$

Put $x = 0$

$$f(0) + g(0) = \frac{1}{2} - 1 + C \Rightarrow c = 0$$

$$f(x) + g(x) = \frac{e^x}{2} + (x-1);\, f(1) + g(1) = \frac{e}{2} = 1.4$$

Hence, the correct answer is (1).

49. $e^x f(x) + e^x f'(x) \le 2e^x \Rightarrow (e^x f(x))' \le 2e^x \Rightarrow e^2 f(2) \le 2(e^2 - 1)$

$\Rightarrow \quad f(2) \le 2 - \dfrac{3}{e^2}$

$\therefore \quad M = 2 - \dfrac{2}{e^2}$

$\therefore \quad [M] = 1$

Hence, the correct answer is (1).

50. $\dfrac{d^2y}{dx^2} = y$

$$y = Ae^x + Be^{-x}$$
$$A + B = 2$$
$$A - B = 3$$

On solving $y = \dfrac{1}{2}\left(5e^x - e^{-x}\right); f(\ln 2) = \dfrac{1}{2}\left(5 \times 2 - \dfrac{1}{2}\right) = 5 - \dfrac{1}{4}$

Hence, the correct answer is (4).

51. $f(x) = 0$ has 3 distinct real roots

$\Rightarrow \quad f'(x) = 0$ has atleast 2 distinct real roots.

$\Rightarrow \quad (f(x).\, f'(x))' = 0$ has atleast 4 distinct real roots.

Hence, the correct answer is (4).

52. $x^2dy + 2xydx = 3x^2dx + dy \quad \Rightarrow \quad d(x^2y) = d(x^3 + y)$

$$x^2y = x^3 + y + c$$

$\Rightarrow \quad c = 0 \quad (\because$ curve passes through 0, 0)

When $x = 2, y = a$

$\Rightarrow \quad 4a = 8 + a \Rightarrow 3a = 8$

Hence, the correct answer is (8).

ARCHIVE: JEE MAIN

1. $\dfrac{dP}{dt} = 0.5P - 450$

$$\Rightarrow \quad \int_0^t \frac{dP}{P - 900} = \int_0^t \frac{dt}{2} \quad \Rightarrow \left[\ln|P(t) - 900|\right]_0^1 = \left[\frac{t}{2}\right]_0^1$$

$$\Rightarrow \quad \ln|P(t) - 900| - \ln|P(0) - 900| = \frac{t}{2}$$

$$\Rightarrow \quad \ln|P(t) - 900| - \ln 50 = \frac{t}{2}$$

For $P(t) = 0$

$$\Rightarrow \quad \ln\left|\frac{900}{50}\right| = \frac{t}{2} \quad \Rightarrow t = 2\ln 18$$

Hence, the correct answer is (A).

2. $f(x)f''(x) - (f'(x))^2 = 0 \Rightarrow \dfrac{f''(x)}{f'(x)} = \dfrac{f'(x)}{f(x)}$

$$\ln(f'(x)) = \ln f(x) + \ln c$$

$$\Rightarrow \quad f'(x) = cf(x) \Rightarrow \frac{f'(x)}{f(x)} = c$$

$$\ln f(x) = cx + k_1$$
$$f(x) = ke^{cx}$$
$$f(0) = 1 = k;\, f'(0) = c = 2$$
$$f(x) = e^{2x}$$
$$f(1) = e^2 \in (6, 9)$$

Hence, the correct answer is (C).

3. $\dfrac{dy}{dx} + \dfrac{y}{x} = bx^3 \quad$ I.F. $= e^{\frac{1}{x}dx} = x$

So, solution of D.E. is given by $y.x = \int b.x^3.x\, dx + c$

$$y = \frac{c}{x} + \frac{bx^4}{5} \text{ passes through } (1, 2)$$

$$2 = c + \frac{b}{5} \qquad \text{…(1)}$$

$$\int_1^2 f(x)dx = \frac{62}{5} \Rightarrow \left[c\ln x + \frac{bx^5}{25}\right]_1^2 = \frac{62}{5}$$

$$C\ln 2 + \frac{32b}{25} - \frac{b}{25} = \frac{62}{5};$$

$$C\ln 2 + \frac{31b}{25} = \frac{62}{5} \qquad \text{…(2)}$$

By equation (1) and (2) $\Rightarrow c = 0$ and $b = 10$

Hence, the correct answer is (B).

4. Given $y(0) = 0$ and $\dfrac{dy}{dx} = \dfrac{(x-2)^2 + y + 4}{x-2}$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x-2} = (x-2) + \frac{4}{x-2}$$

$$\Rightarrow \quad \text{I.F.} = e^{-\int \frac{1}{x-2}dx} = \frac{1}{x-2}$$

Solution of L.D.E.

$$\Rightarrow \quad y\frac{1}{x-2} = \int \frac{1}{x-2}\left((x-2) + \frac{4}{x-2}\right).dx$$

$$\Rightarrow \quad \frac{y}{x-2} = x - \frac{4}{x-2} + C$$

Now, at $x = 0$, $y = 0 \Rightarrow C = -2$

$$y = x\,(x-2) - 4 - 2\,(x-2) \Rightarrow y = x^2 - 4x$$

This curve passes through (5, 5)

Hence, the correct answer is (D).

5. $\dfrac{dB}{dt} = \lambda B \quad \Rightarrow \displaystyle\int_{1000}^{1200} \frac{dB}{B} = \lambda \int_0^2 dt \quad \Rightarrow \lambda = \frac{1}{2}\ln\left(\frac{6}{5}\right)$

$$\int_{1000}^{2000} \frac{dB}{B} = \frac{1}{2}\ln\left(\frac{6}{5}\right)\int_0^T dt \quad \Rightarrow T = \frac{2\ln 2}{\ln\left(\frac{6}{5}\right)}$$

$$\Rightarrow \quad k = 2\ln 2$$

Hence, the correct answer is (A).

6. $\dfrac{dy}{dx} = \dfrac{xy^2 + y}{x}$

$$\Rightarrow \quad \frac{xdy - ydx}{y^2} = x\,dx;$$

$$\Rightarrow \quad -d\left(\frac{x}{y}\right) = x\,dx$$

$$-\frac{x}{y} = \frac{x^2}{2} + c$$

$\because$ curve intersects the line $x + 2y = 4$ at

$x = -2 \Rightarrow$ point of intersection is $(-2, 3) \Rightarrow \dfrac{2}{3} = 2 + c \Rightarrow c = -\dfrac{4}{3}$

$$\Rightarrow \quad \frac{-x}{y} = \frac{x^2}{2} - \frac{4}{3}$$

Now, put $(3, y)$ $\dfrac{-3}{y} = \dfrac{19}{6} \Rightarrow y = \dfrac{-18}{19}$

Hence, the correct answer is (D).

7. $\dfrac{dy}{dx} + 2y\tan x = \sin x$

$\text{I.F.} = e^{\int 2\tan x\,dx} = e^{2\ln\sec x}$

$\text{I.F.} = \sec^2 x$

$$y.(\sec^2 x) = \int \sec x.\sec^2 x\,dx;\ y.(\sec^2 x) = \int \sec x.\tan^2 x\,dx$$

$$y.(\sec^2 x) = \sec x + C$$

$$x = \frac{\pi}{3};\ y = 0 \Rightarrow C = -2$$

$$\Rightarrow \quad y = \frac{\sec x - 2}{\sec^2 x} = \cos x - 2\cos^2 x$$

$$y = t - 2t^2 \Rightarrow \frac{dy}{dx} = 1 - 4t = 0 \Rightarrow t = \frac{1}{4}$$

$$\therefore \quad \max = \frac{1}{4} - \frac{1}{8} = \frac{2-1}{8} = \frac{1}{8}$$

Hence, the correct answer is (D).

8. $\dfrac{dy}{dx} + (\tan x)\,y = \sin x\,;\ 0 \le x \le \dfrac{\pi}{3}$

$\text{I.F.} = e^{\int \tan x dx} = e^{\ln\sec x} = \sec x$

$$y\sec x = \int \tan x\,dx + c \quad \Rightarrow y\sec x = \ln|\sec x| + c$$

Using $y(0) = 0$; $c = 0 \quad \Rightarrow y = \cos x \cdot \ln|\sec x|$

$$y\Big|_{x=\frac{\pi}{4}} = \frac{1}{2\sqrt{2}}\log_c 2$$

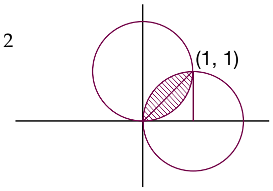
Hence, the correct answer is (B).

9. $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy},\ x \in (0, \infty)$

$$\Rightarrow \quad 2xy\,dy - y^2dx = -x^2\,dx \Rightarrow 2\,(y/x)\,dy - (y^2/x^2)\,dx = -dx$$

$$\Rightarrow \quad \int d\left(\frac{y^2}{x}\right) = \int -dx \Rightarrow y^2/x = -x + c$$

$$\Rightarrow \quad y^2 = -x^2 + cx;\ x > 0$$



Point (1, 1) lies on it $1 = -1 + c \Rightarrow c = 2$

$x^2 + y^2 - 2x = 0$ (Curve C_1)

Similarly, $\dfrac{dy}{dx} = \dfrac{2xy}{x^2 - y^2}$

Put $y = vx$

We get $x^2 + y^2 - 2y = 0$ (Curve C_1)

$$\text{Required area} = 2\left(\frac{\pi(1)^2}{4} - \frac{1}{2}\times 1\times 1\right) = \frac{\pi}{2} - 1$$

Hence, the correct answer is (B).

10. $\dfrac{dy}{dx} = (1 + y)\,(x - 1)$

$$\frac{dy}{(y+1)} = (x-1)\,dx$$

Integrate $\ln(y + 1) = \dfrac{x^2}{2} - x + c$

$$(0, 0) \Rightarrow c = 0 \Rightarrow y = e^{\left(\frac{x^2}{2} - x\right)} - 1$$

Hence, the correct answer is (A).

11. Put $e^{\sin y} = t$

$$e^{\sin y}\cos y\frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \text{D.E is } \frac{dt}{dx} + t\cos x = \cos x$$

$\text{I.F.} = e^{\int \cos x dx} = e^{\sin x}$

$$t.e^{\sin x} = \int \cos x e^{\sin x}\,dx$$

$\Rightarrow \quad e^{\sin y}\,e^{\sin x} = e^{\sin x} + c \qquad \because x = 0, y = 0 \quad \Rightarrow c = 0$

$$\Rightarrow \quad e^{\sin y} = 1 \quad \Rightarrow y = 0$$

$$\Rightarrow \quad 1 + y\left(\frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}y\left(\frac{\pi}{3}\right) + \frac{1}{\sqrt{2}}y\left(\frac{\pi}{4}\right) = 1$$

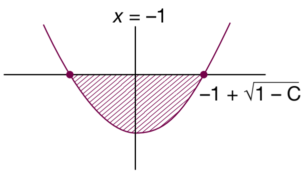
Hence, the correct answer is (1).

12. $\dfrac{dy}{dx} = 2\,(x + 1)$

$$\Rightarrow \quad \int dy = \int 2(x+1)\,dx$$

$$\Rightarrow \quad y(x) = x^2 + 2x + C$$

$$\text{Area} = \frac{4\sqrt{8}}{3} - 1 + \sqrt{1 - C}$$



$$\Rightarrow \quad 2\int_{-1}^{-1+\sqrt{1-C}} (-(x+1)^2 - C + 1)\, dx = \frac{4\sqrt{8}}{3}$$

$$\Rightarrow \quad 2\left[-\frac{(x+1)^3}{3} - Cx + x\right]_{-1}^{-1+\sqrt{1-C}} = \frac{4\sqrt{8}}{3}$$

$$\Rightarrow \quad -\frac{\left(\sqrt{1-C}\right)^3}{3} - C\sqrt{1-C} + \sqrt{1-C} + C - 1 - C + 1 = \frac{2\sqrt{8}}{3}$$

$$\Rightarrow \quad -\frac{(1-C)^{3/2}}{3} + (1-c)^{3/2} = \frac{2\sqrt{8}}{3} \Rightarrow C = -1$$

$\Rightarrow \quad f(x) = x^2 + 2x - 1, f(1) = 2$

Hence, the correct answer is (2).

13. $xdy - ydx = \sqrt{x^2 - y^2}\; dx$

$$\Rightarrow \quad \frac{xdy - ydx}{x^2} = \frac{1}{x}\sqrt{1 - \frac{y^2}{x^2}}\; dx$$

$$\Rightarrow \quad \int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1-\left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x} \Rightarrow \sin^{-1}\left(\frac{y}{x}\right) = \ln|x| + c$$

at $x = 1, y = 0 \Rightarrow c = 0$

$y = x \sin(\ln x)$

$$A = \int_1^{e^n} x\sin(\ln x)\; dx$$

$$x = e^t, dx = e^t dt \quad \Rightarrow \quad e^t dt \quad \Rightarrow \quad \int_0^{\pi} e^{2t}\sin(t)\; dt = A$$

$$ae^{2\pi} + \beta = \left(\frac{e^{2t}}{5}(2\sin t - \cos t)\right)_0^{\pi} = \frac{1+e^{2\pi}}{5}$$

$\alpha = \frac{1}{5}, \beta = \frac{1}{5}$ so $10(\alpha + \beta) = 4$

Hence, the correct answer is (4).

14. $\cos\left(\frac{1}{2}\cos^{-1}(e^{-x})\right) dx = \sqrt{e^{2x} - 1}\; dy$

Put $\cos^{-1}(e^{-x}) = \theta$, $\theta \in [0, \pi]$; $\cos\theta = e^{-x} \Rightarrow 2\cos^2\frac{\theta}{2} - 1 = e^{-x}$

$$\cos\frac{\theta}{2} = \sqrt{\frac{e^{-x}+1}{2}} = \sqrt{\frac{e^{-x}+1}{2}};\; \sqrt{\frac{e^x+1}{2e^x}}\; dx = \sqrt{e^{2x}-1}\; dy$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{e^x}\sqrt{e^x - 1}} = \int dy$$

$$= \frac{1}{\sqrt{2}}\int \frac{e^x\, dx}{e^x\sqrt{e^x}\sqrt{e^x - 1}} = \int dy$$

Put $e^x = t, \frac{dt}{dx} = e^x$;

$$\Rightarrow \quad \int \frac{dt}{t\sqrt{t^2 - t}} = \sqrt{2}\; y$$

Put $t = \frac{1}{z}, \frac{dt}{dz} = -\frac{1}{z^2}$

$$\Rightarrow \quad \int \frac{-\frac{dz}{z^2}}{\frac{1}{z}\sqrt{\frac{1}{z^2} - \frac{1}{z}}} = -\int \frac{dz}{\sqrt{1-z}} = \sqrt{2}y$$

$$\Rightarrow \quad \frac{-2(1-z)^{1/2}}{-1} = \sqrt{2}y + c \quad \Rightarrow 2\left(1 - \frac{1}{t}\right)^{1/2} = \sqrt{2}y + c$$

$\Rightarrow \quad 2(1 - e^{-x})^{1/2} = \sqrt{2}y + c \quad \Rightarrow c = \sqrt{2}$

Put $x = 0; y = -1$

$2(1 - e^{-x})^{1/2} = \sqrt{2}(y + 1)$, passes through $(\alpha, 0)$

$2(1 - e^{-\alpha})^{1/2} = \sqrt{2}$

$$\sqrt{1 - e^{-\alpha}} = \frac{1}{\sqrt{2}} \quad \Rightarrow 1 - e^{-\alpha} = \frac{1}{2}$$

$$e^{-\alpha} = \frac{1}{2} \quad \Rightarrow e^{\alpha} = 2$$

Hence, the correct answer is (2).

15. $2x^2dy = (2xy + y^2)\, dx \Rightarrow \frac{dy}{dx} = \frac{2xy + y^2}{2x^2}$ (Homogeneous D.E.)

$$\Rightarrow \quad t + x\frac{dt}{dx} = t + \frac{t^2}{2} \;\text{ let } y = xt \Rightarrow \frac{dy}{dx} = t + x\frac{dt}{dx}$$

$$\Rightarrow \quad x\frac{dt}{dx} = \frac{t^2}{2} \Rightarrow 2\int\frac{dt}{t^2} = \int\frac{dx}{x} \Rightarrow 2\left(-\frac{1}{t}\right) = \ln(x) + C\left\{\text{Put } t = \frac{y}{x}\right\}$$

$$\Rightarrow \quad -\frac{2x}{y} = \ln(x) + C$$

It passes through (1, 2) $C = -1$

$$\Rightarrow \quad y = f(x) = \frac{2x}{1 - \ln x}$$

$$\text{Now, } f\left(\frac{1}{2}\right) = \frac{1}{1 + \log_e 2}$$

Hence, the correct answer is (B).

16. $x^3dy + xy\, dx = x^2\, dy + 2y\, dx \Rightarrow (x^3 - x^2)\, dy = (2y - xy)\, dx$

$$\Rightarrow \quad -\int\frac{dy}{y} = \int\frac{x-2}{x^2(x-1)}\, dx = \int\left(\frac{1}{x} + \frac{2}{x^2} - \frac{1}{x-1}\right)dx$$

$$\Rightarrow \quad -\ln y = \ln x - \frac{2}{x} - \ln(x-1) + \lambda \text{ given } y(2) = e$$

$-1 = \ln 2 - 1 - 0 + \lambda \quad \therefore \lambda = -\ln 2$

$$\Rightarrow \quad \ln y = -\ln x + \frac{2}{x} + \ln(x-1) + \ln 2$$

Now, put $x = 4$ in equation

$$\Rightarrow \quad \ln y = -\ln 4 + \frac{1}{2} + \ln 3 + \ln 2 = \ln\left(\frac{3}{2}\right) + \frac{1}{2}\ln e \Rightarrow y = \frac{3}{2}\sqrt{e}$$

Hence, the correct answer is (B).

17. $(1 + e^{-x})(1 + y^2)\frac{dy}{dx} = y^2 \Rightarrow \frac{(1+y^2)}{y^2}dy = \left(\frac{e^x}{1+e^x}\right)dx$

$$\Rightarrow \quad \left(y - \frac{1}{y}\right) = \ln(1 + e^x) + c$$

$\therefore$ It passes through (0, 1)

$\Rightarrow \quad c = -\ln 2$

$$\Rightarrow \quad y^2 = 1 + y\ln\left(\frac{1+e^x}{2}\right)$$

Hence, the correct answer is (A).

18. $\ln(y + 3x) = z$ (let)

$$\Rightarrow \quad \frac{1}{y+3x}\cdot\left(\frac{dy}{dx} + 3\right) = \frac{dz}{dx} \qquad \ldots(1)$$

$$\frac{dy}{dx} + 3 = \frac{y+3x}{\ln(y+3x)} \quad \text{(Given)}$$

$$\Rightarrow \quad \frac{dz}{dx} = \frac{1}{z} \Rightarrow z\, dz = dx \Rightarrow \frac{z^2}{2} = x + C$$

$$\Rightarrow \quad \frac{1}{2}\ln^2(y + 3x) = x + C \Rightarrow x - \frac{1}{2}(\ln(y+3x))^2 = C$$

Hence, the correct answer is (C).

19. $x\frac{dy}{dx} - y = x^2(x\cos x + \sin x),\ x > 0$

$\frac{dy}{dx} - \frac{y}{x} = x(x\cos x + \sin x) \Rightarrow \frac{dy}{dx} + Py = Q$

So, I.F. $= e^{\int -\frac{1}{x}dx} = \frac{1}{|x|} = \frac{1}{x}\ (x > 0)$

Thus, $\frac{y}{x} = \int \frac{1}{x}(x(x\cos x + \sin x))\,dx$

$\Rightarrow \quad \frac{y}{x} = x\sin x + C$

$\because \quad y(\pi) = \pi \Rightarrow C = I$

So, $y = x^2\sin x + x \Rightarrow y\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4} + \frac{\pi}{2}$

Also, $\frac{dy}{dx} = x^2\cos x + 2x\sin x + 1 \Rightarrow \frac{d^2y}{dx^2} = -x^2\sin x + 4x\cos x + 2\sin x$

$\Rightarrow \quad y''\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{4} + 2$

Thus, $y''\left(\frac{\pi}{2}\right) + y\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + 2$

Hence, the correct answer is (A).

20. $\cos x\frac{dy}{dx} + 2y\sin x = \sin 2x \Rightarrow \frac{dy}{dx} + \frac{2\sin x}{\cos x}y = 2\sin x$

I.F. $= e^{\int 2\frac{\sin x}{\cos x}dx} = e^{2\ln\sec x} = \sec^2 x$

$y\sec^2 x = \int 2\sin x \cdot \sec^2 x\,dx \Rightarrow y\sec^2 x = 2\int \tan x\sec x\,dx$

$\Rightarrow \quad y\sec^2 x = 2\sec x + c$

At $x = \frac{\pi}{3}$, $y = 0$ we have $0 = 2\sec\frac{\pi}{3} + C \Rightarrow C = -4$

$y\sec^2 x = 2\sec x - 4$

Put $x = \frac{\pi}{4}$ we have $y \cdot 2 = 2\sqrt{2} - 4 \Rightarrow y = \sqrt{2} - 2$

Hence, the correct answer is (A).

21. $\frac{(5+e^x)}{2+y}\frac{dy}{dx} = -e^x \Rightarrow \int \frac{dy}{2+y} = \int \frac{-e^x}{e^x+5}dx$

$\ln(y+2) = -\ln(e^x+5) + k \Rightarrow (y+2)(e^x+5) = C$

$\because \quad y(0) = 1 \Rightarrow C = 18$

$y + 2 = \frac{18}{e^x+5}$

At $x = \ln 13$ we have $y + 2 = \frac{18}{13+5} = 1 \Rightarrow y = -1$

Hence, the correct answer is (B).

22. $x\frac{dy}{dx} + 2y = x^2 \Rightarrow \frac{dy}{dx} + \frac{2y}{x} = x$

I.F. $= e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$y.x^2 = \int x.x^2.dx + c \Rightarrow yx^2 = \frac{x^4}{4} + c \qquad y(1) = 1$

$1.1 = \frac{1}{4} + c \Rightarrow c = 1 - \frac{1}{4} \Rightarrow c = \frac{3}{4}$

$yx^2 = \frac{x^4}{4} + \frac{3}{4}$

$y.\frac{1}{4} = \frac{1}{4}.\frac{1}{16} + \frac{3}{4} = \frac{49}{16}$

Hence, the correct answer is (B)

23. If $f(xy) = f(x)f(y)\ \forall\ x, y \in R$ and $f(0) \neq 0$ put $x = y = 0$

$\Rightarrow \quad f(0) = [f(0)]^2$

$\Rightarrow \quad f(0) = 1$

Put $y = 0 \Rightarrow f(0) = f(x)f(0) \Rightarrow f(x) = 1$

given that $\frac{dy}{dx} = f(x)$

$\therefore \quad \frac{dy}{dx} = 1 \Rightarrow y = x + k$

given that $y(0) = 1$

$\therefore \quad k = 1$ hence $y = x + 1$

$y\left(\frac{1}{4}\right) + y\left(\frac{3}{4}\right) = \left(\frac{1}{4}+1\right) + \left(\frac{3}{4}+1\right) = 3$

Hence, the correct answer is (B)

24. $\frac{dy}{dx} + 3\sec^2 xy = \sec^2 x$

$x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$

IF $= e^{\int 3\sec^2 x dx} = e^{3\tan x}$

$ye^{3\tan x} = \int e^{3\tan x}\sec^2 x dx$

$\Rightarrow \quad ye^{3\tan x} = \int e^t\frac{dt}{3} \quad$ put $3\tan x = t \Rightarrow 3\sec^2 x dx = dt$

$\Rightarrow \quad ye^{3\tan x} = \frac{e^1}{3} + C = \frac{e^{3\tan x}}{3} + C$

$\Rightarrow \quad y\left(\frac{\pi}{4}\right) = \frac{4}{3} \Rightarrow \frac{4}{3}e^3 = \frac{e^3}{3} + C \Rightarrow C = e^3$

$\Rightarrow \quad y\left(\frac{-\pi}{4}\right)e^{-3} = \frac{e^{-3}}{3} \Rightarrow y\left(\frac{-\pi}{4}\right) = \frac{1}{3} + e^6$

Hence, the correct answer is (A)

25. $(x^2 - y^2)dx + 2xydy = 0 \Rightarrow \frac{dy}{dx} = \frac{-(x^2-y^2)}{2xy}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + \frac{xdv}{dx}$

$\Rightarrow \quad v + \frac{xdv}{dx} = \frac{-(x^2 - v^2x^2)}{2x(vx)} \Rightarrow v + \frac{xdv}{dx} = \frac{-1+v^2}{2v}$

$\Rightarrow \quad x\frac{dv}{dx} = \frac{-1+v^2}{2v} - v \Rightarrow x\frac{dv}{dx} = \frac{-1-v^2}{2v}$

$\Rightarrow \quad \int \frac{2vdv}{1+v^2} = -\int \frac{dy}{x} \Rightarrow \log(1+v^2) = -\log x + \log C$

$\Rightarrow \quad l + \frac{y^2}{x^2} = \frac{C}{x} \Rightarrow x^2 + y^2 = Cx$

it passes through (1, 1)

$\therefore \quad C = 2$

$\therefore \quad$ Curve $x^2 + y^2 - 2x = 0$

Hence, the correct answer is (C)

26. $f'(x) = 7 - \frac{3}{4}\frac{f(x)}{x}\ (x > 0) \Rightarrow \frac{dy}{dx} + \frac{3}{4}\frac{y}{x} = 7$ (linear diff. equation)

I.F. $= e^{\int \frac{3}{4x}dx} = e^{\int \frac{3}{4}\log|x|} = x^{3/4}$

$yx^{3/4} = \int 7.x^{3/4}dx \Rightarrow yx^{3/4} = 7\frac{x^{7/4}}{(7/4)} + C$

$\Rightarrow \quad f(x) = y = 4x + Cx^{-3/4} \Rightarrow f\left(\frac{1}{x}\right) = \frac{4}{x} + Cx^{3/4}$

$\lim_{x\to 0^+} xf\left(\frac{1}{x}\right) = \lim_{x\to 0^+}(4 + C.x^{7/4}) = 4$

Hence, the correct answer is (C)

27. $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{2x}, (x > 0)$

$$\text{I.F} = e^{\int \frac{2x+1}{x}dx} = e^{\int\left(2+\frac{1}{x}\right)dx}$$

$$e^{2x + \ln x} = e^{2x}.e^{\ln x} = xe^{2x}$$

Solution given by

$$y.(xe^{2x}) = \int e^{-2x}x.e^{2x}dx + c \Rightarrow yxe^{2x} = \frac{x^2}{2} + c$$

at $x = 1$, $\frac{1}{2}e^{-2} \times 1 \times e^2 = \frac{1}{2} + c \Rightarrow c = 0$

$$yxe^{2x} = \frac{x^2}{2} \text{ Þ} y = \frac{xe^{-2x}}{2}$$

$$\frac{dy}{dx} = \frac{1}{2}e^{-2x}(-2x + 1)$$

$f(x)$ is decreasing in $\left(\frac{1}{2}, 1\right)$

Hence, the correct answer is (D)

28. $\frac{dy}{dx} = (x - y)^2$

Put $x - y = v \Rightarrow 1 - \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$

$$\Rightarrow \quad 1 - \frac{dv}{dx} = v^2 \Rightarrow \frac{dv}{dx} = 1 - v^2 \Rightarrow \frac{dv}{1-v^2} = dx \Rightarrow \int \frac{dv}{1-v^2} = \int dx$$

$\Rightarrow \quad \frac{1}{2\times1}\log\left|\frac{1+v}{1-v}\right| = x + c$ where $x - y = v$

$$\frac{1}{2}\log\left|\frac{1+x-y}{1-x+y}\right| = x + c \qquad \ldots(1)$$

$y(1) = 1$ at $x = 1, y = 1$; $\frac{1}{2}\log\left|\frac{1}{1}\right| = 1 + C \Rightarrow C = -1$

From (1) $\frac{1}{2}\log\left|\frac{1+x-y}{1-x+y}\right| = x - 1 \Rightarrow \log\left|\frac{1+x-y}{1-x+y}\right| = 2(x - 1)$

$$\Rightarrow \quad -\log\left|\frac{1-x+y}{1+x-y}\right| = 2(x - 1)$$

Hence, the correct answer is (D)

29. $\sin x\frac{dy}{dx} + y\cos x = 4x$

$\therefore \quad \frac{dy}{dx} + y \cdot \cot x = \frac{4x}{\sin x}$

This is a linear differential equation.

$$\text{I.F.} = e^{\int Pdx} = e^{\log(\sin x)} = \sin x$$

Therefore, solution is $y \sin x = \int 4x\, dx + C$

$\therefore \quad y \sin x = 2x^2 + C$

Given that $y(\pi/2) = 0$.

$\therefore \quad 0 \times \sin \pi/2 = 2 \times (\pi/2)^2 + C$ or $C = -\pi/2^2$

$\therefore \quad y \sin x = 2x^2 - \pi^2/2$

Putting $x = \pi/6$, we get $y/2 = 2(\pi/6)^2 - \pi^2/2$

$\therefore \quad y = -8\pi^2/9$

Hence, the correct answer is (D)

30. We have $(2 + \sin x)\frac{dy}{dx} + (y + 1)\cos x = 0$

$\Rightarrow \quad \frac{d}{dx}[(2 + \tan x)(y + 1)] = 0$

$\Rightarrow \quad (2 + \sin x)(y + 1) = c$

Since $y(0) = 1$, $c = 4$

So, $y + 1 = \frac{4}{2+\sin x}$

$\therefore \quad y\left(\frac{\pi}{2}\right) = \frac{4}{3} - 1 = \frac{1}{3}$

Hence, the correct answer is (B)

31. $y(1 + xy)dx = xdy$

$\Rightarrow \quad ydx - xdy + xy^2dx = 0 \Rightarrow y^2d\left(\frac{x}{y}\right) + xy^2dx = 0$

$$\Rightarrow \quad \frac{x}{y} + \frac{x^2}{2} = C \qquad \ldots(1)$$

Curve passes through (1, –1)

$\Rightarrow \quad -1 + \frac{1}{2} = C \qquad$ (from (1))

$\Rightarrow \quad C = -\frac{1}{2} \Rightarrow \frac{x}{y} + \frac{x^2}{2} = -\frac{1}{2}$

Now put $x = -\frac{1}{2} \Rightarrow \frac{-1/2}{y} + \frac{1/4}{2} = -\frac{1}{2}$

$\Rightarrow \quad \frac{-1}{2y} = \frac{-1}{2} - \frac{1}{8} \Rightarrow y = \frac{4}{5}$

Hence, the correct answer is (C)

32. We have $\frac{dy}{dx} + \frac{y}{x\log x} = 2$

$$\text{I.F.} = e^{\int\frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$$

Now solution is $y(\log x) = \int 2(\log x)\, dx$

$\Rightarrow \quad y(\log x) = 2[x\log x - x] + c$

when $x = 1$, $c = 2$,

when $x = e$ then $y = 2(e - e) + 2$ or $y = 2$

Hence, the correct answer is (C)

33. $\frac{dp}{dt} = \frac{p-400}{2} \Rightarrow \frac{dp}{p-400} = \frac{1}{2}dt$

Integrating, we get $\ln|p - 400| = \frac{1}{2}t + c$

When $t = 0$, $p = 100$, we have $\ln 300 = c$

$\therefore \quad \ln\left|\frac{p-400}{300}\right| = \frac{t}{2} \Rightarrow |p - 400| = 300\,e^{t/2}$

$\Rightarrow \quad 400 - p = 300\,e^{t/2} \qquad$ (as $p < 400$)

$\Rightarrow \quad p = 400 - 300\,e^{t/2}$

Hence, the correct answer is (A)

ARCHIVE: JEE ADVANCED

Single Correct Choice Type Questions

1. Given differential equation

$$dy = \left(\frac{1}{8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)}\right)dx$$

Let $4+\sqrt{9+\sqrt{x}} = t \Rightarrow \frac{1}{2\sqrt{9+\sqrt{x}}}\cdot\frac{1}{2\sqrt{x}}dx = dt$

$$\Rightarrow \quad dy = \frac{dt}{2\sqrt{t}} \quad \Rightarrow \int dy = \int \frac{1}{2\sqrt{t}}dt \Rightarrow y = \sqrt{t} + c$$

Required curve is $y = \sqrt{4+\sqrt{9+\sqrt{x}}} + c$; $y(0) = \sqrt{7} \Rightarrow c = 0$

$$y = \sqrt{4+\sqrt{9+\sqrt{x}}} \text{ we get } y(256) = 3$$

Hence, the correct answer is (A).

2. $f'(x) = 2 - \frac{f(x)}{x}$ or $\frac{dy}{dx} + \frac{y}{x} = 2$

Integrating factor $= e^{\int \frac{1}{x}dx} = e^{\ell nx} = x$

Solution is $yx = \int 2xdx = x^2 + c$

or $\quad y = x + \frac{c}{x}$ & $c \neq 0$ since $f(1) \neq 1$

$$\lim_{x\to0^+} f'\left(\frac{1}{x}\right) = \lim_{x\to0^+}(1-cx^2) = 1;\ \lim_{x\to0^+} xf\left(\frac{1}{x}\right) = \lim_{x\to0^+} 1+cx^2 = 1$$

$$\lim_{x\to0^+} x^2 f'(x) = \lim_{x\to0^+} x^2 - c = -c \neq 0;$$

$\lim_{x\to0^+} f(x) \to +\infty$ or $-\infty$ it is unbounded function

Hence, the correct answer is (A).

3. I.F. $= e^{\int \frac{x}{x^2-1}dx} = e^{\frac{1}{2}\int\frac{2x}{x^2-1}dx} = e^{\frac{1}{2}\ln|x^2-1|} = e^{\frac{1}{2}\ln(1-x^2)} = \sqrt{1-x^2}$ $x \in (-1, 1)$

$$\therefore \quad y\sqrt{1-x^2} = \int \frac{x^4+2x}{1-x^2}\times\sqrt{1^2-x^2}dx = \int x^4 + 2xdx$$

$$\Rightarrow \quad y\sqrt{1^2-x^2} = \frac{x^5}{5} + x^2 + c$$

Given $f(0) = 0, x = 0, y = 0 \Rightarrow c = 0$ $y = f(x) = \dfrac{\frac{x^5}{5}+x^2}{\sqrt{1-x^2}}$

$$\therefore \quad I = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx = \int_0^{\frac{\sqrt{3}}{2}}\left(\frac{\frac{x^5}{5}+x^2}{\sqrt{1-x^2}} + \frac{\frac{-x^5}{5}+x^2}{\sqrt{1-x^2}}\right)dx = 2\int_0^{\frac{\sqrt{3}}{2}}\frac{x^2}{\sqrt{1-x^2}}dx$$

Put $x = \sin\theta \Rightarrow dx = \cos\theta d\theta$

$$I = 2\int_0^{\frac{\pi}{3}}\frac{\sin^2\theta\cos\theta}{\cos\theta}d\theta$$

$$= \int_0^{\frac{\pi}{3}}(1-\cos 2\theta)d\theta = \left(\theta - \frac{1}{2}\sin 2\theta\right)\Bigg|_0^{\frac{\pi}{3}} = \frac{\pi}{3} - \frac{1}{2}\times\sin\frac{2\pi}{3} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

Hence, the correct answer is (B).

4. $\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}$. Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \quad \frac{dv}{\sec v} = \frac{dx}{x} \quad \Rightarrow \int \cos v\, dv = \int \frac{dx}{x}$$

$$\Rightarrow \quad \sin v = \ln x + c$$

$$\Rightarrow \quad \sin\left(\frac{y}{x}\right) = \ln x + c$$

The curve passes through $\left(1, \frac{\pi}{6}\right)$ then curve is $\sin\left(\frac{y}{x}\right) = \ln x + \frac{1}{2}$

Hence, the correct answer is (A).

Multiple Correct Choice Type Questions

5. Integrating factor $= e^{\alpha x}$

So $ye^{\alpha x} = \int xe^{(\alpha+\beta)x}\, dx$

CASE 1:

If $\alpha + \beta = 0$ $\qquad ye^{\alpha x} = \frac{x^2}{2} + c$

If passes through (1, 1) $\qquad \Rightarrow C = e^{\alpha} - \frac{1}{2}$

So $ye^{\alpha x} = \frac{x^2-1}{2} + e^{\alpha}$ $\qquad$ for $\alpha = 1$

$$y = \frac{x^2}{2}e^{-x} + \left(e - \frac{1}{2}\right)e^{-x}$$

CASE 2:

If $\alpha + \beta \neq 0$

$$ye^{\alpha x} = \frac{x.e^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{1}{\alpha+\beta}\, e^{(\alpha+\beta)x}\, dx$$

$$\Rightarrow \quad ye^{ax} = \frac{xe^{(\alpha+\beta)x}}{\alpha+\beta} - \frac{e^{(\alpha+\beta)x}}{(\alpha+\beta)^2} + c \quad \Rightarrow \text{So } c = e^{\alpha} - \frac{e^{(\alpha+\beta)}}{\alpha+\beta} + \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}$$

$$y = \frac{e^{\beta x}}{(\alpha+\beta)^2}\left((\alpha+\beta)\,x - 1\right) + e^{-\alpha x}\left(e^x - \frac{e^{\alpha+\beta}}{\alpha+\beta} + \frac{e^{\alpha+\beta}}{(\alpha+\beta)^2}\right)$$

If $\alpha = \beta = 1$

$$y = \frac{e^x}{4}\,(2x-1) + e^{-x}\left(e - \frac{e^2}{2} + \frac{e^2}{4}\right); \quad y = \frac{e^x}{2}\left(x - \frac{1}{2}\right) + e^{-x}\left(e - \frac{e^2}{4}\right)$$

Hence, (A) and (C) are correct.

6. $b \neq 0$

$$\int\frac{f'(x)}{f(x)}dx = \int\frac{1}{b^2+x^2}dx$$

$$\ln(x) = \frac{1}{b}\tan^{-1}\frac{x}{b} + C$$

$$f(x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}+C}; \quad f(0) = 1$$

$$f(0) = 1 = e^{o+c} = e^{c} = 1$$

$$\left.\begin{aligned} f(x) &= e^{\frac{1}{b}\tan^{-1}\frac{x}{b}} \\ f(-x) &= e^{\frac{1}{b}\tan^{-1}\left(\frac{-x}{b}\right)} = e^{-\frac{1}{b}\tan^{-1}\frac{x}{b}} \end{aligned}\right] f(x)\,f(-x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}.e^{-\frac{1}{b}\tan^{-1}\frac{x}{b}}$$

$$= e^{o} = 1$$

$$f(x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}} \quad \forall\, x \in R$$

$$f'(x) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}\left(\frac{1}{b}\cdot\frac{1}{1+\left(\frac{x}{b}\right)^2}\times\frac{1}{b}\right) = e^{\frac{1}{b}\tan^{-1}\frac{x}{b}}\left[\frac{1}{b^2+x^2}\right] > 0$$

$\forall\, b \in R; \forall\, x \in R$

Hence, (A) and (C) are correct.

7. Equation tangent $y-y_1=\frac{dy_1}{dx_1}(x-x_1)$, it cuts y-axis at

$$y_p\left(0, y_1-x_1\frac{dy_1}{dx_1}\right)$$

$$Py_p=\sqrt{x_1^2+\left(-x_1\frac{dy_1}{dx_1}\right)^2}=1$$

$$\Rightarrow \frac{dy}{dx}=\pm\frac{\sqrt{1-x^2}}{x}\Rightarrow xy'\pm\sqrt{1-x^2}=0$$

$$\Rightarrow \int dy=\pm\int\frac{\sqrt{1-x^2}}{x}dx$$

Put $1-x^2=t^2, -x\,dx=t\,dt$

$$y=\pm\int\frac{-t^2}{1-t^2}dt=\pm\left(\int 1dt-\int\frac{1}{1-t^2}dt\right)=\pm\left(t-\frac{1}{2}\ell n\left|\frac{1+t}{1-t}\right|\right)+c$$

$$y=\pm\left(\sqrt{1-x^2}-\frac{1}{2}\ln\left|\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right|\right)+c;$$

y is passing through $(1, 0)$ so $c=0$

$$y=\pm\left(\sqrt{1-x^2}-\frac{1}{2}\ell n\left(\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right)\right);$$

$$y=\pm\left(\sqrt{1-x^2}-\ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right)\right)$$

As curve $y=y(x)$ lies in the first quadrant

$$y=\ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right)-\sqrt{1-x^2} \text{ and } xy'+\sqrt{1-x^2}=0$$

So option A and B only satisfy. So AB are correct.

Hence, (A) and (B) are correct.

8. $f'(\text{x})=\text{e}^{f(\text{x})-\text{g}(\text{x})}\cdot g'(\text{x})$

$$\int -f'(\text{x})\cdot e^{-f(\text{x})}dx=\int -e^{-g(x)}g'(x)dx$$

$$e^{-f(\text{x})}=e^{-g(x)}+c$$

Given $f(1)=g(2)=1$

Put $x=1\Rightarrow c=\frac{1}{e}-\frac{1}{e^{g(1)}}$

Put $x=2; e^{-f(2)}=e^{-g(2)}+c\Rightarrow e^{-f(2)}=\frac{2}{e}-\frac{1}{e^{g(1)}}$ …(1)

Now $e^{f(2)}=\frac{e^{1+g(1)}}{2e^{g(1)}-e}>0\Rightarrow 2e^{g(1)}-e>0$

$\Rightarrow\quad 2e^{g(1)}>e\Rightarrow g(1)>1-\log_e 2$

Rewriting the relation (1)

$$e^{-g(1)}=\frac{2}{e}-\frac{1}{e^{f(2)}}\Rightarrow e^{g(1)}=\frac{e^{f(2)+1}}{2e^{f(2)}-e}>0\Rightarrow 2e^{f(2)}-e>0$$

$$2e^{f(2)}>e\Rightarrow f(2)>1-\log_e 2$$

Hence, (B) and (C) are correct.

9. $[(x+2)(x+2+y)]\frac{dy}{dx}-y^2=0$ rewriting it as

$$\frac{dy}{y}=\frac{yd(x+2)-(x+2)dy}{(x+2)^2}=d\left(\frac{-y}{x+2}\right)$$

On integration we get

$$\ln y=\frac{-y}{x+2}+c$$

It passes through the point (1, 3)

$\ln 3=-1+c$ or $c=1+\ln 3$

Required curve is $\ln\frac{y}{3}+\frac{y}{x+2}=1$

Line $y=x+2$ intersect it at point (1, 3)

Solving curve with $y=(x+2)^2$ we get $\ln\frac{(x+2)^2}{3}+\frac{(x+2)^2}{x+2}=1$

$\Rightarrow\quad 2\ln\frac{x+2}{\sqrt{3}}=-1-x$

$\Rightarrow\quad x+2=\sqrt{3}e^{\frac{-1-x}{2}}$ for $x>0$ no solution

Solving curve with $y=(x+2)^3$ we get $\ln\frac{(x+2)^3}{3}+\frac{(x+2)^3}{x+2}=1$

$\Rightarrow\quad \ln\frac{(x+2)^3}{3}+(x+2)^2=1$ but for $x>0$ $\ln\frac{(x+2)^3}{3}+(x+2)^2>4$ so No solution

Hence, (A) and (D) are correct.

10. Let the family of circles be $x^2+y^2-\alpha x-\alpha y+c=0$

On differentiation $2x+2yy'-\alpha-\alpha y'=0$

Again on differentiation and substituting 'α' we get

$$2+2y'^2+2yy''-\left(\frac{2x+2yy'}{1+y'}\right)y''=0$$

$\Rightarrow\quad (y-x)y''+y'(1+y'+y'^2)+1=0$

Now, compare it with given condition $Py''+Qy'+1=0$

We get $P=y-x$, $Q=1+y'+y'^2$

Now, check options

Hence, (B) and (C) are correct.

11. The given differential equation is

$$\frac{dy}{dx}-(\tan x)\,y=2x\sec x \quad \text{…(1)}$$

I.F. $=e^{-\int\tan x dx}=e^{-\ln(\sec x)}=\cos x$

Multiplying (1) by $\cos x$, we get $\frac{d}{dx}[y\cos x]=2x$

$\Rightarrow\quad y\cos x=x^2+c$

When $x=0$, $y=0$, thus

$0=0+c\Rightarrow c=0\quad\therefore\ y(x)=x^2\sec x$

Also, $y'(x)=(\tan x)\,y+2x\sec x=x^2\tan x\sec x+2x\sec x$

Now, $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{16}\left(\sqrt{2}\right)=\frac{\pi^2}{8\sqrt{2}}$ and $y'\left(\frac{\pi}{3}\right)=\frac{\pi^2}{9}\left(\sqrt{3}\right)(2)+2\left(\frac{\pi}{3}\right)(2)$

$$=\frac{4\pi}{3}+\frac{2\pi^2}{3\sqrt{3}}$$

Hence, (A) and (D) are correct.

Integer/Numerical Answer Type Questions

12. Rewriting the given differential equation

$$\int_0^y\frac{dy}{25y^2-4}=\int_0^x dx\Rightarrow\frac{1}{25}\int_0^y\frac{dy}{y^2-\left(\frac{2}{5}\right)^2}=x\Rightarrow\frac{1}{25}\cdot\frac{1}{2\cdot\frac{2}{5}}\ln\left|\frac{\frac{2}{5}-y}{\frac{2}{5}+y}\right|=x+c$$

Given $f(0)=0$; $c=0$

$$\ln\left|\frac{\frac{2}{5}-y}{\frac{2}{5}+y}\right|=20x\Rightarrow\frac{0.4-y}{0.4+y}=e^{20x}\Rightarrow-\frac{0.8}{2y}=\frac{e^{20x}+1}{e^{20x}-1}$$

$\Rightarrow\quad y=-0.4\left(\frac{e^{20x}-1}{e^{20x}+1}\right)$

$\Rightarrow\quad \lim\limits_{x\to-\infty} y=0.4$

Hence, the correct answer is (0.4).

CHAPTER 11